Fig. 1.

as follows (see Fig. 1):

$$\int_0^1 \left| \left(\frac{1}{2}\theta^2 - \mu\theta\right) \right| d\theta = -\int_0^{2\mu} \left(\frac{1}{2}\theta^2 - \mu\theta\right) d\theta + \int_{2\mu}^1 \left(\frac{1}{2}\theta^2 - \mu\theta\right) d\theta = 4/3\mu^3 - \mu/2 + 1/6.$$

Hence (8) becomes

$$|e(\mu)| \le h^2 \left(\frac{1}{2} - \mu\right) |\ddot{X}(t+h)| + h^3 \underset{0\le\theta\le1}{M} \left(4/3\mu^3 - \frac{1}{2}\mu + \frac{1}{6}\right)$$

which is exactly (5′). Equations (6′) and (7′) follow by substituting $\mu = 0$ and $\frac{1}{2}$, respectively, in (5′).

Despite Russo's claim, (7′), and not (7), is a well-known result. (See, for example, [2, p. 31] or [3, p. 165].)

2) The relation $\mu = 1/q - [1/(e^q - 1)]$, and the statement that follows, i.e., "μ is a monotonically decreasing function of $q \,\forall\, q$[2]," are given in [1], and therefore reference to [2] is unnecessary.

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Bounds on the Truncation Error

We shall work with functions of time that are real and periodic with period T. These functions are those which can be expressed by the Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \tag{1}$$

where

$$\omega_0 = 2\pi/T \tag{2}$$

and

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt. \tag{3}$$

We shall approximate $f(t)$ by $f_m(t)$ where

$$f_m(t) = \sum_{n=-m}^{m} C_n e^{jn\omega_0 t}. \tag{4}$$

We wish to determine a value of m which guarantees that the error between $f(t)$ and $f_m(t)$ is less than some specified value.

Two different error criteria will be used. The first is the mean square or Hilbert norm given by

$$\epsilon_{MS}^2 = \|f - f_m\|_2^2 = \int_{-T/2}^{T/2} [f(t) - f_m(t)]^2\, dt. \tag{5}$$

The second error criterion is the maximum error or the Chebyshev norm

$$\epsilon_{\max} = \|f - f_m\| = \sup |f(t) - f_m(t)|, \qquad -\frac{T}{2} \le t \le \frac{T}{2}. \tag{6}$$

This is the maximum pointwise difference between $f(t)$ and $f_m(t)$.

We shall start with the Hilbert norm.

Theorem 1

If a periodic $f(t)$ is of bounded variation whose total variation over one period is bounded by N, that is