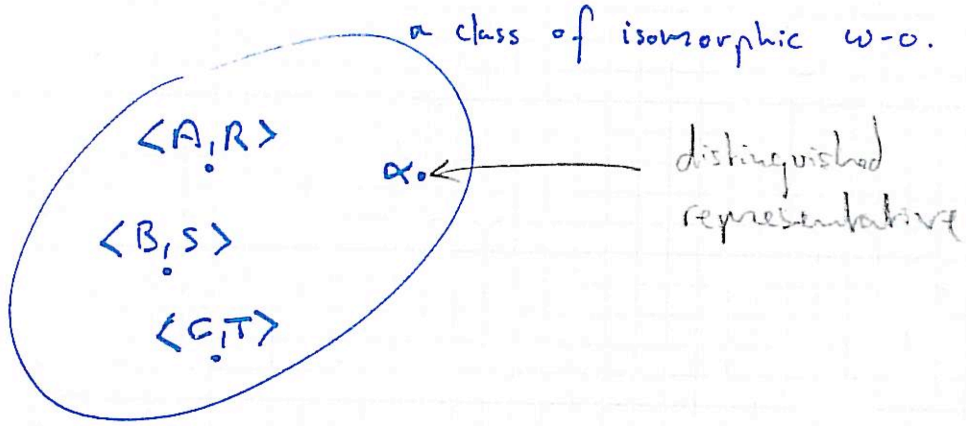
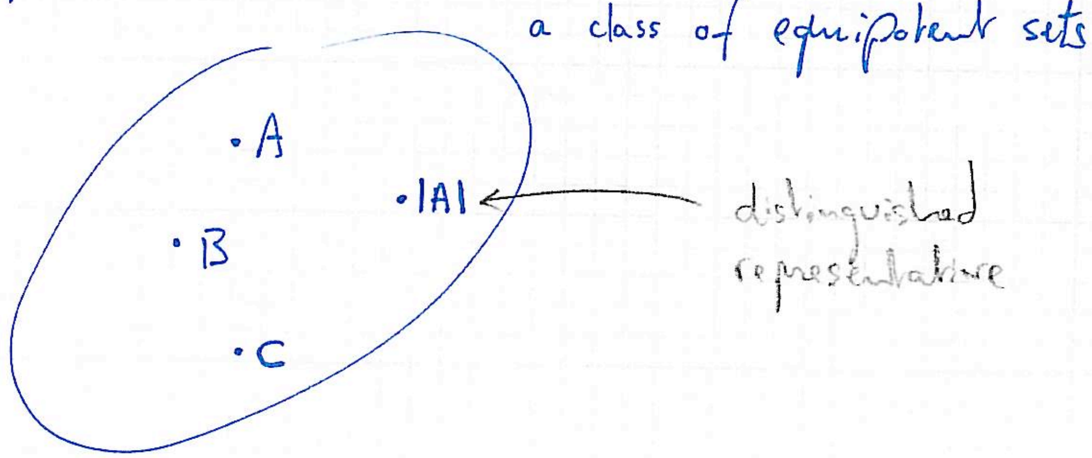
Chapter 3: Cardinals

Ordinals are distinguished representatives of classes of isomorphic w-o:



Cardinals are distinguished representatives of classes of equipotent sets:



Cardinals will be used to measure the size of sets ...

3.1. First definitions

Def: let A, B be two sets, we write:

i) $A \lesssim B$ iff $\exists f$ injective $: A \to B$

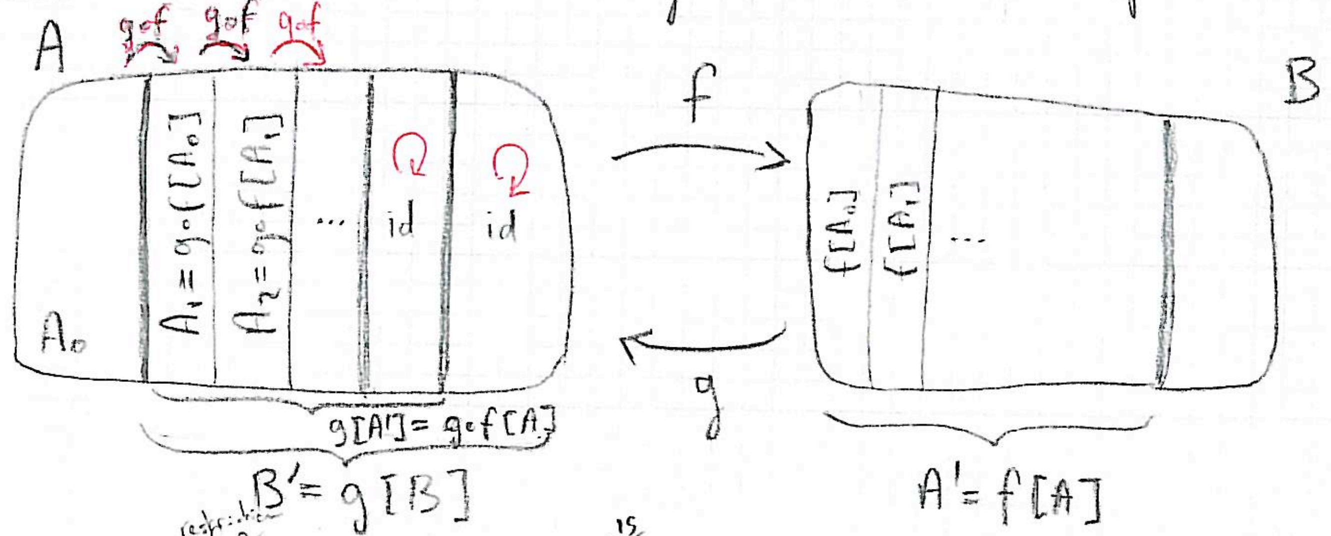
ii) $A \approx B$ (equipotence) iff $\exists f$ bij $: A \to B$

iii) $A < B$ iff $A \lesssim B$ and $B \not\lesssim A$.

Thm 1 (Cantor-Schröder-Bernstein):

$$A \lesssim B \text{ and } B \lesssim A \to A \approx B$$

Proof: Let $f: A \to B$ and $g: B \to A$ be two injections



g inj $\overset{\text{restriction}}{\Rightarrow} g: B \to \operatorname{ran}(g) = B'$ $\overset{\text{is}}{}$ bij $\Rightarrow g^{-1}: B' \to B$ bij also

We build $h: A \to B'$ bij $\Rightarrow$ $g^{-1} \circ h : A \to B$ bij

let $A_0 = A \setminus B'$ and $A_{n+1} \overset{\text{by construction}}{=} g \circ f\,[A_n]$.

Note that ~~def~~ the restr. $g \circ f : \bigcup_{i \geq 0} A_i \longrightarrow \bigcup_{i \geq 1} A_i$ is bijective.

- injective as a composition of injections
- surjective since $\forall y \in A_{n+1}$, $n \geq 0$, $\exists x \in A_n$ s.t. $g \circ f(x) = y$ since $A_{n+1} = g \circ f[A_n]$

intuition: on pousse A pour le faire entrer dans B'

→ let h be def by $h(a) = \begin{cases} g \circ f(a) & \text{if } a \in \bigcup_{i \geq 0} A_i \\ a & \text{if } a \in A \setminus \bigcup A_i \end{cases}$

$: A \to B'$ — everything but A_0

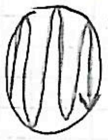
Then $\operatorname{ran}(h) \overset{\text{def}}{=} g \circ f\left[\bigcup_{i \geq 0} A_i\right] \cup A \setminus \bigcup_{i \geq 0} A_i = \bigcup_{i \geq 1} A_i \cup A \setminus \bigcup_{i \geq 0} A_i \overset{i \geq 0}{=} B'$

and thus $h: A \to B'$ bijective (as composed of bijections) ∎

If A can be w-o by some $R \subseteq A \times A$, then (by thm 7, ch2) there is a unique ordinal α s.t. $\langle A, R\rangle \cong \langle \alpha, \in \rangle$

Hence, we have in particular (by forgetting the ordering) that $A \approx \alpha$.

explain with figure



$R \subseteq A \times A \Rightarrow \cong \langle \alpha, \in \rangle$ $\Rightarrow A \approx \alpha$

<u>Def</u>: If A can be w-o, we let $|A|$ be the least ordinal α s.t. $A \approx \alpha$. (i.e. $A \approx |A|$ for all A)

$|A|$ is the <u>cardinality</u> of A

<u>Remark</u>: $||A|| = |A|$

$\boxed{\text{donner ex p.3' ici}}$

<u>Remark</u>: Under (AC), every set A can be w-o $\Rightarrow$ isom to a unique ord $\Rightarrow$ equipotent to at least one ord.
$\Rightarrow$) cardinality $|A|$ well-defined for all A.

<u>Proposition 2</u>: Let A, B two sets that can be w-o.

(i) $|A| = |B|$ iff $A \approx B$

(ii) $|A| \leq |B|$ iff $A \preccurlyeq B$

(iii) $|A| < |B|$ iff $A \prec B$ (i.e. $A \preccurlyeq B$ and $B \not\preccurlyeq A$)

<u>Proof</u>: (i) and (iii) follow from (ii)

(ii): If $|A| \leq |B| \Rightarrow$ ($id : |A| \hookrightarrow |B|$ inj.) $|A| \subseteq |B|$

Then $A \approx |A| \xrightarrow[id]{} |B| \approx B$ implies $A \preccurlyeq B$ by comp of bij and inj

Conversely, suppose $A \preccurlyeq B$ by f

Then $|A| \approx A \underset{(by\ f)}{\preccurlyeq} B \approx |B|$

$$|A| \preccurlyeq |B| \Rightarrow |A| \leq |B|$$

$\square$

not immediate: by contradiction, if $|B| < |A|$ (id injection)

$\Rightarrow |B| \preccurlyeq |A|$ and $|A| \preccurlyeq |B| \underset{\text{Thm 1.}}{\Rightarrow} |A| \approx |B|$

$\Rightarrow ||A|| = ||B||$ i.e. $|A| = |B|$

Contradiction (with $|B| < |A|$)

Hence, comparing the cardinality of sets exactly amounts to finding the possible injections and bijections between these sets!

<u>Remark:</u> For all ord α, we have $|\alpha| \leq \alpha$ (since $\alpha \approx \alpha$)

<u>Def</u>: An ordinal α is a <u>cardinal</u> iff $|\alpha| = \alpha$.

i.e. iff α is the least of its equipotent class

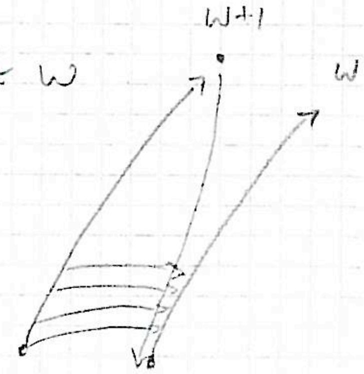
i.e. iff $\forall \beta < \alpha \, (\beta \not\approx \alpha)$

<u>Examples</u>:
- $\forall n \in \omega$, $|n| = n$ i.e. n is a cardinal proof not "so easy", by induction on n that $n \not\approx n+1$

exercices

- $|\omega| = \omega$, hence ω is a cardinal proof not "so easy" also
- $|\omega+1| = \omega$ Clearly, $\omega+1 \approx \omega$
- $|\omega+n| = \omega$
- $|\omega+\omega| = \omega$
- $|\omega \cdot n| = \omega$
- $|\omega \cdot \omega| = \omega$
- $|\omega^n| = \omega$
- $|\omega^\omega| = \omega$

$\vdots$



<u>Def</u>: A is <u>finite</u> iff $|A| < \omega$; <u>countable</u> iff $|A| \leq \omega$; <u>uncountable</u> iff $|A| > \omega$.

oral

So far, we don't have any easy way to build cardinals larger than ω.
(In fact, it is consistent with the axioms presented so far that the only infinite cardinal is ω i.e.

$$\mathrm{Cons}(ZFC) \rightarrow \mathrm{Cons}(ZFC - P + \forall x\,(x \text{ countable}))\ )$$

$\Rightarrow$ We introduce the Power Set Axiom

Axiom 8 : Power set

$$\forall x\, \exists y\, \forall z\, (z \subseteq x \rightarrow z \in y)$$

By comprehension, we can then form

<u>Def</u>: $\mathcal{P}(x) := \{ z \in y : z \subseteq x \}$

Thm 3 (Cantor) $x < \mathcal{P}(x)$

<u>Proof</u>: $x \leq \mathcal{P}(x)$ trivial (e.g. injection $a \mapsto \{a\}$)

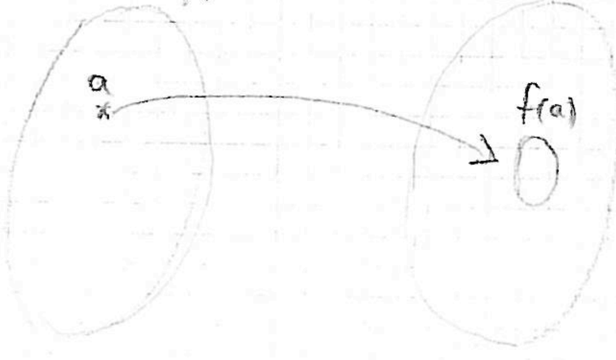
Let $f: x \longrightarrow \mathcal{P}(x)$, we prove that f not surjective (hence $\nexists$ bij $x \to \mathcal{P}(x)$)

Consider $A = \{ a \in x : a \notin f(a) \} \in \mathcal{P}(x)$

Since a belongs to one and not the other

let $a \in x$. If $a \in A \Rightarrow a \notin f(a) \Rightarrow A \neq f(a)$

If $a \notin A \Rightarrow a \in f(a) \Rightarrow A \neq f(a)$



Hence $\forall a \in x$, $f(a) \neq A$, thus f not surjective

By thm 3, $\omega \precnsim P(\omega) \Rightarrow$ (prop 2 (iii))
$|\omega| = \omega < |P(\omega)|$
So we have some card above ω ...

We have something more ...

Thm 4 : $\forall \alpha \exists K$ (K cardinal $> \alpha$)

Proof . • If $\alpha < \omega$, take $K = \alpha + 1$.

• Suppose $\alpha \geq \omega$. By Ax 8 and comprehension,
we can form $W = \{ R \in P(\alpha \times \alpha) : \langle \alpha, R \rangle$ is a w-o $\}$

Chap 2, thm 7

• We have $\forall R \in W \; \exists! \beta_R \; (\beta_R \cong \langle \alpha, R \rangle)$, indeed $\beta_R = type(\langle \alpha, R \rangle)$
$\Rightarrow$ We have a functional relation and by Replacement
we can form $S = \{ type(\langle \alpha, R \rangle) : R \in W \}$ set of all possible ordered types over α

- S set of ordinals $\Rightarrow$ sup(S) ordinal

re-ordering of α s.t. $\langle \alpha, R \rangle \cong \alpha + 1$

- $\alpha \in S$, since $\alpha = type(\langle \alpha, \in \rangle)$ and $\in \in W$ (trivial)
trivial

since $\alpha \approx \alpha + 1 \Rightarrow \alpha + 1 \in S$ also

en transportant le bon ordre de $\alpha+1$ sur α par une bij, on obtient un bon ordre R sur α t.q $type(\langle \alpha, R \rangle) = \alpha + 1$

($\alpha + 1 = type(\langle \alpha, R \rangle)$ where R def by
$aR0 \; \forall a \neq 0$ and aRb iff $a < b$ for all $a, b \neq 0$)

$\Rightarrow \sup(S) \geq \alpha + 1 > \alpha$

c.f. verso

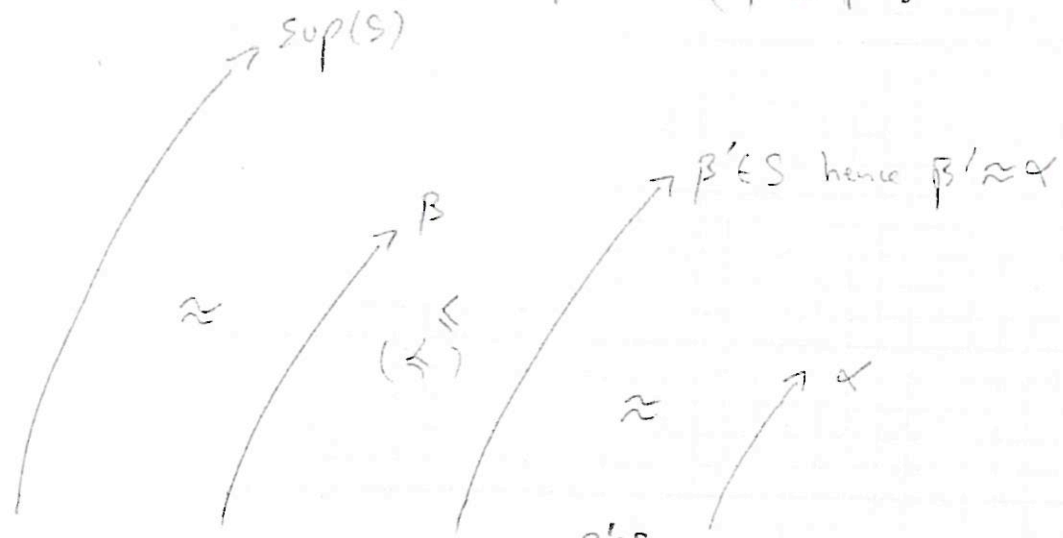
c.f. figure

- Sup(S) is a cardinal : If not, there exists $\beta < \sup(S)$
such that $\beta \approx \sup(S)$. $\beta < \sup(S) \Rightarrow \exists \beta' \in S$ s.t. $\beta \leq \beta'$
$\beta' \in S \Rightarrow \beta' \approx \alpha$. Hence $\sup(S) \approx \beta \overset{(id)}{\precsim} \beta' \approx \alpha \Rightarrow \sup(S) \precsim \alpha$
Conversely, $\alpha \in S \Rightarrow \alpha \leq \sup(S) \Rightarrow \alpha \overset{(id)}{\precsim} \sup(S) \Rightarrow \alpha \approx \sup(S)$
$\Rightarrow \sup(S) \in S \Rightarrow \sup(S) + 1 \in S$, contradiction (by def of sup) $\square$
by def of S

same reasoning as before

sup(S) cardinal: if not $\exists \beta$ s.t. $\sup(S) \approx \underbrace{\beta < \sup(S)}$

$\Rightarrow \exists\, \beta' \in S \ (\beta \leq \beta')$



$\Rightarrow \sup(S) \approx \beta \overset{(id)}{\preccurlyeq} \beta' \overset{\text{since } \beta' \in S}{\approx} \alpha \Rightarrow \sup(S) \preccurlyeq \alpha$

Also $\alpha \in S \Rightarrow \alpha \leq \sup(S) \Rightarrow \alpha \preccurlyeq \sup(S)$ $\Big\} \Rightarrow \sup(S) \approx \alpha$

$\Rightarrow \sup(S) \in S \Rightarrow \sup(S)+1 \in S$, impossible

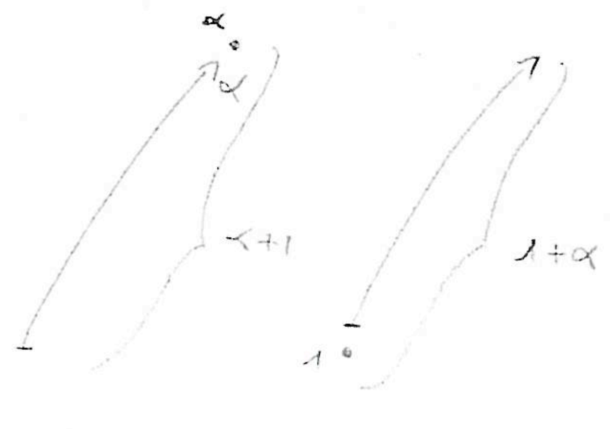
∎

Lemma 5: Every infinite cardinal κ is a limit ordinal

Proof: By contradiction. If $\kappa = \alpha + 1$, with α infinite, then

$\kappa \overset{\text{by def of } \kappa \text{ cardinal}}{=} |\kappa| = |\alpha + 1| = |\overbrace{1+\alpha}^{=\alpha \text{ if } \alpha \text{ infinite}}| = |\alpha| \leq \alpha < \kappa$

contradiction. since $\alpha + 1 \approx 1 + \alpha$ and using Prop 2



∎

<u>Def:</u> If κ cardinal, we let κ^+ be the least cardinal $> \kappa$ (exists by thm 4)

κ is <u>successor</u> iff $\kappa = \lambda^+$ for some cardinal λ
κ is <u>limit</u> iff not successor

<u>Def</u>: for each ordinal α, we define by induction

$$\begin{cases} \aleph_0 = \omega \\ \aleph_{\alpha+1} = \aleph_\alpha^+ \\ \aleph_\lambda = \sup\{\aleph_\xi : \xi < \lambda\}, \text{ for } \lambda \text{ limit} \end{cases}$$

The sequence of "alephs" actually exhaust all possible cardinals (infinite).

<u>Proposition 6</u>: Every infinite cardinal is an "aleph" i.e.
$\forall \kappa$ card $\geq \omega$ $\exists \alpha$ ord s.t. $\kappa = \aleph_\alpha$.

<u>Proof:</u> By induction on κ (infinite)

- If $\kappa = \omega$, then $\kappa = \aleph_0$, o.k.
- If $\kappa = \lambda^+$, for some λ cardinal.
  $\lambda \overset{\lambda^+}{<} \kappa \Rightarrow$ ind hyp $\exists \alpha$ s.t. $\lambda = \aleph_\alpha$
  $\Rightarrow \kappa = \lambda^+ = \aleph_\alpha^+ \underset{def}{=} \aleph_{\alpha+1}$, o.k.
- If κ limit cardinal, let $A = \{\alpha : \aleph_\alpha < \kappa\}$ set of all ord. which enumerate the card. below κ
  (we have $\alpha \leq \aleph_\alpha$, $\forall \alpha$ (by ind.), so, $\aleph_\alpha < \kappa \Rightarrow \underline{\alpha \leq \aleph_\alpha < \kappa}$) $= \{\alpha < \kappa : \aleph_\alpha < \kappa\}$ well defined by Compr.
  Let $\lambda = \sup(A)$. λ ordinal

We have $\underline{\lambda \notin A}$, since $\lambda \in A \Rightarrow \overset{\text{the largest card} < K}{\aleph_\lambda} < K \Rightarrow K = \aleph_\lambda^+$ successor, contradiction

$\underline{\lambda \text{ is limit}}$: If $\lambda \overset{=\sup(A)}{=} \alpha+1$, then $\lambda \in A$ which is impossible (by prec. arg.)

(when the sup is a succ, it belongs to the set (i.e. it is a max), otherwise, its pred would also be a sup.) ⊛

We have $\overset{\text{by hyp}}{\aleph_\alpha < K} \ \forall \alpha < \lambda \overset{\text{sup}(A)}{\Rightarrow} \aleph_\lambda \leq K$ ($\aleph_\lambda = \sup\{\aleph_\alpha : \alpha < \lambda\} \leq K$)

If $\aleph_\lambda < K$, then $\lambda \in A$ by def of A, impossible

$\Rightarrow \underline{\aleph_\lambda = K}$ and K is an aleph. ∎

if all el of the sequence $(\aleph_\alpha)_{\alpha<\lambda}$ are $< K$, then the sup of the sequence is $\leq K$.

⊛ If $\sup(A) = \alpha+1$, then $\alpha+1 \in A$

Indeed, $\forall \beta \in A\ (\beta \leq \alpha+1)$ by def of sup

If $\alpha+1 \notin A$, $\forall \beta \in A\ (\beta \leq \alpha) \Rightarrow \sup(A) = \alpha \neq \alpha+1$

(skip) Lemma 7: (i) $\alpha < \beta \longrightarrow \aleph_\alpha < \aleph_\beta$

(ii) $\aleph_\alpha$ limit cardinal iff α limit ordinal

$\aleph_\alpha$ succ. cardinal iff α succ ordinal

Proof: (Excercice)

(ii) α successor $= \beta+1 \Rightarrow \aleph_\alpha = \aleph_{\beta+1} = \aleph_\beta^+$ (by def)

$= $ successor cardinal

Let $\aleph_\alpha$ successor card i.e. $\exists \lambda$ s.t. $\aleph_\alpha = \lambda^+$

But $\lambda = \aleph_\beta$ for some $\beta \Rightarrow \aleph_\alpha = \lambda^+ = \aleph_\beta^+$

$= \aleph_{\beta+1}$

$\overset{(i)}{\Rightarrow} \alpha = \beta+1$ successor.

(i) By induction on β.

Def: When $\aleph_\alpha$ is considered as an ordinal instead of a cardinal, we denote it ω_α, $\forall \alpha$.

So $\omega_0 = \omega$, $\omega_1 =$ 1st ordinal $\underline{\text{not}}$ equipotent to ω, etc...

3.2. Cardinal Arithmetic

Very different from ordinal arithmetic...

<u>Def</u>: let A, B be sets:

A^B or sometimes ${}^B A$

$= \{ f : f \text{ is a function } \wedge \operatorname{dom}(f) = B \wedge \operatorname{ran}(f) \subseteq A \}$

$= \{ f : f : B \to A \}$ (in short)

Note that $A^B \subseteq \mathcal{P}(B \times A)$. Indeed $f : A \to B \subseteq A \times B$, one has $A^B = \{ f \in \mathcal{P}(B \times A) : f \text{ function } B \to A \}$ exists by Power Set and Comprehension.

A^n = set of functions from n into A, often thinked as the set of sequences of n elements of A.

$A^{<\omega} = \bigcup \{ A^n : n \in \omega \}$ set of finite sequences of elements of A.

<u>Def</u>: let κ, λ be cardinals

1) $\kappa \oplus \lambda = | \kappa \times \{0\} \cup \lambda \times \{1\} |$

2) $\kappa \otimes \lambda = | \kappa \times \lambda |$

3) (AC) $\kappa^\lambda = | {}^\lambda \kappa |$

$\longrightarrow$

oral

Note that 1) and 2) do not need (AC).

① $K \times \{0\} \cup \lambda \times \{1\} \approx \overset{\text{ordinal}}{K + \lambda} \leftarrow$ exists (by ord addition)

$\Rightarrow |K \times \{0\} \cup \lambda \times \{1\}| =_{(def)} K \oplus \lambda$ exists always

② $K \times \lambda \approx \overset{\text{ordinal}}{K \cdot \lambda} \leftarrow$ exists (ordinal multiplication)

$\Rightarrow |K \times \lambda| =_{(def)} K \otimes \lambda$ always exists.

③ Without (AC), nothing tells us that there indeed exists some ordinal α s.t. $\alpha \approx {}^{\lambda}K$

With (AC), ${}^{\lambda}K$ can be well ordered by some R.

$\Rightarrow \langle {}^{\lambda}K, R \rangle \cong \text{type}(\langle {}^{\lambda}K, R \rangle)$

Thus ${}^{\lambda}K \approx \text{type}(\langle {}^{\lambda}K, R \rangle)$ ordinal

$\Rightarrow |{}^{\lambda}K| =_{def} K^{\lambda}$ exists under (AC)

<u>Remark:</u> $\oplus$ and $\otimes$ are commutative (unlike ordinal operations)

i) $K \times \{0\} \cup \lambda \times \{1\} \approx \lambda \times \{0\} \cup K \times \{1\}$, thus

$$K \oplus \lambda = |K \times \{0\} \cup \lambda \times \{1\}| \overset{\text{prop 2.}}{=} |\lambda \times \{0\} \cup K \times \{1\}| = \lambda \oplus K$$

ii) $K \times \lambda \approx \lambda \times K$, thus

$$K \otimes \lambda = |K \times \lambda| = |\lambda \times K| = \lambda \otimes K.$$

Example: $\omega \oplus 1 = |\omega \times \{0\} \cup 1 \times \{1\}|$

$$= |1 \times \{0\} \cup \omega \times \{1\}|$$

since $1+\omega$ (ordinal) $\approx 1\times\{0\} \cup \omega \times \{1\}$

$$= |1+\omega| = |\omega| = \omega$$

$$\omega \otimes 2 = |\omega \times 2| = |2 \times \omega| = |2 \cdot \omega| = |\omega| = \omega$$

lemma 8: For any $n, m \in \omega$, we have (skip)

$$n \oplus m = n+m \text{ and } n \otimes m = n \cdot m$$

ordinal and cardinal operations coincide on natural numbers.

Proof: Excercice.

By induction on m, show that $n+m < \omega$ and $n \cdot m < \omega$

Then show that $\forall k < \omega \; \forall \alpha \; (\alpha \approx k \rightarrow \alpha = k)$

Taking $k = n+m$ and $\alpha = n \oplus m$ leads to concl.
Taking $k = n \cdot m$ and $\alpha = n \otimes m$ leads to concl.

Thm 9: If κ infinite cardinal, then $\kappa \otimes \kappa = \kappa$

i.e. for all α ord, we have $\aleph_\alpha \otimes \aleph_\alpha = \aleph_\alpha$

Proof: Clearly, $\aleph_\alpha = |\aleph_\alpha| \underset{\text{trivial injection}}{\leq} |\aleph_\alpha \times \aleph_\alpha| =_{def} \aleph_\alpha \otimes \aleph_\alpha$.

$\forall \alpha$ ord.

Conversely,

For all ord α, we define "$<^*_\alpha$" on $\aleph_\alpha \times \aleph_\alpha$ by

$\langle \alpha, \beta \rangle <^*_\alpha \langle \gamma, \delta \rangle$ iff

$$\begin{cases} \text{either } \max(\alpha,\beta) < \max(\gamma,\delta) \\ \text{or } \max(\alpha,\beta) = \max(\gamma,\delta) \text{ and } \langle \alpha,\beta\rangle <_{lex} \langle \gamma,\delta \rangle. \end{cases}$$

We can prove that $\langle \aleph_\alpha \times \aleph_\alpha, <^*_\alpha \rangle$ is a w-o (exo.)

We prove by induction on α that

$\text{type}(\aleph_\alpha \times \aleph_\alpha, <^*_\alpha) \leq \aleph_\alpha$ — and thus $\text{type}(\ldots) \leq \aleph_\alpha$ via the identity — which implies

$\aleph_\alpha \otimes \aleph_\alpha =_{def} |\aleph_\alpha \times \aleph_\alpha| = |\text{type}(\aleph_\alpha \times \aleph_\alpha, <^*_\alpha)|$

obvious bij (the isom itself)

$\leq |\aleph_\alpha| = \aleph_\alpha$

by our proof by induction and by prop 7: $A \leq B \Rightarrow |A| \leq |B|$

Thus $\aleph_\alpha = \aleph_\alpha \otimes \aleph_\alpha$.

details: let us call $K_\alpha = \text{type}(\aleph_\alpha \times \aleph_\alpha, <^*_\alpha)$

and f_α isom: $\langle \aleph_\alpha \times \aleph_\alpha, <^*_\alpha \rangle \longrightarrow \langle K_\alpha, \in \rangle$

c.f. figure.

case $\alpha = 0$: If $p, q \in \aleph_0$, $\langle p, q \rangle$ has only finitely many $<^*_0$-pred in $(\aleph_0 \times \aleph_0, <^*_0)$

$\underset{f \text{ isom.}}{\Longrightarrow}$ so does $f_0(\langle p,q \rangle)$ in K_0 $\Rightarrow$ $K_0 \leq \aleph_0$

(since every el of K_0 is finite)

K_0 is an ordinal whose every pred are finite.

case $\alpha \geq 1$: let $\beta, \gamma \in \aleph_\alpha$

$$\left|\operatorname{pred}\left(K_\alpha, f_\alpha(\langle \beta, \gamma \rangle)\right)\right|$$

$$\underset{f_\alpha \text{ isom}}{=} \left|\operatorname{pred}\left(\aleph_\alpha \times \aleph_\alpha, \langle \beta, \gamma \rangle, <^*_\alpha\right)\right|$$

$$\leq \left|\max(\beta,\gamma)+1 \times \max(\beta,\gamma)+1\right|$$

$$\underset{\text{obvious bij}}{=} \Big|\, |\max(\beta,\gamma)+1| \times |\max(\beta,\gamma)+1| \,\Big|$$

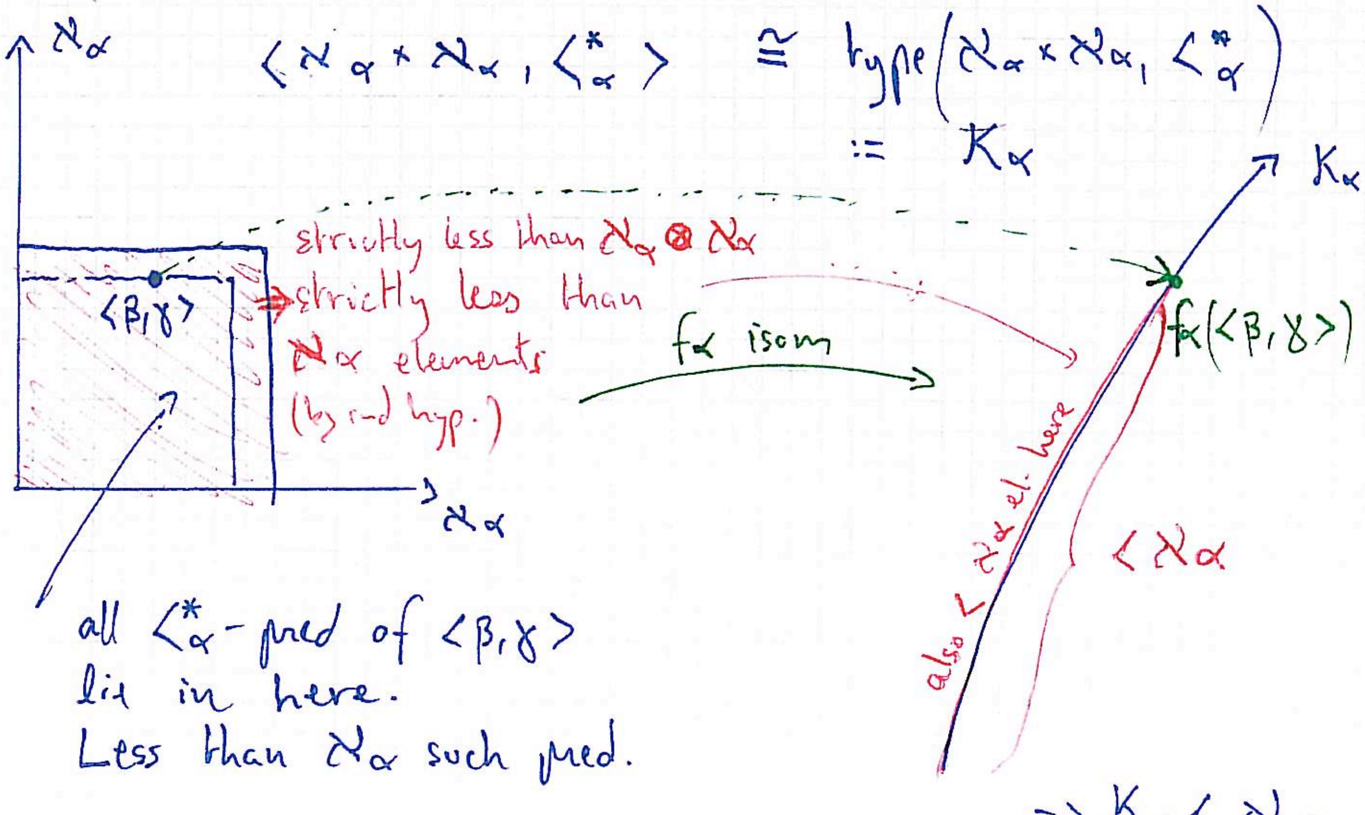
$$\underset{\text{def}}{=} \underbrace{|\max(\beta,\gamma)+1|}_{<\aleph_\alpha \text{ since } \beta,\gamma \in \aleph_\alpha} \otimes \underbrace{|\max(\beta,\gamma)+1|}_{\text{idem}}$$

$$\underset{\text{ind. hyp.}}{=} |\max(\beta,\gamma)+1| < \aleph_\alpha$$

$$\Longrightarrow K_\alpha \leq \aleph_\alpha \quad \blacksquare$$

K_α is an ordinal whose every initial segments have length $< \aleph_\alpha$, thus $K_\alpha \leq \aleph_\alpha$

$$\langle \aleph_\alpha \times \aleph_\alpha, <^*_\alpha \rangle \cong \operatorname{type}(\aleph_\alpha \times \aleph_\alpha, <^*_\alpha) := K_\alpha$$



$\Longrightarrow K_\alpha \leq \aleph_\alpha$

Corollary 10: let κ, λ infinite cardinals

1) $\kappa \oplus \lambda = \max(\kappa, \lambda)$

2) $\kappa \otimes \lambda = \max(\kappa, \lambda)$

facultative →3) $|\kappa^{<\omega}| = \kappa$

obvious inj i.e. $A \preceq B$ so $|A| \leq |B|$ by prop 2.

$\kappa \oplus \lambda = |\kappa \times \{0\} \cup \lambda \times \{1\}| \leq |\lambda \times \{0\} \cup \lambda \times \{1\}|$
$= \lambda \oplus \lambda$

same reasoning for other points

Proof: suppose $\max(\kappa, \lambda) = \lambda$

1) $\kappa \oplus \lambda \leq \lambda \oplus \lambda = \lambda \otimes 2 \leq \lambda \otimes \lambda \overset{\text{thm } 9}{=} \lambda \overset{\text{obvious}}{\leq} \kappa \oplus \lambda$

Thus $\kappa \oplus \lambda = \lambda$

2) $\kappa \otimes \lambda \leq \lambda \otimes \lambda \overset{\text{thm } 9}{=} \lambda \overset{\text{trivial}}{\leq} \kappa \otimes \lambda$

3) Define by induction on n a 1-1 function $f_n : \kappa^n \longrightarrow \kappa$ (c.f. proof thm 9)

Deduce a 1-1 function $f : \bigcup_{n \geq 0} \kappa^n \longrightarrow \omega \times \kappa$

$\Rightarrow |\bigcup_{n \geq 0} \kappa^n| \underset{def}{=} |\kappa^{<\omega}| \leq |\omega \times \kappa| = \omega \otimes \kappa \underset{(2)}{=} \kappa$ ■

- $f_0 = \emptyset$
- for f_{n+1}: $\kappa^{n+1} \approx \kappa \times \kappa^n \approx \kappa \times \kappa \underset{\text{thm } 9}{\approx} \kappa$ i.h.

so take f_n as the composition of these bijections.

Important result... Shows that cardinal arithmetic is very different from ordinal arithmetic.

We also have: $\kappa \otimes (\lambda \oplus \mu) = (\kappa \otimes \lambda) \oplus (\kappa \otimes \mu)$

$\kappa^{\lambda \oplus \mu} = \kappa^\lambda \otimes \kappa^\mu$

$(\kappa^\lambda)^\mu = \kappa^{\lambda \otimes \mu}$

$(\kappa \otimes \lambda)^\mu = \kappa^\mu \otimes \lambda^\mu$

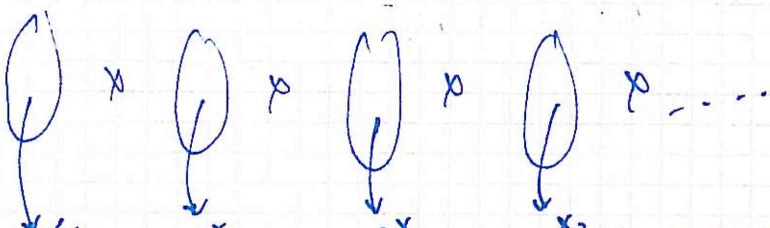
skip

exercise

Lemma 11 (AC): let $\kappa \geq \omega$ cardinal.

If $|X_\alpha| \leq \kappa$ for all $\alpha < \kappa \Rightarrow \left|\bigcup_{\alpha<\kappa} X_\alpha\right| \leq \kappa$

Proof: Equivalent form of (AC):
"every product of non-empty sets is non-empty".
i.e. one can select simultaneously an infinite sequence of elements



$\langle x_0, x_1, x_2, x_3, \ldots \rangle$

$$X_i \neq 0 \ \forall i < \theta \Rightarrow \exists \langle x_i : i < \theta \rangle \in \prod_{i<\theta} X_i$$

Here, let $inj(X_\alpha, \kappa)$ be the set of injective functions from X_α into κ, $\forall \alpha < \kappa$.

$|X_\alpha| \leq \kappa \Rightarrow inj(X_\alpha, \kappa) \neq 0, \forall \alpha < \kappa$.

$\underset{(AC)}{\Longrightarrow} \prod_{\alpha<\kappa} inj(X_\alpha, \kappa) \neq 0$.

Hence let $\langle f_\alpha : \alpha < \kappa \rangle \in \prod_{\alpha<\kappa} inj(X_\alpha, \kappa)$

Let $f : \bigcup_{\alpha<\kappa} X_\alpha \longrightarrow \kappa \times \kappa$ defined by

$$f(x) = \langle \text{least } \alpha \text{ s.t. } x \in X_\alpha,\ f_\alpha(x) \rangle$$

Then f injective, thus $\left|\bigcup_{\alpha<\kappa} X_\alpha\right| \leq |\kappa \times \kappa|$

$$= \kappa \otimes \kappa = \kappa \quad \boxtimes$$

In particular, every countable union of countable sets is countable!

We clearly have $\mathcal{P}(\aleph_\alpha) \approx 2^{\aleph_\alpha}$ by the bij def by $A \longmapsto \chi_A : A \to \{0,1\}$

oral (take as bijection the function which maps every $A \subseteq \aleph_\alpha$ to its characteristic function $\chi_A : \aleph_\alpha \to \{0,1\}$)

Moreover, by Thm 3 (Cantor), we have

$$\aleph_\alpha < \mathcal{P}(\aleph_\alpha) \approx 2^{\aleph_\alpha}, \quad \forall \alpha \in \mathrm{Ord}$$

Thm 3 — obvious bij

écrire direct ceci → Therefore $\aleph_\alpha =_{def} |\aleph_\alpha| < |\mathcal{P}(\aleph_\alpha)| = |2^{\aleph_\alpha}| =_{def} 2^{\aleph_\alpha}$

But $\aleph_\alpha < 2^{\aleph_\alpha}$ implies $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$, $\forall \alpha \in \mathrm{Ord}$

However, do we have $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, $\forall \alpha \in \mathrm{ORD}$?

This is one of the fundamental question of set theory ...

<u>Def:</u> <u>Continuum Hypothesis</u> (CH) is the statement

$$2^{\aleph_0} = \aleph_1$$

<u>The Generalized Continuum Hypothesis</u> (GCH) is the statement:

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}, \quad \forall \alpha \in \mathrm{Ord}$$

These statements are independent from ZF and ZFC (Gödel 1938) and (Cohen 1963), i.e.

ZFC $\nvdash \neg$ HGC (Gödel) <u>and</u> ZFC $\nvdash$ HGC (Cohen)