# BACKPROPAGATION

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# CHAIN RULE

On rappelle le **théorème des fonctions composées** (chain rule).

Soient les fonctions suivantes:

$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R} & \xrightarrow{g} & \mathbb{R} \\ x & \xmapsto{f} & y = f(x) & \xmapsto{g} & z = g(y) \\ & & & & = g\left(f(x)\right) \end{array}$$

alors on a:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

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# CHAIN RULE

**Exemple:**

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "f"] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \\
x \arrow[r, mapsto, "f"] & y = x^2+1 \arrow[r, mapsto, "g"] & \begin{aligned} z &= 5y \\ &= 5(x^2+1) \end{aligned}
\end{tikzcd}$$

On a:

$$\frac{\partial z}{\partial x} = \frac{\partial \left[5(x^2+1)\right]}{\partial x} = 10x = 5 \cdot 2x = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

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# Chain Rule

## Généralisation multidimensionnelle

Soient les fonctions suivantes:

$$\begin{array}{ccccc} \mathbb{R}^m & \xrightarrow{f} & \mathbb{R}^n & \xrightarrow{g} & \mathbb{R} \\ \boldsymbol{x} & \xmapsto{f} & \boldsymbol{y} = f(\boldsymbol{x}) & \xmapsto{g} & z = g(\boldsymbol{y}) \\ & & & & = g\left(f(\boldsymbol{x})\right) \end{array}$$

alors on a:

$$\frac{\partial z}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial z}{\partial y_j} \cdot \frac{\partial y_j}{\partial x_i}, \quad i = 1, \ldots, m$$

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# Chain Rule

## Formulation vectorielle

$$\begin{array}{ccccc} \mathbb{R}^m & \xrightarrow{f} & \mathbb{R}^n & \xrightarrow{g} & \mathbb{R} \\ \boldsymbol{x} & \overset{f}{\longmapsto} & \boldsymbol{y} = f(\boldsymbol{x}) & \overset{g}{\longmapsto} & z = g(\boldsymbol{y}) \\ & & & & \quad = g\left(f(\boldsymbol{x})\right) \end{array}$$

Soient $\nabla_{\boldsymbol{x}} z$ et le *gradient* de $z$ par rapport à $\boldsymbol{x}$, $\nabla_{\boldsymbol{y}} z$ et le gradient de $z$ par rapport à $\boldsymbol{y}$, et $\boldsymbol{J}_f := \left[\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\right]$ le *jacobien* de la fonction $f$:

$$\nabla_{\boldsymbol{x}} z = \left[ \left[\nabla_{\boldsymbol{y}} z\right]^T \left[\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\right] \right]^T = \boldsymbol{J}_f^T \nabla_{\boldsymbol{y}} z$$

$$\begin{pmatrix} \frac{\partial z}{\partial x_1} \\ \vdots \\ \frac{\partial z}{\partial x_m} \end{pmatrix} = \left[ \begin{pmatrix} \frac{\partial z}{\partial y_1} & \cdots & \frac{\partial z}{\partial y_n} \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_m} \end{pmatrix} \right]^T$$

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# Chain Rule

**Exemple:**

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "f"] & \mathbb{R}^n \arrow[r, "g"] & \mathbb{R} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \arrow[r, mapsto, "f"] & \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1x_2 \\ x_1^2 \\ x_2^3 \end{pmatrix} \arrow[r, mapsto, "g"] & z = y_1y_2 + y_3 \\ & & = (x_1x_2)x_1^2 + x_2^3 \end{tikzcd}$$

On a:

$$\begin{aligned} \frac{\partial z}{\partial x_1} &= \frac{\partial\left[(x_1x_2)x_1^2 + x_2^3\right]}{\partial x_1} && = x_1^2x_2 + x_1x_2 2x_1 + 0 \\ & && = \frac{\partial z}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_1} + \frac{\partial z}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_1} + \frac{\partial z}{\partial y_3} \cdot \frac{\partial y_3}{\partial x_1} \\ \frac{\partial z}{\partial x_2} &= \frac{\partial\left[(x_1x_2)x_1^2 + x_2^3\right]}{\partial x_2} && = x_1^2x_1 + 3x_2^2 + 0 \\ & && = \frac{\partial z}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_2} + \frac{\partial z}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_2} + \frac{\partial z}{\partial y_3} \cdot \frac{\partial y_3}{\partial x_2} \end{aligned}$$

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**Exemple (suite):**

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On a:

$$\nabla_{\boldsymbol{x}} z = \begin{pmatrix} \frac{\partial z}{\partial x_1} \\ \frac{\partial z}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 3 x_2 x_1^2 \\ x_1^2 + 3 x_2^2 \end{pmatrix} = \begin{pmatrix} y_2 & y_1 & 1 \end{pmatrix} \begin{pmatrix} x_2 & x_1 \\ 2x_1 & 0 \\ 0 & 3x_2^2 \end{pmatrix} = \left[\nabla_{\boldsymbol{y}} z\right]^T \left[\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\right]$$

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# Neural Network as a Function

- Soit $S = \{(\boldsymbol{x_i}, \boldsymbol{y_i}) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} : i = 1, \ldots, N\}$ un dataset.
- Soit $\mathcal{N}_{\boldsymbol{\Theta}}$ un réseau de neurones (MLP) à $L$ couches donné par les *paramètres* (poids et biais)

$$\boldsymbol{\Theta} := \left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \ldots, L \right\}$$

et par la dynamique

$$\begin{cases} \boldsymbol{a}^{[0]} & = \boldsymbol{x} \\ \boldsymbol{z}^{[l]} & = \boldsymbol{W}^{[l]} \boldsymbol{a}^{[l-1]} + \boldsymbol{b}^{[l]}, \\ \boldsymbol{a}^{[l]} & = \boldsymbol{\sigma} \left( \boldsymbol{z}^{[l]} \right) \end{cases} \qquad l = 1, \ldots, L$$

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# NEURAL NETWORK AS A FUNCTION

- **Remarque:** le réseau $\mathcal{N}_{\boldsymbol{\Theta}}$ peut-être naturellement associé à la fonction

$$\begin{array}{rcl} f_{\boldsymbol{\Theta}} : \mathbb{R}^{d_1} & \longrightarrow & \mathbb{R}^{d_2} \\ \boldsymbol{x} & \longmapsto & f_{\boldsymbol{\Theta}}(\boldsymbol{x}) := \boldsymbol{a}^{[\boldsymbol{L}]} \end{array}$$

- $f_{\Theta}(x)$ est la *prédiction* (output) de $\mathcal{N}_{\Theta}$ associée à l'input $x$.
- Chaque jeu de paramètres $\boldsymbol{\Theta}$ donne lieu à une fonction $f_{\Theta}$ différente.

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# LOSS FUNCTION

- Soit une *fonction de coût* (cost or loss function) qui mesure l'erreur entre la *prédiction* $\hat{\boldsymbol{y}}_{\boldsymbol{i}}$ et la *réalité* $\boldsymbol{y}_{\boldsymbol{i}}$:

$$\begin{aligned} \ell : \mathbb{R}^{d_2} \times \mathbb{R}^{d_2} &\longrightarrow \mathbb{R} \\ (\hat{\boldsymbol{y}}_{\boldsymbol{i}}, \boldsymbol{y}_{\boldsymbol{i}}) &\longmapsto \mathcal{L}(\hat{\boldsymbol{y}}_{\boldsymbol{i}}, \boldsymbol{y}_{\boldsymbol{i}}) \end{aligned}$$

- Typiquement, la fonction de coût pourrait être l'erreur quadratique (distance Euclidienne au carré)

$$\ell(\hat{\boldsymbol{y}}_{\boldsymbol{i}}, \boldsymbol{y}_{\boldsymbol{i}}) = \frac{1}{2} \left\| \hat{\boldsymbol{y}}_{\boldsymbol{i}} - \boldsymbol{y}_{\boldsymbol{i}} \right\|_2^2$$

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# Loss Function

- La *fonction de coût* peut être naturellement généralisée à un ensemble de *prédictions* et de *réalités*:

$$\begin{aligned} \mathcal{L} : \mathbb{R}^{d_2} \times \cdots\cdots \times \mathbb{R}^{d_2} &\longrightarrow \mathbb{R} \\ \left(\hat{\boldsymbol{y}}_{\mathbf{1}}, \ldots, \hat{\boldsymbol{y}}_{\boldsymbol{N}}, \boldsymbol{y}_{\boldsymbol{i}} \ldots, \boldsymbol{y}_{\boldsymbol{N}}\right) &\longmapsto \mathcal{L}\left(\hat{\boldsymbol{y}}_{\mathbf{1}}, \ldots, \hat{\boldsymbol{y}}_{\boldsymbol{N}}, \boldsymbol{y}_{\boldsymbol{i}} \ldots, \boldsymbol{y}_{\boldsymbol{N}}\right) \end{aligned}$$

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# LOSS FUNCTION

- Pour un réseau de neurones $\mathcal{N}_{\boldsymbol{\Theta}}$, l'erreur entre les prédictions et les réalités est

$$\mathcal{L}\left(f_{\boldsymbol{\Theta}}\left(\boldsymbol{x_1}\right), \ldots, f_{\boldsymbol{\Theta}}\left(\boldsymbol{x_N}\right), \boldsymbol{y_1}, \ldots, \boldsymbol{y_N}\right).$$

- Pour différents paramètres $\boldsymbol{\Theta}$, on aura différentes prédictions $f_{\boldsymbol{\Theta}}\left(\boldsymbol{x_1}\right), \ldots, f_{\boldsymbol{\Theta}}\left(\boldsymbol{x_N}\right)$, et donc différentes erreurs $\mathcal{L}\left(\ldots\right)$.
- **Ainsi, $\mathcal{L}$ est une fonction des paramètres $\boldsymbol{\Theta}$ du réseau:**

$$\begin{aligned} \mathcal{L} : \mathbb{R}^{|\boldsymbol{\Theta}|} &\longrightarrow \mathbb{R} \\ \boldsymbol{\Theta} &\longmapsto \mathcal{L}\left(f_{\boldsymbol{\Theta}}\left(\boldsymbol{x_1}\right), \ldots, f_{\boldsymbol{\Theta}}\left(\boldsymbol{x_N}\right), \boldsymbol{y_1}, \ldots, \boldsymbol{y_N}\right). \end{aligned}$$

où $|\boldsymbol{\Theta}|$ est le nombre de paramètres $\boldsymbol{\Theta}$ (poids et biais, souvent plusieurs millions).

# Loss Function

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# Training

- *L'entraînement* du réseau $\mathcal{N}_{\boldsymbol{\Theta}}$ consiste à déterminer des paramètres $\boldsymbol{\Theta}$ qui minimisent l'erreur

$$\mathcal{L}\Big(f_{\boldsymbol{\Theta}}\left(\boldsymbol{x_1}\right), \ldots, f_{\boldsymbol{\Theta}}\left(\boldsymbol{x_N}\right), \boldsymbol{y_1}, \ldots, \boldsymbol{y_N}\Big).$$

- Pour cela, on utilise une descente de gradient: *mini-batch stochastic gradient descent*.
- *Backpropagation* est un algorithme qui permet de calculer les gradients $\nabla_{\boldsymbol{\Theta}}\mathcal{L}$ de manière efficiente.

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# TRAINING

$$\mathcal{L}\left(f_{\boldsymbol{\Theta}}\left(\boldsymbol{x_1}\right),\ldots,f_{\boldsymbol{\Theta}}\left(\boldsymbol{x_N}\right),\boldsymbol{y_1},\ldots,\boldsymbol{y_N}\right)=\mathcal{L}\left(\boldsymbol{\Theta},\ldots\right)$$

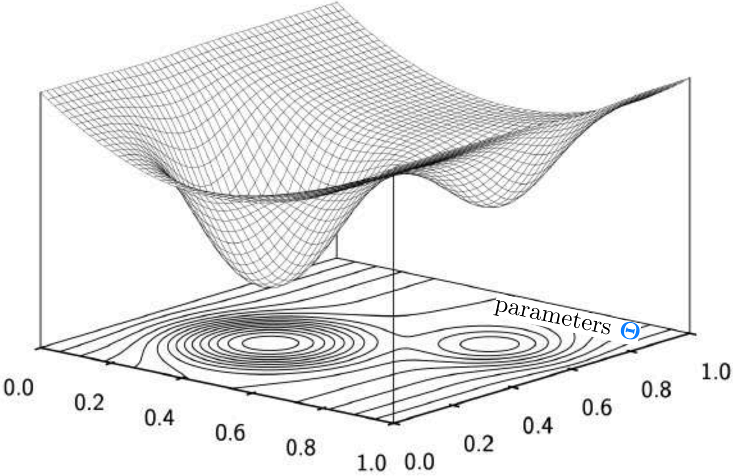

Figure adapted from [?]

# TRAINING

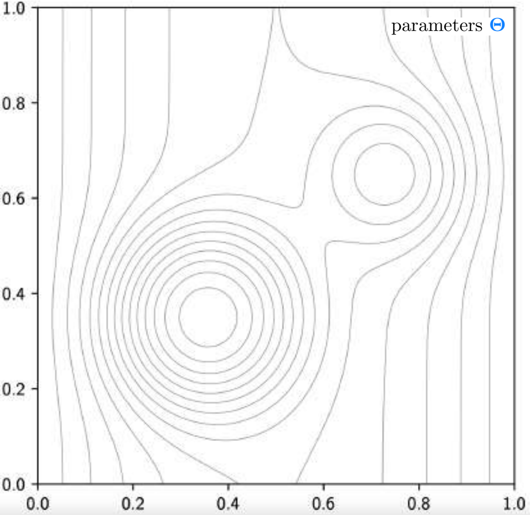

Figure adapted from [?]

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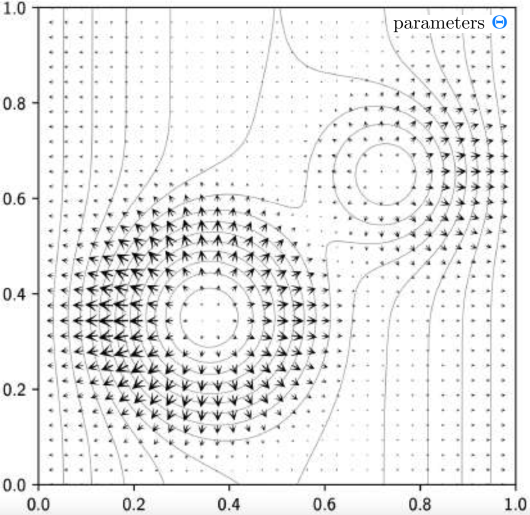

Figure adapted from [?]

# Training

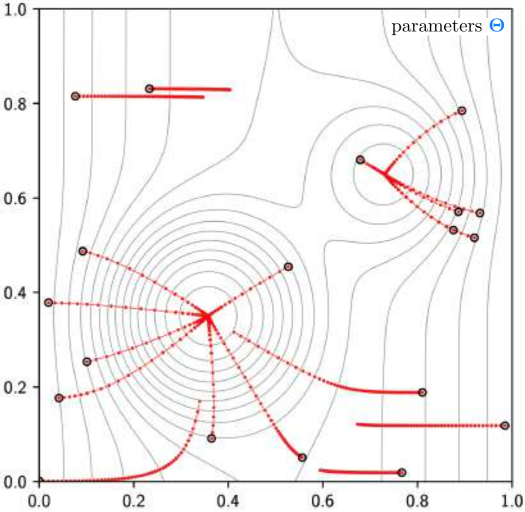

Figure adapted from [?]

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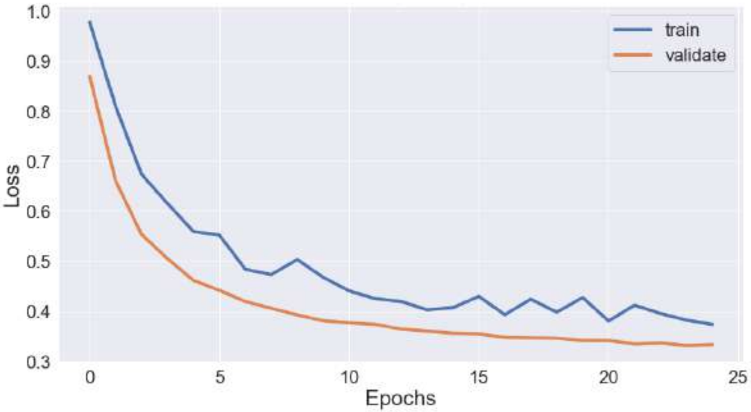
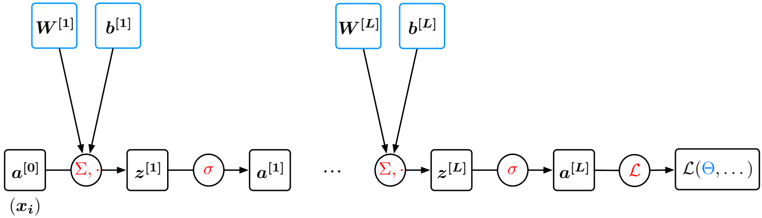


Figure taken from towardsdatascience.com

# Graphe computationnel d'un réseau de neurones (forward pass)

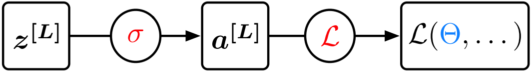


- On veut calculer les gradients:

$$\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) := \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial \boldsymbol{W}^{[l]}} \quad \text{et} \quad \nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) := \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial \boldsymbol{b}^{[l]}}$$

pour $l = 1, \dots, M$

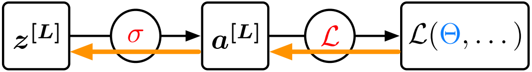
# Calcul des gradients: équation 1



$\boldsymbol{z}^{[\boldsymbol{L}]}$ $\sigma$ $\boldsymbol{a}^{[\boldsymbol{L}]}$ $\mathcal{L}$ $\mathcal{L}(\Theta, \dots)$

$$\begin{aligned}
\delta_j^{[L]} \; := \; \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[L]}} \quad &= \quad \sum_{k=1}^{|\boldsymbol{a}^{[\boldsymbol{L}]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_k^{[L]}} \cdot \frac{\partial a_k^{[L]}}{\partial z_j^{[L]}} \\
&= \quad \sum_{k=1}^{|\boldsymbol{a}^{[\boldsymbol{L}]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_k^{[L]}} \cdot \frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} \\
\left( \frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} = 0 \text{ for } k \neq j \right) \quad &= \quad \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_j^{[L]}} \cdot \sigma'(z_j^{[L]})
\end{aligned}$$

# CALCUL DES GRADIENTS: ÉQUATION 1



$$
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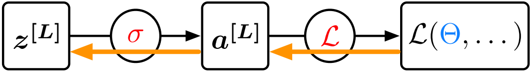

$$\delta_j^{[L]} := \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[L]}} = \sum_{k=1}^{|\boldsymbol{a}^{[\boldsymbol{L}]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_k^{[L]}} \cdot \frac{\partial a_k^{[L]}}{\partial z_j^{[L]}}$$

$$= \sum_{k=1}^{|\boldsymbol{a}^{[\boldsymbol{L}]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_k^{[L]}} \cdot \frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}}$$

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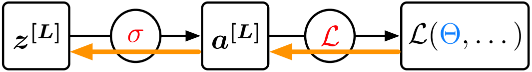

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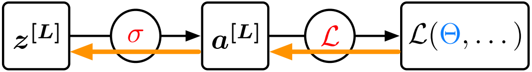
# Calcul des gradients: équation 1



$$\begin{aligned}
\delta_j^{[L]} := \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[L]}} &= \sum_{k=1}^{|\boldsymbol{a}^{[\boldsymbol{L}]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_k^{[L]}} \cdot \frac{\partial a_k^{[L]}}{\partial z_j^{[L]}} \\
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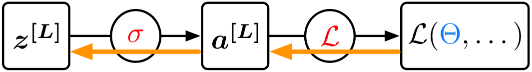
\end{aligned}$$

# Calcul des gradients: équation 1

Formulation vectorielle:



$$\delta^{[L]} := \nabla_{z^{[L]}} \mathcal{L}(\Theta) = \nabla_{a^{[L]}} \mathcal{L}(\Theta) \odot \sigma'(z^{[L]})$$

où $\odot$ est le produit de Hadamard (composante par composante).

# CALCUL DES GRADIENTS: ÉQUATION 1

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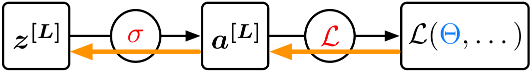

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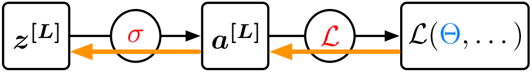
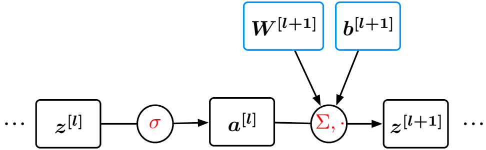


$$\boldsymbol{\delta}^{[L]} \;:=\; \nabla_{\boldsymbol{z}^{[L]}} \mathcal{L}(\boldsymbol{\Theta}) \;=\; \nabla_{\boldsymbol{a}^{[L]}} \mathcal{L}(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[L]})$$

où $\odot$ est le produit de Hadamard (composante par composante).

# CALCUL DES GRADIENTS: ÉQUATION 2

Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous $k$:



$$\begin{aligned}
\delta_j^{[l]} := \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} &= \sum_{k=1}^{|\boldsymbol{a}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_k^{[l]}} \cdot \frac{\partial a_k^{[l]}}{\partial z_j^{[l]}} = \sum_{k=1}^{|\boldsymbol{a}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_k^{[l]}} \cdot \frac{\partial \sigma(z_k^{[l]})}{\partial z_j^{[l]}} \\
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&= \sum_{k=1}^{|\boldsymbol{z}^{[l+1]}|} \delta_k^{[l+1]} \cdot \frac{\partial \left( \sum_{k'} (w_{kk'}^{[l+1]} \cdot a_{k'}^{[l]}) + b_k^{[l+1]} \right)}{\partial a_j^{[l]}} \cdot \sigma'(z_j^{[l]}) \\
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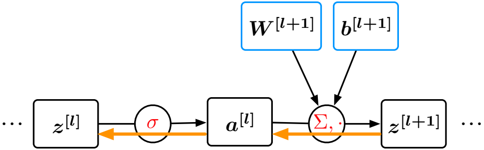

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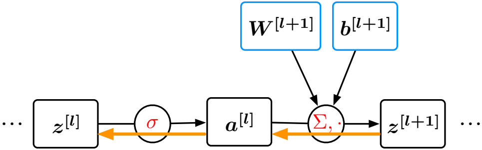

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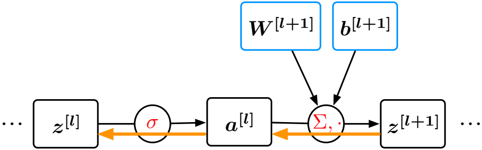

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# Calcul des gradients: équation 2

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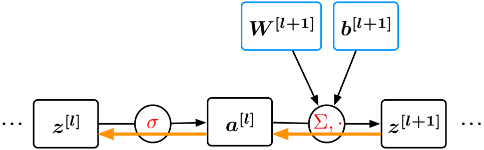

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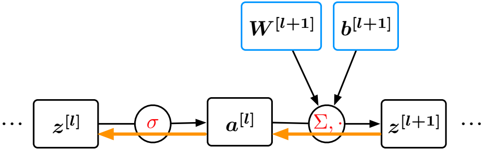
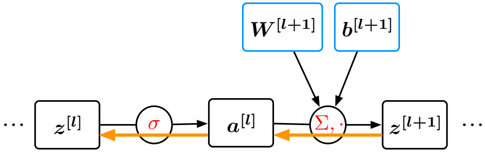


$$
\begin{aligned}
\delta_j^{[l]} := \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} &= \sum_{k=1}^{|\boldsymbol{a}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_k^{[l]}} \cdot \frac{\partial a_k^{[l]}}{\partial z_j^{[l]}} = \sum_{k=1}^{|\boldsymbol{a}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_k^{[l]}} \cdot \frac{\partial \sigma(z_k^{[l]})}{\partial z_j^{[l]}} \\
&= \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_j^{[l]}} \cdot \sigma'(z_j^{[l]}) = \sum_{k=1}^{|\boldsymbol{z}^{[l+1]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_k^{[l+1]}} \cdot \frac{\partial z_k^{[l+1]}}{\partial a_j^{[l]}} \cdot \sigma'(z_j^{[l]}) \\
&= \sum_{k=1}^{|\boldsymbol{z}^{[l+1]}|} \delta_k^{[l+1]} \cdot \frac{\partial \left( \sum_{k'} (w_{kk'}^{[l+1]} \cdot a_{k'}^{[l]}) + b_k^{[l+1]} \right)}{\partial a_j^{[l]}} \cdot \sigma'(z_j^{[l]}) \\
&= \sum_{k=1}^{|\boldsymbol{z}^{[l+1]}|} \delta_k^{[l+1]} \cdot w_{kj}^{[l+1]} \cdot \sigma'(z_j^{[l]})
\end{aligned}
$$

# Calcul des gradients: équation 2

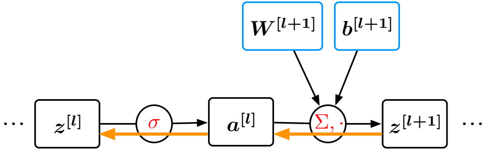
Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous $k$:



$$\begin{aligned}
\delta_j^{[l]} := \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} &= \sum_{k=1}^{|\boldsymbol{a}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_k^{[l]}} \cdot \frac{\partial a_k^{[l]}}{\partial z_j^{[l]}} = \sum_{k=1}^{|\boldsymbol{a}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_k^{[l]}} \cdot \frac{\partial \sigma(z_k^{[l]})}{\partial z_j^{[l]}} \\
&= \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_j^{[l]}} \cdot \sigma'(z_j^{[l]}) = \sum_{k=1}^{|\boldsymbol{z}^{[l+1]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_k^{[l+1]}} \cdot \frac{\partial z_k^{[l+1]}}{\partial a_j^{[l]}} \cdot \sigma'(z_j^{[l]}) \\
&= \sum_{k=1}^{|\boldsymbol{z}^{[l+1]}|} \delta_k^{[l+1]} \cdot \frac{\partial \left( \sum_{k'} (w_{kk'}^{[l+1]} \cdot a_{k'}^{[l]}) + b_k^{[l+1]} \right)}{\partial a_j^{[l]}} \cdot \sigma'(z_j^{[l]}) \\
&= \sum_{k=1}^{|\boldsymbol{z}^{[l+1]}|} \delta_k^{[l+1]} \cdot w_{kj}^{[l+1]} \cdot \sigma'(z_j^{[l]})
\end{aligned}$$

# CALCUL DES GRADIENTS: ÉQUATION 2

Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous $k$:



$$\delta_j^{[l]} := \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} = \sum_{k=1}^{|\boldsymbol{a}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_k^{[l]}} \cdot \frac{\partial a_k^{[l]}}{\partial z_j^{[l]}} = \sum_{k=1}^{|\boldsymbol{a}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_k^{[l]}} \cdot \frac{\partial \sigma(z_k^{[l]})}{\partial z_j^{[l]}}$$

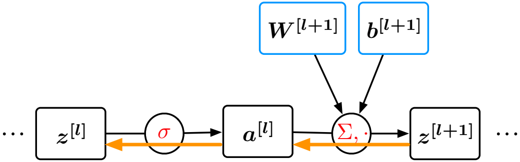
$$= \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial a_j^{[l]}} \cdot \sigma'(z_j^{[l]}) = \sum_{k=1}^{|\boldsymbol{z}^{[l+1]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_k^{[l+1]}} \cdot \frac{\partial z_k^{[l+1]}}{\partial a_j^{[l]}} \cdot \sigma'(z_j^{[l]})$$

$$= \sum_{k=1}^{|\boldsymbol{z}^{[l+1]}|} \delta_k^{[l+1]} \cdot \frac{\partial \left( \sum_{k'} (w_{kk'}^{[l+1]} \cdot a_{k'}^{[l]}) + b_k^{[l+1]} \right)}{\partial a_j^{[l]}} \cdot \sigma'(z_j^{[l]})$$

$$= \sum_{k=1}^{|\boldsymbol{z}^{[l+1]}|} \delta_k^{[l+1]} \cdot w_{kj}^{[l+1]} \cdot \sigma'(z_j^{[l]})$$

# CALCUL DES GRADIENTS: ÉQUATION 2

Formulation vectorielle:

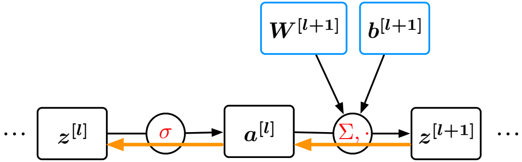


$$\begin{aligned}\boldsymbol{\delta}^{[l]} &:= \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \\ &= \left[\boldsymbol{\delta}^{[l+1]}\right]^T \boldsymbol{W}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[l]}) \\ &= \left[\boldsymbol{W}^{[l+1]}\right]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[l]})\end{aligned}$$

où $\odot$ est le produit de Hadamard (composante par composante).

# Calcul des gradients: équation 2

Formulation vectorielle:

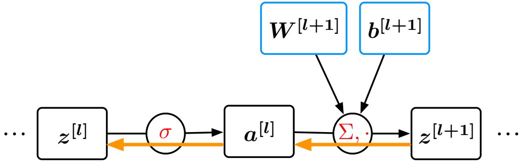


$$
\begin{aligned}
\boldsymbol{\delta}^{[l]} &:= \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \\
&= \left[\boldsymbol{\delta}^{[l+1]}\right]^T \boldsymbol{W}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[l]}) \\
&= \left[\boldsymbol{W}^{[l+1]}\right]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[l]})
\end{aligned}
$$

où $\odot$ est le produit de Hadamard (composante par composante).

# Calcul des gradients: équation 2

Formulation vectorielle:

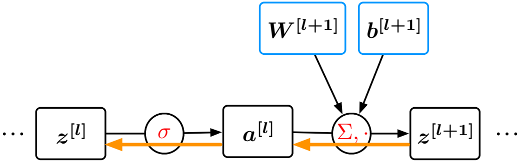


$$
\begin{aligned}
\boldsymbol{\delta}^{[l]} &:= \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \\
&= \left[\boldsymbol{\delta}^{[l+1]}\right]^T \boldsymbol{W}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[l]}) \\
&= \left[\boldsymbol{W}^{[l+1]}\right]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[l]})
\end{aligned}
$$

où $\odot$ est le produit de Hadamard (composante par composante).

# Calcul des gradients: équation 2

Formulation vectorielle:

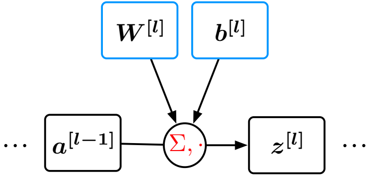


$$\begin{aligned}
\boldsymbol{\delta}^{[l]} \quad &:= \quad \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \\
&= \quad \left[\boldsymbol{\delta}^{[l+1]}\right]^T \boldsymbol{W}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[l]}) \\
&= \quad \left[\boldsymbol{W}^{[l+1]}\right]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[l]})
\end{aligned}$$

où $\odot$ est le produit de Hadamard (composante par composante).

# CALCUL DES GRADIENTS: ÉQUATION 3

Calcul des gradients proprement dits en utilisant les erreurs $\delta_j^l$:

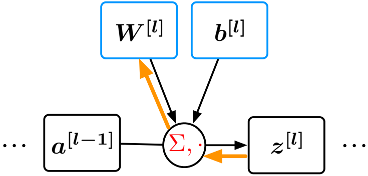


$$\frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial w_{jk}^{[l]}} = \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial w_{jk}^{[l]}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} \cdot \frac{\partial z_j^{[l]}}{\partial w_{jk}^{[l]}}$$

$$= \delta_j^{[l]} \cdot \frac{\partial \left( \sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_j^{[l]} \right)}{\partial w_{jk}^{[l]}} = \delta_j^{[l]} \cdot a_k^{[l-1]}$$

# Calcul des gradients: équation 3

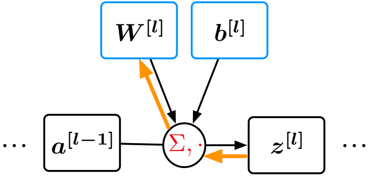
Calcul des gradients proprement dits en utilisant les erreurs $\delta^l_j$:



$$\begin{aligned}\frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial w^{[l]}_{jk}} &= \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z^{[l]}_{k'}} \cdot \frac{\partial z^{[l]}_{k'}}{\partial w^{[l]}_{jk}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z^{[l]}_{j}} \cdot \frac{\partial z^{[l]}_{j}}{\partial w^{[l]}_{jk}} \\ &= \delta^{[l]}_j \cdot \frac{\partial\left(\sum_{k'}(w^{[l]}_{jk'} \cdot a^{[l-1]}_{k'}) + b^{[l]}_j\right)}{\partial w^{[l]}_{jk}} = \delta^{[l]}_j \cdot a^{[l-1]}_k\end{aligned}$$

# CALCUL DES GRADIENTS: ÉQUATION 3

Calcul des gradients proprement dits en utilisant les erreurs $\delta_j^l$:

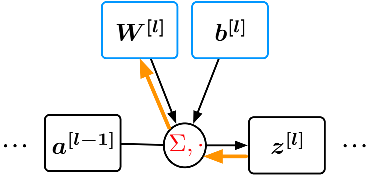


$$\frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial w_{jk}^{[l]}} = \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial w_{jk}^{[l]}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{j}^{[l]}} \cdot \frac{\partial z_{j}^{[l]}}{\partial w_{jk}^{[l]}}$$

$$= \delta_j^{[l]} \cdot \frac{\partial \left(\sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_j^{[l]}\right)}{\partial w_{jk}^{[l]}} = \delta_j^{[l]} \cdot a_k^{[l-1]}$$

# Calcul des gradients: équation 3

Calcul des gradients proprement dits en utilisant les erreurs $\delta_j^l$:

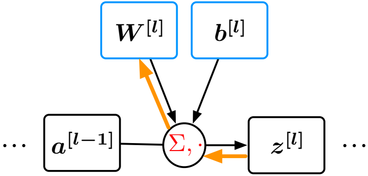


$$\frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial w_{jk}^{[l]}} = \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial w_{jk}^{[l]}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} \cdot \frac{\partial z_j^{[l]}}{\partial w_{jk}^{[l]}}$$

$$= \delta_j^{[l]} \cdot \frac{\partial \left( \sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_j^{[l]} \right)}{\partial w_{jk}^{[l]}} = \delta_j^{[l]} \cdot a_k^{[l-1]}$$

# Calcul des gradients: équation 3

Calcul des gradients proprement dits en utilisant les erreurs $\delta_j^l$:

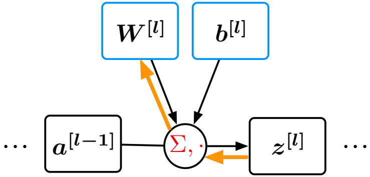


$$\frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial w_{jk}^{[l]}} = \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial w_{jk}^{[l]}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} \cdot \frac{\partial z_j^{[l]}}{\partial w_{jk}^{[l]}}$$

$$= \delta_j^{[l]} \cdot \frac{\partial\left(\sum_{k'}(w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_j^{[l]}\right)}{\partial w_{jk}^{[l]}} = \delta_j^{[l]} \cdot a_k^{[l-1]}$$

# CALCUL DES GRADIENTS: ÉQUATION 3

Calcul des gradients proprement dits en utilisant les erreurs $\delta_j^l$:

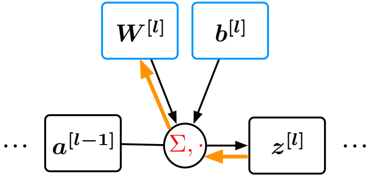


$$\frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial w_{jk}^{[l]}} = \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial w_{jk}^{[l]}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} \cdot \frac{\partial z_j^{[l]}}{\partial w_{jk}^{[l]}}$$

$$= \delta_j^{[l]} \cdot \frac{\partial \left(\sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_j^{[l]}\right)}{\partial w_{jk}^{[l]}} = \delta_j^{[l]} \cdot a_k^{[l-1]}$$

# Calcul des gradients: équation 3

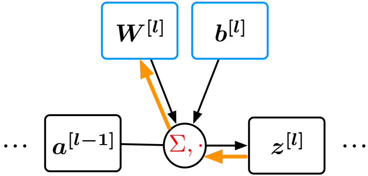
Formulation vectorielle:



$$\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) = \boldsymbol{\delta}^{[l]} \left[\boldsymbol{a}^{[l-1]}\right]^T$$

# CALCUL DES GRADIENTS: ÉQUATION 3

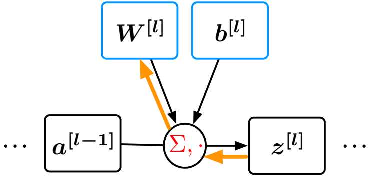
Formulation vectorielle:



$$\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) = \delta^{[l]} \left[ a^{[l-1]} \right]^T$$

# CALCUL DES GRADIENTS: ÉQUATION 3

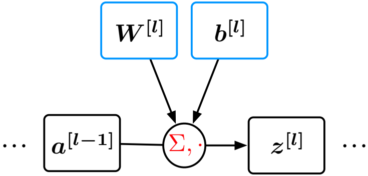
Formulation vectorielle:



$$\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \; = \; \boldsymbol{\delta}^{[l]} \left[ \boldsymbol{a}^{[l-1]} \right]^T$$

# Calcul des gradients: équation 4

Calcul des gradients proprement dits en utilisant les erreurs $\delta_j^l$:

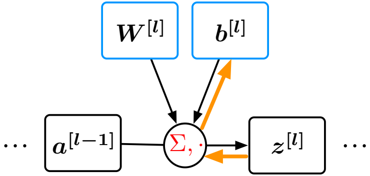


$$\frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial b_j^{[l]}} = \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial b_j^{[l]}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} \cdot \frac{\partial z_j^{[l]}}{\partial b_j^{[l]}}$$

$$= \delta_j^{[l]} \cdot \frac{\partial \left( \sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_j^{[l]} \right)}{\partial b_j^{[l]}} = \delta_j^{[l]}$$

# Calcul des gradients: équation 4

Calcul des gradients proprement dits en utilisant les erreurs $\delta_j^l$:

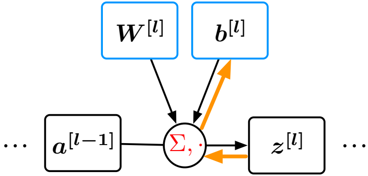


$$\frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial b_j^{[l]}} = \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial b_j^{[l]}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} \cdot \frac{\partial z_j^{[l]}}{\partial b_j^{[l]}}$$

$$= \delta_j^{[l]} \cdot \frac{\partial \left( \sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_j^{[l]} \right)}{\partial b_j^{[l]}} = \delta_j^{[l]}$$

# Calcul des gradients: équation 4

Calcul des gradients proprement dits en utilisant les erreurs $\delta_j^l$:

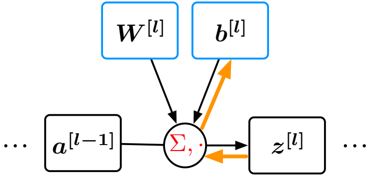


$$\frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial b_j^{[l]}} = \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial b_j^{[l]}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} \cdot \frac{\partial z_j^{[l]}}{\partial b_j^{[l]}}$$

$$= \delta_j^{[l]} \cdot \frac{\partial \left( \sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_j^{[l]} \right)}{\partial b_j^{[l]}} = \delta_j^{[l]}$$

# CALCUL DES GRADIENTS: ÉQUATION 4

Calcul des gradients proprement dits en utilisant les erreurs $\delta_j^l$:

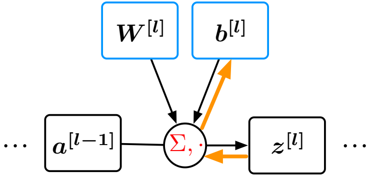


$$\frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial b_j^{[l]}} = \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial b_j^{[l]}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} \cdot \frac{\partial z_j^{[l]}}{\partial b_j^{[l]}}$$

$$= \delta_j^{[l]} \cdot \frac{\partial \left( \sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_j^{[l]} \right)}{\partial b_j^{[l]}} = \delta_j^{[l]}$$

# CALCUL DES GRADIENTS: ÉQUATION 4

Calcul des gradients proprement dits en utilisant les erreurs $\delta_j^l$:

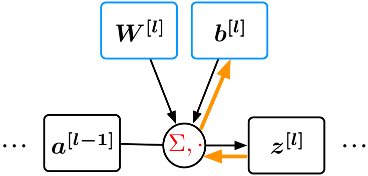


$$\frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial b_j^{[l]}} = \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial b_j^{[l]}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} \cdot \frac{\partial z_j^{[l]}}{\partial b_j^{[l]}}$$

$$= \delta_j^{[l]} \cdot \frac{\partial \left( \sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_j^{[l]} \right)}{\partial b_j^{[l]}} = \delta_j^{[l]}$$

# CALCUL DES GRADIENTS: ÉQUATION 4

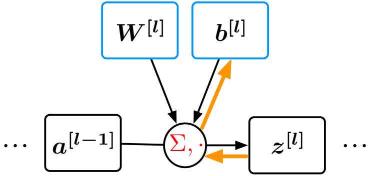
Calcul des gradients proprement dits en utilisant les erreurs $\delta_j^l$:



$$\begin{aligned}
\frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial b_j^{[l]}} &= \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial b_j^{[l]}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_j^{[l]}} \cdot \frac{\partial z_j^{[l]}}{\partial b_j^{[l]}} \\
&= \delta_j^{[l]} \cdot \frac{\partial \left( \sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_j^{[l]} \right)}{\partial b_j^{[l]}} = \delta_j^{[l]}
\end{aligned}$$

# CALCUL DES GRADIENTS: ÉQUATION 4

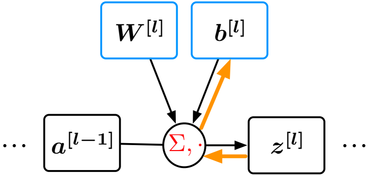
Formulation vectorielle:



$$\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) = \boldsymbol{\delta}^{[l]}$$

# CALCUL DES GRADIENTS: ÉQUATION 4

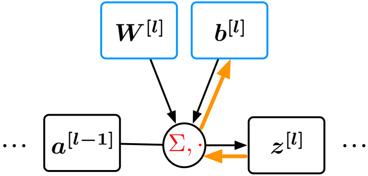
Formulation vectorielle:



$$\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) = \delta^{[l]}$$

# Calcul des gradients: équation 4

Formulation vectorielle:



$$\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \;=\; \boldsymbol{\delta}^{[l]}$$

# ÉQUATIONS

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# ÉQUATIONS: BATCHED

**Remarque:** on peut déduire une version "batched" des équations.

- Soit $B = (X, Y)$ un batch composé de $B$ inputs et outputs $x_k$ et $y_k$ alignés en deux matrices:

$$\boldsymbol{X} = \begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{x_1} & \boldsymbol{x_2} & \cdots & \boldsymbol{x_B} \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix} \text{ et } \boldsymbol{Y} = \begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{y_1} & \boldsymbol{y_2} & \cdots & \boldsymbol{y_B} \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix}$$

- Soit $A^{[L]}$ les outputs du réseau associés aux inputs $X$:

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où $\eta$ est le *learning rate*.

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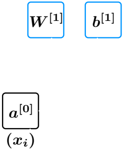
# Forward Pass and Backward Pass

$\boxed{\boldsymbol{a}^{[0]}}$

$(\boldsymbol{x}_i)$

# Forward Pass and Backward Pass



$\boldsymbol{W}^{[1]}$ $\boldsymbol{b}^{[1]}$

$\boldsymbol{a}^{[0]}$

$(\boldsymbol{x}_i)$

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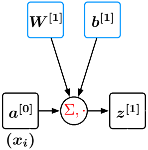

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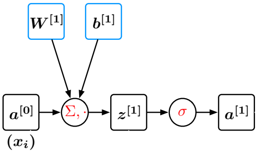
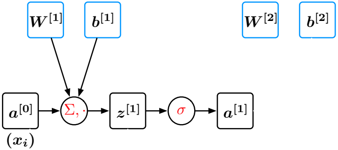
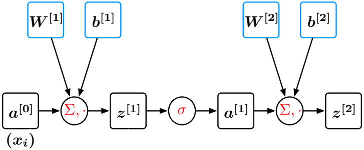


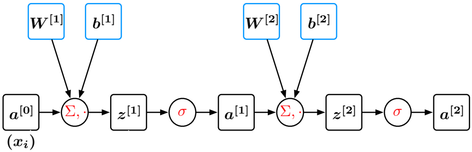
# Forward Pass and Backward Pass



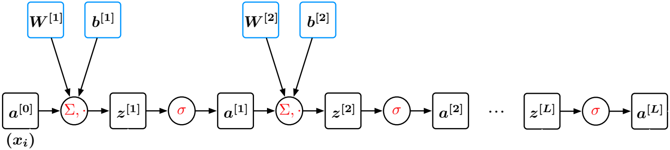
# Forward Pass and Backward Pass



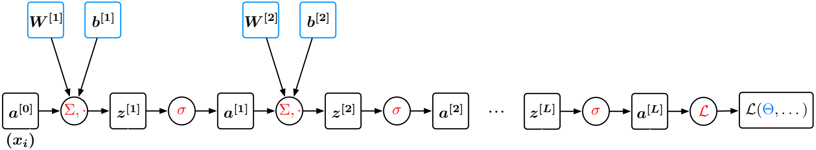
# Forward Pass and Backward Pass



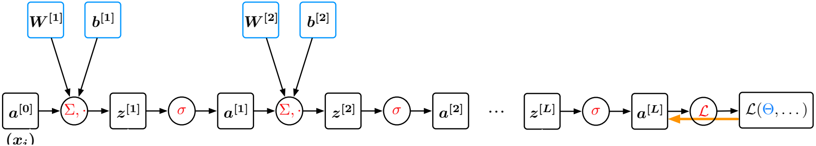
# Forward Pass and Backward Pass



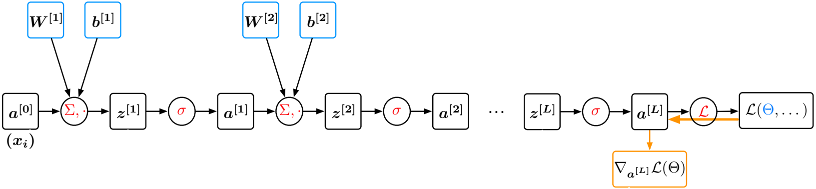
# Forward Pass and Backward Pass



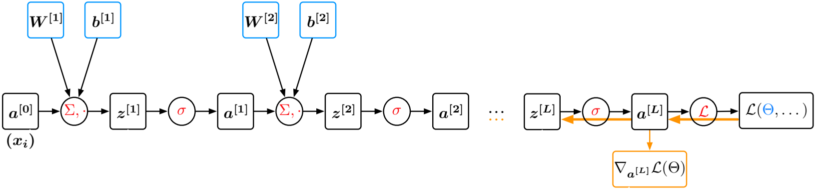
# Forward Pass and Backward Pass



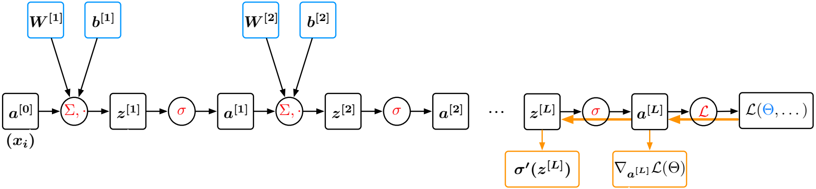
# Forward Pass and Backward Pass



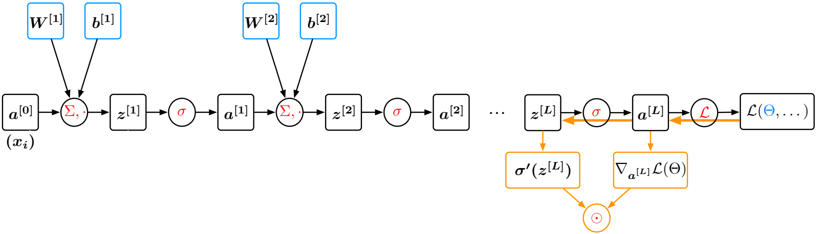
# Forward Pass and Backward Pass



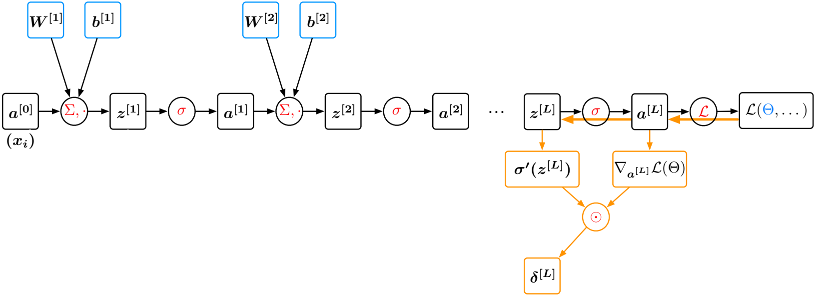
# Forward Pass and Backward Pass



# Forward Pass and Backward Pass



# Forward Pass and Backward Pass



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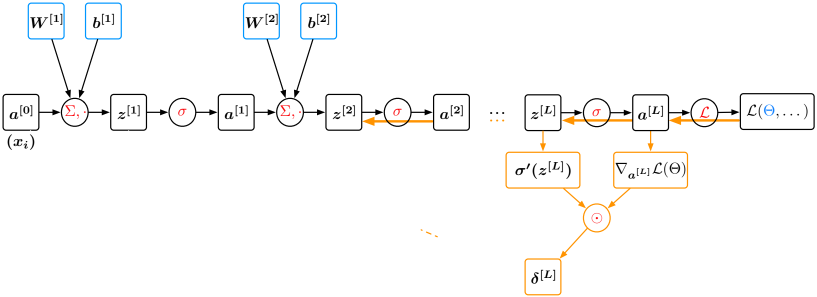
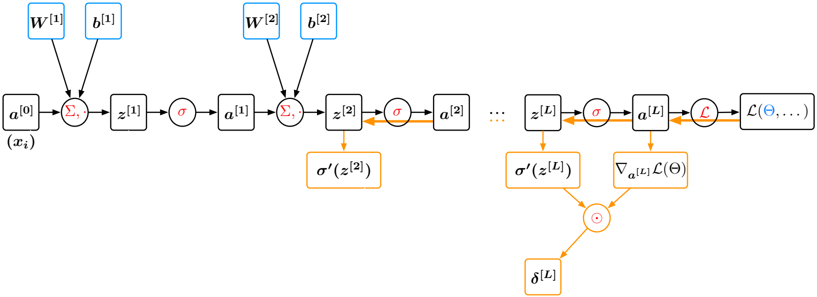


# FORWARD PASS AND BACKWARD PASS



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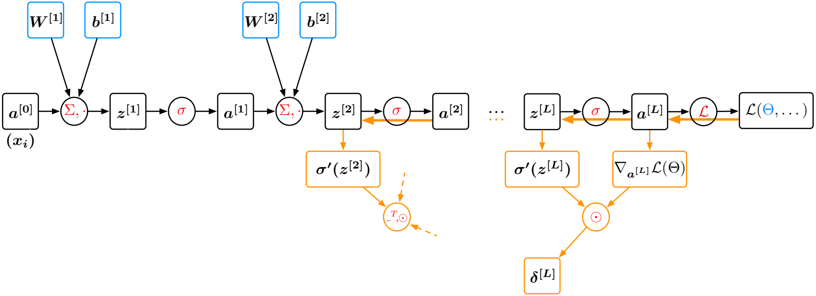

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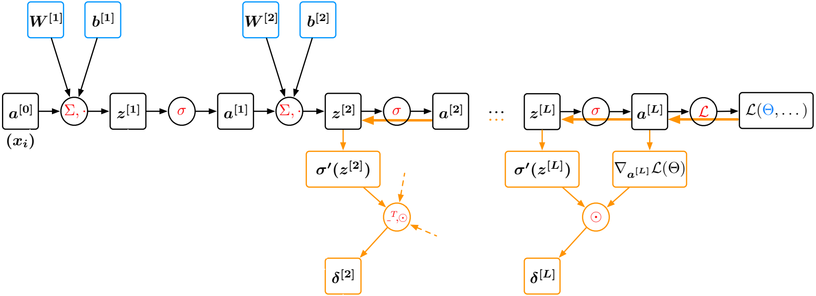

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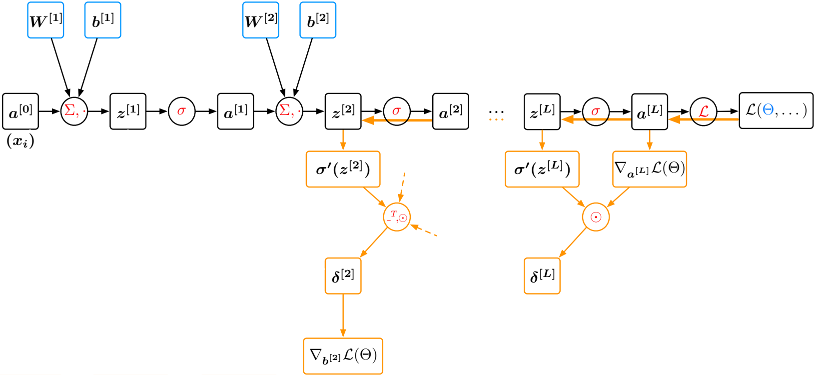

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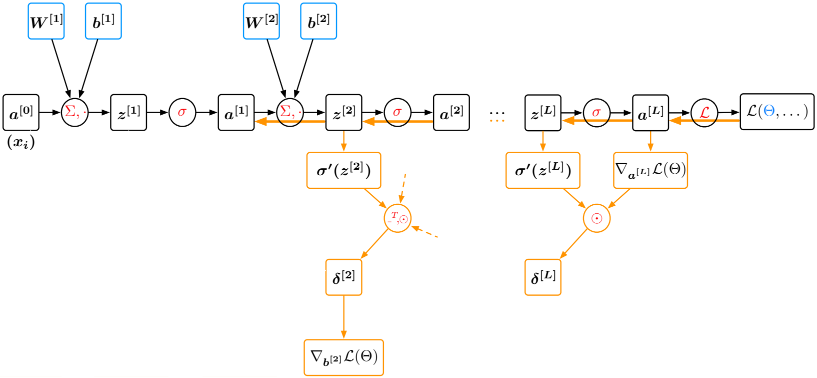

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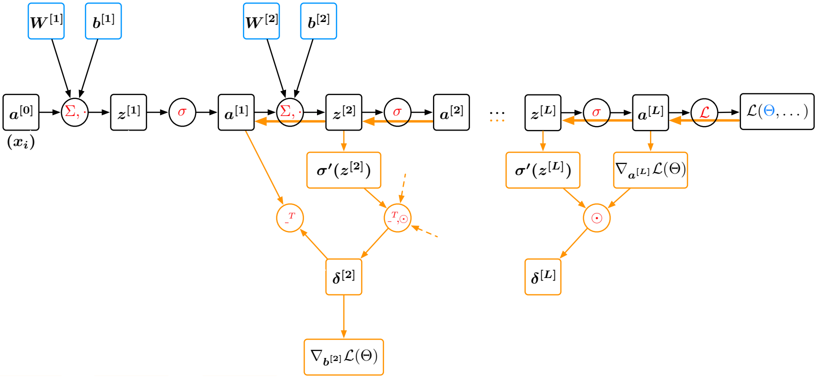
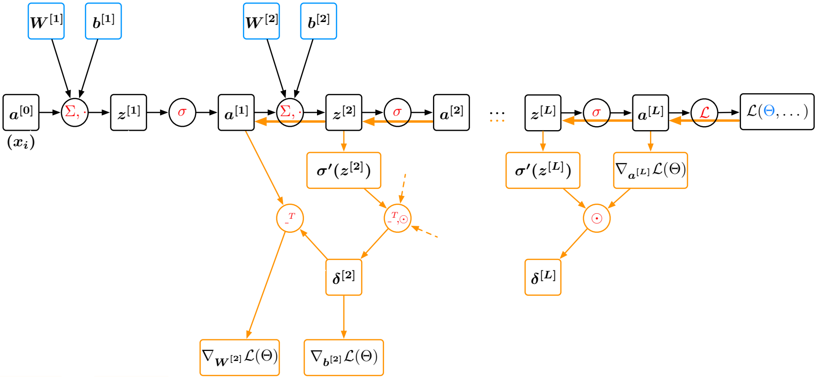


# Forward Pass and Backward Pass



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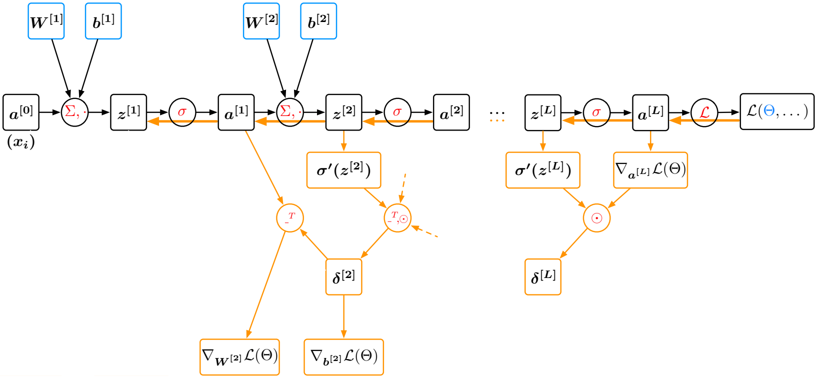

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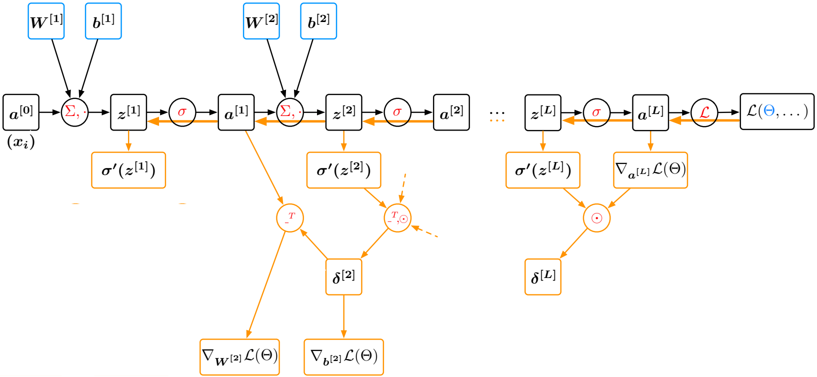
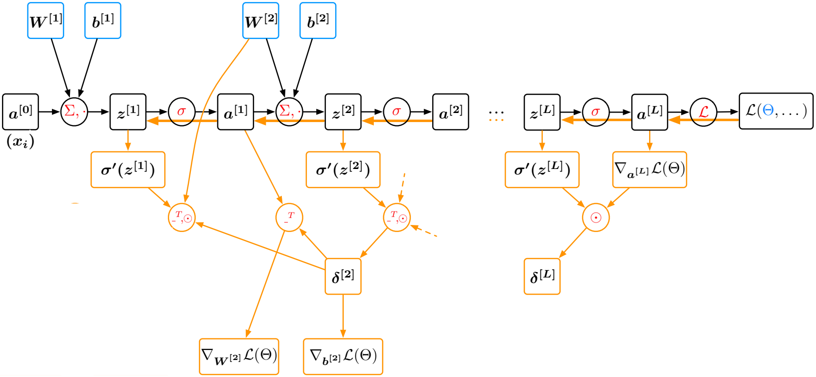


# Forward Pass and Backward Pass



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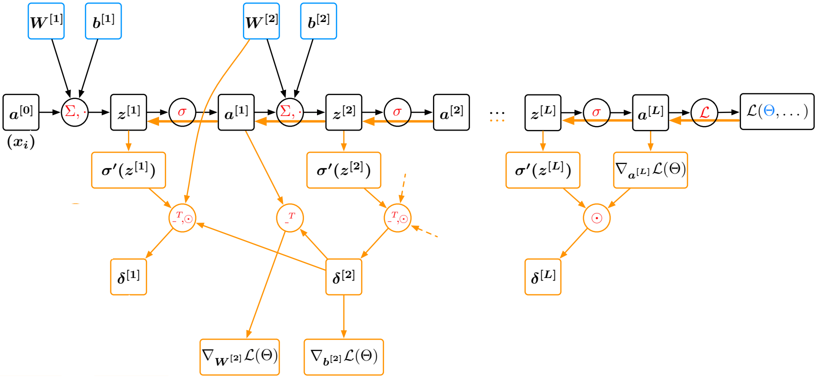

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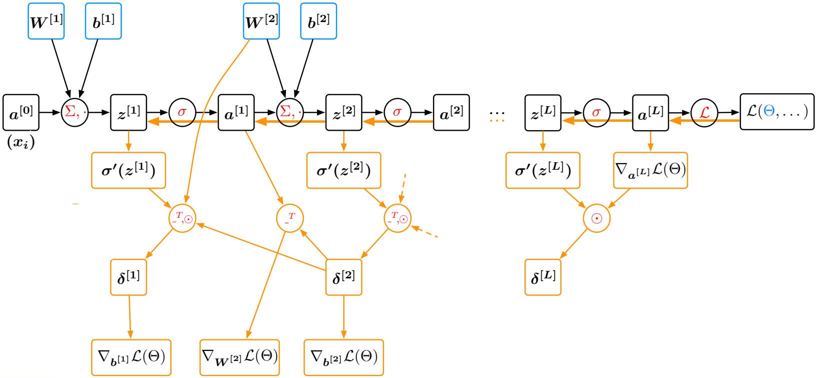

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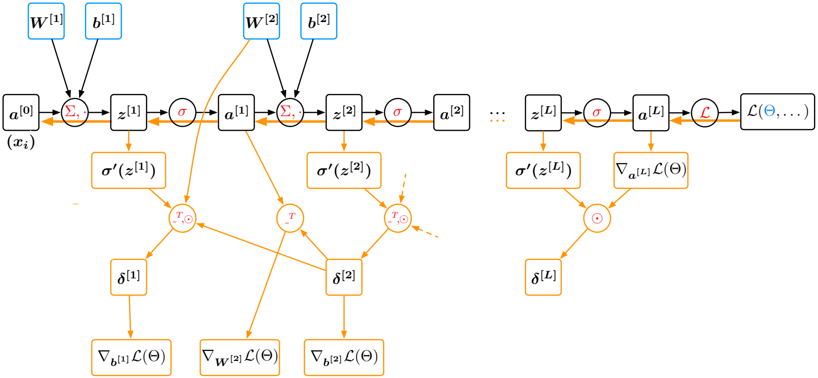

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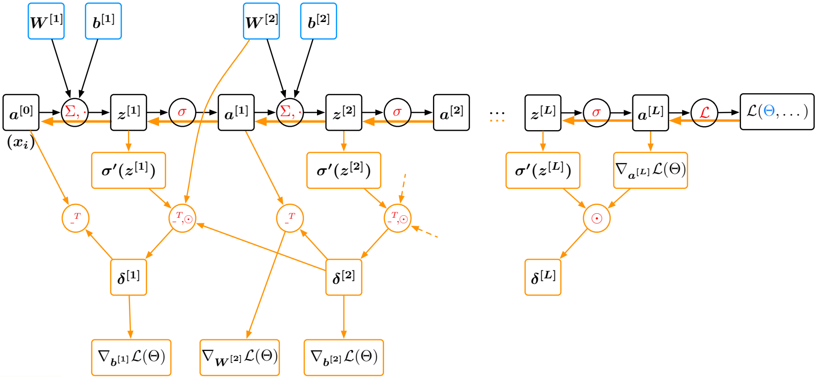

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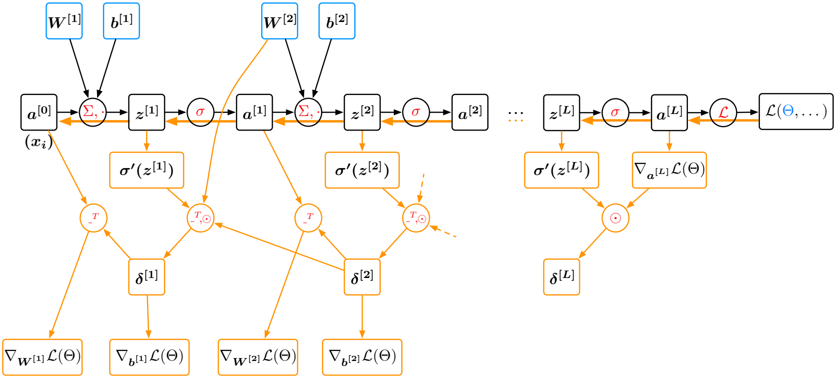

# BACKPROPAGATION (BP)

```
Algorithm 1: Backpropagation (stochastic gradient descent)
Data: dataloader = $\{\boldsymbol{B_i} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \dots, nb\_batches\}$
Inputs: MLP = $\left\{\left(\boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]}\right) : l = 1, \dots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                    // loop over epochs
    for $i = 1$ to $nb\_batches$ do                    // loop over batches
        for $l = 1$ to $L$ do                          // forward pass
            ...
            ...
        end
        for $l = L$ to $1$ do                          // backward pass
            ...                                        // compute error
            ...                                        // update gradient
            ...                                        // update gradient
        end
    end
end
```

# BACKPROPAGATION (BP)

**Algorithm 1:** Backpropagation (stochastic gradient descent)

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Data: dataloader = $\{\boldsymbol{B_i} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \dots, nb\_batches\}$
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for $epochs = 1$ to $nb\_epochs$ do                              // loop over epochs
    for $i = 1$ to $nb\_batches$ do                              // loop over batches
        for $l = 1$ to $L$ do                                      // forward pass
            ...
            ...
        end
        for $l = L$ to $1$ do                                      // backward pass
            ...                                                    // compute error
            ...                                                    // update gradient
            ...                                                    // update gradient
        end
    end
end
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# BACKPROPAGATION (BP)

**Algorithm 1:** Backpropagation (stochastic gradient descent)

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for $epochs = 1$ to $nb\_epochs$ do                    // loop over epochs
    for $i = 1$ to $nb\_batches$ do                    // loop over batches
        for $l = 1$ to $L$ do                          // forward pass
            ...
            ...
        end
        for $l = L$ to $1$ do                          // backward pass
            ...                                        // compute error
            ...                                        // update gradient
            ...                                        // update gradient
        end
    end
end
```

# BACKPROPAGATION (BP)

**Algorithm 1:** Backpropagation (stochastic gradient descent)

**Data:** dataloader = $\{\boldsymbol{B_i} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \dots, nb\_batches\}$

**Inputs:** MLP = $\left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \dots, L \right\}$ initialized randomly

```
for $epochs = 1$ to $nb\_epochs$ do                          // loop over epochs
    for $i = 1$ to $nb\_batches$ do                          // loop over batches
        for $l = 1$ to $L$ do                                // forward pass
            ...
            ...
        end
        for $l = L$ to $1$ do                                // backward pass
            ...                                              // compute error
            ...                                              // update gradient
            ...                                              // update gradient
        end
    end
end
```

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for $epochs = 1$ to $nb\_epochs$ do                          // loop over epochs
    for $i = 1$ to $nb\_batches$ do                          // loop over batches
        for $l = 1$ to $L$ do                                // forward pass
            ...
            ...
        end
        for $l = L$ to $1$ do                                // backward pass
            ...                                              // compute error
            ...                                              // update gradient
            ...                                              // update gradient
        end
    end
end
```

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    for $i = 1$ to $nb\_batches$ do                          // loop over batches
        for $l = 1$ to $L$ do                                // forward pass
            ...
            ...
        end
        for $l = L$ to $1$ do                                // backward pass
            ...                                              // compute error
            ...                                              // update gradient
            ...                                              // update gradient
        end
    end
end
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**Algorithm 1:** Backpropagation (stochastic gradient descent)

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for $epochs = 1$ to $nb\_epochs$ do                              // loop over epochs
    for $i = 1$ to $nb\_batches$ do                              // loop over batches
        for $l = 1$ to $L$ do                                    // forward pass
            ...
            ...
        end
        for $l = L$ to $1$ do                                    // backward pass
            ...                                                  // compute error
            ...                                                  // update gradient
            ...                                                  // update gradient
        end
    end
end
```

# Backpropagation (BP)

**Algorithm 1:** Backpropagation (stochastic gradient descent)

```
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Inputs: MLP = $\left\{\left(\boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]}\right) : l = 1, \ldots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                          // loop over epochs
    for $i = 1$ to $nb\_batches$ do                          // loop over batches
        for $l = 1$ to $L$ do                                // forward pass
            ...
            ...
        end
        for $l = L$ to $1$ do                                // backward pass
            ...                                              // compute error
            ...                                              // update gradient
            ...                                              // update gradient
        end
    end
end
```

# BACKPROPAGATION (BP)

**Algorithm 1:** Backpropagation (stochastic gradient descent)

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Inputs: MLP = $\left\{\left(\boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]}\right) : l = 1, \ldots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                       // loop over epochs
    for $i = 1$ to $nb\_batches$ do                       // loop over batches
        for $l = 1$ to $L$ do                             // forward pass
            ...
            ...
        end
        for $l = L$ to $1$ do                             // backward pass
            ...                                           // compute error
            ...                                           // update gradient
            ...                                           // update gradient
        end
    end
end
```

# Backpropagation (BP)

**Algorithm 1:** Backpropagation (stochastic gradient descent)

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Data: dataloader = $\{\boldsymbol{B_i} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \ldots, nb\_batches\}$
Inputs: MLP = $\left\{\left(\boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]}\right) : l = 1, \ldots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                                   // loop over epochs
    for $i = 1$ to $nb\_batches$ do                                   // loop over batches
        for $l = 1$ to $L$ do                                         // forward pass
            ...
            ...
        end
        for $l = L$ to $1$ do                                         // backward pass
            ...                                                       // compute error
            ...                                                       // update gradient
            ...                                                       // update gradient
        end
    end
end
```

# BACKPROPAGATION (BP)

**Algorithm 2:** Backpropagation (stochastic gradient descent)

**Data:** dataloader $= \{\boldsymbol{B_i} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \dots, nb\_batches\}$

**Inputs:** MLP $= \left\{\left(\boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]}\right) : l = 1, \dots, L\right\}$ initialized randomly

```
for $epochs = 1$ to $nb\_epochs$ do                                        // loop over epochs
    for $i = 1$ to $nb\_batches$ do                                        // loop over batches
        $A^{[0]} = X_i$
        for $l = 1$ to $L$ do                                              // forward pass
            $Z^{[l]} = W^{[l]} A^{[l-1]} + b^{[l]}$
            $A^{[l]} = \sigma\left(Z^{[l]}\right)$
        end
        for $l = L$ to $1$ do                                              // backward pass
            if $l = L$ then                                                // compute error
                $\delta_k^{[l]} = \nabla_{a^{[l]}} \mathcal{L}_k(\Theta) \odot \sigma'(z_k^{[l]})$        for $k = 1, \dots, B$
            else if $L > l \geq 1$ then
                $\delta_k^{[l]} = \left[W^{[l+1]}\right]^T \delta_k^{[l+1]} \odot \sigma'(z_k^{[l]})$     for $k = 1, \dots, B$
            end
            $\nabla_{W^{[l]}} \mathcal{L}_k(\Theta) = \delta_k^{[l]} \left[a_k^{[l-1]}\right]^T$             for $k = 1, \dots, B$
            $\nabla_{b^{[l]}} \mathcal{L}_k(\Theta) = \delta_k^{[l]}$                                   for $k = 1, \dots, B$
            $W^{[l]} := W^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{W^{[l]}} \mathcal{L}_k(\Theta)$        // update gradient
            $b^{[l]} := b^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{b^{[l]}} \mathcal{L}_k(\Theta)$        // update gradient
        end
    end
end
```

# Backpropagation (BP)

**Algorithm 2:** Backpropagation (stochastic gradient descent)

```
Data: dataloader = $\{B_i = (X_i, Y_i) : i = i, \ldots, nb\_batches\}$
Inputs: MLP = $\left\{\left(W^{[l]}, b^{[l]}\right) : l = 1, \ldots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                                   // loop over epochs
    for $i = 1$ to $nb\_batches$ do                                   // loop over batches
        $A^{[0]} = X_i$
        for $l = 1$ to $L$ do                                         // forward pass
            $Z^{[l]} = W^{[l]} A^{[l-1]} + b^{[l]}$
            $A^{[l]} = \sigma\left(Z^{[l]}\right)$
        end
        for $l = L$ to $1$ do                                         // backward pass
            if $l = L$ then                                           // compute error
                $\delta_k^{[l]} = \nabla_{a^{[l]}} \mathcal{L}_k(\Theta) \odot \sigma'(z_k^{[l]})$        for $k = 1, \ldots, B$
            else if $L > l \geq 1$ then
                $\delta_k^{[l]} = \left[W^{[l+1]}\right]^T \delta_k^{[l+1]} \odot \sigma'(z_k^{[l]})$      for $k = 1, \ldots, B$
            end
            $\nabla_{W^{[l]}} \mathcal{L}_k(\Theta) = \delta_k^{[l]} \left[a_k^{[l-1]}\right]^T$           for $k = 1, \ldots, B$
            $\nabla_{b^{[l]}} \mathcal{L}_k(\Theta) = \delta_k^{[l]}$                                for $k = 1, \ldots, B$
            $W^{[l]} := W^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{W^{[l]}} \mathcal{L}_k(\Theta)$      // update gradient
            $b^{[l]} := b^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{b^{[l]}} \mathcal{L}_k(\Theta)$      // update gradient
        end
    end
end
```

# BACKPROPAGATION (BP)

**Algorithm 2:** Backpropagation (stochastic gradient descent)

```
Data: dataloader = $\{\boldsymbol{B_i} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \dots, nb\_batches\}$
Inputs: MLP = $\left\{\left(\boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]}\right) : l = 1, \dots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                                        // loop over epochs
    for $i = 1$ to $nb\_batches$ do                                        // loop over batches
        $\boldsymbol{A}^{[0]} = \boldsymbol{X}_i$
        for $l = 1$ to $L$ do                                              // forward pass
            $\boldsymbol{Z}^{[l]} = \boldsymbol{W}^{[l]} \boldsymbol{A}^{[l-1]} + \boldsymbol{b}^{[l]}$
            $\boldsymbol{A}^{[l]} = \boldsymbol{\sigma}\left(\boldsymbol{Z}^{[l]}\right)$
        end
        for $l = L$ to $1$ do                                              // backward pass
            if $l = L$ then                                                // compute error
                $\delta_k^{[l]} = \nabla_{a^{[l]}} \mathcal{L}_k(\Theta) \odot \boldsymbol{\sigma}'(z_k^{[l]})$          for $k = 1, \dots, B$
            else if $L > l \geq 1$ then
                $\delta_k^{[l]} = \left[\boldsymbol{W}^{[l+1]}\right]^T \delta_k^{[l+1]} \odot \boldsymbol{\sigma}'(z_k^{[l]})$   for $k = 1, \dots, B$
            end
            $\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\Theta) = \delta_k^{[l]} \left[a_k^{[l-1]}\right]^T$                for $k = 1, \dots, B$
            $\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\Theta) = \delta_k^{[l]}$                                 for $k = 1, \dots, B$
            $\boldsymbol{W}^{[l]} := \boldsymbol{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\Theta)$      // update gradient
            $\boldsymbol{b}^{[l]} := \boldsymbol{b}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\Theta)$      // update gradient
        end
    end
end
```

# BACKPROPAGATION (BP)

**Algorithm 2:** Backpropagation (stochastic gradient descent)

```
Data: dataloader = $\{\boldsymbol{B_i} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \dots, nb\_batches\}$
Inputs: MLP = $\left\{\left(\boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]}\right) : l = 1, \dots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                                                    // loop over epochs
    for $i = 1$ to $nb\_batches$ do                                                     // loop over batches
        $\boldsymbol{A}^{[0]} = \boldsymbol{X_i}$
        for $l = 1$ to $L$ do                                                             // forward pass
            $\boldsymbol{Z}^{[l]} = \boldsymbol{W}^{[l]} \boldsymbol{A}^{[l-1]} + \boldsymbol{b}^{[l]}$
            $\boldsymbol{A}^{[l]} = \boldsymbol{\sigma}\left(\boldsymbol{Z}^{[l]}\right)$
        end
        for $l = L$ to $1$ do                                                             // backward pass
            if $l = L$ then                                                               // compute error
                $\delta_k^{[l]} = \nabla_{a^{[l]}} \mathcal{L}_k(\Theta) \odot \boldsymbol{\sigma}'(z_k^{[l]})$          for $k = 1, \dots, B$
            else if $L > l \geq 1$ then
                $\delta_k^{[l]} = \left[\boldsymbol{W}^{[l+1]}\right]^T \delta_k^{[l+1]} \odot \boldsymbol{\sigma}'(z_k^{[l]})$      for $k = 1, \dots, B$
            end
            $\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\Theta) = \delta_k^{[l]} \left[a_k^{[l-1]}\right]^T$                 for $k = 1, \dots, B$
            $\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\Theta) = \delta_k^{[l]}$                               for $k = 1, \dots, B$
            $\boldsymbol{W}^{[l]} := \boldsymbol{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\Theta)$          // update gradient
            $\boldsymbol{b}^{[l]} := \boldsymbol{b}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\Theta)$             // update gradient
        end
    end
end
```

# BACKPROPAGATION (BP)

**Algorithm 2:** Backpropagation (stochastic gradient descent)

```
Data: dataloader = $\{\boldsymbol{B_i} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \ldots, nb\_batches\}$
Inputs: MLP = $\left\{\left(\boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]}\right) : l = 1, \ldots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                                   // loop over epochs
    for $i = 1$ to $nb\_batches$ do                                   // loop over batches
        $\boldsymbol{A}^{[0]} = \boldsymbol{X_i}$
        for $l = 1$ to $L$ do                                         // forward pass
            $\boldsymbol{Z}^{[l]} = \boldsymbol{W}^{[l]} \boldsymbol{A}^{[l-1]} + \boldsymbol{b}^{[l]}$
            $\boldsymbol{A}^{[l]} = \boldsymbol{\sigma}\left(\boldsymbol{Z}^{[l]}\right)$
        end
        for $l = L$ to $1$ do                                         // backward pass
            if $l = L$ then                                           // compute error
                $\boldsymbol{\delta}_k^{[l]} = \nabla_{\boldsymbol{a}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma}'(\boldsymbol{z}_k^{[l]})$        for $k = 1, \ldots, B$
            else if $L > l \geq 1$ then
                $\boldsymbol{\delta}_k^{[l]} = \left[\boldsymbol{W}^{[l+1]}\right]^T \boldsymbol{\delta}_k^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}_k^{[l]})$        for $k = 1, \ldots, B$
            end
            $\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) = \boldsymbol{\delta}_k^{[l]} \left[\boldsymbol{a}_k^{[l-1]}\right]^T$        for $k = 1, \ldots, B$
            $\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) = \boldsymbol{\delta}_k^{[l]}$        for $k = 1, \ldots, B$
            $\boldsymbol{W}^{[l]} := \boldsymbol{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta})$        // update gradient
            $\boldsymbol{b}^{[l]} := \boldsymbol{b}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta})$        // update gradient
        end
    end
end
```

# BACKPROPAGATION (BP)

**Algorithm 2:** Backpropagation (stochastic gradient descent)

```
Data: dataloader = $\{B_i = (X_i, Y_i) : i = i, \dots, nb\_batches\}$
Inputs: MLP = $\left\{ \left( W^{[l]}, b^{[l]} \right) : l = 1, \dots, L \right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                                        // loop over epochs
    for $i = 1$ to $nb\_batches$ do                                       // loop over batches
        $A^{[0]} = X_i$
        for $l = 1$ to $L$ do                                             // forward pass
            $Z^{[l]} = W^{[l]} A^{[l-1]} + b^{[l]}$
            $A^{[l]} = \sigma\left(Z^{[l]}\right)$
        end
        for $l = L$ to $1$ do                                             // backward pass
            if $l = L$ then                                               // compute error
                $\delta_k^{[l]} = \nabla_{a^{[l]}} \mathcal{L}_k(\Theta) \odot \sigma'(z_k^{[l]})$          for $k = 1, \dots, B$
            else if $L > l \geq 1$ then
                $\delta_k^{[l]} = \left[W^{[l+1]}\right]^T \delta_k^{[l+1]} \odot \sigma'(z_k^{[l]})$    for $k = 1, \dots, B$
            end
            $\nabla_{W^{[l]}} \mathcal{L}_k(\Theta) = \delta_k^{[l]} \left[a_k^{[l-1]}\right]^T$            for $k = 1, \dots, B$
            $\nabla_{b^{[l]}} \mathcal{L}_k(\Theta) = \delta_k^{[l]}$                              for $k = 1, \dots, B$
            $W^{[l]} := W^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{W^{[l]}} \mathcal{L}_k(\Theta)$          // update gradient
            $b^{[l]} := b^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{b^{[l]}} \mathcal{L}_k(\Theta)$          // update gradient
        end
    end
end
```

# BACKPROPAGATION (BP)

**Algorithm 2:** Backpropagation (stochastic gradient descent)

```
Data: dataloader = $\{\boldsymbol{B_i} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \dots, nb\_batches\}$
Inputs: MLP = $\left\{\left(\boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]}\right) : l = 1, \dots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                                         // loop over epochs
    for $i = 1$ to $nb\_batches$ do                                         // loop over batches
        $\boldsymbol{A}^{[0]} = \boldsymbol{X_i}$
        for $l = 1$ to $L$ do                                               // forward pass
            $\boldsymbol{Z}^{[l]} = \boldsymbol{W}^{[l]} \boldsymbol{A}^{[l-1]} + \boldsymbol{b}^{[l]}$
            $\boldsymbol{A}^{[l]} = \boldsymbol{\sigma}\left(\boldsymbol{Z}^{[l]}\right)$
        end
        for $l = L$ to $1$ do                                               // backward pass
            if $l = L$ then                                                 // compute error
                $\boldsymbol{\delta}_{\boldsymbol{k}}^{[l]} = \nabla_{\boldsymbol{a}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma}'(\boldsymbol{z}_{\boldsymbol{k}}^{[l]})$        for $k = 1, \dots, B$
            else if $L > l \geq 1$ then
                $\boldsymbol{\delta}_{\boldsymbol{k}}^{[l]} = \left[\boldsymbol{W}^{[l+1]}\right]^T \boldsymbol{\delta}_{\boldsymbol{k}}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}_{\boldsymbol{k}}^{[l]})$     for $k = 1, \dots, B$
            end
            $\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) = \boldsymbol{\delta}_{\boldsymbol{k}}^{[l]} \left[\boldsymbol{a}_{\boldsymbol{k}}^{[l-1]}\right]^T$          for $k = 1, \dots, B$
            $\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) = \boldsymbol{\delta}_{\boldsymbol{k}}^{[l]}$                             for $k = 1, \dots, B$
            $\boldsymbol{W}^{[l]} := \boldsymbol{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta})$          // update gradient
            $\boldsymbol{b}^{[l]} := \boldsymbol{b}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta})$              // update gradient
        end
    end
end
```

# BACKPROPAGATION (BP)

**Algorithm 2:** Backpropagation (stochastic gradient descent)

```
Data: dataloader = $\{\boldsymbol{B_i} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \ldots, nb\_batches\}$
Inputs: MLP = $\left\{\left(\boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]}\right) : l = 1, \ldots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                                         // loop over epochs
    for $i = 1$ to $nb\_batches$ do                                         // loop over batches
        $\boldsymbol{A}^{[0]} = \boldsymbol{X_i}$
        for $l = 1$ to $L$ do                                               // forward pass
            $\boldsymbol{Z}^{[l]} = \boldsymbol{W}^{[l]}\boldsymbol{A}^{[l-1]} + \boldsymbol{b}^{[l]}$
            $\boldsymbol{A}^{[l]} = \boldsymbol{\sigma}\left(\boldsymbol{Z}^{[l]}\right)$
        end
        for $l = L$ to $1$ do                                               // backward pass
            if $l = L$ then                                                 // compute error
                $\boldsymbol{\delta}_{\boldsymbol{k}}^{[l]} = \nabla_{\boldsymbol{a}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma}'(\boldsymbol{z}_{\boldsymbol{k}}^{[l]})$        for $k = 1, \ldots, B$
            else if $L > l \geq 1$ then
                $\boldsymbol{\delta}_{\boldsymbol{k}}^{[l]} = \left[\boldsymbol{W}^{[l+1]}\right]^T \boldsymbol{\delta}_{\boldsymbol{k}}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}_{\boldsymbol{k}}^{[l]})$        for $k = 1, \ldots, B$
            end
            $\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) = \boldsymbol{\delta}_{\boldsymbol{k}}^{[l]} \left[\boldsymbol{a}_{\boldsymbol{k}}^{[l-1]}\right]^T$        for $k = 1, \ldots, B$
            $\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) = \boldsymbol{\delta}_{\boldsymbol{k}}^{[l]}$        for $k = 1, \ldots, B$
            $\boldsymbol{W}^{[l]} := \boldsymbol{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta})$        // update gradient
            $\boldsymbol{b}^{[l]} := \boldsymbol{b}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta})$        // update gradient
        end
    end
end
```

# BACKPROPAGATION (BP)

**Algorithm 2:** Backpropagation (stochastic gradient descent)

```
Data: dataloader = $\{B_i = (X_i, Y_i) : i = i, \dots, nb\_batches\}$
Inputs: MLP = $\left\{\left(W^{[l]}, b^{[l]}\right) : l = 1, \dots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                                   // loop over epochs
    for $i = 1$ to $nb\_batches$ do                                   // loop over batches
        $A^{[0]} = X_i$
        for $l = 1$ to $L$ do                                         // forward pass
            $Z^{[l]} = W^{[l]} A^{[l-1]} + b^{[l]}$
            $A^{[l]} = \sigma\left(Z^{[l]}\right)$
        end
        for $l = L$ to $1$ do                                         // backward pass
            if $l = L$ then                                           // compute error
                $\delta_k^{[l]} = \nabla_{a^{[l]}} \mathcal{L}_k(\Theta) \odot \sigma'(z_k^{[l]})$    for $k = 1, \dots, B$
            else if $L > l \geq 1$ then
                $\delta_k^{[l]} = \left[W^{[l+1]}\right]^T \delta_k^{[l+1]} \odot \sigma'(z_k^{[l]})$    for $k = 1, \dots, B$
            end
            $\nabla_{W^{[l]}} \mathcal{L}_k(\Theta) = \delta_k^{[l]} \left[a_k^{[l-1]}\right]^T$    for $k = 1, \dots, B$
            $\nabla_{b^{[l]}} \mathcal{L}_k(\Theta) = \delta_k^{[l]}$    for $k = 1, \dots, B$
            $W^{[l]} := W^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{W^{[l]}} \mathcal{L}_k(\Theta)$    // update gradient
            $b^{[l]} := b^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{b^{[l]}} \mathcal{L}_k(\Theta)$    // update gradient
        end
    end
end
```

# Backpropagation (BP)

**Algorithm 2:** Backpropagation (stochastic gradient descent)

```
Data: dataloader = $\{\boldsymbol{B_i} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \dots, nb\_batches\}$
Inputs: MLP = $\left\{\left(\boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]}\right) : l = 1, \dots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                                                  // loop over epochs
    for $i = 1$ to $nb\_batches$ do                                                  // loop over batches
        $\boldsymbol{A}^{[0]} = \boldsymbol{X_i}$
        for $l = 1$ to $L$ do                                                        // forward pass
            $\boldsymbol{Z}^{[l]} = \boldsymbol{W}^{[l]} \boldsymbol{A}^{[l-1]} + \boldsymbol{b}^{[l]}$
            $\boldsymbol{A}^{[l]} = \boldsymbol{\sigma}\left(\boldsymbol{Z}^{[l]}\right)$
        end
        for $l = L$ to $1$ do                                                        // backward pass
            if $l = L$ then                                                          // compute error
                $\boldsymbol{\delta_k^{[l]}} = \nabla_{\boldsymbol{a}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma'}(\boldsymbol{z_k^{[l]}})$          for $k = 1, \dots, B$
            else if $L > l \geq 1$ then
                $\boldsymbol{\delta_k^{[l]}} = \left[\boldsymbol{W}^{[l+1]}\right]^T \boldsymbol{\delta_k^{[l+1]}} \odot \boldsymbol{\sigma'}(\boldsymbol{z_k^{[l]}})$          for $k = 1, \dots, B$
            end
            $\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) = \boldsymbol{\delta_k^{[l]}} \left[\boldsymbol{a_k^{[l-1]}}\right]^T$          for $k = 1, \dots, B$
            $\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) = \boldsymbol{\delta_k^{[l]}}$          for $k = 1, \dots, B$
            $\boldsymbol{W}^{[l]} := \boldsymbol{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta})$          // update gradient
            $\boldsymbol{b}^{[l]} := \boldsymbol{b}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta})$          // update gradient
        end
    end
end
```

# BACKPROPAGATION (BP)

**Algorithm 2:** Backpropagation (stochastic gradient descent)

```
Data: dataloader = $\{\boldsymbol{B_i} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \dots, nb\_batches\}$
Inputs: MLP = $\left\{\left(\boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]}\right) : l = 1, \dots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                                    // loop over epochs
    for $i = 1$ to $nb\_batches$ do                                    // loop over batches
        $\boldsymbol{A}^{[0]} = \boldsymbol{X_i}$
        for $l = 1$ to $L$ do                                          // forward pass
            $\boldsymbol{Z}^{[l]} = \boldsymbol{W}^{[l]} \boldsymbol{A}^{[l-1]} + \boldsymbol{b}^{[l]}$
            $\boldsymbol{A}^{[l]} = \boldsymbol{\sigma}\left(\boldsymbol{Z}^{[l]}\right)$
        end
        for $l = L$ to $1$ do                                          // backward pass
            if $l = L$ then                                            // compute error
                $\boldsymbol{\delta_k^{[l]}} = \nabla_{\boldsymbol{a}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma}'(\boldsymbol{z_k^{[l]}})$        for $k = 1, \dots, B$
            else if $L > l \geq 1$ then
                $\boldsymbol{\delta_k^{[l]}} = \left[\boldsymbol{W}^{[l+1]}\right]^T \boldsymbol{\delta_k^{[l+1]}} \odot \boldsymbol{\sigma}'(\boldsymbol{z_k^{[l]}})$        for $k = 1, \dots, B$
            end
            $\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) = \boldsymbol{\delta_k^{[l]}} \left[\boldsymbol{a_k^{[l-1]}}\right]^T$        for $k = 1, \dots, B$
            $\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) = \boldsymbol{\delta_k^{[l]}}$        for $k = 1, \dots, B$
            $\boldsymbol{W}^{[l]} := \boldsymbol{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta})$        // update gradient
            $\boldsymbol{b}^{[l]} := \boldsymbol{b}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta})$        // update gradient
        end
    end
end
```

# BACKPROPAGATION (BP)

**Algorithm 2:** Backpropagation (stochastic gradient descent)

```
Data: dataloader = $\{\boldsymbol{B_i} = (\boldsymbol{X_i}, \boldsymbol{Y_i}) : i = i, \dots, nb\_batches\}$
Inputs: MLP = $\left\{\left(\boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]}\right) : l = 1, \dots, L\right\}$ initialized randomly
for $epochs = 1$ to $nb\_epochs$ do                                          // loop over epochs
    for $i = 1$ to $nb\_batches$ do                                          // loop over batches
        $\boldsymbol{A}^{[0]} = \boldsymbol{X_i}$
        for $l = 1$ to $L$ do                                                // forward pass
            $\boldsymbol{Z}^{[l]} = \boldsymbol{W}^{[l]} \boldsymbol{A}^{[l-1]} + \boldsymbol{b}^{[l]}$
            $\boldsymbol{A}^{[l]} = \boldsymbol{\sigma}\left(\boldsymbol{Z}^{[l]}\right)$
        end
        for $l = L$ to $1$ do                                                // backward pass
            if $l = L$ then                                                  // compute error
                $\boldsymbol{\delta_k^{[l]}} = \nabla_{\boldsymbol{a}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma}'(\boldsymbol{z_k^{[l]}})$    for $k = 1, \dots, B$
            else if $L > l \geq 1$ then
                $\boldsymbol{\delta_k^{[l]}} = \left[\boldsymbol{W}^{[l+1]}\right]^T \boldsymbol{\delta_k^{[l+1]}} \odot \boldsymbol{\sigma}'(\boldsymbol{z_k^{[l]}})$    for $k = 1, \dots, B$
            end
            $\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) = \boldsymbol{\delta_k^{[l]}} \left[\boldsymbol{a_k^{[l-1]}}\right]^T$    for $k = 1, \dots, B$
            $\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta}) = \boldsymbol{\delta_k^{[l]}}$    for $k = 1, \dots, B$
            $\boldsymbol{W}^{[l]} := \boldsymbol{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta})$    // update gradient
            $\boldsymbol{b}^{[l]} := \boldsymbol{b}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}_k(\boldsymbol{\Theta})$    // update gradient
        end
    end
end
```

# Exemple de training via Backrop

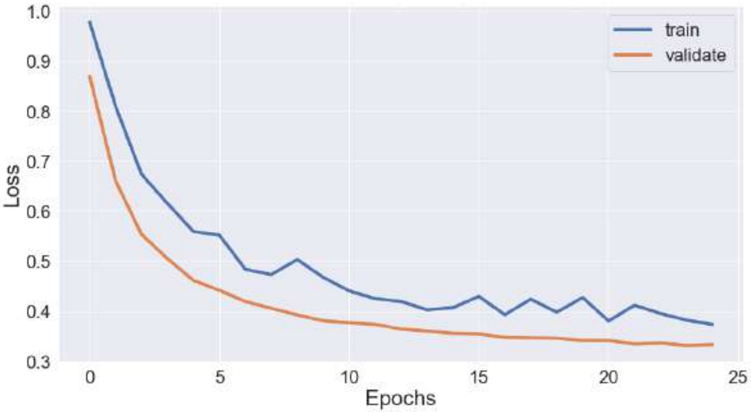

Figure taken from towardsdatascience.com

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