NGSA: NETWORK SCIENCE ANALYTICS

CENTRALESUPÉLEC

Assignment 1

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Due: **December 23, 2018 at 23:00**

How to submit: Please complete the first assignment **individually**. *Typeset* all your answers (**PDF** file only). Submissions should be made on **gradescope** (Entry Code: MYDKND). Make sure that the answer to each question is on a separate page (questions 1-10). Also, include in your solutions the important parts of your code and any reference that you have used. No late assignments will be accepted.

I. Graph Theory and Graph Properties

Question 1 [6 points]

Let $\mathbf{A}$ be the adjacency matrix of an undirected graph (unweighted, with no self-loops) and $\mathbf{1}$ be the column vector whose elements are all 1. In terms of these quantities and simple matrix operations like matrix transpose and matrix trace, write expressions for:

(a) [2 p] The vector $\mathbf{k}$ whose elements are the degrees k_i of the nodes.

(b) [2 p] The number m of edges in the graph.

(c) [2 p] The matrix $\mathbf{N}$ whose element N_{ij} is equal to the number of common neighbors of nodes i and j.

Question 2 [5 points]

Consider a bipartite network, with its two types of nodes (type 1 and 2), and suppose that there are n_1 nodes of type 1 and n_2 nodes of type 2. Show that the mean degrees c_1 and c_2 of the two types are related by

$$c_2 = \frac{n_1}{n_2} c_1.$$

Question 3 [15 points]

Let $G = (V, E)$ be an unweighted, undirected graph with no self-loops. The (i, j)-th element of $\mathbf{A}^\ell$ (i.e., the adjacency matrix raised to the ℓ-th power) counts the number of paths of length ℓ that start from node i and end at node j. A triangle in a graph corresponds to a clique of three nodes.

(a) [4 p] Using simple matrix operations, express the total number of triangles in the graph $\Delta(G)$, as a function of the adjacency matrix $\mathbf{A}$.

(b) [4 p] Similarly, express the total number of triangles in the graph $\Delta(G)$ as a function of the eigenvalues $\lambda_i,\ \forall i \in V$ of $\mathbf{A}$.

(c) [7 p] Let $\Delta_i,\ \forall i \in V$ be the number of triangles that node i participates in. Express Δ_i as a function of the spectrum (i.e., eigenvalues and/or eigenvectors) of the adjacency matrix $\mathbf{A}$.

II. Graph Models

Question 4 [15 points]

Consider the random graph $G_{n,p}$ with average degree c.

(a) [6 p] Show that in the limit of large n, the expected number of triangles in the graph is $\frac{1}{6}c^3$. In other words, show that the number of triangles is constant, neither growing nor vanishing in the limit of large n.

(b) [6 p] A connected triplet is defined as a triplet of nodes uvw, with edges (u, v) and (v, w) (the edge (u, w) can be present or not). Show that the expected number of connected triplets in the graph is $\frac{1}{2}nc^2$.

(c) [3 p] The clustering coefficient of a graph can also be expressed as

$$C = \frac{\text{(number of triangles)} \times 3}{\text{(number of connected triplets)}}.$$

Calculate the clustering coefficient of the $G_{n,p}$ random graph using the above formula based on (a) and (b), and confirm that for large n it agrees with the value shown in class (Lecture 2A; slide number 32).

III. Centrality Criteria

Question 5 [7 points]

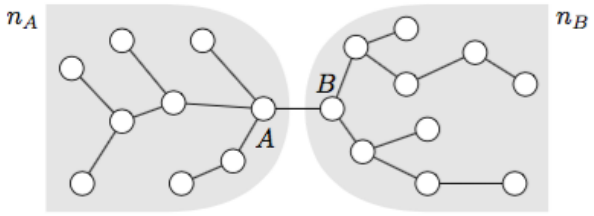
Suppose that we define a new centrality criterion $x_i, \forall i \in V$ to be a sum of contributions as follows: 1 for node i itself, α for each node at (geodesic) distance 1 from i, α^2 for each node at distance 2, and so forth, where $\alpha < 1$ is a given constant.

(a) [4 p] Write an expression for x_i in terms of α and the geodesic distances d_{ij} between node pairs.

(b) [3 p] Describe briefly (max 3 lines) an algorithm for computing this centrality measure. What is the complexity of calculating x_i for all $i \in V$?

Question 6 [7 points]

Consider an undirected, unweighted graph of n nodes that is composed by exactly two subgraphs of size n_A and n_B, which are connected by a single edge (A, B), as shown below:



Show that the closeness centralities C_A and C_B of nodes A and B respectively, are related by

$$\frac{1}{C_A} + \frac{n_A}{n} = \frac{1}{C_B} + \frac{n_B}{n}.$$

Recall that, the closeness centrality is given by: $C_i = \frac{n}{\sum_j d_{ij}}$, where d_{ij} is the length of the geodesic path from i to j.

IV. Analyzing a Real Network

In the last part of the assignment, you will analyze the `CA-GrQc` collaboration network, examining several structural properties. The arXiv GR-QC (General Relativity and Quantum Cosmology) collaboration network has been extracted from the e-print arXiv (`arxiv.org`) and covers scientific collaborations between authors for papers submitted to General Relativity and Quantum Cosmology category. If an author i co-authored a paper with author j, the graph contains an undirected edge from i to j. Here we consider that the graph is unweighted.
The graph is stored in the `ca-GrQc.txt` file[1], as an edge list:

```
# Directed graph (each unordered pair of nodes is saved once): CA-GrQc.txt
# Collaboration network of Arxiv General Relativity category (there is an
edge if authors coauthored at least one paper)
# Nodes: 5242 Edges: 28980
# FromNodeId ToNodeId
3466 937
3466 5233
...
```

For the following questions, feel free to use the graph library of your preference (we strongly encourage you to use Python's NetworkX, igraph or snappy), unless stated otherwise. For the figures, make sure to include axes titles and labels.

Question 7 [19 points]

(a) [3 p] (*Basic properties of the network*). (1) Compute the following statistics: number of nodes, number of edges. (2) If the graph is not connected: (i) find the number of connected components (CCs); (ii) plot the distribution of their sizes (similar to MSN's connectivity shown in Lecture 1B, slide number 81). (3) Extract the largest (i.e., giant) connected component (GCC), and then (i) find the number of nodes and edges of GCC and (ii) the fraction of nodes and edges of the whole graph that belong to GCC (e.g., 5% of the number of nodes are part of GCC). Discuss briefly your observation (2-3 lines).

[1]The data can be downloaded from the following link: `http://snap.stanford.edu/data/ca-GrQc.txt.gz`.

(b) [5 p] (*Analysis of the degree distribution*). (1) Find the minimum, maximum, median and mean degree of the nodes of the graph. What do you observe (please, discuss briefly in 1-2 lines)? (2) Visualize (appropriately) the degree distribution of the graph. What is the type of the degree distribution and what are the parameters? (You can use freely available software that was discussed in class).

(c) [4 p] (*Triangles*). For this one and the following questions, we will focus on the GCC of the graph (in other words, consider that the GCC itself is the graph that we are interested to analyze). As we have discussed in the class, a triangle is a clique of three nodes, i.e., all nodes are connected to each other. Triangle subgraphs play a crucial role in the area of graph mining and social network analysis, since they are closely related to the existence of clustering structures in the graph. (1) Compute the total number of triangles in the GCC of the network. (2) Visualize (plot) the triangle participation distribution, i.e., a histogram that shows the number of triangles that each node participates in (i.e., how many nodes participate in one triangle, how many nodes participate in two triangles, etc). Note that, this process is similar to the computation of the degree distribution. Discuss briefly your observations.

(d) [7 p] (*Spectral counting of triangles*). In Question 3, you have been asked to express the total number of triangles $\Delta(G)$ using information about the eigenvalues of the adjacency matrix of the graph. Here, you should compute the number of triangles in the GCC of the graph using the eigenvalues of the corresponding adjacency matrix (again, consider that your graph G is the GCC). In order to compute exactly $\Delta(G)$, we need to compute the whole spectrum of the adjacency matrix, which can be a computational bottleneck (why?). Here, we argue that we can approximate $\Delta(G)$ using the top-k ($k \ll |V|$) largest eigenvalues of the adjacency matrix, i.e., $\Delta(G) \approx \tilde{\Delta}_k(G)$, where $\tilde{\Delta}_k(G)$ is the spectral computation of the number of triangles, using only the top-k eigenvalues of the adjacency matrix. (1) Why is this happening? Explain your answer, giving specific arguments. [Hint: we can do low rank approximation of the adjacency matrix for the computation of triangles, taking advantage of some of the "power-law" properties of real networks that we have seen in class]. (2) Suppose that we approximate $\Delta(G)$ with $\tilde{\Delta}_k(G)$. Compute and visualize the error of approximation for various values of k (i.e., how the error behaves as we increase the number of eigenvalues used in the computation of $\tilde{\Delta}_k(G)$):

$$\text{error}_k = \frac{|\tilde{\Delta}_k(G) - \Delta(G)|}{\Delta(G)},$$

How many eigenvalues should we retain to achieve good approximation? In the extreme case where all the $|V|$ eigenvalues are used, $\Delta(G) = \tilde{\Delta}_{k=|V|}(G)$.
[**Note:** Depending on the memory of your computer, the computation of the full spectrum of $\mathbf{A}$ may take significant amount of time and resources. Thus, in your experiments, restrict your attention only on the largest 1000 eigenvalues of $\mathbf{A}$, avoiding computing all the eigenvalues.]

Question 8 [6 points]

Generate an Erdös-Rényi random graph $G_{n,p}$ with $n = 1000$ nodes and $p = 0.009$. For this task, you can use the built-in function of NetworkX `nx.fast_gnp_random_graph(n, p)`.

(a) [2 p] What is the mean degree of the graph? Justify theoretically your answer (max 1 line).

(b) [2 p] Is the graph connected? Justify theoretically your answer (max 1 line).

(c) [2 p] Compute the mean degree from the graph that you have generated and visualize the degree distribution of the graph. Discuss briefly your observations.

Question 9 [8 points]

In this question, you will examine the properties of the graphs produced by the *Kronecker model*, which was introduced as a simple generation model for real-world graphs based on the Kronecker product of matrices. More precisely, assuming an initiator adjacency matrix $\mathbf{A}_1$ of size $\ell \times \ell$, the Kronecker graph after k iterations is defined as the graph with the following adjacency matrix:

$$\mathbf{A}_k = \underbrace{\mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \cdots \otimes \mathbf{A}_1}_{k \text{ iterations}} = \mathbf{A}_{k-1} \otimes \mathbf{A}_1.$$

In practice, a stochastic version of the Kronecker model is used, in the sense that the initiator matrix $\mathbf{A}_1$ is not the binary adjacency matrix itself but the probability matrix for the existence of an edge. For example, in the typical case of a 2×2 initiator matrix $\mathbf{A}_1 = [a\ b; c\ d]$, each value represents the probability of existence of the corresponding edge. Starting by such an initiator matrix and applying the Kronecker product for a desired number of iterations k, the resulting adjacency matrix of the graph corresponds to a realization of the matrix $\mathbf{A}_k$, i.e., each edge (i, j) is introduced to the graph with probability $A_k(i, j)$.

Consider the following 2×2 initiator matrix $\mathbf{A}_1 = [0.99, 0.26; 0.26, 0.53]$ (notice that the matrix is in the form $[a\ b; b\ c]$). The values of the initiator matrix are important, in the sense that they are related to the properties of the generated graph. In particular, the above matrix has been produced by the KRONFIT algorithm presented in class, fitting its parameters to the structure of the `CA-GrQc` network. Repeat the Kronecker product for k times and compute the $\mathbf{A}_k$ adjacency matrix (you can use the `kron(A,B)` built-in function of *numpy* for Kronecker multiplication). Lastly, for each entry $\mathbf{A}_k(i, j)$ of matrix $\mathbf{A}_k$, include the edge (i, j) with probability $\mathbf{A}_k(i, j)$. This matrix will be the adjacency matrix of the final graph (consider that the produced Kronecker graph is *undirected*).

(a) [2 p] (i) Is the produced Kronecker graph connected? (ii) Does the graph have a giant connected component of size $\Theta(n)$? Provide *theoretical justification* of your answers. You can also confirm your answers by examining the properties of the graph (e.g., computing the actual size of the GCC).

(b) [6 p] Describe **three** structural properties which show if the produced graph looks similar or not (qualitatively or even quantitatively) to the `CA-GrQc` network. For each of those properties you should examine how they look like in both the original `CA-GrQc` network and produced Kronecker graph (e.g., compute/visualize property A of `CA-GrQc` and repeat the same for the Kronecker graph). Explain briefly your observations.

Question 10 [12 points]

The robustness of a network is related to the capability of the network to retain its structure and connectivity properties, after losing a portion of its nodes and edges. Let us focus our attention to the case where the nodes of a network are deleted according to their degree based on two strategies:

(i) *Random deletion:* delete a randomly selected node.

(ii) *Targeted deletion:* delete a node chosen among the ones with the highest degree in the network.

Depending on the type of networks, the above two strategies can simulate various scenaria. For example, in the case of the Internet graph (nodes correspond to routers and edges capture physical connections between routers), a random deletion can be interpreted as an *error* that occurred in the network, where a router switched off due to technical problems (e.g., electricity problems). On the other hand, the targeted removal of a high-degree nodes can simulate the case of an attack to the network, where the removal of nodes aims to cause big damage to the network.

How the structure of the network is affected after random and targeted deletions of nodes? In the class, we discussed that due to the existence of the heavy-tailed degree distribution, real-world networks tend to be robust under random removal of nodes (e.g., errors) and vulnerable under the attacks to high degree nodes. The notion of robustness can be quantified by structural characteristics of the network, such as the fragmentation of the network into disconnected components. For instance, we can examine how the fraction of nodes that belong to the largest connected component (GCC) and the rest isolated components is affected by the random/targeted removal of nodes. Here, your goal will be to examine the robustness of the `CA-GrQc` network, with respect to the above strategies.

For this question, work with the GCC of the graph (i.e., your initial graph is the GCC of the `CA-GrQc` network). To assess the robustness, you should examine how the size of GCC and the rest components is affected, after removing a fraction of nodes in the range of $[0\%, 20\%]$.

- Plot the size of GCC and the size of the rest components (sum of their sizes) vs. the fraction of deleted nodes, for both strategies (i.e., random and targeted) in the same figure (i.e., four different curves). Discuss briefly your observations.

Note: In order to have a "reference figure", the goal is to reproduce Fig. 3 (c) or (d) of the paper *Error and attack tolerance of complex networks*, by R. Albert, H Jeong, and A.-L. Barabasi (Nature 406, 378-382, 2000)[2]. Notice that in the beginning, the GCC contains the whole graph (thus the rest components are empty), but both curves start from the same point (1) in the vertical axis. Moreover, the produced figure may not be exactly the same as the one of the above paper, since we analyze a different network – but, overall, there should be some resemblance.

[2] `https://www.nature.com/articles/35019019`