Demo Fourier FBM

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Let $H \in [0,1]$ and $x = 0, 1, ..., N-1$ with $N \in \mathbb{N}$. We have $(\phi_i)_{i \in \mathbb{Z}}$ i.i.d. $\sim U([0, 2\pi])$.
We define:

$$W_x^H = \sqrt{2} \sum_{k=-N/2}^{N/2-1} \sqrt{S(f_k)\Delta f} \cos(2\pi f_k x + \phi_k) \quad \text{with } f_k = \frac{k}{N} \text{ , } \Delta f = \frac{1}{N}$$

Where $S(f)$ is the discrete Fourier transform of C defined by:

$$S(f) = \sum_{t \in \mathbb{Z}} C(t) \cos(2\pi t f)$$

We can rewrite W_x^H as:

$$W_x^H = \sum_{k=-N/2}^{N/2-1} X_k \quad \text{with } X_k = \sqrt{S(f_k)\Delta f} \cos(2\pi f_k x + \phi_k)$$

With the X_k independently distributed by composition of a continuous function. We can show that a condition analogous to the Lindeberg condition (Rice et al.) (1954) is satisfied:

$$\sum_{k=0}^{N} \mathbb{E}[X_k^2])^{-\frac{3}{2}} \sum_{k=0}^{N} \mathbb{E}[|X_k|^3] \xrightarrow[N \to \infty]{} 0$$

which ensures that no variable has a predominant importance compared to the others, thus allowing us to dispense with the condition of identical distribution of the variables. Therefore, for sufficiently large N, we have by the central limit theorem:

$$W_x^H \xrightarrow[N \to \infty]{} \mathcal{N}(\mu, \sigma^2)$$

We now want to verify that the entire process is Gaussian. Let $\lambda_0, ..., \lambda_{N-1} \in \mathbb{R}^N$ and define:

$$
\begin{aligned}
W &= \lambda_0 W_0^H + ... + \lambda_{N-1} W_{N-1}^H \\
&= \lambda_0 \sqrt{2} \sum_{k=-N/2}^{N/2-1} \sqrt{S(f_k)\Delta f} \cos(\phi_k) + ... + \lambda_{N-1} \sqrt{2} \sum_{k=-N/2}^{N/2-1} \sqrt{S(f_k)\Delta f} \cos(2\pi f_k (N-1) + \phi_k) \\
&= \sqrt{2} \sum_{k=-N/2}^{N/2-1} \sqrt{S(f_k)\Delta f} (\lambda_0 \cos(\phi_k) + ... + \lambda_{N-1} \cos(2\pi f_k (N-1) + \phi_k)) \\
&= \sum_{k=-N/2}^{N/2-1} X'_k
\end{aligned}
$$

We can show here that the same condition used previously is satisfied, and thus by the central limit theorem:

$$W \xrightarrow[N\to\infty]{} \mathcal{N}(\mu', \sigma'^2)$$

Therefore, $(W_0^H, ...W_{N-1}^H)$ is a Gaussian vector with $(W_x^H)_{x\in\{0,...,N-1\}}$ a series of observations of the Gaussian process W_t^H. This result remains valid for any discrete set where x is defined.

We now want to calculate the expectation and the autocovariance function of our process, for all $x \in \{0, ..., N-1\}$:

$$
\begin{aligned}
\mathbb{E}[W_x^H] &= \sqrt{2} \sum_{k=-N/2}^{N/2-1} \sqrt{S(f_k)\Delta f} \mathbb{E}[\cos(2\pi f_k x + \phi_k)] \\
&= \sqrt{2} \sum_{k=-N/2}^{N/2-1} \sqrt{S(f_k)\Delta f} \int_{-\infty}^{\infty} \cos(2\pi f_k x + u) \mathbf{1}_{[0,2\pi]}(u)\, du \\
&= \sqrt{2} \sum_{k=-N/2}^{N/2-1} \sqrt{S(f_k)\Delta f} [\sin(2\pi f_k x + u)]_{u=0}^{u=2\pi} \\
&= 0
\end{aligned}
$$

Our process is therefore centered. For all $(x, x+t) \in \{0, ..., N-1\}^2$:

$$Cov(W_x^H, W_{x+t}^H) = \mathbb{E}[W_x^H W_{x+t}^H]$$

Since W^H is centered.

$$\mathbb{E}[W_x^H W_{x+t}^H] = \mathbb{E}[\sqrt{2}\sum_{k=-N/2}^{N/2-1}\sqrt{S(f_k)\Delta f}\cos(2\pi f_k x+\phi_k)\sqrt{2}\sum_{k=-N/2}^{N/2-1}\sqrt{S(f_k)\Delta f}\cos(2\pi f_k(x+t)+\phi_k)]$$

$$= 2\mathbb{E}[\sum_{k=-N/2}^{N/2-1} S(f_k)\Delta f\cos(2\pi f_k x+\phi_k)\cos(2\pi f_k(x+t)+\phi_k)$$

$$+ 2\sum_{k=-N/2}^{N/2-1}\sum_{\substack{k'=-N/2\\k'\neq k}}^{N/2-1}\sqrt{S(f_k)\Delta f}\sqrt{S(f_k')\Delta f}\cos(2\pi f_k x+\phi_k)\cos(2\pi f_{k'}(x+t)+\phi_{k'})]$$

$$= 2\sum_{k=-N/2}^{N/2-1} S(f_k)\Delta f\mathbb{E}[\cos(2\pi f_k x+\phi_k)\cos(2\pi f_k(x+t)+\phi_k)]$$

$$+ 4\sum_{k=-N/2}^{N/2-1}\sum_{\substack{k'=-N/2\\k'\neq k}}^{N/2-1}\sqrt{S(f_k)\Delta f}\sqrt{S(f_k')\Delta f}\mathbb{E}[\cos(2\pi f_k x+\phi_k)]\mathbb{E}[\cos(2\pi f_{k'}(x+t)+\phi_{k'})]$$

since ϕ_k and $\phi_{k'}$ are independent

$$= 2\sum_{k=-N/2}^{N/2-1} S(f_k)\Delta f\int_0^{2\pi}\cos(2\pi f_k x+u)\cos(2\pi f_k(x+t)+u)\,du$$

since $\mathbb{E}[\cos(2\pi f_k x+\phi_k)] = \mathbb{E}[\cos(2\pi f_{k'}(x+t)+\phi_{k'})] = 0$

$$= 2\sum_{k=-N/2}^{N/2-1} S(f_k)\Delta f\int_0^{2\pi}\frac{1}{2}[\cos(2\pi f_k t)+\cos(2\pi f_k(2x+t)+2u)]\,du$$

since $\cos(s)\cos(t) = \dfrac{1}{2}[\cos(t-s)+\cos(t+s)]$

$$= 2\sum_{k=-N/2}^{N/2-1} S(f_k)\Delta f[2\pi\frac{1}{2}\cos(2\pi f_k t)+\int_0^{2\pi}\cos(2\pi f_k(2x+t)+2u)\,du]$$

$$= 2\pi\sum_{k=-N/2}^{N/2-1} S(f_k)\Delta f\cos(2\pi f_k t)\quad\text{(inverse discrete Fourier transform of } S)$$

since $\displaystyle\int_0^{2\pi}\cos(2\pi f_k(2x+t)+2u)\,du = 0$

$$= C(t) = \frac{1}{2}(|t+1|^{2H}+|t-1|^{2H}-2|t|^{2H})$$

We identify the mean function and the characteristic autocovariance function of an fBM in our simulated process. Since a Gaussian process is entirely characterized by these, the process simulated with this method is indeed an fBM with exponent H. The convergence occurs for large N, so this method may not be suitable for simulations with few steps.