Notes on the Implicit Solutions Homework

Goals: This problem will give you practice working with implicit solutions. In particular, we will learn how to derive an DE from a given implicit solution. We will also study what the Existence and Uniqueness Theorem tells us about uniqueness of solutions and when uniqueness fails. We will use Maple to plot solutions and see this "bad behavior."

Directions: For this problem you will need to print out and turn in your Maple sheet.

Problem 1. *The curve defined by the equation*

$$x^{\frac{2}{3}} + t^{\frac{2}{3}} = 1$$

is called the astroid.

a) *Use Maple to create a plot of the astroid. Use Maple to plot the equation*

$$|x|^{\frac{2}{3}} + |t|^{\frac{2}{3}} = 1$$

on the interval $-1 \leq x \leq 1$. To do so, first load the 'plots' package in Maple. Use the 'implicitplot' command, and add the option 'numpoints=10000' at the end, otherwise you'll miss some important features.)

To plot an implicit curve $g(t,x) = 1$ in Maple (we'll call $g(t,x)$ 'g'), we type:

```
with(plots):
```
- calls the plots package

```
implicitplot(g=1,t=a..b,x=c..d)
```
- plots $g = 1$ on the range $a \leq t \leq b$ and $c \leq x \leq d$

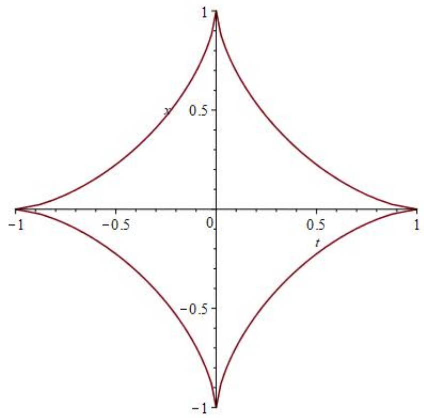
```
implicitplot(g=1,t=a..b,x=c..d,numpoint=X)
```
- plots $g = 1$ on the range $a \leq t \leq b$ and $c \leq x \leq d$ using X points

b) *Find a first order differential equation in normal form (i.e., $x' = f(t,x)$) for which the astroid is an implicit solution. Note that you cannot solve for x explicitly, because the graph in part a) is not a graph of a function.*

c) *Determine where the Existence and Uniqueness Theorem guarantees that the initial value problem to your DE in part a) with initial conditions $x(t_0) = x_0$.*

d) *Use your plot in part a) to help explain where the existence and uniqueness theorem doesn't apply.*

The plot in Maple without the numpoint option can sometimes give a poor picture of the graph. By forcing Maple to plot more points, we can get a better graph such as the one below:



It is not a great idea to try to solve for x as a function of t, the graph shows us that it cannot be a function since it fails the vertical line test. However, we can use *implicit differentiation* on this implicit solution. That is, differentiate both sides of the equation $t^{\frac{2}{3}} + x^{\frac{2}{3}} = 1$ with respect to t, using the Chain Rule. Then, solving for x', we obtain

$$\frac{dx}{dt} = \frac{-x^{1/3}}{t^{1/3}} = f(t,x)$$

The Existence and Uniqueness Theorem doesn't apply when either $x = 0$ or $t = 0$, because $\frac{\partial f}{\partial x}$ is discontinuous when $x = 0$. We can see from the plot that this is where the graph has "sharp turns" (cusps) and is not differentiable. So as long as $x_0 \neq 0$ and $t_0 \neq 0$ we are guaranteed a unique solution to the initial value problem (IVP) with initial condition (IC), $x(t_0) = x_0$.

Instead, if we start with the initial condition $x(-1) = 0$, there are two distinct solutions for $t > -1$: one curve is positive, and one curve is negative. There is no solution when $t < -1$. Since the Existence and Uniqueness Theorem doesn't apply to the initial condition, and the existence of two distinct solutions not violate the theorem.