3.3. Cofinality

Oral /

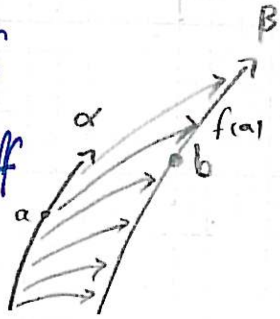
Cardinal exponentiation is a crucial issue of set theory. What is the value of $\aleph_\alpha^{\aleph_\beta}$ $\forall \alpha, \beta \in$ ord ? This is a hard question. The notion of cofinality helps us understanding this issue.

Oral /

Some ordinals and cardinals can be "approached" or "reached" by rather short sequences whereas others cannot. For instance, if we take the set of ord $\{\aleph_n : n < \omega\}$, we have that $\sup\{\aleph_n : n<\omega\} = \aleph_\omega$ which is an uncountable "large" set, but which can be approached by a countable "small" sequence. Some other ordinals or cardinals don't have this property.

Let α, β ordinals.

Def: i) $f : \alpha \to \beta$ is a cofinal map iff $\text{ran}(f)$ is unbounded in β, i.e. iff $\forall b \in \beta\ \exists a \in \alpha$ s.t. $f(a) \geq \beta$ i.e. iff $\sup\{f(a) : a \in \alpha\} = \beta$

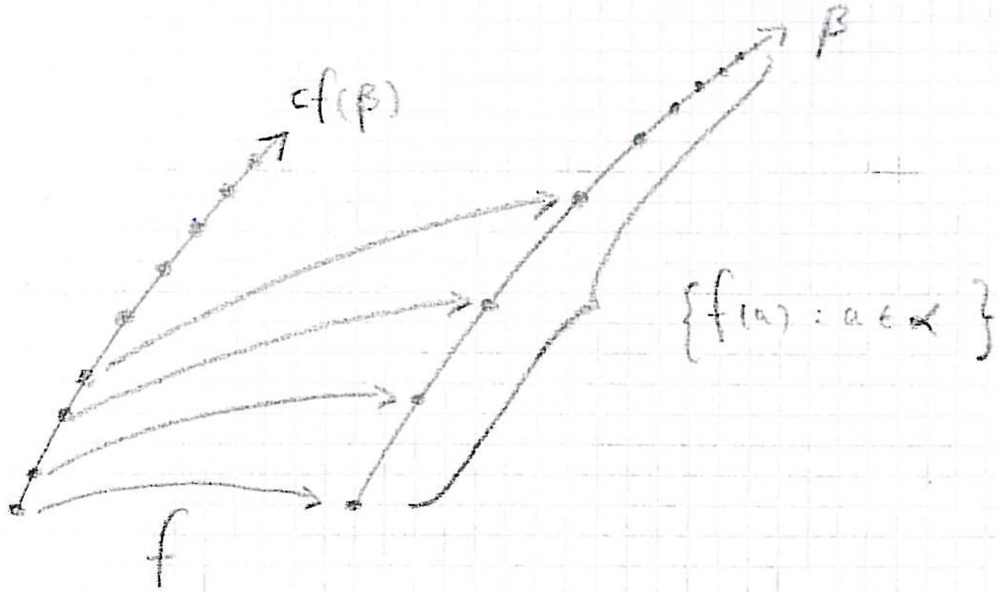


Think of f as a sequence of length α. Then $f : \alpha \to \beta$ cofinal means that "the sequence f can reach the end of β in α-steps".
cover, exhaust, ...

ii) α is cofinal in β iff $\exists f : \alpha \to \beta$ cofinal map.

iii) the cofinality of β, $cf(\beta)$, is the least α s.t. $\exists f : \underset{\underset{cf(\beta)}{\shortparallel}}{\alpha} \to \beta$ cofinal map.

Intuitively, $cf(\beta)$ is the ~~minimal~~ length of all possible sequences whose sup are precisely β.
"which cover β"



After lemma 12

Examples:

i) $cf(\omega) = \omega$, no finite sequence can "exhaust" ω. The minimal length such an "exhausting" sequence is ω.

ii) $cf(\aleph_\omega) = \omega$: · The function $f : \omega \longrightarrow \aleph_\omega$ defined by $f(n) = \aleph_n$ is cofinal
$\Rightarrow cf(\aleph_\omega) \leqslant \omega$.
· $\aleph_\omega$ limit $\Rightarrow cf(\aleph_\omega) \geqslant \omega$.

iii) $cf(\aleph_{\omega+\omega}) = \omega$: · $f : \omega \to \aleph_{\omega+\omega}$ defined by $f(n) = \aleph_{\omega+n}$ is cofinal.
$\Rightarrow cf(\aleph_{\omega+\omega}) \leqslant \omega$
· $\aleph_{\omega+\omega}$ limit $\Rightarrow cf(\aleph_{\omega+\omega}) \geqslant \omega$.

Large cardinals can have small cofinalities ...

lemma 12: i) $cf(\alpha) \leq \alpha$

ii) if α successor, $cf(\alpha) = 1$

iii) if α limit, $cf(\alpha) \geq \omega$

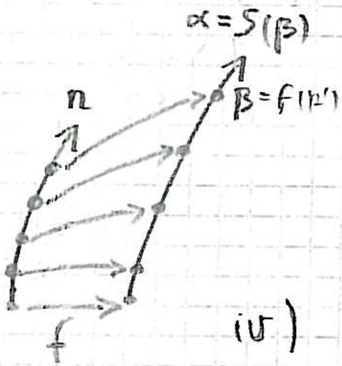
iv) $cf(cf(\alpha)) = cf(\alpha)$

v) $cf(\alpha)$ is a cardinal

Proof: (i) $id : \alpha \rightarrow \alpha$ is cofinal $\Rightarrow cf(\alpha) \leq dom(f) = \alpha$

(ii) if $\alpha = \beta + 1 = \beta \cup \{\beta\}$, then $f : 1 = \{0\} \longrightarrow \alpha = \beta+1$ def by $f(0) = \beta$ is cofinal

(iii) Suppose α limit and $cf(\alpha) = n < \omega$, and Let $f : n < \omega \longrightarrow \alpha$ cofinal. Then



$\forall a \in \alpha \ \exists k < \overset{\substack{dom(f)\\ \downarrow}}{n} \ (f(k) \geq a)$

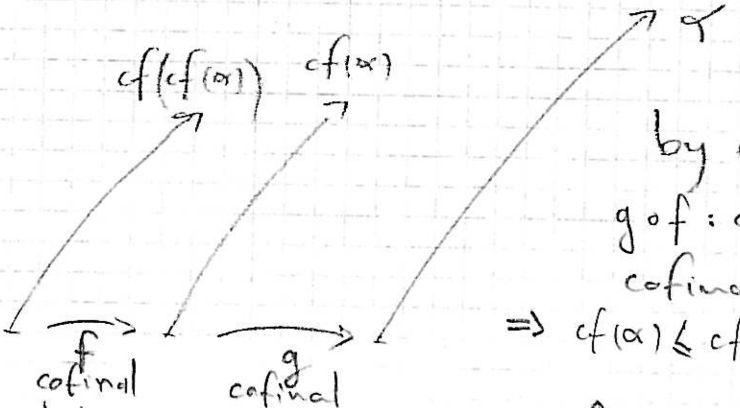
since n finite

let $\beta = \sup\{f(k) : k < n\} = \max\{f(k) : k < n\}$
and $\beta = f(k') \in \alpha$ for some k'

Then $\forall a \in \alpha, a \leq \beta \Rightarrow \alpha = S(\beta)$ (when the sup is in the set, we have a successor ordinal)

iv) By i), $cf(cf(\alpha)) \leq cf(\alpha)$.

Conversely, $\overset{\text{by def of cf}}{,}$ $cf(cf(\alpha))$ is cofinal in $cf(\alpha)$ which is cofinal in $\alpha \Rightarrow cf(\alpha) \leq cf(cf(\alpha))$.



by composition, $g \circ f : cf(cf(\alpha)) \rightarrow \alpha$ cofinal too.
$\Rightarrow cf(\alpha) \leq cf(cf(\alpha))$

v) By contradiction, suppose $\exists \ \theta < cf(\alpha)$ with $\theta \approx cf(\alpha)$
$\Rightarrow \exists f : \theta \rightarrow cf(\alpha)$ bij and $g : cf(\alpha) \longrightarrow \alpha$ cofinal.
Then $g \circ f : \theta \rightarrow \alpha$ cofinal $\Rightarrow cf(\alpha) \leq \theta < cf(\alpha)$, $\perp$

Def: let κ be an infinite cardinal

κ is regular iff $cf(\kappa) = \kappa$

κ is singular otherwise i.e. iff $cf(\kappa) < \kappa$.

↖ i.e. κ can be covered by some "small" sequence

Examples: i) $\aleph_0$ is regular : $cf(\aleph_0) = \omega = \aleph_0$

ii) $\aleph_\omega$ is singular : $cf(\aleph_\omega) = \omega < \aleph_\omega$

- Successor cardinals are regular ...

Lemma 13: If κ is successor cardinal, then κ regular.

Proof: Suppose $\kappa = \lambda^+$ and not regular (i.e. singular)
i.e. $cf(\kappa) < \kappa = \lambda^+$.
Since $cf(\kappa)$ is a cardinal (lemma 12 vi) $\Rightarrow \underline{cf(\kappa) \leq \lambda}$

let $f : cf(\kappa) \longrightarrow \kappa = \lambda^+$ cofinal map.
Since f cofinal, we have

$$\kappa = \lambda^+ = \sup\{f(\xi) \in \lambda^+ : \xi \in cf(\kappa)\}$$
$$= \bigcup\{f(\xi) < \lambda^+ : \xi < cf(\kappa)\}$$

But $f(\xi) < \lambda^+ \Rightarrow |f(\xi)| < \overset{= \lambda^+}{|\lambda^+|},\ \forall \xi < cf(\kappa)$

$\Rightarrow \underline{|f(\xi)| \leq \lambda,\ \forall \xi < cf(\kappa)}$

Thus $\kappa = |\kappa| = \left|\bigcup_{\xi < cf(\kappa) \leq \lambda} f(\xi)\right| \underset{\text{lemma 11}}{\leq} \lambda < \lambda^+ = \kappa$, contradiction ▨

- What about limit cardinals: are they regular or singular?

Limit cardinals are "often" singular; for instance,
$\aleph_\omega$ is singular : $cf(\aleph_\omega) = \omega < \aleph_\omega$.
But do limit (and) regular cardinals also exist?

⟶

Observe that if $\aleph_\alpha$ is limit and regular cardinal, then it must satisfy $\aleph_\alpha = \alpha$. i.e. $\aleph_\alpha$ limit regular $\Rightarrow \aleph_\alpha = \alpha$

Indeed:
- $\alpha \leqslant \aleph_\alpha$ trivial
- $\aleph_\alpha \underset{\text{since regular}}{=} \mathrm{cf}(\aleph_\alpha) \underset{\text{since } \alpha \text{ limit, Kunen 10.38}}{=} \mathrm{cf}(\alpha) \leqslant \alpha$

i.e. $\aleph_\alpha = \alpha \not\Rightarrow \aleph_\alpha$ limit regular.

$\lambda = \sup\{\omega, \aleph_\omega, \aleph_{\aleph_\omega}, \aleph_{\aleph_{\aleph_\omega}}, \dots\}$

But this condition is not sufficient: consider $\sigma_0 = \omega$, $\sigma_{n+1} = \aleph_{\sigma_n}$, $\forall n < \omega$, and $\lambda = \sup\{\sigma_n : n < \omega\}$

Then $\aleph_\lambda = \lambda$: indeed $\aleph_\lambda = \aleph_{\sup\{\sigma_n : n<\omega\}}$

by def, since sup{--} is limit → $= \sup\{\aleph_{\sigma_n} : n < \omega\}$

$= \sup\{\sigma_{n+1} : n < \omega\} = \lambda$

But $\mathrm{cf}(\aleph_\lambda) = \omega < \aleph_\lambda$, since it can be "reached" by the ω-sequence of σ_n's, meaning that $\aleph_\lambda$ is not regular (i.e. singular).

→

So $\aleph_\lambda$ is the 1st limit cardinal s.t. $\aleph_\lambda = \lambda$ and it is not "large enough" to be regular.

$\Rightarrow$ Thus limit regular cardinals must be "rather large".

This motivates the following definition:

Def: let K be an infinite cardinal

K is weakly inaccessible iff K is limit and regular

K is strongly inaccessible iff $K > \omega$, regular, and $\forall \lambda < K \, (2^\lambda < K)$

(i.e. K cannot be reached from below by the power-set operation)

ZF cannot prove the existence of neither weakly nor strongly innaccessible cardinals.

Remark: Strongly inacc. $\Rightarrow$ weakly inacc. (i.e. limit and regular)

let κ str. in. $\Rightarrow$ κ regular (by def). It remains to prove that κ is limit

and suppose κ successor i.e. $\kappa = \lambda^+$. Then we have (by def of str. inacc.)

$\forall \mu < \lambda^+ \; (2^\mu < \lambda^+)$

But $\lambda < \lambda^+$ and $2^\lambda \geqslant \lambda^+$, contradiction.

Under GCH, strongly inacc. $\Leftrightarrow$ weakly inacc.

Suppose κ weak. inacc. and GCH.

let $\lambda < \kappa$, then $2^\lambda \overset{GCH}{=} \lambda^+ \leqslant \kappa$

$\Rightarrow \lambda^+ < \kappa$, otherwise κ would be successor, (contradiction with weak. inacc.)

$\Rightarrow \kappa$ strong. inacc.

As soon as λ becomes larger than $cf(\kappa)$, the exponentiation κ^λ becomes strictly above κ.

lemma 14 (König) (AC): If κ infinite and $\lambda \geqslant cf(\kappa)$, then $\kappa^\lambda > \kappa$ $\left(\begin{array}{l} \text{i.e. } \kappa^\lambda \leqslant \kappa \\ \Rightarrow \lambda < cf(\kappa) \end{array} \right)$

the exp. vanishes...

$\longrightarrow$ illustration

$\kappa \leqslant \kappa^\lambda$ trivial:

Proof: Let $f : \lambda \to \kappa$ cofinal (exists since $\lambda \geqslant cf(\kappa)$) and let $G : \kappa \to {}^\lambda\kappa$.

We show that G is not onto, i.e. $\kappa < {}^\lambda\kappa$ and thus $\kappa < \kappa^\lambda$.

Define $h : \lambda \to \kappa$ by $h(\alpha) =$ least el. of $\kappa \setminus \{G(\mu)(\alpha) : \mu < f(\alpha)\}$

Then $h \notin ran(G)$: suppose it does:

$\exists \mu' < \kappa$ s.t. $G(\mu') = h$ $\Rightarrow G(\mu')(\alpha') = h(\alpha')$. But f cofinal $\Rightarrow$

$\exists \alpha' < \lambda$ s.t. $f(\alpha') > \mu'$ $\longrightarrow \mu' < f(\alpha')$ thus $G(\mu')(\alpha') = h(\alpha')$ belongs to this set

$h(\alpha') =$ least el of $\kappa \setminus \underbrace{\{G(\mu)(\alpha') : \mu < f(\alpha')\}}_{h(\alpha'')}$

$\neq h(\alpha')$, contradiction ∎

Corollary 15: If $\lambda \geqslant \omega$, $cf(2^{\lambda}) > \lambda$

i.e. not only $2^{\lambda} > \lambda$ but already $cf(2^{\lambda}) > \lambda$

as soon a $\lambda \geqslant \omega$, we have that $cf(2^{\lambda}) > \lambda$ and thus also $2^{\lambda} \geqslant cf(2^{\lambda}) > \lambda$

Proof: We have $(2^{\lambda})^{\lambda} = 2^{\lambda \otimes \lambda} = 2^{\lambda}$, thus by lemma 14, $cf(2^{\lambda}) \cancel{\leqslant} \lambda$ $\blacksquare$

(with $K = 2^{\lambda}$)

Under AC and GCH, thanks to the notion of cofinality, we can precisely compute the cardinal exponentiation.

$\boxed{\begin{array}{c}AC\\+\\GCH\end{array}}$ **Proposition 16:** Let $K, \lambda \geqslant 2$ with at least K or λ infinite:

$$K^{\lambda} = \begin{cases} \lambda^{+} & \text{, if } K \leqslant \lambda \\ K^{+} & \text{, if } cf(K) \leqslant \lambda < K \\ K & \text{, if } \lambda < cf(K) \end{cases}$$

when λ too small, exponentiation doesn't work anymore.

Proof: i) $$\lambda^{+} \underset{GCH}{\approx} {}^{\lambda}2 \leqslant \underline{\underline{{}^{\lambda}K}} \leqslant \bigcirc\!\!\!\!\!\!{}^{\lambda}\lambda \leqslant \mathcal{P}(\lambda \times \lambda) \underset{\lambda \text{ infinite}}{\approx} \mathcal{P}(\lambda) \approx {}^{\lambda}2 \underset{GCH}{\approx} \lambda^{+}$$

by. def, a function $\lambda \to \lambda$ is a subset of $\lambda \times \lambda$

Thus $K^{\lambda} = |{}^{\lambda}K| = |\lambda^{+}| = \lambda^{+}$

ii) By lemma 14, $K^{\lambda} \overset{\lambda \geqslant cf(K)}{>} K$, but also

$K^{\lambda} \underset{\substack{\text{since}\\ \lambda < K}}{\leqslant} K^{K} = K^{+}$. Thus $K^{\lambda} = K^{+}$

by (i) (since $K \leqslant K$, i) applies)

iii) $\lambda < cf(K) \Rightarrow$ any $f \in {}^{\lambda}K$ is not cofinal in K and thus has its range bounded by some $\alpha_f < K$ i.e.

${}^{\lambda}K \subseteq \bigcup\{{}^{\lambda}\alpha : \alpha < K\}$ and by (i),

each $|{}^{\lambda}\alpha| \leqslant \max(\alpha, \lambda)^{+} \leqslant K$.

By lemma 11, $K^{\lambda} \leqslant K \Rightarrow K^{\lambda} = K$ $\blacksquare$

$K \leqslant K^{\lambda}$ obvious since $\lambda \geqslant 2$

$\left(a \longmapsto f : 1 = \{0\} \longrightarrow K, \; 0 \longmapsto a \right)$