BACKPROPAGATION

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CHAIN RULE

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On rappelle le théorème des fonctions composées (chain rule).

$$\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$$

$$x \xrightarrow{f} y = f(x) \xrightarrow{g} z = g(y)$$

$$= g(f(x))$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

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Exemple:

$$\frac{\partial z}{\partial x} = \frac{\partial \left[5(x^2+1)\right]}{\partial x} = 10x = 5 \cdot 2x = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

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Exemple:

$$\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$$

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$$= 5(x^2 + 1)$$

On a:

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Généralisation multidimensionnelle

$$\frac{\partial z}{\partial x_i} = \sum_{j=1}^n \frac{\partial z}{\partial y_j} \cdot \frac{\partial y_j}{\partial x_i}, \quad i = 1, \dots, m$$

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Formulation vectorielle

$$\begin{array}{cccc}
\mathbb{R}^m & \xrightarrow{f} & \mathbb{R}^n & \xrightarrow{g} & \mathbb{R} \\
\mathbf{x} & \xrightarrow{f} & \mathbf{y} = f(\mathbf{x}) & \xrightarrow{g} & z = g(\mathbf{y}) \\
& & = g(f(\mathbf{x}))
\end{array}$$

Soient $\nabla_x z$ et le *gradient* de z par rapport à x, $\nabla_y z$ et le gradient de z par rapport à y, et $J_f:=\left[\frac{\partial y}{\partial x}\right]$ le *jacobien* de la fonction f:

$$abla_{m{x}}z = \left[[
abla_{m{y}}z]^T \left[rac{\partial m{y}}{\partial m{x}}
ight]
ight]^T = m{J}_f^T
abla_{m{y}}z$$

$$\begin{pmatrix} \frac{\partial z}{\partial x_1} \\ \vdots \\ \frac{\partial z}{\partial x_m} \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} \frac{\partial z}{\partial y_1} & \cdots & \frac{\partial z}{\partial y_n} \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_m} \end{pmatrix} \end{bmatrix}^T$$

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Formulation vectorielle

Soient $\nabla_{x}z$ et le gradient de z par rapport à x, $\nabla_{y}z$ et le gradient de z par rapport à ${m y}$, et ${m J}_f:=\left[rac{\partial {m y}}{\partial {m x}}
ight]$ le jacobien de la fonction f:

$$\nabla_{\boldsymbol{x}}z = \left[\left[\nabla_{\boldsymbol{y}}z \right]^T \left[\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} \right] \right]^T = \boldsymbol{J}_f^T \nabla_{\boldsymbol{y}} z$$

$$\begin{pmatrix} \frac{\partial z}{\partial x_1} \\ \vdots \\ \frac{\partial z}{\partial x_m} \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} \frac{\partial z}{\partial y_1} & \cdots & \frac{\partial z}{\partial y_n} \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_m} \end{pmatrix} \end{bmatrix}^T$$

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Exemple:

$$\mathbb{R}^{m} \xrightarrow{f} \mathbb{R}^{n} \xrightarrow{g} \mathbb{R}$$

$$\begin{pmatrix} x_{1} \\ x_{2} \end{pmatrix} \xrightarrow{f} \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix} = \begin{pmatrix} x_{1}x_{2} \\ x_{1}^{2} \\ x_{2}^{3} \end{pmatrix} \xrightarrow{g} \qquad z = y_{1}y_{2} + y_{3}$$

$$= (x_{1}x_{2})x_{1}^{2} + x_{2}^{3}$$

$$\frac{\partial z}{\partial x_1} = \frac{\partial \left[(x_1 x_2) x_1^2 + x_2^3 \right]}{\partial x_1} = x_1^2 x_2 + x_1 x_2 2 x_1 + 0$$

$$= \frac{\partial z}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_1} + \frac{\partial z}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_1} + \frac{\partial z}{\partial y_3} \cdot \frac{\partial y_3}{\partial x_1}$$

$$\frac{\partial z}{\partial x_2} = \frac{\partial \left[(x_1 x_2) x_1^2 + x_2^3 \right]}{\partial x_2} = x_1^2 x_1 + 3 x_2^2 + 0$$

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Exemple (suite):

$$\mathbb{R}^{m} \xrightarrow{f} \mathbb{R}^{n} \xrightarrow{g} \mathbb{R}$$

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$$\nabla_{\boldsymbol{x}} z = \begin{pmatrix} \frac{\partial z}{\partial x_1} \\ \frac{\partial z}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 3x_2x_1^2 \\ x_1^2 + 3x_2^2 \end{pmatrix} = \begin{pmatrix} y_2 & y_1 & 1 \end{pmatrix} \begin{pmatrix} x_2 & x_1 \\ 2x_1 & 0 \\ 0 & 3x_2^2 \end{pmatrix} = \begin{bmatrix} \nabla_{\boldsymbol{y}} z \end{bmatrix}^T \begin{bmatrix} \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} \end{bmatrix}$$

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- lacksquare Soit $S=\{(oldsymbol{x_i},oldsymbol{y_i})\in\mathbb{R}^{d_1} imes\mathbb{R}^{d_2}:i=1,\ldots,N\}$ un dataset.

$$\mathbf{\Theta} := \left\{ \left(\mathbf{W}^{[l]}, \mathbf{b}^{[l]} \right) : l = 1, \dots, L \right\}$$

$$egin{cases} m{a^{[0]}} &= m{x} \ m{z^{[l]}} &= m{W^{[l]}}m{a^{[l-1]}} + m{b^{[l]}}, \ m{a^{[l]}} &= m{\sigma\left(m{z^{[l]}}
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- lacksquare Soit $S=\{(oldsymbol{x_i},oldsymbol{y_i})\in\mathbb{R}^{d_1} imes\mathbb{R}^{d_2}:i=1,\ldots,N\}$ un dataset.
- \triangleright Soit \mathcal{N}_{Θ} un réseau de neurones (MLP) à L couches donné par les paramètres (poids et biais)

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et par la dynamique

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ightharpoonup Remarque: le réseau \mathcal{N}_{Θ} peut-être naturellement associé à la fonction

$$egin{array}{lll} f_{m{\Theta}}: \mathbb{R}^{d_1} & \longrightarrow & \mathbb{R}^{d_2} \ x & \longmapsto & f_{m{\Theta}}(x) := m{a}^{[L]} \end{array}$$

- $ightharpoonup f_{\Theta}(x)$ est la *prédiction* (output) de \mathcal{N}_{Θ} associée à l'input x.
- ightharpoonup Chaque jeu de paramètres Θ donne lieu à une fonction f_{Θ} différente.

Remarque: le réseau \mathcal{N}_{Θ} peut-être naturellement associé à la fonction

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- lacktriangle Chaque jeu de paramètres Θ donne lieu à une fonction f_{Θ} différente.

Soit une fonction de coût (cost or loss function) qui mesure l'erreur entre la prédiction \hat{y}_i et la réalité y_i :

$$egin{array}{lll} \ell: \mathbb{R}^{d_2} imes \mathbb{R}^{d_2} & \longrightarrow & \mathbb{R} \ (\hat{m{y}}_{m{i}}, m{y}_{m{i}}) & \longmapsto & \mathcal{L}\left(\hat{m{y}}_{m{i}}, m{y}_{m{i}}
ight) \end{array}$$

▶ Typiquement, la fonction de coût pourrait être l'erreur quadratique (distance Euclidienne au carré)

$$\ell\left(\hat{\boldsymbol{y}}_{\boldsymbol{i}}, \boldsymbol{y}_{\boldsymbol{i}}\right) = \frac{1}{2} \left\| \hat{\boldsymbol{y}}_{\boldsymbol{i}} - \boldsymbol{y}_{\boldsymbol{i}} \right\|_{2}^{2}$$

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► La fonction de coût peut être naturellement généralisée à un ensemble de *prédictions* et de *réalités*:

$$egin{array}{lll} \mathcal{L}: \mathbb{R}^{d_2} imes \cdots imes \mathbb{R}^{d_2} & \longrightarrow & \mathbb{R} \ (\hat{m{y_1}}, \ldots, \hat{m{y_N}}, m{y_i} \ldots, m{y_N}) & \longmapsto & \mathcal{L}\left(\hat{m{y_1}}, \ldots, \hat{m{y_N}}, m{y_i} \ldots, m{y_N}
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► Typiquement, la fonction de coût pourrait être l'erreur quadratique moyenne (mean squared error MSE)

$$\mathcal{L}(\hat{y}_1, \dots, \hat{y}_N, y_i, \dots, y_N) = \frac{1}{N} \sum_{i=1}^N \ell(\hat{y}_i, y_i)$$
$$= \frac{1}{2N} \sum_{i=1}^N ||\hat{y}_i - y_i||$$

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$$= \frac{1}{2N} \sum_{i=1}^{N} \|\hat{\boldsymbol{y}}_{i} - \boldsymbol{y}_{i}\|_{2}^{2}$$

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Pour un réseau de neurones \mathcal{N}_{Θ} , l'erreur entre les prédictions et les réalités est

$$\mathcal{L}\left(f_{\mathbf{\Theta}}\left(\boldsymbol{x_{1}}\right),\ldots,f_{\mathbf{\Theta}}\left(\boldsymbol{x_{N}}\right),\boldsymbol{y_{1}},\ldots,\boldsymbol{y_{N}}\right).$$

$$egin{array}{lll} \mathcal{L}: \mathbb{R}^{|\Theta|} & \longrightarrow & \mathbb{R} \\ & \Theta & \longmapsto & \mathcal{L}\left(f_{\Theta}\left(x_{1}
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- \triangleright Pour différents paramètres Θ , on aura différentes prédictions $f_{\Theta}(x_1), \ldots, f_{\Theta}(x_N)$, et donc différentes erreurs $\mathcal{L}(\ldots)$.

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- \triangleright Ainsi, \mathcal{L} est une fonction des paramètres Θ du réseau:

$$\mathcal{L}: \mathbb{R}^{|\Theta|} \longrightarrow \mathbb{R}$$

$$\Theta \longmapsto \mathcal{L}\left(f_{\Theta}\left(x_{1}\right), \dots, f_{\Theta}\left(x_{N}\right), y_{1}, \dots, y_{N}\right).$$

où $|\Theta|$ est le nombre de paramètres Θ (poids et biais, souvent plusieurs millions).

 \triangleright L'entraînement du réseau \mathcal{N}_{Θ} consiste à déterminer des paramètres Θ qui minimisent l'erreur

$$\mathcal{L}(f_{\Theta}(x_1),\ldots,f_{\Theta}(x_N),y_1,\ldots,y_N).$$

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- Pour cela, on utilise une descente de gradient: *mini-batch stochas-tic gradient descent*.
- ▶ Backpropagation est un algorithme qui permet de calculer les gradients $\nabla_{\Theta} \mathcal{L}$ de manière efficiente.

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- ▶ Pour cela, on utilise une descente de gradient: *mini-batch stochas-tic gradient descent*.
- ▶ Backpropagation est un algorithme qui permet de calculer les gradients $\nabla_{\Theta} \mathcal{L}$ de manière efficiente.

$$\mathcal{L}\left(f_{\Theta}\left(x_{1}\right),\ldots,f_{\Theta}\left(x_{N}\right),y_{1},\ldots,y_{N}\right)=\mathcal{L}\left(\Theta,\ldots\right)$$

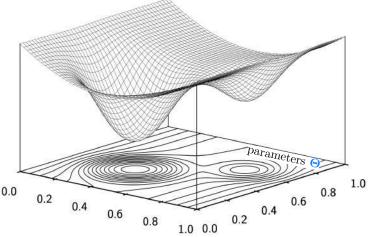
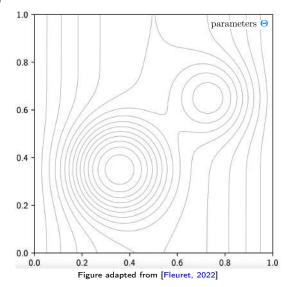
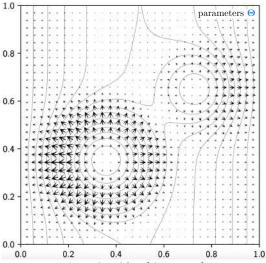


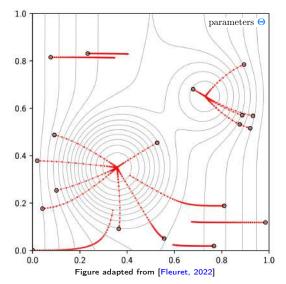
Figure adapted from [Fleuret, 2022]



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TRAINING



TRAINING

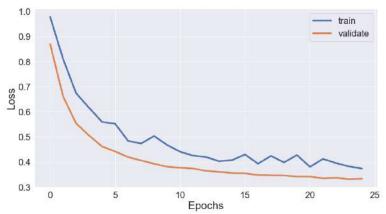
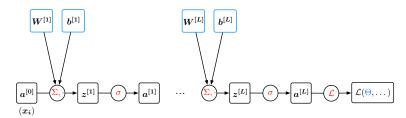


Figure taken from towardsdatascience.com

Graphe computationnel d'un réseau de NEURONES (FORWARD PASS)



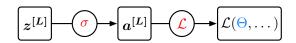
BACKPROPAGATION •00000000000000000

On veut calculer les gradients:

$$\nabla_{\boldsymbol{W}^{[l]}}\mathcal{L}(\boldsymbol{\Theta}) := \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial \boldsymbol{W}^{[l]}} \ \text{ et } \ \nabla_{\boldsymbol{b}^{[l]}}\mathcal{L}(\boldsymbol{\Theta}) := \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial \boldsymbol{B}^{[l]}}$$

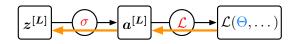
pour
$$l = 1, \ldots, M$$

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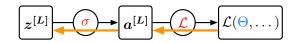


$$\begin{split} \delta_j^{[L]} &:= \frac{\partial \mathcal{L}(\Theta)}{\partial z_j^{[L]}} &= \sum_{k=1}^{|\mathbf{a}^{[L]}|} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \frac{\partial a_k^{[L]}}{\partial z_j^{[L]}} \\ &= \sum_{k=1}^{|\mathbf{a}^{[L]}|} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} \\ \left(\frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} = 0 \text{ for } k \neq j \right) &= \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \sigma'(z_j^{[L]}) \end{split}$$

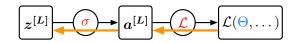
Calcul des gradients: équation 1



$$\begin{split} \delta_{\pmb{j}}^{[L]} &:= \frac{\partial \mathcal{L}(\pmb{\Theta})}{\partial z_{\pmb{j}}^{[L]}} &= \sum_{k=1}^{|a^{[L]}|} \frac{\partial \mathcal{L}(\pmb{\Theta})}{\partial a_{k}^{[L]}} \cdot \frac{\partial a_{k}^{[L]}}{\partial z_{j}^{[L]}} \\ &= \sum_{k=1}^{|a^{[L]}|} \frac{\partial \mathcal{L}(\pmb{\Theta})}{\partial a_{k}^{[L]}} \cdot \frac{\partial \sigma(z_{k}^{[L]})}{\partial z_{j}^{[L]}} \\ \left(\frac{\partial \sigma(z_{k}^{[L]})}{\partial z_{j}^{[L]}} = 0 \text{ for } k \neq j \right) &= \frac{\partial \mathcal{L}(\pmb{\Theta})}{\partial a_{k}^{[L]}} \cdot \sigma'(z_{j}^{[L]}) \end{split}$$

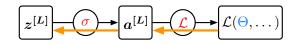


$$\begin{split} \delta_j^{[L]} &:= \frac{\partial \mathcal{L}(\Theta)}{\partial z_j^{[L]}} &= \sum_{k=1}^{|a^{[L]}|} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \frac{\partial a_k^{[L]}}{\partial z_j^{[L]}} \\ &= \sum_{k=1}^{|a^{[L]}|} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} \\ \left(\frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} = 0 \text{ for } k \neq j \right) &= \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \sigma'(z_j^{[L]}) \end{split}$$



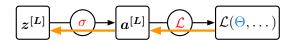
$$\begin{split} \delta_j^{[L]} &:= \frac{\partial \mathcal{L}(\Theta)}{\partial z_j^{[L]}} &= \sum_{k=1}^{|\boldsymbol{a}^{[L]}|} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \frac{\partial a_k^{[L]}}{\partial z_j^{[L]}} \\ &= \sum_{k=1}^{|\boldsymbol{a}^{[L]}|} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} \\ \left(\frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} = 0 \text{ for } k \neq j \right) &= \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \sigma'(z_j^{[L]}) \end{split}$$

Calcul des gradients: équation 1



$$\begin{split} \delta_j^{[L]} &:= \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial z_j^{[L]}} &= \sum_{k=1}^{|\mathbf{a}^{[L]}|} \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial a_k^{[L]}} \cdot \frac{\partial a_k^{[L]}}{\partial z_j^{[L]}} \\ &= \sum_{k=1}^{|\mathbf{a}^{[L]}|} \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial a_k^{[L]}} \cdot \frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} \\ \left(\frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} = 0 \text{ for } k \neq j \right) &= \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial a_k^{[L]}} \cdot \sigma'(z_j^{[L]}) \end{split}$$

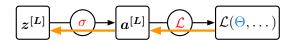
Formulation vectorielle:



$$\delta^{[L]} \; := \; \nabla_{z^{[L]}} \mathcal{L}(\Theta) \;\; = \;\; \nabla_{a^{[L]}} \mathcal{L}(\Theta) \odot \sigma'(z^{[L]})$$

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Formulation vectorielle:

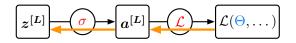


$$\boldsymbol{\delta^{[L]}} \; := \; \nabla_{\boldsymbol{z}^{[L]}} \mathcal{L}(\boldsymbol{\Theta}) \;\; = \;\; \nabla_{\boldsymbol{a}^{[L]}} \mathcal{L}(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[L]})$$

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CALCUL DES GRADIENTS: ÉQUATION 1

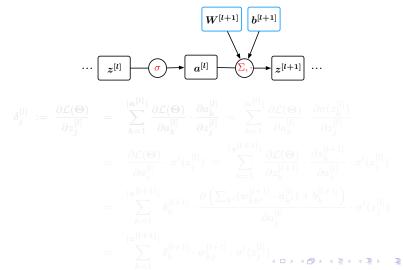
Formulation vectorielle:



$$\delta^{[L]} \; := \; \nabla_{\boldsymbol{z}^{[L]}} \mathcal{L}(\boldsymbol{\Theta}) \;\; = \;\; \nabla_{\boldsymbol{a}^{[L]}} \mathcal{L}(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[L]})$$

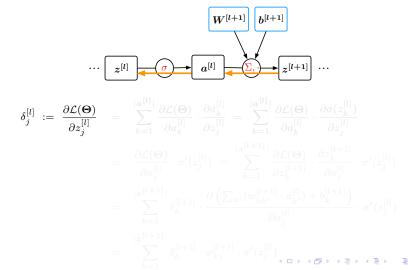
où ⊙ est le produit de Hadamard (composante par composante).

Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous k:



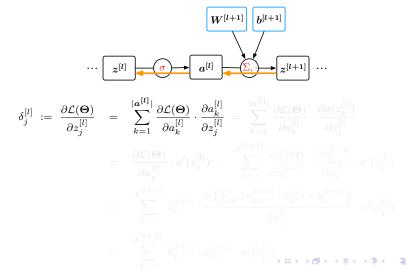
BACKPROPAGATION

Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous k:



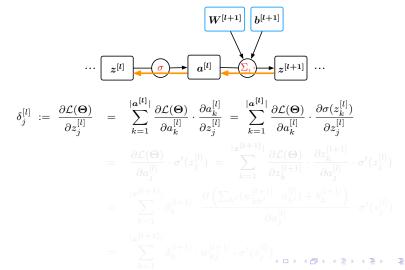
BACKPROPAGATION

Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous k:

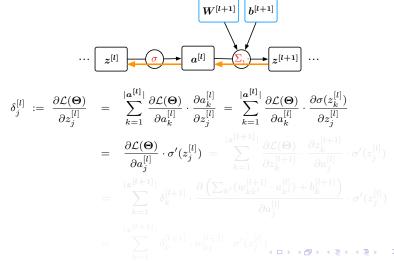


CALCUL DES GRADIENTS: ÉQUATION 2

Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous k:

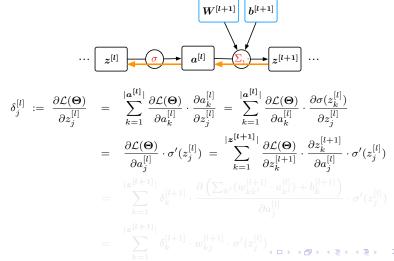


Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous k:



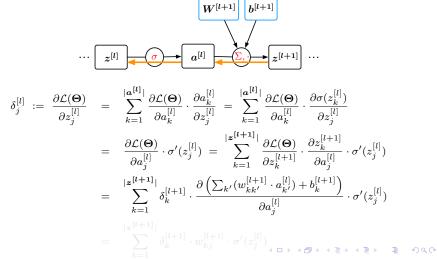
Calcul des gradients: équation 2

Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous k:



Calcul des gradients: équation 2

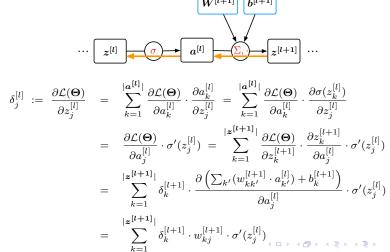
Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous k:



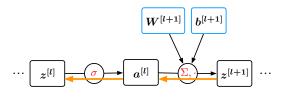
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Calcul des gradients: équation 2

Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous k:



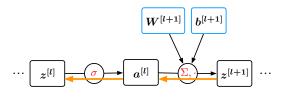
Formulation vectorielle:



$$\begin{split} \boldsymbol{\delta}^{[l]} &:= & \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \\ &= & \left[\boldsymbol{\delta}^{[l+1]}\right]^T \boldsymbol{W}^{[l+1]} \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}) \\ &= & \left[\boldsymbol{W}^{[l+1]}\right]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}) \end{split}$$

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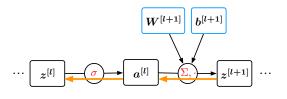
Formulation vectorielle:



$$\begin{split} \boldsymbol{\delta}^{[l]} &:= & \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \\ &= & \left[\boldsymbol{\delta}^{[l+1]} \right]^T \boldsymbol{W}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[l]}) \\ &= & \left[\boldsymbol{W}^{[l+1]} \right]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[l]}) \end{split}$$

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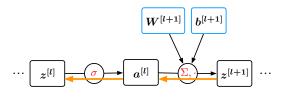
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Formulation vectorielle:

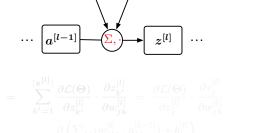


$$\begin{split} \boldsymbol{\delta}^{[l]} &:= \quad \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \\ &= \quad \left[\boldsymbol{\delta}^{[l+1]} \right]^T \boldsymbol{W}^{[l+1]} \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}) \\ &= \quad \left[\boldsymbol{W}^{[l+1]} \right]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}) \end{split}$$

où \odot est le produit de Hadamard (composante par composante).

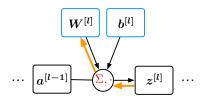
Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :

 $W^{[l]}$



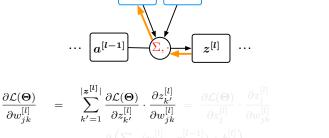
 $b^{[l]}$

Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :



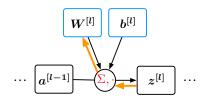
$$\begin{split} \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial \boldsymbol{w}_{jk}^{[l]}} &= \sum_{k'=1}^{\lfloor s-1 \rfloor} \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial w_{jk}^{[l]}} = \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial z_{j}^{[l]}} \cdot \frac{\partial z_{j}^{[l]}}{\partial w_{jk}^{[l]}} \\ &= \delta_{j}^{[l]} \cdot \frac{\partial \left(\sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_{k}^{[l]}\right)}{\partial w_{jk}^{[l]}} = \delta_{j}^{[l]} \cdot a_{k}^{[l-1]} \end{split}$$

Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :

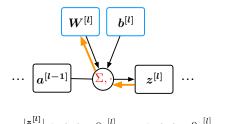


 $b^{[l]}$

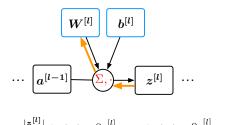
Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :



$$\begin{array}{ll} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial w_{jk}^{[l]}} & = & \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial w_{jk}^{[l]}} = & \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{j}^{[l]}} \cdot \frac{\partial z_{j}^{[l]}}{\partial w_{jk}^{[l]}} \\ & = & \delta_{j}^{[l]} \cdot \frac{\partial \left(\sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_{k}^{[l]}\right)}{\partial w_{jk}^{[l]}} = & \delta_{j}^{[l]} \cdot a_{k}^{[l-1]} \end{array}$$

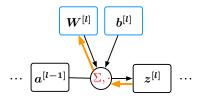


$$\begin{array}{lll} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial w_{jk}^{[l]}} & = & \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial w_{jk}^{[l]}} = & \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{j}^{[l]}} \cdot \frac{\partial z_{j}^{[l]}}{\partial w_{jk}^{[l]}} \\ & = & \delta_{j}^{[l]} \cdot \frac{\partial \left(\sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_{k}^{[l]}\right)}{\partial w_{jk}^{[l]}} = & \delta_{j}^{[l]} \cdot a_{k}^{[l-1]} \end{array}$$



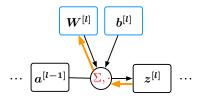
$$\begin{array}{ll} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial w_{jk}^{[l]}} & = & \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial w_{jk}^{[l]}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{j}^{[l]}} \cdot \frac{\partial z_{j}^{[l]}}{\partial w_{jk}^{[l]}} \\ & = & \delta_{j}^{[l]} \cdot \frac{\partial \left(\sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_{k}^{[l]}\right)}{\partial w_{jk}^{[l]}} = \delta_{j}^{[l]} \cdot a_{k}^{[l-1]} \end{array}$$

Formulation vectorielle:



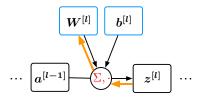
$$abla_{oldsymbol{W}^{[l]}} \mathcal{L}(oldsymbol{\Theta}) = \delta^{[l]} \left[oldsymbol{a}^{[l-1]}
ight]^T$$

Formulation vectorielle:



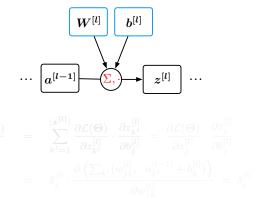
$$abla_{oldsymbol{W}^{[l]}} \mathcal{L}(oldsymbol{\Theta}) \ = \ \delta^{[l]} \left[a^{[l-1]}
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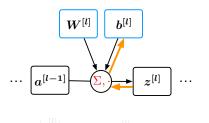
Formulation vectorielle:



$$abla_{oldsymbol{W}^{[l]}} \mathcal{L}(oldsymbol{\Theta}) \ = \ oldsymbol{\delta}^{[l]} \left[oldsymbol{a}^{[l-1]}
ight]^T$$

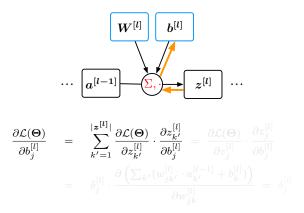
Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :

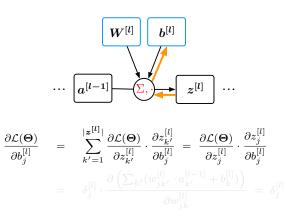




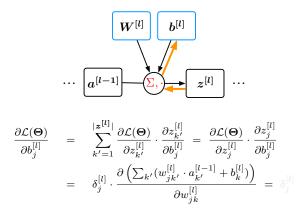
$$\frac{\partial}{\partial z} = \sum_{k'=1}^{\lfloor 2^{l+1} \rfloor} \frac{\partial \mathcal{L}(\Theta)}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial b_{j}^{[l]}} = \frac{\partial \mathcal{L}(\Theta)}{\partial z_{j}^{[l]}} \cdot \frac{\partial z_{j}^{[l]}}{\partial b_{j}^{[l]}}$$

$$= \delta_{j}^{[l]} \cdot \frac{\partial \left(\sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]} + b_{k}^{[l]})\right)}{\partial w_{jk}^{[l]}} = \delta_{j}^{[l]}$$

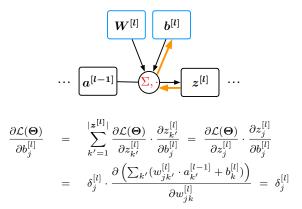




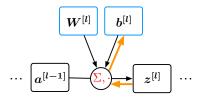
Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :



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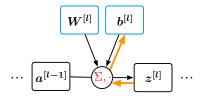


Formulation vectorielle:



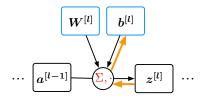
$$abla_{b^{[l]}} \mathcal{L}(\mathbf{\Theta}) = \delta^{[l]}$$

Formulation vectorielle:



$$abla_{m{b}^{[l]}} \mathcal{L}(m{\Theta}) = \delta^{[l]}$$

Formulation vectorielle:



$$\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) = \boldsymbol{\delta}^{[l]}$$

Calcul des erreurs $\delta^{[l]} := \nabla_{z^{[l]}} \mathcal{L}(\Theta)$ et des gradients $\nabla_{W^{[l]}} \mathcal{L}(\Theta)$ et $\nabla_{\mathbf{h}^{[l]}} \mathcal{L}(\mathbf{\Theta})$, pour toute couche $l = L, \ldots, 1$:

$$\delta^{[l]} = \begin{cases} \nabla_{a^{[l]}} \mathcal{L}(\mathbf{\Theta}) \odot \sigma'(z^{[l]}), & \text{si } l = L \\ \left[\mathbf{W}^{[l+1]} \right]^T \delta^{[l+1]} \odot \sigma'(z^{[l]}), & \text{si } L > l \ge 1 \end{cases}$$
(1)

$$\nabla_{W^{[l]}} \mathcal{L}(\Theta) = \delta^{[l]} \left[a^{[l-1]} \right]^T \tag{2}$$

$$\nabla_{b^{[l]}} \mathcal{L}(\mathbf{\Theta}) = \delta^{[l]} \tag{3}$$

Calcul des erreurs $\boldsymbol{\delta}^{[l]} := \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta})$ et des gradients $\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}(\boldsymbol{\Theta})$ et $\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}(\boldsymbol{\Theta})$, pour toute couche $l = L, \dots, 1$:

$$\boldsymbol{\delta}^{[l]} = \begin{cases} \nabla_{\boldsymbol{a}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}), & \text{si } l = L \\ \left[W^{[l+1]} \right]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}), & \text{si } L > l \ge 1 \end{cases}$$
(1)

Pour $l = L, \ldots, 1$:

$$\nabla_{W^{[l]}} \mathcal{L}(\Theta) = \delta^{[l]} \left[a^{[l-1]} \right]^T \tag{2}$$

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$$\nabla_{b^{[l]}} \mathcal{L}(\Theta) = \delta^{[l]} \tag{3}$$

Calcul des erreurs $\boldsymbol{\delta}^{[l]} := \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta})$ et des gradients $\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}(\boldsymbol{\Theta})$ et $\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}(\boldsymbol{\Theta})$, pour toute couche $l = L, \ldots, 1$:

$$\boldsymbol{\delta}^{[l]} = \begin{cases} \nabla_{\boldsymbol{a}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}), & \text{si } l = L \\ \left[\boldsymbol{W}^{[l+1]} \right]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}), & \text{si } L > l \ge 1 \end{cases}$$
(1)

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Calcul des erreurs $\boldsymbol{\delta}^{[l]} := \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta})$ et des gradients $\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}(\boldsymbol{\Theta})$ et $\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}(\boldsymbol{\Theta})$, pour toute couche $l = L, \dots, 1$:

$$\boldsymbol{\delta}^{[l]} = \begin{cases} \nabla_{\boldsymbol{a}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}), & \text{si } l = L \\ \left[\boldsymbol{W}^{[l+1]} \right]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}), & \text{si } L > l \ge 1 \end{cases}$$
(1)

$$\nabla_{W^{[l]}} \mathcal{L}(\mathbf{\Theta}) = \delta^{[l]} \left[a^{[l-1]} \right]^T \tag{2}$$

$$\nabla_{b^{[l]}} \mathcal{L}(\Theta) = \delta^{[l]} \tag{3}$$

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Une fois gradients calculés, on effectue l'update des poids et biais (cf. gradient descent algo):

$$\boldsymbol{W}^{[l]} := \boldsymbol{W}^{[l]} - \eta \cdot \nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \tag{4}$$

$$b^{[l]} := b^{[l]} - \eta \cdot \nabla_{b^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \tag{5}$$

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(5)

où η est le learning rate.

CHAIN RULE

Remarque: on peut déduire une version "batched" des équations.

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Soit B = (X, Y) un batch composé de B inputs et outputs x_k et y_k alignés en deux matrices:

$$egin{aligned} egin{aligned} egin{aligned} egin{aligned} egin{aligned} dallet & dallet & \dots & dallet \ m{x_1} & m{x_2} & \cdots & m{x_B} \ dallet & dallet & \dots & dallet \end{aligned} \end{aligned} egin{aligned} dallet & egin{aligned} dallet & dallet & \dots & dallet \ m{y_1} & m{y_2} & \cdots & m{y_B} \ dallet & dallet & \dots & dallet \end{aligned}$$

$$A^{[L]} = \begin{pmatrix} \vdots & \vdots & \dots & \vdots \\ a_1^{[L]} & a_2^{[L]} & \cdots & a_B^{[L]} \\ \vdots & \vdots & \dots & \vdots \end{pmatrix}$$

CHAIN BILLE

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BACKPROPAGATION

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Soit $A^{[L]}$ les outputs du réseau associés aux inputs X:

$$A^{[L]} = egin{pmatrix} dots & dots & \ldots & dots \ a_1^{[L]} & a_2^{[L]} & \cdots & a_B^{[L]} \ dots & dots & \ldots & dots \end{pmatrix}$$

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CHAIN BILLE

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BACKPROPAGATION

$$egin{aligned} X = egin{pmatrix} dots & dots & \ldots & dots \ x_1 & x_2 & \cdots & x_B \ dots & dots & \ldots & dots \end{pmatrix} & ext{et } Y = egin{pmatrix} dots & dots & \ldots & dots \ y_1 & y_2 & \cdots & y_B \ dots & dots & \ldots & dots \end{pmatrix} \end{aligned}$$

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lackbrack Soit $\mathcal{L}_k(m{\Theta}) := \mathcal{L}(m{\Theta}, m{a_k^{[L]}}, m{y_k})$ la loss associée à l'exemple k, pour $k = 1, \dots, B$. 4□ ト 4 □ ト 4 □ ト 4 □ ト 9 0 ○

CHAIN BILLE

Calcul des erreurs $\delta_{\mathbf{k}}^{[l]} := \nabla_{\mathbf{z}^{[l]}} \mathcal{L}_k(\mathbf{\Theta})$ et des gradients $\nabla_{\mathbf{W}[l]} \mathcal{L}_k(\mathbf{\Theta})$ et $\nabla_{\mathbf{h}[l]} \mathcal{L}_k(\mathbf{\Theta})$, pour tout exemple $k = 1, \dots, B$ pour toute couche $l = L, \ldots, 1$:

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$$\delta_{k}^{[l]} = \begin{cases} \nabla_{a^{[L]}} \mathcal{L}_{k}(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma}'(\boldsymbol{z}_{k}^{[L]}), & \text{si } l = L \\ \left[\boldsymbol{W}^{[l+1]} \right]^{T} \delta_{k}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}_{k}^{[l]}), & \text{si } L > l \ge 1 \end{cases}$$
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$$\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) = \delta_k^{[l]} \left[a_k^{[l-1]} \right]^T \tag{7}$$

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CHAIN BILLE

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$$\mathbf{W}^{[l]} := \mathbf{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\mathbf{W}^{[l]}} \mathcal{L}_k(\mathbf{\Theta})$$
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:

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$$l = L, \ldots, 1$$
:

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 (10)

où η est le *learning rate*.

FORWARD PASS AND BACKWARD PASS



FORWARD PASS AND BACKWARD PASS



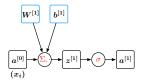


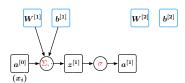


FORWARD PASS AND BACKWARD PASS

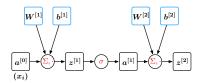


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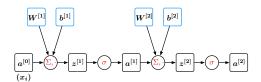




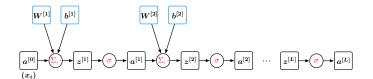
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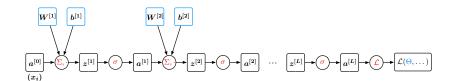


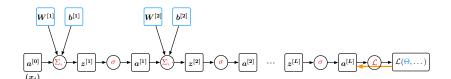
FORWARD PASS AND BACKWARD PASS



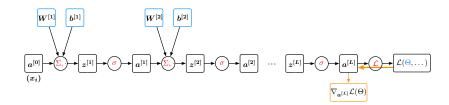
FORWARD PASS AND BACKWARD PASS



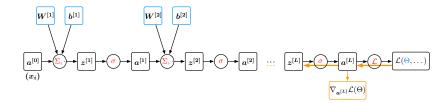




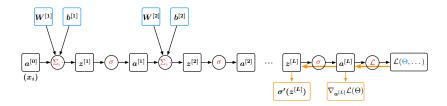
BACKPROPAGATION

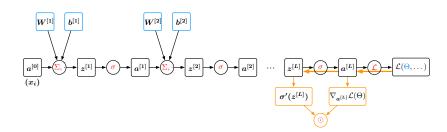


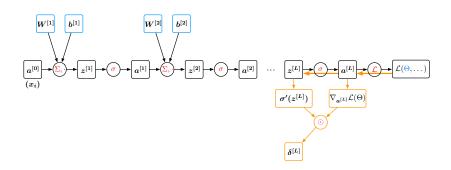
FORWARD PASS AND BACKWARD PASS

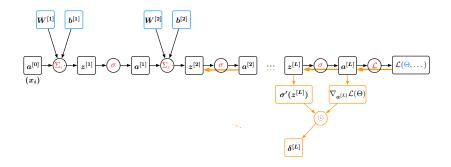


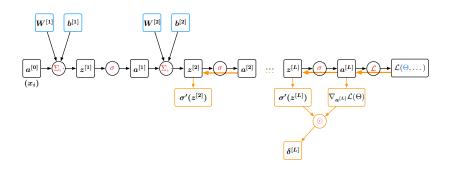
FORWARD PASS AND BACKWARD PASS

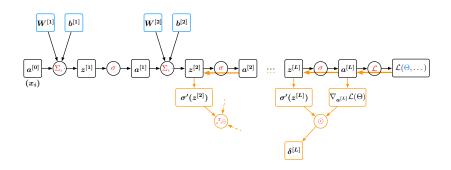


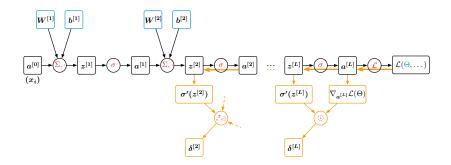


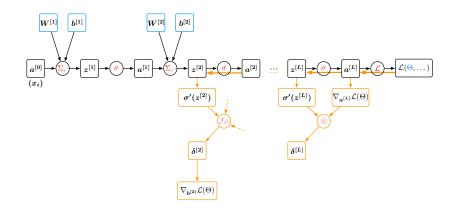


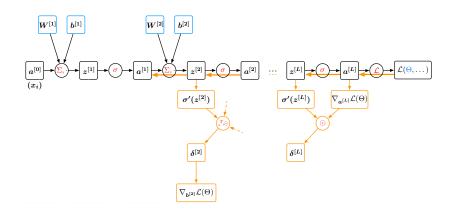


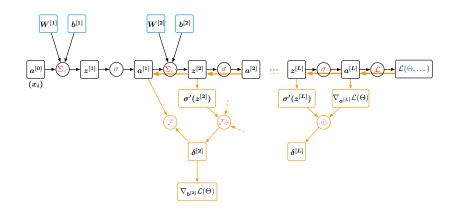


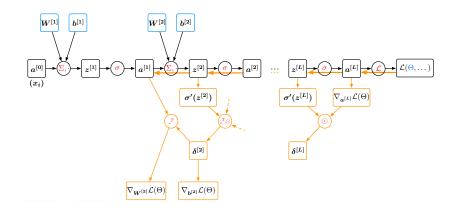


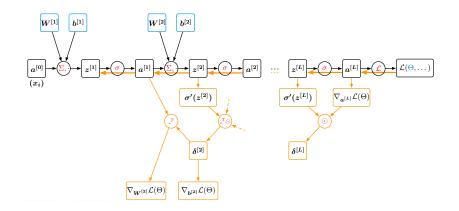


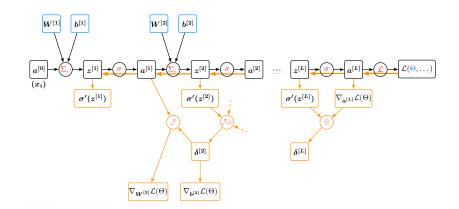


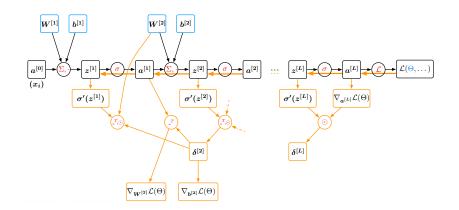


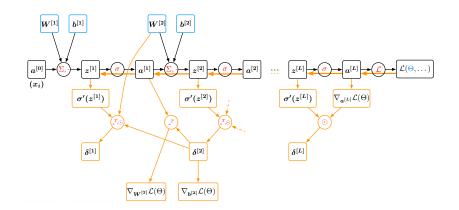


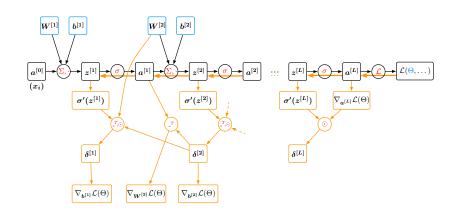


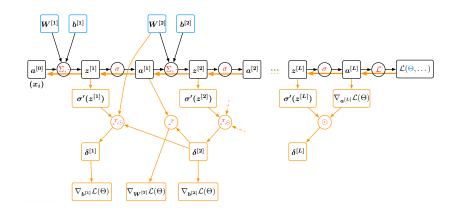


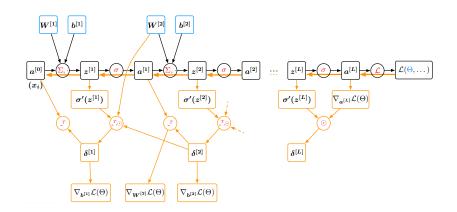


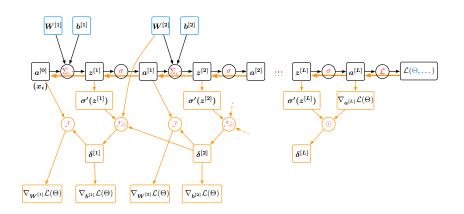












```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb \ batches\}
Inputs: MLP = \left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \dots, L \right\} initialized randomly
```

BACKPROPAGATION 0000000000000000000

Algorithm 1: Backpropagation (stochastic gradient descent)

```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb \ batches\}
Inputs: MLP = \left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \dots, L \right\} initialized randomly
```

```
\mbox{ for } epochs = 1 \mbox{ to } nb \mbox{ } epochs \mbox{ do}
        for i = 1 to nb batches do
```

// loop over epochs

```
\mathsf{Data:}\ \mathsf{dataloader} = \big\{B_{\boldsymbol{i}} = (\boldsymbol{X}_{\boldsymbol{i}}, \boldsymbol{Y}_{\boldsymbol{i}}) : i = i, \dots, nb\_batches\big\}
Inputs: MLP = \left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \dots, L \right\} initialized randomly
\  \, \mathbf{for}\,\,epochs = 1\,\,\mathbf{to}\,\,nb \  \, epochs\,\,\mathbf{do}
                                                                                                                                         // loop over epochs
         for i = 1 to nb batches do
                                                                                                                                        // loop over batches
                    for l=1 to L do
```

Backpropagation 000000000000000000000

BACKPROPAGATION (BP)

```
\mathsf{Data:}\ \mathsf{dataloader} = \big\{B_{\boldsymbol{i}} = (\boldsymbol{X}_{\boldsymbol{i}}, \boldsymbol{Y}_{\boldsymbol{i}}) : i = i, \dots, nb\_batches\big\}
Inputs: MLP = \left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \dots, L \right\} initialized randomly
\  \, \mathbf{for}\,\,epochs = 1\,\,\mathbf{to}\,\,nb \  \, epochs\,\,\mathbf{do}
                                                                                                                                      // loop over epochs
         for i = 1 to nb batches do
                                                                                                                                    // loop over batches
                   for l = 1 to L do
                                                                                                                                              // forward pass
```

```
\mathsf{Data:}\ \mathsf{dataloader} = \big\{B_{\boldsymbol{i}} = (\boldsymbol{X}_{\boldsymbol{i}}, \boldsymbol{Y}_{\boldsymbol{i}}) : i = i, \dots, nb\_batches\big\}
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\  \, \mathbf{for}\,\,epochs = 1\,\,\mathbf{to}\,\,nb \  \, epochs\,\,\mathbf{do}
                                                                                                                               // loop over epochs
         for i = 1 to nb batches do
                                                                                                                              // loop over batches
                  for l = 1 to L do
                                                                                                                                      // forward pass
                  for l = L to 1 do
                                                                                                                                     // backward pass
```

```
\mathsf{Data:}\ \mathsf{dataloader} = \big\{B_{\boldsymbol{i}} = (\boldsymbol{X}_{\boldsymbol{i}}, \boldsymbol{Y}_{\boldsymbol{i}}) : i = i, \dots, nb\_batches\big\}
Inputs: MLP = \left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \dots, L \right\} initialized randomly
\  \, \mathbf{for}\,\,epochs = 1\,\,\mathbf{to}\,\,nb \  \, epochs\,\,\mathbf{do}
                                                                                                                            // loop over epochs
        for i = 1 to nb batches do
                                                                                                                           // loop over batches
                  for l = 1 to L do
                                                                                                                                   // forward pass
                  for l = L to 1 do
                                                                                                                                  // backward pass
                                                                                                                                  // compute error
```

```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb \ batches\}
Inputs: MLP = \left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \dots, L \right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                        // loop over epochs
      for i = 1 to nb batches do
                                                                                       // loop over batches
             for l = 1 to L do
                                                                                             // forward pass
             for l = L to 1 do
                                                                                            // backward pass
                                                                                            // compute error
                                                                                          // update gradient
```

Backpropagation 000000000000000000000

```
\overline{\mathsf{Data: dataloader}} = \big\{ B_{i} = (X_{i}, Y_{i}) : i = i, \dots, nb\_batches \big\}
Inputs: MLP = \left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \dots, L \right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                               // loop over epochs
      for i = 1 to nb batches do
                                                                                              // loop over batches
              for l = 1 to L do
                                                                                                     // forward pass
              for l = L to 1 do
                                                                                                   // backward pass
                                                                                                   // compute error
                                                                                                 // update gradient
                                                                                                 // update gradient
       end
```

BACKPROPAGATION

end

```
\mathsf{Data:}\ \mathsf{dataloader} = \big\{B_{\boldsymbol{i}} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \dots, nb\_batches\big\}
Inputs: MLP = \{(\mathbf{W}^{[l]}, \mathbf{b}^{[l]}) : l = 1, ..., L\} initialized randomly
```

Algorithm 2: Backpropagation (stochastic gradient descent)

```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb \ batches\}
Inputs: MLP = \left\{\left(m{W}^{[l]}, m{b}^{[l]}\right): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                       // loop over epochs
      for i = 1 to nb batches do
```

Algorithm 2: Backpropagation (stochastic gradient descent)

```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb \ batches\}
Inputs: MLP = \left\{\left(m{W}^{[l]}, m{b}^{[l]}\right): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                      // loop over epochs
      for i = 1 to nb\_batches do
                                                                                     // loop over batches
```

Algorithm 2: Backpropagation (stochastic gradient descent)

```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb \ batches\}
Inputs: MLP = \left\{\left(m{W}^{[l]}, m{b}^{[l]}\right): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                        // loop over epochs
      for i = 1 to nb\_batches do
                                                                                      // loop over batches
            A^{[0]} = \overline{X_i}
            for l = 1 to L do
                                                                                            // forward pass
```

```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb \ batches\}
Inputs: MLP = \{(\mathbf{W}^{[l]}, \mathbf{b}^{[l]}) : l = 1, ..., L\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                          // loop over epochs
      for i = 1 to nb\_batches do
                                                                                        // loop over batches
             A^{[0]} = \bar{X_i}
             for l = 1 to L do
                                                                                               // forward pass
                    Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                   A^{[l]} = \sigma\left(Z^{[l]}\right)
             end
```

```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb \ batches\}
Inputs: MLP = \{(\mathbf{W}^{[l]}, \mathbf{b}^{[l]}) : l = 1, ..., L\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                        // loop over epochs
      for i = 1 to nb\_batches do
                                                                                       // loop over batches
             A^{[0]} = \bar{X_i}
             for l = 1 to L do
                                                                                             // forward pass
                   Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                   A^{[l]} = \sigma\left(Z^{[l]}\right)
             end
             for l = L to 1 do
                                                                                            // backward pass
```

Algorithm 2: Backpropagation (stochastic gradient descent)

```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb \ batches\}
Inputs: MLP = \left\{\left(oldsymbol{W}^{[l]}, oldsymbol{b}^{[l]}
ight): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                                                // loop over epochs
        for i = 1 to nb batches do
                                                                                                              // loop over batches
                A^{[0]} = \bar{X}_{i}
                for l = 1 to L do
                                                                                                                      // forward pass
                        Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                        A^{[l]} = \sigma\left(Z^{[l]}\right)
                end
                for l = L to 1 do
                                                                                                                     // backward pass
                        if l = L then
                                                                                                                     // compute error
                                \delta_{\mathbf{h}}^{[l]} = \nabla_{[l]} \mathcal{L}_{k}(\boldsymbol{\Theta}) \odot \sigma'(\boldsymbol{z}_{\mathbf{h}}^{[l]})
                                                                                               for k = 1, \ldots, B
                        else if L > l > 1 then
                                \delta_{h}^{[l]} = \begin{bmatrix} W^{[l+1]} \end{bmatrix}^{T} \delta_{h}^{[l+1]} \odot \sigma'(z_{h}^{[l]})
                                                                                              for k = 1, \dots, B
                        end
```

```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb \ batches\}
Inputs: MLP = \left\{\left(oldsymbol{W}^{[l]}, oldsymbol{b}^{[l]}
ight): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                                                 // loop over epochs
        for i = 1 to nb batches do
                                                                                                               // loop over batches
                A^{[0]} = \bar{X}_{i}
                for l = 1 to L do
                                                                                                                       // forward pass
                        Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                        A^{[l]} = \sigma\left(Z^{[l]}\right)
                end
                for l = L to 1 do
                                                                                                                     // backward pass
                        if l = L then
                                                                                                                     // compute error
                                \delta_k^{[l]} = \nabla_{\sigma^{[l]}} \mathcal{L}_k(\Theta) \odot \sigma'(z_k^{[l]})
                                                                                                for k = 1, \ldots, B
                        else if L > l > 1 then
                                \delta_{h}^{[l]} = \begin{bmatrix} W^{[l+1]} \end{bmatrix}^{T} \delta_{h}^{[l+1]} \odot \sigma'(z_{h}^{[l]})
                                                                                               for k = 1, \dots, B
                         end
                        \nabla_{\mathbf{W}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_k^{[l]} \left[ a_k^{[l-1]} \right]^T
                                                                                                for k = 1, \dots, B
```

BACKPROPAGATION

```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb \ batches\}
Inputs: MLP = \left\{\left(oldsymbol{W}^{[l]}, oldsymbol{b}^{[l]}
ight): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                                                    // loop over epochs
        for i = 1 to nb batches do
                                                                                                                  // loop over batches
                 A^{[0]} = X_i
                 for l = 1 to L do
                                                                                                                          // forward pass
                         Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                         A^{[l]} = \sigma\left(Z^{[l]}\right)
                 end
                 for l = L to 1 do
                                                                                                                         // backward pass
                         if l = L then
                                                                                                                         // compute error
                                 \delta_{k}^{[l]} = \nabla_{[l]} \mathcal{L}_{k}(\Theta) \odot \sigma'(z_{k}^{[l]})
                                                                                                  for k = 1, \ldots, B
                         else if L > l > 1 then
                                 \delta_{h}^{[l]} = \begin{bmatrix} W^{[l+1]} \end{bmatrix}^T \delta_{h}^{[l+1]} \odot \sigma'(z_{h}^{[l]})
                                                                                                  for k = 1, \dots, B
                         end
                         \nabla_{\mathbf{W}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_k^{[l]} \left[ a_k^{[l-1]} \right]^T
                                                                                                   for k = 1, \dots, B
                         \nabla_{\mathbf{h}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_{i}^{[l]}
                                                                                                  for k = 1, \ldots, B
```

```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb\_batches\}
Inputs: MLP = \{(\mathbf{W}^{[l]}, \mathbf{b}^{[l]}) : l = 1, ..., L\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                                                          // loop over epochs
        for i = 1 to nb batches do
                                                                                                                         // loop over batches
                  A^{[0]} = X_i
                  for l = 1 to L do
                                                                                                                                 // forward pass
                          Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                          A^{[l]} = \sigma\left(Z^{[l]}\right)
                  end
                  for l = L to 1 do
                                                                                                                               // backward pass
                          if l = L then
                                                                                                                               // compute error
                                  \delta_{k}^{[l]} = \nabla_{[l]} \mathcal{L}_{k}(\Theta) \odot \sigma'(z_{k}^{[l]})
                                                                                                        for k = 1, \ldots, B
                          else if L > l > 1 then
                                   \delta_{h}^{[l]} = \begin{bmatrix} W^{[l+1]} \end{bmatrix}^{T} \delta_{h}^{[l+1]} \odot \sigma'(z_{h}^{[l]})
                                                                                                       for k = 1, \dots, B
                           end
                          \nabla_{\mathbf{W}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_k^{[l]} \left[ a_k^{[l-1]} \right]^T
                                                                                                        for k = 1, \dots, B
                          \nabla_{\mathbf{h}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_{\mathbf{h}}^{[l]}
                                                                                                        for k = 1, \ldots, B
                          \mathbf{W}^{[l]} := \mathbf{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\mathbf{v}_{\mathbf{v}_{\ell}}[l]} \mathcal{L}_{k}(\mathbf{\Theta})
                                                                                                                            // update gradient
```

Backpropagation 000000000000000000

```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb \ batches\}
Inputs: MLP = \{(\mathbf{W}^{[l]}, \mathbf{b}^{[l]}) : l = 1, ..., L\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                                                              // loop over epochs
         for i = 1 to nb batches do
                                                                                                                             // loop over batches
                  A^{[0]} = X_i
                  for l = 1 to L do
                                                                                                                                     // forward pass
                           Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                           A^{[l]} = \sigma\left(Z^{[l]}\right)
                  end
                  for l = L to 1 do
                                                                                                                                    // backward pass
                           if l = L then
                                                                                                                                    // compute error
                                    \delta_{k}^{[l]} = \nabla_{[l]} \mathcal{L}_{k}(\Theta) \odot \sigma'(z_{k}^{[l]})
                                                                                                           for k = 1, \dots, B
                           else if L > l > 1 then
                                    \delta_{k}^{[l]} = \begin{bmatrix} W^{[l+1]} \end{bmatrix}^{T} \delta_{k}^{[l+1]} \odot \sigma'(z_{k}^{[l]})
                                                                                                          for k = 1, \dots, B
                            end
                           \nabla_{\mathbf{W}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_k^{[l]} \left[ a_k^{[l-1]} \right]^T
                                                                                                            for k = 1, \dots, B
                           \nabla_{\mathbf{h}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_{\mathbf{h}}^{[l]}
                                                                                                           for k = 1, \ldots, B
                           \mathbf{W}^{[l]} := \mathbf{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\mathbf{v}_{\mathbf{v}_{\ell}}[l]} \mathcal{L}_{k}(\mathbf{\Theta})
                                                                                                                                // update gradient
                          b^{[l]} := b^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\mathbf{r}[l]} \mathcal{L}_{k}(\boldsymbol{\Theta})
                                                                                                                                // update gradient
```

```
Data: dataloader = \{B_i = (X_i, Y_i) : i = i, ..., nb \ batches\}
Inputs: MLP = \left\{\left(oldsymbol{W}^{[l]}, oldsymbol{b}^{[l]}
ight): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                                                                 // loop over epochs
         for i = 1 to nb batches do
                                                                                                                               // loop over batches
                  A^{[0]} = \bar{X}_{i}
                  for l = 1 to L do
                                                                                                                                        // forward pass
                            Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                           A^{[l]} = \sigma\left(Z^{[l]}\right)
                  end
                  for l = L to 1 do
                                                                                                                                      // backward pass
                            if l = L then
                                                                                                                                      // compute error
                                    \delta_{k}^{[l]} = \nabla_{[l]} \mathcal{L}_{k}(\Theta) \odot \sigma'(z_{k}^{[l]})
                                                                                                             for k = 1, \dots, B
                            else if L > l > 1 then
                                     \delta_{h}^{[l]} = \begin{bmatrix} W^{[l+1]} \end{bmatrix}^{T} \delta_{h}^{[l+1]} \odot \sigma'(z_{h}^{[l]})
                                                                                                            for k = 1, \dots, B
                            end
                            \nabla_{\mathbf{W}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_k^{[l]} \left[ a_k^{[l-1]} \right]^T
                                                                                                              for k = 1, \dots, B
                           \nabla_{\mathbf{b}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_{\mathbf{b}}^{[l]}
                                                                                                             for k = 1, \dots, B
                            \mathbf{W}^{[l]} := \mathbf{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\mathbf{v}_{\mathbf{v}_{\ell}}[l]} \mathcal{L}_{k}(\mathbf{\Theta})
                                                                                                                                  // update gradient
                           b^{[l]} := b^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\mathbf{r}[l]} \mathcal{L}_k(\boldsymbol{\Theta})
                                                                                                                                  // update gradient
                  end
         end
end
```

EXEMPLE DE TRAINING VIA BACKROP

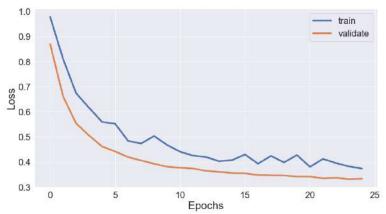


Figure taken from towardsdatascience.com

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