Set Theory - Supplementary

COMP9020 Tutorial

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1 Tutorial Outline

1.1 Definition

Ask yourself these questions below to test your understanding.

- What is the definition of the *subset* and *equality* relations of sets?
  - What is the *basic method* for proving them (respectively)?
  - What other methods do we have for proving or analysing them?
- What are the *empty set* and the *universe*?
  - What is the relationship between them?
- What operations can be performed on sets?
  - What are the definitions of them?
  - (How to do when we encounter *set difference*?)
- What do the following Set Theory Laws entail?
  - **Five basic laws:** Commutative, Associative, Distributive, Identity, and Complement Law
  - **Derive laws:** Idempotence, Double Complement, Annihilation, Absorption, and De Morgan's Law
  - What are **two useful results** mentioned in our lecture?
- How to calculate the *cardinality* of a *finite* set?

1.2 Brainstorming

The following questions are open and have no standard answers.

- What can the elements of a set be? Are there any restrictions?
- What is the connection between *element proof* and *proof by laws*?
- Why aren't Venn Diagrams rigorous for proofs?
- ......

2 Explanation for Some Exercises

This section is intended to supplement the parts I didn't explain clearly or didn't have time to cover in class. If there are any proofs you provided that are more concise than mine, I will include them as well.

Exercise 7: For sets A and B, prove or disprove $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

Analysis[1]:

1. To show $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$:

$$\begin{aligned}
\text{For any element} \quad & x \in \mathcal{P}(A \cup B) \\
& \boxed{\begin{aligned} & \Rightarrow x \subseteq A \cup B \\ (?) & \Rightarrow x \subseteq A \text{ or } x \subseteq B \end{aligned}} \\
& \Rightarrow x \in \mathcal{P}(A) \text{ or } x \in \mathcal{P}(B) \\
& \Rightarrow x \in \mathcal{P}(A) \cup \mathcal{P}(B).
\end{aligned}$$

2. To show $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$:

$$\begin{aligned}
\text{For any element} \quad & x \in \mathcal{P}(A) \cup \mathcal{P}(B) \\
& \Rightarrow x \in \mathcal{P}(A) \text{ or } x \in \mathcal{P}(B) \\
& \boxed{\begin{aligned} & \Rightarrow x \subseteq A \text{ or } x \subseteq B \\ (?) & \Rightarrow x \subseteq A \cup B \end{aligned}} \\
& \Rightarrow x \in \mathcal{P}(A \cup B).
\end{aligned}$$

For the content in these two boxes, we don't have a direct rule to apply. Therefore, we need to consider whether they are correct. (Please note that the essence of x is a set.)

The second box is **correct**.

$$\begin{aligned}
\text{For any element} \quad & x' \in x \\
& \Rightarrow x' \in A \text{ or } x' \in B \\
& \Rightarrow x' \in A \cup B.
\end{aligned}$$

But when we consider the first box, it **cannot be directly proven**.

$$\begin{aligned}
\text{For any element} \quad & x' \in x \\
& \Rightarrow x' \in A \cup B \\
& \Rightarrow x' \in A \text{ or } x' \in B.
\end{aligned}$$

This is because x' is a particular but arbitrary element of x, and we have shown that it may be in A or in B. This means that some elements of the set x are in A, while others are in B. Therefore, $x \subseteq A$ or $x \subseteq B$ may both not hold simultaneously.

Additionally, we have also identified the characteristics of a counterexample—some elements in $A \cup B$ belong **only** to A, while others belong **only** to B.

[1]Please note that if you can directly find a counterexample, such thinking is unnecessary. **This analysis is only for students who totally have no idea how to approach the problem.**

Disproof:

Counterexample: Let $A = \{1\}$, $B = \{2\}$. Then

$$\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \neq \{\emptyset, \{1\}, \{2\}\} = \mathcal{P}(A) \cup \mathcal{P}(B). \square$$

Exercise 12: Let A, B, C be languages with alphabet $\{a, b\}$ where $AC = BC$. Prove or disprove $A = B$.

Analysis:

This is a problem that requires constructing a counterexample, so it is quite challenging. Additionally, we don't have a general approach to thinking about this type of problem. Here, I will just share my own analysis.

I initially tried to **prove** that this was a *correct* conclusion. Starting with a simple case, I assume C is a finite set. Then, I followed with an *element proof*, but the condition $AC = BC$ made it difficult to handle the condition $x \in A$ or $x \in B$.

Therefore, I sought to improve this approach. Fortunately, I quickly discovered a useful property: if $AC = BC$, then the shortest words in A and B must be the same.[2] Thus, I **refined** the element analysis into this algorithm:

Given that $AC = BC$, assume $C = \{c_1, c_2, \ldots, c_n\}$.

- The shortest word w in A and B is the same;
- remove w, obtaining A' and B';
- prove $A'C = B'C$;
- the shortest word w' in A' and B' is the same;
- remove w', obtaining A'' and B'';
- ... and so on.

This provides a method to reduce A and B while maintaining $AC = BC$. Therefore, if A and B are also finite sets, this algorithm comes to an end in a finite step and implies A and B must contain the same element.

However, I found it difficult to prove $A'C = B'C$, because when we remove w, we cannot ensure that $wc_1, wc_2, \ldots, wc_n$ are simultaneously removed or retained in both AC and BC.

After a long time, I realized that this indeed cannot be held. Thus, I **shifted** from proving to disproving and attempted to construct a counterexample through this breakthrough[3], which led to the following disproof.

Disproof:

Counterexample:

Let $A = \{\lambda, aa\}$, $B = \{\lambda, a, aa\}$, and $C = \{a, aa\}$. Then $A \neq B$ and

$$AC = \{a, aa, aaa, aaaa\} = BC. \square$$

[2]This proposition is not obvious, and A, B, and C don't have to be finite sets. I will leave it to you, and if you're interested in exploring, you can try to prove it:)

[3]When constructing, please note that the fact that the shortest words in A and B are the same still holds. Through the proof of this proposition, we can also deduce that if A, B, and C are finite sets, the longest words in A and B are also the same.

Exercise 13: Let L_1 and L_2 be languages over Σ. Prove that

$$(L_1 \cap L_2)^* \subseteq L_1^* \cap L_2^*.$$

Analysis:

As we analyzed in our tutorial, I will directly list the process here.

$$
\begin{aligned}
\text{For any element} \quad & x \in (L_1 \cap L_2)^* \\
& \Rightarrow x \in (L_1 \cap L_2)^0 \cup (L_1 \cap L_2)^1 \cup \dots && \text{(def of Kleene Star)} \\
& \Rightarrow x \in (L_1 \cap L_2)^0 \text{ or } (L_1 \cap L_2)^1 \text{ or } \dots && \text{(def of } \cup\text{)} \\
& \Rightarrow x \in (L_1 \cap L_2)^k \text{ for some integer } k && \text{(def of } \textit{or}\text{)} \\
& \Rightarrow x = w_1 w_2 \cdots w_k, \text{ where each } w_i \in L_1 \cap L_2 && \text{(prop of Concatenation)} \\
& \Rightarrow w_i \in L_1 \text{ and } w_i \in L_2 && \text{(def of } \cap\text{)} \\
& \Rightarrow x \in L_1^k \text{ and } x \in L_2^k && \text{(prop of Concatenation)} \\
& \Rightarrow (x \in L_1^0 \text{ or } L_1^1 \text{ or } \dots) \text{ and } (x \in L_2^0 \text{ or } L_2^1 \text{ or } \dots) && \text{(def of } \textit{or}\text{)} \\
& \Rightarrow (x \in L_1^0 \cup L_1^1 \cup \dots) \text{ and } (x \in L_2^0 \cup L_2^1 \cup \dots) && \text{(def of } \cup\text{)} \\
& \Rightarrow x \in L_1^* \text{ and } x \in L_2^* && \text{(def of Kleene Star)} \\
& \Rightarrow x \in L_1^* \cap L_2^* && \text{(def of } \cap\text{)}
\end{aligned}
$$

The content of *prop of Concatenation*[4] is:

> Given that A is a language and n is an integer.
>
> - For any word $w \in A^n$, w can be expressed as $w_1 w_2 \dots w_n$, where $w_i \in A$.
> - Conversely, if any word w can be expressed as $w_1 w_2 \dots w_n$, where $w_i \in A$, then $w \in A^n$.

Proof:

Please refer to the solution provided in the materials.

[4] We briefly discussed its proof in our tutorial, and I hope you still remember it;) In fact, after learning **mathematical induction**, we can prove this property in a more rigorous and formal way.