5.3. Reflection Principle

- In Prop 9 iii) and iv), we showed that

$$ZF^- \vdash \text{“} WF \models ZF \text{”} \quad \text{and} \quad ZFC^- \vdash \text{“} WF \models ZFC \text{”},$$

and thus obviously

$$ZF \vdash \text{“} WF \models ZF \text{”} \quad \text{and} \quad ZFC \vdash \text{“} WF \models ZFC \text{”}.$$

- But WF is a proper class, not a set;
Hence is there a <u>set</u> M which is a model for all ZFC? Can one, arguing just from ZFC, produce a <u>set</u> model of ZFC?
- Obviously not, since if $ZFC \vdash \exists M \text{“} M \models ZFC \text{”}$, it would follow that $ZFC \vdash Cons(ZFC)$, and ZFC would be inconsistent by incompleteness thm.
- But this is almost the case. We will see that for any finite fragment $\varphi_1, \ldots, \varphi_n$ of **ZFC**, one has

$$ZFC \vdash \exists M \text{“} M \models \bigwedge_{i=1}^{n} \varphi_i \text{”} \quad \text{i.e.}$$

$$ZFC \vdash \exists M (\varphi_1^M \wedge \varphi_2^M \wedge \ldots \wedge \varphi_n^M)$$

Thm 11 (Reflection) (meta-result) and <u>schema</u>. (for each $\varphi_1,\dots,\varphi_n$, we have one instance of the thm)

Let $\varphi_1,\dots,\varphi_n$ be any formulas of set theory, then:

(1) $ZF \vdash \forall\alpha\, \exists\beta > \alpha\, (\varphi_1,\dots,\varphi_n \text{ abs. for } V_\beta)$

and more generally ...

(2) $ZF \vdash \forall M_0\, \exists \underset{\text{set}}{M} \supseteq M_0\, (\varphi_1,\dots,\varphi_n \text{ abs. for } M)$

We say that V_β and M reflect $\varphi_1,\dots,\varphi_n$

In particular, if $\varphi_1,\dots,\varphi_n$ are axioms of ZF, one obviously has $ZF \vdash \varphi_1,\dots,\varphi_n$, and thus

(1') $ZF \vdash \forall\alpha\, \exists\beta > \alpha\, (\text{“}V_\beta \models \bigwedge_{i=1}^{n} \varphi_i\text{”})$ and

(2') $ZF \vdash \forall M_0\, \exists M \supseteq M_0\, (\text{“}M \models \bigwedge_{i=1}^{n} \varphi_i\text{”})$.

In other words, any finite fragment of ZF can be satisfied into some "set model" V_β or M.

The proof of thm 11 relies on the following lemma

Lemma 12 (ZF). Let $\varphi_1,\dots,\varphi_n$ be a subformula closed list of formulas and M be a class.

TFAE:

(a) $\varphi_1,\dots,\varphi_n$ are abs. for M

(b) If $\varphi_i(\bar{y})$ is of the form $\exists x\, \varphi_j(x,\bar{y})$, then

$$\forall y_1,\dots,y_n \in M\, \big(\exists x\, \varphi_j(x,\bar{y}) \overset{\text{trivial}}{\longleftrightarrow} \exists x \in M\, \varphi_j(x,\bar{y})\big).$$

In other words, absoluteness of the φ_i's reduces to checking this existential condition (b).

Proof of the lemma:

We can suppose this since the list is supposed to be subformula closed

(a) → (b) : suppose $\varphi_i(\vec{y})$ of the form $\exists x\, \varphi_j(x,\vec{y})$.

Fix $y_1,\dots,y_n \in M$ and assume $\varphi_i(\vec{y}) = \exists x\, \varphi_j(x,\vec{y})$ holds.

By (a), φ_j is abs. for M, thus $\varphi_i^M(\vec{y}) =$ $\exists x \in M\, \varphi_j^M(x,\vec{y})$ holds, and by abs. of φ_j $\exists x \in M\, \varphi_j(x,\vec{y})$.

(b) → (a) : By induction on the length of φ_i.

- If φ_i is atomic, then it is abs for M (since $\varphi_i^M = \varphi_i$)
- If $\varphi_i = \varphi_j \wedge \varphi_{j'}$ or $\varphi_i = \neg \varphi_j$, then it is abs for M by absoluteness of the shorter formulas φ_j and $\varphi_{j'}$.
- If $\varphi_i(\vec{y}) = \exists x\, \varphi_j(x,\vec{y})$: Fix $y_1,\dots,y_n \in M$, then

$$\begin{aligned}
\varphi_i^M(\vec{y}) &\leftrightarrow \exists x \in M\, \varphi_j^M(x,\vec{y}) \\
&\leftrightarrow \exists x \in M\, \varphi_j(x,\vec{y}) && (\varphi_j \text{ abs by ind. hyp.}) \\
\text{trivial} \rightarrow &\Leftrightarrow \exists x\, \varphi_j(x,\vec{y}) && (\text{by b)}) \\
&\leftrightarrow \varphi_i(\vec{y}) && (\text{by def})
\end{aligned}$$

∎

Proof of the theorem. We argue from ZF, i.e. $ZF \vdash \dots$

(1) We may assume w.l.o.g. that the list $\varphi_1, \dots, \varphi_n$ is subformula closed. If not, expand it.

- For each φ_i of the form $\exists x\, \varphi_j(x, \vec{y})$ [" $y_1, \dots, y_n$ "], we let the functional class $G_i : V^{<\omega} \longrightarrow \mathbf{ON}$ be def. by:

$$G_i(z_1, \dots, z_n) = \begin{cases} 0 & , \text{ if } \neg \exists x\, \varphi_j(x, \vec{z}) \\ \text{least } \eta \text{ s.t. } \exists x \in V_\eta\, \varphi_j(x, \vec{z}) & , \text{ if } \exists x\, \varphi_j(x, \vec{z}) \end{cases}$$

least level of appearance of x satisfying $\varphi_j(x, \vec{z})$

We then define the functional class $H_i : \mathbf{ON} \longrightarrow \mathbf{ON}$ by

sup of levels of appearance of the x's satisfying $\varphi_j(x, \vec{z})$ when $\vec{z}$ vary inside V_ζ

$H_i(\zeta) = \sup\{ G_i(z_1, \dots, z_n) : z_1, \dots, z_n \in V_\zeta \}$ and

i.e. level λ s.t. for any $\vec{z} \in V_\lambda$, if $\exists x\, \varphi_j(x, \vec{z})$, then $\exists x \in V_\lambda\, \varphi_j(x, \vec{z})$

$H_i(\zeta) = 0$ if φ_i is not of the form $\exists x\, \varphi_j$

Note that since G_i is functional, the Replacement axiom ensures that $G_i{}'' V_\zeta{}^n = \{ G_i(z_1, \dots, z_n) : z_1, \dots, z_n \in V_\zeta \}$ is indeed a set of ordinals, so that $H_i(\zeta) = \sup\{ G_i(z_1, \dots, z_n) : z_1, \dots, z_n \in V_\zeta \}$ indeed exists.

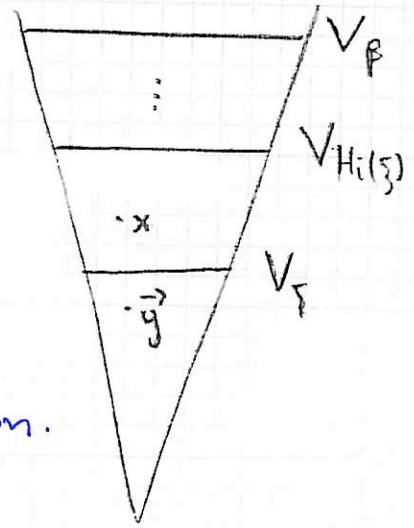
- let $(\beta_p)_{p \in \omega}$ be def by induction by

$\beta_0 = \alpha$ and $\beta_{p+1} = \max\{ \beta_p + 1, H_1(\beta_p), \dots, H_n(\beta_p) \}$

and let $\beta = \sup\{ \beta_p : p \in \omega \}$.

Then $\alpha \leq \beta_0 < \beta_1 < \beta_2 < \dots < \beta$

i.e. $\beta > \alpha$ and β limit, by construction.



- Suppose φ_i of the form $\exists x\, \varphi_j(x, \vec{y})$.

If $\exists x\, \varphi_j(x, \vec{y})$, then $\exists x \in V_{H_i(\zeta)}\, \varphi_j(x, \vec{y})$.

Fix $\vec{y} \in V_\beta$ and suppose $\exists x\, \varphi_j(x, \vec{y})$ holds.

Since β limit, $\exists\, \beta_p < \beta$ s.t. $\vec{y} \in V_{\beta_p}$

By def of H_i, $\exists x \in V_{\underbrace{H_i(\beta_p)}_{\leq \beta_{p+1}}} \varphi_j(x, \vec{y})$ $\quad$ (*) $(\beta > H_i(\beta_p) \Rightarrow V_\beta \supset V_{H_i(\beta_p)})$

Thus(*) $\exists x \in V_\beta\, \varphi_j(x, \vec{y})$. By Lemma 12, all φ_i's are abs. for V_β ∎

(2) Since M_0 is a set, $\exists \alpha$ s.t. $M_0 \subseteq V_\alpha$.
By (1), $\exists V_\beta \supset V_\alpha \supseteq M_0$ s.t. φ_i's are abs for V_β.
Hence $M = V_\beta$ leads to the conclusion ▨

(1') and (2') are direct consequences of (1) and (2).

<u>Remark</u>: From a platonist point of view, we could carry out this argument with all formulas simultaneously and show that $\forall \alpha \exists \beta > \alpha$ $\bigwedge_{i \in \omega} \varphi_i^{V_\beta}$, where the φ_i's list all axioms of ZF or ZFC. According to this point of view, one could argue that ZF and ZFC are consistent. But this argument is not formalizable within ZF or ZFC, since by Gödel's incompleteness theorem, ZF $\nvdash$ cons(ZF) and ZFC $\nvdash$ cons(ZFC) (as long as we suppose ZF and ZFC to be consistent).

Let us argue a bit more precisely about why Reflection thm doesn't hold for infinitely many formulas.
- Firstly, the statement which says:
  "let $\varphi_0, \varphi_1, \varphi_2, \dots$ be infinitely many formulas, then ZF $\vdash \forall \alpha \exists \beta$ "$V_\beta \models \varphi_0, \varphi_1, \varphi_2, \dots$""
  is impossible to formalize in the language of set theory, since the expression "$V_\beta \models \varphi_0, \varphi_1, \varphi_2, \dots$" is an abuse of language when we have infinitely many formulas
- Secondly, the alternative statement of the form:
  "Let $\varphi_0, \varphi_1, \varphi_2, \dots$ be infinitely many formulas, then ZF $\vdash \forall \alpha \exists \beta \forall i$ "$V_\beta \models \varphi_i$""
  is also impossible to formalize since the set-theoretical integers i's involved cannot properly refer to the intuitive integers that are the subscripts of φ_i's. Etc.

Corollary 13: (meta-result) and schema

Let S be any set of axioms extending ZF. If S is consistent, then S is not finitely axiomatizable (i.e. there is no $\varphi_1, \dots, \varphi_n$ axioms of S s.t. $\varphi_1, \dots, \varphi_n$ can prove all axioms of S).

Hence, ZF and ZFC are - if consistent - not finitely axiomatizable.

Proof: Suppose S finitely ax. by $\varphi_1, \dots, \varphi_n$
We argue from S (i.e. $S \vdash \dots$) and produce a contradiction. Thus S is not consistent.
By reflection thm, $\exists \alpha$ "$V_\alpha \models \bigwedge_{i=1}^n \varphi_i$"
Let β be the least such ordinal, i.e. "$V_\beta \models \bigwedge_{i=1}^n \varphi_i$" ①
Since $\varphi_1, \dots, \varphi_n$ finite ax of S, we can deduce from these formulas the same conclusions as from S, hence

$$\bigwedge_{i=1}^n \varphi_i \to \exists \alpha \text{ "} V_\alpha \models \bigwedge_{i=1}^n \varphi_i \text{"}. \quad ②$$

By ① and ②, we have "$V_\beta \models \exists \alpha$ "$V_\alpha \models \bigwedge_{i=1}^n \varphi_i$"" ⊛
i.e. "$V_\beta \models \exists \alpha (\bigwedge \varphi_i)^{V_\alpha}$" hold (from S)
i.e. $\left[\exists \alpha \left(\bigwedge_{i=1}^n \varphi_i\right)^{V_\alpha}\right]^{V_\beta}$ i.e. $\exists \alpha \in V_\beta \bigwedge_{i=1}^n \left[\varphi_i^{V_\alpha}\right]^{V_\beta}$

But since S extends ZF, all basic absoluteness results hold, and in particular, if $\alpha \in V_\beta$, then $\alpha < \beta$, and one can prove (by ind on φ) that $[\varphi^{V_\alpha}]^{V_\beta} = \varphi^{V_\alpha}$.
Thus $\exists \alpha < \beta \left(\bigwedge_{i=1}^n \varphi_i^{V_\alpha}\right)$, contrad with min of β. ∎

⊛ In general, if φ^M and $\varphi \to \psi$ hold, then ψ^M hold (by ind on ψ)

<u>simple proof</u> (via Gödels thms) (informal)

Suppose ZF finitely axiomatizable by $\varphi_1, \dots, \varphi_n$.
Then $\varphi_1, \dots, \varphi_n \vdash ZF$ i.e. $\varphi_1, \dots, \varphi_n \models ZF$ by completeness.
By reflexion thm., $ZF \vdash \exists \beta$ "$V_\beta \models \varphi_1, \dots, \varphi_n$", hence
$ZF \vdash \exists \beta$ "$V_\beta \models ZF$" i.e. by completeness
$ZF \vdash \operatorname{cons}(ZF)$, thus by incompleteness
$\neg \operatorname{cons}(ZF)$. ▩

This proof uses Gödel's thms and is "informal" since we don't really have that
$ZF \vdash \exists \beta$ "$V_\beta \models ZF$", since we saw that sentences of the form "$V_\beta \models ZF$" are not really formalizable in ZF but are abus of language. Nevertheless, this proof gives the precise intuition of the formal proof.

By slightly modifying the proof of points (2) and (2') of Thm 11, one obtains the following version of the downward Löwenheim-Skolem theorem. Note that the thm requires the choice to be proved.

Thm 14 (Löwenheim-Skolem) (meta-result) and schema

(3) Let $\varphi_1, \dots, \varphi_n$ be formulas of set theory, then:

$$\begin{aligned} ZFC \vdash\ & \forall M_0 \text{ transitive } \exists M \supseteq M_0 \\ & [M \text{ transitive } \wedge \\ & |M| \leq \max(\omega, |M_0|) \wedge \\ & \varphi_1, \dots, \varphi_n \text{ are abs. for } M]. \end{aligned}$$

we need to be in ZFC since we use the choice in the proof.

(3') Let $\varphi_1, \dots, \varphi_n$ be axioms of ZFC, then:

$$\begin{aligned} ZFC \vdash\ & \forall M_0 \text{ transitive } \exists M \supseteq M_0 \\ & [M \text{ transitive } \wedge \\ & |M| \leq \max(\omega, |M_0|) \wedge \\ & \text{“} M \models \bigwedge_{i=1}^{n} \varphi_i \text{”}] \end{aligned}$$

Proof: (3') is an obvious consequence of (3).

(3) We argue from ZFC, i.e. ZFC $\vdash$ ...

Add the axiom of extensionality to the list if not already inside, and suppose w.l.o.g. that the list is subformula closed.

Let M_0 be a transitive set. Then $\exists \alpha (M_0 \subseteq V_\alpha)$. By Thm 11 (1), let $\beta > \alpha$ s.t. $\varphi_1, \dots, \varphi_n$ abs. for V_β. One has $M_0 \subseteq V_\beta$

By (AC), fix a well-ordering $\triangleleft$ on V_β.

If φ_i has l_i free variables, define the function $H_i : V_\beta^{\,l_i} \to V_\beta$ as follows:

- If φ_i is $\exists x\, \varphi_j(x, y_1, \dots, y_{l_i})$, then

$$H_i(y_1,\dots,y_{l_i}) = \begin{cases} \text{the } \Delta\text{-first el } x \text{ of } V_\beta \text{ s.t. } \exists x \in V_\beta\ \varphi_j(x,y_1,\dots,y_{l_i}) & \text{if } \exists x \in V_\beta\ \varphi_j(x,y_1,\dots,y_{l_i}) \\ \text{the } \Delta\text{-first el of } V_\beta & \text{if } \neg\exists x \in V_\beta\ \varphi_j(x,y_1,\dots,y_{l_i}) \end{cases}$$

- If φ_i is not existential formula, then

$$H_i(y_1,\dots,y_{l_i}) = \Delta\text{-first el. of } V_\beta .$$

Let $\overline{M}$ be the closure of M_0 under $H_1,\dots,H_n$, i.e. the least $\overline{M}$ s.t. $M_0 \subseteq \overline{M} \subseteq V_\beta$ and for all i and $y_1,\dots,y_{l_i} \in \overline{M}$, $H_i(y_1,\dots,y_{l_i}) \in \overline{M}$ (i.e. $\overline{M}$ closed under all functions H_i's).

By definition of $\overline{M}$, if φ_i is of the form $\exists x\, \varphi_j$ then $\forall y_1,\dots,y_{l_i} \in \overline{M}\ [\exists x\, \varphi_j(x,\bar{y}) \leftrightarrow \exists x \in V_\beta\ \varphi_j(x,\bar{y}) \leftrightarrow \exists x \in \overline{M}\ \varphi_j(x,\bar{y})]$.

By Lemma 12, it follows that $\underline{\varphi_1,\dots,\varphi_n \text{ are abs. for } \overline{M}}$.

We prove that $|\overline{M}| \leq \max(\omega, |M_0|)$. Note that $\overline{M}$ is obtained as follows: let $\overline{M}_0 = M_0$ and $\overline{M}_{n+1} = \overline{M}_n \cup \{H_i[\overline{M}_n^{\,l_i}] : i=1,\dots,n\}$, then $\overline{M} = \bigcup_{n\in\omega} \overline{M}_n$. By induction, $|\overline{M}_n| = |M_0|$, $\forall n \in \omega$, thus $\underline{|\overline{M}| \leq \max(\omega, |M_0|)}$.

Finally, we supposed that the axiom of extensionality was one of the formula, say φ_k. Since φ_k is true in ZFC and abs. for $\overline{M}$, then $\varphi_k^{\overline{M}}$ hold, i.e. $\overline{M}$ is extensional. Hence, we can apply the Mostowsky collapsing thm to $\overline{M}$ and produce a transitive set M $\in$-isom to $\overline{M}$. Thus: $\underline{M \text{ transitive}}$; M isom to $\overline{M}$ implies M satisfies the same formulas as $\overline{M}$ (by induction on φ), thus $\varphi_1,\dots,\varphi_n$ abs. for $\overline{M}$ implies $\underline{\varphi_1,\dots,\varphi_n \text{ abs. for } M}$; Also, $\underline{|M| = |\overline{M}| = \max(\omega, |M_0|)}$;

We prove that $M \supset M_0$. One has $G(x) = \{G(y) : y \in x \wedge y \in \overline{M}\}$, hence for $x \in M_0$, since $M_0 \subset \overline{M}$ and M_0 transitive, one has $G(x) = \{G(y) : y \in x\}$. Then, for all $x \in M_0$ one has $G(x) = x$ (by $\in$-induction, if $G(y) = y$, $\forall y \in x$, then $G(x) = \{y : y \in x\} = x$). Thus, $\underline{M = G(\overline{M}) \supseteq G(M_0) = M_0}$.