Chapter 1: First Axioms

1.1. Introduction

Set theory provides a foundational approach to general mathematics: i.e. it is a theory where all mathematical concepts (function, real numbers, topological spaces, ...) are defined in terms of the primitive notions of sets and membership relation ($\in$).

In axiomatic set theory, we formulate a few simple axioms in an attempt to capture the basic set-theoretic principles that we consider to be "obviously true" (this is subjective of course).

From such axioms, all known mathematics can be derived. However, there are some questions which the axioms fail to settle, like the continuum hypothesis for instance.

Speak a bit more in details about the continuum hypothesis:

Gödel (1940): There exists a model M s.t.

$$M \models ZFC + \underset{AC}{CH} \quad \text{(choice)}$$

Hence $ZFC \nvdash \underset{\neg AC}{\neg CH}$

Cohen (1963): There exists a model M s.t.

$$M \models ZFC + \underset{\neg AC}{\neg CH}$$

Hence $ZFC \nvdash \underset{AC}{CH}$

independence of CH (AC) from ZFC

The axiomatic system ZFC that we are going to present provides a relevant way to capture the set-theoretic principles that considered to be "obviously true", and most of all, this system permits to derive all of current mathematical principles.

For example, since Fermat's thm is true in "conventional mathematics", then it holds that ZFC $\vdash$ Fermat's thm.

The axiomatic system ZFC will be stated in first-order logic predicate calculus with only $=$ and $\in$ as binary relation symbols.

With this formal logical approach, one has the following advantages:

- precise formal language to state the axioms
- provides a rigorous definition of the notion of "property"
- rigorous definition of the notion of "formal deduction".

1.2. The philosophy of mathematics.

Platonist: believes that the set-theoretic universe has an existence outside ourselves, i.e. outside the sensible world, existence which lies in the intelligible world. In this intelligible world, CH is either true or false. However, our distorted sensible perception of the intelligible word missed out some (non-recursive) set of axioms from which one could derive this truth or falsity. Hence, according to our distorted sensible perception, CH remains undecidable for us.

Finitist: believes only in finite objects. This point of view discards much modern mathematics.

Formalist: believes that everything we are doing in set theory is only juggling with syntax, finite sequences of symbols. When challenged about the validity of infinite objects, he replies that all he is really doing is juggling with finite sequences of symbols.

We will develop set theory from a platonist point of view.

1.3. First axioms

Axiom 0: set existence : says that the universe is non-void

$$\exists x\ (x = x)$$

Axiom 1: Extensionality : a set is fully determined by its members

$$\forall x \forall y \left[\forall z \left(z \in x \leftrightarrow z \in y \right) \longrightarrow x = y \right]$$

Axiom 3: Comprehension scheme

idea: given some property $P(x)$ of x we would like to formalize $\{x : P(x)\}$.

It is tempting to set the axiom sheme:

$$\left[\begin{array}{l} \text{for any formula } \varphi(x) \text{ with } x \text{ free} \\ \exists y \ \ \forall x \left(x \in y \leftrightarrow \varphi(x) \right) \end{array} \right.$$

This set y will thus be by def $\{x : \varphi(x)\}$.

Unfortunately, if we take $\varphi(x)$ as $x \notin x$ we would have $\exists y \ \forall x \left(x \in y \leftrightarrow x \notin x \right)$.

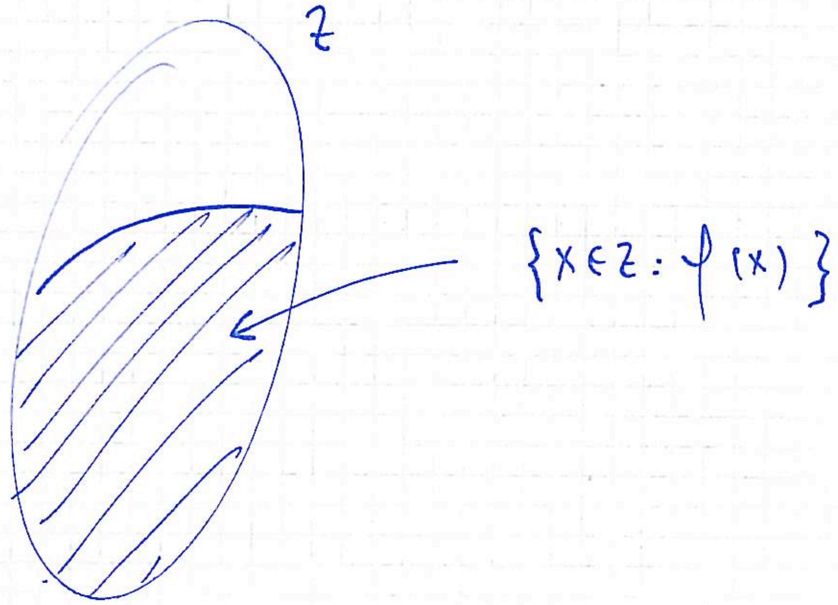
But for $x = y$, one has $\left(y \in y \leftrightarrow y \notin y \right)$, contradiction

This is the Russel Paradox

So instead we consider an axiom scheme which allows to build $\{x \in z : \varphi(x)\}$ for any property $\varphi(x)$ and set z, i.e.

For each φ with free variables among $x, z, w_1, -, w_n$ (but y not free)

$$\forall z \forall w_1 \ldots \forall w_n \; \exists y \; \forall x \left[x \in y \Longleftrightarrow (x \in z \wedge \varphi(x, z, w_1, -, w_n)) \right]$$



Remark: By Ax 0, some set z exists.
By Ax 3, we can form $\{x \in z : x \neq x\}$
This set has no member.
By Ax 1, it is unique, denoted by 0 (abbreviation)
(or sometimes $\emptyset$).

clarification →

There is no universal set (the collection of all sets is not a set!)

Thm 1: $\neg \exists z \, \forall x \, (x \in z)$

Proof: By contradiction, suppose there is a universal set z.
Then by Ax.3, we can form the set
$y = \{x \in z : x \notin x\} = \{x : x \notin x\}$
Thus $(y \in y \leftrightarrow y \notin y)$, contradiction ∎

Axiom 4: Pairing
$$\forall x \, \forall y \, \exists z \, (x \in z \wedge y \in z)$$

By Ax 4, for any x, y, there exists z s.t. $x \in z \wedge y \in z$
By Ax. 3, we can form $\{v \in z : v = x \vee v = y\}$
By Ax. 1, this set is unique, denoted $\{x, y\}$. (abbreviation again.)

Hence, for any x, we can form $\{x, x\}$, which by Ax 1. is equal to $\{x\}$.

Given x and y, we can form $\{x\}$ and $\{x, y\}$, and then also $\{\{x\}, \{x, y\}\} := \langle x, y \rangle$ called the <u>ordered pair</u> of x and y

The remark means precisely:

$$Ax.0, Ax.1, Ax.3 \vdash \left[\exists z \; \forall x(\neg x \in z) \wedge \forall z' \left(\forall x(\neg x \in z') \rightarrow z' = z\right)\right]$$

existence of the empty set

uniqueness of the empty set.

We introduce the abbreviation 0 (i.e. extension by definition) to denote the empty set, but formally it would be possible to do (almost) everything without any abbreviation, just in the initial language.

lemma 2: $\forall x \forall y \forall x' \forall y' \left[\langle x, y \rangle = \langle x', y' \rangle \longleftrightarrow (x = x' \wedge y = y') \right].$

can be written without abbreviation "$\langle \cdot, \cdot \rangle$"

Proof: "$\leftarrow$" If $x = x'$ and $y = y'$, then by Ax 1, $\{x\} = \{x'\}$ and $\{x, y\} = \{x', y'\}$.
Thus $\langle x, y \rangle = \langle x', y' \rangle$

exercice

"$\rightarrow$" Suppose $\langle x, y \rangle = \langle x', y' \rangle$

- Case 1: if $x = y$

Then $\langle x, y \rangle = \langle x, x \rangle = \langle x \rangle = \{\{x\}\} \overset{hyp}{=} \langle x', y' \rangle$

$\Rightarrow \{x\} = \{x'\} = \{x', y'\} \Rightarrow x = x' = y'$

- Case 2: if $x \neq y$

We have by hyp $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$ ⊛

- If $\{x\} = \{x', y'\} \Rightarrow y' = x'$ and ⊛ becomes $\{\{x\}, \{x, y\}\} = \{\{x'\}\}$
$\Rightarrow \{x, y\} = \{x'\} \Rightarrow x = y = x'$, contradiction.

Thus $\{x\} \neq \{x', y'\} \Rightarrow \{x\} = \{x'\} \Rightarrow \underline{x = x'}$.

- If $\{x, y\} = \{x'\} \Rightarrow x = y = x'$, contradiction

Thus $\{x, y\} \neq \{x'\} \Rightarrow \{x, y\} = \{x', y'\}$

- If $y = x'$, since $x = x' \Rightarrow y = x' = x$, contrad.

Thus $\underline{y = y'}$

▨

Axiom 5: Union : for every set $\mathcal{F}$, there a set which contains the elements of elements of $\mathcal{F}$.

$$\forall \mathcal{F}\, \exists U\, \forall y\, \forall x \left[(x \in y \wedge y \in \mathcal{F}) \to x \in U \right]$$

Given a (family of) set(s) $\mathcal{F}$, by Ax.3 and 1, we can form :

$$\bigcup \mathcal{F} := \{ x \in U : \exists y \in \mathcal{F}\, (x \in y) \}$$

let B be any el. of $\mathcal{F}$, we let

$$\bigcap \mathcal{F} := \{ x \in B : \forall y \in \mathcal{F}\, (x \in y) \} \text{ (for any } B \in \mathcal{F})$$
$$= \{ x : \forall y \in \mathcal{F}\, (x \in y) \}$$

Example: $\bigcup \{ \{a,b\}, \{a\}, \{b,a,d\} \} = \{a,b,d\}$

$\bigcap \{ \{a,b\}, \{a\}, \{b,a,d\} \} = \{a\}$

We set $A \cup B := \bigcup \{A, B\}$ and

$A \cap B := \bigcap \{A, B\}$

Axiom 6: Replacement scheme.

For any relation $\varphi(x,y)$ which is functional on some domain X, there is a set which contains the "image of X by φ".

For any φ without Y free (with free var. among $x, y, w_1, \dots, w_n$)

$$\forall X\, \forall w_1 \dots \forall w_n \left[\forall x \in X\, \exists!\, y\, \varphi(x,y,w_1,\dots,w_n) \longrightarrow \exists Y\, \forall x \in X\, \exists y \in Y\, \varphi(x,y,w_1,\dots,w_n) \right]$$

By Ax 1, 3, 6, we can form $\{ y \in Y : \exists x \in X\, \varphi(x,y) \}$

We want to define the <u>cartesian product</u>

$A \times B = \{ \langle x, y \rangle : x \in A \wedge y \in B \}$

We use Replacement twice:

functional total on A

First: Fix y in B, then $\forall x \in A \, \exists! z \, (z = \langle x, y \rangle)$

Thus by Ax. 6, we form

image: $\text{prod}(A, y) = \{ z : \exists x \in A \, (z = \langle x, y \rangle) \}$ set of ordered pairs whose second component is y

functional total on B

Second: $\forall y \in B \, \exists! z \, (z = \text{prod}(A, y))$

Thus by Ax. 6, we form

image: $\text{prod}'(A, B) = \{ z : \exists y \in B \, (z = \text{prod}(A, y)) \}$ set of set of ordered pairs whose second components are some y in B

Finally, we let $A \times B = \bigcup \text{prod}'(A, B)$

take the elements of elements of...

Definition: • A <u>relation</u> is a set of ordered pairs R

$\text{dom}(R) = \{ x \in \bigcup\bigcup R : \exists y \, (\langle x, y \rangle \in R) \}$

$\text{ran}(R) = \{ y \in \bigcup\bigcup R : \exists x \, (\langle x, y \rangle \in R) \}$

(legitimate by Ax. 3 since $\text{dom}(R)$ and $\text{ran}(R) \subseteq \bigcup\bigcup R$)

$R^{-1} = \{ \langle y, x \rangle \in \text{ran}(R) \times \text{dom}(R) : \langle x, y \rangle \in R \}$

• A <u>function</u> f is a relation s.t.

$\forall x \in \text{dom}(f) \, \exists! y \in \text{ran}(f) \, (\langle x, y \rangle \in f)$

$f : A \to B$ means $f \subseteq A \times B$ relation, $\text{dom}(f) = A$ and $\text{ran}(f) \subseteq B$

If $x \in A$, $f(x)$ means the unique y s.t. $\langle x, y \rangle \in f$.
If $C \subseteq A$, $f''C = \{ f(x) : x \in C \}$

$f : A \to B$ is <u>injective</u> iff f^{-1} is a function.
$f : A \to B$ is <u>surjective</u> iff $\mathrm{ran}(f) = B$
f is <u>bijective</u> iff injective and surjective.