Remark: For comprehension axiom to hold in $M$, it suffices that $\forall z \in M \left(\mathcal{P}(z) \subseteq M\right)$,
(Indeed $\{x \in z : \varphi^M\} \in \mathcal{P}(z) \to \{x \in z : \varphi^M\} \in M$ )

Corollary 3: "$WF \models ZF - Inf$"
(ZF$^-$) "$V_\omega \models ZF - Inf$"

série 8 exo 3.

But what about the axiom of Infinity?
We can easily anticipate that it will be satisfied in WF but not satisfied in $V_\omega$.

Inf: $\exists x \left[ 0 \in x \wedge \forall y \in x (S(y) \in x) \right]$

But this would require to rewrite the axiom without the successor symbol and then see if the relativised version of this obtained formula is satisfied in the classes that we consider.

In order to do this in a more general way, we introduce a new tool: absoluteness.

Proof: WF and $V_\omega$ are transitive and all the conditions of Prop 2 are satisfied.

For replacement in $V_\omega$, let $\varphi(x, y, A, \vec{z})$, and $A, \vec{z} \in V_\omega$ s.t. $\forall x \in A \ \exists! y \in V_\omega \ \varphi^{V_\omega}(x, y, A, \vec{z})$. Then $Y = \{y : \exists x \in A \ \varphi^M\}$ exists by Rempl. and $\subseteq V_\omega$

But $A$ finite and "$Y = \varphi^M(A)$" implies $Y$ finite thus $\subseteq V_n$ for some $n$, thus $\in V_{n+1} \subseteq V_\omega$

<u>Definition:</u> let $\varphi(x_1, \dots, x_n)$ be a formula with at most $x_1, \dots, x_n$ free, and $M, N$ be classes s.t. $M \subseteq N$, then:

a) $\varphi$ is <u>up-absolute</u> for $M, N$ iff
$$\forall x_1, \dots, x_n \in M \left[ \varphi^M(x_1, \dots, x_n) \rightarrow \varphi^N(x_1, \dots, x_n) \right]$$

b) $\varphi$ is <u>down-absolute</u> for $M, N$ iff
$$\forall x_1, \dots, x_n \in M \left[ \varphi^N(x_1, \dots, x_n) \longrightarrow \varphi^M(x_1, \dots, x_n) \right]$$

c) $\varphi$ is <u>absolute</u> for $M, N$ iff
$$\forall x_1, \dots, x_n \in M \left[ \varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi^N(x_1, \dots, x_n) \right]$$

<u>Example:</u> let $M = \{ 0, \overbrace{\{\{0\}\}}^{a} \} = \{0; a\}$. Then $\varphi(x, y) = x \subseteq y = \forall z (z \in x \rightarrow z \in y)$ is not absolute for $M$.

Indeed, $\varphi^M(a, 0) = \forall z \in M (\underbrace{z \in a}_{\text{always false: no } z \in M \text{ is in } a} \rightarrow z \in 0)$ holds, whereas $\varphi(a, 0) = \forall z (z \in a \rightarrow z \in 0)$ doesn't hold.

We will see that specific kinds of formulas are always absolute for transitive classes.

d) $\varphi$ is <u>absolute</u> for $M$ iff
$$\forall x_1, \dots, x_n \in M \left[ \varphi^M(x_1, \dots, x_n) \Leftrightarrow \varphi(x_1, \dots, x_n) \right]$$

<u>Convention:</u> p. 5'

Definition: The $\Delta_0$-formulas are defined by induction as follows:

i) $x \in y$ and $x = y$ are $\Delta_0$

ii) if $\varphi$ is $\Delta_0$, $\neg\varphi$ is $\Delta_0$

iii) if $\varphi, \psi$ are $\Delta_0$, then $\varphi \wedge \psi$ is $\Delta_0$

v) if $\varphi$ is $\Delta_0$, then $\exists x \in y\ \varphi$ is $\Delta_0$

bounded quantifications

i.e. $\exists x\,(x \in y \wedge \varphi)$ is $\Delta_0$

The formula "$x \subseteq y$" will be absolute for transitive classes.

But "$x \subseteq y$" is $\forall z\,(z \in x \rightarrow z \in y)$ i.e. $\neg\exists z\,\neg(z \in x \rightarrow z \in y)$ which is not $\Delta_0$, but is logically equivalent to the $\Delta_0$-formula $\neg\exists z \in x\,(\neg z \in y)$. This motivates the following def:

If $T$ is a theory, the formula $\varphi(x_1, \dots, x_n)$ is said to be $\Delta_0^T$ iff there exists $\varphi'(x_1, \dots, x_n) \in \Delta_0$ s.t. $T \vdash \forall x_1 \dots x_n\,\big(\varphi(x_1, \dots, x_n) \leftrightarrow \varphi'(x_1, \dots, x_n)\big)$.

before def

Convention: If $F(x_1, \dots, x_n)$ is a defined operation on $x_1, \dots, x_n$ (e.g. $F(x) = \bigcup x$ or $F(x) = S(x)$), we say that $F(x_1, \dots, x_n)$ is absolute for $M$ iff the formula $y = F(x_1, \dots, x_n)$ is.

See Kunen p. 142 for more formal treatment

Example: $\bigcup x$ is abs for $M$ means $\forall x, y \in M\left([y = \bigcup x]^M \leftrightarrow y = \bigcup x\right)$

**Lemma 4 (ZF):** Let $T$ be a theory $\subseteq ZF$ and let $M$ be a transitive model of $T$. Then any $\varphi \in \Delta_0^T$ is absolute for $M$.

we mention this just for clarifying the proof

Proof: i) We first prove that all $\Delta_0$-formulas are absolute for $M$ transitive, by induction on $\varphi \in \Delta_0$

the cases of $\varphi$ atomic or $\varphi = \neg\varphi'$ or $\varphi = \varphi' \wedge \varphi''$ are easy

- If $\varphi$ is quantifier free, then $\varphi^M = \varphi$, thus $\forall \vec{x} \, (\varphi(\vec{x}) \leftrightarrow \varphi^M(\vec{x}))$ holds, i.e. $\varphi$ abs. for $M$.
- The case of logical connectives $\neg$ and $\wedge$ is easy.
- suppose $\varphi(y,\vec{z}) = \exists x \in y \, \varphi'(x,y,\vec{z})$, with $\varphi'(x,y,z)$ abs. for $M$; then $\forall y, \vec{z} \in M$

$$\varphi^M(y,\vec{z}) = \left[\exists x \left(x \in y \wedge \varphi'(y,\vec{z})\right)\right]^M = \exists x \in M \left(x \in y \wedge \varphi'^M(y,\vec{z})\right)$$

$$\leftrightarrow \exists x \left(x \in y \wedge \varphi'^M(y,\vec{z})\right) \quad \text{by transitivity of } M \;\; x \in y \in M \to x \in M$$

$$\leftrightarrow \exists x \left(x \in y \wedge \varphi'(y,\vec{z})\right) \quad \text{since } \varphi' \text{ is } M\text{-abs (I.H.)}$$

Thus $\underset{= \varphi(y,\vec{z})}{\exists x \in y \, \varphi'}$ is absolute for $M$.

ii) - $\varphi \in \Delta_0^T \Rightarrow \exists \psi \in \Delta_0$ s.t. $T \vdash \forall \vec{x} \, (\varphi \leftrightarrow \psi)$

But "$M \models T$" $\Rightarrow$ "$M \models \forall \vec{x} \, (\varphi \leftrightarrow \psi)$" i.e.

$[\forall \vec{x} \, (\varphi \leftrightarrow \psi)]^M$ i.e. $\forall \vec{x} \in M \, (\varphi^M \leftrightarrow \psi^M)$ holds ①

- $T \subseteq ZF \Rightarrow ZF \vdash \forall \vec{x} \, (\varphi \leftrightarrow \psi)$ also holds ②
- $\psi \in \Delta_0$ and $M$ transitive $\overset{(i)}{\Rightarrow} \forall \vec{x} \, (\psi^M \leftrightarrow \psi)$ ③

Thus we have, $\forall \vec{x} \in M$

$$\varphi^M(\vec{x}) \overset{①}{\Longleftrightarrow} \psi^M(\vec{x}) \overset{③}{\Longleftrightarrow} \psi(\vec{x}) \overset{②}{\Longleftrightarrow} \varphi(\vec{x})$$

i.e. $\varphi$ is absolute for $M$. ∎

<u>Lemma 5</u>: Absolute notions are closed under composition i.e. if $\varphi(\vec{x})$, $F(\vec{x})$ and $G_i(\vec{y})$ are all absolute for $M$, then so are formula

$$\varphi\left(G_1(\vec{y}), \ldots, G_n(\vec{y})\right) \text{ and function}$$
$$F\left(G_1(\vec{y}), \ldots, G_m(\vec{y})\right)$$

<u>Proof</u>: Case $n = m = 1$. Let $y \in M$, then

since $G^M(y) = G(y)$ and $\varphi$ abs for $M$

$$\left[\varphi(G(y))\right]^M = \varphi^M\left(G^M(y)\right) \longleftrightarrow \varphi\left(G(y)\right)$$

$$\left[F(G(y))\right]^M = F^M\left(G^M(y)\right) = F^M\left(G(y)\right)$$
$$= F\left(G(y)\right) \quad \blacksquare$$

**Prop 6** : The following relations and functions are absolute for any transitive model $M$ of $ZF^- - P - Inf$:

By Lemma 4
$$\begin{cases} x \in y, \quad x = y, \quad x \subseteq y, \\ z = \{x, y\}, \quad z = \{x\}, \quad z = \langle x, y \rangle, \\ z = 0, \quad x \cup y, \quad x \cap y, \\ x \setminus y, \quad z = S(x), \quad x \text{ is transitive}, \\ z = \bigcup x, \quad z = \bigcap x \ (\text{with } \bigcap 0 = 0) \end{cases}$$

By Lemmas 4 and 5
$$\begin{cases} z \text{ is an ordered pair} \\ z = A \times B \quad \text{i.e. } z = A \times B \\ R \text{ is a relation} \\ z = \mathrm{dom}(R) \\ z = \mathrm{ran}(R) \\ f \text{ is a function} \\ f \text{ is a 1-1 function} \\ f \text{ is a bijection} \end{cases}$$

Every time we consider a defined operation $F$ (e.g. $F(x) = S(x)$), abs. of $F(\vec{x})$ means abs of the formula $z = F(\vec{x})$.

**Proof:** We prove that each such relation and function is $\Delta_0^{ZF^- - P - Inf}$.

ex: $z = S(x) = x \cup \{x\} \iff [x \in z \wedge x \subseteq z \wedge \forall w \in z \, (w = x \vee w \in x)] \in \Delta_0$

ex: $z$ is an ordered pair $\iff \exists x \in \bigcup z \, \exists y \in \bigcup z \, (z = \langle x, y \rangle)$

i.e. $\varphi(G_1(z), G_2(z), G_3(z))$, where

$G_1(z) = G_2(z) = \bigcup z$ absolute,

$G_3(z) = z$, and $\varphi(a, b, c)$ is

$\exists x \in a \, \exists y \in b \, (c = \langle x, y \rangle)$ which is absolute (since $c = \langle x, y \rangle$ abs.)

# More absoluteness... (Facultative) ~~skip~~

<u>Definition</u>: • A formula $\varphi$ (over $\{\neg, \wedge, \exists\}$) is $\Sigma_1$ if $\varphi$ is logically equivalent to $\exists x_1 \dots \exists x_n \psi$, for some $n > 0$ and $\psi \in \Delta_0$

voir la def de Debray

• $\varphi$ is $\Pi_1$ if $\varphi \equiv \neg \psi$ with $\psi \in \Sigma_1$ (logically equivalent)

• $\varphi$ is $\Delta_1$ if $\varphi \in \Sigma_1 \cap \Pi_1$

Moreover, as before, $\varphi(\vec{x}) \in \Sigma_1^T$ (resp. $\Pi_1^T$, $\Delta_1^T$) iff there exists $\varphi'(\vec{x}) \in \Sigma_1$ (resp. $\Pi_1$, $\Delta_1$) s.t. $T \vdash \forall \vec{x} \left( \varphi(\vec{x}) \leftrightarrow \varphi'(\vec{x}) \right)$.

<u>Lemma 7 (ZF)</u>: a) Let $M$ be a transitive class, then:

mentioned just for clarity of the proof

$\varphi \in \Sigma_1 \Rightarrow \varphi$ is up-abs for $M$

$\varphi \in \Pi_1 \Rightarrow \varphi$ is down-abs for $M$

b) If $T \subseteq ZF$ and $M$ transitive model of $T$, then any $\varphi \in \Delta_1^T$ is absolute for $M$.

<u>Proof</u>: a) By induction on the number $k$ of unbounded quantifiers $\exists$, and for a given $k$, by induction on the length of the formula

exercice

- If $k=0$, then $\varphi \in \Delta_0$ and by L.4 i), $\varphi$ is abs. for $M$
- If $\varphi(\vec{x}) = \exists y\, \psi(\vec{x}, y)$ with $k-1$ "$\exists$"s in $\psi$
  let $\vec{a} \in M$ and suppose "$M \models \exists y\, M(\vec{a}, y)$", i.e.
  $\exists y \in M\, \psi^M(\vec{a}, y)$. Then $\psi^M(\vec{a}, y \to b)$ for some $b \in M$
  $\Rightarrow \psi(\vec{a}, b)$ by up-abs of $\psi$ (I.H.)
  $\Rightarrow \exists y \in M\, \psi(\vec{a}, y) \Rightarrow \exists y\, \psi(\vec{a}, y)$.

$\Pi_1$ case is similar.

ii) Similar to L.4.
let $M$ be transitive model of $T \subseteq ZF$ and
$\varphi \in \Delta_1$.
let $\vec{x}' \in M$, then

$$\varphi^M(\vec{x}') \Longleftrightarrow \underbrace{\psi^M(\vec{x}')}_{\in \Sigma_1} \xrightarrow{i} \psi(\vec{x}') \Longleftrightarrow \varphi(\vec{x}') \quad \text{and}$$

$$\varphi(\vec{x}') \Longleftrightarrow \underbrace{\psi(\vec{x}')}_{\in \Pi_1} \xrightarrow{i} \psi^M(\vec{x}') \Longleftrightarrow \varphi^M(\vec{x}')$$

Thus $\forall \vec{x}' \in M \left( \varphi^M(\vec{x}') \leftrightarrow \varphi(\vec{x}') \right)$ i.e. $\varphi$ is abs for $M$.

∎

As a consequence, we have the following useful property.

Lemma 8 (ZF⁻): If $M$ is a transitive model of $ZF^- - P - Inf$, then "$R$ well-orders $A$" is down-absolute for $M$.

Proof: let $\phi(A,R)$ be "$R$ well-orders $A$".

Then $\phi(A,R) =$ "$R$ totally orders $A$"

$$\wedge \ \forall X \left[ (X \subseteq A \wedge X \neq 0) \rightarrow \exists y \in X \ \forall z \in X \ (\langle z, y \rangle \notin R) \right]$$

$$\in \Pi_1^{ZF^- - P - Inf} \quad (\text{by Prop 6})$$

Thus $\phi(A,R)$ down-abs for $M$ by L. 7. ∎

This means that we have in $ZF^-$:

$$\forall A \forall R \left( \text{"}R \text{ well-orders } A\text{"} \longrightarrow \left[ \text{"}R \text{ well-orders } A\text{"} \right]^M \right)$$

In fact, "$R$ well-orders $A$" is absolute for any transitive $M$ model of $ZF - P$ (so we need a bit more axioms), c.f. Kunen chap. IV, thm 5.4.

Without using L.7

$$\phi(A,R) = \varphi_1(A,R) \wedge \forall X \ \varphi_2(X,A,R) \quad \text{with } \varphi_1, \varphi_2 \in \Delta_0^{ZF^- - P - Inf}$$

Let $A, R \in M$. Hence $\phi(A,R) := \varphi_1(A,R) \wedge \forall X \ \varphi_2(X,A,R)$

implies $\varphi_1(A,R) \wedge \underline{\forall X \in M} \ \varphi_2(X,A,R)$

which is equiv to $\varphi_1^M(A,R) \wedge \forall X \in M \ \varphi_2^M(A,X,R)$

i.e. $\phi^M(A,R)$.

## 5.2. Basic Relative Consistency Results

Using the tool of absoluteness, we can now deduce our relative consistency results.

**Theorem 9:** (important)

recall that this means: for all $\varphi \in ZF - Inf + \neg Inf$, one has $ZF^- \vdash$ "$V_\omega \models \varphi$"

i) $(ZF^-)$ "$V_\omega \models ZF - Inf + \neg Inf$"

ii) $(ZF^-)$ "$V_\omega \models ZFC - Inf + \neg Inf$"

iii) $(ZF^-)$ "$WF \models ZF$"

iv) $(ZFC^-)$ "$WF \models ZFC$"

v) $(ZF^-)$ "$V_{\omega+\omega} \models ZF - Rempl + \neg Rempl$"

vi) $(ZFC^-)$ "$V_{\omega+\omega} \models ZFC - Rempl + \neg Rempl$"

Proof: i) By Cor 3, $V_\omega \models ZF - Inf$.

Suppose "$V_\omega \models Inf$" i.e.

$\exists x \in V_\omega \left[ 0^{V_\omega} \in x \wedge \forall y \in x \, (S^{V_\omega}(y) \in x) \right]$ holds

By abs of $0$ and $S(\cdot)$ in $V_\omega$, one has

$\exists x \in V_\omega \left[ 0 \in x \wedge \forall y \in x \, (S(y) \in x) \right]$

i.e. $x \supseteq \omega$, thus $\mathrm{rank}(x) \geq \omega$,

contradiction with $x \in V_\omega$

Thus $(\neg Inf)^{V_\omega}$ holds

ii) We need to prove that $(AC)^{V_\omega}$ holds, i.e.

$\forall A \in V_\omega \exists R \in V_\omega \, [R \text{ well-orders } A]^{V_\omega}$

Let $A \in V_\omega$. $A$ finite $\overset{\text{without AC}}{\Longrightarrow} \exists R \subseteq A \times A \; (R \text{ w-o } A)$

But $A \in V_\omega$ and $R \subseteq A \times A \Rightarrow R \in V_\omega$.

By L. 8, $(R \text{ w-o } A) \to (R \text{ w-o } A)^{V_\omega}$.

Thus $\forall A \in V_\omega \exists R \in V_\omega (R \text{ w-o } A)^{V_\omega}$ i.e. $(AC)^{V_\omega}$

iii) By Cor 3, "$WF \models ZF - Inf$".

We show that $(Inf)^{WF}$ holds.

Since $\omega \in WF$, $\exists x \in WF \left[ 0 \in x \wedge \forall y (y \in x \to S(y) \in x) \right]$

Thus, by abs. of $0$ and $S$ in $WF$

$\exists x \in WF \left[ 0^{WF} \in x \wedge \forall y (y \in x \to S^{WF}(y) \in x) \right]$

i.e. $(Inf)^{WF}$ holds.

(iv) We are in $ZFC^-$ and need to show $(AC)^{WF}$

i.e. $\forall A \in WF \exists R \in WF \, (R \text{ w-o } A)^{WF}$.

Let $A \in WF$. By AC, $\exists R \subseteq A \times A \; (R \text{ w-o } A)$

$R \subseteq A \times A \Rightarrow R \in WF$

By Lemma 8, $(R \text{ w-o } A) \to (R \text{ w-o } A)^{WF}$

Thus $(AC)^{WF}$ holds.

v) and vi) It is clear that "$V_{\omega+\omega} \models ZFC - Rempl$".

We need to prove that "$V_{\omega+\omega} \models \neg Rempl$".

$\longrightarrow$

let $F(n,\alpha)$ be the formula

$$\exists f \, (\underbrace{n\in\omega \wedge \alpha \in ON \wedge \text{"}f \text{ isom}: \omega+n \to \overset{=\omega+n}{\alpha}\text{"}}_{G(n,\alpha,f)})$$

One can show that $G(n,\alpha,f)$ is absolute for $V_{\omega+\omega}$ i.e. $\forall n,\alpha,f\in V_{\omega+\omega}\left(G(n,\alpha,f) \leftrightarrow G(n,\alpha,f)^{V_{\omega+\omega}}\right)$

Since $\omega+n\in V_{\omega+\omega}$ and $id_n : \omega+n \to \omega+n \in V_{\omega+\omega}$, $\forall n\in\omega$ one has:

$\forall n\in\omega\ \exists! \overset{=\omega+n}{\alpha} \in V_{\omega+\omega}\ \exists f\in V_{\omega+\omega}\ G(n,\alpha,f)$ i.e.

$\forall n\in\omega\ \exists! \alpha \in V_{\omega+\omega}\ \exists f\in V_{\omega+\omega}\ G^{V_{\omega+\omega}}(n,\alpha,f)$ i.e.

$\forall n\in\omega\ \exists! \alpha\in V_{\omega+\omega}\ F^{V_{\omega+\omega}}(n,\alpha)$,

thus $F^{V_{\omega+\omega}}$ functional on $\omega$, and $F^{V_{\omega+\omega}}(n)=\omega+n$.

If $(\text{Repl})^{V_{\omega+\omega}}$ holds, $\exists Y\in V_{\omega+\omega}$ s.t.

$Y \supseteq \{\alpha : \exists n\in\omega\ F^{V_{\omega+\omega}}(n,\alpha)\}$
$= \{\omega+n : n\in\omega\}$

thus $\omega+\omega \subseteq Y\cup\omega \in V_{\omega+\omega}$ (since $Y$ and $\omega \in V_{\omega+\omega}$)

contradiction. $\blacksquare$

As a corollary and using Lemma 1, we have the following relative consistency results.

⚠ meta-result

**Thm 10:** (important)

result of the meta theory, i.e. doesn't need ZF axioms to be proved

i) $\mathrm{cons}(ZF^-) \longrightarrow \mathrm{cons}(ZF - \mathrm{Inf} + \neg\mathrm{Inf})$

In particular, also

$\mathrm{cons}(ZF) \longrightarrow \underbrace{\mathrm{cons}(ZF - \mathrm{Inf} + \neg\mathrm{Inf})}_{\equiv\ ZF - \mathrm{Inf}\ \nvdash\ \mathrm{Inf}}$

if ZF consistent, they cannot prove Inf, i.e. Inf cannot be deduced from previous axioms, and Inf is legitimate as an axiom of ZF

ii) $\mathrm{cons}(ZF^-) \longrightarrow \mathrm{cons}(ZFC - \mathrm{Inf} + \neg\mathrm{Inf})$

In particular,

$\mathrm{cons}(ZFC) \longrightarrow \underbrace{\mathrm{cons}(ZFC - \mathrm{Inf} + \neg\mathrm{Inf})}_{\equiv\ ZFC - \mathrm{Inf}\ \nvdash\ \mathrm{Inf}}$

iii) $\mathrm{cons}(ZF^-) \longleftrightarrow \mathrm{cons}(ZF)$

iv) $\mathrm{cons}(ZFC^-) \longleftrightarrow \mathrm{cons}(ZFC)$

i.e. we may assume w.l.o.g that every set is w-f, which facilitates many proofs.

v, vi) $\mathrm{cons}(ZF^-) \longrightarrow \mathrm{cons}(ZF - \mathrm{Rempl} + \neg\mathrm{Rempl})$

$\mathrm{cons}(ZFC^-) \longrightarrow \mathrm{cons}(ZFC - \mathrm{Rempl} + \neg\mathrm{Rempl})$

In particular,

$\mathrm{cons}(ZF) \longrightarrow \underbrace{\mathrm{cons}(ZF - \mathrm{Rempl} + \neg\mathrm{Rempl})}_{ZF - \mathrm{Rempl}\ \nvdash\ \mathrm{Rempl}}$

$\mathrm{cons}(ZFC) \longrightarrow \underbrace{\mathrm{cons}(ZFC - \mathrm{Rempl} + \neg\mathrm{Rempl})}_{ZFC - \mathrm{Rempl}\ \nvdash\ \mathrm{Rempl}}$

**Remark:** The axiom of Foundation has a special status:

$$\mathrm{Cons}(ZF^-) \longleftrightarrow \mathrm{Cons}(ZF)$$

This property doesn't hold for other axioms, indeed:

**Claim:** $\mathrm{Cons}(ZF-\mathrm{Inf}) \not\rightarrow \mathrm{Cons}(ZF)$
$\mathrm{Cons}(ZFC-\mathrm{Inf}) \not\rightarrow \mathrm{Cons}(ZFC)$

$\mathrm{Cons}(ZF-\mathrm{Rempl}) \not\rightarrow \mathrm{Cons}(ZF)$
$\mathrm{Cons}(ZFC-\mathrm{Rempl}) \not\rightarrow \mathrm{Cons}(ZFC)$

**Proof:** We prove $\mathrm{Cons}(ZF-\mathrm{Inf}) \not\rightarrow \mathrm{Cons}(ZF)$. The other proofs are similar.

By Thm 9 (i), one has

$$ZF \vdash \mathrm{Cons}(ZF-\mathrm{Inf}) \qquad (1)$$

Now suppose that $\mathrm{Cons}(ZF-\mathrm{Inf}) \rightarrow \mathrm{Cons}(ZF)$ holds. It means that there exists a proof of this fact. Hence we can carry this proof inside ZF (formally, the proof of $\mathrm{Cons}(ZF-\mathrm{Inf}) \rightarrow \mathrm{Cons}(ZF)$ inside ZF is done by contraposition – i.e. $\neg\mathrm{Cons}(ZF) \rightarrow \neg\mathrm{Cons}(ZF-\mathrm{Inf})$ – in order to involve only finite expressions and avoid non-formalizable expressions of the form "$M \models ZF$"), and thus

$$ZF \vdash \mathrm{Cons}(ZF-\mathrm{Inf}) \rightarrow \mathrm{Cons}(ZF) \qquad (2)$$

By (1) and (2), one has $ZF \vdash \mathrm{Cons}(ZF)$, contradiction with Gödels incompleteness theorem. ▨

(This proof can be formalized more precisely)