Chapter 6: The Constructible Sets

The goal of this chapter is to introduce the class of constructible sets L, to prove that L is the minimal inner model of ZF inside V (in some specific sense), and most importantly, that L is a model of $ZF + AC + GCH$. These results provide the relative consistency of AC and GCH from ZF, i.e.

$$\mathrm{Cons}(ZF) \rightarrow \underbrace{\mathrm{Cons}(ZF + AC + GCH)}_{\text{i.e.}} .$$

$$ZF \nvdash \neg AC \text{ and } ZF \nvdash \neg GCH$$

6.1. Definability

Idea: we would like to formalize a notion of definability such that a set a is called definable iff there exists some property $P(x)$, expressed in the language of set theory, such that a is the unique object satisfying P, i.e. $\forall x \left(P(x) \leftrightarrow x = a \right)$.

More generally, for any A and $n \in \omega$, we introduce the set $Df(A,n)$ of all n-ary relations on A "definable" by a n-free variable formula relativized to A, i.e.

$R \in Df(A,n)$ iff there exists $\varphi(x_0, -, x_{n-1})$ s.t. $R = \{ \langle a_0, -, a_{n-1} \rangle \in A^n : \varphi^A(a_0, -, a_{n-1}) \}$

A relation R will be called definable iff there exist A and n s.t. $R \in Df(A,n)$.
An element a will be calle definable iff $\{a\}$ is, i.e. iff there exists A s.t. $\{a\} \in Df(A,1)$.

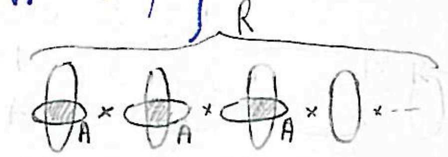
Since there are only countably many formulas over the language of set theory, there are only countably many definable relations and elements.

<u>Definition</u>: For any $n \in \omega$ and $i,j < n$, we set:

these are relations

(a) $\mathrm{Proj}(A, R, n) = \{ s \in A^n : \exists t \in R \ (t \upharpoonright n = s) \}$

projection of the relation R on A^n



$= \mathrm{proj}(A,R,3)$

(b) $\mathrm{Diag}_\in (A, n, i, j) = \{ s \in A^n : s(i) \in s(j) \}$

(c) $\mathrm{Diag}_= (A, n, i, j) = \{ s \in A^n : s(i) = s(j) \}$

(d) By recursion on $k \in \omega$, we define $Df'(k, A, n)$ – the <u>set</u> of n-ary relations on A of order k – simultaneously for all n by:

(i) $Df'(0, A, n) = \{ \mathrm{Diag}_\in (A, n, i, j) : i,j < n \} \cup \{ \mathrm{Diag}_= (A, n, i, j) : i,j < n \}$

(ii) $Df'(k+1, A, n) = Df'(k, A, n)$

these are sets of relations

complements of relations of order k $\cup \{ A^n \setminus R : R \in Df'(k, A, n) \}$

intersections of relations of order k $\cup \{ R \cap S : R, S \in Df'(k, A, n) \}$

projections of relations of order k $\cup \{ \mathrm{Proj}(A, R, n) : R \in Df'(k, A, n+1) \}$

Intuitively, the set of relations $Df'(k+1, A, n)$ is obtained by applying the logical operations of "$\neg$", "$\wedge$", and "$\exists$" to the set of relations $Df'(k, A, n)$ and $Df'(k+1, A, n)$, respectively.

(e) Finally, we define the set of n-ary definable relations over A as:

$$Df(A, n) = \bigcup \{ Df'(k, A, n) : k \in \omega \}$$

→

<u>Remark:</u> $Df(A,n)$ is closed under the "logical" operations of complementation, intersection and projection, i.e.

if $R, S \in Df(A,n)$, then $A^n \setminus R \in Df(A,n)$, $R \cap S \in Df(A,n)$, and if $R \in Df(A,n+1)$, then $Proj(A,R,n) \in Df(A,n)$.

<u>Proof:</u> immediate by def of $Df(A,n)$:
- If $R, S \in Df(A,n)$, there exists $k \in \omega$ s.t. $R, S \in Df'(k,A,n)$, thus
  $A^n \setminus R \in Df(k+1, A, n) \subseteq Df(A,n)$
  $R \cap S \in Df(k+1, A, n) \subseteq Df(A,n)$
- If $R \in Df(A, n+1)$, there exists $k \in \omega$ s.t. $R \in Df'(k, A, n+1)$, thus
  $Proj(A,R,n) \in Df'(k+1, A, n) \subseteq Df(A,n)$. ■

By defining the set of relations $Df(A,n)$, the goal was to have

$\forall A \forall R \, [R \in Df(A,n)$ iff there is a formula $\varphi(x_0, \dots, x_{n-1})$ such that $R = \{ s \in A^n : \varphi^A(s(0), \dots, s(n-1)) \}]$

i.e. intuitively, "R is definable over A by some formula $\varphi(x_0, \dots, x_{n-1})$"

In fact, we have only the "sufficient" following implication:

Proposition 1: schema

If $\varphi(x_0, \dots, x_{n-1})$ is any formula whose free variables are among $x_0, \dots, x_{n-1}$, then

$$\forall A \left[\{ s \in A^n : \varphi^A(s(0), \dots, s(n-1)) \} \in Df(A,n) \right]$$

i.e. for any formula φ and set A, the "n-ary relation on A "defined" by φ" $\in Df(A,n)$.

Proof: By induction on the length of φ.

- If φ is $x_i \in x_j$ or $x_i = x_j$, then we have
$\{ s \in A^n : s(i) \in s(j) \} = Diag_\in(A,n,i,j) \in Df(A,n)$ and
$\{ s \in A^n : s(i) = s(j) \} = Diag_=(A,n,i,j) \in Df(A,n)$.

- If φ is $\varphi_0 \wedge \varphi_1$, then by the I.H.
$\{ s \in A^n : \varphi_k^A(\vec{s(i)}) \} \in Df(A,n)$ for $k = 1,2$,
thus $\{ s \in A^n : \varphi^A(\vec{s(i)}) \} =$
$\{ s \in A^n : \varphi_0^A(\vec{s(i)}) \wedge \varphi_1^A(\vec{s(i)}) \} =$
$\{ s \in A^n : \varphi_0^A(\vec{s(i)}) \} \cap \{ s \in A^n : \varphi_1^A(\vec{s(i)}) \}$
$\in Df(A,n)$

- If φ is $\neg \psi$, then by the I.H.
$\{ s \in A^n : \psi^A(\vec{s(i)}) \} \in Df(A,n)$, thus
$\{ s \in A^n : \varphi^A(\vec{s(i)}) \} = \{ s \in A^n : \neg \psi^A(\vec{s(i)}) \}$
$= A^n \setminus \{ s \in A^n : \psi^A(\vec{s(i)}) \} \in Df(A,n)$.

- If $\varphi(x_0, \dots, x_{n-1})$ is $\exists y\, \psi(x_0, \dots, x_{n-1}, y)$, with $y \notin \{x_0, \dots, x_{n-1}\}$, then by the I.H., $\{t \in A^{n+1} : \psi^A(\overrightarrow{t(i)})\} \in Df(A, n+1)$, thus

$$\{s \in A^n : \varphi^A(\overrightarrow{s(i)})\} = \{s \in A^n : \exists y \in A\ \psi^A(\overrightarrow{s(i)}, y)\}$$
$$= \{s \in A^n : \exists t \in \{t \in A^{n+1} : \psi^A(\overrightarrow{t(i)})\}\ (t \upharpoonright n = s)\}$$
$$= Proj(A, \underbrace{\{t \in A^{n+1} : \psi^A(\overrightarrow{t(i)})\}}_{\in Df(A, n+1)}, n) \in Df(A, n)$$

- If $\varphi(x_0, \dots, x_{n-1})$ is $\exists x_j\, \psi(x_0, \dots, x_{n-1})$, i.e. x_j not free in φ.
Let y be different from all variables of φ, and let $\psi'(x_0, \dots, x_{n-1}, y)$ be $\psi(x_0, \dots, x_{j-1}, y, x_{j+1}, \dots, x_{n-1})$, (i.e. ψ' is ψ where x_j have been substituted by y) and let $\varphi'(x_0, \dots, x_{n-1})$ be $\exists y\, \psi'(x_0, \dots, x_{n-1}, y)$.
Then we can prove that φ and φ' are logically equivalent, and the preceding argument shows that $\{s \in A^n : \varphi'^A(\overrightarrow{s(i)})\} \in Df(A, n)$, thus $\{s \in A^n\ \varphi^A(\overrightarrow{s(i)})\} \in Df(A, n)$ also.

formally, there are only $n-1$ elements here.

Even if x_j doesn't appear anymore in φ', the free variables of φ' are still <u>among</u> $x_0, \dots, x_{n-1}$, and we can apply I.H.

The "converse" of proposition 1 intuitively says that

$$\forall A\, \forall R\, [R \in Df(A, n) \rightarrow \exists \varphi\, (R = \{s \in A^n : \varphi^A(\overrightarrow{s(i)})\})]$$

i.e. $$\forall A\, \forall R\, \left[R \in Df(A, n) \rightarrow \bigvee_{i=0}^{\infty} (R = \{s \in A^n : \varphi_i^A(\overrightarrow{s(i)})\})\right]$$

where $\varphi_0, \varphi_1, \varphi_2 \dots$ would be an enumeration of all formulas with free variables among $x_0, \dots, x_{n-1}$.
It cannot be formally proved for it is not a sentence of set theory, but is considered to be "platonistically true".

<u>Intuitively, $Df(A, n)$ has to be thought of as precisely the set of n-ary relations definable by some formula.</u>

We now prove that for any A and n, the set $Df(A,n)$ is countable. We give an explicit enumeration of it.

<u>Definition</u>: We define $En(m,A,n)$ by recursion on $m \in \omega$ (and simultaneously for all $n \in \omega$) as follows:

(a) If $m = 2^i \cdot 3^j$ and $i,j < n$, then $En(m,A,n) = Diag_\in(A,n,i,j)$

(b) If $m = 2^i \cdot 3^j \cdot 5$ and $i,j < n$, then $En(m,A,n) = Diag_=(A,n,i,j)$

(c) If $m = 2^i \cdot 3^j \cdot 5^2$, then $En(m,A,n) = A^n \setminus En(i,A,n)$

(d) If $m = 2^i \cdot 3^j \cdot 5^3$, then $En(m,A,n) = En(i,A,n) \cap En(j,A,n)$

(e) If $m = 2^i \cdot 3^j \cdot 5^4$, then $En(m,A,n) = Proj(A, En(i,A,n+1), n)$

(f) If m is not of a form specified above, then $En(m,A,n) = 0$ (empty set)

$En(m,A,n)$ has to be thought of as the "m-th relation of $Df(A,n)$".

We have the following properties:

Lemma 2:

(a) The relations $En(m, A, n)$ $(m \in \omega)$ precisely exhaust the set of relations $Df(A, n)$ i.e.:

For any A and n, $Df(A,n) = \{En(m, A, n) : m \in \omega\}$.

(b) For any A and n, $|Df(A,n)| \leq \aleph_0$

(c) The functions Df and En are absolute for transitive models of $ZF-P$.

Proof: (a) Let A be a set.

$(\supseteq)$ We prove by induction on m that for all $m \in \omega$,

$\forall n \in \omega \, [En(m, A, n) \in Df(A, n)]$

(i.e. we prove it simultaneously for all n.)

It follows that for any n, one has

$\{En(m, A, n) : m \in \omega\} \subseteq Df(A, n)$

almost trivial from here →

- If $m = 0$, then m is not of the form (a)-(e) of def of En, thus $En(m, A, n) = 0 \in Df(A, n)$, $\forall n \in \omega$, since $Df(A,n)$ closed under intersection and complement.
- Let $m > 0$, and assume that for all $i < m$

  $\forall n \in \omega \, [En(i, A, n) \in Df(A, n)]$ (ind. hyp.)

  Fix $n \in \omega$. If m is of the form (a)-(e) of def of En, then by the I.H., we have $En(i, A, n) \in Df(A, n)$, $\forall i < m$ and $En(i, A, n+1) \in Df(A, n+1)$, $\forall i < m$. It follows that $En(m, A, n) \in Df(A, n)$

If m is of the form (f), similarly to $m=0$, we have $En(m, A, n) \in Df(A, n)$. 6'

($\subseteq$) We prove by induction on k that for all $k \in \omega$,

$$\forall n \in \omega \left[Df'(k, A, n) \subseteq \{En(m, A, n) : m \in \omega\} \right]$$

(i.e. we prove it simultaneously for all $n \in \omega$.)

It follows that for any n, one has

$$Df(A, n) = \bigcup_{k \in \omega} Df'(k, A, n) \subseteq \{En(m, A, n) : m \in \omega\}$$

almost trivial from here →

Fix $n \in \omega$. If $k=0$, $Df'(k, A, n) \subseteq \{En(m, A, n) : m \in \omega\}$ by clauses (a) and (b) of def of En.

Suppose $k = k'+1$. By the I.H., one has
$Df'(k', A, n) \subseteq \{En(m, A, n) : m \in \omega\}$ and
$Df'(k', A, n+1) \subseteq \{En(m, A, n+1) : m \in \omega\}$.
It follows by def of $Df'(k'+1, A, n)$ that
$Df(\underbrace{k'+1}_{=k}, A, n) \subseteq \{En(m, A, n) : m \in \omega\}$

(b) By (a), the relations $En(m, A, n)$, $m \in \omega$ provide an enumeration of $Df(A, n)$, thus $|Df(A, n)| \leq \aleph_0$

(c) Follows from the methods of Chapter 5. Uses the fact that functions defined recursively by means of absolute notions are absolute. Uses also the absoluteness of ordinal exponentiation.

∎

We have the counterpart of proposition 1 using the notion of En

Proposition 3: If $\varphi(x_0, -, x_{n-1})$ is any formula whose free variable are among $x_0, -, x_{n-1}$, then for some $m \in \omega$,

schema

$$\forall A\left[\{s \in A^n : \varphi^A(s(0), -, s(n-1))\} = En(m, A, n)\right]$$

Proof: By Proposition 1, $\{s \in A^n : \varphi^A(\overline{s(i)})\} \in Df(A, n)$.
By Lemma 2 (a), there exists thus an $m \in \omega$ s.t. $\{s \in A^n : \varphi^A(\overline{s(i)})\} = En(m, A, n)$. ▨

6.2. The Constructible Universe L

We introduce the constructible universe L of Gödel and prove some basic properties.

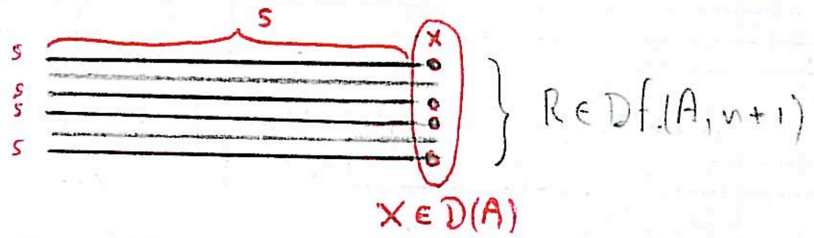
We first introduce the definable power set operation $\mathcal{D}$. Intuitively, $\mathcal{D}(A)$ consist of the set of subsets of A which are definable from a finite number of elements of A by a formula relativized to A.

<u>Definition:</u> For any set A, we define:

$$\mathcal{D}(A) = \{ X \subseteq A : \exists n \in \omega \; \exists R \in Df(A, n+1) \; \exists s \in A^n \; (X = \{ x \in A : s^\frown\langle x \rangle \in R \}) \}$$

set of suffixes of el. of R who begin by s

illustration:



We have the counterpart of Proposition 1 and 3 for the case of definable subsets:

<u>Proposition 4:</u> If $\varphi(v_0, v_1, \ldots, v_{n-1}, x)$ is any formula whose free variables are among $v_0, v_1, \ldots, v_{n-1}, x$, then

<u>schema</u>

$$\forall A \; \forall v_0, \ldots, v_{n-1} \in A \; \left[\{ x \in A : \varphi^A(v_0, v_1, \ldots, v_{n-1}, x) \} \in \mathcal{D}(A) \right]$$

This is how to think of $\mathcal{D}(A)$: the set of subsets of A definable by some formula relativized to A

→

<u>Proof:</u> Let A be a set and $v_0, \ldots, v_{n-1} \in A$.

One has $\{x \in A : \varphi^A(\vec{v}, x)\} =$

$$\{x \in A : \vec{v}^\frown\langle x\rangle \in \underbrace{\{s \in A^{n+1} : \varphi^A(\vec{s})\}}_{\substack{:= R \\ \in Df(A, n+1) \text{ by Prop. 1}}}\}$$

Hence, for $n \in \omega$, $\vec{v} \in A^n$, and $R \in Df(A, n+1)$,

one has $\{x \in A : \varphi^A(\vec{v}, x)\} =$

$\{x \in A : \vec{v}^\frown\langle x\rangle \in R\} \in D(A)$ by def of D.

(Therefore $\{x \in A : \varphi^A(\vec{v}, x)\} \in D(A)$ (by def of D))

$\square$

<u>Remark:</u> Note that, as opposed to the classical power set operation P, the definable power set operation D does not increase the cardinality (for infinite cardinalities).

i.e. under (AC), one has

$|A| \geq \aleph_0 \to |D(A)| = |A|$

<u>Proof:</u>

- For any $y \in A$, one has

  $\{y\} = \{x \in A : (x = y)^A\} \in D(A)$ by Prop. 4

  Thus $|A| \leq |D(A)|$.

- Under (AC) and $|A| \geq \aleph_0$, we have

$$|D(A)| \leq \aleph_0 \cdot \underbrace{|A^n|}_{\substack{= \\ |A| \geq \aleph_0}} \cdot \underbrace{|Df(A, n+1)|}_{\leq \aleph_0} = |A| \qquad \square$$

We define the constructible universe:

<u>Definition</u>: We define by transfinite induction on $\alpha \in ON$,

$$\begin{cases} L_0 = 0 \\ L_{\alpha+1} = \mathcal{D}(L_\alpha) \\ L_\alpha = \bigcup_{\xi<\alpha} L_\xi \text{, for } \alpha \text{ limit} \end{cases}$$

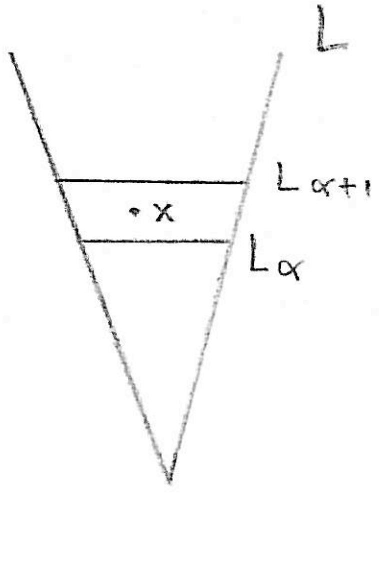
We then set $L = \bigcup \{ L_\alpha : \alpha \in ON \}$

L is defined in a similar way as WF, yet by means of the definable power set operation instead of the usual power set operation.

Like in WF, if $x \in L$, the least α s.t. $x \in L_\alpha$ is a successor ordinal (if α limit $x \in L_\alpha = \bigcup_{\xi<\alpha} L_\xi \rightarrow x \in L_\xi$ for some $\xi < \alpha$, contrad with minimality of α).

<u>Definition</u>: If $x \in L$, the least α s.t. $x \in L_{\alpha+1}$ is the <u>L-rank</u> of x, denoted $\rho(x)$.

$\rho(x)$ is the level that precedes the first appearance of x.



lemma 5: For each $\alpha \in ON$:

(b) $\zeta \leq \alpha \to L_\zeta \subseteq L_\alpha$

(a) L_α is transitive

(c) $L_\alpha = \{x \in L : \rho(x) < \alpha\}$

(d) $L_\alpha \in L_{\alpha+1}$

Proof: We need a claim: A transitive $\to A \subseteq D(A)$.

exercise

Proof: By prop. 4, we have

$$\forall v \in A\left[\{x \in A : x \in v\} \in D(A)\right]$$

which, if A transitive, reduces to

$$\forall v \in A\,[v \in D(A)] \text{ i.e. } A \subseteq D(A).$$

(b) By induction on α:

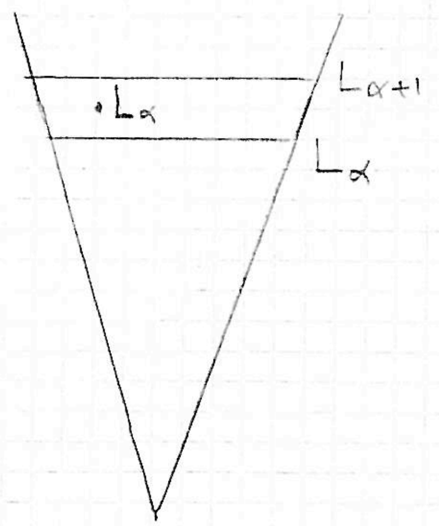
- If $\alpha = 0$, o.k.
- If α limit and $\zeta \leq \alpha$.
  If $\zeta = \alpha$, $L_\zeta \subseteq L_\alpha$.
  If $\zeta < \alpha \Rightarrow \zeta < \zeta' < \alpha \Rightarrow L_\zeta \underset{I.H.}{\subseteq} L_{\zeta'} \underset{def}{\subseteq} L_\alpha$
- If $\alpha = \beta + 1$ and $\zeta \leq \alpha$
  If $\zeta = \alpha$, $L_\zeta \subseteq L_\alpha$
  If $\zeta < \alpha \Rightarrow \zeta \leq \beta \Rightarrow L_\zeta \underset{I.H.}{\subseteq} L_\beta \underset{\text{by claim since } L_\beta \text{ transitive}}{\subseteq} D(L_\beta) = L_\alpha$

(a) By induction on α:

- If $\alpha = 0$ or limit, o.k., by the I.H. and the fact that a union of transitive set is transitive
- If $\alpha = \beta + 1 \underset{IH}{\Rightarrow} L_\beta$ transitive $\Rightarrow P(L_\beta)$ transitive $\Rightarrow L_\alpha = D(L_\beta) \subseteq P(L_\beta)$ transitive also

(c) Same proof as for WF (almost immediate)

(d) $L_\alpha = \{ x \in L_\alpha : (x = x)^{L_\alpha} \}$, thus by Prop 4,
$L_\alpha \in \mathcal{D}(L_\alpha) = L_{\alpha+1}$.



Lemma 6: (a) $\forall \alpha \in ON \left(\alpha \in L \wedge \rho(\alpha) = \alpha \right)$
(b) $\forall \alpha \in ON \left(L_\alpha \cap ON = \alpha \right)$

Proof: (b) By induction on α:

- If $\alpha = 0$, $L_0 \cap ON = 0 \cap ON = 0$
- If α limit, $L_\alpha \cap ON = \bigcup_{\xi < \alpha} L_\xi \cap ON =$
$$\bigcup_{\xi<\alpha} (L_\xi \cap ON) \underset{I.H.}{=} \bigcup_{\xi < \alpha} \xi = \alpha$$
- If $\alpha = \beta + 1$. We first need to prove that $\beta \in L_\alpha$.

We recall that there is Δ_0 formula $\varphi(x)$ s.t.
$\forall x \left(\varphi(x) \leftrightarrow x \text{ is an ordinal} \right)$.
Since Δ_0 formulas are abs. for transitive sets, and using the I.H., one has
$$\beta \underset{I.H.}{=} L_\beta \cap ON = \{ x \in L_\beta : \varphi(x) \}$$
$$= \{ x \in L_\beta : \varphi^{L_\beta}(x) \} \quad \left(\begin{smallmatrix}\text{by abs.}\\ \text{of } \varphi\end{smallmatrix} \right)$$
By Prop. 4, $\beta \in \mathcal{D}(L_\beta) = L_\alpha$. ⊛

$(\subseteq)$: let $\xi \in L_\alpha \cap ON = \mathcal{D}(L_\beta) \cap ON$
$\Rightarrow \xi \subseteq L_\beta$ and $\xi \subseteq ON$ (obvious)
$\Rightarrow \xi \subseteq L_\beta \cap ON \underset{I.H.}{=} \beta < \beta + 1 = \alpha$
$\Rightarrow \xi \in \alpha$

$(\supseteq)$ let $\xi \in \alpha \Rightarrow \xi \leq \beta \underset{⊛}{\in} L_\alpha \underset{L_\alpha \text{ transitive}}{\Rightarrow} \xi \in L_\alpha$
$\Rightarrow \xi \in L_\alpha \cap ON$

(a) let $\zeta > \alpha$, one has:

$\alpha \in \zeta \underset{(b)}{\subseteq} L_\zeta \subseteq L \Rightarrow \underline{\alpha \in L}$

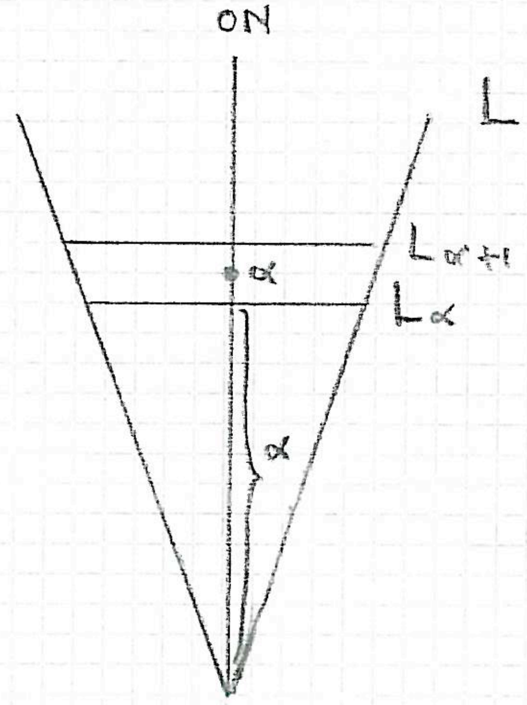
Also, $\alpha \in \alpha+1 \underset{(b)}{\subseteq} L_{\alpha+1} \Rightarrow \alpha \in L_{\alpha+1}$

$\Rightarrow \rho(\alpha) < \alpha+1 \Rightarrow \underline{\rho(\alpha) \leq \alpha}$.

And if $\alpha \in L_\alpha \Rightarrow \alpha \in L_\alpha \cap ON \underset{(b)}{=} \alpha$ impossible,

thus $\underline{\rho(\alpha) \geq \alpha}$. $\alpha \notin L_\alpha$,

$\blacksquare$



Lemma 7: L and WF coincide up to level ω, i.e.
$\forall n \in \omega \, (L_\alpha = V_\alpha)$ and $L_\omega = V_\omega$.

Proof: By induction on $n \in \omega$.

- If $n = 0$, $L_0 = 0 = V_0$.
- If $n = m+1$, $L_n = \mathcal{D}(L_m) \underset{I.H.}{=} \mathcal{D}(V_m) \subseteq \mathcal{P}(V_m) = V_n$

Conversely, $X \in V_n = \mathcal{P}(V_m) \underset{I.H.}{=} \mathcal{P}(L_m)$.
$\Rightarrow X \subseteq L_m$ and X finite (since $m < \omega \Rightarrow L_m$ finite)
Thus $X = \{v_0, --, v_k\}$.
Hence $X = \{x \in L_m : (x = v_0 \vee \dots \vee x = v_k)^{L_m}\}$,
Thus $X \in \mathcal{D}(L_m) = L_n$ by Prop. 4.
Thus $V_n \subseteq L_n$. Therefore $L_n = V_n$, $\forall n \in \omega$.

Now, $L_\omega = \bigcup_{n \in \omega} L_n = \bigcup_{n \in \omega} V_n = V_\omega$

Above level ω, one has the following important property (under AC):

Lemma 8 (AC): $\forall \alpha \geqslant \omega \quad |L_\alpha| = |\alpha|$

Proof: let $\alpha \geqslant \omega$. One has

- $\alpha = L_\alpha \cap On \subseteq L_\alpha \Rightarrow |\alpha| \leqslant |L_\alpha|$
- By induction on α. By the I.H., suppose $\forall \beta$ s.t. $\omega \leqslant \beta < \alpha$, one has $|L_\beta| \leqslant |\beta| \leqslant |\alpha|$

  Then $\forall \beta < \alpha \left(|L_\beta| \leqslant |\alpha| \right)$, since $|L_n| < \omega$ for all $n < \omega$.

  - If α is limit, $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ is a union of $|\alpha|$ sets of cardinalities $\leqslant |\alpha|$, thus (by AC) $|L_\alpha| \leqslant |\alpha|$
  - If $\alpha = \beta+1$, then

$$\begin{aligned} |L_\alpha| &= |D(L_\beta)| \\ &= |L_\beta| \quad \text{(by remark p. 8)} \\ &\leqslant |\beta| \quad \text{(by the I.H.)} \\ &= |\alpha| \end{aligned}$$

$\blacksquare$

So the cardinalities of the L_α's increase very much slower than the cardinalities of the V_α's.