6.5. AC and GCH in L

We prove that "$L \models AC$" and "$L \models GCH$" which yields to the desired relative consistency of AC and GCH from ZF.

AC:

Definition: We define by induction on α the well-ordering Δ_α on L_α as follows:

- $\Delta_0 = 0$
- If α limit, $\Delta_\alpha = \{ \langle x, y \rangle \in L_\alpha \times L_\alpha :$

$$\rho(x) < \rho(y) \quad \vee$$

level of appearance of x

$$(\rho(x) = \rho(y) \wedge x \, \Delta_{\underbrace{\rho(x)+1}_{\substack{<\alpha \text{ since } x \in L_\alpha \Rightarrow \rho(x) < \alpha \\ \Rightarrow (\alpha \text{ limit}) \ \rho(x)+1 < \alpha}}} \, y) \}$$

- Now, given Δ_α, we define Δ_α^n as the lex-ord on L_α^n:

$$s \, \Delta_\alpha^n \, t \iff \exists k < n \, (s \restriction k = t \restriction k \wedge s(k) \, \Delta_\alpha \, t(k)).$$

Then, if $X \in L_{\alpha+1} = D(L_\alpha)$, let n_X be the least n s.t.

$$\exists s \in L_\alpha^n \; \exists R \in Df(L_\alpha, n+1) \; \left(X = \{ x \in L_\alpha : s^\frown\langle x \rangle \in R \} \right) ;$$

then let s_X be the $\Delta_\alpha^{n_X}$-least $s \in L_\alpha^{n_X}$ s.t.

$$\exists R \in Df(L_\alpha, n_X+1) \; \left(X = \{ x \in L_\alpha : s^\frown\langle x \rangle \in R \} \right) ;$$

then let m_X be the least m s.t.

$$X = \{ x \in L_\alpha : s^\frown\langle x \rangle \in En(m, L_\alpha, n_X+1) \}$$

- If $\alpha = \beta+1$ (successor) then

$$\begin{aligned}
\triangleleft_\alpha = \triangleleft_{\beta+1} = \{ \langle x,y\rangle \in L_{\beta+1} \times L_{\beta+1} :\ & (x,y \in L_\beta \wedge x \triangleleft_\beta y) \vee \\
& (x \in L_\beta \wedge y \in L_{\beta+1}) \vee \\
& x,y \in \underbrace{L_{\beta+1}}_{\overset{\shortparallel}{D(L_\beta)}} \wedge [n_x < n_y \vee \\
& \quad (n_x = n_y \wedge s_x \triangleleft^{n_x} s_y) \vee \\
& \quad (n_x = n_y \wedge s_x = s_y \wedge m_x < m_y)] \}
\end{aligned}$$

Theorem 15 (ZF): "$L \models AC$" i.e. $(AC)^L$

Proof: **Lemma (ZF):** $V=L \to AC$. Suppose $V=L$ and let X be a set.
Then $X \in L$, hence $X \in L_\alpha$ for some α,
$\Rightarrow X \subseteq L_\alpha$ since L_α transitive
$\Rightarrow X$ is well-ordered by $\triangleleft_\alpha$
$\Rightarrow AC$ holds. □

We now prove the thm. Since "$L \models ZF$" (thm 9), $ZF \vdash V=L \to AC$, then "$L \models V=L \to AC$", ⊛
i.e. $(V=L)^L \to (AC)^L$. But $(V=L)^L$ holds by thm 10,
therefore $(AC)^L$ holds. ■

meta-result

Corollary 16 (ZF): $Cons(ZF) \to Cons(ZFC)$

Proof: By theorems 9 and 15, "$L \models ZFC$".
By Chap. 5-Lemma 1, $Cons(ZF) \to Cons(ZFC)$. ■

⊛ This is an abuse of language...
We have more precisely that "$L \models \varphi$", where φ is the finite conjunction of axioms of ZF necessary to prove $V=L \to AC$.
Thus "$L \models V=L \to AC$", (We can prove that "$M \models \varphi$" and $\varphi \vdash \psi$ implies "$M \models \psi$", by ind on the length of the proof of $\varphi \vdash \psi$)
i.e. $(V=L)^L \to (AC)^L$.
But $(V=L)^L$ holds (thm 10),
thus $(AC)^L$ holds.

GCH:

Theorem 17 (ZF): "$L \models GCH$" i.e. $(GCH)^L$

Proof: lemma (ZF): $V=L \to \forall \alpha \geq \omega (\mathcal{P}(L_\alpha) \subseteq L_{\alpha^+})$

Proof: Suppose $V=L$. Then AC holds.
Let $\alpha \geq \omega$ and let φ be a sufficiently large frament of ZF needed to obtain Prop 14 (b) i.e.

$$\forall M \left(M \text{ transitive and } "M \models \varphi + V=L" \to M = L_{o(M)} \right) \circledast$$

Let $A \in \mathcal{P}(L_\alpha)$ and let $X = L_\alpha \cup \{A\}$.
Then $|X| = |L_\alpha| = |\alpha|$, by lemma 8 (uses AC).
By Chap 5 - thm 14 (Löwenheim-Skolem argument)

$$\exists M \left[M \text{ transitive } \wedge M \supseteq X \wedge |M| = |X| = |\alpha| \right.$$
$$\left. \wedge \ (\varphi + V=L)^M \leftrightarrow \varphi + V=L \right]$$

But φ fragment of ZF and $V=L$ is supposed, so $\varphi + V=L$ holds, thus $(\varphi + V=L)^M$ holds.
Hence by $\circledast$, $M = L_{o(M)}$.
But $|o(M)| \underset{\text{lemma 8}}{=} |L_{o(M)}| = |M| = |X| = |\alpha| < \alpha^+$,
$\Rightarrow o(M) < \alpha^+ \Rightarrow L_{o(M)} \subseteq L_{\alpha^+}$.
Now $A \in X \subseteq M = L_{o(M)} \subseteq L_{\alpha^+}$, thus $A \in L_{\alpha^+}$.
Therefore $\mathcal{P}(L_\alpha) \subseteq L_{\alpha^+}$ $\square$

lemma (ZF): $V=L \to GCH$

Proof: Suppose $V=L$ and let $\kappa \geq \omega$ cardinal.
Then $2^\kappa = |\mathcal{P}(\kappa)| \underset{\substack{\text{since} \\ \kappa \subseteq L_\kappa}}{\leq} |\mathcal{P}(L_\kappa)| \underset{\substack{\text{previous} \\ \text{lemma}}}{\leq} |L_{\kappa^+}| \underset{\text{lemma 8}}{=} \kappa^+$

Conversely, $\kappa < 2^\kappa \Rightarrow \kappa^+ \leq 2^\kappa$.
Thus $2^\kappa = \kappa^+$, GCH holds $\square$

Finally, one has $V=L \to GCH$ provable in ZF.
But "$L \models ZF$" (thm 9), thus "$L \models V=L \to GCH$"
i.e. $(V=L)^L \to (GCH)^L$.
But $(V=L)^L$ holds (by thm 10), thus GCH holds ■

meta-result

Corollary 18 (ZF): $\mathrm{Cons}(ZF) \to \mathrm{Cons}(ZF + GCH)$

Proof: By theorems 9 and 17, "$L \models ZF + GCH$".
By Chap 5 - Lemma 1, $\mathrm{Cons}(ZF) \to \mathrm{Cons}(ZF + GCH)$. ∎