4.3. Induction on well-founded relations

<u>Def</u>.: <u>schema</u> let $\mathbf{C}$ be a class and $\mathbf{R}$ be a relational class on $\mathbf{C}$.

i). $\mathbf{R}$ is <u>well-founded</u> on $\mathbf{C}$ iff:

$$\forall X \subseteq \mathbf{C} \left[X \neq 0 \rightarrow \exists y \in X \left(\neg \exists z \in X \left(z \mathbf{R} y \right) \right) \right]$$

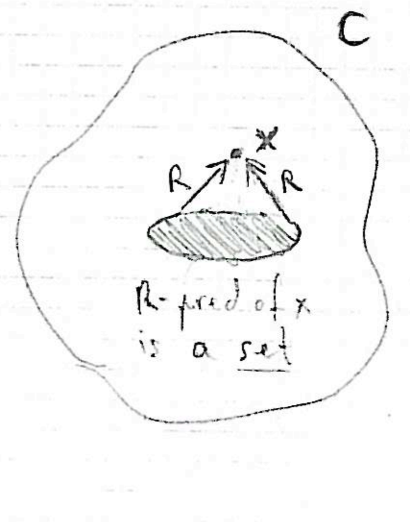
ii) $\mathbf{R}$ is <u>extentional</u> on $\mathbf{C}$ iff
"the axiom of extentionality is true in $(\mathbf{C}, \mathbf{R})$" i.e.

$$\forall x, y \in \mathbf{C} \left[\forall z \in \mathbf{C} \left(z \mathbf{R} x \leftrightarrow z \mathbf{R} y \right) \rightarrow x = y \right].$$

iii) $\mathbf{R}$ is <u>set-like</u> on $\mathbf{C}$ iff
$\forall x \in \mathbf{C}$, the class $\{ y \in \mathbf{C} : y \mathbf{R} x \}$ is a set

Intuition: $\mathbf{R}$ is "small", the predecessor chain of an element is never too big to be a proper class.

Note that this is a <u>def schema</u> i.e. one def for each pair of formula φ_C and φ_R corresponding to $\mathbf{C}$ and $\mathbf{R}$.



set-like relational class $\mathbf{R}$ on $\mathbf{C}$

<u>Def</u> : (Generalisation of transitive closure to the relational class context.)

schema

let $\mathbf{R}$ be set-like on $\mathbf{C}$.
Since the R-predecessors of any element of $\mathbf{C}$ is a set, we can define by induction on n:

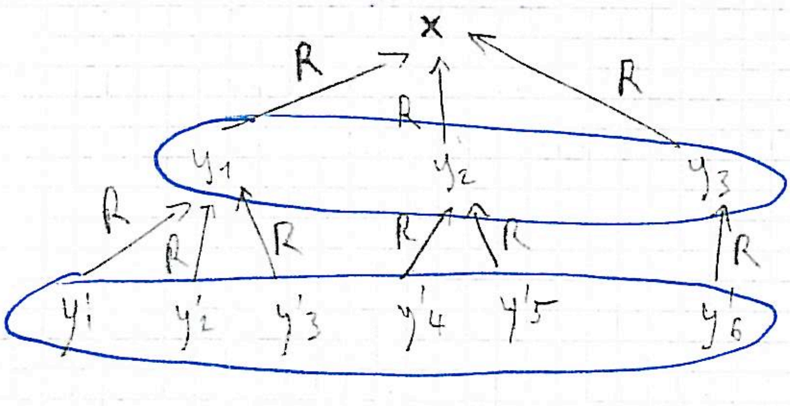
We set : $pred(\mathbf{C}, x, \mathbf{R}) := \{ y \in \mathbf{C} : y \mathbf{R} x \}$ $\longleftarrow$ is a <u>set</u> !

by ind on n

i) $\begin{cases} pred^0(\mathbf{C}, x, \mathbf{R}) := pred(\mathbf{C}, x, \mathbf{R}) \\ pred^{n+1}(\mathbf{C}, x, \mathbf{R}) := \bigcup \{ pred(\mathbf{C}, y, \mathbf{R}) : y \in pred^n(\mathbf{C}, x, \mathbf{R}) \} \end{cases}$

ii) $cl(\mathbf{C}, x, \mathbf{R}) := \bigcup \{ pred^n(\mathbf{C}, x, \mathbf{R}) : n \in \omega \}$

Intuitively, $cl(C, x, R)$ contains the R-el of x
+ the R-el of R-el of x
+ the R-el of R-el of R-el of x
+ ...



$pred^0(\mathbf{C}, x, \mathbf{R})$

$pred^1(\mathbf{C}, x, \mathbf{R})$

⋮

schema

Thm 7 (ZF$^-$ - P): If R is w-f and set-like on C, then every non-empty subclass X of C has an R-least element.

Proof: Claim: Suppose R set-like on C. Then for all $x \in C$ and $y \in cl(C,x,R)$ we have $pred(C,y,R) \subseteq cl(C,x,R)$

Proof: +/- obvious from the picture

Similar to Chap 2, Thm 13

$y \in cl(C,x,R)$

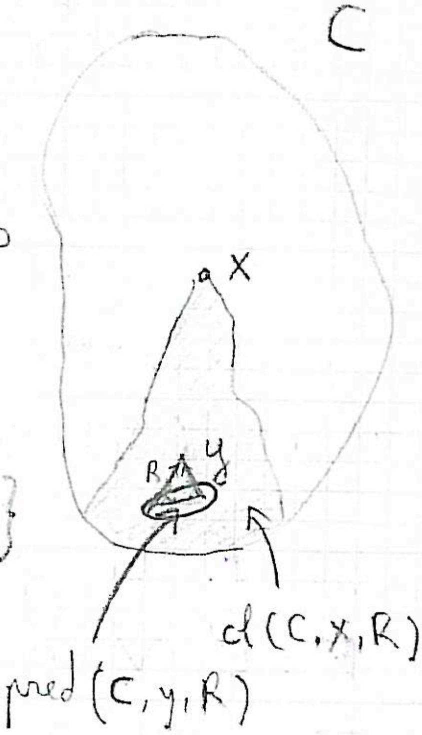
$\Rightarrow y \in pred^n(C,x,R)$ f.s. $n \in \omega$

Let $z \in pred(C,y,R)$

$\Rightarrow z \in \bigcup \{ pred(C,y,R) : y \in pred^n(C,x,R) \}$

$\underset{def}{=} pred^{n+1}(C,x,R)$

$\Rightarrow z \in cl(C,x,R)$ ☐



Let $x \in X$. If x R-min in X, finished.

Otherwise, $\exists y \in X \, (y R x) \Rightarrow X \cap cl(C,x,R) \neq 0$ since y belongs to both

$\Rightarrow$ (R is w-f) $\exists y'$ R-min in $X \cap cl(C,x,R)$

Thus y' R-min in X.

Otherwise, let $z \in X$ s.t. $z R y'$. $\Rightarrow z \in pred(C,y',R)$ and $y' \in cl(C,x,R)$

$\Rightarrow$ (claim) $z \in cl(C,x,R)$, contradicts minimality of y' in $X \cap cl(C,x,R)$. ☐

As a corollary, we can do proofs by induction on <u>well-founded</u> <u>set-like</u> relational classes.

Thm 8 (Proofs by induction) let $\mathbf{R}$ <u>w-f</u> and <u>s-l</u> on $\mathbf{C}$

i.e. $\forall x \in \mathbf{C} \left[\left(\forall y \left(y \in \mathbf{C} \wedge y \mathbf{R} x \right) \rightarrow \varphi(y) \right) \rightarrow \varphi(x) \right]$

schema

$$\left[\forall x \in \mathbf{C} \left[\left(\forall y \in \mathbf{C} \left(y \mathbf{R} x \rightarrow \varphi(y) \right) \right) \rightarrow \varphi(x) \right] \rightarrow \forall x \in \mathbf{C} \left(\varphi(x) \right) \right]$$

if $\varphi(x)$ holds as soon as $\varphi(y)$ holds for all $\mathbf{R}$-pred y of x in $\mathbf{C}$, then $\forall x \in \mathbf{C}\, \varphi(x)$ holds

If $\varphi(x)$ is true as soon as $\varphi(y)$ is true for all $\mathbf{R}$-pred of x, then $\varphi(x)$ is true for all $x \in \mathbf{C}$.

<u>Proof</u>: A corollary of thm 7.
Same proof as thm 14, chap 3.

Thm 9 (defs by induction) <u>schema</u>

let $\mathbf{R}$ be <u>w-f</u> and <u>s-l</u> on $\mathbf{C}$, and let $\mathbf{F} : \mathbf{C} \times \mathbf{V} \rightarrow \mathbf{V}$ functional class.

Then there exists a unique $\mathbf{G} : \mathbf{C} \rightarrow \mathbf{V}$ funct. class s.t.

$$\forall x \in \mathbf{C} \left[\mathbf{G}(x) = \mathbf{F}\left(x, \mathbf{G} \restriction \mathrm{pred}(\mathbf{C}, x, \mathbf{R}) \right) \right]$$

<u>Proof</u>: Similar to thm 16, chap 2.

Note: we need $\mathbf{R}$ set-like in order for the expression $\mathbf{G} \restriction \mathrm{pred}(\mathbf{C}, x, \mathbf{R})$ to be meaningful (since $\mathrm{pred}(\mathbf{C}, x, \mathbf{R})$ is a set).

4.4. Mostowsky collapsing theorem.

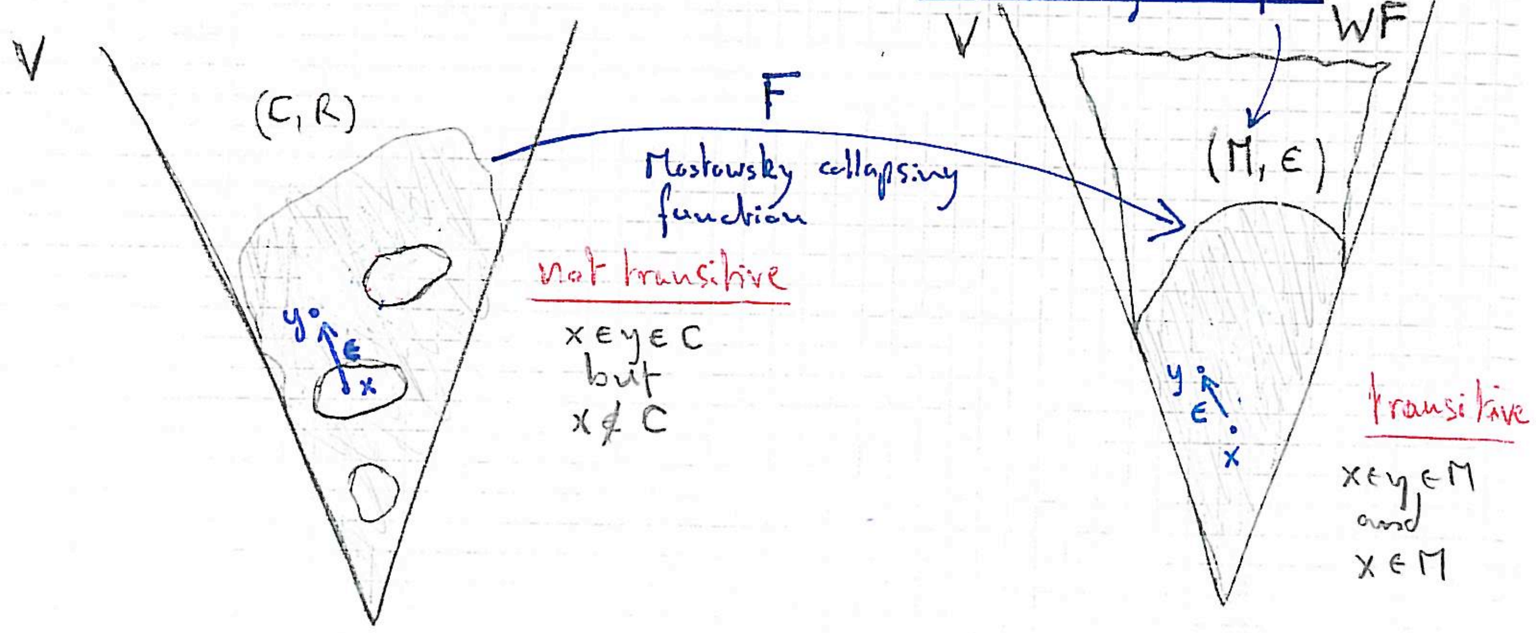
Intro (oral)

We prefer working with transitive relational classes, since they have many suitable properties: no circularities ($x R y$ and $y R x$), no self-predecessing ($x R x$), ...

We prove that without loss of much generality, we can restrict our attention to transitive relational classes (C, R).

This will be important when we will consider models of ZFC: we prefer transitive models, ...

To summarize, we can "transitivize" the relational classes that are not too pathological (set-like, well-founded, extensional). The function that performs this "transitivization" is the Mostowsky collapsing function, and the obtained transitive class is the Mostowsky collapse.



Thm 10 (ZF$^-$) Mostowsky collapsing thm <u>schema</u>

let $\mathbf{R}$ be an extensional, well-founded, set-like relational class on a class $\mathbf{C}$.

Then there exists a unique transitive class $\mathbf{M} \subseteq \mathbf{WF}$ and a unique isom functional class $\mathbf{F}$ such that
$\mathbf{F} : \mathbf{C}, \mathbf{R} \longrightarrow \mathbf{M}, \in\restriction_{\mathbf{M}}$

($\mathbf{F}$ isom means $\forall x, y \in \mathbf{C}, \ x \mathbf{R} y \leftrightarrow \mathbf{F}(x) \in \mathbf{F}(y)$)

$\mathbf{M}$ is the Mostowsky collapse of $\mathbf{C}, \mathbf{R}$
$\mathbf{F}$ is the Mostowsky collapsing function.

faire le dessin après énoncer le thm.

Proof: R w-f and s-l $\overset{\text{thm 7}}{\Longrightarrow}$ We can define by transfinite induction $\mathbf{F} : \mathbf{C} \longrightarrow \mathbf{V}$ as follows:

$$F(x) = \{ F(y) : y \in \text{pred}(C, x, R) \}$$
$$\text{i.e.} \quad = \{ F(y) : y \in C \text{ and } y R x \}$$

i.e. $\varphi_M(x)$ iff $\exists x' (\varphi_C(x') \wedge \varphi_F(\langle x, x' \rangle))$

And let M be the image of F

if x has no R-pred, then $F(x) = 0$, which gives the basis of the inductive construction.

M transitive: let $y \in y' \in M$.

$y' \in M \Rightarrow \exists x' \in C$ s.t. $y' = F(x')$.

$y \in y' = F(x') = \{ F(x) : x \in C \wedge x R x' \} \Rightarrow y = F(x)$ f.s. $x \in C$

i.e. $y \in F(C) = M$

$M \subseteq WF$: By transfinite induction, we prove $y \in WF, \forall y \in M$.

let $F(x) \in M$ (all el. of M are of this form) and suppose $\forall y' \in M \ \Big[\underbrace{(y' \in F(x))}_{\text{iff } y' = F(x') \text{ with } x' R x} \longrightarrow (\underset{= F(x')}{y'} \in WF) \Big]$

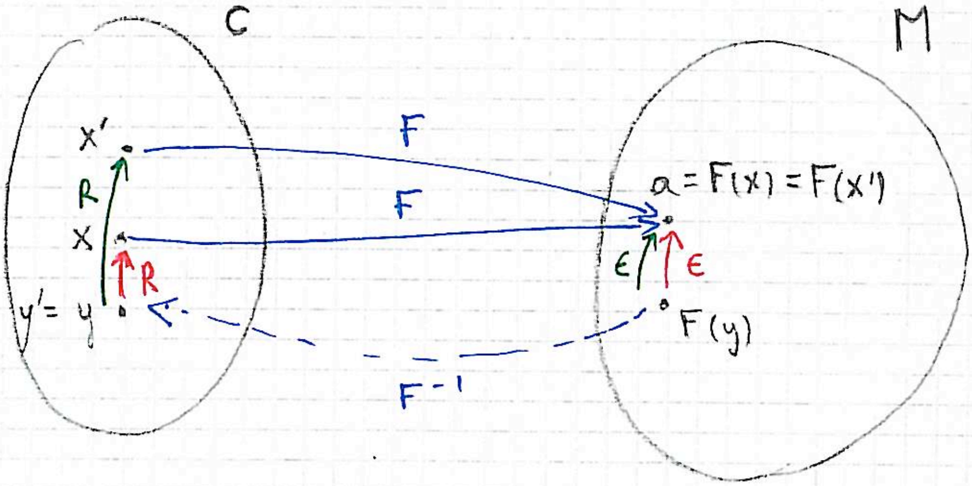
iff $\forall x' \in C \ \Big[(x' R x) \longrightarrow (F(x') \in WF) \Big]$

i.e. $\underbrace{\{F(x') : x' \in C \wedge x' R x\}}_{= F(x) \text{ by def}} \subseteq WF$, hence $\in WF$.

So $\forall y \in M \, (y \in WF)$ i.e. $M \subseteq WF$

<u>F isom</u>: By construction, $F : C \to V$ is onto and and $\forall x, y \in C$, $F(y) \in F(x) \leftrightarrow y R x$

Suppose F not 1-1 (and remember that $M \subseteq WF$)
$\Rightarrow \exists a$ of min rank $\in M$ and $x \neq x'$ both $\in C$ s.t.
$F(x) = F(x') = a$



If $\underline{y R x}$, then $F(y) \in F(x) = a = F(x')$

But $F(y) \in F(x') \Rightarrow \exists\, \underline{y' R x'}$ s.t. $F(y') = F(y)$

Since $F(y) \in a$, the minimality of a ensures that only y has $F(y)$ as image i.e. $\underline{y' = y}$.

direct, by $\in$-min of a

Hence, $y R x \Rightarrow y = y' R x'$ i.e. $pred(C, x, R) \subseteq pred(C, x', R)$

Symmetrically $pred(C, x', R) \subseteq pred(C, x, R)$

Thus equality and by <u>extensionality</u> of R $x = x'$, contradiction

<u>Unicity of M</u>: Suppose F' and M' also satisfy the thm.

F' isom means by def $F'(y) \in F'(x) \leftrightarrow y R x$, $\forall x, y \in C$
i.e. $F'(x) = \{F'(y) : y \in C \wedge y R x\}$
The unicity of def by induction implies $F' = F$, thus $M' = M$.

Chapter 5 : Relative Consistency

Given a theory T that we assume to be consistent (typically Z, ZF^-, ZF-Inf, ZF), we would like to know if the theory remains consistent if we add to it a novel axiom φ, i.e. we would like to prove results of the form:

$$Cons(T) \longrightarrow Cons(T+\varphi)$$

Such kinds of result guarantee that assuming φ as a novel axiom is not problematic

<u>Remark</u> : $Cons(T+\varphi)$ iff $T \nvdash \neg\varphi$

$(\Leftarrow)$: $T \vdash \neg\varphi \Rightarrow T+\varphi \vdash \neg\varphi$

$\Rightarrow T+\varphi \vdash \neg\varphi \wedge \varphi$

$\Rightarrow \neg Cons(T+\varphi)$

$(\Rightarrow)$: $\neg Cons(T+\varphi) \Rightarrow$

$\exists \psi$ s.t. $T+\varphi \vdash \psi$ and $T+\varphi \vdash \neg\psi$

$\Rightarrow T+\varphi \vdash \bot \underset{\text{reductio ad absurdum}}{\Rightarrow} T \vdash \neg\varphi$

Thus, relative consistency results like

$$Cons(T) \rightarrow Cons(T+\varphi)$$

are equivalent to

$$Cons(T) \rightarrow T \nvdash \neg\varphi$$

The big advances in descriptive set theory was to prove <u>independence</u> results i.e.
We say that φ is independent from T iff

$$\text{Cons}(T) \to \text{Cons}(T+\varphi) \quad \text{and}$$

$$\text{Cons}(T) \to \text{Cons}(T+\neg\varphi)$$

i.e. $\text{Cons}(T) \to (T \nvdash \neg\varphi \text{ and } T \nvdash \varphi)$

We have the following important independence results:

$\text{Cons}(ZF) \longrightarrow \text{cons}(ZF+AC)$ (Gödel's model)

$\text{Cons}(ZF) \longrightarrow \text{cons}(ZF+\neg AC)$ (Forcing)

also

$\text{cons}(ZF) \to \text{cons}(ZF+CH)$ (Gödel's model)

$\text{cons}(ZF) \to \text{cons}(ZF+\neg CH)$ (Forcing)

Hence AC, CH, (GCH also) are independent of ZF.

In this chapter, we will prove easier relative consistency results about axioms of ZF, like

$$\text{Cons}(ZF^-) \to \underbrace{\text{Cons}(ZF-\text{Inf}+\neg\text{Inf})}_{\text{i.e. } ZF-\text{Inf} \nvdash \text{Inf}}$$

$$\text{cons}(ZF^-) \to \underbrace{\text{cons}(ZF-\text{Rempl}+\neg\text{Rempl})}_{\text{i.e. } ZF-\text{Rempl} \nvdash \text{Rempl}}$$

$$\text{cons}(ZF^-) \longleftrightarrow \text{Cons}(ZF)$$

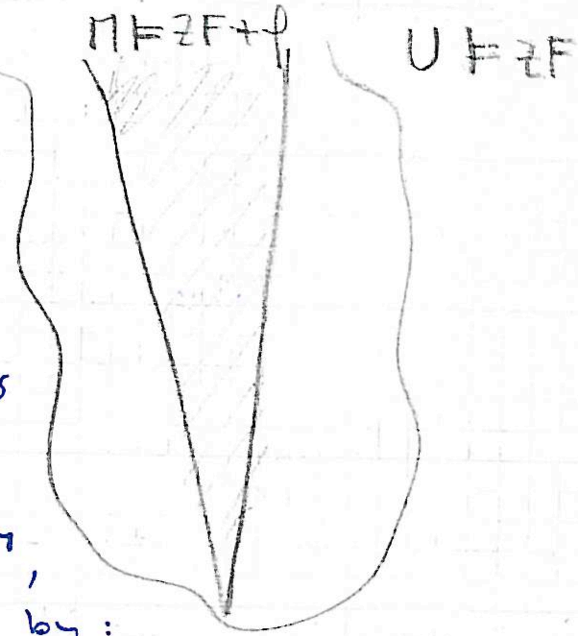
5.1. Relativization and Absoluteness

In order to prove relative consistency results, we will, given a model $\mathbf{U}$ of ZF, build another model $\mathbf{M}$ inside $\mathbf{U}$ of $ZF + \varphi$.

This would imply $\text{cons}(ZF) \rightarrow \text{cons}(ZF + \varphi)$

We need the tools of relativization and absoluteness.

We will "talk" semantically, but everything can be done syntactically.



<u>Definition</u>: Let $\mathbf{M}$ be any class and φ a formula.

The <u>relativization</u> of φ to $\mathbf{M}$, $\varphi^{\mathbf{M}}$, is defined by induction on φ by:

i) $(x = y)^{M}$ is $x = y$

ii) $(x \in y)^{M}$ is $x \in y$

iii) $(\varphi \wedge \psi)^{M}$ is $\varphi^{M} \wedge \psi^{M}$

iv) $(\neg \varphi)^{M}$ is $\neg \varphi^{M}$

→ v) $(\exists x\, \varphi)^{M}$ is $\exists x\, (x \in \mathbf{M} \wedge \varphi^{M})$

i.e. $\exists x\, (M(x) \wedge \varphi^{M})$, where $\mathbf{M}(\cdot)$ is the formula defining the class $\mathbf{M}$

<u>Definition</u>: let M be any class

a) "<u>φ is true in M</u>" (or "$M \models \varphi$") means φ^M

b) for a theory S, "S is true in M" or "<u>M is a model of S</u>" (or "$M \models S$") means that each φ in S is true in M.

<u>Remark</u>: φ^M is just one sentence, whereas "$M \models S$" means that each φ in S can be proved from the axioms we are presently using (e.g. ZF), but cannot be expressed by a single formula (c.f. Kunen p. 134, 143-146 for more details).
Thus, results involving such considerations are not technically provable from ZF or ZFC, this is a certain abuse of notation.

the following lemma will be useful

⚠ meta-result ⊛

<u>Lemma 1</u>: let S and T be two theories, and let M be a class s.t. $S \vdash (M \neq 0 \wedge "M \models T")$
Then $\text{cons}(S) \to \text{cons}(T)$

⚠ abuse of language since this is not one formula

If we can prove from S that M is a non-void model of T, then $\text{cons}(S) \to \text{cons}(T)$.

⊛ we do not argue from ZF (but from nothing)

<u>Proof</u>: Suppose S consistent and T not.
Then $T \vdash \varphi \wedge \neg\varphi$ for some (any) φ
But the hyp $S \vdash "M \models T"$ together with $T \vdash \varphi \wedge \neg \varphi$ implies $S \vdash \varphi^M \wedge \neg \varphi^M$ thus inconsistent, contrad.
Thus T consistent.

∎

(See Kunen p. 141-142 for more formal.)

In this chapter, S will usually be ZF, and T be ZF, ZF-Inf+ $\neg$I etc...

We like transitive classes because they have "good properties". We look at the validity of the axioms inside transitive classes.

Prop 2: (ZF$^-$) Let M be a transitive class

i) "$M \models$ Extensionality" (i.e. formally Ext^M holds)

ii) If $M \subseteq WF$, then "$M \models$ Foundation"

iii) "$M \models$ Pairing, Union, Power Set" iff

$\forall a, b \in M \ \exists c \in M \ (\{a,b\} \subseteq c)$,

$(\bigcup a \subseteq c)$, and $(\mathcal{P}(a) \cap M \subseteq c)$ resp.

iv) "$M \models$ Comprehension" iff for every $\varphi(x, y, \vec{z})$ one has

free variables among $x, y, \vec{z}$

$$\forall y, \vec{z} \in M \ (\{x \in y : \varphi^M(x, y, \vec{z})\} \in M)$$

v) "$M \models$ Replacement" iff for every formula

$\varphi(x, y, A, \vec{z})$ and every $A, \vec{z} \in M$ satisfying

$\forall x \in A \ \exists! y \in M \ \varphi^M(x, y, A, \vec{z})$, then

$\exists Y \in M (\{y : \exists x \in A \ \varphi^M(x, y, A, \vec{z})\} \subseteq Y)$

not necessary to say $\forall x \in A \cap M$ which would be the relativised version.

Proof: i) We have to prove that Ext^M holds.

Ext^M is $\forall x, y \in M [\forall z \in M (z \in x \leftrightarrow z \in y) \rightarrow x = y]$

Exercise

Just give hints

But for any $x, y \in M$, M transitive implies

$\forall z \in M (z \in x \leftrightarrow z \in y) \overset{\text{trivial}}{\underset{\text{by transitivity of } M}{\Longleftrightarrow}} \forall z (z \in x \leftrightarrow z \in y)$

Thus Ext^M iff $\forall x, y \in M [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y]$

which is true by Ext.

ii) Suppose $M \subseteq WF$. $(\text{Foundation})^M$ is

$$\forall x \in M \left[\exists y \in M (y \in x) \to \exists y \in M \left(y \in x \wedge \neg \exists z \in M (z \in x \wedge z \in y) \right) \right] \quad \circledast$$

take y of minimal rank in $x \cap M$, then y satisfies $\circledast$.

iii) $(\text{Pairing})^M$ is $\forall a, b \in M \, \exists c \in M \, (a \in c \wedge b \in c)$

i.e. $\forall a, b \in M \, \exists c \in M \, (\{a, b\} \subseteq c)$ (doesn't use transitivity of M)

$(\text{Union})^M$ is $\forall F \in M \, \exists A \in M \, \forall Y \in M \, \forall x \in M$

$(x \in Y \in F \to x \in A)$

trivial $\Leftarrow$ $\Rightarrow$ by transitivity of M

$\forall F \in M \, \exists A \in M \, \forall Y, x \, (x \in Y \in F \to x \in A)$

i.e. $\forall F \in M \, \exists A \in M \, (\bigcup F \subseteq A)$

prove this one as example $\longrightarrow$ $(\text{Power Set})^M$ is $\forall x \in M \, \exists y \in M \, \forall z \in M \left(\underbrace{[z \subseteq x]^M}_{\forall t \in M (t \in z \to t \in x)} \to z \in y \right)$

$\underbrace{\forall t \in M (t \in z \to t \in x)}_{\equiv \; z \cap M \subseteq x}$

$\underbrace{z \cap M}_{= z}$ since M transitive (and $z \in M$)

- $u \in z \cap M \to u \in z$
- $u \in z$: since $z \in M, u \in M \to u \in z \cap M$

iff $\forall x \in M \, \exists y \in M \, \forall z \in M \, (z \subseteq x \to z \in y)$

iff $\forall x \in M \, \exists y \in M \, \forall z \left[(z \in M \wedge z \subseteq x) \to z \in y \right]$

iff $\forall x \in M \, \exists y \left[\mathcal{P}(x) \cap M \subseteq y \right]$

iv) $(\text{Comp})^M$ is $\forall y, \vec{z} \in M \, \exists Y \in M \, \underbrace{\forall x \in M \, (x \in Y \leftrightarrow x \in y \wedge \varphi^M)}$

trivial $\Leftarrow$ $\Rightarrow$ by transitivity of M

$\forall x \, (x \in Y \leftrightarrow x \in y \wedge \varphi^M)$

iff $\forall y, \vec{z} \in M \left(\{x \in y : \varphi^M\} \in M \right)$ $\quad Y := \{x \in y : \varphi^M\}$

v) similar: the transitivity of M ensures that elements of A are in M (i.e. $x \in A \in M \to x \in M$).