Binary Relation - Supplementary

COMP9020 Tutorial

JIAPENG WANG

24T3 Week4, H18B & F15A

1 Tutorial Outline

1.1 Definition

Ask yourself these questions below to test your understanding.
(If you find the content overwhelming, please check the sections in red first.)

1. What is the definition of the *Binary Relation*?
   - How to express it?
   - What properties does it have?
     - What is *Function*?
     - What are *Equivalence Relation* and *Partial Order*?
2. As for the **Equivalence Relation**:
   - Are there any special ways to represent it?
     - What is *Equivalence Class*?
   - What are the features of *Equivalence Relations*?
     - **Partitionable**
       * What is *partition*?
3. As for the **Partial Order**:
   - Are there any special ways to represent it?
     - What is the term *poset* (i,e., partial order set)?
     - What is *Hasse Diagram*?
   - What are the features of *Partial Orders*?
     - **Comparable** or not
       * What is *Total Order*? → Special Notation: *Linearity Graph*
       * What are *Minimal* and *Maximal* element in poset $(S, \preceq)$? Exist?
       * What are *Minimum* and *Maximum* element in poset $(S, \preceq)$? Exist?
         - Minimum → What is the term *woset* (i,e., well-ordered set)?
       * What are *Upper Bound* and *Lower Bound* for set A in poset $(S, \preceq)$?
         - What are $ub(A)$ and $lb(A)$?
         - What are *lub* and *glb*? → What is *Lattice*? *Complete Lattice*?
     - **Compatible** or not
       * What is *Topological Sorting*?
   - Special *Partial Orders* on *Cartesian Products* and *Languages*:
     - What is *Product Order*?
     - What is *Lexicographic Order*? → What is *Lenlex Order*?
4. What operations can be performed on binary relations?
   - Converse, (Pre-)Image; Composition

1.2 Brainstorming

The following questions are open and have no standard answers.

- What is the connection between *relations*, *functions*, and the 'operations'?
- Why do we study equivalence relations and partial orders?
- Why is it rigorous for a specific partial-order set to use Hasse diagrams to find the *upper bounds*, *lower bounds*(, etc.)?
- What kind of mathematical *relations* do you prefer? Why?
- ......

2 Explanation for Some Exercises

There are a lot of concepts this week. I hope some of them have already sunk in during our tutorials. This section is intended to supplement the parts I didn't explain clearly or didn't have time to cover in class.

In this supplementary material, I will aim to present the following as concisely and clearly as possible:

1. Key points on five properties (Ex4(c), Ex5(a))
2. Equivalence classes
3. Compatibility and Topological Sorting (Ex12)
4. Lattices (Ex10(b), Ex11(f))

2.1 Key Points on Five Properties

In actual teaching, I noticed that you might have the following misconceptions about (R), (AR), (S), (AS), and (T), so I'd like to clarify them here.

1. Generally, we assume the domain is non-empty.

   An empty domain is an extraordinary case, where the only possible relation is the empty relation, and due to *vacuous truth*, all these five properties are satisfied.

2. (S), (AS), (T) are conditional statements.[1]

   A conditional statement will hold if there is nothing which satisfies the 'if' part.

3. Properties have to hold for all elements.

   Don't make a *hasty generalization*.

   For the elements a, b, and c mentioned in the definition, they are arbitrary elements from the domain, which means they can also be the same.

4. Equivalence relations and partial orders are binary relations which satisfy specific properties.

   Using the definition, prove that a binary relation is
   Equivalence relations: Prove (R), (S), (T)
   Partial orders: Prove (R), (AS), (T)

Exercise 4c Which of the properties (R), (AR), (S), (AS), and (T) does R satisfy, where $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : |a - b| \leq 2\}$?

Analysis:
We just need to verify whether the following proposition is correct.

- (R): For any $a \in \mathbb{Z}$, $|a - a| \leq 2$?
- (AR): For any $a \in \mathbb{Z}$, $|a - a| \nleq 2$?
- (S): For any $a, b \in \mathbb{Z}$, if $|a - b| \leq 2$, then $|b - a| \leq 2$?
- (AS): For any $a, b \in \mathbb{Z}$, if $|a - b| \leq 2$ and $|b - a| \leq 2$, then $a = b$?
- (T): For any $a, b, c \in \mathbb{Z}$, if $|a - b| \leq 2$ and $|b - c| \leq 2$, then $|a - c| \leq 2$?

For the last property (T), we know from the *triangle inequality* that

$$|a - c| = |(a - b) + (b - c)| \leq |(a - b)| + |(b - c)| \leq 2 + 2 = 4.$$

By using the condition for equality, we can quickly identify a counterexample.

[1] In fact, (R) and (AR) are also if statements; however, since the domain is usually non-empty, we don't write them in the 'if..., then...' form.

Exercise 5a If a binary relation R on set A is (S) and (T), is R (R)?

Analysis:

To analyze this problem, let's start with the given conditions and consider: If a relation is symmetric (S), what does this relation look like?

Based on the definition of symmetry (don't forget vacuous truth), a relation that satisfies (S) can be divided into the following two cases:

- **Case 1:** There exists some pairs $(a, b) \in R$;
- **Case 1:** There doesn't exist any pair $(a, b) \in R$.

From **Case 1**, we have

$$\begin{aligned} \text{For any pair} \quad & (a, b) \in R \\ \Rightarrow\ & (b, a) \in R \quad \text{(By (S))} \\ \Rightarrow\ & (a, a) \in R \quad \text{(By (T))} \end{aligned}$$

Now, we need to consider whether such an a can be any element in the domain in order to prove whether the relation is reflexive.

However, we find that these a are only those elements which have this relation with some elements, ... **may not cover the entire domain**, which leads us to our first counterexample:

Counterexample 1.1: R on $\{1, 2\}$: $\{(1, 1)\}$, or
Counterexample 1.2: R on $\{1, 2, 3\}$: $\{(1, 2), (2, 1), (1, 1), (2, 2)\}$.

From **Case 2**,

Since there doesn't exist any pair $(a, b) \in R$, both 'if' parts of (S) and (T) are false, which implies the whole statements for both (S) and (T) are true.

At the same time, we observe that as long as **the domain is non-empty**, since no relations exist in R, (a, a) will certainly not hold. Therefore, this relation cannot be reflexive, which leads us to our second counterexample:

Counterexample 2: R on $\{1\}$: $\varnothing$.

With this, we have completed the analysis. If you provide any counterexample that is identical or similar to these, it will correctly disprove the claim.

2.2 Equivalence Classes

Suppose A is a set, R is an **equivalence relation** on A, and a, b are elements of A.

In the lecture, we proved the following property using *element proof*:

Lemma1: If $(a, b) \in R$, then $[a] = [b]$.

In the tutorial, we derived a deeper property based on this *Lemma1*.

Lemma2: Either $[a] \cap [b] = \varnothing$ or $[a] = [b]$.

Furthermore, based on these two properties, we can arrive at a very elegant property concerning equivalence relations. (The proof section is optional to review.)

Lemma3: The distinct equivalence classes of R form a **partition** of A; i.e., the union of the equivalence classes is all of A, and the intersection of any two distinct classes is empty.

Proof:

For notational simplicity, we assume that R has only a finite number of distinct equivalence classes, which we denote

$$A_1, A_2, \ldots, A_n,$$

where n is a positive integer.[a]

1. **Proof that** $A = A_1 \cup A_2 \cup \cdots \cup A_n$: (by *element proof*)

To show that $A \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$, suppose x is any element of A. By reflexivity of R, $x \, R \, x$. And this implies that $x \in [x]$ by definition of class. Since x is in *some* equivalence class, it must be in one of the distinct equivalence classes $A_1, A_2, \ldots, A_n$. Thus $x \in A_i$ for some index i, and hence $x \in A_1 \cup A_2 \cup \cdots \cup A_n$ by definition of union.

To show that $A_1 \cup A_2 \cup \cdots \cup A_n \subseteq A$, suppose $x \in A_1 \cup A_2 \cup \cdots \cup A_n$. Then $x \in A_i$ for some $i = 1, 2, \ldots, n$, by definition of union. Now each A_i is an equivalence class of R, and equivalence classes are subsets of A. Hence $A_i \subseteq A$ and so $x \in A$.

Thus, by definition of set equality, $A = A_1 \cup A_2 \cup \cdots \cup A_n$.

2. **Proof that the distinct classes of** R **are mutually disjoint**:

Suppose that A_i and A_j are any two distinct equivalence classes of R. Since A_i and A_j are distinct, then $A_i \neq A_j$. And since A_i and A_j are equivalence classes of R, there must exist elements a and b in A such that $A_i = [a]$ and $A_j = [b]$. By *Lemma2*, either $[a] \cap [b] = \varnothing$ or $[a] = [b]$.

Now $[a] \neq [b]$ because $A_i \neq A_j$, and hence $[a] \cap [b] = \varnothing$. Thus $A_i \cap A_j = \varnothing$, and so A_i and A_j are disjoint.

Thus, by the definition of partition, the proof is complete. $\square$

[a] When the number of classes is infinite, the proof is identical except for notation.

Lemma3 presents us with a very important point:

- It is always the case that the equivalence classes of an equivalence relation **partition the domain** of the relation into a union of mutually disjoint subsets.

This point is extremely useful when analyzing and thinking about problems related to equivalence relations - just like *'birds of a feather flock together'*.

An Example for Equivalence Classes

To give a simple and easily understandable example, suppose the domain A represents all humans, and the relation R is having the same personality type. Though the definition of this personality type is somewhat vague, we can quantify it, for instance, by **using a test and classifying individuals based on their final score**, such as MBTI personality type.

Let's use it as an example here, i.e., if two people, a and b, fall into the same score range (as defined by the official categorization) in the MBTI test, then $(a, b) \in R$.

It is not difficult to prove that R is an **equivalence relation** on A. (Why?)

Thus, by *Lemma2*, if you and I are related by R (i.e., we have the same personality type), then I must be in your equivalence class, and you must be in mine; if not related, there must be no intersection between our respective classes.

Lemma3 tells us that such an equivalence relation necessarily creates a partition of all humans. In the MBTI framework, personalities are divided into 16 types, such as INFP, ENFP, etc.[2] (similar to the sets $A_1, A_2, \ldots, A_{16}$ in our proof of *Lemma3*). As for me, after taking the test, my result corresponds to INFP, so I belong to this category, and [Jiapeng Wang] = {INFP}. If you are the same, you also fall into this category, and [You] = {INFP}. Thus, we can use a single label *'INFP'* to refer to this personality type, avoiding unnecessary repetition.

I hope this example helps you better understand *equivalence classes*.

[2]Additionally, such partitions can be described by certain classes of elements. In other words, for example, if there are 16 people with different personality types based on their test results, we could also 'name' these 16 personality types after their names.

2.3 Compatibility and Topological Sorting

The term *'Compatibility'* did not appear in our lecture; I am using it here only to help you better understand the concept of topological sorting.

> **Definition:**
> Given partial order relations $\preceq$ and $\preceq'$ on a set A, $\preceq'$ is **compatible** with $\preceq$ if, and only if, for every a and b in A, if $a \preceq b$, then $a \preceq' b$.

Intuitively, if one partial order **includes all the pairs** in another partial order, then the former must be compatible with the latter.

In our tutorial, I gave the following example:

Suppose $A = \{1, 2, 3, 4\}$ is a set, $\preceq_1, \preceq_2$ is two **binary relations** on A, satisfying:

$$\preceq_1 = \{(1,1),(2,2),(3,3),(4,4),(1,2),(3,4)\};$$

$$\preceq_2 = \{(1,1),(2,2),(3,3),(4,4),(1,2),(3,4),(2,3)\}.$$

It is not difficult to prove that $\preceq_1, \preceq_2$ are both **partial orders** on A. (Why?)

Then based on the definition provided above, we can conclude that:

- $\preceq_2$ is compatible with $\preceq_1$;
- $\preceq_1$ is not compatible with $\preceq_2$ (because $(2,3) \in \preceq_2$ but $(2,3) \notin \preceq_1$).

Through the concept of compatibility, we can gain a better understanding of **topological sorting**.

> **Note:**
> Given partial order relations $\preceq$ and $\preceq'$ on a set A, $\preceq'$ is a **topological sorting** for $\preceq$ if, and only if, $\preceq'$ is a total order that is compatible with $\preceq$.

In other words, if a total order is compatible with a partial order, then the total order is a topological sorting of the partial order. (Please note that the essence of topological sorting is **a total order relation**.)

Topological sorting is an important concept in computer science because our input is sequential, and time is linear. As a result, the input order in relation to the sequence of events over time must be a total order. Therefore, for a partial order, it is crucial to find a compatible total order. (For example, in data management, we have the concept of *schedule*.)

Additionally, for this very reason, **topological sorting is not necessarily unique**; it is sufficient to find one feasible solution.

Exercise 12 For the poset $(Pow(\{a,b,c\}), \subseteq)$, find a topological sort.

Analysis:

From the perspective of compatibility, this problem is equivalent to this scenario:

- Is it possible to input the elements of $\mathcal{P}(\{a,b,c\})$ into a computer in a way that is *compatible* with the subset relation $\subseteq$ in the sense that if set U is a subset of set V, then U is input before V?

The answer, as it turns out, is yes. For instance, the following input order satisfies the given condition:

$$\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}.$$

Another input order that satisfies the condition is

$$\varnothing, \{a\}, \{b\}, \{a,b\}, \{c\}, \{b,c\}, \{a,c\}, \{a,b,c\}.$$

Of course, there may be other cases as well. You only need to provide one (note that a total order can be represented by a *linear graph*), i.e.:

Answer 1: $\varnothing \leq \{a\} \leq \{b\} \leq \{c\} \leq \{a,b\} \leq \{a,c\} \leq \{b,c\} \leq \{a,b,c\}$.
Answer 2: $\varnothing \leq \{a\} \leq \{b\} \leq \{a,b\} \leq \{c\} \leq \{a,c\} \leq \{b,c\} \leq \{a,b,c\}$.
Answer 3: ...

Personally, I prefer the *second answer* because it is easy to make the computer recognize it by assigning values, such as letting a = 001, b = 010, and c = 100 (based on the *binary system*).

We also have general methods for computers to find a topological sort, though they are not very friendly for humans (much like the Hasse diagram shown in tutorial slides). However, if you are completely unsure while solving a such question, you might consider the following steps:

Algorithm:

Let $\preceq$ be a partial order relation on a **nonempty finite** set A.
To construct a topological sorting:

1. Pick any minimal element x in A.(Such x exists since A is nonempty.)
2. Set $A' := A - \{x\}$.
3. Repeat steps a–c while $A' \neq \varnothing$.
   (a) Pick any minimal element y in A'.
   (b) Define $x \preceq' y$.
   (c) Set $A' := A' - \{y\}$ and $x := y$.

2.4 Lattices

I'm planning to mention the concept of *lattices* in next week's tutorial since it's closely related to *Boolean algebra*. For now, please focus on understanding its definition and being able to use that definition to determine whether a given partial order is a lattice.

Exercise 10b Is this poset $(\{2, 4, 6, 9, 12, 36, 72\}, |)$ a lattice?

Analysis:

From 10 a), we know glb($\{6, 9\}$) doesn't exist in this poset. Thus, this poset is not a lattice.

Exercise 11f Is this poset $(\mathrm{Pow}(\{a, b, c\}), \subseteq)$ a lattice?

Analysis 1:

For this problem, a brute-force approach is possible. Since there are only eight elements in this poset, we have a total of 64 cases. (Fortunately, they're not too much!) By listing each one and confirming that their *lub* and *glb* exist, we can complete the proof.

Another way to approach this, after the setup in the previous questions, we're led to wonder whether

> Let A be a set. For any two elements in poset $(\mathrm{Pow}(A), \subseteq)$,
>
> - $\mathrm{lub}(\{X, Y\}) = X \cup Y$, and $\mathrm{glb}(\{X, Y\}) = X \cap Y$.

Such hypotheses are always sparked by certain conclusions; once we have a hypothesis, we need to either prove or disprove it. In fact, this hypothesis is correct. **Let's analyze** $\mathrm{lub}(\{X, Y\}) = X \cup Y$ **as an example.**

Analysis 2:

The most natural approach is, of course, an *element proof*, since this is essentially a proof of set equality. Therefore, we break this problem down into:

- To show $\mathrm{lub}(\{X, Y\}) \subseteq X \cup Y$;
- To show $X \cup Y \subseteq \mathrm{lub}(\{X, Y\})$.

However, when we consider an element $x \in \mathrm{lub}(\{X, Y\})$, we realize from the definition of lub that $\mathrm{lub}(\{X, Y\})$ is a (minimum) element of upper-bound set. Then x is an element of an element, what does it mean? Also, there are no other further conditions to reason from.

Therefore, we must abandon using the *element proof* proving subset relation and start analyzing from the definition of lub itself: (Please focus on the parts in red. They correspond to the relevant properties.)

Property 1: (least/ minimum)

$$\begin{aligned} & \text{lub}(\{x,y\}) \text{ is the minimum elements in the } ub \text{ set} && \text{(def of lub)} \\ \Rightarrow\ & \text{If } a \in ub(\{x,y\}), \text{ then } \text{lub}(\{x,y\}) \preceq a && \text{(def of minimum)} \end{aligned}$$

Property 2: (upper bound)

$$\begin{aligned} & \text{lub}(\{x,y\}) \text{ is one of the upper bound} && \text{(def of lub)} \\ \Rightarrow\ & \text{lub}(\{x,y\}) \in \{t : a \preceq t \text{ for all } a \in \{x,y\}\} && \text{(def of } ub \text{ set)} \\ \Rightarrow\ & x \preceq \text{lub}(\{x,y\}) \text{ and } y \preceq \text{lub}(\{x,y\}) && \text{(def of set)} \end{aligned}$$

Notice that the partial order relation $\preceq$ in this question is $\subseteq$, we can proceed with the following reasoning:

1. To show $\text{lub}(\{X,Y\}) \subseteq X \cup Y$**:**

$$\begin{aligned} \text{Notice that}\quad & X \subseteq (X \cup Y) \text{ and } Y \subseteq (X \cup Y) && \text{(def of lub)} \\ \Rightarrow\ & (X \cup Y) \in \{x : a \subseteq x \text{ for all } a \in \{X,Y\}\} && \text{(def of set)} \\ \Rightarrow\ & (X \cup Y) \in \text{ub}(\{X,Y\}) && \text{(def of } ub \text{ set)} \\ \Rightarrow\ & \text{lub}(\{X,Y\}) \subseteq X \cup Y && \text{(Property 1)} \end{aligned}$$

2. To show $X \cup Y \subseteq \text{lub}(\{X,Y\})$**:**

$$\begin{aligned} \text{For any}\quad & x \in (X \cup Y) \\ \Rightarrow\ & x \in X \text{ or } x \in Y && \text{(def of } \cup) \\ \Rightarrow\ & x \in \text{lub}(\{X,Y\}) \text{ or } x \in \text{lub}(\{X,Y\}) && \text{(Property 2, def of } \subseteq) \\ \Rightarrow\ & x \in \text{lub}(\{X,Y\}) && \text{(def of } \mathit{or}) \end{aligned}$$

Thus, we finish the proof for $\text{lub}(\{X,Y\}) = X \cup Y$. Our approach to thinking about $\text{glb}(\{X,Y\}) = X \cap Y$ is similar.

Having established these two conclusions, we only need to prove that if X and Y are elements of $\text{Pow}(A)$, then $X \cup Y$ and $X \cap Y$ are also in $\text{Pow}(A)$. If so, this poset is a lattice. (With the knowledge of set theory, this doesn't seem too difficult...)

As for how to write it, please refer to the solution provided in the materials. I hope this analysis helps you understand the proof in the solution;)