SGA - Linear algebra - Isupov Ilya

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1 The first task

A linear operator $f : R_2 \to R_2$ maps the vectors $a = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $b = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ to the vectors $p = f(a) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $q = f(b) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$. Find the matrix of f in the basis $\{a, q\}$.

Do not use any machine tools, provide a full solution with all required computations.

Let's find the matrix of the linear operator in the standard basis:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$\begin{cases} 2a - b = 2 \\ 2c - d = 4 \\ -3a + 2b = -1 \\ -3c + 2d = -2 \end{cases} \quad \begin{cases} b = 2a - 2 \\ d = 2c - 4 \\ -3a + 2b = -1 \\ -3c + 2d = -2 \end{cases} \quad \begin{cases} a = 3 \\ b = 4 \\ c = 6 \\ d = 8 \end{cases}$$

Matrix of linear operator in the standard basis:

$$F = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$$

Let's find the transformation matrix T from the standard basis to basis (a,q):

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot T = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}; T = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}$$

Let's find the matrix of linear operator (F') in the new basis (a,q):

$$F' = T^{-1} \cdot F \cdot T$$

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & -3 & 1 & 1 \\ -1 & -2 & 0 & 1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & -3 & 1 & 1 \\ 0 & -5 & 1 & 2 \end{array}\right) \left(\begin{array}{cc|cc} 1 & 0 & 0.4 & -0.2 \\ 0 & 1 & -0.2 & -0.4 \end{array}\right)$$

$$T^{-1} = \begin{pmatrix} 0.4 & -0.2 \\ -0.2 & -0.4 \end{pmatrix}$$

$$F' = \begin{pmatrix} 0.4 & -0.2 \\ -0.2 & -0.4 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 11 \end{pmatrix}$$

2 The second task

Find a decomposition $A = U\Sigma V^T$ of the matrix A = $\begin{bmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

, where Σ is a rectangular diagonal matrix of size $4 \times 3, U$ and V are orthogonal matrices.

Do not use any machine tools for finding SVD or for finding eigenvalues/eigenvectors, provide a full solution with all required computations. You may multiply matrices and/or solve systems using machine but beware: the resulting answers should be exact (for example, $\sqrt{2}$, not 1.41).

Let's find the matrix B:

$$B = A \cdot A^T = \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 & 4 & 2 \\ 1 & -2 & 2 & 1 \\ 2 & -2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 9 & -9 & 18 & 9 \\ -9 & 9 & -18 & -9 \\ 18 & -18 & 36 & 18 \\ 9 & -9 & 18 & 9 \end{pmatrix}$$

Let's find the eigenvalues of the matrix B:

$$\begin{vmatrix} 9-\lambda & -9 & 18 & 9 \\ -9 & 9-\lambda & -18 & -9 \\ 18 & -18 & 36-\lambda & 18 \\ 9 & -9 & 18 & 9-\lambda \end{vmatrix} = (9-\lambda)\begin{vmatrix} 9-\lambda & -18 & -9 \\ -18 & 36-\lambda & 18 \\ -9 & 18 & 9-\lambda \end{vmatrix} + 9\begin{vmatrix} -9 & 18 & 9 \\ -18 & 36-\lambda & 18 \\ -9 & 18 & 9-\lambda \end{vmatrix} +$$

$$+18\begin{vmatrix} -9 & 18 & 9 \\ 9-\lambda & -18 & -9 \\ -9 & 18 & 9-\lambda \end{vmatrix} - 9\begin{vmatrix} -9 & 18 & 9 \\ 9-\lambda & -18 & -9 \\ -18 & 36-\lambda & 18 \end{vmatrix} = (x^2 - 18x + 81) \cdot \begin{vmatrix} 36-\lambda & 18 \\ 18 & 9-\lambda \end{vmatrix} +$$

$$+(-18x+162)\begin{vmatrix} -18 & -9 \\ 18 & 9-\lambda \end{vmatrix} + (9x-81)\begin{vmatrix} -18 & -9 \\ 36-\lambda & 18 \end{vmatrix} - 81\begin{vmatrix} 36-\lambda & 18 \\ 18 & 9-\lambda \end{vmatrix} +$$

$$+162\begin{vmatrix} 18 & 9 \\ 18 & 9-\lambda \end{vmatrix} - 81\begin{vmatrix} 18 & 9 \\ 36-\lambda & 18 \end{vmatrix} - 162\begin{vmatrix} -18 & -9 \\ 18 & 9-\lambda \end{vmatrix} + (18x-162)\begin{vmatrix} 18 & 9 \\ 18 & 9-\lambda \end{vmatrix} -$$

$$-162\begin{vmatrix} 18 & 9 \\ -18 & -9 \end{vmatrix} + 81\begin{vmatrix} -18 & -9 \\ 36-\lambda & 18 \end{vmatrix} + (-9x+81)\begin{vmatrix} 18 & 9 \\ 36-\lambda & 18 \end{vmatrix} + 162\begin{vmatrix} 18 & 9 \\ -18 & -9 \end{vmatrix} =$$

$$= \lambda^4 - 63\lambda^3 = \lambda^3(\lambda - 63)$$

$$\lambda_1 = 63$$

$$\lambda_2 = 0$$

$$\lambda_3 = 0$$

$$\lambda_4 = 0$$

$$\sum = \begin{pmatrix} \sqrt{63} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let's find the eigenvectors:

$$\lambda_1 = 63$$

$$\begin{pmatrix} -54 & -9 & 18 & 9 \\ -9 & -54 & -18 & -9 \\ 18 & -18 & -27 & 18 \\ 9 & -9 & 18 & -54 \end{pmatrix} \begin{pmatrix} -1 & -6 & -2 & -1 \\ -6 & -1 & 2 & 1 \\ 0 & -2 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & -6 & -2 & -1 \\ 0 & 35 & 14 & 7 \\ 0 & -2 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & -6 & -2 & -1 \\ 0 & 5 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 - x_4 = 0 \\ x_2 + x_4 = 0 \\ x_3 - 2x_4 = 0 \end{cases} \quad \begin{cases} x_1 = x_4 \\ x_2 = -x_4 \\ x_3 = 2x_4 \\ x_4 = x_4 \end{cases}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ -x_4 \\ 2x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix} x_4$$

$$eigenvector_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = \lambda_3 = \lambda_4 = 0$$

$$\begin{pmatrix} 9 & -9 & 18 & 9 \\ -9 & 9 & -18 & -9 \\ 18 & -18 & 36 & 18 \\ 9 & -9 & 18 & 9 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 - x_2 + 2x_3 - x_4 = 0$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 - 2x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_4$$

$$eigenvector_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} eigenvecotr_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} eigenvecotr_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Orthonormalize the set of eigenvectors using the Gram-Schmidt process and find the matrix U:

$$a_1 = b_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix} a_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} a_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} a_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$b_2 = a_2 - \frac{< a_2, b_1 >}{< b_1, b_1 >} b_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 0 \cdot b_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$b_3 = a_3 - \frac{< a_3, b_2 >}{< b_2, b_2 >} b_2 - \frac{< a_3, b_1 >}{< b_1, b_1 >} b_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{-2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 0 \cdot b_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$b_4 = a_4 - \frac{< a_4, b_3 >}{< b_3, b_3 >} b_3 - \frac{< a_4, b_2 >}{< b_2, b_2 >} b_2 - \frac{< a_4, b_1 >}{< b_1, b_1 >} b_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{-1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 0 \cdot b_1 = \begin{pmatrix} -\frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

$$e_1 = \frac{b_1}{||b_1||} = \begin{pmatrix} \frac{\sqrt{7}}{7} \\ -\frac{\sqrt{7}}{7} \\ \frac{2\sqrt{7}}{7} \\ \frac{\sqrt{7}}{7} \end{pmatrix} e_2 = \frac{b_2}{||b_2||} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{pmatrix} e_3 = \frac{b_3}{||b_3||} = \begin{pmatrix} -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ 0 \end{pmatrix} e_4 = \frac{b_4}{||b_4||} = \begin{pmatrix} -\frac{\sqrt{42}}{42} \\ \frac{\sqrt{42}}{42} \\ -\frac{\sqrt{42}}{21} \\ \frac{\sqrt{42}}{7} \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{\sqrt{7}}{7} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{42}}{42} \\ -\frac{\sqrt{7}}{7} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{42}}{42} \\ \frac{2\sqrt{7}}{7} & 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{42}}{21} \\ \frac{\sqrt{7}}{7} & 0 & 0 & \frac{\sqrt{42}}{7} \end{pmatrix}$$

Let's find V1:

$$V^1 = \frac{1}{\sqrt{63}} \cdot \begin{pmatrix} 2 & -2 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & -2 & 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{7}}{7} \\ -\frac{\sqrt{7}}{7} \\ \frac{2\sqrt{7}}{7} \\ \frac{\sqrt{7}}{7} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$\frac{2}{3}x_1 + \frac{1}{3}x_2 + \frac{2}{3}x_3 = 0$$

$$2x_1 + 1x_2 + 2x_3 = 0$$

$$x_1 + \frac{1}{2}x_2 + x_3 = 0$$

$$\begin{cases} x_1 = -\frac{1}{2}x_2 - x_3 \\ x_2 = -x_2 \\ x_3 = x_3 \end{cases}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} x_3$$

Orthonormalize the set of vectors using the Gram-Schmidt process and find V2 and V3:

$$a_1 = b_1 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} a_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$b_2 = a_2 - \frac{< a_2, b_1 >}{< b_1, b_1 >} b_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{5} \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ -\frac{2}{5} \\ 1 \end{pmatrix}$$

$$e_1 = \frac{b_1}{||b_1||} = \begin{pmatrix} \frac{\sqrt{5}}{5} \\ -\frac{2\sqrt{5}}{5} \\ 0 \end{pmatrix} e_2 = \frac{b_2}{||b_2||} = \begin{pmatrix} \frac{-4\sqrt{5}}{15} \\ \frac{-2\sqrt{5}}{15} \\ \frac{\sqrt{5}}{3} \end{pmatrix}$$

$$V^2 = \begin{pmatrix} \frac{\sqrt{5}}{5} \\ -\frac{2\sqrt{5}}{5} \\ 0 \end{pmatrix} V^3 = \begin{pmatrix} \frac{-4\sqrt{5}}{15} \\ \frac{-2\sqrt{5}}{15} \\ \frac{\sqrt{5}}{3} \end{pmatrix}$$

$$V = \begin{pmatrix} V^1 & V^2 & V^3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{5}}{5} & \frac{-4\sqrt{5}}{15} \\ \frac{1}{3} & -\frac{2\sqrt{5}}{5} & \frac{-2\sqrt{5}}{15} \\ \frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \end{pmatrix}$$

$$A = U \cdot \sum \cdot V^T =$$

$$\begin{pmatrix} \frac{\sqrt{7}}{7} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{42}}{42} \\ -\frac{\sqrt{7}}{7} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{42}}{42} \\ \frac{2\sqrt{7}}{7} & 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{42}}{21} \\ \frac{\sqrt{7}}{7} & 0 & 0 & \frac{\sqrt{42}}{7} \end{pmatrix} \begin{pmatrix} \sqrt{63} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} & 0 \\ \frac{-4\sqrt{5}}{15} & \frac{-2\sqrt{5}}{15} & \frac{\sqrt{5}}{3} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix}$$

3 The third task

Let a directed graph be defined by the adjacency matrix $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$. (For the definition of the adjacency matrix see Notes: PageRank). Consider a random walk with the probabilities on equal among vertices.

a) Find the Markov transition matrix.

b) Apply regularization with $\alpha = 0.25$.

c) Use PageRank to find the most influencial vertex (the vertex with maximal weight). You are allowed to use a machine to compute the approximate solution of the system you obtained.

Do not use any machine tools in a) and b), provide a full solution with all required computations.

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Let's find the Markov transition matrix (P):

$$A^T = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} P = \begin{pmatrix} 0 & 0 & 1 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{3} \end{pmatrix}$$

Apply regularization with a = 0.25:

$$P_a = (1-a)P + aQ$$

$$P_a = (1-0.25)P + 0.25Q = 0.75P + 0.25Q$$

$$P_a = \begin{pmatrix} \frac{1}{16} & \frac{1}{16} & \frac{13}{16} & \frac{5}{16} \\ \frac{1}{16} & \frac{7}{16} & \frac{1}{16} & \frac{1}{16} \\ \frac{7}{16} & \frac{1}{16} & \frac{1}{16} & \frac{5}{16} \\ \frac{7}{16} & \frac{7}{16} & \frac{1}{16} & \frac{5}{16} \end{pmatrix}$$

Using the numpy library, we will find the vertex with the maximum weight by a large number of iterations of multiplying the matrix Pa by the vector X:

```
import numpy as np


Pa = np.array([[1/16, 1/16, 13/16, 5/16],
               [1/16, 7/16, 1/16, 1/16],
               [7/16, 1/16, 1/16, 5/16],
               [7/16, 7/16, 1/16, 5/16]])

x = np.array([[1, 1, 1, 1]]).T
tolerance = 1e-6
max_iter = 1000
x_prev = 0

for i in range(max_iter):
    x = np.matmul(Pa, x) / np.linalg.norm(np.matmul(Pa, x))
    x_cur = x
    if np.linalg.norm(x_cur - x_prev) < tolerance:
        break
    x_prev = x

x_norm = x / sum(x)
print(x_norm)
```

Thus, after N iterations through the graph, we got a normalized vector showing the probabilities of the further path:

```
[[0.33541676]
 [0.1       ]
 [0.26354162]
 [0.30104162]]
```

Based on the values of the resulting vector, the most influential vertex is the first vertex with the weight **0.33541676**.

This vector can also be found if we find the eigenvector of the matrix Pa with an eigenvalue equal to 1:

```python
import numpy as np


Pa = np.array([[1/16, 1/16, 13/16, 5/16],
               [1/16, 7/16, 1/16, 1/16],
               [7/16, 1/16, 1/16, 5/16],
               [7/16, 7/16, 1/16, 5/16]])

eigenvalues = np.linalg.eig(Pa).eigenvalues
eigenvectors = np.linalg.eig(Pa).eigenvectors
sum_x = sum([i[0] for i in eigenvectors])
result = [round((i[0]/sum_x), 7) for i in eigenvectors]

print(f'Eigenvalues:\n {eigenvalues}')
print(f'Eigenvectors:\n {eigenvectors}')
print('Result:', *result, sep='\n')
```

```
Eigenvalues:
 [ 1.00000000e+00 -5.00000000e-01  3.75000000e-01  4.09952835e-18]
Eigenvectors:
 [[ 6.30972405e-01  8.16496581e-01 -5.72077554e-01 -5.34522484e-01]
 [ 1.88115997e-01 -1.38315100e-17  6.67423812e-01 -2.09426474e-16]
 [ 4.95764033e-01 -4.08248290e-01 -3.81385036e-01 -2.67261242e-01]
 [ 5.66307531e-01 -4.08248290e-01  2.86038777e-01  8.01783726e-01]]
Result:
0.3354167
0.1
0.2635417
0.3010417
```

4 The fourth task

A dangerous virus spreads on an island with a population of 10,000. Every day, island authorities collect statistics and want to understand if they have enough health system resources. Doctors report that every day 15 percent of healthy people become infected and have a mild illness (which does not require hospitalization), 12 percent of healthy people become infected and have a difficult illness (they need to get to the hospital). At the same time, 12 percent of people with a mild form of the disease recover completely, and 15 percent go into the category of seriously ill patients. In the category of seriously ill patients, the situation is as follows: 20 percent go into the category of patients with a mild form of the disease and 10 percent completely recover. Recovered patients may become infected again. At the initial time on the island, 500 patients were identified in a mild form of the disease and 100 patients in a severe form. Luckily, the virus is not lethal. How will the number of patients behave with increasing time? From a mathematical point of view, find the limits of the number of patients in mild and severe forms, if they exist. Please do not forget that with real viruses everything is not so simple.

You may use any machine tools here in the first part, provided that you attach their output, explain what you ask them to find and how to interpret the result. Do not find the limit numerically (by considering many steps), instead find it mathematically. Provide a full solution with all required computations.

Based on the conditions of the task, let's draw up an equation for the distribution of the island's inhabitants into 3 categories (a - healthy, b - patients in mild form, c - patients in severe form, k - day):

$$\begin{cases} a_{k+1} = 0.73a_k + 0.12b_k + 0.1c_k \\ b_{k+1} = 0.15a_k + 0.73b_k + 0.2c_k \\ c_{k+1} = 0.12a_k + 0.15b_k + 0.7c_k \end{cases}$$

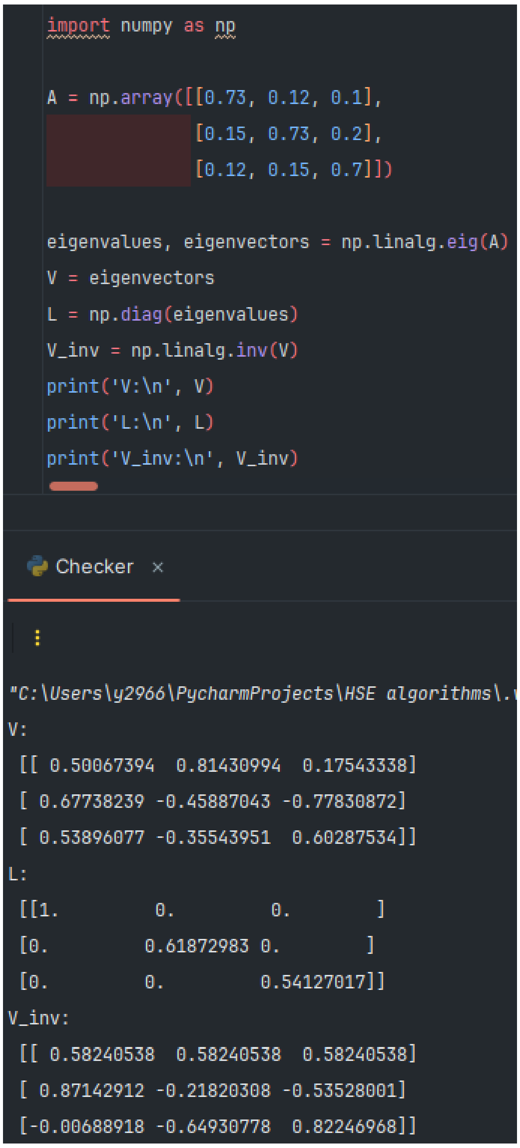
Let's write down our system of linear equations in the form of a matrix of a linear operator:

$$A = \begin{pmatrix} 0.73 & 0.12 & 0.1 \\ 0.15 & 0.73 & 0.2 \\ 0.12 & 0.15 & 0.7 \end{pmatrix}$$

By exponentiating the matrix using the formula eigenvalue decomposition, let's look at the state of the island's inhabitants over time:

$$A^n = V \cdot L^n \cdot V^{-1}$$

Using the library Numpy, we define the matrix V, L and V-1:



```
import numpy as np

A = np.array([[0.73, 0.12, 0.1],
              [0.15, 0.73, 0.2],
              [0.12, 0.15, 0.7]])

eigenvalues, eigenvectors = np.linalg.eig(A)
V = eigenvectors
L = np.diag(eigenvalues)
V_inv = np.linalg.inv(V)
print('V:\n', V)
print('L:\n', L)
print('V_inv:\n', V_inv)
```

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```
"C:\Users\y2966\PycharmProjects\HSE algorithms\.
V:
 [[ 0.50067394  0.81430994  0.17543338]
 [ 0.67738239 -0.45887043 -0.77830872]
 [ 0.53896077 -0.35543951  0.60287534]]
L:
 [[1.         0.         0.        ]
 [0.         0.61872983 0.        ]
 [0.         0.         0.54127017]]
V_inv:
 [[ 0.58240538  0.58240538  0.58240538]
 [ 0.87142912 -0.21820308 -0.53528001]
 [-0.00688918 -0.64930778  0.82246968]]
```

As we can see, the elements of the diagonal matrix L are the eigenvalues equal to 1, 0.61872983,

0.54127017. When the matrix L is raised to the power, all elements on the main diagonal are also raised to the power, which means that when the matrix L is raised to the power of "n" ("n" tends to infinity), elements less than 1 will tend to zero.
Therefore, if we want to raise our matrix A to an infinitely large degree (to find out the state of our island after an infinite amount of time), we can use the matrix L with zero elements on the main diagonal except for the element equal to 1:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let's calculate the matrix A to the power of "n" ("n" tends to infinity):

```python
import numpy as np

A = np.array([[0.73, 0.12, 0.1],
              [0.15, 0.73, 0.2],
              [0.12, 0.15, 0.7]])

eigenvalues, eigenvectors = np.linalg.eig(A)
V = eigenvectors
L = np.array([[1, 0, 0],
              [0, 0, 0],
              [0, 0, 0]])
V_inv = np.linalg.inv(V)

A_pow_n = np.dot(np.dot(V, L), V_inv)
print('Matrix A to the power of "n" ("n" tends to infinity):\n', A_pow_n)
```

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```
"C:\Users\y2966\PycharmProjects\HSE algorithms\.venv\Scripts\python.exe" "C:\
Matrix A to the power of "n" ("n" tends to infinity):
 [[0.2915952  0.2915952  0.2915952 ]
 [0.39451115 0.39451115 0.39451115]
 [0.31389365 0.31389365 0.31389365]]
```

As we can see, the values in the rows are equal to each other, which means that the state of our matrix (island) has stabilized and the number of healthy, mildly ill and severely ill people will not change over time.
The number of healthy people will be 29.16 percent, patients in mild form 39.45 percent, patients in severe form 31.39 percent. This means that compared with the initial number of healthy, mild and severe patients, the number of healthy people will decrease, the number of mild and severe patients will increase.

$$P(T=0|H \geq 1) = \frac{P(T=0, H \geq 1)}{P(H \geq 1)} = \frac{P(H=2, H \geq 1)}{P(H \geq 1)} = \frac{P(H=2)}{3/4} = \frac{1/4}{3/4} = 1/3$$

$$P(35nC) = P(3SC)P(C) = 0.1259P(35nC) = P(C|39)P(35) = 0.25p ==$$

So we get 0.125q=0.25p