6.3. ZF and $V = L$ in L

First of all, we prove (in ZF) that L is a model of ZF.

Theorem 9 (ZF): "$L \models ZF$"

Proof: Extensionality: by Chap 5 - Prop. 2 (i), one has "$L \models$ Extensionality" since L is transitive.

Comprehension: to prove that "$L \models$ Comprehension" we need to show by Chap 5 - Prop. 2 (iv) that for each $\varphi(x, z, v_1, \dots, v_n)$ whose free variables are among $x, z, v_1, \dots, v_n$, one has

$$\forall z, v_1, \dots, v_n \in L \left(\{ x \in z : \varphi^L(x, z, \vec{v}) \} \in L \right)$$

So let $z, v_1, \dots, v_n \in L$. There exists α s.t. $z, v_1, \dots, v_n \in L_\alpha$.
By the reflection thm (Chap 5, thm 11) there exists $\beta > \alpha$ s.t. φ is absolute for L_β, L, i.e.

$$\varphi^L(x, z, \vec{v}) \leftrightarrow \varphi^{L_\beta}(x, z, \vec{v}), \ \forall x \in L_\beta.$$

Moreover, for any x, if $x \in z$, then since $z \in L_\alpha \subseteq L_\beta$ and L_β transitive, one has $x \in L_\beta$.

Hence $\{ x \in z : \varphi^L(x, z, \vec{v}) \} =$ (since we have the equiv between φ^L and φ^{L_β} for the x's in L_β, and $x \in z \Rightarrow x \in L_\beta$)

$$\{ x \in z : \varphi^{L_\beta}(x, z, \vec{v}) \} =$$

$$\{ x \in L_\beta : x \in z \wedge \varphi^{L_\beta}(x, z, \vec{v}) \} =$$

$$\{ x \in L_\beta : (x \in z)^{L_\beta} \wedge \varphi^{L_\beta}(x, z, \vec{v}) \} =$$

$$\{ x \in L_\beta : (x \in z \wedge \varphi(x, z, \vec{v}))^{L_\beta} \}$$

$$\in \mathcal{D}(L_\beta) = L_{\beta+1} \subseteq L \quad \text{by Prop. 4.}$$

So $\{ x \in z : \varphi^L(x, \vec{v}) \} \in L \Rightarrow$ "$L \models$ Compr".

Replacement: to prove "$L \models$ Replacement", we need to show by Chap 5 - Prop. 2 (v) that:
for each formula $\varphi(x, y, A, w_1, \dots, w_n)$ and each $A, w_1, \dots, w_n \in L$,

if $\forall x \in A \, \exists! y \in L \; \varphi^L(x, y, A, \vec{w})$ ⊛

then $\exists Y \in L \left(\{ y : \exists x \in A \; \varphi^L(x, y, A, \vec{w}) \} \subseteq Y \right)$

So assume ⊛ and let

$$\alpha = \sup \{ \rho(y) + 1 : \exists x \in A \; \varphi^L(x, y, A, \vec{w}) \}$$

Note that this is indeed a set by Replacement, this is $f[A]$ for $f : A \to ON$, $x \mapsto \rho(y)+1$ where y s.t. $\varphi^L(x,y)$

and take $Y = L_\alpha$.

Then Y contains the required set, and by Lemma 5 (d),
$Y = L_\alpha \underset{\uparrow}{\in} L_{\alpha+1} \subseteq L$, thus $Y \in L$.

by Lemma 5 (d)

Pairing, Union, and Power Set are similarly proved.

Infinity: since $\omega \in L$, one has

$$\exists x \in L \left[0 \in x \wedge \forall y \, (y \in x \to S(y) \in x) \right]$$

which, by absoluteness of 0 and $S(\cdot)$ in L is equivalent to $(\text{Infinity})^L$, i.e. "$L \models$ Infinity".

Foundation: By Foundation, one has $V = WF$
Thus $L \subseteq V = WF$, so "$L \models$ Foundation".

■

We now introduce the axiom of constructibility $V = L$ and prove that $V = L$ is true in L. In other words, if we place ourself inside L, then our perception of the constructible universe inside L coincide with our perception of the entire universe.

Definition: The axiom of constructibility is the statement $V = L$, i.e $\forall x \exists \alpha (x \in L_\alpha)$. (which formally means $V \subseteq L$, the other inclusion being obvious).

Theorem 10 (ZF): " $L \models V = L$ "

Proof. The proof relies on the following important property:

lemma: The function L_α (i.e $\alpha \mapsto L_\alpha$) is absolute for transitive models of ZF − P.

Proof: By lemma 2 (c), the function Df is absolute. It follows by methods of Chap 5 that the function $\mathcal{D}$ is also absolute, and hence so is L_α. ▨

We show $(V=L)^L$ i.e. $\forall x \in L \, \exists \alpha \in L \, (x \in L_\alpha)^L$.

Let $x \in L$. Then $\exists \alpha \in ON \, (x \in L_\alpha)$.
Yet by lemma 6 (a), $\alpha \in L$ (since $L \supseteq ON$) and by the above lemma $(x \in L_\alpha) \leftrightarrow (x \in L_\alpha)^L$.
Thus $\forall x \in L \, \exists \alpha \in L \, (x \in L_\alpha)^L$ holds. ▨

meta-result
Corollary 11: $Cons(ZF) \rightarrow Cons(ZF + V = L)$

Proof: Thms 9 and 10 say " $L \models ZF + V = L$ ", and Chap 5 - lemma 1 leads to the conclusion ▨

→

However, the axiom $V = L$ is generally considered <u>not</u> to be a plausible axiom to add to ZF, since there is no reason to believe that all mathematical objects lie in $\mathbf{L}$.

6.4. Minimality and uniqueness properties of L

We prove that L is the least inner model of V in the sense that every other model of ZF necessarily includes L.

schema

Proposition 12: If M is a transitive proper class s.t. "$M \models ZF-P$", then $M \supseteq L$ and $L^M = L$.

the perception of L inside M is the "real" L

Proof: We first prove that $ON \subseteq M$.
Let $\alpha \in ON$. M proper class $\Rightarrow M \not\subseteq V_\alpha$
$\Rightarrow \exists x \in M$ s.t. $rank(x) \geqslant \alpha$
By absoluteness of the rank (can be proved properly), has $\alpha \leqslant rank(x) \underset{\text{by abs}}{=} rank^M(x) \underset{\text{by def}}{\in} M$, thus $\alpha \in M$ by transitivity of M. Hence $ON \subseteq M$.

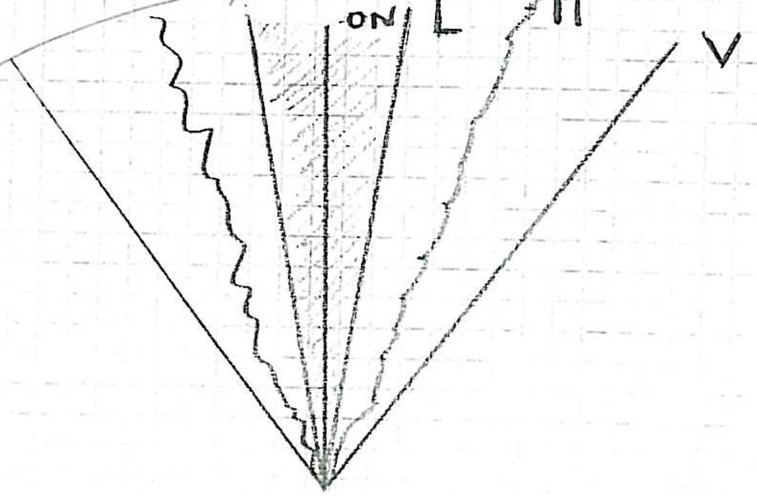
i.e. $\forall x, \alpha \in M\, [(\alpha = rank(x))^M \leftrightarrow \alpha = rank(x)]$

Now by absoluteness of L_α (sub-lemma thm 10) and ON, one has:

$$L^M = \bigcup_{\alpha \in ON^M} L_\alpha^M = \bigcup_{\alpha \in ON} L_\alpha = L,$$

and thus $L = L^M \underset{\text{by def}}{\subseteq} M$. ∎

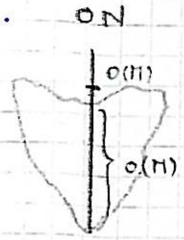
$L^M = \{x \in M : [\exists \alpha \in ON\ (x \in L_\alpha)]^M\}$



We have the analogous of Prop. 12 for transitive set models of $ZF-P$.

<u>Definition</u>: For any M, let $o(M) = M \cap ON$.

If M transitive set, $o(M)$ is the first ordinal not in M.
if $o(M) \in M$, then $o(M) \in M \cap ON =_{def} o(M)$ contradiction.



schema

<u>Proposition 13</u>: If M is a transitive set s.t. "$M \models ZF-P$", then $M \supseteq L_{o(M)}$ and $L^M = L_{o(M)}$

M cannot be thinner than $L_{o(M)}$

the perception of L inside M is $L_{o(M)}$

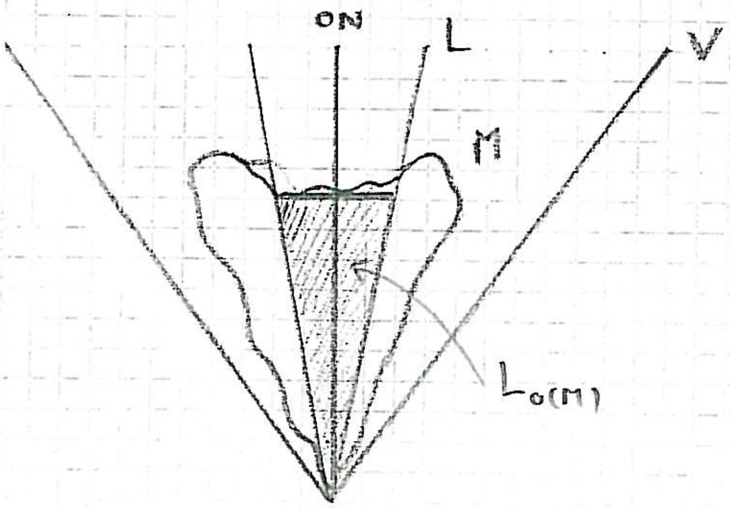
<u>Proof</u>: Note that $o(M)$ is limit, otherwise $o(M) = \alpha + 1$ first ordinal not in M
$\Rightarrow \alpha \in M \Rightarrow$ (since "$M \models ZF-P$")
$\alpha + 1 = \alpha \cup \{\alpha\} \in M$, contradiction.

Then, one has by abs. of L_α and ON

$$L^M = \bigcup_{\alpha \in ON^M} L_\alpha^M = \bigcup_{\alpha \in M} L_\alpha = L_{o(M)}$$ since $o(M)$ is limit

and thus $L_{o(M)} = L^M \subseteq M$ by def. ▨

$$L^M = \{x \in M : [\exists \alpha \in ON (x \in L_\alpha)]^M\}$$



M cannot be thinner than $L_{o(M)}$

L is the unique transitive model of ZF that satisfies $V = L$ in the following sense:

schema

Proposition 14:

(a) If M is a transitive proper class s.t. "$M \models ZF - P + V = L$", then $M = L$

(b) If M is a transitive set s.t. "$M \models ZF - P + V = L$", then $M = L_{o(M)}$.

Proof: If "$M \models V = L$", then "$M \models V \subseteq L$" i.e. $(V \subseteq L)^M = (\forall x (x \in L))^M = \forall x \in M (x \in L^M) =$ $M \subseteq L^M$ holds. Conversely, $L^M \subseteq M$ obvious. Thus $M = L^M$ ⊛

Moreover, if M transitive s.t. "$M \models ZF - P$", then:

- M proper class $\Rightarrow M \overset{⊛}{=} L^M = L$ by Prop. 12.
- M set $\Rightarrow M \overset{⊛}{=} L^M = L_{o(M)}$ by Prop. 13.

∎

Remark: Formally, the conclusions of Propositions 12, 13, 14 (the "then" part) do not need the whole assumption that "$M \models ZF - P + V = L$ to be achieved, but only that "$M \models \varphi$" for some finite conjuction φ of axioms of $ZF - P + V = L$.

the axioms in φ are the ones needed to prove absoluteness of rank, L_α, etc.

Accordingly, Propositions 12, 13, 14 can be reformulated as follows: there exists φ conj of axioms of $ZF + P$ s.t.:

- If M transitive pr. cl. s.t. "$M \models \varphi$", then $M \supseteq L$ and $L^M = L$.
- If M transitive set s.t. "$M \models \varphi$", then $M \supseteq L_{o(M)}$ and $L^M = L_{o(M)}$.
- If M transitive pr. cl. s.t. "$M \models \varphi + V = L$", then $M = L$.
- If M transitive set s.t. "$M \models \varphi + V = L$", then $M = L_{o(M)}$.

→

In this sense, Propositions 12, 13, 14's reformulations are propositions schemas that are really formilazable in ZF, for their statements do not involve some kind of infinitary formula of set theory of the form "$M \models ZF - P + V = L$".

In other words, for Propositions 12, 13, 14 to hold, we don't need to assume the unformalizable statement that "$M \models ZF - P + V = L$," but only need to assume that M satisfies a sufficiently large finite fragment of $ZF - P + V = L$.