Chapter 4 : The Well-Founded Sets

4.1. The class WF of well-founded sets

<u>Def</u>: We define by transfinite induction on $\alpha \in ON$

$$\begin{cases} V_0 = 0 \\ V_{\alpha+1} = \mathcal{P}(V_\alpha) \\ V_\alpha = \bigcup\{V_\xi : \xi < \alpha\}, \text{ for } \alpha \text{ limit} \end{cases}$$

We set $WF = \bigcup\{V_\alpha : \alpha \in ON\}$ called the class of well-founded sets: this name will become clear later

So we have: $V_0 = 0$

$V_1 = \mathcal{P}(0) = \{0\}$

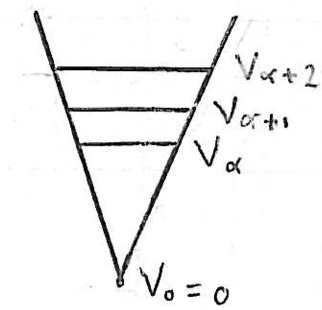
$V_2 = \mathcal{P}(V_1) = \{0, \{0\}\} = \{0, 1\}$

$V_3 = \mathcal{P}(V_2) = \{0, 1, 2, \{1\}\}$

etc.

<u>Lemma 1</u>: i) $\forall \alpha \in ON$, V_α is transitive

ii) $\alpha \leq \beta \Rightarrow V_\alpha \subseteq V_\beta$.



<u>Proof</u>: i) by induction on α:

trivial (skip)

- $\alpha = 0 \Rightarrow V_0 = 0$ transitive
- $\alpha = \beta + 1$ and V_β transitive $\Rightarrow V_\alpha = V_{\beta+1} = \mathcal{P}(V_\beta)$ transitive
- α limit and V_ξ transitive $\forall \xi < \alpha$.
  $\Rightarrow V_\alpha = \bigcup_{\xi<\alpha} V_\xi$ transitive. (as a union of transitive sets)

ii) By induction on β: $\longrightarrow$

- $\beta = 0 \Rightarrow \alpha = 0 \Rightarrow V_\alpha = V_\beta = 0$ o.k.
- $\beta = \gamma + 1$: If $\alpha = \beta$, then $V_\alpha = V_\beta$.
  If $\alpha < \beta$, then $\alpha \leq \gamma \underset{\text{I.H.}}{\Rightarrow} V_\alpha \subseteq V_\gamma \underset{\subseteq}{\in} \mathcal{P}(V_\gamma) = V_{\gamma+1} = V_\beta$.
  since $V_{\gamma+1}$ transitive by (i).
- β limit: $V_\beta = \bigcup_{\zeta < \beta} V_\zeta$
  If $\alpha = \beta$, $V_\alpha = V_\beta$
  If $\alpha < \beta$, $V_\alpha \subseteq V_\beta$ by def of V_β. ∎

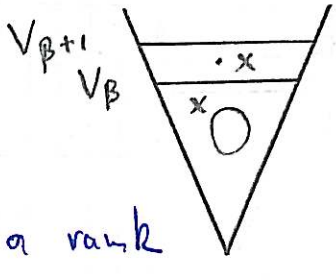
<u>Remark</u>: If $x \in WF$, then the least α s.t. $x \in V_\alpha$ is a <u>successor</u> ordinal

Indeed: suppose it is limit, then by def $V_\alpha = \bigcup_{\zeta < \alpha} V_\zeta$, but then $x \in V_\alpha$ means $x \in V_\zeta$ for some $\zeta < \alpha$, contradiction with the minimality of α.

<u>Def</u>: If $x \in WF$, let <u>rank(x)</u> or <u>$\rho(x)$</u> be the least α s.t. $x \in V_{\alpha+1}$ ← we know that the least ordinal γ s.t. $x \in V_\gamma$ is successor

The rank of x is the level that precedes the "first" appearance of x.

Thus rank$(x) = \beta$ implies:



$$\begin{cases} x \in V_{\beta+1} = \mathcal{P}(V_\beta) \Rightarrow x \subseteq V_\beta \\ x \notin V_\beta \quad \text{by minimality of } \beta \text{ as a rank} \\ x \in V_\alpha, \ \forall \alpha > \beta \end{cases}$$

, since the V_α's are increasing with respect to inclusion.

<u>Remark:</u> characterization of each set V_α:

$$V_\alpha = \{ x \in WF : \operatorname{rank}(x) < \alpha \}$$

since the rank is the level that precedes the first appearance of x, and $x \in V_\alpha$

<u>Proof:</u> ($\subseteq$) $x \in V_\alpha \Rightarrow x \in WF$ and $\operatorname{rank}(x) < \alpha$

($\supseteq$) Let $x \in WF$ s.t. $\operatorname{rank}(x) = \underline{\beta < \alpha}$

Then $x \in V_{\beta+1}$ and $\beta + 1 \leq \alpha$.

Thus $\underset{\text{lemma 1 (ii)}}{V_{\beta+1} \subseteq V_\alpha}$ and hence $x \in V_\alpha$. ∎

<u>Proposition 2:</u> Let $x, y \in WF$

i) $x \in y \Rightarrow \operatorname{rank}(x) < \operatorname{rank}(y)$

ii) $x \subseteq y \Rightarrow \operatorname{rank}(x) \leq \operatorname{rank}(y)$

iii) $\operatorname{rank}(\mathcal{P}(x)) = \operatorname{rank}(x) + 1$

iv) $\operatorname{rank}(\bigcup x) = \operatorname{rank}(x)$

v) $\operatorname{rank}(\bigcap x) < \operatorname{rank}(x)$

vi) $\operatorname{rank}(x \times y)$, $\operatorname{rank}(x \cup y)$, $\operatorname{rank}(x \cap y)$, $\operatorname{rank}(\{x, y\})$, $\operatorname{rank}(\langle x, y \rangle)$ and $\operatorname{rank}({}^{y}x)$ are all bounded by $\max(\operatorname{rank}(x), \operatorname{rank}(y)) + 3$

<u>Proof:</u> excercices

<u>Remark</u>: - In WF, there is no x s.t. $x \in x$

otherwise: we would have $\text{rank}(x) < \text{rank}(x)$, $\perp$

- No circularities like $x \in y \in x$

otherwise: we would have

$\text{rank}(x) < \text{rank}(y) < \text{rank}(x)$, $\perp$

$\rightarrow$ WF is <u>transitive</u>, (i.e $x \in y \in WF \rightarrow x \in WF$)

Let $x, y \in WF$ s.t. $x \in y$. Then $\exists \alpha$ s.t. $x \in y \in V_\alpha \Longrightarrow (V_\alpha \text{ transitive})$ lemma 1. $x \in V_\alpha \Rightarrow x \in WF$

So **WF** "look nice", no pathological properties...

Lemma 3: $\forall \alpha \in ON \left(\alpha \in WF \wedge \operatorname{rank}(\alpha) = \alpha \right)$.

Proof: We show by induction on α that $\alpha \in V_{\alpha+1}$ and $\alpha \notin V_\alpha$. Thus $\operatorname{rank}(\alpha) = \alpha$.

- If $\alpha = 0$, then $\underline{\alpha \in V_1} = \{0\}$ and $\underline{\alpha \notin V_0} = 0$
- If $\alpha = \beta + 1 = \beta \cup \{\beta\}$.

  By I.H. we have $\beta \in V_{\beta+1}$. Hence:

  1°) $\beta \in V_{\beta+2} = V_{\alpha+1}$ (since $V_{\beta+1} \subseteq V_{\beta+2}$)

  2°) $\{\beta\} \in \mathcal{P}(V_{\beta+1}) = V_{\beta+2} = V_{\alpha+1}$. Thus

  $\underline{\alpha = \beta \cup \{\beta\} \in V_{\alpha+1}}$. $\left(\begin{array}{l} \beta \subseteq V_\alpha \text{ and } \{\beta\} \subseteq V_\alpha \\ \Rightarrow \beta \cup \{\beta\} \subseteq V_\alpha \end{array} \right)$

  Now assume $\alpha \in V_\alpha$ i.e. $\beta \cup \{\beta\} \in \mathcal{P}(V_\beta)$

  Then $\{\beta\} \subseteq V_\beta \Rightarrow \beta \in V_\beta$, contradiction with I.H.

  Thus $\underline{\alpha \notin V_\alpha}$.

- If α limit:

  If $\beta \in \alpha$ (i.e. $\beta < \alpha$) then by I.H., $\beta \in V_{\beta+1} \subseteq V_\alpha$ (since $\beta + 1 \leq \alpha$), thus $\beta \in V_\alpha$

  Hence $\alpha \subseteq V_\alpha \Longrightarrow \underline{\alpha \in V_{\alpha+1}}$

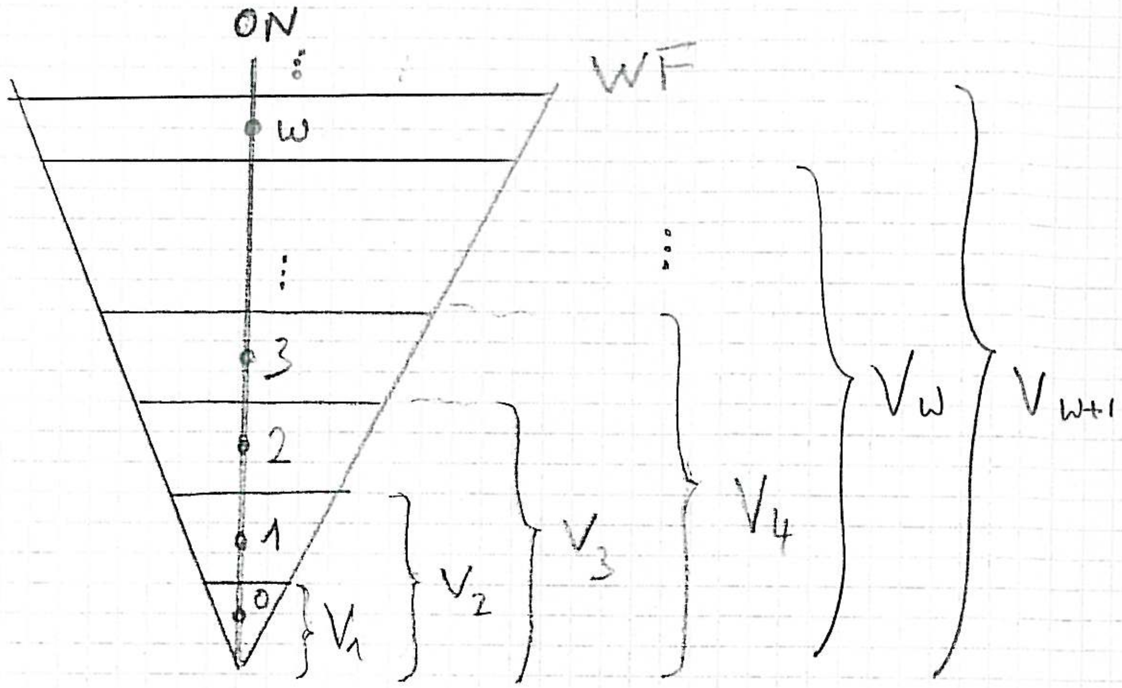
  Now assume $\alpha \in V_\alpha = \bigcup_{\xi < \alpha} V_\xi$ (since α limit)

  Then $\exists \xi \overset{\in}{<} \alpha$ s.t. $\alpha \in V_\xi \Rightarrow \xi \in V_\xi$ (since V_ξ transitif)

  contradiction with I.H. Thus $\underline{\alpha \notin V_\alpha}$

$\blacksquare$

We have the following picture:



<u>Remark</u>: i) We can prove that $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are all in $V_{\omega+\omega}$.

ii) We have $\forall n < \omega \, (|V_n| < \omega)$ and
$|V_\omega| = \omega = \aleph_0$

iii) The cardinalities of V_α increase exponentially
$|V_{\omega+1}| = |\mathcal{P}(V_\omega)| = |\mathcal{P}(\omega)| = 2^{\aleph_0}$
$|V_{\omega+2}| = 2^{2^{\aleph_0}}$, etc...
More precisely, we can prove by induction on α that $|V_{\omega+\alpha}| = \beth_\alpha$

All mathematics take place in WF !

where $\beth_\alpha$ is defined by induction on α by:

$$\begin{cases} \beth_0 = \omega = \aleph_0 \\ \beth_{\alpha+1} = 2^{\beth_\alpha} \\ \beth_\lambda = \sup\{\beth_\xi : \xi < \lambda\}, \text{ for } \lambda \text{ limit} \end{cases}$$

We will introduce the axiom of foundation and see that assuming it is equivalent to stating $V = WF$, i.e. every set is well-founded.

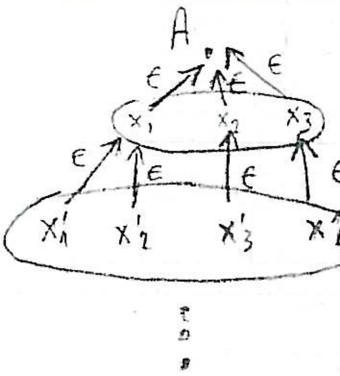
We need the following definitions ...

<u>Def ($ZF^- - P$)</u>: A relation R is <u>well-founded</u> on a set A iff

$$\forall X \subseteq A \left[X \neq 0 \to \exists y \in X \left(\forall z \in X (\neg z R y) \right) \right]$$

i.e. every non-empty subset X of A has an R-minimal element.

<u>Def ($ZF^- - P$)</u>: Let A be a set. Define by induction on n:



set of el. of el. of the previous A_n

$A = U^0 A$ $\quad U^0 A = A \quad , \quad U^{n+1} A = \bigcup (U^n A)$

We set $cl(A) = \bigcup \{ U^n A : n \in \omega \}$

$UA = U^1 A$ $\quad$ the <u>transitive closure</u> of A.

So $cl(A) = A \cup UA \cup U^2 A \cup \ldots$ i.e. contains the el. of A + the el. of el. of A + el. of el. of el. of A + ...

<u>Lemma 4</u>: $cl(A)$ is the least transitive set containing A.
($ZF^- - P$)

Proof: • If $x \in y \in cl(A)$, then $\exists n$ s.t. $y \in U^n A$
$\Rightarrow x \in U(U^n A) = U^{n+1} A \Rightarrow x \in cl(A)$

• Thus $cl(A)$ transitive.

$cl(A) \supseteq A$ by def

• let T transitive s.t. $T \supseteq A$
An induction on n shows $U^n A \subseteq T$, $\forall n \in \omega$. Thus $cl(A) = U\{U^n A : n \in \omega\} \subseteq T$ ☐

(details of the induction :

$U^0 A = A \subseteq T$

let $x \in U^{n+1} A = U(U^n A)$
$\Rightarrow \exists y$ s.t. $x \in y \in U^n A$ and $\underset{\text{I.H.}}{U^n A \subseteq T}$
So $x \in y \in T \underset{(T \text{ transitive})}{\Rightarrow} x \in T$)

We have $cl(A) = A \cup \{cl(x) : x \in A\}$ (exercise)

let $T = A \cup \{cl(x) : x \in A\}$

• Then T transitive $\supseteq A \Rightarrow cl(A) \subseteq T$

• since $A \subseteq cl(A)$ and $x \in A \Rightarrow x \in cl(A)$
$\Rightarrow x \subseteq cl(A)$ (by transitivity)
$\Rightarrow cl(x) \subseteq cl(A)$ (by minimality of $cl(x)$)

Thus $A \cup \{cl(x) : x \in A\} = T \subseteq cl(A)$.

Thm 5: Let A be a set. The following are equivalent: (ZF$^-$)

i) $A \in WF$

ii) $cl(A) \in WF$

iii) $\in$ is well-founded on $cl(A)$.

Proof: i) $\Rightarrow$ ii): If $A \in WF$, then by induction on n, $\bigcup^n A \in WF$, $\forall n \in \omega$ (since WF closed under $\bigcup$).

$\bigcup^0 A = A \in WF$
$\bigcup^{n+1} A = \bigcup(\underbrace{\bigcup^n A}_{\in V_\alpha})$, for some α
$\in V_{\alpha+1}$

By transitivity of WF, $\bigcup^n A \subseteq WF$, $\forall n \in \omega$

Thus $cl(A) \subseteq WF$ i.e.

$cl(A) \subseteq V_\alpha$ for some α, thus

$cl(A) \in V_{\alpha+1} \Rightarrow cl(A) \in WF$.

ii) $\Rightarrow$ iii): Let $X \subseteq cl(A)$, $X \neq 0$.

By (ii), $cl(A) \in WF \Rightarrow cl(A) \subseteq WF \Rightarrow X \subseteq WF$ also, $\Rightarrow$ we can consider

$\alpha = \min\{rank(y) : y \in X\}$

$X \subseteq WF \Rightarrow \forall y \in X, y \in WF$
$\Rightarrow rank(y)$ exist, $\forall y \in X$.

and let $y \in X$ with $rank(y) = \alpha$.

If y not $\in$-minimal in X,

$\exists y' \in X$ s.t. $y' \in y$.

Thus $rank(y') < rank(y) = \alpha$ (Prop-2)

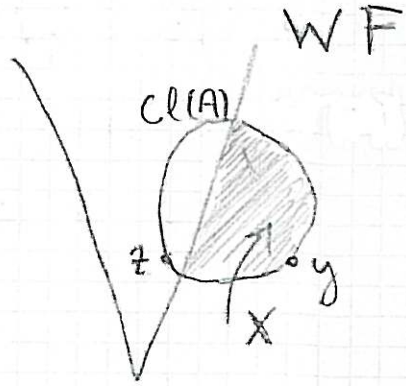
contradiction with minimality of α.

$\longrightarrow$

iii) ⇒ i) We first show that $cl(A) \subseteq WF$

Suppose $cl(A) \nsubseteq WF$.

Let $X = cl(A) \setminus WF \neq 0$



Let y $\in$-min in X (exists by hyp.)

If $\underline{z \in y} \in X \subseteq cl(A)$, then

$\left\{ \begin{array}{l} z \notin X \text{ (by min of } y \text{ in } X\text{)} \\ \text{and} \\ z \in cl(A) \text{ (since } cl(A) \text{ transitive.)} \end{array} \right.$

$\Rightarrow \underline{z \in WF}$ (c.f. figure, $z \in cl(A)$ and $z \notin X \Rightarrow z \in WF$)

$\Rightarrow \underline{\underline{y \subseteq WF}} \Rightarrow y \in WF$ $(y \subseteq V_\alpha \Rightarrow y \in V_{\alpha+1} \subseteq WF)$,

contradiction with $y \in X = cl(A) \setminus WF$.

Thus $cl(A) \subseteq WF$.

Thus $A \subseteq cl(A) \subseteq WF \Rightarrow A \subseteq WF$

$\Rightarrow A \in WF$ $(A \subseteq V_\alpha \Rightarrow A \in V_{\alpha+1} \Rightarrow A \in WF)$

▨

Remark: the def of WF uses the power set axiom, but the equivalent formulation via transitive closure and well-foundation of $\in$ doesn't!

4.2. The Axiom of Foundation

Since all mathematics take place in **WF**, it is "reasonable" to adopt an axiom stating $\mathbf{V} = \mathbf{WF}$ (i.e. every set is well-founded).

It restricts our domain of discourse to "not pathological" sets, but still permits to do all mathematics ...

Since the statement $\mathbf{V} = \mathbf{WF}$ is highly non-elementary to write, we give an equivalent formulation.
in the pure language of set theory

Axiom 2: Foundation

$$\forall x \left[\exists y\, (y \in x) \rightarrow \exists y\, (y \in x \wedge \neg \exists z\, (z \in x \wedge z \in y)) \right]$$

i.e. if $x \neq 0$, then $\exists y \in x\ (x \cap y = 0)$

i.e. every set $x \neq 0$ has an $\in$-minimal element.

The axiom of foundation is equivalent to $\mathbf{V} = \mathbf{WF}$.

Thm 6 (ZF^-) The following are equivalent:

i) the Axiom of Foundation

ii) $\forall A$ ($\in$ is well-founded on A)

iii) $\mathbf{V} = \mathbf{WF}$

→

Proof: i) $\Leftrightarrow$ ii) immediate by def of AF.

ii) $\Rightarrow$ iii): Suppose ii)

We have obviously $WF \subseteq V$

let $A \in V$, then by hyp, $\in$ is w-f on $cl(A)$.

By thm 5, $A \in WF$. Thus $V \subseteq WF$

iii) $\Rightarrow$ ii) Suppose $V = WF$.

let $A \in V$ and $X \subseteq A$ s.t. $X \neq 0$.

Then $A \in WF$ by hyp. Thus $A \subseteq WF \Rightarrow X \subseteq WF$

This ensures the existence of

$\alpha = \min \{ \text{rank}(y) : y \in X \}$

Let $y \in X$ s.t. $\text{rank}(y) = \alpha$.

Then y $\in$-min in X $\blacksquare$

(otherwise $z \in X$ s.t. $z \in y$ implies $\text{rank}(z) < \text{rank}(y) = \alpha$ contradiction with minimality of α)

So under AF, the universe looks like this