BACKPROPAGATION

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CHAIN RULE

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On rappelle le théorème des fonctions composées (chain rule).

$$\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R}$$

$$x \xrightarrow{f} y = f(x) \xrightarrow{g} z = g(y)$$

$$= g(f(x))$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

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Exemple:

$$\frac{\partial z}{\partial x} = \frac{\partial \left[5(x^2+1)\right]}{\partial x} = 10x = 5 \cdot 2x = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

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Généralisation multidimensionnelle

$$\frac{\partial z}{\partial x_i} = \sum_{j=1}^n \frac{\partial z}{\partial y_j} \cdot \frac{\partial y_j}{\partial x_i}, \quad i = 1, \dots, m$$

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Formulation vectorielle

$$\begin{array}{cccc}
\mathbb{R}^m & \xrightarrow{f} & \mathbb{R}^n & \xrightarrow{g} & \mathbb{R} \\
\mathbf{x} & \xrightarrow{f} & \mathbf{y} = f(\mathbf{x}) & \xrightarrow{g} & z = g(\mathbf{y}) \\
& & = g(f(\mathbf{x}))
\end{array}$$

Soient $\nabla_x z$ et le *gradient* de z par rapport à x, $\nabla_y z$ et le gradient de z par rapport à y, et $J_f:=\left[\frac{\partial y}{\partial x}\right]$ le *jacobien* de la fonction f:

$$abla_{m{x}}z = \left[[
abla_{m{y}}z]^T \left[rac{\partial m{y}}{\partial m{x}}
ight]
ight]^T = m{J}_f^T
abla_{m{y}}z$$

$$\begin{pmatrix} \frac{\partial z}{\partial x_1} \\ \vdots \\ \frac{\partial z}{\partial x_m} \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} \frac{\partial z}{\partial y_1} & \cdots & \frac{\partial z}{\partial y_n} \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_m} \end{pmatrix} \end{bmatrix}^T$$

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Formulation vectorielle

Soient $\nabla_{x}z$ et le gradient de z par rapport à x, $\nabla_{y}z$ et le gradient de z par rapport à ${m y}$, et ${m J}_f:=\left[rac{\partial {m y}}{\partial {m x}}
ight]$ le jacobien de la fonction f:

$$\nabla_{\boldsymbol{x}}z = \left[\left[\nabla_{\boldsymbol{y}}z \right]^T \left[\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} \right] \right]^T = \boldsymbol{J}_f^T \nabla_{\boldsymbol{y}} z$$

$$\begin{pmatrix} \frac{\partial z}{\partial x_1} \\ \vdots \\ \frac{\partial z}{\partial x_m} \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} \frac{\partial z}{\partial y_1} & \cdots & \frac{\partial z}{\partial y_n} \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_m} \end{pmatrix} \end{bmatrix}^T$$

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Soient $\nabla_x z$ et le gradient de z par rapport à x, $\nabla_y z$ et le gradient de z par rapport à y, et $J_f := \left\lceil \frac{\partial y}{\partial x} \right\rceil$ le jacobien de la fonction f:

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Exemple:

$$\mathbb{R}^{m} \xrightarrow{f} \mathbb{R}^{n} \xrightarrow{g} \mathbb{R}$$

$$\begin{pmatrix} x_{1} \\ x_{2} \end{pmatrix} \xrightarrow{f} \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix} = \begin{pmatrix} x_{1}x_{2} \\ x_{1}^{2} \\ x_{2}^{3} \end{pmatrix} \xrightarrow{g} \qquad z = y_{1}y_{2} + y_{3}$$

$$= (x_{1}x_{2})x_{1}^{2} + x_{2}^{3}$$

$$\frac{\partial z}{\partial x_1} = \frac{\partial \left[(x_1 x_2) x_1^2 + x_2^3 \right]}{\partial x_1} = x_1^2 x_2 + x_1 x_2 2 x_1 + 0$$

$$= \frac{\partial z}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_1} + \frac{\partial z}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_1} + \frac{\partial z}{\partial y_3} \cdot \frac{\partial y_3}{\partial x_1}$$

$$\frac{\partial z}{\partial x_2} = \frac{\partial \left[(x_1 x_2) x_1^2 + x_2^3 \right]}{\partial x_2} = x_1^2 x_1 + 3 x_2^2 + 0$$

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Exemple (suite):

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$$\nabla_{\boldsymbol{x}} z = \begin{pmatrix} \frac{\partial z}{\partial x_1} \\ \frac{\partial z}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 3x_1^2 x_2 \\ x_1^3 + 3x_2^2 \end{pmatrix} = \begin{pmatrix} y_2 & y_1 & 1 \end{pmatrix} \begin{pmatrix} x_2 & x_1 \\ 2x_1 & 0 \\ 0 & 3x_2^2 \end{pmatrix} = \left[\nabla_{\boldsymbol{y}} z\right]^T \begin{bmatrix} \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} \end{bmatrix}$$

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- lacksquare Soit $S=\{(oldsymbol{x_i},oldsymbol{y_i})\in\mathbb{R}^{d_1} imes\mathbb{R}^{d_2}:i=1,\ldots,N\}$ un dataset.

$$\mathbf{\Theta} := \left\{ \left(\mathbf{W}^{[l]}, \mathbf{b}^{[l]} \right) : l = 1, \dots, L \right\}$$

$$egin{cases} m{a^{[0]}} &= m{x} \ m{z^{[l]}} &= m{W^{[l]}}m{a^{[l-1]}} + m{b^{[l]}}, \ m{a^{[l]}} &= m{\sigma\left(m{z^{[l]}}
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- lacksquare Soit $S=\{(oldsymbol{x_i},oldsymbol{y_i})\in\mathbb{R}^{d_1} imes\mathbb{R}^{d_2}:i=1,\ldots,N\}$ un dataset.
- \triangleright Soit \mathcal{N}_{Θ} un réseau de neurones (MLP) à L couches donné par les paramètres (poids et biais)

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et par la dynamique

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ightharpoonup Remarque: le réseau \mathcal{N}_{Θ} peut-être naturellement associé à la fonction

$$egin{array}{lll} f_{m{\Theta}}: \mathbb{R}^{d_1} & \longrightarrow & \mathbb{R}^{d_2} \ & m{x} & \longmapsto & f_{m{\Theta}}(m{x}) := m{a}^{[m{L}]} \end{array}$$

- $ightharpoonup f_{\Theta}(x)$ est la *prédiction* (output) de \mathcal{N}_{Θ} associée à l'input x.
- ightharpoonup Chaque jeu de paramètres Θ donne lieu à une fonction f_{Θ} différente.

Remarque: le réseau \mathcal{N}_{Θ} peut-être naturellement associé à la fonction

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- lacktriangle Chaque jeu de paramètres Θ donne lieu à une fonction f_{Θ} différente.

Soit une fonction de coût (cost or loss function) qui mesure l'erreur entre la prédiction \hat{y}_i et la réalité y_i :

$$egin{array}{lll} \ell: \mathbb{R}^{d_2} imes \mathbb{R}^{d_2} & \longrightarrow & \mathbb{R} \ (\hat{m{y_i}}, m{y_i}) & \longmapsto & \ell\left(\hat{m{y_i}}, m{y_i}
ight) \end{array}$$

▶ Typiquement, la fonction de coût pourrait être l'erreur quadratique (distance Euclidienne au carré)

$$\ell\left(\hat{\boldsymbol{y}}_{i}, \boldsymbol{y}_{i}\right) = \frac{1}{2} \left\| \hat{\boldsymbol{y}}_{i} - \boldsymbol{y}_{i} \right\|_{2}^{2}$$

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► La fonction de coût peut être naturellement généralisée à un ensemble de *prédictions* et de *réalités*:

$$egin{array}{lll} \mathcal{L}: \mathbb{R}^{d_2} imes \cdots imes \mathbb{R}^{d_2} & \longrightarrow & \mathbb{R} \ (\hat{m{y_1}}, \ldots, \hat{m{y_N}}, m{y_i} \ldots, m{y_N}) & \longmapsto & \mathcal{L}\left(\hat{m{y_1}}, \ldots, \hat{m{y_N}}, m{y_i} \ldots, m{y_N}
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► Typiquement, la fonction de coût pourrait être l'erreur quadratique moyenne (mean squared error MSE)

$$\mathcal{L}(\hat{y}_1, \dots, \hat{y}_N, y_i, \dots, y_N) = \frac{1}{N} \sum_{i=1}^N \ell(\hat{y}_i, y_i)$$
$$= \frac{1}{2N} \sum_{i=1}^N ||\hat{y}_i - y_i||$$

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$$= \frac{1}{2N} \sum_{i=1}^{N} \|\hat{\boldsymbol{y}}_{i} - \boldsymbol{y}_{i}\|_{2}^{2}$$

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Pour un réseau de neurones \mathcal{N}_{Θ} , l'erreur entre les prédictions et les réalités est

$$\mathcal{L}\left(f_{\mathbf{\Theta}}\left(\boldsymbol{x_{1}}\right),\ldots,f_{\mathbf{\Theta}}\left(\boldsymbol{x_{N}}\right),\boldsymbol{y_{1}},\ldots,\boldsymbol{y_{N}}\right).$$

$$\mathcal{L}: \mathbb{R}^{|\Theta|} \longrightarrow \mathbb{R}$$

$$\Theta \longmapsto \mathcal{L}\left(f_{\Theta}\left(\boldsymbol{x}_{1}\right), \ldots, f_{\Theta}\left(\boldsymbol{x}_{N}\right), \boldsymbol{y}_{1}, \ldots, \boldsymbol{y}_{N}\right).$$

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- Pour différents paramètres Θ , on aura différentes prédictions $f_{\Theta}(x_1), \ldots, f_{\Theta}(x_N)$, et donc différentes erreurs $\mathcal{L}(\ldots)$.
- ightharpoonup Ainsi, $\mathcal L$ est une fonction des paramètres Θ du réseau

où $|\Theta|$ est le nombre de paramètres Θ (poids et biais, souvent plusieurs millions).

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- \triangleright Ainsi, \mathcal{L} est une fonction des paramètres Θ du réseau:

$$\mathcal{L}: \mathbb{R}^{|\Theta|} \longrightarrow \mathbb{R}$$

$$\Theta \longmapsto \mathcal{L}\left(f_{\Theta}\left(x_{1}\right), \dots, f_{\Theta}\left(x_{N}\right), y_{1}, \dots, y_{N}\right).$$

où $|\Theta|$ est le nombre de paramètres Θ (poids et biais, souvent plusieurs millions).

ightharpoonup L'entraînement du réseau \mathcal{N}_{Θ} consiste à déterminer des paramètres Θ qui minimisent l'erreur

$$\mathcal{L}(f_{\Theta}(x_1),\ldots,f_{\Theta}(x_N),y_1,\ldots,y_N).$$

- Pour cela, on utilise une descente de gradient: mini-batch stochastic gradient descent.
- ▶ Backpropagation est un algorithme qui permet de calculer les gradients $\nabla_{\Theta} \mathcal{L}$ de manière efficiente.

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- Pour cela, on utilise une descente de gradient: *mini-batch stochas-tic gradient descent*.
- ▶ Backpropagation est un algorithme qui permet de calculer les gradients $\nabla_{\Theta} \mathcal{L}$ de manière efficiente.

$$\mathcal{L}\left(f_{\Theta}\left(x_{1}\right),\ldots,f_{\Theta}\left(x_{N}\right),y_{1},\ldots,y_{N}\right)=\mathcal{L}\left(\Theta,\ldots\right)$$

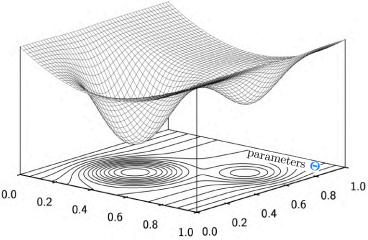
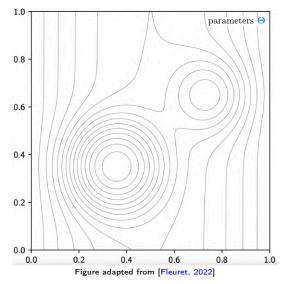
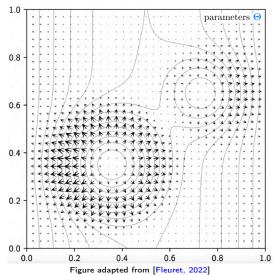


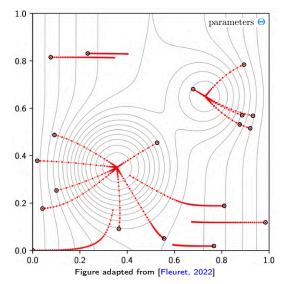
Figure adapted from [Fleuret, 2022]



CHAIN RULE



TRAINING



TRAINING

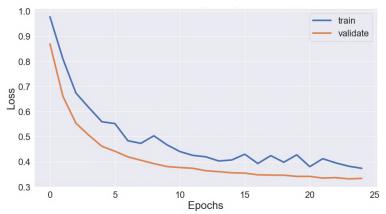
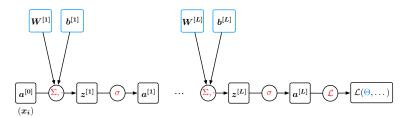


Figure taken from towardsdatascience.com

Graphe computationnel d'un réseau de NEURONES (FORWARD PASS)



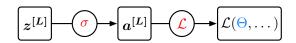
BACKPROPAGATION •00000000000000000

On veut calculer les gradients:

$$\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) := \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial \boldsymbol{W}^{[l]}} \ \ \text{et} \ \ \nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) := \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial \boldsymbol{b}^{[l]}}$$

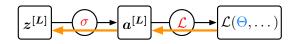
pour
$$l = 1, \ldots, M$$

CHAIN BILLE

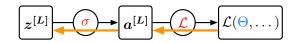


$$\begin{split} \delta_j^{[L]} &:= \frac{\partial \mathcal{L}(\Theta)}{\partial z_j^{[L]}} &= \sum_{k=1}^{\lfloor a^{[L]} \rfloor} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \frac{\partial a_k^{[L]}}{\partial z_j^{[L]}} \\ &= \sum_{k=1}^{\lfloor a^{[L]} \rfloor} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} \\ \left(\frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} = 0 \text{ for } k \neq j \right) &= \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \sigma'(z_j^{[L]}) \end{split}$$

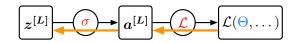
Calcul des gradients: équation 1



$$\begin{split} \delta_{\pmb{j}}^{[L]} &:= \frac{\partial \mathcal{L}(\pmb{\Theta})}{\partial z_{\pmb{j}}^{[L]}} &= \sum_{k=1}^{|a^{[L]}|} \frac{\partial \mathcal{L}(\pmb{\Theta})}{\partial a_{k}^{[L]}} \cdot \frac{\partial a_{k}^{[L]}}{\partial z_{j}^{[L]}} \\ &= \sum_{k=1}^{|a^{[L]}|} \frac{\partial \mathcal{L}(\pmb{\Theta})}{\partial a_{k}^{[L]}} \cdot \frac{\partial \sigma(z_{k}^{[L]})}{\partial z_{j}^{[L]}} \\ \left(\frac{\partial \sigma(z_{k}^{[L]})}{\partial z_{j}^{[L]}} = 0 \text{ for } k \neq j \right) &= \frac{\partial \mathcal{L}(\pmb{\Theta})}{\partial a_{k}^{[L]}} \cdot \sigma'(z_{j}^{[L]}) \end{split}$$

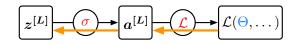


$$\begin{split} \delta_j^{[L]} &:= \frac{\partial \mathcal{L}(\Theta)}{\partial z_j^{[L]}} &= \sum_{k=1}^{|a^{[L]}|} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \frac{\partial a_k^{[L]}}{\partial z_j^{[L]}} \\ &= \sum_{k=1}^{|a^{[L]}|} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} \\ \left(\frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} = 0 \text{ for } k \neq j \right) &= \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \sigma'(z_j^{[L]}) \end{split}$$



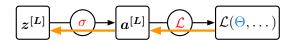
$$\begin{split} \delta_j^{[L]} &:= \frac{\partial \mathcal{L}(\Theta)}{\partial z_j^{[L]}} &= \sum_{k=1}^{|\boldsymbol{a}^{[L]}|} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \frac{\partial a_k^{[L]}}{\partial z_j^{[L]}} \\ &= \sum_{k=1}^{|\boldsymbol{a}^{[L]}|} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} \\ \left(\frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} = 0 \text{ for } k \neq j \right) &= \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[L]}} \cdot \sigma'(z_j^{[L]}) \end{split}$$

Calcul des gradients: équation 1



$$\begin{split} \delta_j^{[L]} &:= \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial z_j^{[L]}} &= \sum_{k=1}^{|\mathbf{a}^{[L]}|} \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial a_k^{[L]}} \cdot \frac{\partial a_k^{[L]}}{\partial z_j^{[L]}} \\ &= \sum_{k=1}^{|\mathbf{a}^{[L]}|} \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial a_k^{[L]}} \cdot \frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} \\ \left(\frac{\partial \sigma(z_k^{[L]})}{\partial z_j^{[L]}} = 0 \text{ for } k \neq j \right) &= \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial a_k^{[L]}} \cdot \sigma'(z_j^{[L]}) \end{split}$$

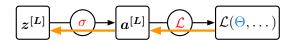
Formulation vectorielle:



$$\delta^{[L]} \; := \; \nabla_{z^{[L]}} \mathcal{L}(\Theta) \;\; = \;\; \nabla_{a^{[L]}} \mathcal{L}(\Theta) \odot \sigma'(z^{[L]})$$

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Formulation vectorielle:



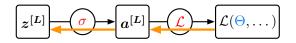
$$\boldsymbol{\delta^{[L]}} \; := \; \nabla_{\boldsymbol{z}^{[L]}} \mathcal{L}(\boldsymbol{\Theta}) \;\; = \;\; \nabla_{\boldsymbol{a}^{[L]}} \mathcal{L}(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[L]})$$

CHAIN BILLE

BACKPROPAGATION

CALCUL DES GRADIENTS: ÉQUATION 1

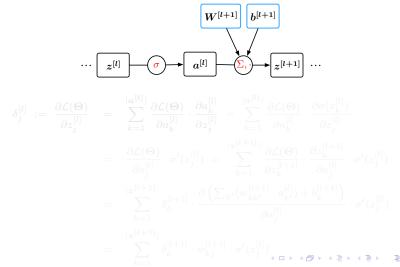
Formulation vectorielle:



$$\delta^{[L]} \; := \; \nabla_{\boldsymbol{z}^{[L]}} \mathcal{L}(\boldsymbol{\Theta}) \;\; = \;\; \nabla_{\boldsymbol{a}^{[L]}} \mathcal{L}(\boldsymbol{\Theta}) \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[L]})$$

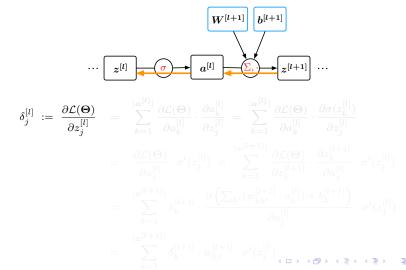
où ⊙ est le produit de Hadamard (composante par composante).

Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous k:

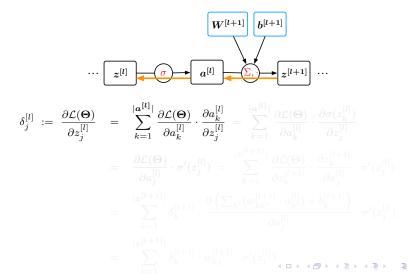


BACKPROPAGATION

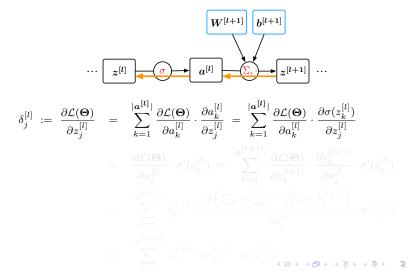
Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous k:



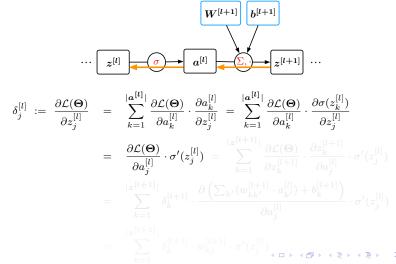
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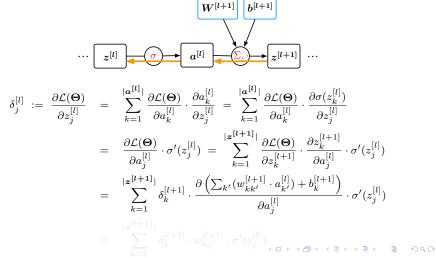
Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous k:

$$\begin{split} \delta_j^{[l]} &:= \frac{\partial \mathcal{L}(\Theta)}{\partial z_j^{[l]}} &= \sum_{k=1}^{|\mathbf{a}^{[l]}|} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[l]}} \cdot \frac{\partial a_k^{[l]}}{\partial z_j^{[l]}} = \sum_{k=1}^{|\mathbf{a}^{[l]}|} \frac{\partial \mathcal{L}(\Theta)}{\partial a_k^{[l]}} \cdot \frac{\partial \sigma(z_k^{[l]})}{\partial z_j^{[l]}} \\ &= \frac{\partial \mathcal{L}(\Theta)}{\partial a_j^{[l]}} \cdot \sigma'(z_j^{[l]}) = \sum_{k=1}^{|\mathbf{z}^{[l+1]}|} \frac{\partial \mathcal{L}(\Theta)}{\partial z_k^{[l+1]}} \cdot \frac{\partial z_k^{[l+1]}}{\partial a_j^{[l]}} \cdot \sigma'(z_j^{[l]}) \\ &= \sum_{k=1}^{|\mathbf{z}^{[l+1]}|} \delta_k^{[l+1]} \cdot \frac{\partial \left(\sum_{k'} (w_{kk'}^{[l+1]} \cdot a_{k'}^{[l]}) + b_k^{[l+1]}\right)}{\partial a_j^{[l]}} \cdot \sigma'(z_j^{[l]}) \\ &= \sum_{k=1}^{|\mathbf{z}^{[l+1]}|} \delta_k^{[l+1]} \cdot w_{kj}^{[l+1]} \cdot \sigma'(z_j^{[l]}) \\ &= \sum_{k=1}^{|\mathbf{z}^{[l+1]}|} \delta_k^{[l+1]} \cdot w_{kj}^{[l+1]} \cdot \sigma'(z_j^{[l]}) \\ \end{split}$$

BACKPROPAGATION

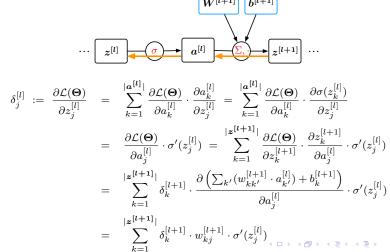
Calcul des gradients: équation 2

Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous k:



Calcul des gradients: équation 2

Supposons que les $\delta_k^{[l+1]}$ ont été calculés pour tous k:

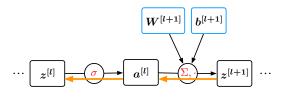


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CALCUL DES GRADIENTS: ÉQUATION 2

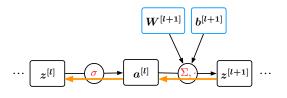
Formulation vectorielle:



$$\begin{split} \boldsymbol{\delta}^{[l]} &:= & \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \\ &= & \left[\boldsymbol{\delta}^{[l+1]}\right]^T \boldsymbol{W}^{[l+1]} \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}) \\ &= & \left[\boldsymbol{W}^{[l+1]}\right]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}) \end{split}$$

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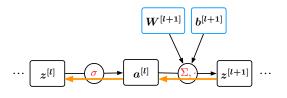
Formulation vectorielle:



$$egin{array}{lll} \boldsymbol{\delta}^{[l]} &:= & \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \ &= & \left[\boldsymbol{\delta}^{[l+1]}
ight]^T W^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[l]}) \ &= & \left[\boldsymbol{W}^{[l+1]}
ight]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma}'(\boldsymbol{z}^{[l]}) \end{array}$$

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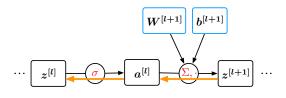
Formulation vectorielle:



$$\begin{split} \boldsymbol{\delta}^{[l]} &:= & \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \\ &= & \left[\boldsymbol{\delta}^{[l+1]}\right]^T \boldsymbol{W}^{[l+1]} \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}) \\ &= & \left[\boldsymbol{W}^{[l+1]}\right]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}) \end{split}$$

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Formulation vectorielle:

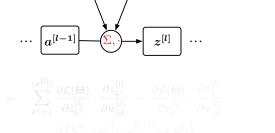


$$\begin{split} \boldsymbol{\delta}^{[l]} &:= & \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \\ &= & \left[\boldsymbol{\delta}^{[l+1]} \right]^T \boldsymbol{W}^{[l+1]} \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}) \\ &= & \left[\boldsymbol{W}^{[l+1]} \right]^T \boldsymbol{\delta}^{[l+1]} \odot \boldsymbol{\sigma'}(\boldsymbol{z}^{[l]}) \end{split}$$

où \odot est le produit de Hadamard (composante par composante).

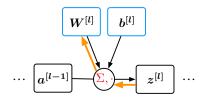
Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :

 $W^{[l]}$



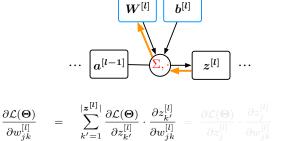
 $b^{[l]}$

Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :

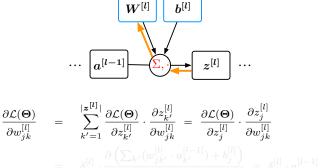


$$\begin{split} \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial \boldsymbol{w}_{jk}^{[l]}} &= \sum_{k'=1}^{\lfloor s-1 \rfloor} \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial w_{jk}^{[l]}} = \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial z_{j}^{[l]}} \cdot \frac{\partial z_{j}^{[l]}}{\partial w_{jk}^{[l]}} \\ &= \delta_{j}^{[l]} \cdot \frac{\partial \left(\sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_{j}^{[l]}\right)}{\partial w_{jk}^{[l]}} = \delta_{j}^{[l]} \cdot a_{k}^{[l-1]} \end{split}$$

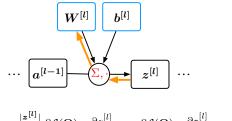
Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :



Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :



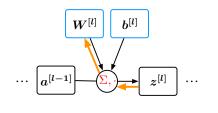
Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :



BACKPROPAGATION

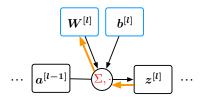
$$\begin{array}{ll} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial w_{jk}^{[l]}} & = & \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial w_{jk}^{[l]}} = \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{j}^{[l]}} \cdot \frac{\partial z_{j}^{[l]}}{\partial w_{jk}^{[l]}} \\ & = & \delta_{j}^{[l]} \cdot \frac{\partial \left(\sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_{j}^{[l]}\right)}{\partial w_{jk}^{[l]}} = \delta_{j}^{[l]} \cdot a_{k}^{[l-1]} \end{array}$$

Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :



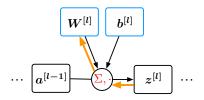
$$\begin{array}{ll} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial w_{jk}^{[l]}} & = & \sum_{k'=1}^{|\boldsymbol{z}^{[l]}|} \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial w_{jk}^{[l]}} = & \frac{\partial \mathcal{L}(\boldsymbol{\Theta})}{\partial z_{j}^{[l]}} \cdot \frac{\partial z_{j}^{[l]}}{\partial w_{jk}^{[l]}} \\ & = & \delta_{j}^{[l]} \cdot \frac{\partial \left(\sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_{j}^{[l]}\right)}{\partial w_{jk}^{[l]}} = & \delta_{j}^{[l]} \cdot a_{k}^{[l-1]} \end{array}$$

Formulation vectorielle:



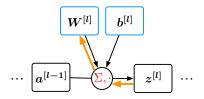
$$abla_{W^{[l]}} \mathcal{L}(\mathbf{\Theta}) = \delta^{[l]} \left[a^{[l-1]} \right]^{T}$$

Formulation vectorielle:



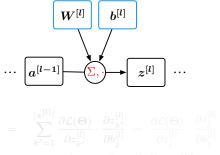
$$abla_{oldsymbol{W}^{[l]}} \mathcal{L}(oldsymbol{\Theta}) \ = \ \delta^{[l]} \left[a^{[l-1]}
ight]^T$$

Formulation vectorielle:



$$abla_{oldsymbol{W}^{[l]}} \mathcal{L}(oldsymbol{\Theta}) \ = \ oldsymbol{\delta}^{[l]} \left[oldsymbol{a}^{[l-1]}
ight]^T$$

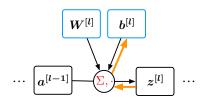
Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :



$$= \sum_{k'=1} \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}}{\partial b_j^{[l]}} = \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial z_j^{[l]}} \cdot \frac{\partial z_j}{\partial b_j^{[l]}}$$

$$= \delta_j^{[l]} \cdot \frac{\partial \left(\sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_j^{[l]}\right)}{\partial b_j^{[l]}} = \delta_j^{[l]}$$

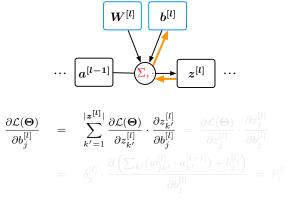
Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :



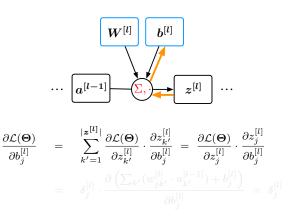
$$\frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial b_{j}^{[l]}} = \sum_{k'=1}^{|\mathbf{z}^{[l]}|} \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial z_{k'}^{[l]}} \cdot \frac{\partial z_{k'}^{[l]}}{\partial b_{j}^{[l]}} = \frac{\partial \mathcal{L}(\mathbf{\Theta})}{\partial z_{j}^{[l]}} \cdot \frac{\partial z_{j}^{[l]}}{\partial b_{j}^{[l]}}$$

$$= \delta_{j}^{[l]} \cdot \frac{\partial \left(\sum_{k'} (w_{jk'}^{[l]} \cdot a_{k'}^{[l-1]}) + b_{j}^{[l]}\right)}{\partial b_{j}^{[l]}} = \delta_{j}^{[l]}$$

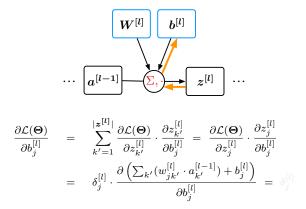
Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :



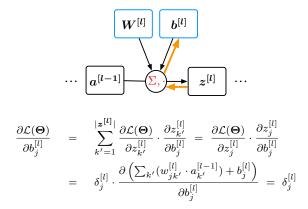
Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :



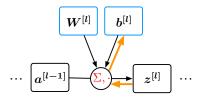
Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :



Calcul des gradients proprement dits en utilisant les erreurs δ_i^l :

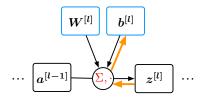


Formulation vectorielle:



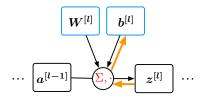
$$\nabla_{b^{[l]}} \mathcal{L}(\mathbf{\Theta}) = \delta^{[l]}$$

Formulation vectorielle:



$$abla_{m{b}^{[l]}} \mathcal{L}(m{\Theta}) = \delta^{[l]}$$

Formulation vectorielle:



$$\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) = \boldsymbol{\delta}^{[l]}$$

Calcul des erreurs $\delta^{[l]} := \nabla_{z^{[l]}} \mathcal{L}(\Theta)$ et des gradients $\nabla_{W^{[l]}} \mathcal{L}(\Theta)$ et $\nabla_{\mathbf{h}^{[l]}} \mathcal{L}(\mathbf{\Theta})$, pour toute couche $l = L, \ldots, 1$:

$$\delta^{[l]} = \begin{cases} \nabla_{a^{[l]}} \mathcal{L}(\mathbf{\Theta}) \odot \sigma'(z^{[l]}), & \text{si } l = L \\ \left[\mathbf{W}^{[l+1]} \right]^T \delta^{[l+1]} \odot \sigma'(z^{[l]}), & \text{si } L > l \ge 1 \end{cases}$$
(1)

$$\nabla_{W^{[l]}} \mathcal{L}(\Theta) = \delta^{[l]} \left[a^{[l-1]} \right]^T \tag{2}$$

$$\nabla_{b^{[l]}} \mathcal{L}(\mathbf{\Theta}) = \delta^{[l]} \tag{3}$$

Calcul des erreurs $\boldsymbol{\delta}^{[l]} := \nabla_{\boldsymbol{z}^{[l]}} \mathcal{L}(\boldsymbol{\Theta})$ et des gradients $\nabla_{\boldsymbol{W}^{[l]}} \mathcal{L}(\boldsymbol{\Theta})$ et $\nabla_{\boldsymbol{b}^{[l]}} \mathcal{L}(\boldsymbol{\Theta})$, pour toute couche $l = L, \dots, 1$:

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BACKPROPAGATION

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Une fois gradients calculés, on effectue l'update des poids et biais (cf. gradient descent algo):

Pour
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$$W^{[l]} := W^{[l]} - \eta \cdot \nabla_{W^{[l]}} \mathcal{L}(\boldsymbol{\Theta}) \tag{4}$$

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où η est le *learning rate*

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CHAIN RULE

Remarque: on peut déduire une version "batched" des équations.

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Soit B = (X, Y) un batch composé de B inputs et outputs x_k et y_k alignés en deux matrices:

BACKPROPAGATION

$$egin{aligned} egin{aligned} egin{aligned} egin{aligned} egin{aligned} dallet & dallet & \dots & dallet \ m{x_1} & m{x_2} & \cdots & m{x_B} \ dallet & dallet & \dots & dallet \end{aligned} \end{aligned} egin{aligned} dallet & egin{aligned} dallet & dallet & \dots & dallet \ m{y_1} & m{y_2} & \cdots & m{y_B} \ dallet & dallet & \dots & dallet \end{aligned}$$

$$A^{[L]} = \begin{pmatrix} \vdots & \vdots & \dots & \vdots \\ a_1^{[L]} & a_2^{[L]} & \cdots & a_B^{[L]} \\ \vdots & \vdots & \dots & \vdots \end{pmatrix}$$

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Soit $A^{[L]}$ les outputs du réseau associés aux inputs X:

$$A^{[L]} = egin{pmatrix} dots & dots & \ldots & dots \ a_1^{[L]} & a_2^{[L]} & \cdots & a_B^{[L]} \ dots & dots & \ldots & dots \end{pmatrix}$$

4□ → 4回 → 4 回 → 4 回 → 9 へ ○

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lackbrack Soit $\mathcal{L}_k(m{\Theta}) := \mathcal{L}(m{\Theta}, m{a_k^{[L]}}, m{y_k})$ la loss associée à l'exemple k, pour $k = 1, \ldots, B$. 4□ ト 4 □ ト 4 □ ト 4 □ ト 9 0 ○

CHAIN BILLE

Calcul des erreurs $\delta_{\mathbf{k}}^{[l]} := \nabla_{\mathbf{z}^{[l]}} \mathcal{L}_k(\mathbf{\Theta})$ et des gradients $\nabla_{\mathbf{W}[l]} \mathcal{L}_k(\mathbf{\Theta})$ et $\nabla_{\mathbf{h}[l]} \mathcal{L}_k(\mathbf{\Theta})$, pour tout exemple $k = 1, \dots, B$ pour toute couche $l = L, \ldots, 1$:

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où η est le *learning rate*.

FORWARD PASS AND BACKWARD PASS



FORWARD PASS AND BACKWARD PASS



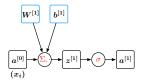


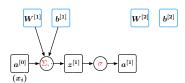


FORWARD PASS AND BACKWARD PASS

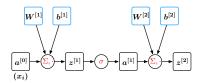


FORWARD PASS AND BACKWARD PASS

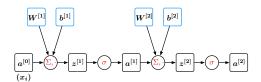


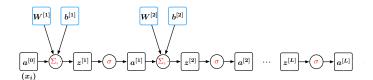


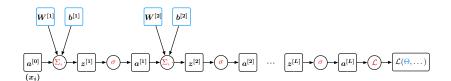
FORWARD PASS AND BACKWARD PASS



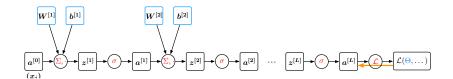
FORWARD PASS AND BACKWARD PASS

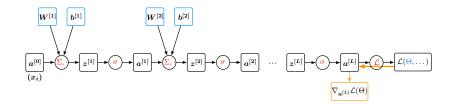




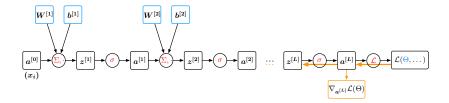


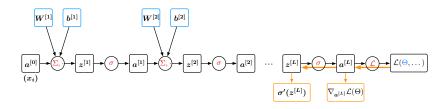
BACKPROPAGATION



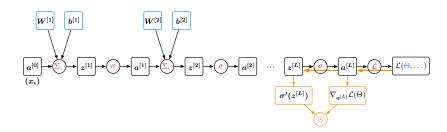


BACKPROPAGATION

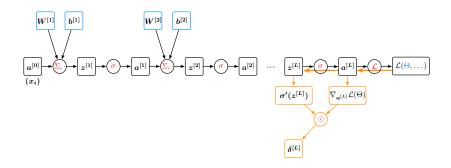


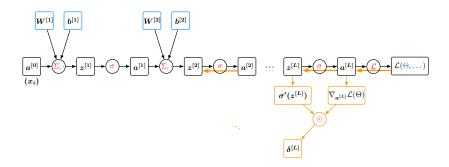


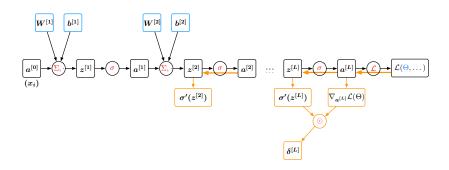
BACKPROPAGATION



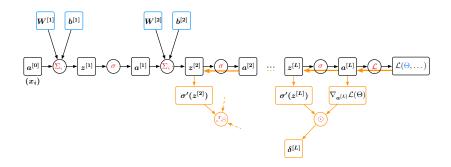
BACKPROPAGATION

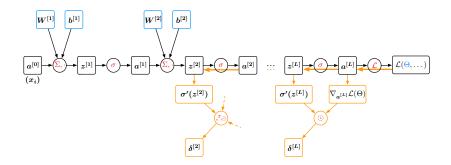


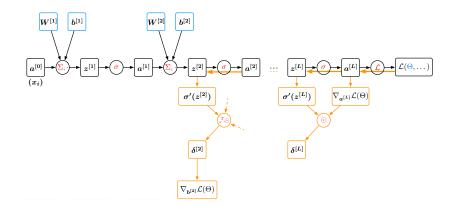


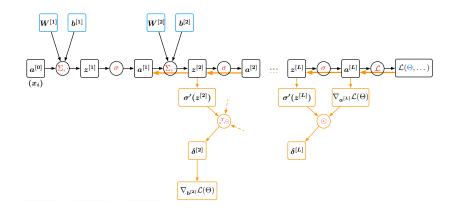


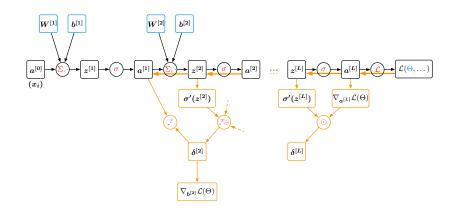
BACKPROPAGATION

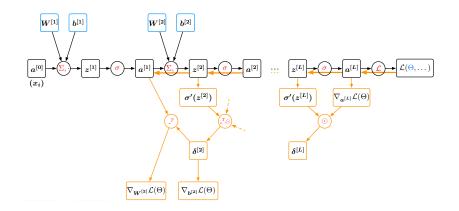


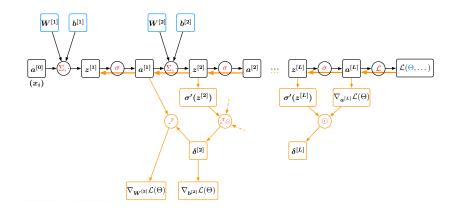


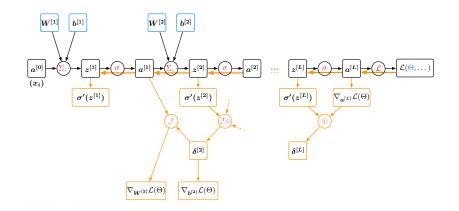


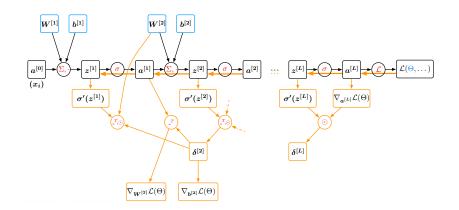


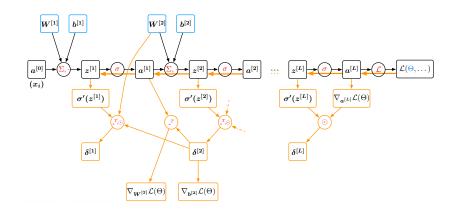


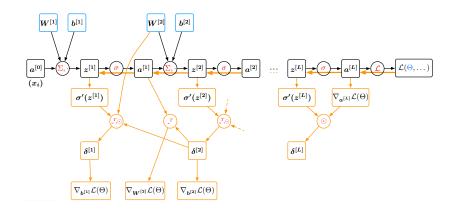


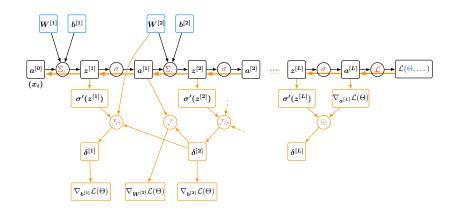


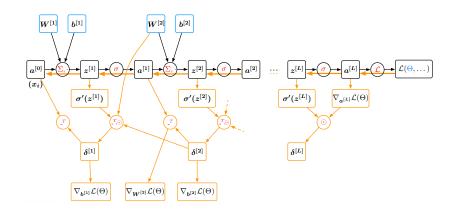


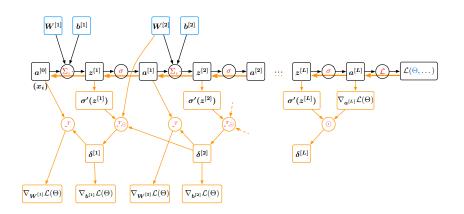












```
\mathsf{Data:}\ \mathsf{dataloader} = \big\{ \boldsymbol{B_b} = (\boldsymbol{X_b}, \boldsymbol{Y_b}) : b = 1, \dots, nb\_batches \big\}
Inputs: MLP = \left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \dots, L \right\} initialized randomly
```

Algorithm 1: Backpropagation (stochastic gradient descent)

```
\mathsf{Data:}\ \mathsf{dataloader} = \big\{ \boldsymbol{B_b} = (\boldsymbol{X_b}, \boldsymbol{Y_b}) : b = 1, \dots, nb\_batches \big\}
Inputs: MLP = \left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \dots, L \right\} initialized randomly
```

```
\mbox{ for } epochs = 1 \mbox{ to } nb \mbox{ } epochs \mbox{ do}
        for k = 1 to nb batches do
```

// loop over epochs

```
\mathsf{Data:}\ \mathsf{dataloader} = \big\{ \boldsymbol{B_b} = (\boldsymbol{X_b}, \boldsymbol{Y_b}) : b = 1, \dots, nb\_batches \big\}
Inputs: MLP = \left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \dots, L \right\} initialized randomly
\  \, \mathbf{for}\,\,epochs=1\,\,\mathbf{to}\,\,nb\  \, epochs\,\,\mathbf{do}
                                                                                                                               // loop over epochs
         for k = 1 to nb batches do
                                                                                                                             // loop over batches
                  for l=1 to L do
```

BACKPROPAGATION 0000000000000000000

BACKPROPAGATION

```
\mathsf{Data:}\ \mathsf{dataloader} = \big\{ \boldsymbol{B_b} = (\boldsymbol{X_b}, \boldsymbol{Y_b}) : b = 1, \dots, nb\_batches \big\}
Inputs: MLP = \left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \dots, L \right\} initialized randomly
\  \, \mathbf{for}\,\,epochs=1\,\,\mathbf{to}\,\,nb\  \, epochs\,\,\mathbf{do}
                                                                                                                            // loop over epochs
        for k = 1 to nb batches do
                                                                                                                          // loop over batches
                  for l = 1 to L do
                                                                                                                                   // forward pass
```

Backpropagation 000000000000000000000

```
\mathsf{Data:}\ \mathsf{dataloader} = \big\{ \boldsymbol{B_b} = (\boldsymbol{X_b}, \boldsymbol{Y_b}) : b = 1, \dots, nb\_batches \big\}
Inputs: MLP = \left\{ \left( \boldsymbol{W}^{[l]}, \boldsymbol{b}^{[l]} \right) : l = 1, \dots, L \right\} initialized randomly
\  \, \mathbf{for}\,\,epochs=1\,\,\mathbf{to}\,\,nb\  \, epochs\,\,\mathbf{do}
                                                                                                                       // loop over epochs
        for k = 1 to nb batches do
                                                                                                                     // loop over batches
                 for l = 1 to L do
                                                                                                                             // forward pass
                 for l = L to 1 do
                                                                                                                            // backward pass
```

Backpropagation 000000000000000000000

BACKPROPAGATION (BP)

Backpropagation 000000000000000000000

BACKPROPAGATION

Algorithm 1: Backpropagation (stochastic gradient descent)

```
\overline{\mathsf{Data: dataloader}} = \big\{ B_b = (X_b, Y_b) : b = 1, \dots, nb\_batches \big\}
Inputs: MLP = \{(\mathbf{W}^{[l]}, \mathbf{b}^{[l]}) : l = 1, ..., L\} initialized randomly
\  \, \mathbf{for}\,\,epochs=1\,\,\mathbf{to}\,\,nb\  \, epochs\,\,\mathbf{do}
                                                                                                // loop over epochs
      for k = 1 to nb batches do
                                                                                               // loop over batches
              for l = 1 to L do
                                                                                                     // forward pass
              for l = L to 1 do
                                                                                                    // backward pass
                                                                                                    // compute error
                                                                                                  // update gradient
                                                                                                  // update gradient
       end
```

end

```
Data: dataloader = \{B_b = (X_b, Y_b) : b = 1, \dots, nb\_batches\}
Inputs: MLP = \left\{\left(oldsymbol{W}^{[l]}, oldsymbol{b}^{[l]}
ight): l=1,\ldots,L\right\} initialized randomly
```

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Algorithm 2: Backpropagation (stochastic gradient descent)

```
Data: dataloader = \{B_b = (X_b, Y_b) : b = 1, \dots, nb \ batches\}
Inputs: MLP = \left\{\left(m{W}^{[l]}, m{b}^{[l]}\right): l=1,\ldots,L\right\} initialized randomly
```

```
for epochs = 1 to nb epochs do
                                                                       // loop over epochs
     for b = 1 to nb batches do
```

Algorithm 2: Backpropagation (stochastic gradient descent)

```
Data: dataloader = \{B_b = (X_b, Y_b) : b = 1, \dots, nb \ batches\}
Inputs: MLP = \left\{\left(oldsymbol{W}^{[l]}, oldsymbol{b}^{[l]}
ight): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                           // loop over epochs
      for b = 1 to nb\_batches do
                                                                                         // loop over batches
```

Algorithm 2: Backpropagation (stochastic gradient descent)

```
Data: dataloader = \{B_b = (X_b, Y_b) : b = 1, \dots, nb \ batches\}
Inputs: MLP = \left\{\left(oldsymbol{W}^{[l]}, oldsymbol{b}^{[l]}
ight): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                          // loop over epochs
      for b=1 to nb\_batches do
                                                                                         // loop over batches
             A^{[0]} = \bar{X_b}
             for l = 1 to L do
                                                                                               // forward pass
```

Algorithm 2: Backpropagation (stochastic gradient descent)

```
Data: dataloader = \{B_b = (X_b, Y_b) : b = 1, \dots, nb \ batches\}
Inputs: MLP = \left\{\left(oldsymbol{W}^{[l]}, oldsymbol{b}^{[l]}
ight): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                              // loop over epochs
      for b=1 to nb\_batches do
                                                                                             // loop over batches
             A^{[0]} = \bar{X_b}
             for l = 1 to L do
                                                                                                    // forward pass
                     Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                    A^{[l]} = \sigma\left(Z^{[l]}\right)
             end
```

```
Data: dataloader = \{B_b = (X_b, Y_b) : b = 1, \dots, nb \ batches\}
Inputs: MLP = \left\{\left(oldsymbol{W}^{[l]}, oldsymbol{b}^{[l]}
ight): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                             // loop over epochs
      for b=1 to nb\_batches do
                                                                                           // loop over batches
             A^{[0]} = \bar{X_b}
             for l = 1 to L do
                                                                                                  // forward pass
                    Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                    A^{[l]} = \sigma\left(Z^{[l]}\right)
             end
             for l = L to 1 do
                                                                                                 // backward pass
```

```
Data: dataloader = \{B_b = (X_b, Y_b) : b = 1, \dots, nb\_batches\}
Inputs: MLP = \{(\mathbf{W}^{[l]}, \mathbf{b}^{[l]}) : l = 1, ..., L\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                                      // loop over epochs
       for b = 1 to nb\_batches do
                                                                                                     // loop over batches
               A^{[0]} = \bar{X_b}
               for l = 1 to L do
                                                                                                            // forward pass
                      Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                      A^{[l]} = \sigma\left(Z^{[l]}\right)
               end
               for l = L to 1 do
                                                                                                          // backward pass
                      if l = L then
                                                                                                          // compute error
                             \delta_{k}^{[l]} = \nabla_{[l]} \mathcal{L}_{k}(\Theta) \odot \sigma'(z_{k}^{[l]})
                                                                                       for k = 1, \ldots, B
                      else if L > l > 1 then
                             \delta_{h}^{[l]} = \begin{bmatrix} W^{[l+1]} \end{bmatrix}^{T} \delta_{h}^{[l+1]} \odot \sigma'(z_{h}^{[l]})
                                                                                      for k = 1, \dots, B
                      end
```

```
Data: dataloader = \{B_b = (X_b, Y_b) : b = 1, \dots, nb\_batches\}
Inputs: MLP = \left\{\left(oldsymbol{W}^{[l]}, oldsymbol{b}^{[l]}
ight): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                                               // loop over epochs
        for b = 1 to nb batches do
                                                                                                             // loop over batches
                A^{[0]} = X_b
                for l = 1 to L do
                                                                                                                     // forward pass
                        Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                        A^{[l]} = \sigma\left(Z^{[l]}\right)
                end
                for l = L to 1 do
                                                                                                                    // backward pass
                        if l = L then
                                                                                                                    // compute error
                               \delta_{k}^{[l]} = \nabla_{[l]} \mathcal{L}_{k}(\Theta) \odot \sigma'(z_{k}^{[l]})
                                                                                              for k = 1, \ldots, B
                        else if L > l > 1 then
                                \delta_{h}^{[l]} = \begin{bmatrix} W^{[l+1]} \end{bmatrix}^{T} \delta_{h}^{[l+1]} \odot \sigma'(z_{h}^{[l]})
                                                                                              for k = 1, \dots, B
                        end
                        \nabla_{\mathbf{W}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_k^{[l]} \left[ a_k^{[l-1]} \right]^T
                                                                                               for k = 1, \dots, B
```

```
Data: dataloader = \{B_b = (X_b, Y_b) : b = 1, \dots, nb\_batches\}
Inputs: MLP = \left\{\left(oldsymbol{W}^{[l]}, oldsymbol{b}^{[l]}
ight): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                                                    // loop over epochs
        for b = 1 to nb batches do
                                                                                                                  // loop over batches
                 A^{[0]} = X_b
                 for l = 1 to L do
                                                                                                                          // forward pass
                         Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                         A^{[l]} = \sigma\left(Z^{[l]}\right)
                 end
                 for l = L to 1 do
                                                                                                                        // backward pass
                         if l = L then
                                                                                                                        // compute error
                                \delta_{k}^{[l]} = \nabla_{[l]} \mathcal{L}_{k}(\Theta) \odot \sigma'(z_{k}^{[l]})
                                                                                                  for k = 1, \ldots, B
                         else if L > l > 1 then
                                 \delta_{h}^{[l]} = \begin{bmatrix} W^{[l+1]} \end{bmatrix}^{T} \delta_{h}^{[l+1]} \odot \sigma'(z_{h}^{[l]})
                                                                                                  for k = 1, \dots, B
                         end
                         \nabla_{\mathbf{W}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_k^{[l]} \left[ a_k^{[l-1]} \right]^T
                                                                                                  for k = 1, \dots, B
                         \nabla_{\mathbf{h}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_{i}^{[l]}
                                                                                                  for k = 1, \dots, B
```

Algorithm 2: Backpropagation (stochastic gradient descent)

```
Data: dataloader = \{B_b = (X_b, Y_b) : b = 1, \dots, nb \ batches\}
Inputs: MLP = \left\{\left(oldsymbol{W}^{[l]}, oldsymbol{b}^{[l]}
ight): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                                                             // loop over epochs
        for b = 1 to nb batches do
                                                                                                                           // loop over batches
                  A^{[0]} = X_b
                  for l = 1 to L do
                                                                                                                                    // forward pass
                           Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                          A^{[l]} = \sigma\left(Z^{[l]}\right)
                  end
                  for l = L to 1 do
                                                                                                                                  // backward pass
                           if l = L then
                                                                                                                                  // compute error
                                   \delta_{k}^{[l]} = \nabla_{[l]} \mathcal{L}_{k}(\Theta) \odot \sigma'(z_{k}^{[l]})
                                                                                                          for k = 1, \ldots, B
                           else if L > l > 1 then
                                   \delta_{h}^{[l]} = \begin{bmatrix} W^{[l+1]} \end{bmatrix}^{T} \delta_{h}^{[l+1]} \odot \sigma'(z_{h}^{[l]})
                                                                                                         for k = 1, \dots, B
                           end
                           \nabla_{\mathbf{W}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_k^{[l]} \left[ a_k^{[l-1]} \right]^T
                                                                                                          for k = 1, \dots, B
                           \nabla_{\mathbf{h}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_{\mathbf{h}}^{[l]}
                                                                                                          for k = 1, \dots, B
                           \mathbf{W}^{[l]} := \mathbf{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\mathbf{v}_{\mathbf{v}_{\ell}}[l]} \mathcal{L}_{k}(\mathbf{\Theta})
                                                                                                                              // update gradient
```

```
Data: dataloader = \{B_b = (X_b, Y_b) : b = 1, \dots, nb\_batches\}
Inputs: MLP = \{(\mathbf{W}^{[l]}, \mathbf{b}^{[l]}) : l = 1, ..., L\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                                                              // loop over epochs
         for b = 1 to nb batches do
                                                                                                                            // loop over batches
                  A^{[0]} = X_b
                  for l = 1 to L do
                                                                                                                                     // forward pass
                           Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                           A^{[l]} = \sigma\left(Z^{[l]}\right)
                  end
                  for l = L to 1 do
                                                                                                                                   // backward pass
                           if l = L then
                                                                                                                                   // compute error
                                   \delta_{k}^{[l]} = \nabla_{[l]} \mathcal{L}_{k}(\Theta) \odot \sigma'(z_{k}^{[l]})
                                                                                                           for k = 1, \dots, B
                           else if L > l > 1 then
                                    \delta_{h}^{[l]} = \begin{bmatrix} W^{[l+1]} \end{bmatrix}^{T} \delta_{h}^{[l+1]} \odot \sigma'(z_{h}^{[l]})
                                                                                                          for k = 1, \dots, B
                            end
                           \nabla_{\mathbf{W}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_k^{[l]} \left[ a_k^{[l-1]} \right]^T
                                                                                                           for k = 1, \dots, B
                           \nabla_{\mathbf{h}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_{\mathbf{h}}^{[l]}
                                                                                                           for k = 1, \dots, B
                           \mathbf{W}^{[l]} := \mathbf{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\mathbf{v}_{\mathbf{v}_{\ell}}[l]} \mathcal{L}_{k}(\mathbf{\Theta})
                                                                                                                                // update gradient
                          b^{[l]} := b^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\mathbf{r}[l]} \mathcal{L}_{k}(\boldsymbol{\Theta})
                                                                                                                                // update gradient
```

BACKPROPAGATION

```
\mathsf{Data:}\ \mathsf{dataloader} = \big\{ \boldsymbol{B_b} = (\boldsymbol{X_b}, \boldsymbol{Y_b}) : b = 1, \dots, nb\_batches \big\}
Inputs: MLP = \left\{\left(oldsymbol{W}^{[l]}, oldsymbol{b}^{[l]}
ight): l=1,\ldots,L\right\} initialized randomly
for epochs = 1 to nb epochs do
                                                                                                                                      // loop over epochs
         for b = 1 to nb batches do
                                                                                                                                     // loop over batches
                   A^{[0]} = X_b
                   for l = 1 to L do
                                                                                                                                              // forward pass
                             Z^{[l]} = W^{[l]}A^{[l-1]} + b^{[l]}
                             A^{[l]} = \sigma\left(Z^{[l]}\right)
                   end
                   for l = L to 1 do
                                                                                                                                            // backward pass
                             if l = L then
                                                                                                                                            // compute error
                                      \delta_{k}^{[l]} = \nabla_{[l]} \mathcal{L}_{k}(\Theta) \odot \sigma'(z_{k}^{[l]})
                                                                                                                  for k = 1, \dots, B
                             else if L > l > 1 then
                                      \delta_{h}^{[l]} = \begin{bmatrix} W^{[l+1]} \end{bmatrix}^{T} \delta_{h}^{[l+1]} \odot \sigma'(z_{h}^{[l]})
                                                                                                                 for k = 1, \dots, B
                             end
                             \nabla_{\mathbf{W}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_k^{[l]} \left[ a_k^{[l-1]} \right]^T
                                                                                                                   for k = 1, \dots, B
                             \nabla_{\mathbf{b}[l]} \mathcal{L}_k(\mathbf{\Theta}) = \delta_{\mathbf{b}}^{[l]}
                                                                                                                  for k = 1, \dots, B
                             \mathbf{W}^{[l]} := \mathbf{W}^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\mathbf{v}_{\mathbf{v}_{\ell}}[l]} \mathcal{L}_{k}(\mathbf{\Theta})
                                                                                                                                        // update gradient
                             b^{[l]} := b^{[l]} - \frac{\eta}{B} \cdot \sum_{k=1}^{B} \nabla_{\mathbf{r}[l]} \mathcal{L}_k(\boldsymbol{\Theta})
                                                                                                                                        // update gradient
                   end
          end
end
```

EXEMPLE DE TRAINING VIA BACKROP

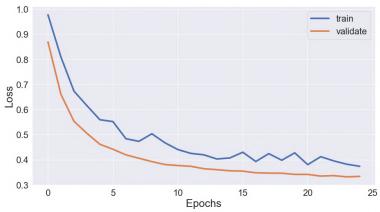


Figure taken from towardsdatascience.com

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