Soit $H \in (0,1)$ et $x = 0, \dots, N-1$ $x \in \mathbb{N}$ ϕ_i iid $\sim \mathcal{U}([0, 2\pi])$

On pose

- $W_x^H = \sqrt{2} \sum_{k=-\frac{N}{2}}^{N/2-1} \sqrt{S(f_k)\Delta f} \cos(2\pi f_k x + \phi_k) \sim \mathcal{N}(0, \sigma_1^2)$ TCL

as ϕ_k iid $\to \cos(2\pi f_k x + \phi_k)$ iid

~~$\forall x \in [0, N-1]$~~ With $S(f)$ spectral power density of fBM^H (→ fourier transform of autocovel fBM^H)

- $\forall \lambda_0, \dots, \lambda_{N-1} \in \mathbb{R}^N$ $\Delta f = \frac{1}{N}$ et $f_k = \frac{k}{N}$

$S(f) = \sum_{k \in \mathbb{Z}} C(k) \cos(2\pi k f)$ (discrete fourier transform)

$\lambda_0 W_0^H + \dots + \lambda_{N-1} W_{N-1}^H$

$= \lambda_0 \sqrt{2} \sum_{k=-\frac{N}{2}}^{N/2-1} \sqrt{S(f_k)\Delta f} \cos(\phi_k) + \dots + \lambda_{N-1} \sqrt{2} \sum_{k=-\frac{N}{2}}^{N/2-1} \sqrt{S(f_k)\Delta f} \cos(2\pi f_k (N-1) + \phi_k)$

$= \sqrt{2} \sum_{k=-\frac{N}{2}}^{N/2-1} \sqrt{S(f_k)\Delta f} \left(\lambda_0 \cos(\phi_k) + \dots + \lambda_{N-1} \cos(2\pi f_k (N-1) + \phi_k) \right) \sim \mathcal{N}(0, \sigma_2^2)$ TCL

$\Rightarrow (W_0^H, \dots, W_{N-1}^H)$ Gaussian vector

We can shown the same result for every t_x wich is the definition of a gaussian process. ~~A fBm~~ An fBM is a gaussian process centred with autocovariance C

$C(k) = \frac{1}{2}\left(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H} \right)$

- $E(W_x^H) = \sqrt{2} \sum_{k=-\frac{N}{2}}^{N/2-1} \sqrt{S(f_k)\Delta f}\, E(\cos(2\pi f_k x + \phi_k))$

$= \sqrt{2} \sum_{k=-\frac{N}{2}}^{N/2-1} \sqrt{S(f_k)\Delta f} \int_{\mathbb{R}} \cos(2\pi f_k x + u) \mathbb{1}_{[0, 2\pi]}(u)\, du$

$= \sqrt{2} \sum_{k=-N/2}^{N/2-1} \sqrt{S(f_k)\Delta f} \underbrace{\left[\sin(2\pi f_k x + u) \right]_0^{2\pi}}_{=0}$

$= 0$

$(x, x+t) \in [0, N-1]^2$

$\mathrm{Cov}(W_x^H, W_{x+t}^H) = E(W_x^H W_{x+t}^H)$ as W_x^H is centered ie $E(W_x^H) = 0$

$E(W_x^H W_{x+t}^H) = E\left(\sqrt{2} \sum_{k=-\frac{N}{2}}^{N/2-1} \sqrt{S(f_k)\Delta f} \cos(2\pi f_k x + \phi_k)\ \sqrt{2} \sum_{k=-\frac{N}{2}}^{N/2-1} \sqrt{S(f_k)\Delta f} \cos(2\pi f_k (x+t) + \phi_k) \right)$

$= 2 E\left(\sum_{k=-\frac{N}{2}}^{N/2-1} S(f_k)\Delta f \cos(2\pi f_k x + \phi_k) \cos(2\pi f_k (x+t) + \phi_k) + 2 \sum_{\substack{k, k' = -\frac{N}{2} \\ k \neq k'}}^{N/2-1} \sqrt{S(f_k)\Delta f} \sqrt{S(f_{k'})\Delta f} \cos(2\pi f_k x + \phi_k) \cdot \cos(2\pi f_{k'}(x+t) + \phi_{k'}) \right)$

$= 2 \sum S(f_k)\Delta f\, E(\cos(2\pi f_k x + \phi_k) \cos(2\pi f_k (x+t) + \phi_k)) + 2 \sum\sum_{k \neq k'} \sqrt{S(f_k) S(f_{k'})}\, \Delta f\, E\left(\cos(2\pi f_k x + \phi_k) \cos(2\pi f_{k'}(x+t) + \phi_{k'})\right)$

$= 2 \sum S(f_k)\Delta f \int_0^{2\pi} \cos(2\pi f_k x + u) \cos(2\pi f_k (x+t) + u)\, du + 2 \sum\sum_{k \neq k'} \sqrt{S(f_k) S(f_{k'})}\, \Delta f\, \underbrace{E(\cos(2\pi f_k x + \phi_k))}_{=0} \underbrace{E(\cos(2\pi f_{k'}(x+t) + \phi_{k'}))}_{=0}$ as independent $\phi_k, \phi_{k'}$

$= 2 \sum S(f_k)\Delta f \int_0^{2\pi} \frac{1}{2}\left[\cos(2\pi f_k t) + \underbrace{\cos(2\pi f_k (2x+t) + u)}_{=0} \right] du$ as $\cos(s)\cos(t) = \frac{1}{2}(\cos(t-s) + \cos(t+s))$

$= 2 \sum_{k=-\frac{N}{2}}^{N/2-1} S(f_k)\Delta f\, \pi \cos(2\pi f_k t) \to$ Inverse fourier transform

$= C(t) = \frac{1}{2}\left(|t+1|^{2H} + |t-1|^{2H} - 2|t|^{2H} \right)$