SGA - Linear algebra - Isupov Ilya

January 2023

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1 The first task

Find out if the rows of each of the matrices form

$$A = \begin{pmatrix} 4 & 8 & 14 & 0 \\ 2 & 4 & 7 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 & 0 & 1 \\ 4 & 8 & 14 & 0 \end{pmatrix}$$

fundamental system of solutions (i.e. basis in the solution space) for the system of linear equations.

$$\begin{cases} 5x_1 + x_2 - 2x_3 + 6x_4 = 0 \\ x_1 + 3x_2 - 2x_3 + 4x_4 = 0 \\ 3x_1 + 2x_2 - 2x_3 + 5x_4 = 0 \\ 4x_1 + 5x_2 - 4x_3 + 9x_4 = 0 \end{cases}$$

Let's find the rank of the system of linear equations.

$$\begin{pmatrix} 5 & 1 & -2 & 6 \\ 1 & 3 & -2 & 4 \\ 3 & 2 & -2 & 5 \\ 4 & 5 & -4 & 9 \end{pmatrix} \begin{pmatrix} 1 & 3 & -2 & 4 \\ 1 & -4 & 2 & -3 \\ 0 & -7 & 4 & -7 \\ 1 & 3 & -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & -7 & 4 & -7 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{2}{7} & 0 \\ 0 & 1 & -\frac{4}{7} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The rank of the system of linear equations is **2**.
Let's find the rank of the matrix A.

$$\begin{pmatrix} 4 & 8 & 14 & 0 \\ 2 & 4 & 7 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -\frac{7}{2} & 0 \\ 1 & 2 & -\frac{7}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -\frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The rank of the matrix A is **1**.
It means that **the rows of the matrix A don't form a fundamental system of solutions** for a system of linear equations.
Let's find the rank of the matrix B.

$$\begin{pmatrix} -1 & -1 & 0 & 1 \\ 4 & 8 & 14 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 4 & 14 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{7}{2} & -2 \\ 0 & 1 & \frac{7}{2} & 1 \end{pmatrix}$$

The rank of the matrix B is **2**.
We also see that the sum of the rank of the solution matrix (2) and the rank of the matrix of the system (2) is equal to the number of unknowns (4):

$$2 + 2 = 4$$

It means that **the rows of the matrix B form a fundamental system of solutions** for a system of linear equations.

2 The second task

Prove that the following system of linear equations is inconsistent. Find a least squares solution of the system.

$$\begin{cases} x + 2y + 3z = 8, \\ x + 3z = -2, \\ 2x + 4z = 0, \\ x - y + 2z = 16. \end{cases}$$

Let's try to solve the system of linear equations using the Gauss method, for this we will make up a matrix and bring it to the echelon form.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 1 & 0 & 3 & -2 \\ 2 & 0 & 4 & 0 \\ 1 & -1 & 2 & 16 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & -2 & 0 & -10 \\ 0 & -4 & -2 & -16 \\ 0 & -3 & -1 & 8 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & -4 & -2 & -16 \\ 0 & -3 & -1 & 8 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 3 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & -1 & 23 \end{array}\right) =$$
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 21 \end{array}\right)$$

Let's write down the last row of the matrix.

$$0 \cdot x + 0 \cdot y + 0 \cdot z = 21$$

Thus, the system of linear equations is **inconsistent**.
Let's find a solution to the system using the least squares method.

$$\hat{x} = pr_L b = (A^T \cdot A)^{-1} \cdot A^T \cdot b$$

$$A^T \cdot A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 0 & -1 \\ 3 & 3 & 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 2 & 0 & 4 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 1 & 16 \\ 1 & 5 & 4 \\ 16 & 4 & 38 \end{pmatrix}$$

$$(A^T \cdot A)^{-1} = \left(\begin{array}{ccc|ccc} 7 & 1 & 16 & 1 & 0 & 0 \\ 1 & 5 & 4 & 0 & 1 & 0 \\ 16 & 4 & 38 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & \frac{1}{7} & \frac{16}{7} & \frac{1}{7} & 0 & 0 \\ 0 & \frac{34}{7} & \frac{12}{7} & -\frac{1}{7} & 1 & 0 \\ 0 & \frac{12}{7} & \frac{10}{7} & -\frac{16}{7} & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & \frac{1}{7} & \frac{16}{7} & \frac{1}{7} & 0 & 0 \\ 0 & 1 & \frac{6}{17} & -\frac{1}{34} & \frac{7}{34} & 0 \\ 0 & 0 & \frac{14}{17} & -\frac{38}{17} & -\frac{6}{17} & 1 \end{array}\right) =$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{38}{17} & \frac{5}{34} & -\frac{1}{34} & 0 \\ 0 & 1 & 0 & \frac{13}{34} & \frac{5}{14} & -\frac{3}{7} \\ 0 & 0 & 1 & -\frac{19}{7} & -\frac{3}{7} & \frac{17}{14} \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{87}{14} & \frac{13}{14} & -\frac{19}{7} \\ 0 & 1 & 0 & \frac{13}{14} & \frac{5}{14} & -\frac{3}{7} \\ 0 & 0 & 1 & -\frac{19}{7} & -\frac{3}{7} & \frac{17}{14} \end{array}\right)$$

$$(A^T \cdot A)^{-1} \cdot A^T = \begin{pmatrix} \frac{87}{14} & \frac{13}{14} & -\frac{19}{7} \\ \frac{13}{14} & \frac{5}{14} & -\frac{3}{7} \\ -\frac{19}{7} & -\frac{3}{7} & \frac{17}{14} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 0 & -1 \\ 3 & 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{87}{14} & \frac{13}{14} & -\frac{19}{7} \\ \frac{13}{14} & \frac{5}{14} & -\frac{3}{7} \\ -\frac{19}{7} & -\frac{3}{7} & \frac{17}{14} \end{pmatrix} = \begin{pmatrix} -\frac{1}{14} & -\frac{27}{14} & \frac{11}{7} & -\frac{1}{7} \\ \frac{5}{14} & -\frac{5}{14} & \frac{1}{7} & -\frac{2}{7} \\ \frac{1}{14} & \frac{13}{14} & -\frac{4}{7} & \frac{1}{7} \end{pmatrix}$$

$$\hat{x} = pr_L b = (A^T \cdot A)^{-1} \cdot A^T \cdot b = \begin{pmatrix} -\frac{1}{14} & -\frac{27}{14} & \frac{11}{7} & -\frac{1}{7} \\ \frac{5}{14} & -\frac{5}{14} & \frac{1}{7} & -\frac{2}{7} \\ \frac{1}{14} & \frac{13}{14} & -\frac{4}{7} & \frac{1}{7} \end{pmatrix} \cdot \begin{pmatrix} 8 \\ -2 \\ 0 \\ 16 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

3 The third task

Let A=(1, 1, 1) be vector in R3, and let L be a vector subspace of R3 spanned by the vectors (1, 0, 1) and (3, 5, x) (where x is a real parameter). Find the values of x such that the distance between A and L is maximal. Find also the square of this maximal distance.

$$L = \begin{pmatrix} 1 & 3 \\ 0 & 5 \\ -1 & x \end{pmatrix}$$

$$d(A, L) = ||A - pr_L A||$$

$$pr_L A = L(L^T L)^{-1} L^T A$$

$$L^T L = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 5 & x \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ 0 & 5 \\ -1 & x \end{pmatrix} = \begin{pmatrix} 2 & 3-x \\ 3-x & 34+x^2 \end{pmatrix}$$

$$det(L^T L) = (68 + 2x^2) - (3-x)(3-x) = x^2 + 6x + 59$$

$$(L^T L)^{-1} = \frac{1}{x^2+6x+59} \cdot \begin{pmatrix} 34+x^2 & x-3 \\ x-3 & 2 \end{pmatrix}$$

$$L(L^T L)^{-1} = \frac{1}{x^2+6x+59} \cdot \begin{pmatrix} 1 & 3 \\ 0 & 5 \\ -1 & x \end{pmatrix} \cdot \begin{pmatrix} 34+x^2 & x-3 \\ x-3 & 2 \end{pmatrix} = \frac{1}{x^2+6x+59} \cdot \begin{pmatrix} x^2+3x+25 & x+3 \\ 5x-15 & 10 \\ -3x-34 & x+3 \end{pmatrix}$$

$$L(L^T L)^{-1} L^T = \frac{1}{x^2+6x+59} \cdot \begin{pmatrix} x^2+3x+25 & x+3 \\ 5x-15 & 10 \\ -3x-34 & x+3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 \\ 3 & 5 & x \end{pmatrix} =$$

$$\frac{1}{x^2+6x+59} \cdot \begin{pmatrix} x^2+6x+34 & 5x+15 & -25 \\ 5x+15 & 50 & 5x+15 \\ -25 & 5x+15 & x^2+6x+34 \end{pmatrix}$$

$$L(L^T L)^{-1} L^T A = \frac{1}{x^2+6x+59} \cdot \begin{pmatrix} x^2+6x+34 & 5x+15 & -25 \\ 5x+15 & 50 & 5x+15 \\ -25 & 5x+15 & x^2+6x+34 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$\frac{1}{x^2+6x+59} \cdot \begin{pmatrix} x^2+11x+24 \\ 10x+80 \\ x^2+11x+24 \end{pmatrix}$$

$$d(A, L) = ||A - pr_L A|| = (1 - \frac{x^2+11x+24}{x^2+6x+59})^2 + (1 - \frac{10x+80}{x^2+6x+59})^2 +$$

$$(1-\frac{x^2+11x+24}{x^2+6x+59})^2=\frac{x^2-14x+49}{x^2+6x+59}$$

Using the derivative, we will find the extremum points of the function:

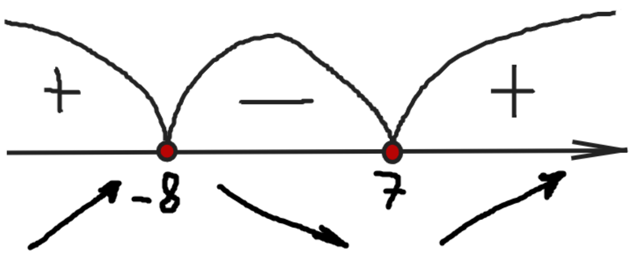
$$(\frac{x^2-14x+49}{x^2+6x+59})'=0$$

$$\frac{(2x-14)(x^2+6x+59)-(x^2-14x+49)(2x+6)}{(x^2+6x+59)^2}=0$$

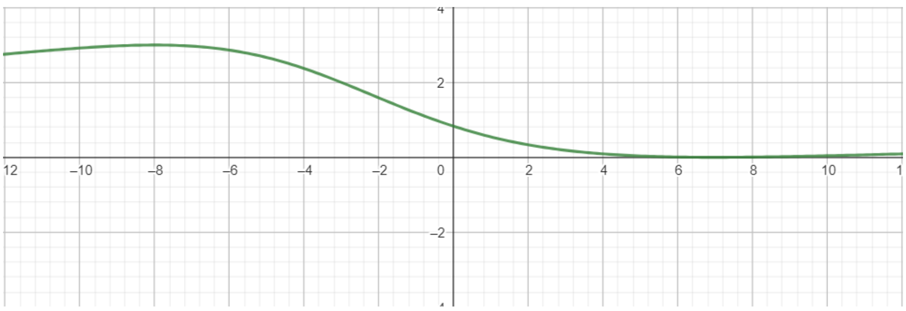
$$\frac{20x^2+20x-1120}{(x^2+6x+59)^2}=0$$

$$20x^2+20x-1120=0$$

$$x_1=7; x_2=-8$$



x = -8 is the maximum point. It means that at x = -8, the distance between A and L will be maximum.



Let's find the square of the maximum distance:

$$(\frac{x^2-14x+49}{x^2+6x+59})=\frac{(-8)^2-14\cdot(-8)+49}{(-8)^2+6\cdot(-8)+59})=\frac{64+112+49}{64-48+59}=3$$

4 The fourth task

Find a rank of a matrix A as a function of "a". For a=7 find a rank p of matrix A and a decomposition of A as a product of two matrices B and C, such that B has p columns and C has p rows.

$$A=\begin{pmatrix} \mathbf{a} & 1 & 2 \\ 5 & 5 & -2 \\ 3 & 3 & 6 \\ 1 & \mathbf{a} & 2 \end{pmatrix}$$

The size of the matrix A is 4x3. So the maximum rank of the matrix can be equal to **3** (the rank of the matrix doesn't exceed its minimum dimension).
Let's try to bring the matrix A into the echelon form.

$$A = \begin{pmatrix} \mathbf{a} & 1 & 2 \\ 5 & 5 & -2 \\ 3 & 3 & 6 \\ 1 & \mathbf{a} & 2 \end{pmatrix} = \begin{pmatrix} \mathbf{a} & 1 & 2 \\ 1 & 1 & -0.4 \\ 1 & 1 & 2 \\ 1 & \mathbf{a} & 2 \end{pmatrix} = \begin{pmatrix} \mathbf{a} & 1 & 2 \\ 0 & 0 & -2.4 \\ 1 & 1 & 2 \\ 1 & \mathbf{a} & 2 \end{pmatrix} = \begin{pmatrix} \mathbf{a} & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & \mathbf{a} & 2 \end{pmatrix} = \begin{pmatrix} \mathbf{a} & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & \mathbf{a} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} \mathbf{a-1} & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & \mathbf{a-1} & 0 \end{pmatrix}$$

After analyzing the 2nd and 3rd rows of the resulting matrix, we can say that the minimum rank of the matrix will be 2, if the other 2 rows are zero.
Thus, **the rank of the matrix will be equal to 2 if 'a' is equal to 1**. **In all other cases, the rank of the matrix will be equal to 3.**
Let's perform the decomposition of the matrix A and find the rank at a=7.

$$A = B \cdot C$$

$$\begin{pmatrix} \mathbf{7-1} & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & \mathbf{7-1} & 0 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 6 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of the matrix A (a=7) will be equal to 3.

$$B = \begin{pmatrix} 7 & 1 & 2 \\ 5 & 5 & -2 \\ 3 & 3 & 6 \\ 1 & 7 & 2 \end{pmatrix} C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 7 & 1 & 2 \\ 5 & 5 & -2 \\ 3 & 3 & 6 \\ 1 & 7 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 1 & 2 \\ 5 & 5 & -2 \\ 3 & 3 & 6 \\ 1 & 7 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$