Measurement of Q Factor, Damping and Phase for RLC Circuits

Jeremy Reiff*

University of Pittsburgh, Pittsburgh, PA 15213

(Dated: March 28, 2019)

Abstract

In this experiment, I use an RLC circuit and attempt to measure the damping factor, angular frequency, resonant frequency, and Q-factor in the case of transient response, and I sweep across AC frequencies, mainly in the kHz range to measure the Q-factor, damping factor, and resonant frequency in the steady-state case. I also use measurements of the Full Width at Half-Maximum to estimate the Q-factor and damping factor. The transient response data yielded imprecise but consistent estimates for the aforementioned quantities, the automated frequency sweep yielded results which adhered to the predicted transfer function for RLC circuits, and the predictions made using the Full Width at Half-Maximum also were supported by the data.

I. INTRODUCTION

RLC circuits are commonly called *Resonant circuits* because they exhibit a sharp peak, or resonance in the frequency spectrum around a frequency ω_0:

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{1}$$

This is fundamentally due to the nature of capacitors and inductors, both of which have complex impedance, the phase of which depends on input frequency. When we use a step function or square wave for the input function, the output exhibits *transient* responses, with a unique frequency and an amplitude which decreases over time. We relate this to a damping factor γ:

$$\gamma = \frac{2D}{\tau} = \frac{R}{L} \tag{2}$$

where D is the natural logarithm of the ratio of succesive voltage peaks' magnitudes, and τ is the period of oscillation. The resonance peak is most easily seen in the steady state (smooth AC input), and the sharpness of the peak can be described by the Q-factor:

$$Q = \frac{\omega_0}{\gamma} \tag{3}$$

Another important quantity for an RLC circuit is the Full Width at Half Maximum, which is the difference between the two frequencies at which the response is $\frac{1}{\sqrt{2}}$ times the maximum response. This quantity, known as $\Delta\omega$, can be calculated by:

$$\Delta\omega = \gamma = \frac{\omega_0}{Q} = \frac{R}{L} \tag{4}$$

II. APPARATUS

The transient response of the RLC series circuit was measured through a setup as in Figure 2. The function generator sends a signal to the RLC circuit and sends that same signal directly to the oscilloscope for comparison to the output. The RLC circuit consists of a variable inductor sending a signal to a variable capacitor, which sends its signal to the positive pin of a variable resistor. From this positive pin, the output is read on the oscilloscope. The ground from the function generator, the negative pin on the resistor, and the negative oscilloscope contact on the channel reading the RLC output are all in contact to establish a common ground.

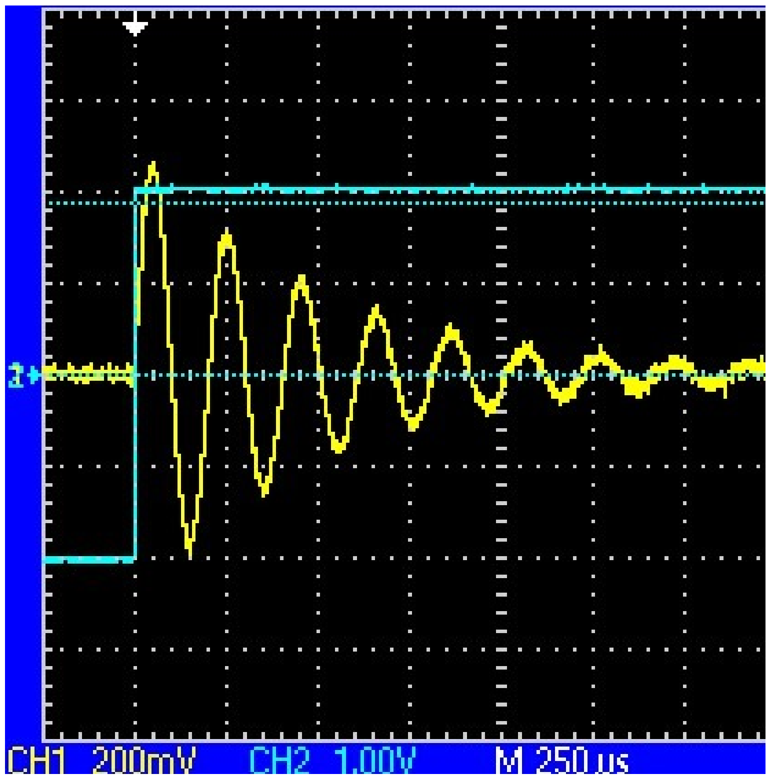


FIG. 1: The oscillating output is a transient oscillation resulting from the square wave input, and the peaks of the output decay according to the damping factor. Note that the output's vertical spacing is 5 times finer than the input's.

For the lock-in amplifier, which takes transfer and phase measurements at a single frequency, and for the network analyzer, which takes transfer and phase measurements over a range of frequencies, the oscilloscope and function generator are removed from the apparatus. A single Arduino board is used, which can be used as a function generator to send the desired frequency to the RLC circuit. This frequency is also used to compare to the output of the RLC circuit, which is also sent back to the Arduino. The board takes voltage measurements and we used Pythics software to give us output voltage and phase measurements.

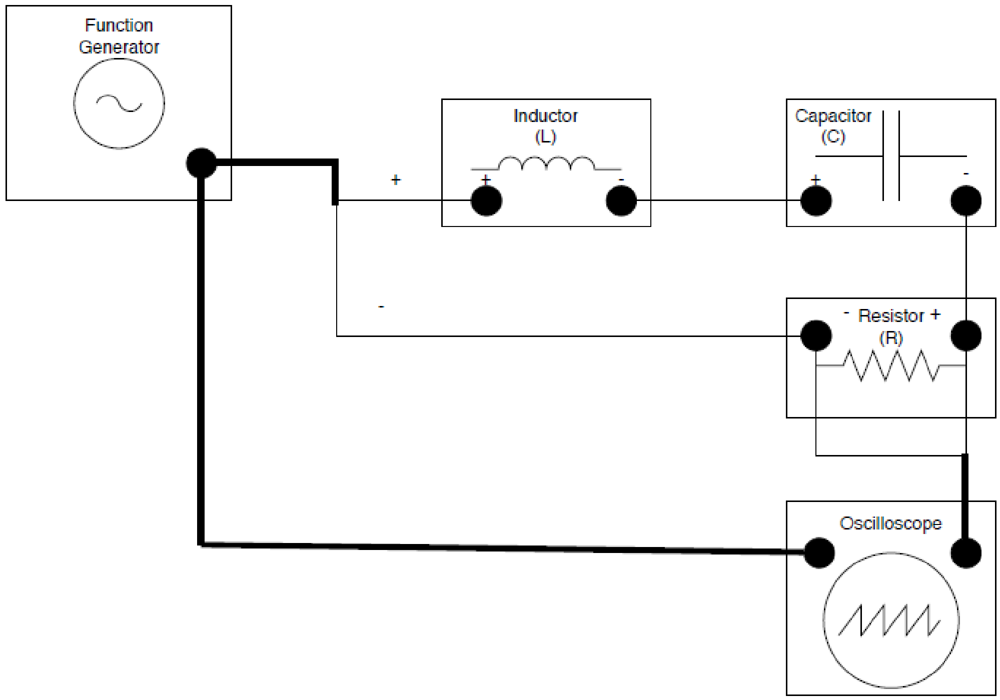


FIG. 2: RLC configuration for measuring the input (from the function generator) and the output of the RLC circuit. Heavily weighted lines are coaxial cables.

In these cases, the RLC circuit was connected in series just as before.

III. METHODS

The first measurements we took were real value measurements for the settings we would use on all of the instruments. The resistor, capacitor, and inductor's real values were measured at each of the settings we used in the experiment. For the capacitor and inductor, we took measurements of **C** and **L** respectively in the 100Hz, 1kHz, and 10kHz ranges. The effects over these ranges were significant in many of the cases we chose to analyze, but their impact (usually $\leq 1\%$) didn't seem to warrant more attention than averaging their values and treating them as constant. The series resistance of the inductor was also measured.

From these measurements, predictions for the resonant frequency and the damping factor were made for the RLC circuits we analyzed using the function generator and oscilloscope, predictions for the resonant frequency and quality factor were made for the RLC circuits we analyzed using the network analyzer, and predictions for the quality factor and damping

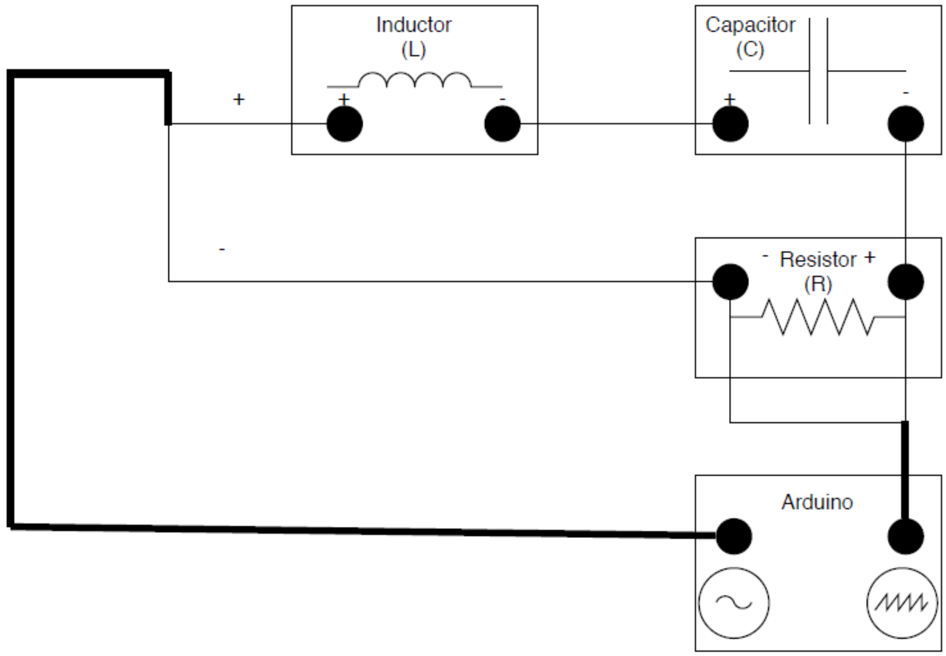


FIG. 3: For lock-in amplifier and network analyzer measurements, the function generator and oscilloscope are replaced by a single Arduino board. Heavy lines are coaxial cables.

factor were made for the RLC circuits we analyzed using the lock-in amplifier.

Data was first taken using the setup from Figure 2. A square wave of 100Hz frequency and 1 Volt peak-to-peak magnitude was sent from the function generator to both the oscilloscope and the RLC circuit. The output of the RLC circuit was displayed on the oscilloscope as well, and an image of the trace at the beginning of the step was taken from the digital oscilloscope and analyzed.

We then took data from the system using an Arduino board for analysis in a few ways, two of which were efficient and accurate enough to perform analysis. The first method was by using the external function generator, and simply using the Arduino in place of the oscilloscope, and we measured $\frac{V_{out}}{V_{in}}$, but it was much easier and more efficient to use the Arduino as the source of the frequency, like in the case where we used the lock-in amplifier.

The lock-in amplifier data collection was done using the Arduino as the source of a single frequency, running it through the RLC circuit, sending it back to the Arduino, and comparing this signal to the source signal, yielding $\frac{V_{out}}{V_{in}}$ information as well as phase in degrees. We tested a few combinations of **R**, **L**, and **C**, changing the frequency by small amounts until $\frac{V_{out}}{V_{in}}$ reached a maximum, and then recording its value at increasing frequencies until we reach -3dB, where $\frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{2}}$.

For taking data with the network analyzer, the wiring was nearly the same as for the lock-in amplifier, except the Arduino now sweeps through frequencies, the range for which was 0–100kHz. Frequencies are sent, in ascending order with a spacing of 40Hz, for .137 seconds each. Voltage and phase information is calculated within the Pythics software and recorded.

IV. RESULTS & ANALYSIS

A. Transient Oscillations

The case we analyzed for transient oscillations is an underdamped RLC circuit, the response for which can be seen in Figure 1. After adjusting for real measurements, the component values were $R = 2022.7\Omega \pm 0.1\Omega$, $L = 495.5\text{mH} \pm 0.1\text{mH}$, $C = 2.03\text{nF} \pm 0.02\text{nF}$, with the errors coming from variation on the multimeter's display. Using Eq. 1, I predicted $\omega_0 = 31517.7\frac{\text{rad}}{\text{s}} \pm 155.292\frac{\text{rad}}{\text{s}}$. In order to predict Q, I first found $\gamma = 4078.85 \pm 0.85\frac{\Omega}{\text{H}}$ using Eq. 2. Then I predicted $Q = 7.7271 \pm 0.038$ using Eq. 3. In order to measure ω_0, a quantity α, the angular frequency of transient oscillations, must be introduced:

$$\alpha = \frac{2 \cdot \pi}{\tau} \tag{5}$$

I measured tau on the oscilloscope trace in Figure 1 to be 190 ± 10µs, with the uncertainty's magnitude dominated by the difficulty of measuring maxima on an oscilloscope trace. Then, $\alpha = 33069.4 \pm 277.008\frac{\text{rad}}{\text{s}}$. I can then obtain ω_0 by the equation:

$$\omega_0 = \sqrt{\alpha^2 + \frac{\gamma^2}{4}} \tag{6}$$

The damping factor γ is found by $\gamma = \frac{2D}{\tau}$, where the constant $D = \ln\frac{V_n}{V_{n+1}}$, and V_n is the voltage at the peak of the nth transient oscillation. I measured $D = 0.408 \pm 0.02$ by using the oscilloscope trace. Then $\gamma = 4298.47 \pm 631.168\frac{\Omega}{\text{H}}$. Finally, I used Eq. 6 to find $\omega_0 = 33139.2 \pm 277.182\frac{\text{rad}}{\text{s}}$ and Eq. 3 to find $Q = 7.710 \pm 1.13$. The resonance frequency is 5.85 times its uncertainty away from the predicted value of $31517.7\frac{\text{rad}}{\text{s}}$, which I attribute to poor estimation of **D** or τ, because it is difficult to determine exact maximum voltages and time positions with a thick weighted line. I think accuracy was likely more of an issue than precision in this case. However, my measured value of the Q factor was only 0.015 times its uncertainty away from its predicted value, but this is likely due to the large uncertainty in the measured values and not necessarily extreme accuracy.

B. Lock-In Amplifier

Using the Lock-in Amplifier, I gathered data on $\frac{V_{out}}{V_{in}}$ and the phase for a few RLC setups at many frequencies. One of the RLC combinations was notable, and the analysis I performed was the prediction and measurement of the Q factor and the damping factor γ. To find gamma in this case, I used Eq. 4 and the relation:

$$\Delta\omega = \omega_2 - \frac{\omega_0^2}{\omega_2} \tag{7}$$

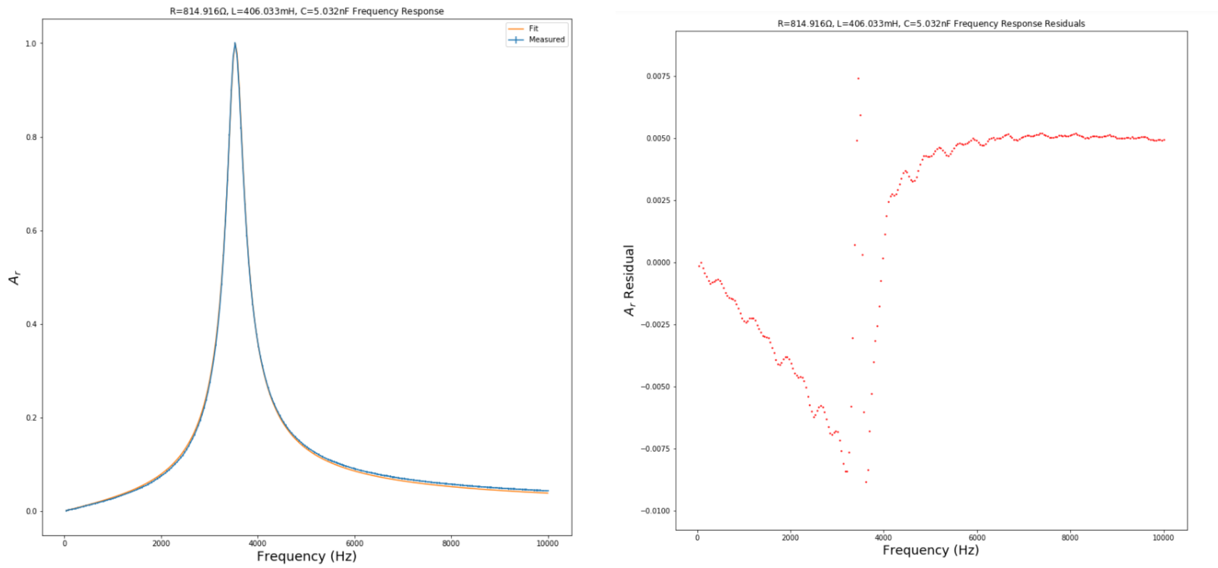
where ω_2 is the frequency above resonance at which $\frac{V_{out}}{V_{in}}$ is reduced to $\frac{1}{\sqrt{2}}$ its value at ω_0. I measured a value for ω_0 by scanning frequencies every 100Hz, scanned frequencies for ω_2 and then calulated $\Delta\omega = \gamma$. A circuit with values $R = 2022.606 \pm 0.1\Omega$, $C = 2.034 \pm 0.01$nF, and $L = 511.533 \pm 0.1$mH produced the experimental results seen in Table I. Even though

TABLE I: First RLC Lock-In Results

Frequency (kHz)	$\frac{V_{out}}{V_{in}}$ (± 0.0005)
4.92615	0.892 (max)
4.72534	0.746
4.62708	0.633
4.61639	0.627
5.12695	0.753
5.22522	0.649
5.2359	0.643
5.3000	0.631

$\frac{V_{out}}{V_{in}}$ is only 0.892, this is its maximum, and so I estimated $\omega_0 = 30951.9 \pm 100\pi\frac{\text{rad}}{\text{s}}$ and $\omega_2 = 33300.9 \pm 140\pi\frac{\text{rad}}{\text{s}}$. Using Eq. 7 I calculated $\gamma = 4530.18 \pm 823.9\frac{\Omega}{\text{H}}$ and using Eq. 3 I calculated $Q = 6.832 \pm 1.245$. While ω was very far off the predicted $33300\frac{\text{rad}}{\text{s}}$, the estimate for γ was only .699 times its uncertainty away from the predicted $3954.2\frac{\Omega}{\text{H}}$, leading me to believe there was some sort of systemic shift in the values for the voltage transfer which shifted the resonance measurement but preserved the FWHM, but I am not sure what it is.

FIG. 4: The response and residuals for the circuit $R = 814.916\Omega$, $L = 406.033\text{mH}$, $C = 5.032\text{nF}$.



Our estimate of Q was also only 0.810 times its uncertainty from the predicted 7.84, but it's important to note that because we didn't scan frequecies with fine spacing, we have large uncertainties in this case.

C. Network Analyzer

With the network analyzer performing a sweep over frequencies, its possible to reliably model the response of the RLC circuit over time. The magnitude of the response of the RLC circuit follows the equation:

$$\frac{V_{out}}{V_{out(max)}} = A_r = [1 + Q^2(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})^2]^{-\frac{1}{2}} \tag{8}$$

If we know ω_0 and the Q factor, we can also model the phase angle:

$$\phi_r = -arctan[Q(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})] \tag{9}$$

The network analyzer gives values for ϕ_r and A_r according to frequency, so I predicted the Q-factor and resonance frequency of a few different circuits and then ran a program to fit the network analyzer data to Eq. 8, and compared those parameters to my predictions. I also fit those generated parameters to the phase function to see how well my fit predicts the phase that I measured.

FIG. 5: The phase and residuals for the circuit $R = 702.559\Omega$, $L = 49.77$mH, $C = 40.06$nF.

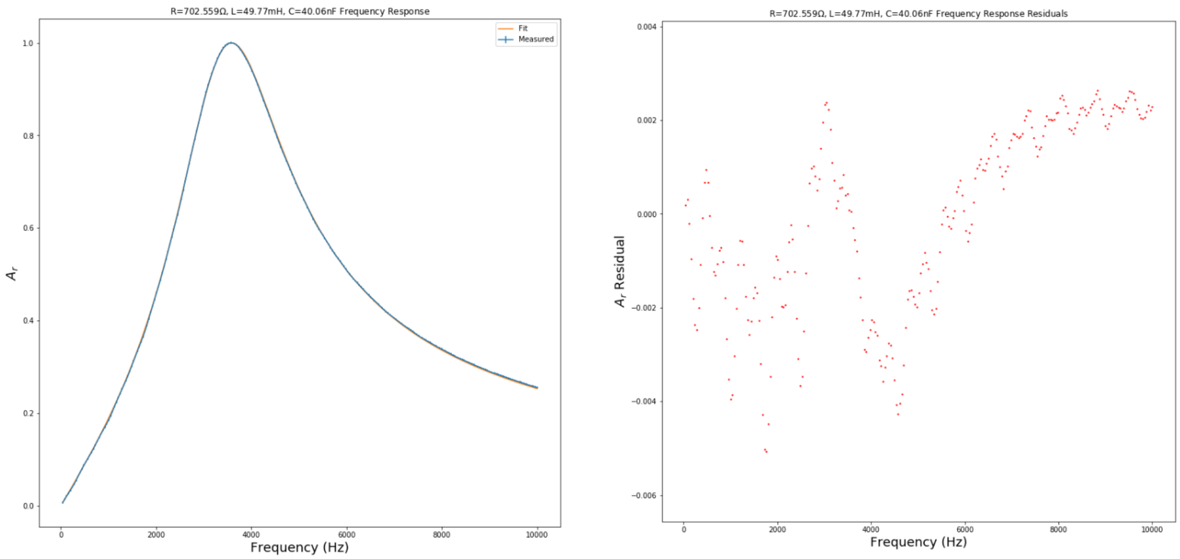


FIG. 6: The phase and residuals for the circuit $R = 6011\Omega$, $L = 251.63$mH, $C = 50.14$nF.

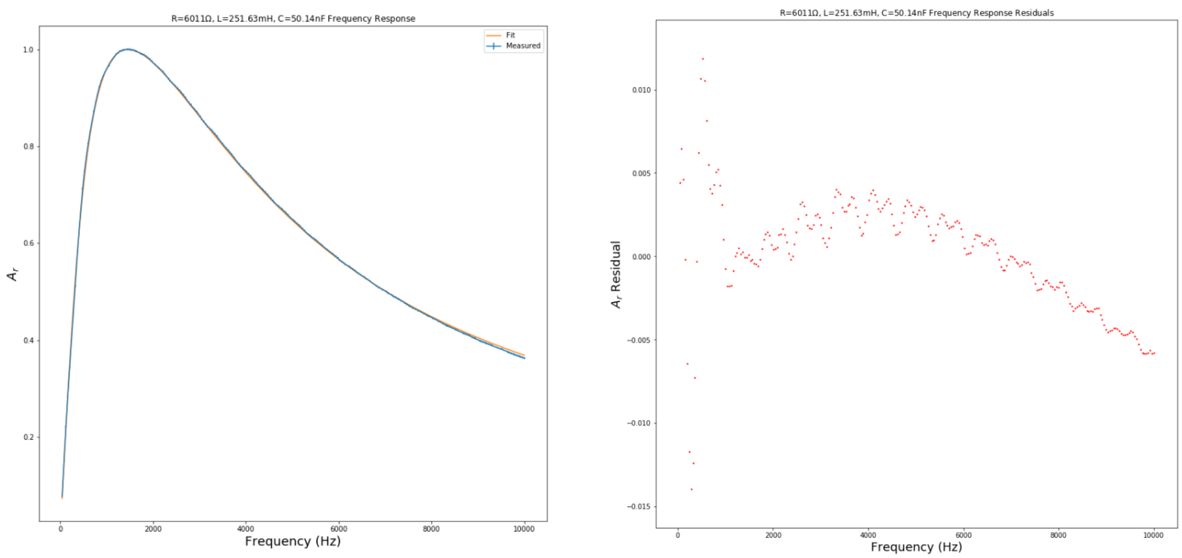


TABLE II: Fit parameters for the RLC circuits

$R(\Omega)$	L(mH)	C(nF)	$\omega_0(Hz)$	Q	χ^2	χ^2_{red}
702.559	49.77	40.06	3585.87 ± 4.33	1.57 ± 0.01	1398.15	5.66
814.916	406.033	5.032	3538.314 ± 3.83	10.49 ± 0.02	262.23	1.06
6011	251.63	50.14	1459.517 ± 9.46	$0.376 \pm .01$	737.91	2.99

FIG. 7: The phase and residuals for the circuit $R = 814.916\Omega$, $L = 406.033$mH, $C = 5.032$nF.

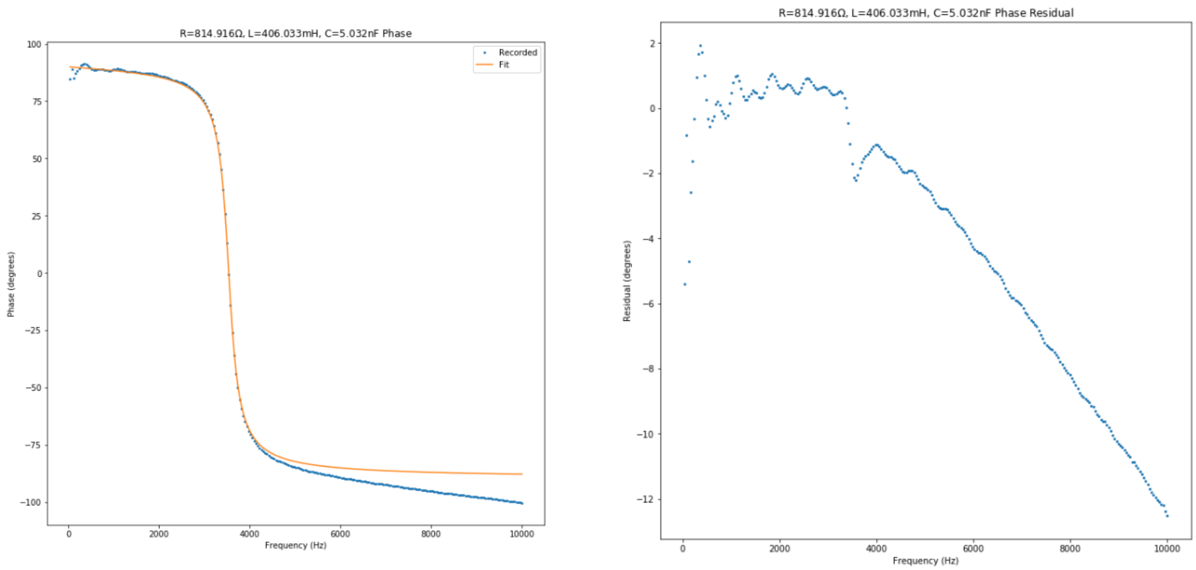
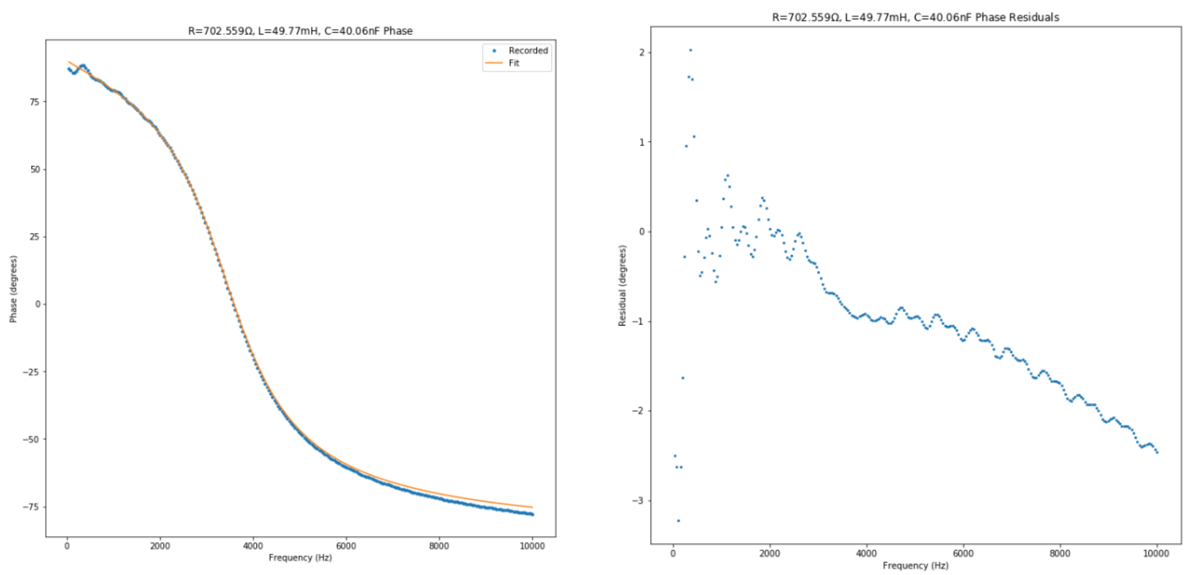
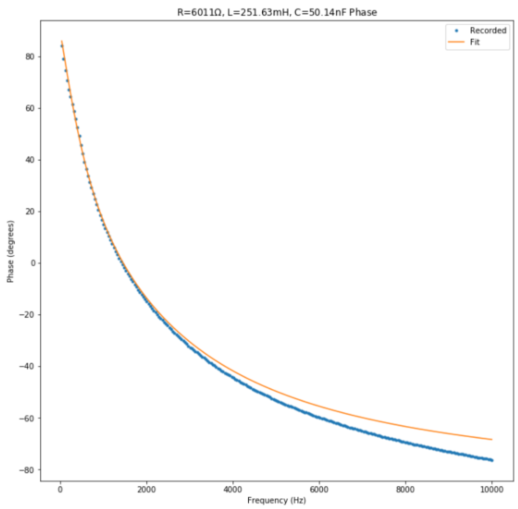


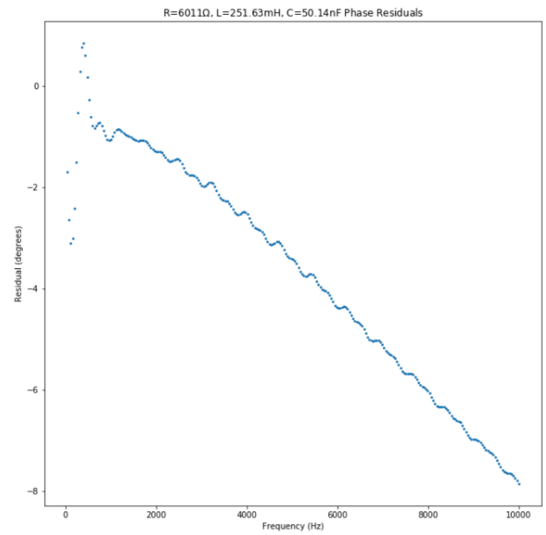
FIG. 8: The phase and residuals for the circuit $R = 702.559\Omega$, $L = 49.77$mH, $C = 40.06$nF.



I examined three cases and fit curves to those cases' data using the 3 sets of fit parameters for Q and ω_0 found in Table II. The case in the first row of the table corresponds to the images in Figures 5 and 8, with the first set depicting the fit for the transfer function, and the second showing the phase function using our generated fit parameters. The data fit the function form pretty well, and it's hard to tell from looking at the fit image that both real data and a fit are on the plot. The residuals do have a clear pattern which peaks just

FIG. 9: The phase and residuals for the circuit $R = 6011\Omega$, $L = 251.63$mH, $C = 50.14$nF.





before resonance and dips into the negatives afterward, indicating that the resonance of the fit curve was maybe just a bit higher than it should have been, but none of the residuals were larger than 0.03, so I believe this is a good fit. The uncertainties in the parameters are very low, and the ${\chi^2}_{reduced}$ of 5.66 seems high for how well the data fit, so I think that we underestimated the uncertainty in the value of A_r given by the Pythics software. I used a suggested value of 2 times the least significant bit (Arduino ADC is 10-bit) so there was approximately a 1/500 error, but this seems to be a best-case scenario for the equipment working perfectly. We also didn't have any reliable estimates for uncertainties in our control variable: the frequency, and this may have reduced the goodness of the fit. The measured resonance frequency of 3585.87 ± 4.33Hz was 4.97 times its uncertainty away from the predicted 3564.35Hz while the measured Q factor of $1.57 \pm .01$ was just over one times its uncertainty value from the predicted 1.586. The poor accuracy in the resonance frequency is, in my opinion, not terrible poor accuracy, and more likely a case of greatly overstated precision. Judging from the oscillatory nature of the residuals in some of the cases, there was more random variation in voltage measurement than we accounted for. When using our fit parameters from this case, the phase function Eq. 9 produced a good fit until high frequencies, where the fit underestimates the magnitude of the negative phase. I think this is due to underestimation of the inductance and overestimation of the capacitance at high frequencies, since I used their average value over the 0–10kHz range.

The next case was that in row 2 of Table II, seen in Figures 4(response) and 7(phase). This set of data fit extremely well, yielding a low $chi^2{}_{reduced} = 1.06$, indicating a very good fit despite the extremely low stated uncertainties in measurements. The residuals, slightly negative until just after the resonance frequency, indicate that their nonzero value comes from a slight overestimation of the resonance frequency. This is confirmed by the measured $\omega_0 = 3538.314 \pm 3.83$Hz which was about 4.5 uncertainties away from the predicted 3521Hz and the measured $Q = 10.49 \pm 0.02$ which was a full 26 uncertainties from the predicted 11.02 (another confirmation that our uncertainties were probably too low and that this would be otherwise good data) . The phase plots for this case look the worst out of all of them, and but it only looks poor near the high frequencies, just like the other cases. I think that the reason the negative phase's magnitude is so underestimated by the model in this case is that this setup had the highest value of inductance by about 60%, which means the absolute inductance value varies more, even though I still treated it as constant.

The last case I examined was in row 3 of Table II, seen in Figures 6(response) and9(phase). I am pleased with the fit for this data set, because our $\chi^2{}_{reduced} = 2.99$, indicating a pretty good fit to the model, despite our low uncertainties. The residual plot for this case looks very nice for most of the frequency range, but at high frequencies, the model overestimated the transfer value. I'm not entirely sure why the values are overestimated at the end, but I assume that it's due to the underestimation of the inductance at high frequencies and the low Q-factor which make the limitations of our assumptions of constant inductance and capacitance apparent. The measured value of $\omega_0 = 1459.517 \pm 9.46$Hz was over 9 times its uncertainty away from the predicted 1417Hz, but I believe this is entirely due to the underestimation of uncertainty in the measured voltage and ignored uncertainty in frequency. The measured $Q = 0.376 \pm 0.01$ was about 0.3 times its uncertainty from the predicted value of 0.373, so I believe we were fairly accurate in our measurements. The phase plots for this case highlight the limitation of treating L as constant over the range of 0–10kHz, as we clearly see the fit underestimating the magnitude of the phase for high frequencies.

V. CONCLUSION

The resonance frequency and Quality factor predictions for the transient response of the RLC circuit were, as I expected, not extremely accurate because their source was the oscilloscope screen, but I consider them to be good data because I was fairly accurate while maintaining acceptable precision for non-automated measurements. The damping factor and quality factor predictions for the steady-state response in the case where we used a lock-in amplifier were confirmed by the experiment, though the resonance frequency was far from its prediction. I attribute this to either a systemic shift or poor choice of frequency spacing, as we only used about a 100Hz spacing to scan for ω_0. The predictions for the Quality factor and resonance frequency in the steady-state case with a network analyzer were very accurate, fitting the model for the transfer function very well in every case, and fitting the phase function very well near the resonance frequency (limitations exist far from ω_0 due to the assumption of constant L), when in reality it follows a difficult-to-fit concave up relationship with frequency over our range of interest. The parameters obtained from the experiment, however, were assigned much too low of an uncertainty, due to me using the best-case uncertainty of 2 times the least significant bit in the estimation of $\frac{V_{out}}{V_{in}}$ and resulted in a seemingly too restrictive estimate of these parameters. It's also possible that parasitic capacitance due to the proximity of the resistor plays a role in underestimating our uncertainty of C, but this is unlikely as the components are physically separated from each other by a considerable distance. Diefenderfer[1] indicates that if present, stray capacitance would be most important across the resistor, as it would shunt some of the high-frequency signal to ground and reduce the measurement of A_r at high frequencies, which we don't see, so I don't believe this to be an issue.

ACKNOWLEDGMENTS

---

* jar257@pitt.edu; permanent address: 4610 Truro Pl, Pittsburgh, PA 15213

1 A. James Diefenderfer, /textitPrinciples of Electronic Instrumentation, (West Washington Square, Philadelphia, PA, 1972), pp. 235–236.