SGA - Midterm Quiz - Isupov Ilya

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1 The first task

Can it be that $|A| = 5$, $|B| = 3$ and $|A \cup B| = 6$? If it is possible, provide an example, otherwise provide a proof that this is impossible.

It is possible. For example, we have the following two sets: $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6\}$, $|A| = 5$ and $|B| = 3$.
Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$, $|A \cup B| = 6$.
We can also show this solution graphically - using a Venn diagram:

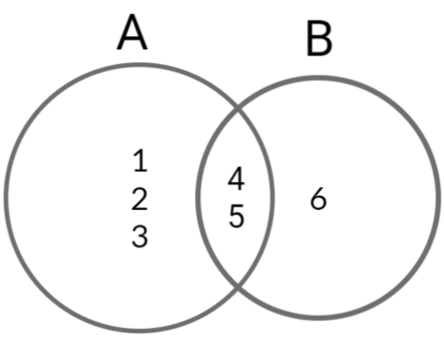


Figure 1: Set A and set B

2 The second task

Can it be that $|A| = 5$, $|B| = 3$, $|A \cup B| = 6$ and $|A \cap B| = 1$? If it is possible, provide an example, otherwise provide a proof that this is impossible.

It is impossible. For example, we can use data and Venn diagram from **Task 1**: $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6\}$, $|A| = 5$ and $|B| = 3$.
Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$, $|A \cup B| = 6$, $A \cap B = \{5, 6\}$, $|A \cap B| = 2$
We can also use the following formula:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Let's substitute our data into the formula:

$$6 = 5 + 3 - X$$

Calculate X and get:

$$|A \cap B| = X = 2 \neq 1$$

3 The third task

Check whether the following equivalence on sets holds by considering elements of each side and checking whether they are contained in the other side:

$$(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$$

For example, we have the following three sets: $A = \{0, 1, 2, 3\}$, $B = \{0, 2, 4, 5\}$ and $C = \{0, 3, 5, 6\}$.

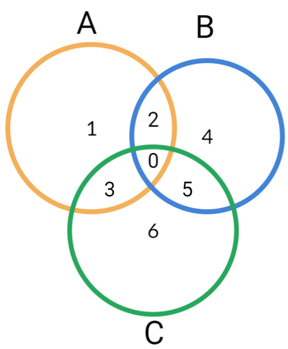


Figure 2: Set A, set B and set C

Then $A \cup B = \{0, 1, 2, 3, 4, 5\}$.
$(A \cup B) \backslash C = \{0, 1, 2, 3, 4, 5\} \backslash \{0, 3, 5, 6\} = \{1, 2, 4\}$.
Calculate the right side:

$(A\backslash C) = \{0, 1, 2, 3\}\backslash\{0, 3, 5, 6\} = \{1, 2\}$
$(B\backslash C) = \{0, 2, 4, 5\}\backslash\{0, 3, 5, 6\} = \{2, 4\}$
$(A\backslash C) \cup (B\backslash C) = \{1, 2\} \cup \{2, 4\} = \{1, 2, 4\}$
Finally, let's compare the left and right sides:

$$\{1, 2, 4\} = \{1, 2, 4\}$$

Conclusion: the equivalence on sets holds.

4 The fourth task

Suppose mobile company has 6 mobile plans. They make a survey among their clients asking for the client's favorite mobile plan, second favorite plan (that should be different from the first one) and the most overpriced mobile plan in client's opinion (that can be the same as his two favorite plans, or can be some other plan). What is the number of possible different outcomes of the survey? Provide a detailed explanation of your calculation. If you are using some of the rules (including the rule of sum and the rule of product), please explain why are you using these rules.

The number of possible different outcomes is equal to

$$6 \cdot 5 \cdot 6 = \mathbf{180}$$

or

$$\frac{6!}{(6-2)!} \cdot 6 = \mathbf{180}$$

In the solution, we used the multiplication rule, since we choose options from one set {1, 2, 3, 4, 5, 6}. First, we need to choose our favorite mobile plan out of six - so we have **6 options**. Then we choose the second favorite plan - we have **5 options** left. And finally we choose the most overpriced mobile plan - we have **6 options**, since the most overpriced plan can also be one of our favorites.

5 The fifth task

Suppose mortgage company has 10 houses. They want to calculate distance between each two (distinct) houses among them (just usual distance on the *map). We want to count how many distances we will calculate. What is wrong with the following solution attempt? Explain the mistake and provide a correct answer to the problem.*

Solution attempt: *We need to count the number of pairs of houses. The first house can be picked in 10 ways. The second house can be picked in 9 possible ways, since one of the houses was already picked. By the rule of product the pairs of houses can be picked in 10 times 9 = 90 possible ways. So we will need to make 90 calculations.*

The above solution takes into account duplicate pairs. To solve the problem, it is necessary to use the formula of unordered combinations without repetitions:

$$\frac{n!}{k!(n-k)!} = \frac{10!}{2!(10-8)!} = \mathbf{45}$$

For additional verification, we can also represent the solution graphically - draw 10 houses and connect them together with lines. Then the solution to the problem will be the number of lines between houses:

$$9+8+7+6+5+4+3+2+1 = \mathbf{45}$$

6 The sixth task

How many ways are there to write down numbers from 0 to 9 in a sequence in such a way that even numbers are positioned in the sequence in the increasing order and odd numbers are positioned in the decreasing order? Provide a detailed explanation for your solution.

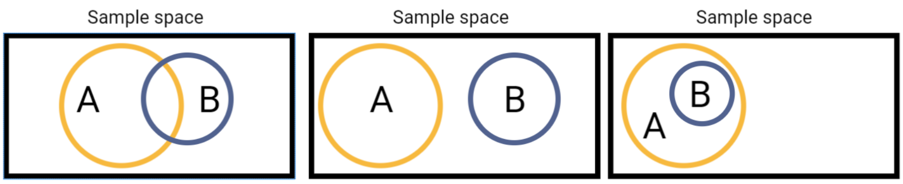
We have 10 positions to place a set of numbers $\{0,1,2,3,4,5,6,7,8,9\}$. This set is divided into two subsets - even numbers $\{0,2,4,6,8\}$ and odd numbers $\{1,3,5,7,9\}$. If we place one of these sets, the other set will not need to be placed.
For example: let's place even numbers in the following positions - (_0__2_46_8), then there will be no other options for placing odd numbers, except - (9075234618).
It turns out that we need to place one of the subsets in 10 possible positions:

$$C_{10}^{5} = \frac{10!}{5!(10-5)!} = \mathbf{252}$$

7 The seventh task

Let ***A*** *and* ***B*** *be events associated with some random experiment. Is it possible that these events never occur simultaneously and* ***P(A)=1/2, P(B)=1/3****? If yes, give an example of random experiment and events that satisfy these conditions. If no, give a proof.*

It is possible. For example, we throw a fair dice. The probability that we get a number less than 4 is equal to $\{1, 2, 3\}/\{1, 2, 3, 4, 5, 6\} = 1/2$. The probability of getting a number greater than 4 is $\{5, 6\}/\{1, 2, 3, 4, 5, 6\} = 1/3$.
These events never occur simultaneously.
If we know that the sum of the probabilities of events A and B is less than 1 ($P(A) + P(B) < 1$), then we can state that these events never occur simultaneously.
In the case of $P(A) + P(B) < 1$ and $P(A) > P(B)$ there are three possible outcomes:



The condition of the problem is satisfied by option number 2.

8 The eighth task

There is a small village with only ten adult people living in it. Instead of elections, they form their "government" using random choice. Every adult villager has equal probability to be chosen. Consider two random experiments:
1. The villagers want to select a Village Council that consists of three persons. The researcher is interested in the list of members of one Council.
2. Every four months villagers select a President. The same person can be President arbitrary number of times. The same person can be a President and a member of Village Council at the same time. The researcher is interested in the ordered list of Presidents during one year.
Describe the sample spaces (sets of all elementary outcomes) for both experiments. Use the corresponding notions from combinatorics. Find the number of elements in both sample spaces. Provide all calculations and detailed explanations. What is the difference between these experiments?

Let's consider case No. 1 - residents choose a village council consisting of 3 people, in this case all three members will be different people. We are considering a group of people who are chosen once - the order is not important. To determine the number of elements in the sample space:

$$C_n^k = \frac{10!}{3!(10-3)!} = \mathbf{120}$$

Consider case No. 2 - residents elect a president every 4 months, the same person can be president again. In this case, the selection takes place in stages, so the order of the elements is important. The choice is always made of 10 elements, so repetitions are allowed. To determine the number of elements in the sample space:

$$n^k = 10^3 = \mathbf{1000}$$

The difference between the experiments is that **in the first case, unordered combinations without repetitions** are obtained, **in the second case, ordered combinations with repetitions** are obtained.

9 The ninth task

Consider the following random experiment. We roll two identical indistinguishable symmetric dices once and record the result of this tossing (i.e. "on one dice we obtained 1 and on another we obtained 2").

Describe sample space (set of all outcomes) of this experiment. What is the probability of event "on one dice we obtained 1 and on another we obtained 2"? What is the probability of event "we obtained two 1's"? What kind of sample space you consider in this problem: sample space with equal probabilities of outcomes or sample space with non-equal probabilities of outcomes? Explain your answer.

In this problem, it is necessary to consider a sample space with unequal probabilities, since the probability of two different numbers falling out will be twice as large as the same ones, since the combinations (2, 1) and (1, 2) will be considered as "on one dice we obtained 1 and on another we obtained 2", but for a combination of "we obtained two 1's" there is only one option in the sample space - (1, 1).

The probability of event "on one dice we obtained 1 and on another we obtained 2":

$$P = \frac{|A|}{|\Lambda|} = \frac{|(2,1),(1,2)|}{|n^k|} = \frac{2}{6^2} = \frac{1}{18}$$

The probability of event "we obtained two 1's":

$$P = \frac{|A|}{|\Lambda|} = \frac{|((1,1)|}{|n^k|} = \frac{1}{6^2} = \frac{1}{36}$$