SGA - Graphs - Isupov Ilya

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Telegram: @Jeremix

1 The first task

In a country there are several airports. Airport A is directly connected to 23 other airports. Airport B has a direct connection to 3 other airports. Each airport, except A and B, is directly connected to 10 other airports. Prove that there is an airline route (maybe with flight changes) between A and B.

Let's consider this problem with the help of graphs: cities - vertices, the connection between cities - edges. It is required to prove that there is a way from city A to city B. In other words, we need to prove that we have a connected graph.
Suppose that airport A and B are not connected, then we get several connectivity components. In one of these connectivity components there are N vertices with degree 10 and one vertex with degree 23 (city A). The other connectivity component also has N vertices with degree 10 and one vertex with degree 3 (city B). It turns out that we have at least two subgraphs with one vertex with odd degree and several graphs with vertices with degree 10 (optional).
Let's apply the lemma about handshakes. The graph cannot have an odd number of vertices with odd degrees, so **vertices (cities) A and B must be in the same subgraph.**

2 The second task

Consider an Erdős–Rényi random graph on 4 vertices with p=1/2. Calculate the probability that this graph is connected.
2.1. Determine the maximum number of edges in a graph with 4 vertices:

$$\frac{n(n-1)}{2} = \frac{4 \cdot 3}{2} = 6$$

2.2. Determine the number of possible graphs with 4 vertices:
- number of graphs with 0 edge:

$$(\frac{6}{0}) = \frac{6!}{0!(6-0)!} = 1$$

- number of graphs with 1 edges:

$$(\frac{6}{1}) = \frac{6!}{1!(6-1)!} = 6$$

- number of graphs with 2 edges:

$$(\frac{6}{2}) = \frac{6!}{2!(6-2)!} = 15$$

- number of graphs with 3 edges:

$$(\frac{6}{3}) = \frac{6!}{3!(6-3)!} = 20$$

- number of graphs with 4 edges:

$$(\frac{6}{4}) = \frac{6!}{4!(6-4)!} = 15$$

- number of graphs with 5 edges:

$$(\frac{6}{5}) = \frac{6!}{5!(6-5)!} = 6$$

- number of graphs with 6 edges:

$$(\frac{6}{6}) = \frac{6!}{6!(6-6)!} = 1$$

Total number of graphs with 4 vertices:

$$1 + 6 + 15 + 20 + 15 + 6 + 1 = \mathbf{64}$$

2.3. Let's determine which of these options satisfy the condition that the graph is disconnected. A graph with 4 vertices is not connected in the following cases: if it has no edges, or it has 1 edge, if it has 2 edges and if 3 edges form a triangle (a triangle between 4 vertices can be formed only in 4 ways).

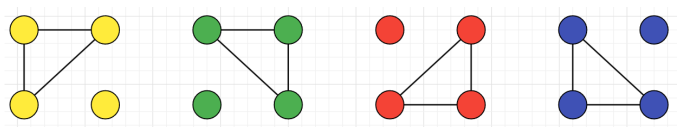


Figure 1: Triangles between 4 vertices

Number of variants of disconnected graphs with 4 vertices:

$$1 + 6 + 15 + 4 = 26$$

Number of connected graphs with 4 vertices:

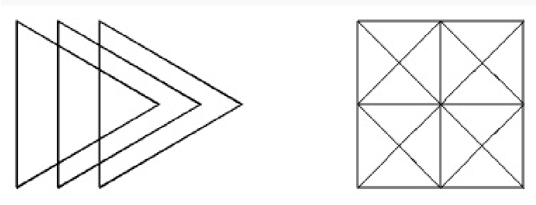
$$64 - 26 = 38$$

The probability that a graph with 4 vertices is connected:

$$P = \frac{38}{64} = \frac{19}{32}$$

3 The third task

Which of the following pictures can be drawn in one stroke of pen, without traversing a line twice (like a Euler path in a graph)?



Let's imagine the pictures in the form of graphs: vertices are intersection points, edges are lines. To draw a picture with one stroke of the pen, you need to enter each vertex (except the initial and final) as many times as you exit . This means that no more than 2 vertices of the graph can be odd.
Let's look at our pictures:
- in the first picture, all vertices are even. **This means it can be drawn with one stroke of the pen.**
- the second picture has more than 2 odd vertices. **This means it cannot be drawn with one stroke of the pen.**

4 The fourth task

Does there exist a graph with 5 vertices which have the following degrees: 2, 4, 4, 4, 4?

The graph does not exist. To prove it, let's try to sequentially remove the vertices of the graph (2, 4, 4, 4, 4):
- remove a vertex with degree 4 - reduce the degree in other vertices by 1 (since we remove 4 connections), we get a graph (1, 3, 3, 3)
- remove another vertex with degree 3 - we get a graph (0, 2, 2)
- remove another vertex with degree 2 - we get the sequence (-1, 1).
A graph with a vertex with a negative degree -1 and a vertex with a degree 1 cannot be graphically represented, **so such a graph and a graph (2, 4, 4, 4, 4) they do not exist** But we could stop at the sequence (0, 2, 2), because such a graph is also impossible to represent graphically.

5 The fifth task

A connected graph on 10 vertices has 15 edges. What is the maximal number of edges one can remove so that the graph remains connected? Note that if your answer is N, then you need to explain that:
a) after removing some N edges from any connected graph on 10 vertices with 15 edges the resulting graph remains connected;
b) there exists a connected graph on 10 vertices with 15 edges such that after removing any (N + 1) edges it becomes disconnected.

To solve this problem, we will use the circuit rank. A graph is not connected if its connectivity component is greater than or equal to 2. Find the extreme connectivity condition - a graph in which, when any edge is removed, the connectivity component becomes equal to 2.
a) use the circuit rank formula from the lecture (circuit rank is 0, vertices is 10, connectivity component is 1, edges is m):

$$r = m - n + c$$

$$0 = m - 10 + 1$$

$$m = 9$$

If the graph has 9 edges and 10 vertices, it can still remain connected, but if we remove any edge, it will become disconnected.

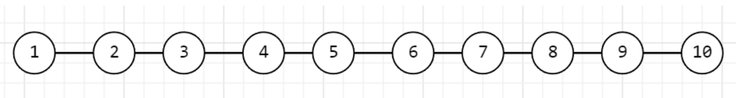
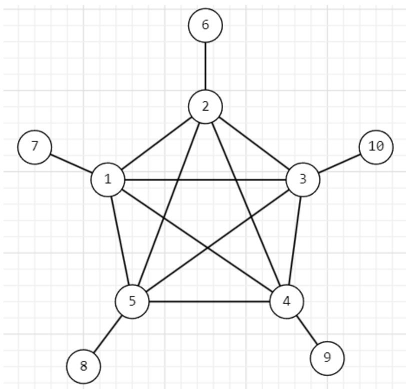


Figure 2: Graph with 10 vertices and 9 edges

The maximum number of edges that we can remove so that the graph remains connected (N = 6).
However, there are graphs in which, after removing 1 certain edge, they become disconnected.
Example:



When one of the 5 edges (1-7, 2-6, 3-10, 4-9, 5-8) is removed, the graph becomes disconnected.

b) In order for the graph to remain connected, all its vertices must be connected by at least 1 edge. The minimum number of edges for a connected graph is determined by the formula (n is the number of vertices of the graph):

$$Edges(minimum) = n - 1$$

Minimum number of edges for 10 vertices (Figure 1):

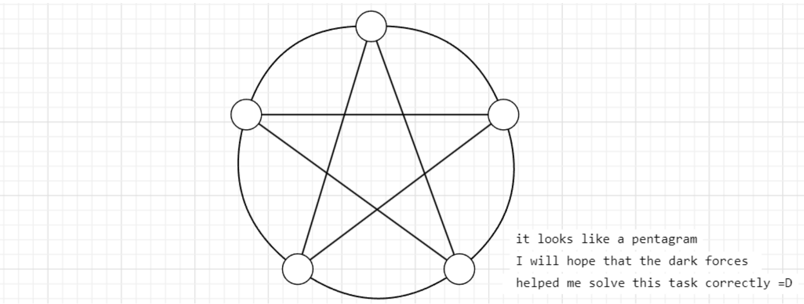
$$Edges(minimum) = 10 - 1 = 9$$

If there is a graph with 10 vertices and 15 edges, then after removing any 7 edges, the graph will become disconnected.

6 The sixth task

A graph on 6 vertices has 11 edges. Prove that this graph is connected.

To prove this, we first consider a complete graph with 5 vertices. This graph has 5 vertices and 10 edges. Each vertex is connected to other vertices.



In order to get a graph with 6 vertices and 11 edges, you need to add 1 more vertex and 1 edge to the existing graph. Since all the vertices in the complete graph have already been connected, it means that when adding 1 more vertex and 1 edge, we must connect this vertex to one of the 5 available ones and get a connected graph.

7 The seventh task

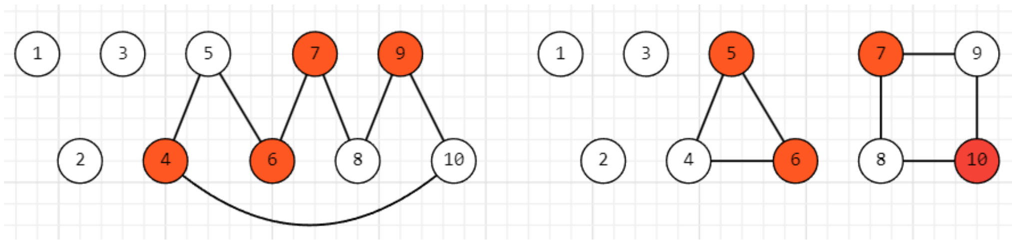
A graph on 10 vertices has 3 isolated vertices (degree 0) and 7 vertices of degree 2. Could such a graph be bipartite? How many vertices are there in an optimal vertex cover for this graph? (Consider all possible cases.)

We know the number of vertices and their degrees, calculate the number of edges in a subgraph with 7 vertices with degree 2:

$$\frac{7 \cdot 2}{2} = 7$$

A bipartite graph does not contain an odd-length cycle. We have 7 vertices and 7 edges, that is, we have an odd number of vertices and edges, so we cannot split the graph into two disjoint subsets (without adjacent vertices in each subset), therefore **a bipartite graph with such parameters does not exist.**

Examples of graphs:



The optimal vertex cover is the minimum set of vertices of a graph that each edge of the graph has at least one end included in this set.
Since the degree of each vertex in the subgraph is 2, **the optimal vertex cover is 4** (marked in red in the figures).