THE RANDOM GRAPH AND ITS DICHOTOMOUS NATURE

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Abstract. The (countable) random graph is fascinating combinatorial structure with model-theoretic relevance. This paper investigates various aspects of the random graph, following the theme of *witnessing dichotomy*. The theory of the random graph witnesses the dichotomy between countable and uncountable categoricity, plays a central role in a dichotomous characterization of the order property, and breaks the seeming dichotomy between "syntactic complexity" and "semantic complexity."

Contents

1. Where is the randomness in the random graph?

The term "random graph" may mean different things to different mathematicians. To some it might mean a finite probability space of finite graphs. To model theorists, it means a model of a theory T_R in the language of graphs. I'd like to first briefly discuss the connection between the probabilistic notion of randomness and the model theoretic notion, proving along the way some basic facts about this theory.

The model theoretic definition of "random graph" is a model of the following theory T_R: the theory of (simple) graphs and for each $r, s < \omega$ the sentence

$$A_{r,s}: \quad \forall x_1 \cdots \forall x_r \forall y_1 \cdots \forall y_s \exists z \left(\text{"all variables distinct"} \wedge \bigwedge_i (x \not\sim x_i) \wedge \bigwedge_i (x \sim y_i) \right).$$

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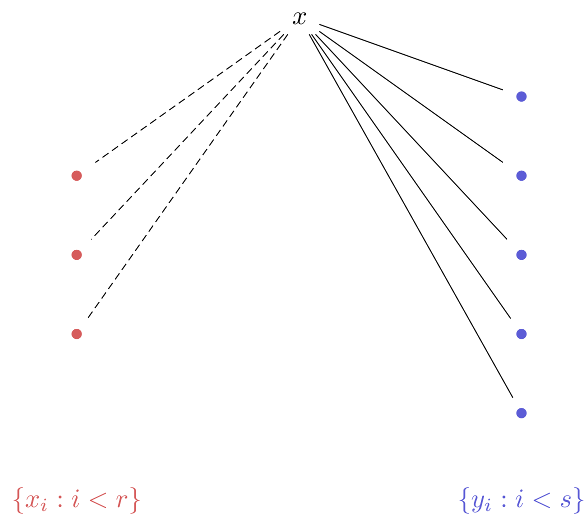


Remark 1.1. It is worth remarking that the very definition of the random graph is an instance of it witnessing dichotomy. The $A_{r,s}$ axioms literally state that in a random graph *any two* disjoint finite subsets are witnessed.

So far it may not seem clear where all the "randomness" is. To remedy this, I'll show that T_R is (1) the theory of sentences holding "almost surely" on finite random gaphs and (2) that T_R thus has a model by compactness.

Definition 1.2. We say some first-order sentence σ in the language of graphs holds **almost surely** if

$$\lim_{n\to\infty} P_n(G \models \sigma) = 1,$$

where P_n is the uniform probability measure on $\mathcal{G}_n$ the space of graphs on a fixed set of n vertices and the event $G \models \sigma$ is subset of $\mathcal{G}_n$.

Lemma 1.3. *Each $A_{r,s}$ axiom holds almost surely.*

Proof. Fix $n < \omega$. Let $\mathcal{G}_n$ be the set of graphs on the set of vertices $\{0, 1, ..., n-1\} = n < \omega$ and let P_n be the uniform distribution on $\mathcal{G}_n$. Let $w, v \subseteq n$ be disjoint with $r = r$ and $s = s$. The probability a particular $z \in n$ is not a w, v-witness is

$$p \coloneqq \left(1 - \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^s\right) = 1 - \left(\frac{1}{2}\right)^{r+s}.$$

The probability that there is no w, v-witness is

$$p^{n-r-s}.$$

There are $\binom{n}{r}\binom{n-r}{s}$ possible choices for w and v so the probability in $\mathcal{G}_n$ that $A_{r,s}$ fails is

$$\binom{n}{r}\binom{n-r}{s} p^{n-r-s} \to 0$$

so $A_{r,s}$ fails almost never and thus holds almost surely. $\square$

Corollary 1.4. *The theory T_R has a model (which is infinite).*

Proof. Since each $A_{r,s}$ axiom holds almost surely, for any finite set $\mathcal{A}$ of $A_{r,s}$ axioms, there is an $N > 0$ sufficiently large that

$$P_N\left[\bigwedge_{A\in\mathcal{A}} A\right] > 0.$$

This is a consequence of the union bound and Lemma 1.3:

$$P_n\left[\bigvee_{A\in\mathcal{A}} \neg A\right] \leq \sum_{A\in\mathcal{A}} P_n[\neg A] \to 0.$$

This implies that there exists a model $G \in \mathcal{G}_N$ satisfying each $A \in \mathcal{A}$ (as well as graph axioms). By compactness, there is a model satisfying T_R. □

Remark 1.5. A simple consequence of the axioms $A_{r,s}$ is that any model of T_R is infinite.

Remark 1.6. The Lemma 1.3 combined with the argument in 2.1 also shows that if you take the probability space of graphs on the vertices $V = \omega$ with probability distribution generated by independently flipping coins for edges ($\Pr[i \sim j] = 1/2$) then with probability 1, G and H sampled from this space are isomorphic.

Hopefully you're convinced that a random graph is indeed "random" and if not, then at the very least we know that we're dealing with a consistent theory!

2. Categoricity

The theory of the random graph witnesses a dichotomy when it comes to categoricity. The theory T_R is $\aleph_0$-categorical yet it is the farthest it can be from any uncountable categoricity.

2.1. **Countably Categorical.**

Fact 2.1. The theory T_R is $\aleph_0$-categorical.

The Fact 2.1 is a corollary of a fairly strong property possessed by the random graph called "ultrahomogeneity".

Definition 2.1. A structure in which any partial automorphism between finite substructures extends to a full automorphism is said to be **ultrahomogeneous**.

Lemma 2.2. *Let $G, H \models T_R$ be countable. Let $A \subseteq G$ and $B \subseteq H$ such that the submodels are elementarily equivalent: $A \equiv B$, say by the isomorphism $\Phi : A \to B$. Then there is an isomorphism $\Psi : G \to H$ extending Φ.*

Proof of Lemma. Let $G, H \models T_R$ be countable models. A "back-and-forth" argument yields an isomorphism $G \to H$, taking A to B. Enumerate the vertices other than A and B: $G \setminus A = \langle g_n : n < \omega\rangle$ and $H \setminus B = \langle h_n : n < \omega\rangle$. We'll build partial isomorphisms $\Psi_0, \Psi_1, ...$ such that $\bigcup_{i<\omega} \Psi_i$ is a desired isomorphism.

Let $\Psi_0 = \Phi$. Suppose Ψ_i is defined. On "forth" steps (odd i), find least n such that Ψ_i is not defined on g_n. Let J be the set of indexes j such that $g_j \sim g_n$, then by the axioms of T_R, there is a point $h \in H$ such that

$$\text{“}h \text{ distinct from all } h_n \in \operatorname{im}(\Psi_i)\text{”} \wedge \bigwedge_{j\in J} h \sim \Psi_i(g_j) \wedge \bigwedge_{j\in \operatorname{dom}\Psi_i \setminus J} h \not\sim \Psi_i(g_j).$$

Define $\Psi_{i+1}(g_n) = h$ and this is still a partial isomorphism.

On the "back" steps (even $i > 0$) we can apply the same argument going from $H \to G$ via Ψ_i^{-1} allows us to not miss any elements in either graph.

The union $\Psi \coloneqq \bigcup_{i<\omega} \Psi_i$ is an isomorphism because for any $g_i, g_j \in G$ by construction $\Psi(g_i) \sim \Psi(g_j) \iff \Psi_n(g_i) \sim \Psi_n(g_j) \iff g_i \sim g_j$ where n is the least such that $g_i, g_j \in \mathrm{dom}\Psi_n$. □

Corollary 2.3. *The theory T_R is $\aleph_0$-categorical.*

Proof. The back-and-forth proof in Lemma 2.2 works starting with $A, B = \emptyset$. □

Corollary 2.4. *The unique countable model R of T_R is ultrahomogeneous.*

Theorem 2.5 (Łoś-Vaught Test)**.** *If T is a consistent theory with only infinite models and is $\kappa-$categorical for some $\kappa > \aleph_0$ then T is complete.*

Corollary 2.6. *The theory T_R is complete.*

Proof. By Fact 2.1, Corollary 1.4 and Remark 1.5 the theory of the Random Graph is complete. □

2.2. **Uncountably Uncategorical.** Let T be a consistent theory in the language $\mathcal{L}$. A little bit of cardinal arithmetic shows that there are at most 2^κ non-isomorphic models of T of size κ. We can make the rough approximation that a difference in interpretation leads to nonisomorphic models. This will give an upper bound. Then there are at most

$$(\#\text{ constants}) \cdot \kappa + (\#\text{ relation symbols}) \cdot 2^\kappa + (\#\text{ function symbols}) \cdot 2^\kappa$$
$$\leq |\mathcal{L}| \cdot 2^\kappa$$

non-isomorphic models of cardinality κ. The theory of the Random Graph T_R is $\aleph_0$-categorical, i.e. it minimizes the number of nonisomorphic models of size $\aleph_0$. This is in sharp contrast to any $\kappa > \aleph_0$ where it attains the maximum: 2^κ.[1] We'll see a simpler argument that shows there are two non-isomorphic random graphs of given size $\kappa > \aleph_0$.

Theorem 2.7. *The theory T_R is not categorical in any $\kappa > \aleph_0$.*

Proof. We'll show that any graph G of size κ can be extended to a Random Graph. Enumerate the finite subsets of G, say $\langle A_\alpha : \alpha < \kappa \rangle$. Note that this is a κ-long list because the set of finite subsets is a countable disjoint union of κ-large sets, so by the fundamental theorem of cardinal arithmetic we get κ-many finite subsets. For each $\alpha < \kappa$, add a vertex v_α connected to everything in A_α and nothing else. Let $X_\alpha \coloneqq \{v_\beta : \beta < \alpha\}$ and let $Y_\alpha \coloneqq \{A_\gamma : \gamma < \alpha\}$. For each of the following sets B add a vertex v_B connected to everything in B and nothing else: for $X \subseteq X_\alpha$ and $Y \subseteq Y_\alpha$ nonempty sets the set

$$B = X \cup Y.$$

Call the graph we get in the end of this process $\mathtt{cl}\, G$. By construction the graph $\mathtt{cl}\, G$ satisfies the $A_{r,s}$ axioms of the random graph, so $\mathtt{cl}\, G$ is a random graph on κ vertices.

Let G be a clique on κ vertices and let H be an anti-clique on κ vertices. A moment of thought shows that $\mathtt{cl}\, H$ cannot have any uncountable cliques, whereas $\mathtt{cl}\, G$ does have an uncountable clique, so there are at least two non-isomorphic random graphs of cardinality $\kappa > \aleph_0$.

[1] This is true in general of unstable theories. See this entry in the Encyclopedia of Mathematics.

□

Remark 2.8. This sort of argument could possibly be extended to an elementary proof that there are 2^κ isomorphism classes of random graphs of size κ, but it is not as simple as producing 2^κ non-isomorphic graphs on κ vertices. For instance, if G and H are infinite graphs that only disagree on a countable subgraph then $\mathtt{cl}\, G \cong \mathtt{cl}\, H$ because every countable graph can be found in a random graph.

Remark 2.9. The back-and-forth argument we used to show countable categoricity fails for uncountable models at a particular place because of the finitary nature of first-order logic. We can well-order the vertices of any random graph no matter the cardinality (assuming AC), so that isn't an issue. We run into trouble when we need to start making infinite partial isomorphisms. We'd need partial isomorphisms of sizes up to the size of the graphs we're considering and axioms written in first-order logic cannot force witnesses to infinite sets of vertices.

3. Dichotomy of Asymmetry: Randomness or Strict Order

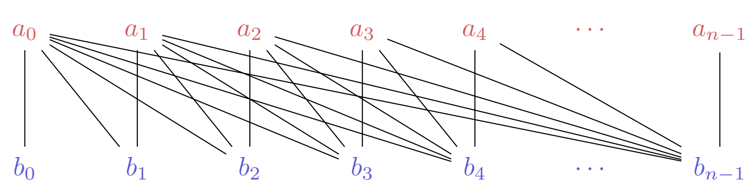
In this section we prove a fundamental result in stability theory in which the random graph is a central player in a dichotomous characterization of stability (aka the order property): the order property comes in two (not mutually exclusive) flavors, namely "randomness" or "strict order." For reasons to be discussed shortly, we associate the order property with "asymmetry," hence the title of this section.

In order to avoid talking about many different models which have particular properties we will now consider a "monster model" which is sufficiently saturated. Fix a complete first-order theory T in countable language $\mathcal{L}$. If σ is a sentence then by $\models \sigma$ we mean that σ is satisfied by the monster model of the theory T. If we say there is a(n) (indiscernible) sequence, we mean in the monster model.

3.1. What is Asymmetry?

Definition 3.1. An n**-half-graph** is a bipartite graph on $2n$ vertices $\{a_0, ..., a_{n-1}\} \sqcup \{b_0, ..., b_{n-1}\}$ such that

$$a_i \sim b_j \iff i \leq j.$$



Continuing the construction so that there are infinitely many a_i and b_i we get an infinite **half graph**, say H. A moment of thought shows that H has trivial automorphism group, which is why this combinatorial object captures "asymmetry".

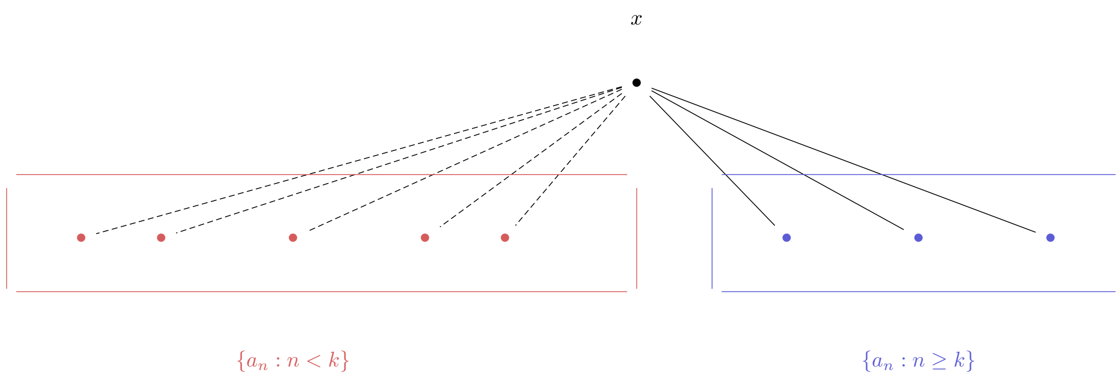
The order property captures the notion of "half graph" being encoded in a particular theory.

Definition 3.2. A formula $\varphi(\overline{x}, \overline{y})$ has the **order property** if there are $\overline{a}_n$ $(n < \omega)$ such that for every $k < \omega$,

$$\{\varphi(\overline{x}, \overline{a}_n)^{\text{if } k \leq n} : n < \omega\}$$

$$= \left\{ \bigwedge_{n<k} \neg\varphi(\overline{x}, \overline{a}_n) \wedge \bigwedge_{n \geq k} \varphi(\overline{x}, \overline{a}_n) : n < \omega \right\}$$

is consistent. In other words, by compactness, the monster model has a half-graph.



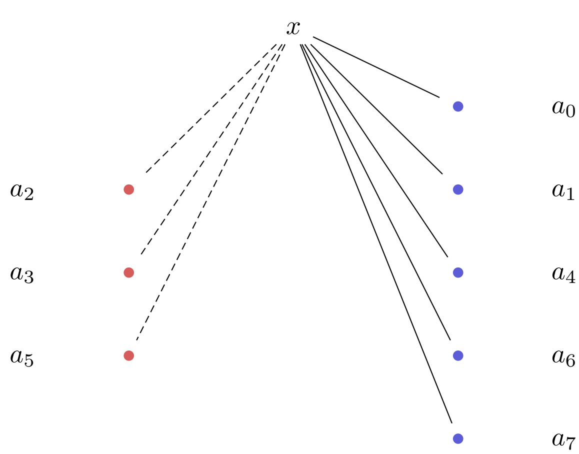
Remark 3.3. All that matters for a half-graph is that the edges interact with the order on the a_i and b_i. The order, however, need not be a well-order. For instance, we could (*and, in fact, do!*) take sequences indexed by the rationals instead.

3.2. **Randomness and Strict Order.** We've already seen a bit of "randomness" in the form of the $A_{r,s}$ axioms for the random graph. These axioms can be generalized to essentially be what is called the *independence property.*

Definition 3.4. A formula $\varphi(\overline{x}, \overline{a}_l)$ has the **independence property** if for every $n < \omega$ there are sequences $\overline{a}_l$ $(l < n)$ such that for every $w \subseteq n$,

$$\models (\exists \overline{x}) \left[\bigwedge_{l \in w} \varphi(\overline{x}, \overline{a}_l) \wedge \bigwedge_{l \in n \setminus w} \neg\varphi(\overline{x}, \overline{a}_l) \right].$$

Example 3.5. Here $w = \{0, 1, 4, 6, 7\}$ and $n = 8$. A line means that $\varphi(x, a_i)$ holds and a dotted line means that $\neg\varphi(x, a_i)$ holds.

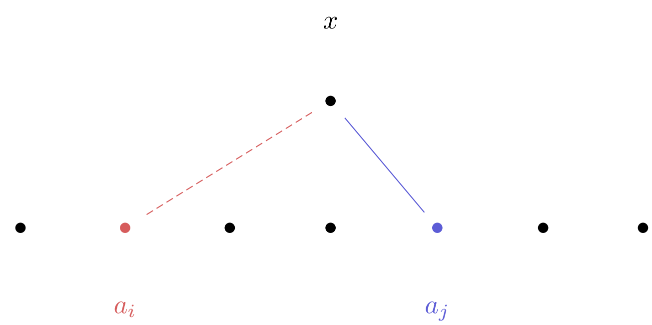


On the opposite end of asymmetry (the order property) there is strict order.

Definition 3.6. A formula $\varphi(\overline{x}, \overline{y})$ has the **strict order property** if for every $n < \omega$ there are $\overline{a}_l$ with $(l < n)$ such that for any $k, l < n$,

$$\models (\exists \overline{x})[\neg(\varphi(\overline{x}, \overline{a}_k) \wedge \varphi(\overline{x}, \overline{a}_l)] \iff k < l.$$

Example 3.7. For instance, if $i < j$ we get following picture where the blue line means $\varphi(x, a_j)$ holds and the red dotted line means $\neg\varphi(x, a_i)$ holds.



Remark 3.8. For any of these properties defined in Definitions 3.2, 3.4, or 3.6, we say that a theory possesses such property if some formula has that property.

Remark 3.9. The difference between *strict* order property and just the order property can be hard to spot. For the strict order property, there is a witness to a pair $(\overline{a}_k, \overline{a}_l)$ *only* when $k < l$. In other words, there is no witness to the opposite order, so $\models \neg(\exists \overline{x})[\neg(\varphi(\overline{x}, \overline{a}_l) \wedge \varphi(\overline{x}, \overline{a}_k)]$ when $k < l$. For the order property, there is a witness to every ordered pair, but there might also be a witness to the other direction.

3.3. **Order Property $\iff$ Independence or Strict Order.** We are now ready for the big theorem that connects asymmetry, randomness, and strict order.

Theorem 3.10. *A formula $\varphi(\overline{x}, \overline{y})$ has the order property if and only if it has either the independence property or a related formula has the strict order property. In particular, if φ does not have the independence property then for some $n < \omega$ and $\eta \in {}^n2$,*

$$\psi_\eta(\overline{x}, \overline{y}_0, ..., \overline{y}_{n-1}) = \bigwedge_{l<n} \varphi(\overline{x}, \overline{y}_l)^{\eta[l]}$$

has the strict order property where in the conjunction we mean $\varphi(\overline{x}, \overline{y}_l)$ if $\eta[l] = 1$ and $\neg\varphi(\overline{x}, \overline{y}_l)$ if $\eta[l] = 0$.

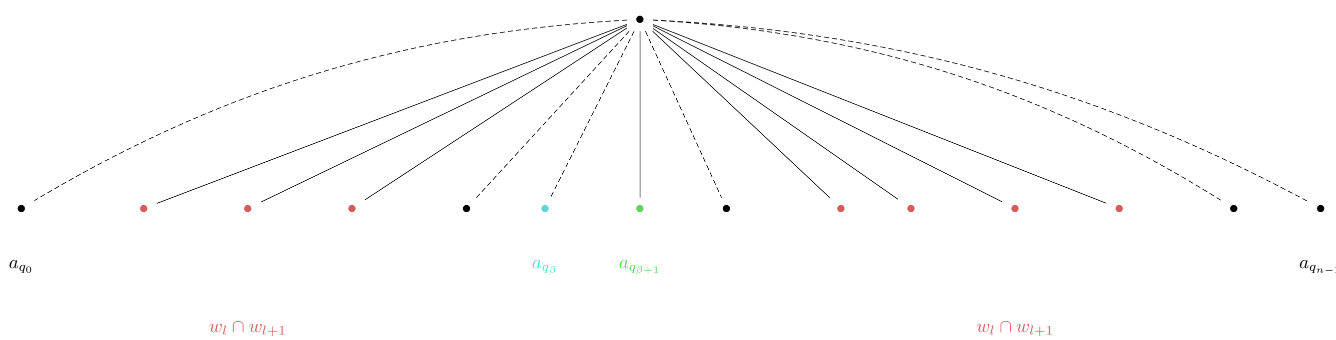
One direction of this theorem is relatively easy. As was remarked in 3.9, the strict order property implies the order property. It is a simple exercise to show that the independence property implies the order property. We prove the "hard" direction. Unfortunately, the notation gets a little dense at times, so to mitigate the trouble that can cause I've included a corresponding picture.

Proof of ($\implies$). Assume $\varphi(\overline{x}, \overline{y})$ has the order property. If $\varphi(\overline{x}, \overline{y})$ has the independence property then we are done, so assume it does not have the independence property and we will produce a ψ_η as above with the strict order property.

Since φ does not have the independence property, there is an indiscernible sequence indexed by the rationals $(\overline{a}_q)_{q \in \mathbb{Q}}$ (we can take such an indiscernible sequence by compactness) and some $n < \omega$ with $a_{q_0}, ..., a_{q_n-1}$ and $w \subseteq n$ such that

$$\models \neg(\exists \overline{x}) \left[\bigwedge_{k<n} \varphi(\overline{x}, \overline{a}_{q_k})^{\text{if } k \in w} \right]. \tag{3.11}$$

An observation we can immediately make is that if w is the complement of an initial segment (a *final segment, if you will*) of n then (3.11) would not be possible because φ has the order property. This is the motivation behind the indiscernible sequence! The idea is to shift elements of $w_0 := w$ one at a time by one step ($q_k \mapsto q_{k+1}$), getting sets w_{l+1} from w_l such that $|w_{l+1} \cap w_l| = |w| - 1$ so that at the end we get w_α which is the $|w|$-long "final" segment of n. At some l there will be a "break" and there will stop being a witness $\overline{x}$ to w_l. We can now construct ψ_η. Let $q_\beta \in w_l \setminus w_{l+1}$ be the "breakpoint" where there is no witness to w_l but when $\overline{a}_{q_\beta}$ is moved to $\overline{a}_{q_{\beta+1}}$, there is a witness to w_{l+1}. Here is a depiction of the x which witnesses w_{l+1}.



Consider $\psi(\overline{x}, \overline{y}, \overline{y}_{q_0}, ..., \widehat{\overline{y}_{q_\beta}}, \widehat{\overline{y}_{q_{\beta+1}}}, ..., \overline{y}_{q_{n-1}})$ given by

$$\bigwedge_{n>k\neq q_\beta, q_{\beta+1}} \varphi(\overline{x}, \overline{y}_k)^{\text{if } k\in w_l \cap w_{l+1}} \wedge \varphi(\overline{x}, \overline{y}).$$

Then the following should be more or less apparent from the picture:

$$\models (\exists \overline{x}) \left[\bigwedge_{n>k\neq q_\beta, q_{\beta+1}} \varphi(\overline{x}, \overline{a}_k)^{\text{if } k\in w_l \cap w_{l+1}} \wedge \varphi(\overline{x}, \overline{a}_{q_{\beta+1}}) \wedge \neg\varphi(\overline{x}, \overline{a}_{q_\beta}) \right]$$

$$\iff: \qquad (\exists \overline{x})[\psi(\overline{x}, \overline{a}_{q_{\beta+1}}, (\overline{a}_{q_k})_{k\in n\setminus\{\beta,\beta+1\}}) \wedge \neg\varphi(\overline{x}, \overline{a}_{q_\beta})].$$

Similarly,

$$\models \neg(\exists \overline{x}) \left[\bigwedge_{n>k\neq q_\beta, q_{\beta+1}} \varphi(\overline{x}, \overline{a}_k)^{\text{if } k\in w_l \cap w_{l+1}} \wedge \varphi(\overline{x}, \overline{a}_{q_\beta}) \wedge \neg\varphi(\overline{x}, \overline{a}_{q_\beta+1}) \right]$$

$$\iff: \qquad \neg(\exists \overline{x})[\psi(\overline{x}, \overline{a}_{q_\beta}, (\overline{a}_{q_k})_{k\in n\setminus\{\beta,\beta+1\}}) \wedge \neg\varphi(\overline{x}, \overline{a}_{q_{\beta+1}})]$$

This is in particular true because there is no x witnessing w_l, so adding the additional condition that $\neg\varphi(\overline{x}, \overline{a}_{q_{\beta+1}})$ changes nothing. The additional term $\neg\varphi(\overline{x}, \overline{a}_{q_j})$ bit ($j \in \{\beta, \beta+1\}$) gives us that

$$\models (\exists \overline{x})[\psi(\overline{x}, \overline{a}_{q_{\beta+1}}, (\overline{a}_{q_k})_{k\in n\setminus\{\beta,\beta+1\}}) \wedge \neg\psi(\overline{x}, \overline{a}_{q_\beta}, (\overline{a}_{q_k})_{k\in n\setminus\{\beta,\beta+1\}})] \tag{3.12}$$

so ψ (nearly) has the order property, and moreover that

$$\models \neg(\exists \overline{x})[\psi(\overline{x}, \overline{a}_{q_\beta}, (\overline{a}_{q_k})_{k\in n\setminus\{\beta,\beta+1\}}) \wedge \neg\psi(\overline{x}, \overline{a}_{q_{\beta+1}}, (\overline{a}_{q_k})_{k\in n\setminus\{\beta,\beta+1\}})] \tag{3.13}$$

so ψ (nearly) has the *strict* order property! Admittedly, I cheated and said "nearly," but this is where the indiscernible sequence comes in. Applying the fact that (a_q) is an indiscernible sequence, we get that if $(b_{q_k})_{k<n} \subseteq (a_q)_{q\in\mathbb{Q}}$ is order-isomorphic to (a_{q_k}) then (3.12) and (3.13) hold of the $(b_{q_k})_{k<n}$ as well, so ψ has the strict order property with the sequence $(a_q)_{q\in Q}$ and we are done! □

4. Quantifier Elimination: Breaking the Seeming Dichotomy Between Syntactic and Semantic Complexity

Quantifier elimination is syntactic simplicity: any formula can be written simply. This does not imply semantic simplicity, i.e. model-theoretic stability.

Definition 4.1. A theory is **stable** if no formula has the order property.

As we saw in Theorem 3.10, the random graph is very much unstable. The theory of the random graph breaks the seeming dichotomy between syntactic and semantic simplicity.

Definition 4.2. A theory T **admits quantifier elimination** if for any $\phi(\overline{x})$ in one or more free variables, there is a quantifier-free formula $\theta(\overline{x})$ such that

$$T \vdash [\phi(\overline{x}) \longleftrightarrow \theta(\overline{x})].$$

Quantifier elimination says that global information (as in information gained from quantifiers) is determined locally. It should be natural then that quantifier elimination is a consequence of ultrahomogenity (Lemma 2.2). We will show that T_R admits quantifier elimination.

Theorem 4.3. *The theory T_R admits quantifier elimination.*

Proof. Fix $n < \omega$ and take two finite substructures $A = \{a_i : i < n\}$ and $B = \{b_i : i < n\}$ of R that are elementarily equivalent. By the ultrahomogeneity of R, there is an automorphism $f : R \to R$ such that $f(a_i) = b_i$ for each $0 \leq i < n$. This means that for any n-ary formula $\varphi(\overline{x})$,

$$T \vdash \varphi(\overline{a}) \longleftrightarrow \varphi(f \cdot \overline{a}) \longleftrightarrow \varphi(\overline{b}).$$

So two n-tuples have the same type if they are elementarily equivalent as finite structures. But for a finite structure there is a single quantifier-free formula $\theta(\overline{x})$ that determines a given elementary equivalence class (just write out how each vertex relates to another). It follows that there are finitely many complete n-types and moreover each n-type is generated by a quantifier-free $\theta(\overline{x})$. So for any $\varphi(\overline{x})$ that can be realized, we have $\varphi(\overline{x}) \in p(\overline{x}) = \{\varphi(\overline{x}) : T \vdash \theta(\overline{x}) \longrightarrow \varphi(\overline{x})\}$ a complete n-type, and

$$T \vdash \varphi(\overline{x}) \longleftrightarrow \theta(\overline{x}).$$

□

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