# Ordinals are good representatives of well-orderings

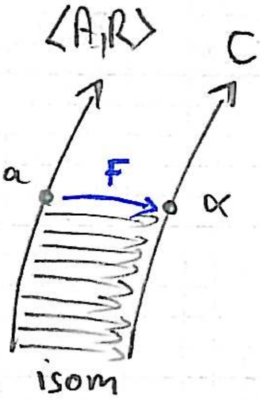
**Thm 7:** If $\langle A, R\rangle$ is a w-o, then there is a unique ordinal $C$ s.t. $\langle A, R\rangle \cong C$
(i.e. $\langle A, R\rangle \cong \langle C, \in_C\rangle$ )

**Proof:** Uniqueness: Suppose $\langle A, R\rangle \cong C$ and $C'$.
Then $C \cong C' \Rightarrow$ (L. 5(ii)) $C = C'$

Existence: Consider the set

$$B = \{ a \in A : \exists \text{ ordinal } \alpha \text{ s.t. } \underbrace{\langle \mathrm{pred}(A,a,R), R\rangle \cong \alpha}_{\phi(a,\alpha)} \}$$

By uniqueness (above),




faire le dessin

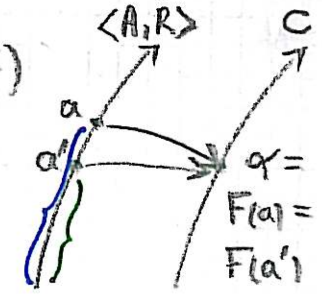
$\forall a \in B\ \exists!\, \alpha\ \phi(a, \alpha)$ i.e. $\phi$ is functional

$\Rightarrow$ By replacement (and comprehension)

we can form $C = \{\alpha : \exists a \in B\ \phi(a,\alpha)\}$

and also $\underset{\subseteq B\times C}{F} : B \longrightarrow C$ def by $F(a) = \alpha$ iff $\phi(a, \alpha)$

- By def, $F$ is surjective, $C = \mathrm{ran}(F)$
  By lemma 1, $F$ is injective
  (otherwise $\{$ isom to $\{$, impossible)




$\Rightarrow$ F bijective

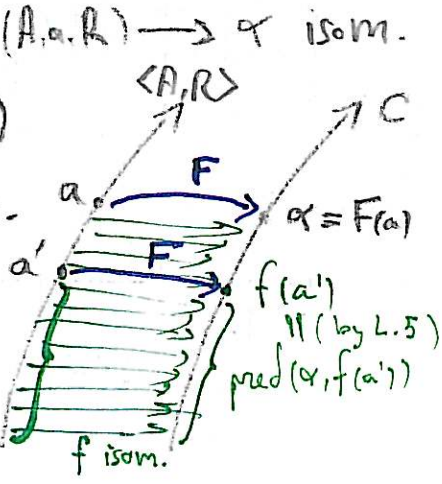
- $B$ is closed by $R$-predecessors, i.e. $a \in B$, $a'Ra \Rightarrow a' \in B$

faire le dessin

Suppose $\overset{a \in B \text{ i.e. } \exists \alpha \in C \text{ s.t.}}{F(a) = \alpha}$. Then let $f : \mathrm{pred}(A,a,R) \longrightarrow \alpha$ isom.

$\Rightarrow f \restriction \mathrm{pred}(A, a', R)$ (restriction of $f$)

$: \mathrm{pred}(A, a', R) \longrightarrow \underbrace{\mathrm{pred}(\alpha, f(a'))}_{= f(a') \text{ (by L.5(i))}}$ isom.

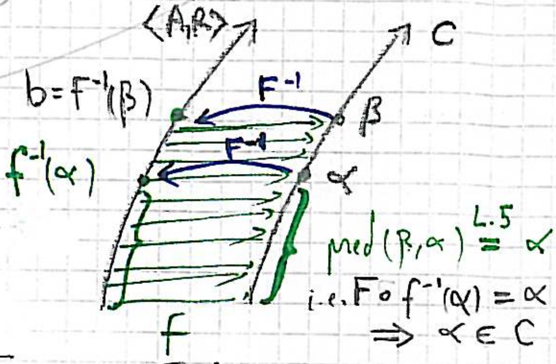



$\Rightarrow F(a') = f(a') \Rightarrow a' \in B$
$\qquad \| \qquad \| \text{ (by L.5)}$
$\quad \mathrm{pred}(\alpha, f(a'))$

- $B$ closed by pred $\Rightarrow$ $\underline{B = A \text{ or some initial segment of } A}$ g'

  Dekerving lemma 1.16 chap 2.

- Similar reasoning $C$ is closed by pred i.e. $C$ transitive set
  $(\alpha \in \beta \in C \Rightarrow \alpha \in C)$
  Moreover $C$ set of ordinals $\Rightarrow$ $C$ w-o by $\in$ (Thm 6)
  $\Rightarrow$ $\underline{C \text{ ordinal}}$




We prove that

- $\underline{F : B \to C \text{ isom}}$ i.e. $aRb \Rightarrow F(a) \in F(b)$

Indeed let $\alpha = F(a)$ and $\beta = F(b)$. Suppose $a, b \in B$ s.t. $aRb$ and
i.e. $\langle \text{pred}(A,a,R), R\rangle \underset{\text{isom } f}{\cong} \alpha$ and $\langle \text{pred}(A,b,R), R\rangle \underset{\text{isom } g}{\cong} \beta$

If $\alpha = \beta$, $\langle \text{pred}(A,a,R),R\rangle \cong \langle \text{pred}(A,b,R),R\rangle$, contrad with L.1.

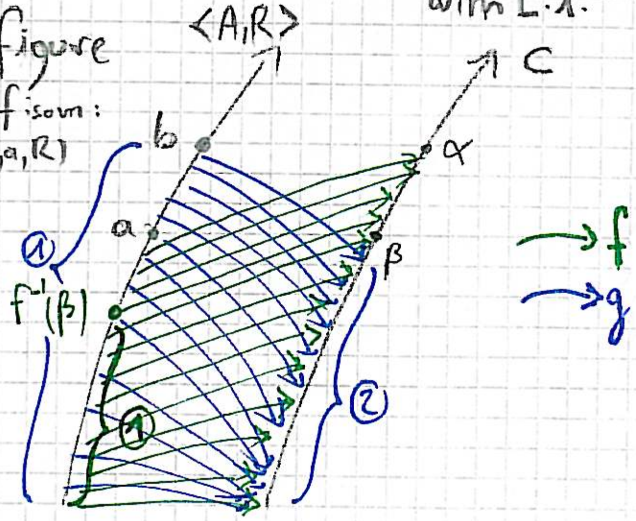
If $\beta \in \alpha$, $\exists f, g$ isom like in figure

$\beta \in \alpha \Rightarrow f^{-1}(\beta) \in \text{pred}(A,a,R)$ since $f$ isom: $\text{pred}(A,a,R) \to \alpha$
i.e. $f^{-1}(\beta)\, R\, a$

$\}① \cong \}② \cong \}①$

(But $f^{-1}(\beta)\, R\, b$), contrad with L.1.

Thus $\alpha \in \beta$ i.e. $F(a) \in F(b)$




So we have $F : \langle B, R\rangle \longrightarrow C$ isom
ordinal

- Either $B = A$ (done!) ✓
- or $B = \text{pred}(A,b,R)$ for some $b \in A$
  $\Rightarrow$ $\text{pred}(A,b,R) \cong C$ ordinal i.e. $b \in B$ by def
  $\Rightarrow$ $bRb$, contradiction ✓ $\text{pred}(A,b,R)$

**Def:** If $\langle A,R\rangle$ w-o, we note $\underline{\text{type}(A,R)}$ the unique $C$ s.t. $\langle A,R\rangle \cong C$.

The class of all ordinals is not a set.

**Thm 8:** $\neg \exists x \, \forall \alpha \, (\alpha \text{ ordinal} \rightarrow \alpha \in x)$

Proof: By contradiction, suppose $x$ exists, then we could form

$$ON = \{\alpha \in x : \alpha \text{ ordinal}\}$$

which is the set of all ordinals.

- By lemma 5 (ii), $\alpha \in \beta \in ON \Rightarrow \beta \in ON$
  i.e. $ON$ transitive
- By Thm 6, $ON$ would be w-o by $\in$

$\Rightarrow$ $ON$ is an ordinal
i.e. $ON \in ON$, contradiction with the Remark (p. 5). ∎

Successor and limit ordinals, sup and min of a set of ordinals

<u>Def</u>: - let $\alpha$ ordinal, then $S(\alpha) := \alpha \cup \{\alpha\}$ is called the successor of $\alpha$

- let $X$ be a set of ordinals.
  $\sup(X) := \bigcup X$, and if $X \neq \emptyset$
  $\min(X) := \bigcap X$

These def are justified by the following lemma

<u>lemma 9</u>: i) $\forall \alpha, \beta$ ord $(\alpha \leq \beta \rightarrow \alpha \subseteq \beta)$

$S(\alpha)$ is the "immediate successor" of $\alpha$ (nothing in between) $\longrightarrow$ ii) If $\alpha$ ord, then $S(\alpha)$ <u>ord</u> s.t. $S(\alpha) > \alpha$ and $\forall \beta \, (\beta < S(\alpha) \leftrightarrow \beta \leq \alpha)$

iii) $\sup(X)$ is the least <u>ord</u> $\geqslant$ all el. of $X$ (<u>not necessarily in $X$</u>!)

iv) If $X \neq \emptyset$, $\min(X)$ is the least <u>ord</u> <u>in $X$</u>

<u>Proof</u>: i) trivial
ii), iii), iv) exercises.
Dehornoy prop 2.12 (iii), 2.13, 2.16 chap 2.

<u>Def</u>: · $\alpha$ is a <u>successor ordinal</u> iff $\exists \beta \, (\alpha = S(\beta))$
· $\alpha$ is a <u>limit ordinal</u> iff $\alpha \neq 0$ and $\alpha$ not successor

## 2.3. Integers (and above)

we define the set-theoretic representatives of natural numbers.

Def: we let $1 := S(0)$, $2 := S(1)$, $3 := S(2)$, etc.

$\alpha$ is a natural number iff $\forall \beta \leq \alpha$ ($\beta = 0$ $\vee$ $\beta$ is a successor ordinal) all its el and itself are successors.

So we have $0 = \emptyset$, $1 = S(0) = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\}$

$2 = S(1) = 1 \cup \{1\} = \{0\} \cup \{\{0\}\} = \{0, \{0\}\} = \{0, 1\}$.

$3 = S(2) = 2 \cup \{2\} = \{0,1\} \cup \{\{0,1\}\} = \dots = \{0,1,2\}$

$4 = S(3) = \{0,1,2,3\}$ etc...

The natural numbers form an initial segment of the ordinals

We would like to climb above in the transfinite

$\Rightarrow$ we introduce Axiom of Infinity

Axiom 7: Infinity

$$\exists x \left(0 \in x \wedge \forall y \in x \, (S(y) \in x)\right)$$

Remark: This set $x$ contains all natural numbers

proof: By contradiction, let $n$ natural $\notin x$.

Then $n \neq 0$ (since $0 \in x$) and $n = S(m)$

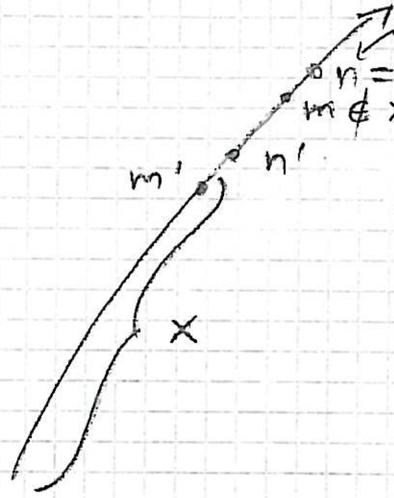
$\Rightarrow$ $m < n$ and $m$ natural (since all its predecessors are also predecessors of $n$, hence successors or $0$ by def), and $m \notin x$ (otherwise $S(m) = n \in x$).

exercise

Hence $n \setminus x \neq 0$. So let $n'$ min in $n \setminus x$

Then (same argument) $n' = S(m')$ with $m' < n' < n$ and $m' \notin x$, contradiction with min of $n'$

illustration:




$\left.\begin{array}{l} n = S(m) \notin X \\ m \notin X \text{ and } m \in n \end{array}\right\} \Rightarrow n \setminus X \neq 0$

Let $n'$ min in $n \setminus X$ ie. $\underline{n' \notin X}$

Then $n' = S(m') \Rightarrow m' < n'$

$\Rightarrow m' \in X \Rightarrow S(m') = \underline{n' \in X}$

contradiction

So by comprehension, we can form the set of all natural numbers

<u>Def</u>: $\omega = \{ n \in \underline{\underline{X}} : n \text{ is natural number} \}$

given by axiom 7

intercaller p.13 ici ⚠

These set-theoretic integers behave like the usual ones

<u>Thm 10</u>: Peano Postulates

(1) $0 \in \omega$
(2) $\forall n \in \omega \, (S(n) \in \omega)$
(3) $\forall n, m \in \omega \, (n \neq m \rightarrow S(n) \neq S(m))$
(4) (Induction scheme)
$\forall X \subseteq \omega \left[ (0 \in X \wedge \forall n \in X (S(n) \in X)) \longrightarrow X = \omega \right]$

<u>Proof</u>: (1) trivial (0 is a natural by def, hence $\in \omega$)

(2) $m \leqslant S(n) \Rightarrow m \leqslant n \vee m = S(n)$
$\Rightarrow S(n)$ natural — 0 or successor by def of $n$ nat — successor
$\Rightarrow S(n) \in \omega$

exercice

(3) Dehornoy prop. 2.12 (iii)

(4) By contradiction, suppose $X \neq \omega$ (i.e. $X \subsetneq \omega$) and consider $\gamma$ min in $\omega \setminus X$.
$\gamma \in \omega \Rightarrow \gamma = S(\gamma') \Rightarrow \gamma' < \gamma \Rightarrow \gamma' \in X$
$\Rightarrow$ (def of $X$) $S(\gamma') = \gamma \in X$, contradiction ▨

**Remark**: i) $\omega$ is an ordinal !

exercise

proof: - $\omega$ is a set of ordinals, hence it is w-o by "$\in$" (Thm. 6)

- let $m \in n \in \omega$.
This means $m < n \in \omega \Rightarrow m \in \omega$
Hence $\omega$ transitive ■

by def of n natural, m is 0 or succ and all el of m also $\Rightarrow$ m natural

see precisely

ii) $\omega = \sup(\omega)$ ($\omega$ is the sup of all nat numbers)

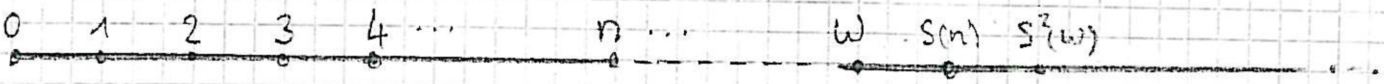
proof (informal): $\sup(\omega) = \bigcup \omega$

$= \bigcup \{0, 1, 2, 3, 4, \dots\}$

$= \bigcup \{0, \{0\}, \{0,1\}, \{0,1,2\}, \dots\}$

$= \{0, 1, 2, 3, 4, \dots\} = \omega$

Up to now, we have ...




iii) $\omega$ is limit

Otherwise, $\omega = S(n) \Rightarrow n < \omega$ (since $\omega = S(n)$)
i.e $n \in \omega \Rightarrow$ (def) $\underbrace{S(n)}_{\omega} \in \omega$, contrad.

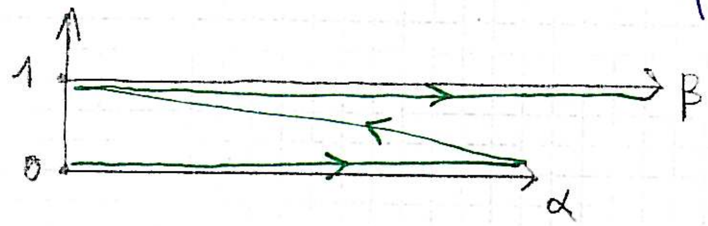
## 2.4. Ordinal arithmetic

We define arithmetical operations on ordinals

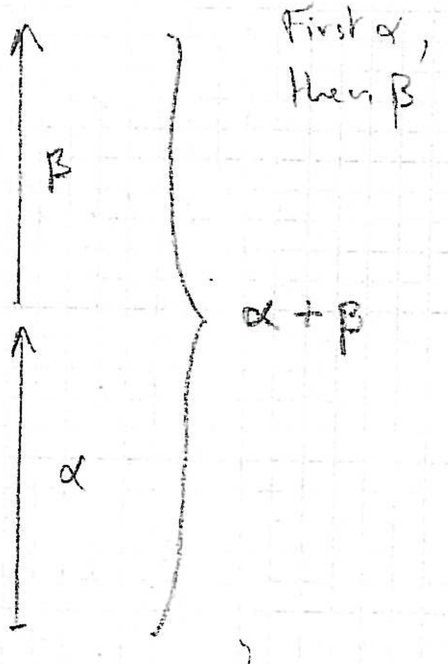
to render the union disjoint.

**Def**: $\alpha + \beta = \text{type}\left(\alpha \times \{0\} \cup \beta \times \{1\}, R\right)$,
where $R$ defined by

$$\langle \xi, i \rangle \, R \, \langle \eta, j \rangle \text{ iff } \begin{cases} i < j \text{ or} \\ i = j \text{ and } \xi < \eta \end{cases}$$




but think of it like that




**lemma 11**: for any $\alpha, \beta, \gamma$:

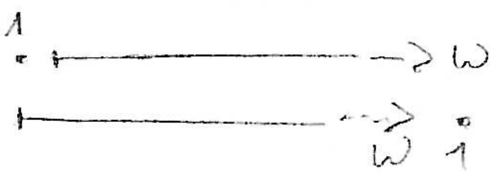
1) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
2) $\alpha + 0 = \alpha$
3) $\alpha + 1 = S(\alpha)$
4) $\alpha + S(\beta) = S(\alpha + \beta)$
5) If $\beta$ limit, $\alpha + \beta = \sup\{\alpha + \xi : \xi < \beta\}$

important!

**Proof**: exercice (construct the isomorphisms)

**Remark**: the sum is not commutative:

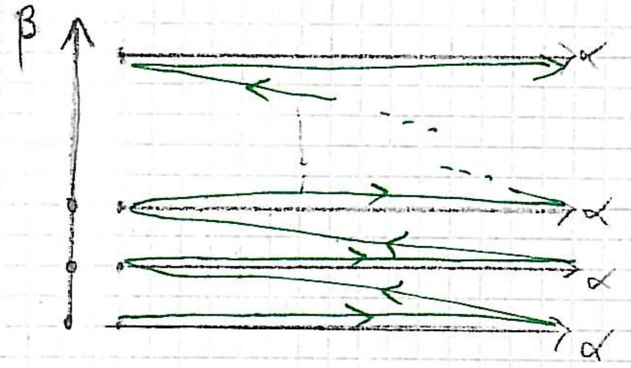
$$1 + \omega = \sup\{1 + n : n < \omega\} = \omega$$
$$\neq \omega + 1 = S(\omega) > \omega.$$




$1 + \omega = \omega$
$\omega + 1 = S(\omega)$

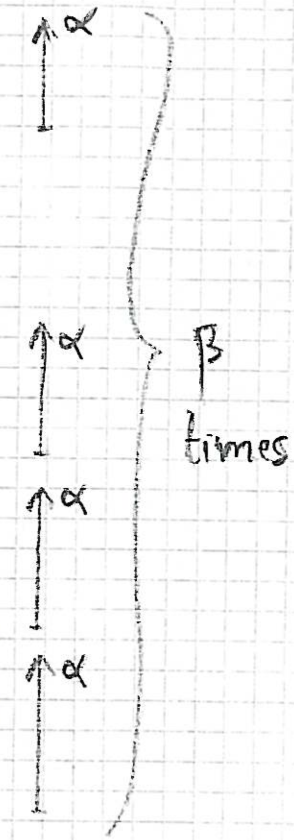
**Def:** $\alpha \cdot \beta = \text{type}(\alpha \times \beta, R)$ where $R$ is

$(\eta, \zeta) R (\eta', \zeta')$ iff $\begin{cases} \zeta < \zeta' \text{ or} \\ \zeta = \zeta \text{ and } \eta < \eta' \end{cases}$




align $\alpha$ $\beta$ times

But think of it like that




**Lemma 12:** let $\alpha, \beta, \gamma$ ord:

(1) $\alpha \cdot (\beta \cdot \gamma) = \alpha \cdot (\beta \cdot \gamma)$

(2) $\alpha \cdot 0 = 0$

(3) $\alpha \cdot 1 = \alpha$

(4) $\alpha \cdot S(\beta) = \alpha \cdot \beta + \alpha$

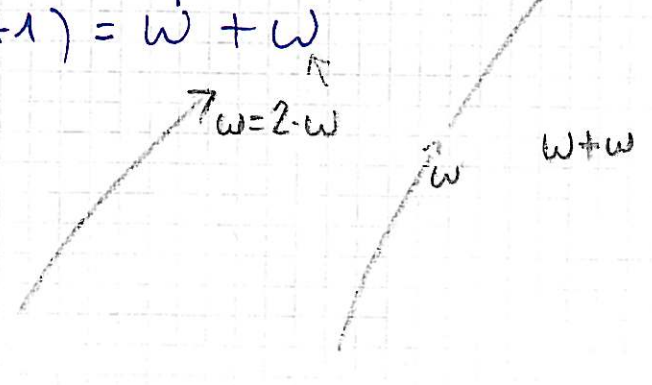
(5) $\beta$ limit $\Rightarrow$ $\alpha \cdot \beta = \sup\{\alpha \cdot \zeta : \zeta < \beta\}$

(6) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$

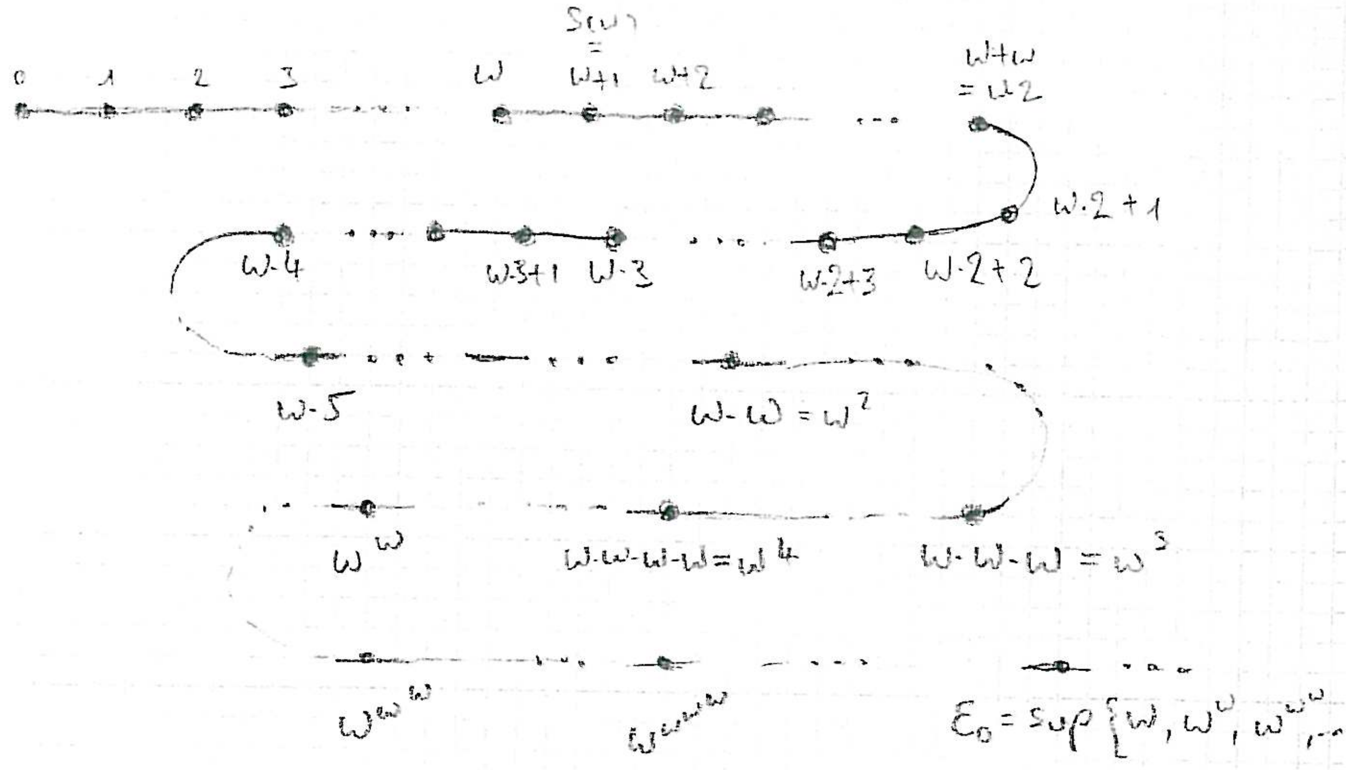
**Remark:** multiplication not commutative:

$$2 \cdot \omega = \sup\{2 \cdot n : n < \omega\} = \omega$$

$$\neq \omega \cdot 2 = \omega \cdot (1+1) = \omega + \omega$$



With addition and multiplication, we can reach the following ordinals:

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