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Research internship report

Lot Sizing with remanufacturing and consideration of new and used sales

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Contents

1 Introduction

The subject of the course was initially formulated as follows:

Over the last ten years or so, classic lot sizing models, often used for supply or production planning or production planning, have been enriched by taking into account various constraints. These include remanufacturing constraints. The problem is to plan the production of and remanufactured products. The remanufactured products are products that are returned to the manufacturer after use. The aim is to plan these two types of production together, sharing common launch costs (setups) or or common production capacities.

In all current work, we make the strong assumption that remanufactured products are of the same quality as new products, and therefore meet customer demand in the same way. same quality as new products, and therefore meet customer demand in the same way. In this work, we explore problems in which the demand for remanufactured products and for new products are distinct. Remanufactured products are sold as second-hand products second-hand products, to meet specific demands for this type of product.

The aim of the internship is to model one of the variants of this problem, look for the properties that optimal solutions must satisfy, and look for polynomial or pseudo-polynomial solution algorithms, and eventually to program them. In this kind of problem, dynamic programming is often the most appropriate technique, but other approaches are also possible.

You can see the code of the project on the github repository.

2 Notations

2.1 Parameters

Parameters in each period t (T the length of the horizon):

- D_t^n: Demand of manufactured products (new) (satisfied at the end of the period);
- D_t^s: Demand of remanufactured products (second-hand) (satisfied at the end of the period);
- R_t: Returns (at the beginning of the period);
- f_t: Fixed joint setup cost;
- p_t^s: Variable remanufacturing cost;
- p_t^n: Variable manufacturing cost;
- p_t^r: Variable cost to dispose returns;
- h_t^s: Holding cost of remanufactured products;
- h_t^n: Holding cost of manufactured products;

- h_t^r: Holding cost of returns;
- C_t^s: Remanufacturing capacity;
- C_t^n: Manufacturing capacity;
- C_t: Manufacturing and remanufacturing capacity (In case of common capacity);

2.2 Decision variables

Decision variables in each period t:

- x_t^s: Number of remanufactured products;
- x_t^n: Number of manufactured products;
- x_t^r: Number of disposed return products;
- y_t: 1 if $x_t^s + x_t^n > 0$, 0 otherwise;
- s_t^s: Inventory of remanufactured products (at the end of the period);
- s_t^n: Inventory of manufactured products (at the end of the period);
- s_t^r: Inventory of returns (at the end of the period);

3 Data structure constraints

We consider the lot sizing problem under different constraints

3.1 Non-speculative costs

Definition 1 *Non-speculative costs are define by the following property :*

$$p_t + h_t > p_{t+1} \qquad \forall t \in [\![1, T-1]\!]$$

It means that it is always better to not predict the production.
By extension, with our notation, we have : $\forall t \in [\![1, T-1]\!]$

$$p_k^s + \sum_{j=k}^{t} h_j^s > p_{t+1}^s \qquad \forall k < t+1 \qquad (2.1.s)$$

$$p_k^n + \sum_{j=k}^{t} h_j^n > p_{t+1}^n \qquad \forall k < t+1 \qquad (2.1.n)$$

$$(1)$$

There is a difference for returns ; non-speculative costs means that it is always better to dispose earlier. We can translate it by : $\forall t \in [\![1, T-1]\!]$

$$p_k^r > p_{t+1}^r + \sum_{j=k}^{t} h_j^r \qquad \forall k < t+1 \qquad (2.1.r)$$

3.2 Return-advantage costs

An other constraint is to admit that it is always better to stock returns then produce remanufactured products than the opposite.

We can translate this by the following property :

$$p_t^s + \sum_{j=t}^{t+k-1} h_j^s > p_{t+k}^s + \sum_{j=t}^{t+k-1} h_j^r \qquad \forall t \in [\![1, T-1]\!], k \geq 1 \tag{2.2}$$

3.3 Constraints on stock

We suppose that we do not have stock at the beginning : $s_0^s = s_0^r = s_0^n = 0$.

More, in the case when we include dispose (dispose = we are able to sell/throw returns), we suppose that we are not able to stock returns at the end of the horizon : $s_T^r = 0$

3.4 Constraints on demand

We suppose that the demand is strictly positive for each period :

$$D_t^s > 0 \text{ and } D_t^n > 0 \qquad \forall t \in [\![1, T]\!]$$

4 Uncapacitated lot sizing problem with returns

4.1 Feasibility condition

Proposition 1 *The uncapacitated lot sizing problem with returns admits a solution iff*

$$\sum_{t=1}^{T'} R_t \geq \sum_{t=1}^{T'} D_t^s \qquad \forall T' \in [\![1, T]\!] \tag{3.1}$$

Proof.

- ($\Leftarrow$) Suppose that we have $\forall T' \in [\![1, T]\!] : \sum_{t=1}^{T'} R_t \geq \sum_{t=1}^{T'} D_t^s$.

We want to show that we can answer to the demand for each period $\forall t \in [\![1, T]\!]$. Then we want to show that we can produce more than the demand for each period with $x_t^s = R_t$, $\forall t \in [\![1, T]\!]$ (we remanufacture every return).

Let $T' \in [\![2, T]\!]$. We have :

$$\begin{aligned}
& \sum_{t=1}^{T'} R_t \geq \sum_{t=1}^{T'} D_t^s \\
\Leftrightarrow & \sum_{t=1}^{T'} R_t - \sum_{t=1}^{T'-1} D_t^s \geq \sum_{t=1}^{T'} D_t^s - \sum_{t=1}^{T'-1} D_t^s \\
\Leftrightarrow & R_{T'} + \sum_{t=1}^{T'-1} (x_t^s - D_t^s) \geq D_{T'}^s && \text{Because } x_t^s = R_t, \forall t \in [\![1, T]\!] \\
\Leftrightarrow & x_{T'}^s + s_{T'-1} \geq D_{T'}^s
\end{aligned}$$

More, for $T' = 1$, it is also verified by (3.1) ($x_1^s = R_1 \geq D_1$) .

Then we can produce enough for each period. We can build a solution by stock every excess of production. So the problem admits a solution.

- ($\Rightarrow$) Suppose that (3.1) is not verified, i.e. : $\exists T' \in [\![1, T]\!]$ as $\sum_{t=1}^{T'} R_t < \sum_{t=1}^{T'} D_t^s$.

If $T' = 1$, then $R_1 < D_1^s$ and we can't produce enough for the first period, so there isn't any solution.

If $T' \in [\![2, T]\!]$, then we suppose that we are producing the maximum that we can ($x_t^s = R_t, \forall t \in [\![1, T]\!]$), and so in the same as before we have : $x_{T'}^s + s_{T'-1} < D_{n+1}^s$.

Then we can't produce enough for the period T'.

So the problem doesn't admit any solution. □

4.2 Without dispose problem

4.2.1 Wagner-Within property

Proposition 2 *There exists at least an optimal solution to the Lot sizing problem with remanufacturing and returns without dispose that satisfies the Wagner-Within property :*

$$s_{t-1}^s \times x_t^s = 0 \text{ and } s_{t-1}^n \times x_t^n = 0 \qquad \forall t \in [\![1, T]\!] \tag{3.2.1}$$

Proof.

- $s_{t-1}^n \times x_t^n = 0$

We suppose for the sake of contradiction that for all optimal solutions $\exists t \in [\![1, .., T]\!] : s_{t-1}^n \times x_t^n \neq 0$
Since $s_{t-1}^n \neq 0$ then, there is a $k_j < t$ for each new product j in storage at time $t-1$ such that it was produced at time k_j.
However, due to the non-speculative property of costs, that is (following a trivial induction):
$p_{k_j}^n + \sum_{i=k_j}^{t-1} h_i^n > p_t^n$.
We obtain a better solution by producing those stored products (at time $t-1$) at time t rather, since that would lead to a total change of cost of $\sum_{j=1}^{s_{t-1}^n} (p_t^n - \sum_{i=k_j}^{t-1} h_i^n) < 0$ or even less if there is no more production in one of the times of production k_j, thus resulting in a better solution where $s_{t-1}^n = 0$

- $s_{t-1}^s \times x_t^s = 0$

We suppose for the sake of contradiction that for all optimal solutions $\exists t \in [\![1, .., T]\!] : s_{t-1}^s \times x_t^s \neq 0$.
Following the exact same line of reasoning, except that the property being used this time would be the "return-advantage costs" property, that is: $p_{k_j}^s + \sum_{i=k_j}^{t-1} h_i^s > p_t^s + \sum_{i=k_j}^{t-1} h_i^r$, we obtain a better solution by storing each returned product j from time k_j to time t then producing it rather than producing it first at time k_j then storing it, all while obtaining $s_{t-1}^s = 0$. □

4.2.2 Same-period production

Proposition 3 *There exists at least an optimal solution to the Lot sizing problem with remanufacturing and returns without dispose that satisfies the following property :*

$$x_t^n > 0 \Leftrightarrow x_t^s > 0 \qquad \forall t \in [\![1, T]\!]$$

Proof.

• (⇒) We are going to prove this property by finite strong recursion with the following property $\forall t \in [\![1,T]\!]$: $P(t) = (x_t^n > 0 \Rightarrow x_t^s > 0)$.

⋆ $t = 0$: $D_0^n > 0$ and $s_0^n = 0$ then $x_0^n > 0$. Same, $D_0^s > 0$ and $s_0^s = 0$ then $x_0^s > 0$. So $P(0)$ is verified.

⋆ Let $t \in [\![2,T]\!]$. Suppose that $P(t')$ is verified $\forall t' < t$.

Suppose that $x_t^n > 0$.

Let's reason by the absurd and suppose that $x_t^s = 0$.

As $D_t^s > 0$, we have $s_t^s > 0$. We now consider a small production quantity ϵ_s. Let k the period where the quantity ϵ_s is produced. We chose ϵ_s as small than we have $\forall t' \in [\![k+1,t]\!]$: $s_{t'}^s > 0$ and $x_{t'}^s = 0$ (ϵ_s is produced at the last period of production).

By $P(k)$ we got $x_k^n > 0$. And we supposed that $x_t^n > 0$. Then we ignore setup cost f_k and f_t. Then we have :

$$p_k^s + \sum_{j=k}^{t-1} h_j^s \le p_t^s + \sum_{j=k}^{t-1} h_j^r$$

We got this because otherwise we would produce ϵ_s at period t (it is at best equivalent). But this is absurd by the constraint (2.2).

Then $x_t^s > 0$ and $P(t)$ is verified.

• (⇐) We are going to prove this property by finite strong recursion with the following property $\forall t \in [\![1,T]\!]$: $P(t) = (x_t^s > 0 \Rightarrow x_t^n > 0)$.

⋆ $t = 0$: $D_0^s > 0$ and $s_0^s = 0$ then $x_0^s > 0$. Same, $D_0^n > 0$ and $s_0^n = 0$ then $x_0^n > 0$. So $P(0)$ is verified.

⋆ Let $t \in [\![2,T]\!]$. Suppose that $P(t')$ is verified $\forall t' < t$.

Suppose that $x_t^s > 0$.

Let's reason by the absurd and suppose that $x_t^n = 0$.

As $D_t^n > 0$, we have $s_t^n > 0$. We now consider a small production quantity ϵ_n. Let k the period where the quantity ϵ_n is produced. We chose ϵ_n as small than we have $\forall t' \in [\![k+1,t]\!]$: $s_{t'}^n > 0$ and $x_{t'}^n = 0$ (ϵ_n is produced at the last period of production).

By $P(k)$ we got $x_k^s > 0$. And we supposed that $x_t^s > 0$. Then we ignore setup cost f_k and f_t. Then we have :

$$p_k^n + \sum_{j=k}^{t-1} h_j^n \le p_t^n$$

We got this because otherwise we would produce ϵ_n at period t (it is at best equivalent). But this is absurd by the non-speculative costs constraint (2.1.n).

Then $x_t^n > 0$ and $P(t)$ is verified. □

4.3 With dispose problem

4.3.1 Wagner-Within property

Proposition 4 *There exists at least an optimal solution to the Lot sizing problem with remanufacturing and returns with dispose that satisfies the Wagner-Within property :*

$$s_{t-1}^s \times x_t^s = 0 \text{ and } s_{t-1}^n \times x_t^n = 0 \qquad \forall t \in [\![1,T]\!]$$

Proof. Including dispose does not affect the proof of the property.

Then, the proof is the same as before (3.2.1). □

4.3.2 Same-period production and storage

Proposition 5 *There exists at least an optimal solution to the Lot sizing problem with remanufacturing and returns with dispose that satisfies the following property :*

$$x_t^n > 0 \Leftrightarrow x_t^s > 0 \qquad \forall t \in [\![1, T]\!]$$

Proof. Including dispose does not affect the proof of the property.
Then, the proof is the same as before (3.2.2). □

We notice that it follows immediately due to the Wagner-Within property and the fact that all demands must be satisfied that:

$$s_t^n > 0 \Leftrightarrow s_t^s > 0 \qquad \forall t \in [\![1, T]\!]$$

4.4 Subplan structure

In this section, we focus on the uncapacitated lot sizing problem with returns and **dispose**, the case without dispose can be easily treated with a classical 2-index subplan structure.

We can define, as the uncapacitated single-item lot sizing problem a structure of subplans that would be interesting for dynamic programming.

Definition 2 *A subplan $S_{uv,nm}$ $(u \leq n \leq v \leq m)$ between the periods u and v under the demand between the periods n and m is a defined such as :*

- $s_{u-1}^r = 0,\ s_v^r = 0$
- $\forall t$ *such as* $n < t \leq v$: $s_{t-1}^r > 0$
- $x_n^s > 0$ *and* $\forall t$ *such as* $u \leq t < n$: $x_t^s = 0$ *(n first period of production)*
- $s_m^s = 0$ *and* $\forall t$ *such as* $v < t < m$: $s_{t-1}^s > 0$ *(m last period of storage)*

A subplan $S_{uv,nm}$ corresponds to idea of : "The interval $[\![u, v]\!]$ meets to the demand in the interval $[\![n, m]\!]$".

4.4.1 Subplan independence

We will now prove that finding the optimal solution to each subplan independently leads to an optimal solution to the entire problem.

Let $Y = (X^r, S^r, X^s, S^s, X^n, S^n)$ be an optimal solution to the whole problem, satisfying all properties in Section 3.3 (Wagner-Within and Same-period properties). Let's denote $Y_{k,l}$ the tuple Y where X^r and S^r are restricted to their prefix up to period k (inclusive), and the rest are restricted to their prefix up to period l (inclusive)

Proposition 6 *For all $0 \leq k \leq l \leq T$, such that $s_k^r = 0$, $s_l^s = 0$, $x_k^s > 0$ and $\forall i \in]\!]k, l]\!] : x_i^s = 0$, $Y_{k,l}$ can be partitioned into Subplans.*

Proof. We will prove the partition on the returns and second-hand products only, since the partition as a whole follows it immediately due to the same-period production and storage property. We will proceed with a proof by induction:

- **Base case:** $k = 0$ and $l = 0$: It's a given.
- **Induction:** Let $0 \leq k \leq l \leq T$. If $k = 0$, then $l = 0$ too because otherwise if $l > 0$, the first l demands would be unsatisfied since $\forall i \in]\!]0, l]\!] : x_i^s = 0$.

  Now if $k > 0$, let:

  - $w = \min\{1 \leq i \leq k \text{ where } \forall i \leq t < k : s_t^r > 0\}$
  - $u = \min\{1 \leq i \leq w \text{ where } \forall i \leq t < w : x_t^s = 0\}$
  - $n = \min\{i \text{ where } u \leq i \leq k \text{ and } x_i^s > 0\}$

  Hence, we have that $S_{uk,nl}$ is a subplan because:

  - $s_k^r = 0$ and $s_{u-1}^r = 0$, because if $u = w$, then by definition of w we have $s_{w-1}^r = 0$, otherwise if $u < w$, then an incoming stock of returned products at time u would have to be disposed at some time $u \leq t < w$ since $s_{w-1}^r = 0$ and there is no production in the period $[\![u, w[\![$, which is sub-optimal because of the constraint on non-speculative costs
  - $\forall t$ such that $n < t \leq k : s_{t-1}^r > 0$, which follows from the fact that $n \geq w$.
  - $x_n^s > 0$ and $\forall t$ where $u \leq t < n : x_t^s = 0$
  - $s_l^s = 0$, and $\forall t$ where $k \leq t < l : s_t^s > 0$ because otherwise there would be some unsatisfied demands since $\forall t$ where $k < t \leq l : x_t^s = 0$

  Let's note $k' = u - 1$ and $l' = n - 1$.

  We have that $x_n^s > 0$, then following the Wagner-Within property we have that $s_{l'}^s = s_{n-1}^s = 0$, we also have that $s_{k'}^r = s_{u-1}^r = 0$ and that $x_{k'}^s > 0$ by definition of u, and for all i where $k' < i \leq l' : x_i^s = 0$ and $k' \leq l'$, then inductively we obtain that $Y_{k',l'}$ can be partitioned into subplans, then by adding the subplan $S_{uk,nl}$ to it we obtain a partition of $Y_{k,l}$.

□

Proposition 7 *$\exists k \in [\![1, T]\!] : Y_{k,T}$ can be partitioned into subplans, and $\forall t > k$, the returns at time t are disposed immediately.*

Proof. Let $k = \max\{i \in [\![1, T]\!] \text{ such that } x_i^s > 0\}$, then $s_k^r = 0$ because otherwise that stock would have to be disposed later on since there is no more production after k, which would be sub-optimal given the constraint on non-speculative costs. Additionally we have that $s_T^s = 0$, $x_k^s > 0$, and $\forall i \in]\!]k, T]\!] : x_i^s = 0$. Then according to Proposition 6, we have that $Y_{k,T}$ can be partitioned into subplans, and $\forall t > k$, the returns at time t are disposed immediately because they will be disposed eventually since there is no production left and it is optimal to dispose of them immediately due to the non-speculative constraint on costs. □

We show now that the partitioning of the optimal solution into subplans and the eventual immediate disposals, allows us to immediately conclude the subplans' independence. That is because, the cost of the whole problem is equal to the sum of costs of all subplans, and the cost of the disposals at the end. That is true because the cost of the problem is the sum of the costs for storage, production and disposal, and production setup, furthermore, we know that the total cost for storage (whether for returned, remanufactured or new products) for the entire problem is the sum of storage cost for each subplan (because subplans make up a partition of the solution), and the same thing goes for the total cost for production and disposal. It remains to show that the total cost for production setup is the sum of costs for production setup for each subplan, but then again that is also quite evident, since for every subplan $S_{uv,nm}$ (immediate disposal subplans excluded), production only occurs during the period $[n, v]$, and we know that those periods are disjoint among subplans because two consecutive subplans $S_{u_1v_1,n_1m_1}$ and $S_{u_2v_2,n_2m_2}$ verify that: $v_2 \geq n_2 \geq u_2 > v_1 \geq n1$

4.4.2 Dispose at the beginning

Proposition 8 *Let a subplan $S_{uv,nm}$. An optimal solution of the subplan verify :*

$$\exists t \in [\![u, n]\!] \text{ such as } \forall k \in]\!]t, v]\!] : x_t^r = 0 \tag{3.4.2}$$

Proof. Suppose an optimal solution of a subplan $S_{uv,nm}$.

Suppose that $\exists k$ such as $n < k \leq v$ and $x_k^r > 0$.

By definition of a subplan, we have $s_{k-1}^r > 0$.

The solution is optimal then we have $p_{k-1}^r > h_{k-1}^r + p_k^r$ which is absurd by non-speculative costs constraint (2.1.r).

It is the same for the case when $\exists k$ such as $u < k \leq n$ and $x_k^r > 0$ and $s_{k-1}^r > 0$: it is better to dispose s_{k-1}^r at period $k-1$ than at period k. □

4.4.3 Cost of a subplan

We can calculate the cost of each subplan $S_{uv,nm}$ by dynamic programming.

We define :

- $D_{i,j}^s = \sum_{t=i}^{j} D_t^s$ the sum of the remanufactured products demand between periods i and j
- $D_{i,j}^n = \sum_{t=i}^{j} D_t^n$ the sum of the new products demand between periods i and j
- $R_{i,j} = \sum_{t=i}^{j} R_t$ the sum of returns between periods i and j
- $Q^r = R_{u,v} - D_{n,m}^s$ The dispose quantity at the beginning of the subplan (cf 3.4.2)

Also we define the following cost functions :

- $C(P_{t,k}) = f_t + (p_t^s D_{t,k-1}^s + \sum_{j=t}^{k-2} h_j^s D_{j+1,k-1}^s) + (p_t^n D_{t,k-1}^n + \sum_{j=t}^{k-2} h_j^n D_{j+1,k-1}^n)$ the cost of production of new and remanufactured products in answer to demand between periods t and k
- $C_s(t,k) = \sum_{j=t}^{k-1} ((R_{u,t} - Q^r - D_{n,k-1})h_j^r + R_{t,j} h_j^r)$ the cost associated to storage between periods t and k. The quantity $R_{u,t} - Q^r - D_{n,k-1}$ is equal to the incoming stock at the beginning of this storage period.

Then we define the objective function (the cost of an optimal solution) by the recursion formula (Bellman equation) :

$$f^*(t) = \min_{k=t+1}^{v} \{C(P_{t,k}) + C_s(t,k) + f^*(k)\}$$

With $f^*(v) = C(P_{v,m})$ and $f^*(u) = \sum_{j=u}^{n-1}(p_j^r x_j^r + s_j^r h_j^r) + f^*(n)$

with $x_j^r = \mathbb{1}_{R_{u,j} \leq Q^r} R_j + \mathbb{1}_{R_{u,j+1} > Q^r > R_{u,j}}(Q^r - R_{u,j})$ the quantity of disposed products at period i

and $s_j^r = \max(0, R_{u,j} - Q^r)$ the stock quantity at period j

.

Then we want to calculate $f^*(u)$.
We need to calculate all the $f^*(t)$ for $n \leq t \leq v$.

4.4.4 Feasibility condition to subplan construction

There is also a feasibility condition to a subplan $S_{uv,nm}$. A subplan only exists if :

$$\bullet R_{u,k} \geq D_{n,k}^s \qquad \forall k \text{ such as } n \leq k \leq v$$
$$\bullet R_{u,v} \geq D_{n,m}^s$$

Then when these conditions are not verified, we assume that $C(S_{uv,nm}) = +\infty$.
Same when $v < n$ or $n < u$ or $m < v$ (it is not a subplan) : $C(S_{uv,nm}) = +\infty$.

4.5 Cost of the problem

With 3.4.1 we can find an optimal solution to the problem by building independent subplans.
Then we define the Bellman equation for the global problem :

$$f^*(t,k) = \min_{\substack{t \leq i \leq T \\ k \leq j \leq T}} \{C(S_{ti,kj}) + f^*(i+1, j+1)\}$$

with :

$$f^*(t, T+1) = \sum_{j=t}^{T} R_j p_j^r \text{ the cost of all immediate dispose at the end}$$

The cost of the global problem is $f^*(1,1)$.

5 Lot sizing problem with uncommon capacity

We keep the notations defined in 3.4.3 section.
More, we keep the capacity constant. It means $\forall t \in [\![1,T]\!]$: $C_t^n = C^n$ and $C_t^s = C^s$.

5.1 Feasibility condition

Proposition 9 *The capacitated lot sizing problem with returns and uncommon capacity admits a solution iff*

$$M_t^s \geq D_{1,t}^s \qquad \forall t \in [\![1,T]\!] \tag{4.1.s}$$
$$\text{and } tC^n \geq D_{1,t}^n \qquad \forall t \in [\![1,T]\!] \tag{4.1.n}$$

with M_t^s verifying $M_{t+1}^s = \min(R_{1,t+1} - M_t^s, C^s) + M_t^s$ and $M_1^s = \min(R_1, C^s)$.

M_t^s represents the maximum of production of remanufactured products until the period t.

Proof. We are going to demonstrate that M_t^s is the maximum of production of remanufactured products between periods 1 and t.

Let X_t^s the maximum of production at period t admitting that all the production until $t-1$ is maximized.

We have $X_1^s = \min(R_1, C^s) = M_1^s$ by definition of X_1^s and M_1^s and because we can't produce more than the capacity and more than the returns we have.

By definition of M_t^s, we have $M_{t+1}^s = M_t^s + X_{t+1}^s$.

But we have $X_{t+1}^s = \min(C^s, R_{t+1} + S_t^r)$ with S_t^r the stock of remanufactured products admitting the production until period t is maximized.

But we have $S_t^r = R_{1,t} - M_t^s$ because it's the number of returns that we have minus the number of remanufactured products we produced.

Then we have the relation $M_{t+1}^s = \min(R_{1,t+1} - M_t^s, C^s) + M_t^s$.

So M_t^s is the maximum of production of remanufactured products. And the proposition (4.1.s) can be demonstrated with same arguments than proposition (3.1).

Same for (4.1.n), tC^n is the maximum of production of new products between period 1 and t then we prove (4.1.n) like we proved (3.1). □

5.2 Without dispose problem

5.2.1 Non-fractional productions

Definition 3 *We define a fractional period of production a period t when $0 < x_t < C$.*

In the context of finding an optimal solution, we also define a pseudo-fractional period of production a period t where $x_t > 0$ and $s_{t-1} = 0$ (in practice, it means that this period of production "can be at a non full capacity" in an optimal solution, this will be justified by the following proposition).

Proposition 10 *For the lot sizing problem without dispose with uncommon capacity, there is an optimal solution that verifies :*

$$\bullet s_{t-1}^n > 0 \Rightarrow x_t^n \in \{0, C^n\} \qquad \forall t \in [\![2, T]\!]$$
$$\bullet s_{t-1}^s > 0 \Rightarrow x_t^s \in \{0, C^s\} \qquad \forall t \in [\![2, T]\!]$$

Proof.

• Let $t \in [\![2, T]\!]$ such that $s_{t-1}^n > 0$, suppose that $0 < x_t^n < C^n$. Since $s_{t-1}^n > 0$, there is a time $t' < t$ where one of the new products in storage arriving at time t was produced and stored during the period $[t', t[$, but since $x_t^n < C^n$, we can choose to produce it at time t rather which would lead to a better solution due to the constraint on non-speculative costs, which contradicts the optimality of our initial solution.

• Let $t \in [\![2, T]\!]$ such that $s_{t-1}^s > 0$, suppose that $0 < x_t^s < C^s$. Since $s_{t-1}^s > 0$, there is a time $t' < t$ where one of the remanufactured products in storage arriving at time t was remanufactured from a returned product and stored during the period $[t', t[$, but since $x_t^s < C^s$, we can choose to store it as a returned product from time during the period $[t', t[$ then remanufacture it at time t rather which would lead to a better solution due to the constraint on return-advantage costs, which contradicts the optimality of our initial solution. □

5.2.2 Same-period production

Proposition 11 *There exists at least an optimal solution to the lot sizing problem with remanufacturing and returns without dispose with uncommon capacity that satisfies the following property :*

$$x_t^n > 0 \Leftrightarrow x_t^s > 0 \qquad \forall t \in [\![1, T]\!]$$

Proof. The proof is the same as the uncapacitated case (3.2.2). □

5.2.3 Cost of the problem

We are going to calculate the optimal cost of the problem by dynamic programming.

For $t \in [1, T+1]$ and $k_s, k_n \geq t$, we define the cost function $f^*(t, k_s, k_n, N_s, N_n)$ as the sum of all costs after period t for the optimal solution of the problem verifying the following constraints:

- The first period of non storage after period t is $k_s - 1$ for remanufactured products and $k_n - 1$ for new products, that is:

$$\begin{cases} \forall i \in [t, k_s - 1[: s_i^s > 0 \\ s_{k_s-1}^s = 0 \end{cases} \quad \text{and} \quad \begin{cases} \forall i \in [t, k_n - 1[: s_i^n > 0 \\ s_{k_n-1}^n = 0 \end{cases} \qquad \textit{(4.2.3.i)}$$

- There are N_s periods of remanufactured production in the time period $[t, k_s[$, and N_n periods of new production in the time period $[t, k_n[$ *(4.2.3.ii)*
- The "Non-fractional productions" and "Same-period production" properties are verified *(4.2.3.iii)*

Following this definition, we get that the optimal solution for the entire problem is nothing but $f^*(1, 1, 1, 0, 0)$.

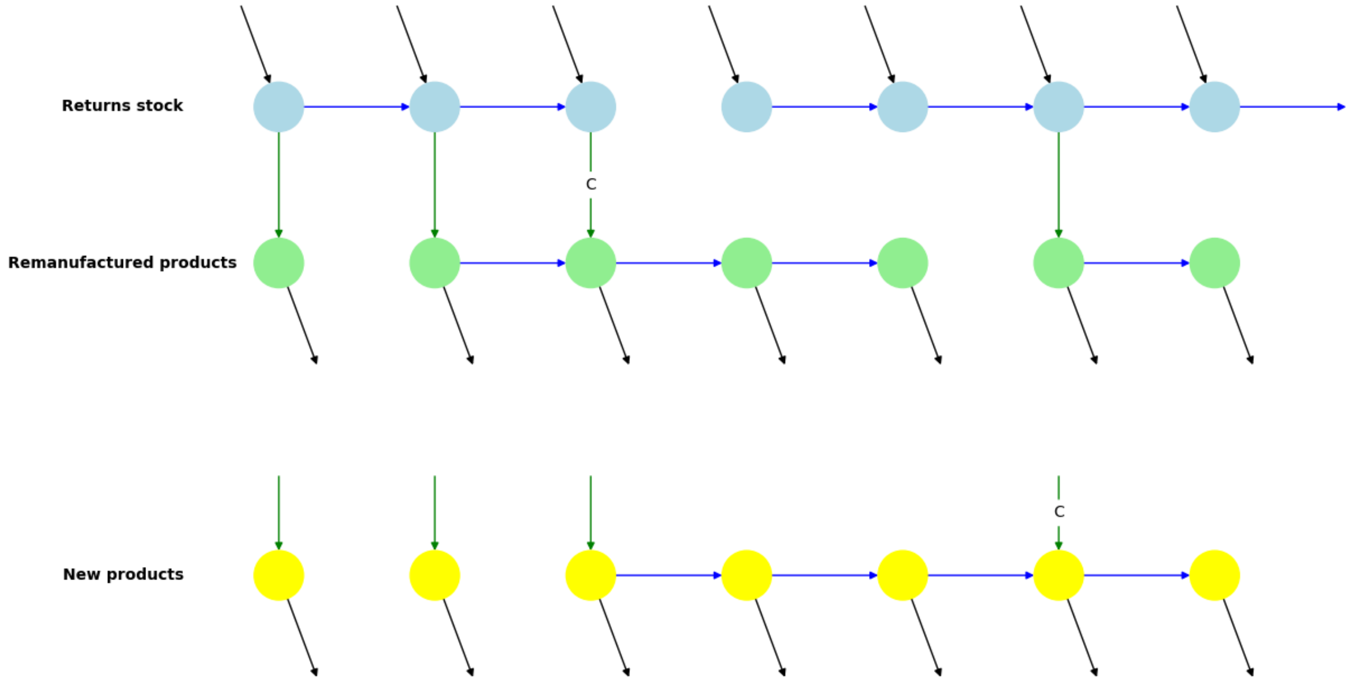
Then we can establish Bellman equation for the problem :

$$f^*(t, k_s, k_n, N_s, N_n) = \begin{cases} \min\limits_{t \leq i \leq T} \left(f^*(t, i+1, k_n, \left\lceil \frac{D_{t,i}^s}{C^s} \right\rceil, N_n) \right) \text{ if } t = k_s \text{ and } t < k_n \\ \\ \min\limits_{t \leq j \leq T} \left(f^*(t, k_s, j+1, N_s, \left\lceil \frac{D_{t,j}^n}{C^n} \right\rceil) \right) \text{ if } t < k_s \text{ and } t = k_n \\ \\ \min\limits_{t \leq i \leq T} \left(\min\limits_{t \leq j \leq T} \left(f^*(t, i+1, j+1, \left\lceil \frac{D_{t,i}^s}{C^s} \right\rceil, \left\lceil \frac{D_{t,j}^n}{C^n} \right\rceil) \right) \right) \text{ if } t = k_s = k_n \\ \\ \min \Big(f^*(t+1, k_s, k_n, N_s - 1, N_n - 1) + C_p(t, k_s, k_n, N_s, N_n), \\ f^*(t+1, k_s, k_n, N_s, N_n) + C_{np}(t, k_s, k_n, N_s, N_n) \Big) \text{ otherwise} \end{cases}$$

We define C_p and C_{np} respectively as the costs in case of production or non-production at period t.

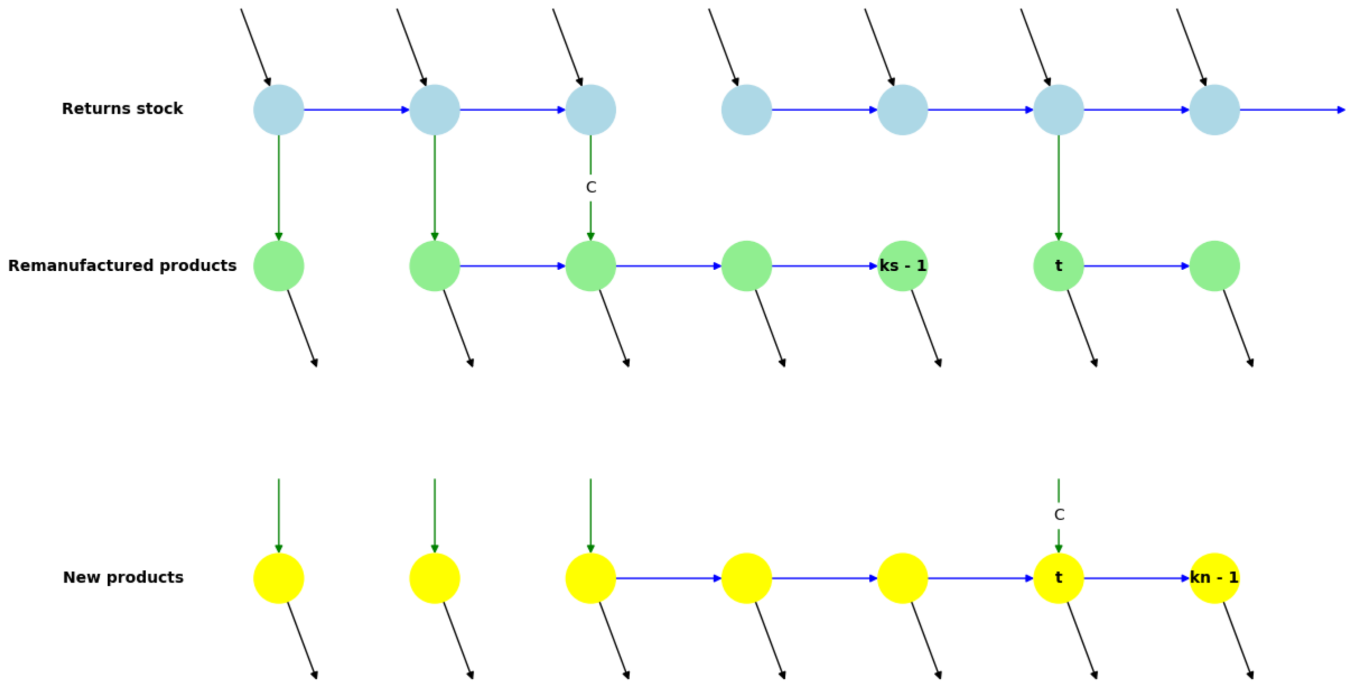
In order to explain how we build an optimal solution by this method, we are going to explain every cases of the Bellman equation.

Then we will explain the construction of the solution of the following instance (the solution is already drawn) :



The shape of an optimal solution constructed using dynamic programming

First, let's take a look at case 1, where $t = k_s$ and $t < k_n$. For example, here with $t = k_s = 6$ and $k_n = 8$:



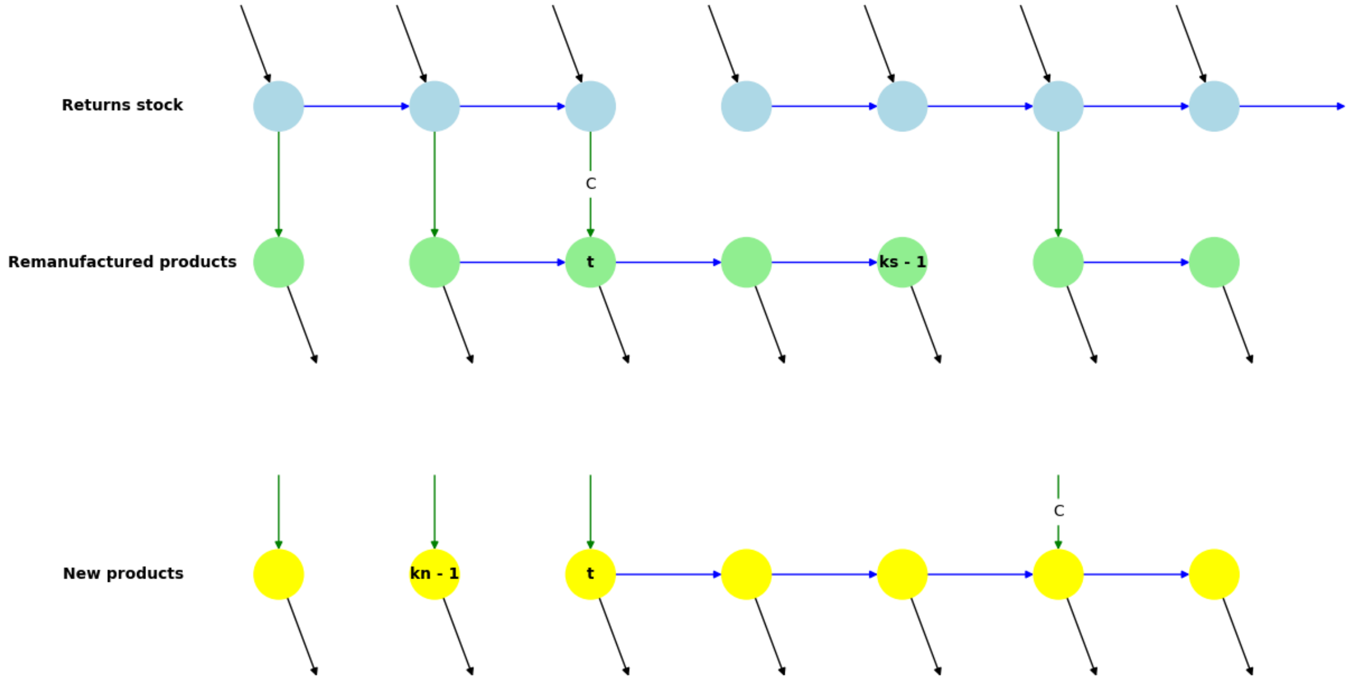
Example where case 1 appears

This case is about the end of a succession of remanufactured products stocks (we will call that a "chain") and not for the new products. Then at this moment we will determine the end of the next stock chain that minimize the cost of the global problem.

More, we know by the Non-fractional productions property that we have only one fractional production per chain which is at the beginning of the chain. And we need to meets to the demand of each chain then we know the optimal number of production that will occur in the chain which is $\left\lceil \frac{D^s_{t,k_s-1}}{C^s} \right\rceil$ for a given k_s.

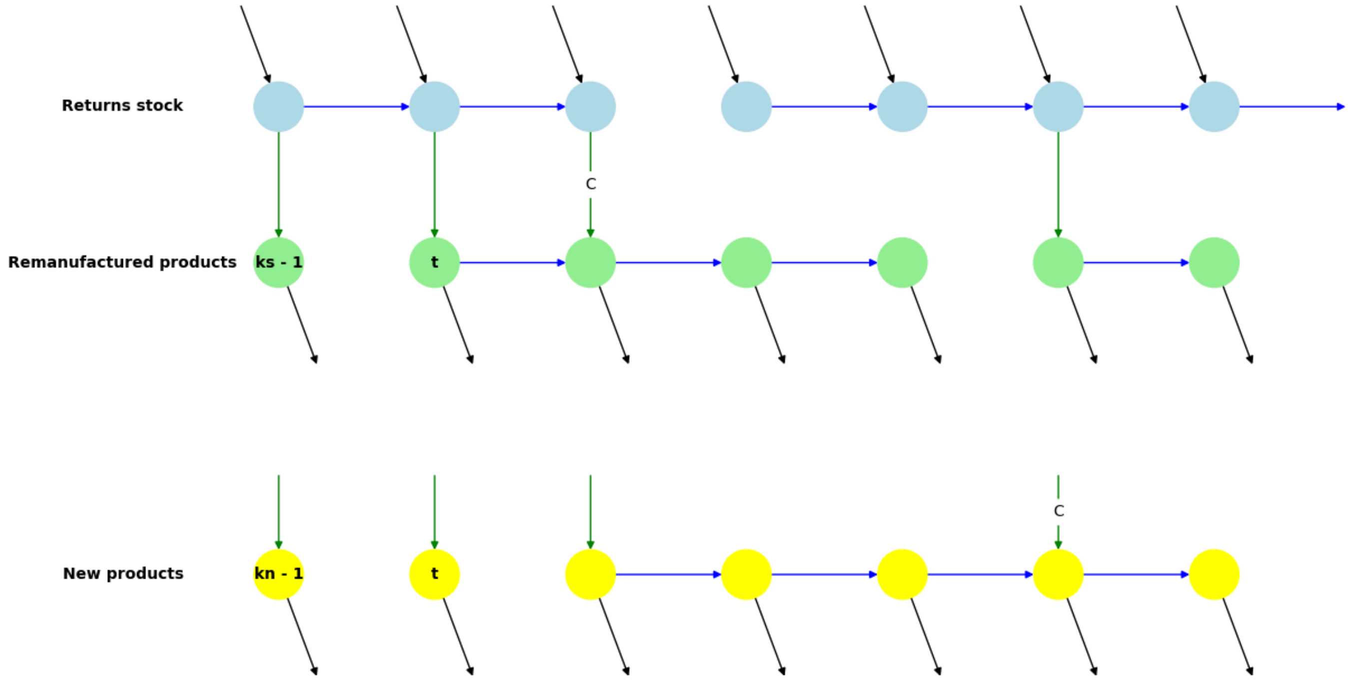
This case happens $O(T^4)$ times and has a cost in $O(T)$. Then it has a complexity in $O(T^5)$.

The second case is the same as the first but the end of the chain is for the new products. It has a complexity in $O(T^5)$ too. Here, this case takes place for $t = k_n = 3$ and $k_s = 6$:



Example where case 2 appears

The third case is a combination of both of cases 1 and 2. It's the case when the and chain of new and remanufactured products happens at the same period. For example, here with $t = k_s = k_n = 2$:



Example where case 3 appears

As same as case 1 and 2, we need to determine the length of both of remanufactured and new chains of stock. Then the calculation of one case is in $O(T^2)$, but there are $O(T^3)$ cases of this kind. Then the complexity is in $O(T^5)$ too.

Now we will treat the 4$^{\text{th}}$ case. Which represent the case when we are in the middle of both of remanufactured and new chains ($t < k_s$ and $t < k_n$). This case calculate the minimum of the case when we decide to produce at period t and the case when we decide to not produce at period t.

We first take a look at the costs in case of production. We need to know from a given period of production t if it is a pseudo-fractional period or a non-fractional period.

Proposition 12 *In case of a production at period $t \in [\![1, T]\!]$. Let k the next period without incoming stock and N the remaining number of production between the periods t and k.*

In an optimal solution, if t verifies $D_{t,k-1} \leq NC$, then t is pseudo fractional.

Proof. By the proposition 10, we have that the $N-1$ further production are at full capacity. Then production and stock at t need to be $D_{t,k-1} - (N-1)C \leq C$ in order to meets to the demand between t and $k-1$.

Suppose that $s_{t-1} > 0$ and $x_t = C$. Then the production and stock is $> C$: this is absurd because of $D_{t,k-1} - (N-1)C \leq C$.

Suppose that $s_{t-1} > 0$ and $0 < x_t < C$. Then by the non-speculative costs property, this is not optimal, so this is absurd.

Then we have $s_{t-1} = 0$ and t is pseudo-fractional. □

From this proposition, we can define the production C_p the cost in the case of a production at period t.

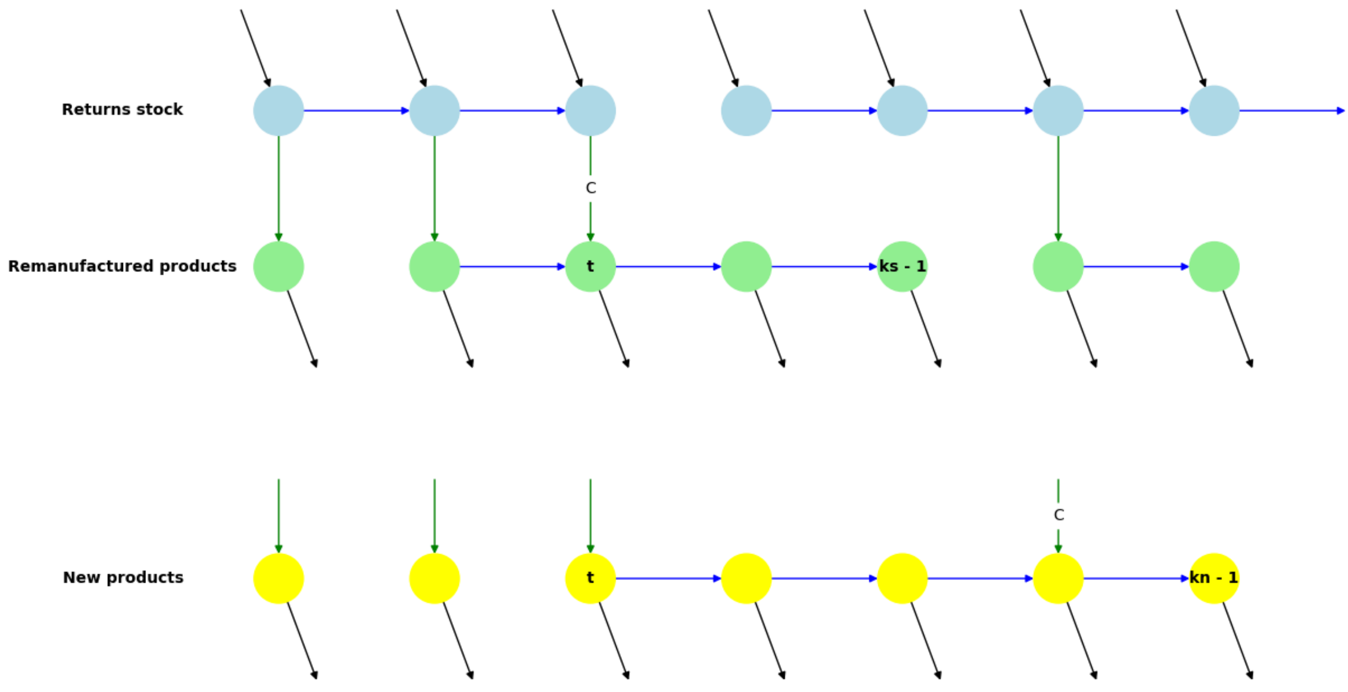
- The storage cost of remanufactured products is $h_t^s \times (D^s_{t+1,k_s-1} - (N_s - 1) \times C^s)$ because we know that it remains $N_s - 1$ periods of production at full capacity until period $k_s - 1$ (non-fractional productions property) and we need to meet to the demand between periods $t+1$ and $k_s - 1$.
- The storage cost of new products is $h_t^n \times (D^n_{t+1,k_n-1} - (N_n - 1) \times C^n)$ for the same reasons as the storage costs of remanufactured products.
- The storage cost of returned products is $h_t^r \times ((R_{1,t} - D^s_{1,t}) - (D^s_{t+1,k_s-1} - (N_s - 1) \times C^s))$ because we know that we store $D^s_{t+1,k_s-1} - (N_s - 1) \times C^s$ of remanufactured products (they are already produced), we know that we have already met to the demand until t ($-D^s_{1,t}$) and that we have collected all the returns until t ($+R_{1,t}$).
- The cost of remanufacturing is $p_t^s \times (\alpha(D^s_{t,k_s-1} - (N_s - 1) \times C^s) + (1-\alpha)C^s)$ where $\alpha = \mathbb{1}_{D^s_{t,k_s-1} \leq N_s \times C^s}$, because when $\alpha = 1$ it means that t is pseudo-fractional (Proposition 12) and then we produce only what will not be produced by the following full-capacity periods of production ($D^s_{t,k_s-1} - (N_s - 1) \times C^s$), otherwise if $\alpha = 0$, it means that t is a non-fractional period of production then we produce at full capacity C^s.
- The cost of new production is $p_t^n \times (\zeta(D^n_{t,k_n-1} - (N_n - 1) \times C^n) + (1-\zeta)C^n)$ where $\zeta = \mathbb{1}_{D^n_{t,k_n-1} \leq N_n \times C^n}$ for the same reasons as the remanufacturing.
- We also have the setup cost f_t.

If any of the before mentioned storage costs is negative, that means that the storage would be negative which is not possible, so we instead set it to $+\infty$.

And that:

$$C_p(t, k_s, k_n, N_s, N_n) =$$
$$h_t^s \times (D^s_{t+1,k_s-1} - (N_s - 1) \times C^s) + h_t^n \times (D^n_{t+1,k_n-1} - (N_n - 1) \times C^n) + h_t^r \times ((R_{1,t} - D^s_{1,t}) - (D^s_{t+1,k_s-1} - (N_s - 1) \times C^s)) + p_t^s \times (\alpha(D^s_{t,k_s-1} - (N_s - 1) \times C^s) + (1-\alpha)C^s) + p_t^n \times (\zeta(D^n_{t,k_n-1} - (N_n - 1) \times C^n) + (1-\zeta)C^n) + f_t$$

An example of this case appears for $t = 3$, $k_s = 6$ and $k_n = 8$:



Example where a case of production appears

We also define C_{np} the cost in case of no production at period t.

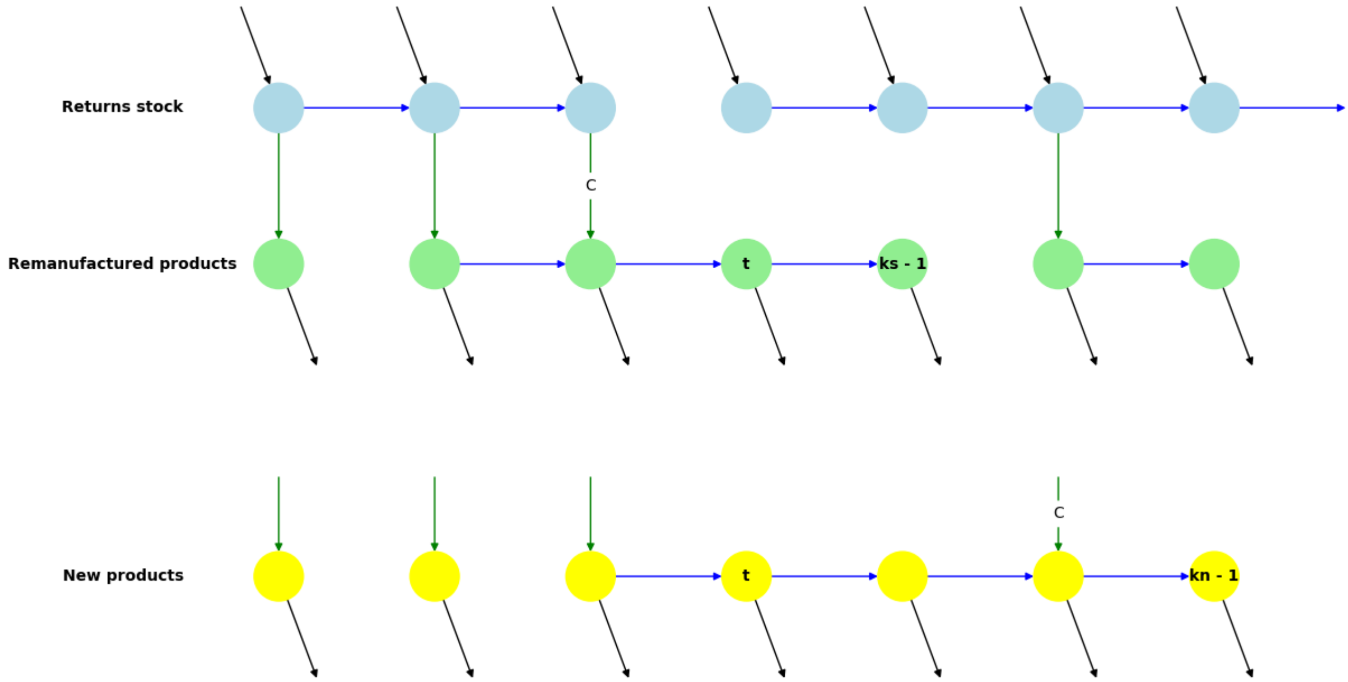
- The storage cost for remanufactured products is $h_t^s \times (D^s_{t+1,k_s-1} - N_s \times C^s)$ because we know that it remains N_s periods of production at full capacity until period $k_s - 1$ (non-fractional productions property) and we need to meet to the demand between periods $t+1$ and $k_s - 1$.
- The storage cost for new products is $h_t^n \times (D^n_{t+1,k_n-1} - N_n \times C^n)$ for the same reasons as the storage costs of remanufactured products.
- The storage cost for returned products is $h_t^r \times ((R_{1,t} - D^s_{1,t}) - (D^s_{t+1,k_s-1} - N_s \times C^s))$ because we know that we store $D^s_{t+1,k_s-1} - N_s \times C^s$ of remanufactured products (they are already produced), we know that we have already met to the demand until t $(-D^s_{1,t})$ and that we have collected all the returns until t $(+R_{1,t})$.

If any of the before mentioned storage costs is negative, that means that the storage would be negative which is not possible, so we instead set it to $+\infty$.

And that:

$$C_{np}(t, k_s, k_n, N_s, N_n) =$$
$$h_t^s \times (D^s_{t+1,k_s-1} - N_s \times C^s) + h_t^n \times (D^n_{t+1,k_n-1} - N_n \times C^n) + h_t^r \times ((R_{1,t} - D^s_{1,t}) - (D^s_{t+1,k_s-1} - N_s \times C^s))$$

An example of this case of non-production appears for $t = 4$, $k_s = 6$ and $k_n = 8$:



Example where a case of non-production appears

This 4^{th} costs $O(1)$ if we already calculate the $D_{i,j}$ and $R_{i,j}$ and we have $O(T^5)$ cases of this kind. Then the complexity of this case is in $O(T^5)$.

Then the complexity of this Bellman equation is in $\boxed{O(T^5)}$.

5.2.4 Base cases of dynamic programming

We reject some cases when we know that they are not optimal.

Then we have $f^*(t, k_s, k_n, N_s, N_n) = +\infty$ in these cases :

- $t = k_s$ and $N_s > 0$ (we did not produce enough to respond to the demand of remanufactured products)
- $t = k_n$ and $N_n > 0$ (we did not produce enough to respond to the demand of new products)

Also we have $C_p(t, k_s, k_n, N_s, N_n) = +\infty$ in these cases:

- If $D^s_{t+1,k_s-1} \leq (N_s - 1)C^s$ when $t \leq k_s - 2$ ($t+1$ pseudo-fractional period for remanufactured products in case of production in t)
- If $D^n_{t+1,k_n-1} \leq (N_n - 1)C^n$ when $t \leq k_n - 2$ ($t+1$ pseudo-fractional period for new products in case of production in t)
- If $N_s < 0$ or $N_n < 0$ (we produce too much)

We reject the first 2 cases because of the proposition 12 : it means that the period $t+1$ is a pseudo-fractional period which in case of an optimal solution is not possible because by definition $\forall t' \in [\![t+1, k-1]\!]$, $s_{t'} = 0$. And we have that if $t+1$ is pseudo-fractional, then t' is pseudo-fractional because $D_{t',k-1} < D_{t+1,k-1} \leq (N-1)C$, then it means that we can't build an optimal solution form this property on t and $t+1$. So we reject these cases.

Same as for the case of production, we have $C_{np}(t, k_s, k_n, N_s, N_n) = +\infty$ in these cases :

- If $D^s_{t+1,k_s-1} \leq N_s C^s$ ($t+1$ pseudo-fractional period for remanufactured products in case of no production in t)
- If $D^n_{t+1,k_n-1} \leq N_n C^n$ ($t+1$ pseudo-fractional period for new products in case of no production in t)
- If $N_s < 0$ or $N_n < 0$ (we produce too much)

For the base cases, that is for $t = T+1$, let $f^*(T+1, T+1, T+1, 0, 0) = 0$.

5.2.5 Proof of the Bellman equation

We will now prove that the equation above ensures that f^* verifies the definition that we gave initially.

Proposition 13 *$f^*(t, k_s, k_n, N_s, N_n)$ is the sum of all costs after period t for the optimal solution of the problem verifying the constraints (4.2.3.i), (4.2.3.ii) and (4.2.3.iii) if it exists, $+\infty$ if it doesn't.*

Proof.

- **Base case:** The base case for $t = T+1$ is evident

- **Induction:** Let $t \in [\![0, T]\!]$, $t \leq k_s, k_n \leq T+1$, and $N_s, N_n \in \mathbb{N}$. And we suppose that the proposition is true for all $t' > t$, and for all $f^*(t, k'_s, k'_n, N_s, N_n)$ where we have at least one of $k'_s > k_s$ or $k'_n > k_n$.

  Let's look at all the possible cases:

  - **If $t = k_s$ and $N_s > 0$, or $t = k_n$ and $N_n > 0$:**
    then $f^*(...) = +\infty$ follows the definition because there is no solution that verifies such conditions.
  - **If $t = k_s$ and $t < k_n$:**
    then $f^*(t, t, k_n, 0, N_n) = \min_{t \leq i \leq T}\left(f^*(t, i+1, k_n, \left\lceil \frac{D^s_{t,i}}{C^s} \right\rceil, N_n)\right)$ follows the definition because, since $k_s = t$, t is a pseudo-fractional period for remanufactured products, meaning that $f^*(t, t, k_n, 0, N_n)$ is the best among the $f^*(t, t', k_n, N_s, N_n)$ where $t' > t$ is the next pseudo-fractional period for remanufactured products and such that N_s is such that t is a pseudo-fractional period for remanufactured products, which means that $N_s = \left\lceil \frac{D^s_{t,t'-1}}{C^s} \right\rceil$.
  - **If $t < k_s$ and $t = k_n$:** We follow the exact same reasoning as for the previous case.
  - **If $t = k_s = k_n$:** We also follow the same reasoning as before.
  - **Otherwise:** For an optimal solution X where $k_s > t$ is the next pseudo fractional period for remanufactured products, $k_n > t$ is the next pseudo-fractional period for new products, N_s productions occurred between t and k_s, and N_n productions occurred between t and k_n, and such that the "Non fractional productions" and "Same period-production" properties are verified, we either produce both new and remanufactured products, or we don't produce either at time t, so the optimal solution is the best between the two.
    We first suppose that we do produce at time t, let's show that $C_p(t, k_s, k_n, N_s, N_n)$ corresponds indeed to the optimal solution that we want. From now on we'll refer to remanufactured products as **s**, and new products as **n**.
    We first consider that $t \leq k_s - 2$ and $t \leq k_n - 2$. Since we produce both **s** and **n** at time t, then for X, there are $N_s - 1$ **s** productions from $t+1$ to $k_s - 1$, all of which are at full capacity and have incoming storage because the solution verifies the "Non-fractional productions" property. Similarly for **n**, there are $N_n - 1$ productions from $t+1$ to $k_n - 1$, all of which are at full capacity and have incoming storage. So immediately if we get that $N_s < 0$ or $N_n < 0$ then that is not possible and thus we assign the $+\infty$ value to it, otherwise if $D_{t+1,k_s-1} \leq (N_s - 1)C^s$, then that means that the $N_s - 1$ productions from $t+1$ to $k_s - 1$, which are at full capacity C^s fulfill all the demands from $t+1$ to $k_s - 1$, meaning that there can't be an incoming storage at time $t+1$ since $s^s_{k_s-1} = 0$, which contradicts *4.2.3.i*, thus we assign $+\infty$ to it. And we obtain the same result for **n**. So then now we have that, $(N_s - 1)C^s < D^s_{t+1,k_s-1}$, $(N_n - 1)C^n < D^n_{t+1,k_n-1}$ and $N_s, N_n > 0$. Then we are guaranteed that $t+1$ can't be a pseudo-fractional period, and we get that $C_p(t, k_s, k_n, N_s, N_n) = f^*(t+1, k_s, k_n, N_s - 1, N_n - 1) +$ costs at time t.
    Now if $t > k_s - 2$, that is $k_s - 1 = t$, then the result is even more easily obtained.
    The reasoning is very similar for the case of no-production at time t, thus finally obtaining that $f^*(t, k_s, k_n, N_s, N_n) = \min(C_p(t, k_s, k_n, N_s, N_n), C_{np}(t, k_s, k_n, N_s, N_n))$

□

5.3 With dispose problem

5.3.1 Non-fractional productions

Proposition 14 *For the lot sizing problem with dispose and uncommon capacity, there is an optimal solution that verifies :*

$$\bullet s^n_{t-1} > 0 \Rightarrow x^n_t = C^n \qquad \forall t \in [\![2, T]\!]$$
$$\bullet s^s_{t-1} > 0 \Rightarrow x^s_t = C^s \qquad \forall t \in [\![2, T]\!]$$

Proof. The proof is the same as the without dispose case (4.2.1). □

5.3.2 Same-period production

Proposition 15 *There exists at least an optimal solution to the lot sizing problem with remanufacturing and returns with dispose and uncommon capacity that satisfies the following property :*

$$x^n_t > 0 \Leftrightarrow x^s_t > 0 \qquad \forall t \in [\![1, T]\!]$$

Proof. The proof is the same as the uncapacitated case (3.2.2). □

5.3.3 Case of dispose

Proposition 16 *For the lot sizing problem with dispose and uncommon capacity, an optimal solution verifies:*

$$s^r_{t-1} > 0 \Rightarrow x^r_t = 0 \qquad \forall t \in [\![2, T]\!]$$

Proof. Let an optimal solution where $\exists t \in [\![2, T]\!]$ such as $s^r_{t-1} > 0$. Suppose that $x^r_t > 0$.
The solution is optimal then we have :

$$p^r_{t-1} > h^r_{t-1} + p^r_t$$

Otherwise we would dispose at period $t-1$. But this is absurd because of the non-speculative costs constraint (2.1.r).
Then $x^r_t = 0$. □

5.3.4 Dynamic Programming

We are going to introduce notations about what could be a polynomial or pseudo-polynomial algorithm which can solve the lot sizing problem with uncommon capacity and dispose.

We can define a pseudo structure of a subplan by a a chain of multiple periods with a dispose at the beginning of the subplan.

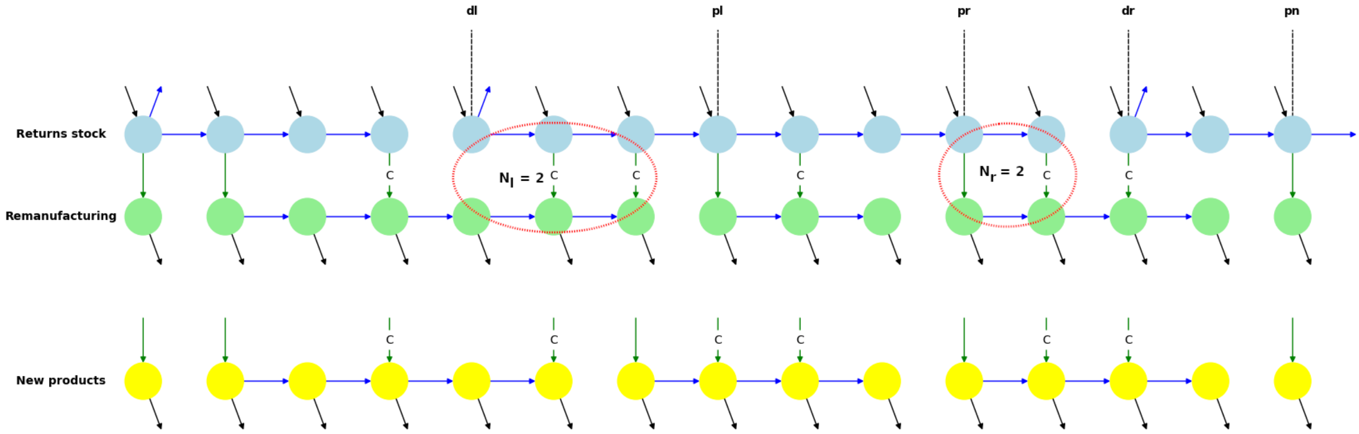
We can distinguish between two types of subplans: the first is one where there is at least one pseudo-fractional production, and the second is one where there are none.

We can characterize a subplan of type 1 by the following attributes :

- d_l : period of the dispose at the beginning of the subplan
- d_r : period of next dispose (and beginning of the next subplan)

- p_l : period of the first pseudo-fractional production period between d_l and d_r excluded
- p_r : period of the last pseudo-fractional production period between d_l and d_r excluded
- p_n : period of the first pseudo-fractional production period after d_r
- N_l : number of production in $[\![d_l, p_l[\![$ (number of production at full capacity we need to finish the last "chain" of production)
- N_r : number of production in $[\![p_r, d_r[\![$ (number of production we do at the beginning of the last chain of production)

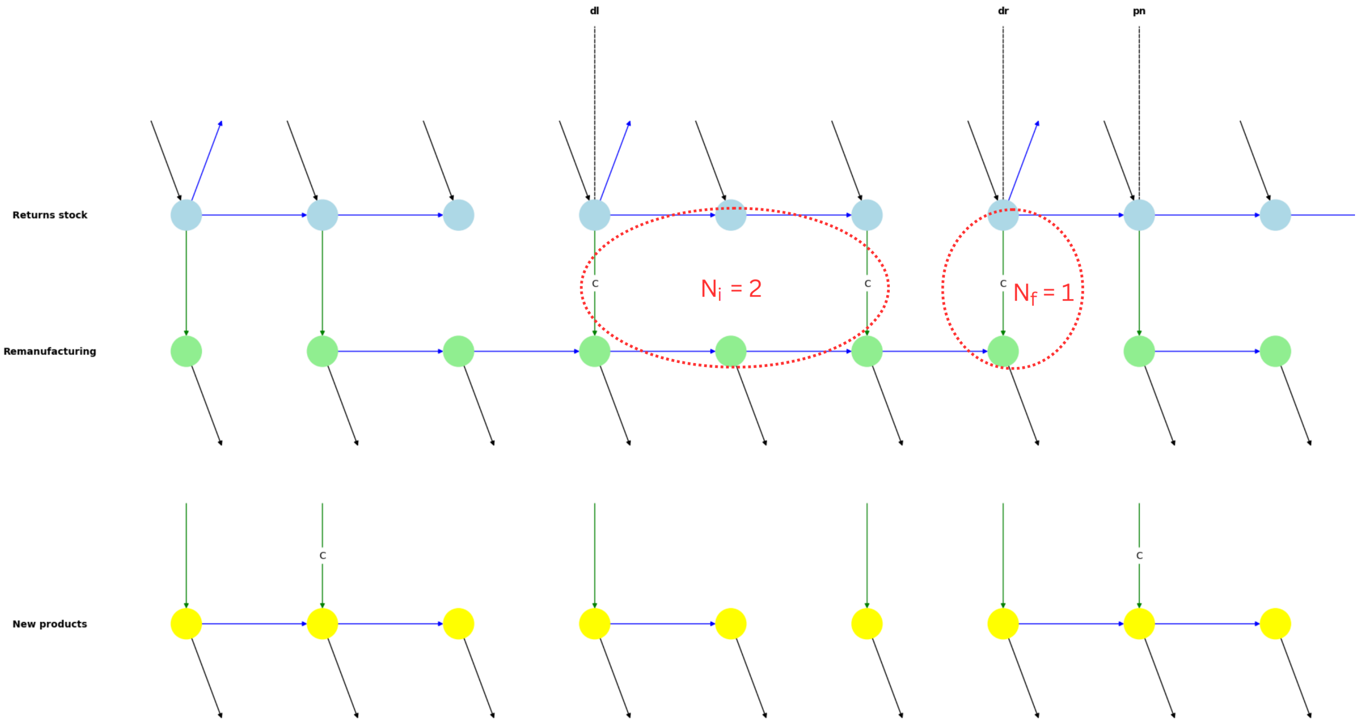
To understand the notations, we can illustrate this with the following example :



And we can characterize a subplan of type 2 by the following attributes:

- d_l: same as the subplan of type 1
- d_r: same as the subplan of type 1
- p_n: same as the subplan of type 1
- N_i: Number of productions between d_l and d_r excluded
- N_f: Number of productions between d_r and p_n excluded

We can see these attributes in the example below:



The resolution of a subplan is nearly the same as the case without dispose.

The first difference is that we are not considering the same quantity of return as the case with dispose: before we considered all the returns we have collected minus all we have already produced but here, we consider all the returns we collected from the period of dispose minus the quantity of disposed returns.

The other difference is the fact that the subplans are overlapping. Then before the period p_l, we need to remember how many production periods (at full capacity in facts) it remains to meets to the demand before p_l. Moreover, as the case without dispose, the new products line is just linked to the remanufacturing line by the production at the same time, then we solve this line as before.

We also know the quantity of disposed returns, and then the cost of these returns is counted at the beginning of a new subplan :

- For a subplan of type 1 : we know that we meets to the demands between p_l and p_r, that we need to product $N_l \times C^s$ remanufactured products before p_l and that we have N_r periods of productions after p_r which the first is pseudo-fractional (which its production is equals to $D_{p_r,p_n-1}\%C^s$ with % as the modulo operator) and partially meets to the demand until $p_n - 1$. We also know that we collect all the returns between d_l and d_r excluded. Then the quantity of dispose is :

$$R_{d_s,d_r-1} - (N_l \times C^s + D_{p_l,p_r-1} + (N_r - 1) \times C^s + D_{p_r,p_n-1}\%C^s) \quad \text{with \% as the modulo operator}$$

- for a subplan of type 2 : we know that we produce $N_i \times C^s$ products between d_l and $d_r - 1$. We also collect all the returns from d_l and $d_r - 1$. Then the quantity of dispose is :

$$R_{d_l,d_r-1} - N_i \times C^s$$

Having this in mind, solving for a certain subplan is done pretty much exactly as before. When making the transition from t to $t-1$ inside a subplan, the deduced storage for remanufactured products and new products, as well as the quantity produced, are deduced using the exact same formula as the problem without dispose, the difference however lies in the storage for returned products. So here's how we obtain it:

- **First Case: The subplan is of type 1 and $p_l \leq t$**
  Knowing the quantity of disposed returns, we know that the returns up to time t are going to satisfy the demands in the period range from p_l to t, in addition to $N_l \times C^s$ demands before p_l, and so we know from that that the quantity $s_t^r + s_t^s$ is equal to the difference between the returns up to time t and the demands satisfied by them. Thus allowing us to conclude s_t^r since we know how to calculate s_t^s.

- **Second Case: The subplan is either of type 2, or type 1 and $d_l \leq t < pl$**
  In this period range, we know exactly how many productions have been done from d_l until t, and we also know that they're all at full capacity, and so we can easily deduce that the stored returns are the returns so far minus the number of productions times the capacity.

Now we take a look into the remaining two cases, that is when we are at of one of the subplan's borders (either d_l or $d_r - 1$).

- **$t = d_l$:**
  In this case, we have to make sure that the constraints on the subplan are being respected. So for a subplan of type 1, we need to make sure that the next pseudo-fractional production is at time p_l and that there are N_l productions up until p_l, thus for all k_s different that p_l, we set the cost function to $+\infty$, and similarly when N_s is not equal to the desired value. As for a subplan of type 2, we must have that $k_s = p_n$, and $k_s = N_i + N_f$, otherwise we set the cost function to $+\infty$.

- $t = d_r - 1$: To complete