Chapter 2: Ordinals

Ordinals are set-theoretic objects that extend the concept of natural numbers. They allow to count in the finite and in the transfinite.

2.1. Well-Orderings.

<u>Def</u>: • A <u>(strict) total ordering</u> is a pair $\langle A, R \rangle$ where A is a set and R is a relation s.t.

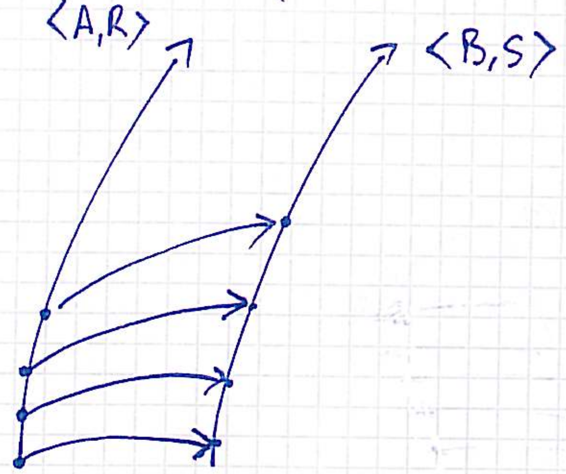
1) transitivity
$\forall x, y, z \in A \, (x R y \wedge y R z \to x R z)$
2) trichotomy
$\forall x, y \in A \, (x R y \vee y R x \vee x = y)$
3) Irreflexivity
$\forall x \in A \, (\neg x R x)$

<A,R>

• A <u>well-ordering</u> $\langle A, R \rangle$ is a total ordering s.t. every non-empty subset of A has an R-least element
i.e. $\forall X \subseteq A \, [X \neq 0 \to \exists m \in X \, \forall y \in X (\neg y R m)]$

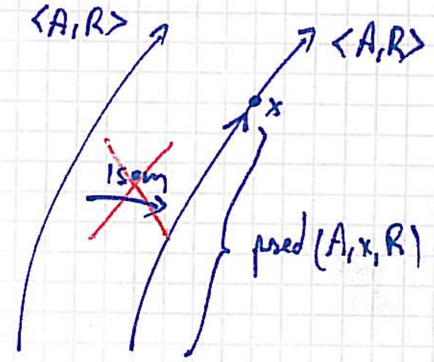
• We note $\operatorname{pred}(A, x, R) = \{ y \in A : y R x \}$

• $f: \langle A,R\rangle \longrightarrow \langle B,S\rangle$ isomorphism iff f is bijection and strictly increasing i.e. $\forall x,y \in A\ \big(xRy \to f(x)\,S\,f(y)\big)$.



If $\langle A,R\rangle$ isom to $\langle B,S\rangle$ we note
$\langle A,R\rangle \cong \langle B,S\rangle$

Lemma 1: If $\langle A,R\rangle$ w-o, then $\forall x \in A$ $\langle A,R\rangle \not\cong \langle pred(A,x,R), R\rangle$.



Proof: By contradiction.

idea: Consider m minimal in $\{z\in A : f(z)\neq z\}$ and derive a contradiction

let $f: \langle A,R\rangle \xrightarrow{\text{isom.}} \langle pred(A,x,R), R\rangle$

$\langle A,R\rangle$ w-o $\Rightarrow \neg\, xRx$ (trivial)

$\Rightarrow x \notin pred(A,x,R) = ran(f)$

$\Rightarrow$ we have (in particular) $f(x) \neq x$ (since $x \notin ran(f)$)

$\Rightarrow X = \{z \in A : f(z) \neq z\} \neq \emptyset$ (x belongs to it)

So let m R-least el. of X. We have $f(m) \neq m$.

Thus: - Either $f(m)\,R\,m \Rightarrow$ (f isom) $f\circ f(m)\,R\,f(m)$

$\Downarrow$ (min of m in X) $f\circ f(m) = f(m)$

contradiction

- Or $m\,R\,f(m) \Rightarrow$ (f^{-1} isom) $f^{-1}(m)\,R\,f^{-1}\circ f(m) = m$

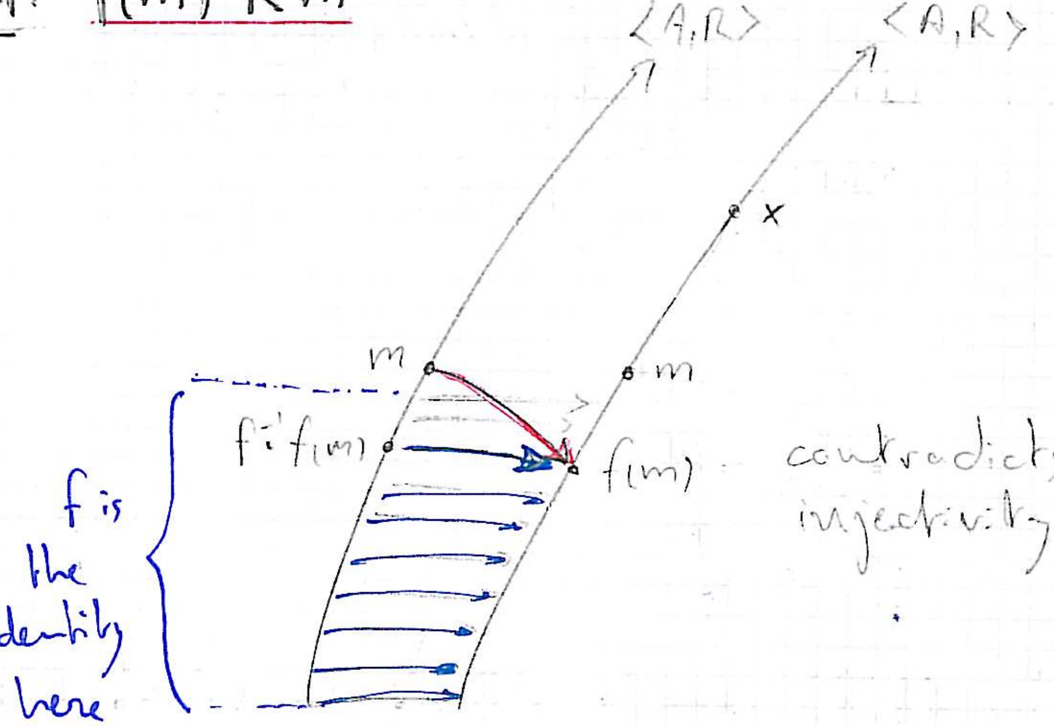
$\Rightarrow$ (min of m in X) $f\circ f^{-1}(m) = m = f^{-1}(m)$

contradiction

⚠ c.f. figures (explanation) $\longrightarrow$

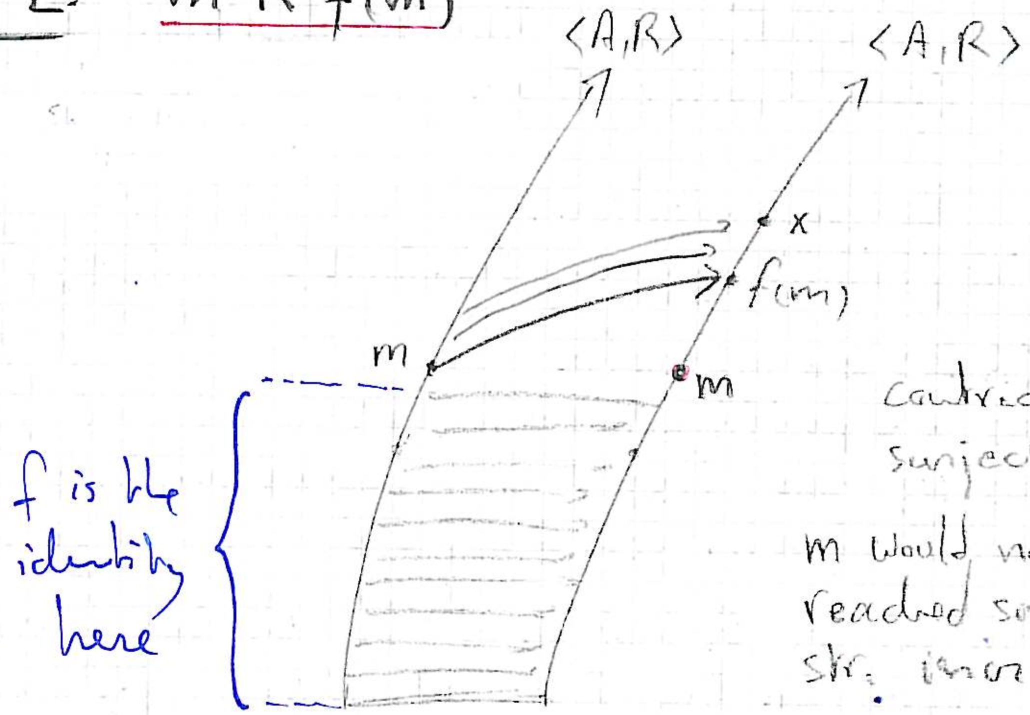
∎

case 1: $f(m)\ R\ m$



contradicts injectivity of f.

case 2: $m\ R\ f(m)$



contradicts surjectivity of f.

m would not be reached since f str. incr.

<u>Lemma 2:</u> If $\langle A, R\rangle$ and $\langle B, S\rangle$ are isom, then the isom is unique

<u>Proof:</u> By contradiction, let f, g be two distinct isom.

<u>idea:</u> consider m minimal in $\{x \in A : f(x) \neq g(x)\}$ and derive a contradiction.

Then $X = \{x \in A : f(x) \neq g(x)\} \neq \emptyset$

let m R-least el. of X. Then $f(m) \neq g(m)$.

- If $f(m)\, R\, g(m) \Rightarrow g^{-1} \circ f(m)\, R\, g^{-1} \circ g(m) = m$

  By min of m, $g^{-1} \circ f(m) \notin X$

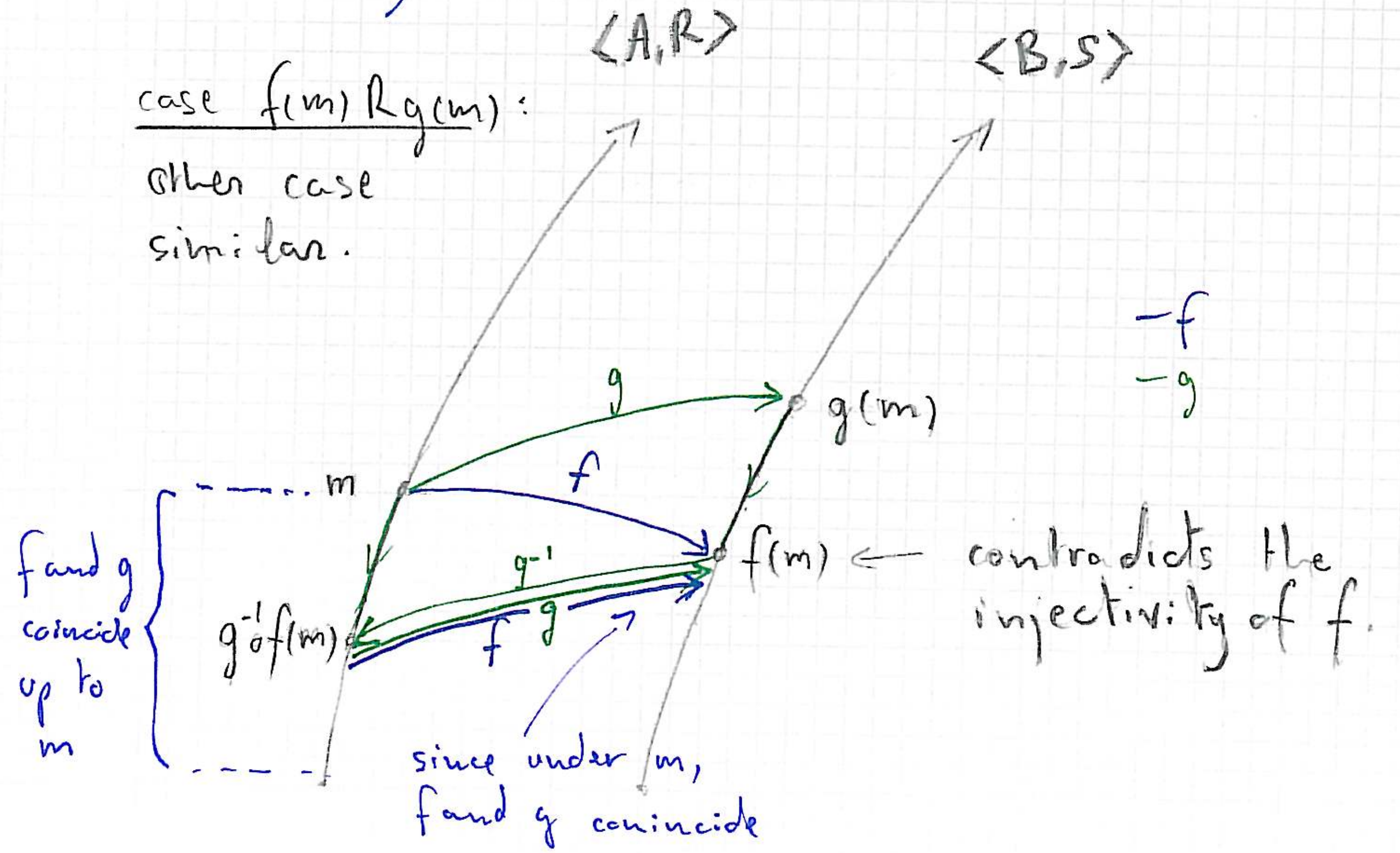
  $\Rightarrow f \circ g^{-1} \circ f(m) = g \circ g^{-1} \circ f(m) = f(m)$

  $\Rightarrow$ (f^{-1} isom) $g^{-1} \circ f(m) = m$, contradiction

- If $g(m)\, R\, f(m)$, similar. ∎

This lemma states the so-called rigidity of well-orderings.

<u>case $f(m)\, R\, g(m)$</u>: other case similar.



Thm 3 (Comparison) Let $\langle A,R\rangle$ and $\langle B,S\rangle$ be w-o.

Then exactly one of the following holds.

1) $\langle A,R\rangle \cong \langle B,S\rangle$

2) $\exists y\in B \quad \langle A,R\rangle \cong \langle \text{pred}(B,y,S),S\rangle$ ← initial segment of B

3) $\exists x\in A \quad \langle \text{pred}(A,x,R),R\rangle \cong \langle B,S\rangle$

↑ initial segment of A.

Proof: Consider (the relation)

$$F=\{\langle a,b\rangle\in A\times B : \langle \text{pred}(A,a,R),R\rangle \cong \langle \text{pred}(B,b,S),S\rangle\}$$

By lemma 1, F is functional partial (i.e. $\forall a\in A\,(\exists b\ \langle a,b\rangle\in F \to \exists!\, b\ \langle a,b\rangle\in F)$) — illustration →

Hence $F:\text{dom}(F)\longrightarrow \text{ran}(F)$ is a function.

It is onto by def
It is 1-1 by lemma 1 again } $F:\text{dom}(F)\to\text{ran}(F)$ is bijective.

Moreover, F is str. increasing (not immediate)
$\Rightarrow$ $F:\text{dom}(F)\longrightarrow \text{ran}(F)$ isom ! } exercise

Also, dom(F) and ran(F) are closed by predecessors (not immediate) i.e.
$\forall a\in\text{dom}(F),\ a'Ra \to a'\in\text{dom}(F)$
$\forall b\in\text{ran}(F),\ b'Rb \to b'\in\text{ran}(F)$ } exercise

$\Rightarrow$ $\text{dom}(F)=A$ entire or $\text{dom}(F)=\text{pred}(A,x,R)$ for some $x\in A$.

only possibilities since closed by pred.

$\text{ran}(F)=B$ or $\text{ran}(F)=\text{pred}(B,y,S)$ for some $y\in A$.

$\Rightarrow$ 4 cases:

i) $\mathrm{dom}(F) = A$ and $\mathrm{ran}(F) = B$
$\Rightarrow \langle A, R\rangle \cong \langle B, S\rangle$

ii) $\mathrm{dom}(F) = A$ and $\mathrm{ran}(F) = \mathrm{pred}(B, y, S)$
$\Rightarrow \langle A, R\rangle \cong \langle \mathrm{pred}(B, y, S), S\rangle$

iii) symmetric of ii)

iv) $\mathrm{dom}(F) = \mathrm{pred}(A, x, R)$ and $\mathrm{ran}(F) = \mathrm{pred}(B, y, S)$
$\Rightarrow \langle \mathrm{pred}(A, x, R), R\rangle \cong \langle \mathrm{pred}(B, y, S), S\rangle$
$\Rightarrow$ (def of F) $F(x) = y$ (i.e. $\langle x, y\rangle \in F$)
i.e. $x \in \underbrace{\mathrm{dom}(F)}_{= \mathrm{pred}(A, x, R)}$ and $y \in \underbrace{\mathrm{ran}(F)}_{= \mathrm{pred}(B, y, S)}$

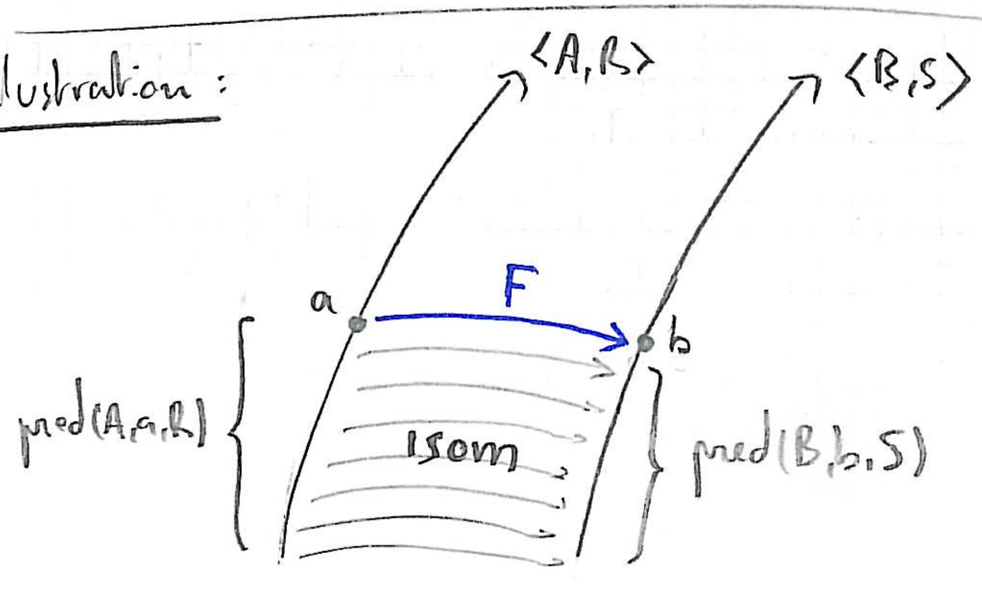
impossible since $\neg xRx$ and $\neg ySy$,
i.e. $x \notin \mathrm{pred}(A, x, R)$, i.e. $y \notin \mathrm{pred}(B, y, S)$ ∎

We can state the axiom of choice (AC)

Axiom 9. Choice

every set can be well-ordered i.e.
$\forall A \exists R$ (R well-orders A)

illustration:



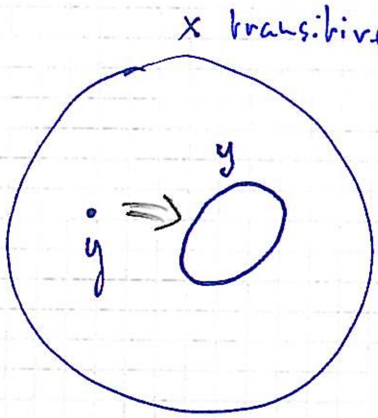
2.2. Ordinals

<u>Def</u>: A set x is <u>transitive</u> iff every element of x is a subset of x i.e.

iff $\forall y \, (y \in x \to y \subseteq x)$ i.e.

we use this definition → iff $\forall y \forall y' \, (y' \in y \in x \to y' \in x)$

means exactly $y \subseteq x$



i.e. $\langle \alpha, \in_\alpha \rangle$ is a w-o where $\in_\alpha = \{ \langle x, y \rangle \in \alpha \times \alpha : x \in y \}$

A set α is an <u>ordinal</u> iff α is transitive and well-ordered by $\in$, i.e. iff

1) $x \in y \in \alpha \to x \in \alpha$

2) $\forall x, y, z \in \alpha$ $(x \notin x)$
$(x \in y \vee y \in x \vee x = y)$
$(x \in y \in z \to x \in z)$

$\forall A \subseteq \alpha, A \neq 0, \exists \beta \in A \; \forall \gamma \in A \, (\gamma \notin \beta)$

<u>Examples</u>: $0, \{0\}, \{0, \{0\}\}, \{0, \{0\}, \{0, \{0\}\}\}$ are ordinals.

<u>Remark</u>: α ordinal $\Rightarrow \alpha \notin \alpha$ (for if $\alpha \in \alpha$, then $\alpha \notin \alpha$ by irreflexivity of $\in$, contradiction).

Lemma 4: Let α ordinal and X be a set of ordinals. Then

1) $\alpha \cup \{\alpha\}$ is an ordinal
2) $\bigcup X$ is an ordinal
3) if $X \neq 0$, $\bigcap X$ is an ordinal.

Proof: (1)

- If $\underline{x \in y \in \alpha \cup \{\alpha\}} \Rightarrow y \in \alpha$ or $y = \alpha$

transitivity of $\alpha \cup \{\alpha\}$

- If $y \in \alpha$, $x \in y \in \alpha \Rightarrow x \in \alpha$ (α transitif)
$\Rightarrow \underline{x \in \alpha \cup \{\alpha\}}$
- If $y = \alpha$, $x \in y$ is $x \in \alpha \Rightarrow \underline{x \in \alpha \cup \{\alpha\}}$.

$\Rightarrow \alpha \cup \{\alpha\}$ transitif.

irreflexivity of $\in$

- $x \in \alpha \cup \{\alpha\} \Rightarrow x \in \alpha$ or $x = \alpha$
  - if $x \in \alpha$, $\underline{x \notin x}$ (since α ord.)
  - if $x = \alpha$, $\underline{x \notin x}$, since α ordinal (c.f. Remark)

- If $x, y, z \in \alpha \cup \{\alpha\}$ with $\underline{x \in y \in z}$

transitivity of $\in$

- If $x, y, z \in \alpha \Rightarrow \underline{x \in z}$ since α ordinal
- If $x = \alpha \Rightarrow y \neq \alpha$ (otherwise $x \in y$ is $\alpha \in \alpha$)
$\Rightarrow y \in \alpha$
Hence $x = \alpha \in y \in \alpha$, $\underline{\text{contradiction}}$ ($\Rightarrow x \in \alpha$ but $x = \alpha \Rightarrow \alpha \in \alpha$)
- If $y = \alpha \Rightarrow z \neq \alpha$ (otherwise $y \in z$ is $\alpha \in \alpha$)
$\Rightarrow z \in \alpha$
Hen $y = \alpha \in z \in \alpha$, $\underline{\text{contradiction}}$
- If $z = \alpha$, we have $x \in y \in z = \alpha$
$\Rightarrow$ (α transitif) $\underline{x \in \alpha = z}$

trichotomy of $\in$ $\Rightarrow$

- let $X \subseteq \alpha \cup \{\alpha\}$ with $X \neq \emptyset$.
  - If $X \cap \alpha \neq \emptyset$, α ordinal $\Rightarrow \exists\, m \in \alpha$ $\in$-min el. in $X \cap \alpha$. But $m \in \alpha$ and m $\in$-min in $X \cap \alpha$
  $\Rightarrow m$ $\in$-min in X (otherwise $\underline{m' \in X}$ s.t. $m' \in m \in \alpha \Rightarrow (\alpha \text{ ord.})$ $\underline{m' \in \alpha}$, contrad min of m in $X \cap \alpha$)

- If $X \cap \alpha = \emptyset$, then $X \cap \alpha \cup \{\alpha\} \neq \emptyset \Rightarrow X = \{\alpha\}$ and $\underbrace{}_{\text{thus}}$ α is $\in$-minimal in X. ■

(2) and (3) : Exercices.

Def: If α ordinal and X set of ordinals

- $S(\alpha) := \alpha \cup \{\alpha\}$ is called the successor of α (ordinal)
- $\sup(X) := \bigcup X$ (ordinal)
- If $X \neq \emptyset$, $\min(X) := \bigcap X$ (ordinal)

We will see later why we use these names.

give this def after, in page 11.

- let $x, y \in \alpha \cup \{\alpha\}$. (trivial)

trichotomy of $\in$

- if $x = \alpha$ and $y = \alpha$, then $\underline{x = y}$
- if $x, y \in \alpha$, $\underline{\text{trichotomy holds}}$ since α ordinal
- if $x \in \alpha$ and $y = \alpha$, then $\underline{x \in y}$
- if $y \in \alpha$ and $x = \alpha$, then $\underline{y \in x}$

Lemma 5: i) If α ordinal and $\beta \in \alpha$, then β ordinal and $\beta = \mathrm{pred}(\alpha, \beta)$

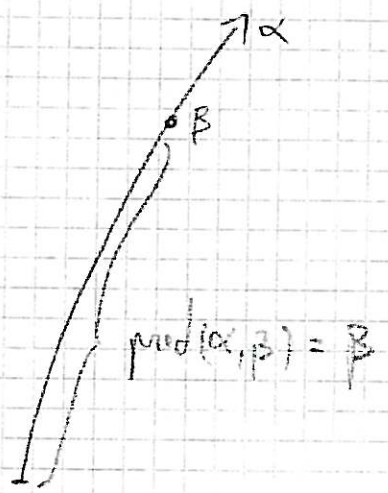
ii) If α, β ordinals s.t. $\alpha \cong \beta$, then $\alpha = \beta$.

Proof: i) α ordinal and $\beta \in \alpha \Rightarrow \beta$ ordinal is easy

(exercice)

- If $\gamma \in \beta \Rightarrow \gamma \in \beta \in \alpha \Rightarrow \gamma \in \alpha$
$\Rightarrow \gamma \in \mathrm{pred}(\alpha, \beta)$ $(= \{x \in \beta : x \in \alpha\})$
- If $\gamma \in \mathrm{pred}(\alpha, \beta) \Rightarrow \gamma \in \alpha$ and $\gamma \in \beta$

Thus $\beta = \mathrm{pred}(\alpha, \beta)$



(exercice only give intuition)

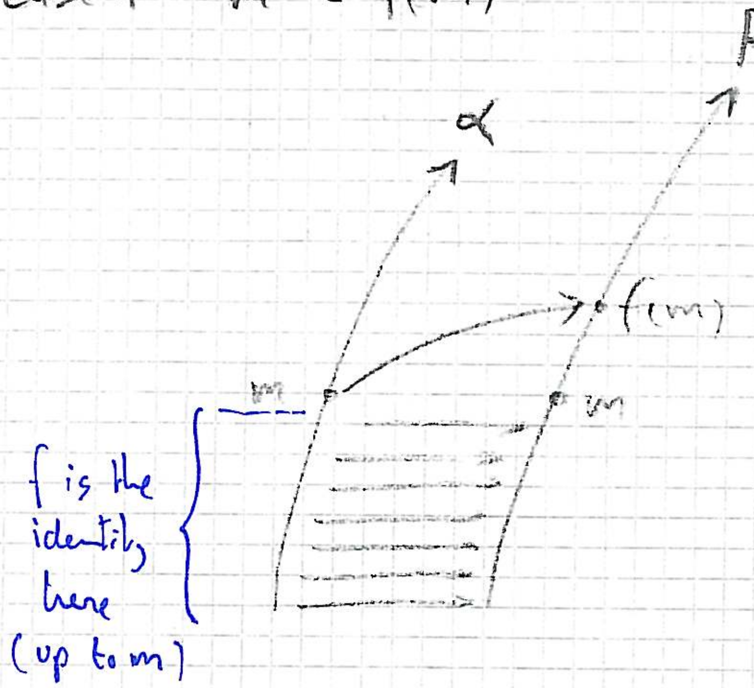
ii) By contradiction, let $f : \alpha \to \beta$ isom and suppose $\alpha \neq \beta$.

(If f bij would be the **identity**, one would have by extensionality
$\alpha = \mathrm{dom}(f) = \mathrm{ran}(f) = \beta$)

Hence $X = \{x \in \alpha : f(x) \neq x\} \neq \emptyset$.

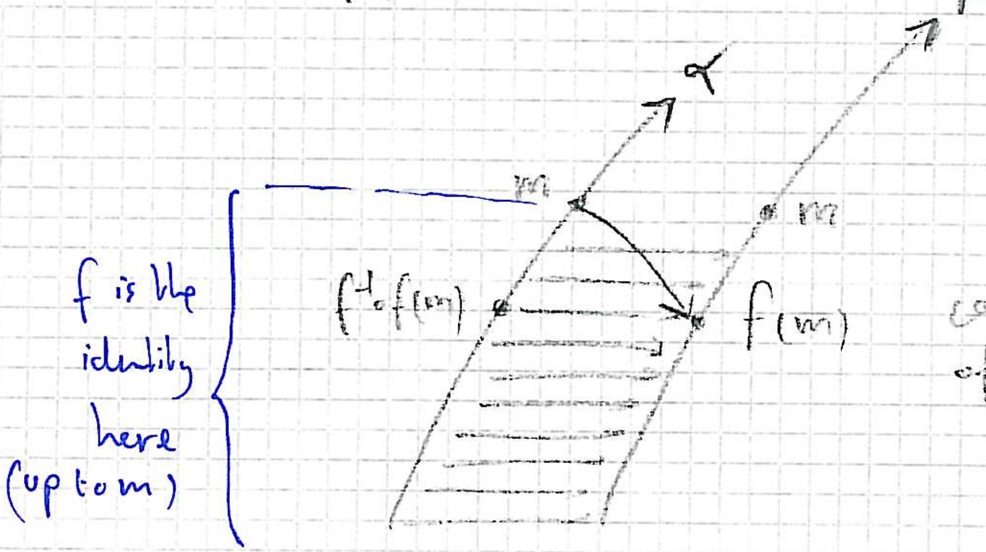
let m min in $X \Rightarrow f(m) \neq m$

$\Rightarrow$ 2 cases (same proof as lemma 1) case 1: $m \in f(m)$



contradicts surjectivity of f (since f str. incr.)

case 2: $f(m) \in m$



contradicts injectivity of f.

The "class" of ordinals is "well-ordered" by $\in$. (informal statement).

Thm 6: let α, β, γ ordinals:

so "$\in$" is some kind of w-o on the class of ordinals

(1) $\alpha \notin \alpha$ (antireflexivity)

(2) exacly one of the following is true: (trichotomy) either $\alpha \in \beta$ or $\beta \in \alpha$ or $\alpha = \beta$

(3) If $\alpha \in \beta \in \gamma$, then $\alpha \in \gamma$ (transitivity)

(4) If C is a non-empty set of ordinals, the C has an $\in$-minimal element. (i.e. $\exists m \in C\ \forall x \in C (\neg x \in m)$)

Proof (1) By contradiction, if $\alpha \in \alpha$, then $\alpha \notin \alpha$ since α ordinal, $\lightning$.

(2) Since α, β w-o, by thm 3, we have:

- Either $\alpha \cong \beta \Rightarrow$ (lemma 5) $\alpha = \beta$
- Or $\alpha \cong \underbrace{\operatorname{pred}(\beta, \beta')}_{= \beta' \text{ (by lemma 5)}}$, for some $\beta' \in \beta$

$\Rightarrow \alpha \cong \beta' \Rightarrow$ (L.5) $\alpha = \beta' \in \beta$

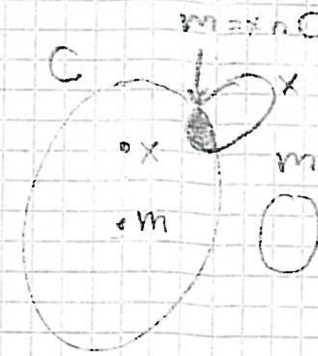
- Or symmetrically $\beta \in \alpha$.

The mutual exclusivity of these cases if $\alpha = \beta$ and $\alpha \in \beta \Rightarrow \beta \in \beta$, contrad.

(3) Trivial, since γ ordinal (hence transitive).

(4) Equivalent formulation:

$\exists m \in C \,(m \cap C = 0)$ ⊛



let $x \in C$ arbitrary.

If $x \cap C = 0$, we are done by taking $m = x$

If $x \cap C \neq 0$, then x ord $\Rightarrow x \cap C$ has an $\in$-min element m, i.e. $m \in x \cap C$

Then $m \cap C = 0$

Otherwise, $z \in m \cap C$, then $\boxed{z \in m \in x} \Rightarrow z \in x$

$\Rightarrow \boxed{z \in x \cap C}$

contradicts minimality of m in $x \cap C$

Since "$\in$" "well-orders" the ordinals, we write $\alpha < \beta$ for $\alpha \in \beta$ and and $\alpha \leq \beta$ for $(\alpha \in \beta \vee \alpha = \beta)$.

⊛ Indeed, if $m \cap C \neq 0 \Leftrightarrow \exists m' \in m \cap C$
$\Leftrightarrow m' \in m \wedge m' \in C$, i.e. contrad with $\in$-minimality of m in C.

END