SGA - Probability Theory - Isupov Ilya

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1 The first task

Consider two random events A and B defined for the same random experiment:
1. Is it possible that A and B are independent and mutually exclusive (disjoint) at the same time?
2. Does the answer change if given that P(A) bigger than 0 and P(B) bigger than 0?

Let's write definitions:
Events A and B are **independent** if the probability of event B is the same whether A occurs or not, and the probability of event A is the same whether B occurs or not.
Events A and B are **mutually exclusive** if they cannot occur at the same time.
Let's write some formulas for independent events:

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A|B) = P(A)$$

$$P(B|A) = P(B)$$

Let's write some formulas for mutually exclusive events:

$$P(A \cap B) = 0$$

$$P(A|B) = 0$$

$$P(B|A) = 0$$

Thus, if two independent events are considered, then:
1. P(A) doesn't depend on whether B occurs or not.
2. P(B) doesn't depend on whether A occurs or not.
If two mutually exclusive events are considered, then:
1. If B occurs, A cannot also occur.
2. If A occurs, B cannot also occur.
If events A and B are independent and at the same time mutually exclusive, then the following conditions must be met:

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \cap B) = 0$$

Thus:

$$P(A \cap B) = P(A) \cdot P(B) = 0$$

Conclusions:
1. Events A and B can be independent and mutually exclusive at the same if P(A)=0 or P(B)=0 or P(A) and P(B) equals to 0.
2. If P(A) bigger than 0 and P(B) bigger than 0 then events A and B can't be independent and mutually exclusive at the same because:

$$P(A \cap B) = P(A) \cdot P(B) \neq 0$$

2 The second task

The main prize at some TV show is a car. At the beginning the car is placed behind one door randomly chosen out of three (with equal probabilities). Participant has to guess the door with a car. If the guess is correct, the participant wins the car, otherwise she wins nothing. After the participant picks the door and announces her choice, host opens one of the two remaining doors and shows that there are no car there. Then the participant can either switch her decision and pick the remaining closed door or keep the door she picked initially. After the final decision is made, the chosen door opens and the participant get her prize (if any). Assume that the host never opens a door that is picked by the participant initially and never opens a door with a car. If the host can chose between several doors, the choice is random (with equal probabilities). Let us enumerate doors in such a way that the door initially picked by the participant has number 1.

Consider events:
H1: the car is behind door number 1;
H2: the car is behind door number 2;
H3: the car is behind door number 3.
Consider also event A: the host opened door number 2.

1. What can you say about probability of H3 before any door is opened?
Let's write the probabilities before opening any door:

$$P(H_1) = P(H_2) = P(H_3) = \frac{1}{3}$$

We know that host never opens a door that is picked by the participant initially (pick the door number 1) and never opens a door with a car, it means that:
- if the car behind the door number 1:

$$P(A|H_1) = \frac{1}{2}$$

- if the car behind the door number 2:

$$P(A|H_2) = 0$$

- if the car behind the door number 3:

$$P(A|H_3) = 1$$

We know that after the participant picks the door and announces her choice, host opens one of the two remaining doors and shows that there are no car there. Let's find the probability of event A (the host opened door number 2) using law of total probability:

$$P(A) = P(H_1) \cdot P(A|H_1) + P(H_2) \cdot P(A|H_2) + P(H_3) \cdot P(A|H_3) = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

2. Assume that the host opened door number 2 (events A happened). What would you say about probability of H3 after you observe that?
Let's calculate $P(H_3|A)$ using Bayes' rule:

$$P(H_3|A) = \frac{P(A|H_3) \cdot P(H_3)}{P(A)} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

After event A occurred, the probability of event H3 increased to 2/3
3. Are events A and H1 independent? A and H2? A and H3?
Let's check this using one of the independent event formulas:

$$P(C|B) = P(C) \cdot P(B)$$

- Check that events A and H1 independent:

$$P(A|H_1) = \frac{1}{2}, P(A) = \frac{1}{2}, P(H_1) = \frac{1}{3}$$

$$\frac{1}{2} \neq \frac{1}{2} \cdot \frac{1}{3}$$

Events A and H1 aren't independent.
- Check that events A and H2 independent:

$$P(A|H_2) = 0, P(A) = \frac{1}{2}, P(H_2) = \frac{1}{3}$$

$$0 \neq \frac{1}{2} \cdot \frac{1}{3}$$

Events A and H2 aren't independent.
- Check that events A and H3 independent:

$$P(A|H_3) = 1, P(A) = \frac{1}{2}, P(H_3) = \frac{1}{3}$$

$$1 \neq \frac{1}{2} \cdot \frac{1}{3}$$

Events A and H3 aren't independent.
4. Use Bayes' rule to find $P(H_1|A)$ and $P(H_3|A)$ (find all necessary probabilities that are used in Bayes' rule first). Compare with your previous answers.

$$P(H_1|A) = \frac{P(A|H_1) \cdot P(H_1)}{P(A)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

$$P(H_3|A) = \frac{P(A|H_3) \cdot P(H_3)}{P(A)} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

5. Should the participant change the initial decision to increase probability of winning?
If the participant initially chose door № 1, and the host opened door № 2, then taking into account all the above conditions, the probability of winning will increase if the participant changes his mind and chooses door № 3.
Thus, **the participant should change his initial decision in order to increase the probability of winning.**

3 The third task

I can buy one lottery ticket out of two available. In the first lottery I can win 100 with probability 0.1, and the price of ticket is 10. In the second lottery, I can win 50 with probability 0.1 and 500 with probability 0.01. The price of ticket is 20. To decide which ticket to buy I toss a fair coin once. I chose first ticket in case of head and second otherwise. Let X be random variable that denotes my net payout (taking into account price of a ticket). Find probability mass function of X (hint: use law of total probability). Show that expected value of X is an average of expected values of net payouts for each of two lotteries. Explain, why. Will it still hold if lotteries has different payouts or probabilities? Prove it.

X - random variable that denotes net payout (taking into account price of a ticket).
Let's denote the events as follows:
1. A - as a result of the coin toss, the first lottery is selected.
- A1 - winning 100 in the first lottery.
- A2 - loss in the first lottery.
2. B - as a result of the coin toss, the second lottery is selected.
- B1 - winning 50 in the second lottery.
- B2 - winning 500 in the second lottery.
- B3 - loss in the second lottery.
Since we are throwing a fair coin, then:

$$P(A) = P(B) = \frac{1}{2}$$

$$P(A1) = \frac{1}{2} \cdot 0.1 = 0.05$$

$$P(A2) = \frac{1}{2} \cdot 0.9 = 0.45$$

$$P(B1) = \frac{1}{2} \cdot 0.1 = 0.05$$

$$P(B2) = \frac{1}{2} \cdot 0.01 = 0.005$$

$$P(B3) = \frac{1}{2} \cdot 0.89 = 0.445$$

The probability of all possible events:

$$P = P(A) + P(B) = \frac{1}{2} + \frac{1}{2} = 1$$

$$P = P(A1) + P(A2) + P(B1) + P(B2) + P(B3) = 0.05 + 0.45 + 0.05 + 0.005 + 0.445 = 1$$

Knowing the size of the winnings and probabilities, let's write the probability mass function in the form of a table:

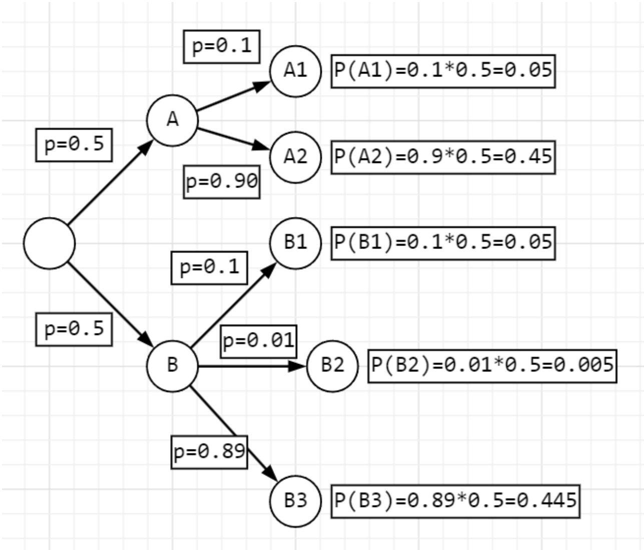
X	90	-10	30	480	-20
P	P(A1)=0.05	P(A2)=0.45	P(B1)=0.05	P(B2)=0.005	P(B3)=0.445

Let's find the expected value of X:

$$EX = 90 \cdot 0.05 - 10 \cdot 0.45 + 30 \cdot 0.05 + 480 \cdot 0.005 - 20 \cdot 0.445 = -5$$

The expected value of the winnings when playing the lottery with a preliminary toss of a fair coin will be -5 dollars.

We also can represent it like a tree (for our case):



Let's look at each lottery separately and write the probability mass function in the form of a table:

1. The first lottery:

X	90	-10
P	0.1	0.9

$$EX_1 = 90 \cdot 0.1 - 10 \cdot 0.9 = 0$$

2. The second lottery:

X	30	480	-20
P	0.1	0.01	0.89

$$EX_2 = 30 \cdot 0.1 + 480 \cdot 0.01 - 20 \cdot 0.89 = -10$$

Note that we are considering an experiment in which further actions depend on previous ones, which means that when determining the probability of a particular event, we must multiply the probability of this event by the probability of the previous event allowing this final action to be performed. Since we have only two outcomes when tossing a coin and we know that the total probability should be one, then:

$$EX = P_1 \cdot EX_1 + P_2 \cdot EX_2$$

$$P_2 = 1 - P_1$$

$$EX = P_1 \cdot EX_1 + (1 - P_1) \cdot EX_2 = P_1 \cdot EX_1 + EX_2 - P_1 \cdot EX_2$$

In our case $\mathrm{P}_1 = \frac{1}{2}$:

$$EX = P_1 \cdot EX_1 + (1 - P_1) \cdot EX_2 = \frac{1}{2} \cdot EX_1 + EX_2 - \frac{1}{2} \cdot EX_2 = \frac{1}{2} \cdot EX_1 + \frac{1}{2} \cdot EX_2 = \frac{EX_1 + EX_2}{2}$$

$$EX = \frac{EX_1 + EX_2}{2} = \frac{0 - 10}{2} = -5$$

This expression will be valid for any probability of winning/losing and the amount of payouts in each individual lottery, BUT the above expression indicates that the expected value of X is the average of the expected value of net payouts for each of the two lotteries only if the probability of a head is 0.5. If we change the probability of choosing a certain lottery when tossing a coin (toss unfair coin), then the expected value of X won't be the average of the expected net payout values for each of the two lotteries.

4 The fourth task

Space Company is preparing to launch a spaceship. If the launch is successful, the company earns 100 million in profit. In case of failure, the company loses 200 million (i.e. negative profit). Probability of failure is 1/10. The company can buy insurance for this launch. Cost of insurance is 30 million (paid before the launch). In case of failure the insurer will pay Space Company 200 million (thus compensating all the damages). Consider two cases: 1. Space Company decided not to buy insurance. 2. Space Company decided to buy insurance. Denote its profit in the first case by X and in the second case by Y. (In the second case profit includes payments to/from the insurer taken with appropriate sign.) Find expected values and variances of X and Y. Describe how buying of insurance affects the profit, its expected value and variance? Does buying insurance is cost-efficient in the long run? In which case Space Company can decide to buy insurance and why?

1. Let's consider the first case and find expected value and variance:

X	100	-200
P	0.9	0.1
X-EX	30	-270
$(\text{X-EX})^2$	900	72900

$$EX = 0.9 \cdot 100 + 0.1 \cdot (-200) = 70$$

$$Var_1 = 900 \cdot 0.9 + 72900 \cdot 0.1 = 810 + 7290 = 8100$$

2. Let's consider the second case and find expected value and variance:

Y	70	-30
P	0.9	0.1
Y-EY	10	-90
$(\text{Y-EY})^2$	100	8100

$$EY = 0.9 \cdot 70 + 0.1 \cdot (-30) = 60$$

$$Var_2 = 100 \cdot 0.9 + 8100 \cdot 0.1 = 90 + 810 = 900$$

Conclusions:
Buying insurance is not cost-efficient in the long run, since with a large number of launched spacecraft, the average benefit value will approach the expected value, which means that launching a spacecraft without insurance will be more cost-efficient than with insurance, since $EX > EY$.
But with a small number of spacecraft launches, there may be cases when our average benefit for launching with insurance will be greater than the average benefit for launching without insurance.
Thus, it is cost-efficient to buy insurance with a small number of spacecraft launches, since the cost of an error is not comparable with the cost of insurance, but in the long term (with a large number of launches), buying insurance is not cost-efficient.

To further confirm the above conclusions, let's conduct a series of random experiments using Python:

```python
from numpy.random import choice

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4 usages
def profit(insurance=False, n=10):
    if not insurance:
        data = choice( a: [100, -200], p=[0.9, 0.1], size=n)
        return sum(data) / len(data)
    else:
        data = choice( a: [70, -30], p=[0.9, 0.1], size=n)
        return sum(data) / len(data)

print(f'Number of launches = 10, the average value of the'
      f' benefit for starting without insurance = {profit(insurance=False, n=10)}')
print(f'Number of launches = 10, the average value of the'
      f' benefit for starting with insurance = {profit(insurance=True, n=10)}')
print(f'Number of launches = 10000, the average value of the'
      f' benefit for starting without insurance = {profit(insurance=False, n=10000)}')
print(f'Number of launches = 10000, the average value of the'
      f' benefit for starting with insurance = {profit(insurance=True, n=10000)}')
```

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```
C:\Users\y2966\PycharmProjects\Probability_theory\.venv\Scripts\python.exe C:\Users\y2966\PycharmPro
Number of launches = 10, the average value of the benefit for starting without insurance = 40.0
Number of launches = 10, the average value of the benefit for starting with insurance = 70.0
Number of launches = 10000, the average value of the benefit for starting without insurance = 70.39
Number of launches = 10000, the average value of the benefit for starting with insurance = 59.87
```