2.5 Transfinite induction

oral — The class of ordinals: Some collections of sets are "too big" to form sets, e.g. the collection of all sets or all ordinals.
But we would like to talk about these collections, we introduce the concept of "class".

Def (informal): For any formula $\varphi(x)$ with x free variable, and possibly other free variables called parameters, the (informal) collection $\{x : \varphi(x)\}$ is called a class.
It is a proper class if it is not a set.

oral — Remark: By comprehension, any subclass of a set is a set

Examples: $V = \{x : \underbrace{x = x}_{\text{true}}\}$ class of all sets

$ON = \{x : x \text{ ordinal}\}$ class of all ordinals

We write "$\alpha \in ON$" or "$ON(\alpha)$" for an abbreviation of the formula "α is an ordinal".

Formally, proper classes do not exist.
(Remember the point of view where everything is syntax) an expression involving a class is just an abbreviation of an expression not involving it! ex: $\alpha \in ON$ abbreviates $\varphi_{ON}(\alpha)$

ex: "$\alpha \in ON \Rightarrow \alpha$ transitive" abbreviates
$$\forall \alpha \left(\underbrace{\text{"}\alpha \text{ ordinal"}}_{\varphi(\alpha)} \to \forall \beta, \gamma \, (\beta \in \gamma \in \alpha \to \beta \in \alpha) \right)$$

→

When we do proofs by induction on the integers, we prove

$$P(0) \wedge \forall n \in \mathbb{N}\left(P(n) \to P(n+1)\right) \quad (*)$$

and from this we conclude

$$\forall n \in \mathbb{N}\left(P(n)\right). \quad (**)$$

Why can we conlude this? Precisely because the usual ordering "$<$" well-orders $\mathbb{N}$.

Indeed: by contradiction, suppose $(*)$ holds and $(**)$ doesn't hold, i.e. $\exists n \in \mathbb{N}(\neg P(n))$

$\Rightarrow X = \{n \in \mathbb{N} : \neg P(n)\} \neq 0$

$\Rightarrow$ (since $\langle \mathbb{N}, < \rangle$ is a w-o) X has a $<$-least el. m.

Then $m \in X$ means $\underline{\neg P(m) \text{ holds}}$.

$(*) \Rightarrow m \neq 0$, thus $m = m'+1$, for some $m' \in \mathbb{N}$.

By $<$-min of m, $P(m')$ holds

$\Rightarrow$ by $(*)$ $P(m'+1) = \underline{P(m) \text{ also holds}}$, contrad.

Hence the possibility to perform proofs by induction comes from the property that "every non-empty set has a $<$-least el".

Principle of induction on ON:

Theorem 13: If $C \subseteq ON$ and $C \neq \emptyset$, then C has a $\in$-least el. (classes) (schema)

Proof: Identical to thm 6 (4).
(Take $\alpha \in C$. If α not minimal, then let β min in $\alpha \cap C$. Then β is $\in$-min in C.)

⚠ Thm 13 is a thm schema (i.e. an infinite family of thms): For any formula φ_C defining the class C, the thm

$$\forall x\left[(\varphi_C(x) \to x \text{ ordinal}) \wedge \exists x\, \varphi_C(x)\right]$$
$$\to \exists m\left[\varphi_C(m) \wedge (\forall y\ \varphi_C(y) \to y \geq m)\right]$$

is provable in ZF.

It provides a thm for each formula $\varphi(x)$.

Proofs by induction:

Usual proofs by induction on the integers

$$\left.\begin{array}{l} P(0) \text{ is true} \\ \forall n \; P(n) \text{ true} \to P(n+1) \text{ true} \end{array}\right\} \Rightarrow \forall n \, P(n)$$

i.e. $\big(P(0) \wedge \forall n \in \omega (P(n) \to P(n+1))\big) \to \forall n \in \omega \, P(n)$.

There is a general formulation of induction:

$$[\forall n \in \omega \, (\forall k < n \, P(k) \to P(n))] \to \forall n \in \omega \, P(n)$$

We can easily prove that this formula is true (in Peano arithmetic, ZF, etc.).

This fact can be generalized to all ordinals $\Rightarrow$ we can do proofs by transfinite induction on ordinals

skip

Thm 14 (proofs by transfinite induction) (thm schema)

$$[\forall \alpha \in ON \, (\forall \beta < \alpha \, P(\beta) \to P(\alpha))] \to \forall \alpha \in ON \, P(\alpha)$$

if, for all α, $P(\alpha)$ is true as soon as $P(\beta)$ is true for all $\beta < \alpha$, then $P(\alpha)$ is true for all α.

Proof: By contradiction: suppose the premise true and that $\exists \alpha \in ON \, (\neg P(\alpha))$.
Then the class $C = \{\beta \in ON : \neg P(\beta)\} \neq 0$.
By thm 13, C has a $\in$-least el. m

m min in C $\Rightarrow \underline{\neg P(m)} \wedge \forall \beta < m \, P(\beta)$

$\Rightarrow$ contradiction

$$\forall \beta < m \, P(\beta) \overset{\text{premise}}{\Downarrow} \underline{P(m)}$$

$\boxtimes$

$\longrightarrow$

Corollary 15: (proofs by induction, usual way) schema

$$\Big[P(0) \wedge \forall \alpha \in ON \big(P(\alpha) \to P(S(\alpha))\big) \leftarrow ① $$
$$\wedge\ \forall \lambda \in \text{"ON limit"} \big(\forall \beta < \lambda\ P(\beta) \to P(\lambda)\big)\Big] \leftarrow ②$$
$$\longrightarrow \forall \alpha \in ON\ P(\alpha)$$

Proof: like thm 14. By contradiction, if not true, then $C = \{\beta \in ON : \neg P(\beta)\} \neq 0$

let m ordinal $\in$-min in C (by thm 13)
m is not 0 by ①
m cannot be successor by ①
m cannot be limit by ②
$\Rightarrow$ m is not an ordinal, contradiction.

$m \in C \Rightarrow \neg P(m)$
$m = 0$ contrad. with ①
$m = S(m') \Rightarrow P(m') \overset{①}{\Rightarrow} P(S(m'))$ i.e. $P(m)$, contrad
m limit $\Rightarrow \forall \zeta < m\ P(\zeta) \underset{①}{\Rightarrow} P(m)$, contrad.

∎

This is exactly how we use proofs by transfinite induction in practice:
- We prove that the property holds for 0.
- We prove that the property is stable by the successor and limit operations

$\Rightarrow$ We deduce that the property holds for all ordinals.

definitions by induction:

example of a def by induction on integers:

fact(0) = 1

fact(n+1) = (n+1) · fact(n).

i.e. fact (n+1) = $\underbrace{\text{function}}_{=F}$ (n+1, previously defined values)

We can prove that the function "fact" is well-defined, i.e. that given the function $F : \mathbb{N} \times \mathbb{N}^{<\omega} \to \mathbb{N}$, the function "fact" whose definition is based on F really exists. The def of "fact" is legitimate.

We extend this result for all ordinals.

i.e. $\forall x \exists! y\, \varphi_F(x,y)$ This unique y is denoted F(x).

Thm 16 (defs by transfinite induction) (Thm schema)

Let F be a functional class : $V \to V$, then there is a unique funct class G : ON → V s.t.

$$\forall \alpha \in ON \left[G(\alpha) = F(\langle \alpha, \underbrace{G\restriction_\alpha}_{\text{set}} \rangle) \right] \quad (*)$$

This ordered pair is a set so no need to write $F: V^2 \to V$

$\alpha = \text{pred}(ON, \alpha) = [0, \alpha[$ intuitively

(more formal statement in verso →)

Proof: (sketch) c.f. Dehornoy chap III, prop 3.11 (p.94) (and prop. 3.2 and 3.4 also). (dire à la fin)

unicity: let G_1 and G_2 satisfying (*).

We prove $\forall \alpha (G_1(\alpha) = G_2(\alpha))$ by induction on α

By contrad, suppose $G_1 \neq G_2$ and let α be the min ord s.t. $G_1(\alpha) \neq G_2(\alpha)$. (exists by thm 13)

Then $G_1(\beta) = G_2(\beta)\ \forall \beta < \alpha$, i.e. $G_1\restriction\alpha = G_2\restriction\alpha$

since F functional $\Rightarrow$ $F(\langle \alpha, G_1\restriction\alpha \rangle) = F(\langle \alpha, G_2\restriction\alpha \rangle)$

i.e. by (*) $G_1(\alpha) = G_2(\alpha)$, contradiction.

→

$$\underbrace{\forall x\, \exists! y\; \varphi_F(x,y)}_{\text{i.e. } F \text{ functional}} \to \Big[\underbrace{\forall \alpha\, \exists! y\; \varphi_G(\alpha,y)}_{G \text{ functional}} \wedge$$

$$\forall \alpha\, \exists x\, \exists y \Big(\underbrace{\varphi_G(\alpha,y) = \varphi_F(x,y)}_{\text{i.e. } G(\alpha) = F(x) \text{ where } x = \langle \alpha,\, G\restriction\alpha \rangle} \wedge\; x = \langle \alpha,\, G\restriction_{\alpha} \rangle \Big) \Big]$$

note that this is a set by replacement

existence: call g a δ-approximation iff g is a function of domain δ s.t.

$$\forall \alpha < \delta \quad g(\alpha) = F(\langle \alpha, g\restriction\alpha \rangle)$$ i.e. g satisfies ⊛ before δ

- (like the unicity proof) we have that if g, g' are δ and δ' approx. resp., then

$$g\restriction(\delta \cap \delta') = g'\restriction(\delta\cap\delta')$$

- By induction, (cor 15) we prove that for all δ, there exists a δ-approximation g_δ (which will be unique by above arg.)

If $\delta = 0$, take $g_\delta = \emptyset$

If $\delta = \beta+1$, take $g_\delta = g_\beta \cup \{\langle \beta, F(\langle \beta, g_\beta\rangle)\rangle\}$ exists by ind. hyp. ($g_\beta\restriction\beta$ ‖ remark)

If δ limit, take $g_\delta = \bigcup \{g_\beta : \beta < \delta\}$ ← exist by replacement; exists by union; exist by ind hyp

- For each α, define $G(\alpha) = g(\alpha)$ where g is the δ-approx. for some $\delta > \alpha$: $G(\alpha)$ will satisfy ⊛

◻

Corollary 17 (def by transfinite induction, usual ~~way~~ scheme)

Let F, F' be two functional classes: $V \to V$. Then there exists a unique functional class G: $ON \to V$ s.t.

$$G(0) = a, \quad G(\alpha+1) = F(\langle \alpha, G(\alpha)\rangle),$$

last value of G only needed

$$G(\lambda) = F'(\langle \lambda, G\restriction\lambda\rangle) \quad \text{for all } \lambda \text{ limit}$$

Proof: Apply thm 16 to functional $\tilde{F}$ defined by

then $G(0) = $ (thm 16) $\tilde{F}(\langle 0, G\restriction 0\rangle) = \tilde{F}(\langle 0, \emptyset\rangle) = a$ which is what we want

$$\tilde{F}(\langle 0, d\rangle) = a, \quad \tilde{F}(\langle \alpha+1, g\rangle) = F(\langle \alpha, g(\alpha)\rangle) \text{ etc.}$$

for $\alpha \geq 0$ and g of domain α, and

$$\tilde{F}(\langle \lambda, g\rangle) = F'(\langle \lambda, g\rangle),$$ for all λ limit and g with domain λ

◻

<u>Definitions</u> (by induction on β): let α, β ordinals

$$\begin{cases} \alpha + 0 = \alpha \\ \alpha + S(\beta) = S(\alpha + \beta) \\ \alpha + \beta = \sup\{\alpha + \zeta : \zeta < \beta\}, \text{ if } \beta \text{ limit} \end{cases}$$

$$\begin{cases} \alpha \cdot 0 = 0 \\ \alpha \cdot S(\beta) = \alpha \cdot \beta + \beta \\ \alpha \cdot \beta = \sup\{\alpha \cdot \zeta : \zeta < \beta\}, \text{ if } \beta \text{ limit} \end{cases}$$

$$\begin{cases} \alpha^{0} = 1 \\ \alpha^{\beta+1} = \alpha^{\beta} \cdot \alpha \\ \alpha^{\beta} = \sup\{\alpha^{\zeta} : \zeta < \beta\}, \text{ if } \beta \text{ limit} \end{cases}$$

So now we can complete the figure p. 15.
We can reach new ordinals with the exponentiation.