Diff Eq 1: Lecture Notes

Jeremy D. Harris, PhD Applied Math

January 5, 2024

Contents

Separable 1st order Differential Equns (DEs)

Def'n: A 1st order DE has the general form:

$$F(t, x, x') = 0,$$

where F is an nonlinear ftn of t, x, and x'.

Here, x and x' also depend on t (implicitly)

. . .

For our Purposes, we restict our focus to eqns (DEs) that can be written as:

$$x' = f(t, x),$$

where f is a nonlinear function of t and x.

o o o

Def'n: A separable 1st order DE has the form:

$$\frac{dx}{dt} = g(t) \cdot h(x),$$

where g is a ftn of t and h is a ftn of x.

See in-class Worksheets (Days 1-2) for example problems.

(Section 2.1) Autonomous 1^{st} order DEs

Def'n: An autonomous 1^{st} order DE is of the form:

$$x' = h(x) \quad (\text{"}t\text{" not explicit})$$

whereas a nonautonomous 1^{st} order DE is:

$$x' = h(t, x) \quad (\text{"}t\text{" is explicit})$$

Ex (from Ex 2.1.1)

Determine whether the following DEs are autonomous/nonautonomous? Separable vs. Nonseparable?

1. $x' = xt + 2x$ (nonaut; separable)
   $= x(t+2)$

2. $x' = x + \cos(t)$ (nonaut; Not separable)

3. $x' = \underbrace{xt^2 + t^2}_{} - \underbrace{tx - t}_{}$ (nonaut; Separable!!) e.g.
   $= t^2(\underline{x+1}) - t(\underline{x+1})$
   $= (t^2 - t)\cdot(x+1)$ whoa!!

4. $x' = x^2 + x + 3$ (auto; separable!!)

Exercise: Try solving using separation. Hint: complete the square.

Given a separable 1st order DE,

$$\frac{dx}{dt} = g(t) \cdot h(x),$$

we can find an implicit general sol'n:

$$\Longleftrightarrow \int \frac{dx}{h(x)} = \int g(t)\,dt$$

$$\Longleftrightarrow H(x) = G(t) + \underbrace{C}_{\text{constant of integration}}.$$

where $H'(x) = \frac{1}{h(x)}$ and $G'(t) = g(t)$.

<u>Def'n</u>: A sol'n is <u>explicit</u> if x can be solved for in terms of t, i.e., $H(x(t)) = G(t) + C$.

$$\Longleftrightarrow x(t) = H^{-1}(G(t) + C).$$

Ex

$$x' = xt + 2x.$$

From previous example: nonaut, separable.

$$\Longleftrightarrow \frac{dx}{dt} = \underbrace{(t+2)}_{g(t)} \cdot \underbrace{x}_{h(x)}.$$

$$\Longleftrightarrow \int \frac{dx}{x} = \int (t+2)\, dt$$

$$\Longleftrightarrow \boxed{\underbrace{\ln(x)}_{H(x)} = \frac{t^2}{2} + 2t + C.}$$

(Implicit solution)

$(H^{-1}(x) = e^x)$

$$\Longleftrightarrow \boxed{x(t) = A\, e^{\left(\frac{t^2}{2} + 2t\right)}}$$

(Explicit sol'n).

(Section 1.1) DEs: Order & sol'ns

Def'n: The Order of a DE is the highest # of derivatives.

Def'n: A sol'n to a DE is a function that satisfies the DE over some interval of t, i.e., the independent variable.

① ∞ c

$\boxed{Ex}$ (Ex 1.1.1) Show that $f(t) = Ae^{-2t}$ (A constant)

is a sol'n to the DE:

$$x'' + 3x' + 2x = 0.$$

$f(t) = Ae^{-2t}$
$f'(t) = -2Ae^{-2t}$
$f''(t) = 4Ae^{-2t}$

Substitute:

$$f''(t) + 3f'(t) + 2f(t)$$

$$= 4\underbrace{Ae^{-2t}} + 3(-2\underbrace{Ae^{-2t}}) + 2\underbrace{Ae^{-2t}}$$

$$= (\underbrace{4 - 6 + 2}_{0}) \cdot \underbrace{Ae^{-2t}}$$

$$= 0 \quad \checkmark \quad \text{(satisfies DE)}$$

$\Longrightarrow$ $f(t) = Ae^{-2t}$ is a family of sol'ns parametrized by A...

Where does "-2" in $f(t) = Ae^{-2t}$ come from?

Some foreshadowing...

Let's take a good guess, ~~the~~ "Ansatz": $g(t) = e^{rt}$.

$$\left.\begin{aligned} g(t) &= e^{rt} \\ g'(t) &= re^{rt} \\ g''(t) &= r^2 e^{rt} \end{aligned}\right\}$$

Substitute:

$$g''(t) + 3g'(t) + 2g(t)$$
$$= r^2 e^{rt} + 3re^{rt} + 2e^{rt}$$
$$= (r^2 + 3r + 2)\underbrace{e^{rt}}_{\neq 0 \text{ for all } t} \overset{\text{Want}}{=} 0$$

Quadratic Equ (in r):

$\Longrightarrow$ $r^2 + 3r + 2 = 0$

$\Longrightarrow$ $(r+2)(r+1) = 0 \Longrightarrow \left\{ \begin{array}{l} \text{2 sol'ns are:} \\ A_1 e^{-2t},\ A_2 e^{-t}. \end{array} \right.$

$\Longrightarrow$ $\boxed{r_1 = -2}$, $r_2 = -1$.

Ex

Exercises 1.1.

10 $x'' + 4x = 0.$ $\quad x_1(t) = \sin(2t)$, $x_2(t) = \cos(2t)$

Show that $x_1(t)$, $x_2(t)$ are sol'ns.

$$\left.\begin{aligned} x_1(t) &= \sin(2t) \\ x_1'(t) &= 2\cos(2t) \\ x_1''(t) &= -4\sin(2t) \end{aligned}\right\}$$

Substitute:

$x_1''(t) + 4x_1(t)$

$= -4\sin(2t) + 4\sin(2t)$

$= 0$ ✓ (satisfies the DE)

$$\left.\begin{aligned} x_2(t) &= \cos(2t) \\ x_2'(t) &= -2\sin(2t) \\ x_2''(t) &= -4\cos(2t) \end{aligned}\right\}$$

Substitute:

$x_2''(t) + 4x_2(t)$

$= -4\cos(2t) + 4\cos(2t)$

$= 0$ ✓ (satisfies the DE).

11 $t^2x'' + 3tx' + x = 0$

Show that $x(t) = 1/t$ is a sol'n.

$X(t) = t^{-1}$, $x'(t) = -t^{-2}$, $x''(t) = 2t^{-3}$.

Substitute :

$$t^2x'' + 3tx' + x$$

$$= t^2\Big(\overbrace{2t^{-3}}^{x''}\Big) + 3t\Big(\overbrace{-t^{-2}}^{x'}\Big) + \Big(\overbrace{t^{-1}}^{x}\Big)$$

$$= \frac{2}{t} + -\frac{3}{t} + \frac{1}{t}$$

$= 0$ for all $t \neq 0$.

14 Check that $P(t) = \frac{1}{1+Ce^{-rt}}$

(C is a constant).

is a sol'n to the DE:

$$\frac{dP}{dt} = r \cdot P(1-P)$$

(logistic Growth Model)

(where r is constant paramter equal to the growth rate)

Want to Show: $\frac{dP}{dt} = rP(1-P)$

$\iff \frac{dP}{dt} - rP(1-P) \overset{(WTS)}{=} 0.$

Calculate:

$\frac{dP}{dt} = \frac{rCe^{-rt}}{(1+Ce^{-rt})^2}$

Substitute: $\frac{dP}{dt} - rP(1-P)$

$$= \cancel{\frac{rCe^{-rt}}{(1+Ce^{-rt})^2}} - \cancel{r \cdot \left(\frac{1}{1+Ce^{-rt}}\right) \cdot \underbrace{\left(1 - \frac{1}{1+Ce^{-rt}}\right)}_{Ce^{-rt}/(1+Ce^{-rt})}} = 0 \checkmark$$

(Section 1.2) Initial Value Problems:

First, Consider

$$x' = x.$$

The general sol'n can be written: $x(t) = Ae^t$, where A is a constant. (one free parameter, since 1st order DE.).

Satisfies DE, since $x'(t) = Ae^t = x(t)$ for all t.

○ ○ ○

<u>Def'n</u>: An <u>initial value problem</u> (IVP) is a differential equation with initial condition.

Ex Now, consider

$$\begin{cases} x' = x & \text{(D.E.)} \\ & + \\ x(0) = \frac{1}{2} & \text{(I.C.)} \end{cases}$$

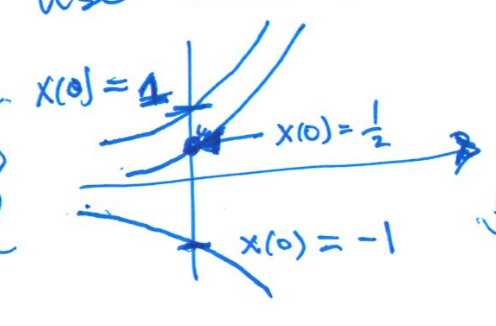
General Sol'n: $x(t) = Ae^t$.

Use I.C. to solve for A: $\frac{1}{2} = x(0) = Ae^0$

$\Longrightarrow A = \frac{1}{2} \Longrightarrow \boxed{x(t) = \frac{1}{2}e^t}$

$\Longrightarrow$ I.C. scales y-intercept.

family of sol'ns



picture.

Ex (Exercises 1.2)

① $x' = x - 2$. Find general sol'n.

We know that: $x_h(t) = Ae^t$ satisfies

$$x_h'(t) = x_h(t).$$

Suppose we think in terms of

"input-response" and

write: $\underbrace{x' - x}_{\text{response}} = \underbrace{-2}_{\text{input}} =: f(t)$

A good guess of response due to input of $f(t) = -2$ is the constant ftn: $x_p(t) = A \Rightarrow x_p'(t) = 0$.

substitute: $x_p' - x_p = 0 - A \overset{\text{Want}}{=} -2 \Rightarrow \underline{A = 2}$!!

We can "Piece together" the general sol'n: $\boxed{x(t) = x_h(t) + x_p(t) = Ae^t + 2}$

3 $x' = x + t$. Show that

$x(t) = Ce^t - 1 - t$ is

the general sol'n.

Calculate:

LHS: $x'(t) = Ce^t - 1$. RHS: $x + t = (Ce^t - 1 - t) + t$
$= Ce^t - 1$

LHS = RHS ✓.

$\Longrightarrow \begin{cases} x(t) = Ce^t - 1 - t \text{ satisfies} \\ x'(t) = x(t) + t. \\ \text{It's general because of scale parameter, } C. \end{cases}$

8 Solve the IVP $\begin{cases} x' = 2tx & \text{(separable)} \\ x(0) = 1 \end{cases}$

$\frac{dx}{dt} = 2tx$

$\Leftrightarrow \int \frac{dx}{x} = \int 2t \, dt$

$\Leftrightarrow \ln|x| = t^2 + C.$

$\Leftrightarrow x(t) = Ae^{t^2} \quad (A = e^C)$

Substitute I.C. $x(0) = 1$ to find A.

$1 = x(0) = Ae^0 = A.$

$\Longrightarrow A = 1 \Rightarrow \boxed{x(t) = e^{t^2}}$

The sol'n to IVP

Def'n: A 1^{st} order DE that is linear (in $x'(t)$, $x(t)$) can be written as:

$$a_1(t)\,x'(t) + a_0(t)\,x(t) = b(t).$$

$(a_1(t) \neq 0)$

$$\Rightarrow \quad x'(t) + \underbrace{\left(\frac{a_0(t)}{a_1(t)}\right)}_{p(t)} x(t) = \underbrace{\left(\frac{b(t)}{a_1(t)}\right)}_{q(t)}$$

$$\Rightarrow \quad \boxed{x'(t) + p(t)\,x(t) = q(t).}$$

Def'n: The above eqn is the form of a 1^{st} order linear DE in standard form.

The DE is homogeneous, if $q(t) = 0$ for all t, i.e.,

$$x'(t) + p(t)\,x(t) = 0.$$

The DE is inhomogeneous (OR Nonhomogeneous), if $q(t) \neq 0$ for some t.

<u>Note</u>: the DE is <u>linear</u> in x (and x'), because x (and its derivative) only appears in the eqn. multiplied by a function of t.

<u>NO TERMS LIKE</u>: x^2, x^3, e^x, $\ln(x)$, $\sin(x)$, $(x')^2$, $(x')^3$, ...

The Coefficients: $p(t)$, $q(t)$ can be any ftn (including nonlinear) of t.

Ex (Ex 2.3.2, page 37) Solve the homogeneous linear eqn: $x' + \underbrace{\cos(t)}_{p(t)} x = \underbrace{0}_{q(t)}$.

$\Rightarrow$ homogeneous

$$\frac{dx}{dt} + \cos(t) \cdot x = 0$$

$$\Rightarrow \frac{dx}{dt} = -\cos(t)\, x \quad \text{(separable)}.$$

$$\Rightarrow \int \frac{dx}{x} = -\int \cos(t)\, dt$$

$$\Rightarrow \ln|x| = -\sin(t) + C.$$

$$\Rightarrow \boxed{x(t) = A e^{-\sin(t)}}$$

The Homogeneous Eqn:

$$x' + p(t)\,x = 0.$$

$$\Leftrightarrow \frac{dx}{dt} = -p(t)\,x.$$

$$\Leftrightarrow \int \frac{dx}{x} = -\int p(t)\,dt.$$

$$\Leftrightarrow \ln|x| = -\int p(t)\,dt + C. \quad \text{(Implicit form of sol'n)}$$

$$\underline{x(t) = A\,e^{-\int p(t)\,dt}} \quad \text{(Explicit form of sol'n)}$$

Hence, $e^{\int p(t)\,dt}\,x(t) = A.$ derivative of constant.

$$\frac{d}{dt}\left(e^{\int p(t)\,dt}\,x(t)\right) = e^{\int p(t)\,dt}\,x'(t) + p(t)\,e^{\int p(t)\,dt}\,x(t) = 0$$

(product rule)

...

The Integrating Factor Technique for 1st order linear DEs,

$$X' + p(t)\,X = q(t)$$

We use the SIMPI mnemonic.

① Standard Form: $X' + p(t)X = q(t)$.

② Integrating Factor (I.F.): $e^{\int p(t)\,dt}$.

③ Multiply Both Sides by I.F.:

$$e^{\int p(t)\,dt}\left(X' + p(t)X\right) = q(t)\,e^{\int p(t)\,dt}$$

④ Product Rule:

$$\int \frac{d}{dt}\left(e^{\int p(t)\,dt}\,X\right)dt = \int q(t)\,e^{\int p(t)\,dt}\,dt.$$

⑤ Integrate:

$$X(t) = e^{-\int p(t)\,dt}\left(\int q(t)\,e^{\int p(t)\,dt}\,dt + C\right)$$

$\boxed{\text{Ex}}$ (Example 2.3.3)

$$x' + \underbrace{2}_{p(t)} x = \underbrace{te^{-2t}}_{q(t)}.$$

Standard Form: $x' + 2\cancel{x} = te^{-2t}$

Integrating Factor: $e^{\int 2\,dt} = e^{2t}$.

Multiply Both SIDES by I.F.:

$$e^{2t}(x' + 2x) = t\cancel{e^{-2t}} \cdot \cancel{e^{2t}}$$

$$\Rightarrow e^{2t}x' + 2e^{2t}x = t$$

$$\Rightarrow \int \frac{d}{dt}\left(e^{2t}x\right)dt = \int t\,dt.$$

Integrate:

$$\Rightarrow e^{2t}x = t^2/2 + C.$$

$$\Rightarrow \boxed{x(t) = e^{-2t}\left(\frac{t^2}{2} + C\right)}$$ general Sol'n

Reflecting...

$$x(t) = e^{-2t}\left(\frac{t^2}{2} + C\right)$$

$$= \boxed{\frac{t^2}{2} e^{-2t}} + \boxed{Ce^{-2t}}$$

solves nonhomogeneous DE

$x' + 2x = te^{-2t}$

(No general constants)

solves homogeneous DE

$x' + 2x = 0$

contains (constant of integration.)

Ex

$$x' + 2tx - t = 0$$

linear, 1st order inhomogeneous.

Standard Form: $x' + \underbrace{2t}_{p(t)} x = \underbrace{t}_{q(t)}$

Integrating Factor: $e^{\int 2t\,dt} = e^{t^2}$.

Multiply Both SIDES by I.F.:

$$e^{t^2}(x' + 2tx) = te^{t^2}$$

$$\Rightarrow e^{t^2}x' + 2te^{t^2}x = te^{t^2}$$

$$\Rightarrow \int \frac{d}{dt}\left(e^{t^2}x\right)dt = \int te^{t^2}\,dt \quad \begin{pmatrix} u\text{-substitution} \\ u = t^2 \\ \frac{du}{2} = t\,dt \end{pmatrix}$$

Integrate:

$$e^{t^2}x = \frac{1}{2}e^{t^2} + C.$$

$$\Rightarrow \boxed{x(t) = \frac{1}{2} + Ce^{-t^2}}$$ general sol'n

$\boxed{\text{Ex}}$ (Example 2.3.4, page 39)

$$x' = x + te^t.$$

Standard Form: $x' - x = te^t$

$p(t) = -1 \quad q(t) = te^t$

Integrating Factor: $e^{\int p(t)dt} = e^{-t}$.

Multiply Both Sides by I.F.:

$$e^{-t}(x' - x) = t\cancel{e^{t}} \cdot \cancel{e^{-t}}$$

Product Rule: $e^{-t}x' - e^{-t}x = t$.

$$\int \frac{d}{dt}(e^{-t}x)\,dt = \int t\,dt.$$

Integrate: $\boxed{x(t) = \left(\frac{t^2}{2} + C\right)e^t}$ general sol'n.

More Practice: Integrating Factor Method

Problem 5:

① $x' = x \implies x' - x = 0$

$\implies e^{-t}(x' - x) = 0$

$\implies \frac{d}{dt}(e^{-t}x) = \int 0\,dt$

$\implies e^{-t}x = A$

$\implies \boxed{x(t) = Ae^{t}}$

2 $x' = x + 1$

Before we apply I.F. method, we decompose via input-response:

The inhomogeneous part

$x' = x \iff x' - x = 0$ has

~~[illegible]~~ general Sol'n : $x(t) = Ae^t$.

Guess : $x_p = A$, $x_p' = 0$.

$\Rightarrow \quad 0 = A + 1 \Rightarrow A = -1$

$\Rightarrow \quad x_p(t) = -1$

Gen Sol'n : $x(t) = x_h(t) + x_p(t)$
$= Ae^t - 1$.

3 $$x' = t(x+1)$$ (Note: separable).

Standard Form: $x' - tx = t$.

Integrating Factor: $e^{-\int t\,dt} = e^{-t^2/2}$

Multiply both sides by I.F.:

$$e^{-\frac{t^2}{2}}\left(x' - tx\right) = te^{-t^2/2}$$

$$\Rightarrow \quad e^{-t^2/2}x' - te^{-t^2/2} = te^{-t^2/2}$$

Product Rule:

$$\Rightarrow \int \frac{d}{dt}\left(e^{-t^2/2}x\right)dt = \int te^{-t^2/2}\,dt.$$

$\Rightarrow$

Integrate: $e^{-t^2/2}x = -e^{-t^2/2} + C$

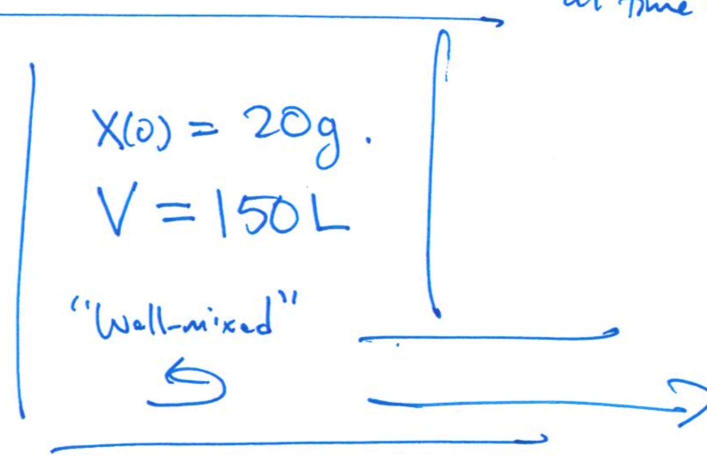
$$\Rightarrow \quad \boxed{x(t) = -1 + Ce^{t^2/2}}$$ general sol'n.

"Salt Tank Problems"

2.3.1: Applications — single-compartment mixing.

Rate of Change = Rate in − Rate out
(per unit time)

$X(t)$: amt. of salt in tank at time t. (grams)



Rate in =

$$= \left(\frac{2\,L}{min}\right)\cdot\left(\frac{3\,g}{L}\right)$$

$$= 6 \ g/min$$

Rate out

$$= \left(\frac{2\,L}{min}\right)\cdot\left(\frac{X(t)\,g}{150\,L}\right)$$

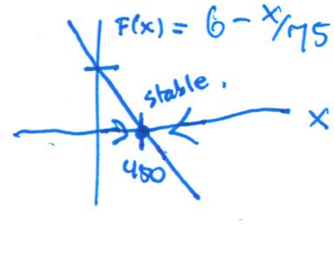
$$= \frac{X(t)}{75} \ g/min.$$

$$\frac{dx}{dt} = \underbrace{6}_{\text{Rate in}} - \underbrace{X/75}_{\text{Rate out}}$$

$$= \frac{1}{75}\left(450 - X\right) =: F(x)$$

Equilibrium: $\bar{X} = 450 (g)$.

Phase line:



Solve $\frac{dx}{dt} = 6 - \frac{x}{75}$ using I.F. method.

Standard Form: $\frac{dx}{dt} + \overset{p(t)}{\overbrace{\frac{1}{75}}} \cdot x = \overset{q(t)}{\overbrace{6}}.$

Integrating Factor: $e^{\int \frac{dt}{75}} = e^{t/75}.$

Multiply: $e^{t/75}\left(\frac{dx}{dt} + \frac{1}{75}x\right) = 6e^{t/75}$

$$\Longrightarrow \quad e^{t/75}\frac{dx}{dt} + \frac{1}{75}e^{t/75}x = 6e^{t/75}.$$

Product Rule; Integrate.

$$\Longrightarrow \quad \int \frac{d}{dt}\left(e^{t/75} \cdot x\right)dt = \int 6e^{t/75}\,dt$$

$$\Longrightarrow \quad \overset{FTC}{=} \; e^{t/75}x = 450e^{t/75} + C.$$

$$\Longrightarrow \quad X(t) = 450 + e^{-t/75}$$

IVP: $20 = X(0) = 450 + C \iff C = -430.$

$$\boxed{X(t) = 450 - 430e^{-t/75}}$$

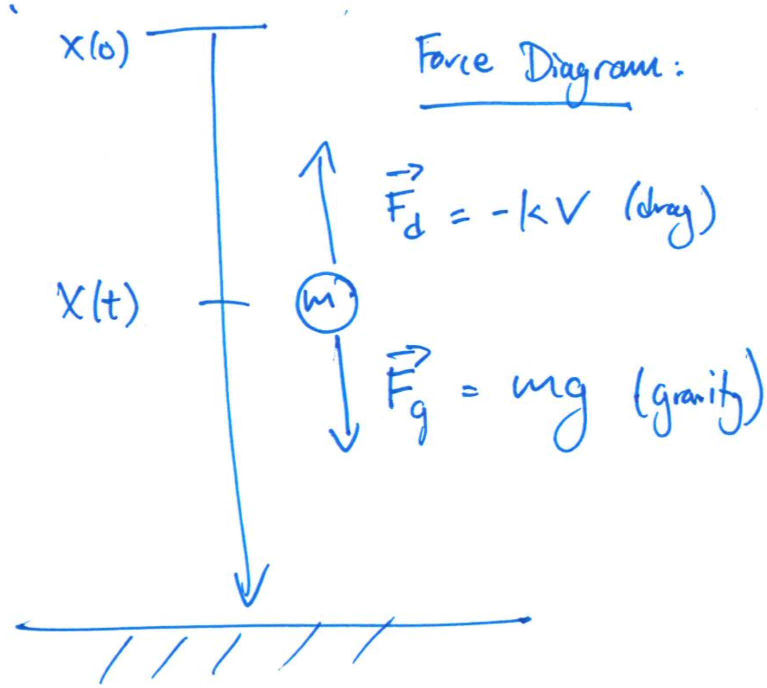
Section 2.3 : Modeling Applications

Object in Free Fall :

Position : $x(t)$.

Velocity : $v(t) = x'(t)$.

accelleration : $a(t) = v'(t)$.



Newton's 2nd Law of Motion:

$$\underbrace{\text{mass}}_{m} \cdot \underbrace{\text{accellention}}_{a(t)} = \Sigma' \text{ net forces.}$$

$$(\text{kg}) \quad (\text{m}/\text{s}^2) = \vec{F}_g + \vec{F}_d$$

$$\Longrightarrow \quad m\frac{dv}{dt} = mg - kv$$

$$\Longrightarrow \quad \frac{dv}{dt} = g - \left(\frac{k}{m}\right)v$$

$g = 9.8 \text{ m}/\text{s}^2$

k : kg/s

m : kg.

$$\frac{dV}{dt} = g - \left(\frac{k}{m}\right) V$$

$$\frac{dV}{dt} = 9.8 - \frac{V}{5}$$

$g = 9.8 \, m/s^2$

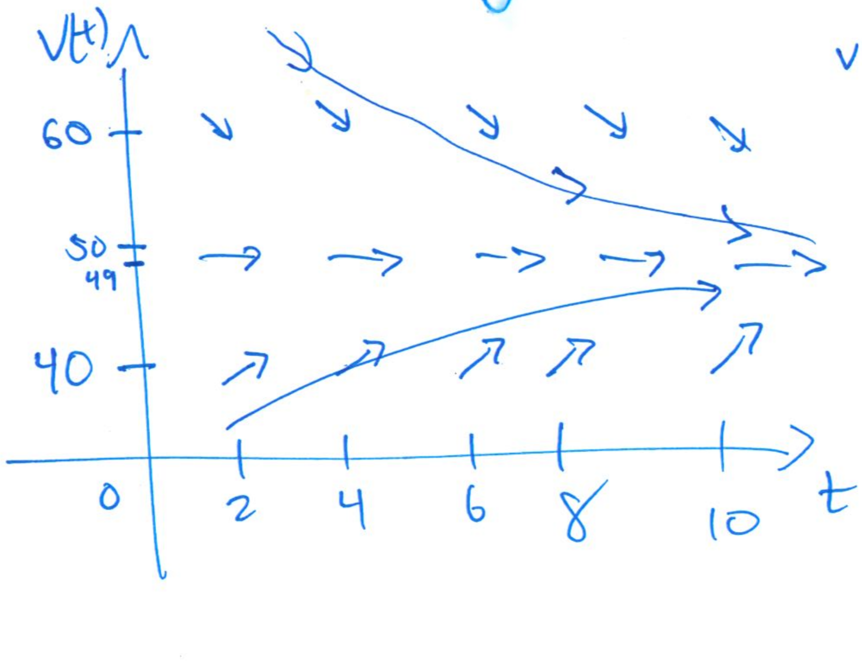
assume $\begin{cases} k = 2 \text{ kg/s} \\ m = 10 \text{ kg} \end{cases}$

Equilibrium: $\frac{dV}{dt} = 0$

$$0 = \frac{dV}{dt} = 9.8 - \frac{\bar{V}}{5}.$$

$$\Longrightarrow \bar{V} = 49 \text{ m/s} \quad \left(\text{terminal velocity}\right)$$

Visualize using Direction (slope) Field.



$V = 60$: $\frac{dV}{dt} = -2.2 < 0$

$\bar{V} = 49$: $\frac{dV}{dt} = 0$.

$V = 40$: $\frac{dV}{dt} = 1.8 > 0$

Let's Solve: $\frac{dv}{dt} = 9.8 - (\frac{1}{5})v$.

vs.

Separation of variables

$$\frac{dv}{9.8 - \frac{v}{5}} = dt$$

$$\Leftrightarrow \int \frac{dv}{49 - v} = \int \frac{1}{5}\, dt$$

$$\Leftrightarrow -\ln|49 - v| = \frac{t}{5} + C.$$

$$\Leftrightarrow \ln|49 - v| = -t/5 - C.$$

$$\Leftrightarrow 49 - v = Ae^{-t/5}$$

$$\Leftrightarrow \boxed{v(t) = 49 - Ae^{-t/5}}$$

Integrating Factor Method.

$$\frac{dv}{dt} + \left(\frac{1}{5}\right)v = 9.8.$$

$$\Leftrightarrow e^{t/5}\left(\frac{dv}{dt} + \left(\frac{1}{5}\right)v\right) = 9.8e^{t/5}$$

$$\Leftrightarrow \int \frac{d}{dt}\left(e^{t/5}v\right) dt = \int 9.8e^{t/5} dt$$

$$\Leftrightarrow e^{t/5}v = 49e^{t/5} + C.$$

$$\Leftrightarrow \boxed{v(t) = 49 + Ce^{-t/5}}$$

Note: $\lim_{t \to \infty} v(t) = 49 = \bar{v}$, (terminal velocity is stable Equilibrium pt)

since $e^{-t/5} \xrightarrow{t \to \infty} 0$

Suppose "Dropped" so that

$$v(0) = 0.$$

$$\Longrightarrow \quad \text{IVP}: \begin{cases} \frac{dv}{dt} = 9.8 - \left(\frac{1}{5}\right)v & \text{(DE)} \\ v(0) = 0. & \text{(I.C.)}. \end{cases}$$

$$\Longrightarrow \quad 0 = v(0) = 49 - A \Longrightarrow A = 49.$$

$$\Longrightarrow \quad \boxed{\begin{aligned} v(t) &= 49 - 49e^{-t/5} \\ &= 49\left(1 - e^{-t/5}\right) \end{aligned}}$$

<u>Question:</u> What if we want position, $x(t)$?

Integrate!! $\frac{dx}{dt} = v(t)$ ...

Application: Logistic Model of Population Dynamics.

Logistic Model: $\dfrac{dP}{dt} = rP\left(1 - P/K\right)$

$P(t)$ = population size

Parameters:

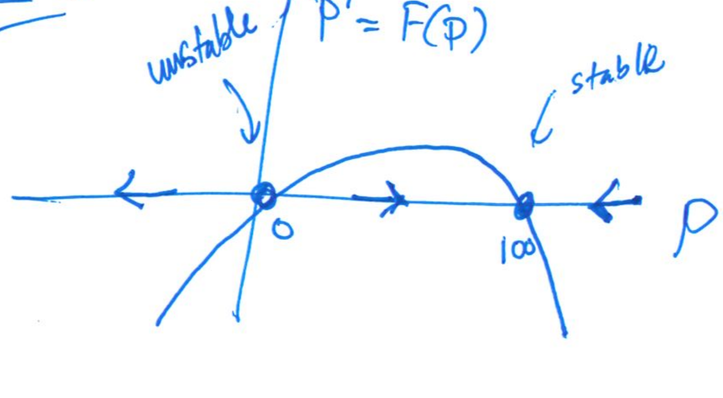
$$\frac{dP}{dt} = P\cdot\left(1 - P/100\right) = F(P). \quad \begin{cases} K = 100 & \text{(Carrying Capacity)} \\ r = 1 & \text{(growth rate)}. \end{cases}$$

Eq Pts: $\dfrac{dP}{dt} = 0 \iff \underline{\bar{P}_1 = 0},$

OR $1 - P/100 = 0$

$\iff \underline{\bar{P}_2 = 100}$

Stability:



Ex (Example 2.7.3, $\alpha = 1$)

$$x' = \alpha x - x^3 =: F(x)$$

$$= x(\alpha - x^2)$$

(autonomous 1st order Nonlinear)

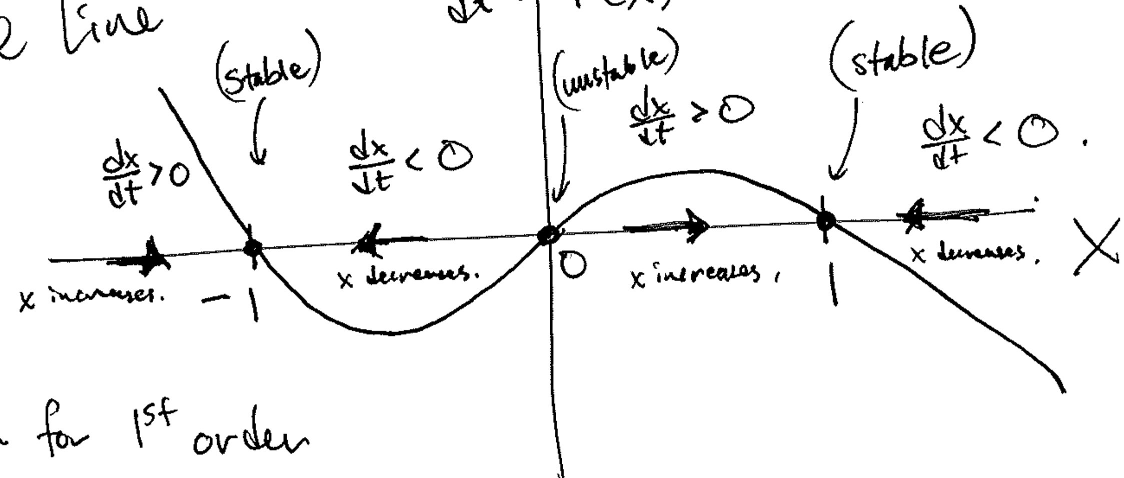
$$= x(\sqrt{\alpha} - x)(\sqrt{\alpha} + x).$$

$$= x(1-x)(1+x) \quad (\alpha = 1).$$

Equilibrium Pts satisfy: $\frac{dx}{dt} = 0 = x(1-x)(1+x)$

$$\bar{x}_1 = -1, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 1.$$

Phase line $\frac{dx}{dt} = F(x) = x - x^3$



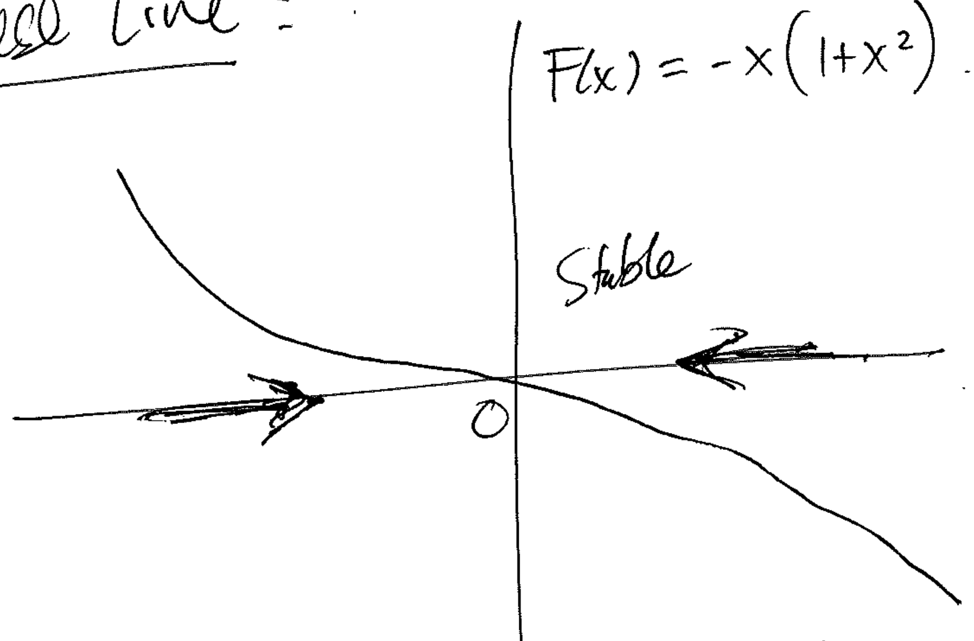
Approach for 1st order autonomous DEs.

When $\alpha = -1$:

$$x' = -x - x^3$$
$$= -x\underbrace{(1+x^2)}_{>0 \text{ for all } x.}$$

$\Longrightarrow$ Only Equilibrium pt: $\bar{x} = 0$.

<u>Phase Line</u>:



This indicates a change in the number of equilibria & ~~stability~~ stability as α varies.

(Help w/ Homework)

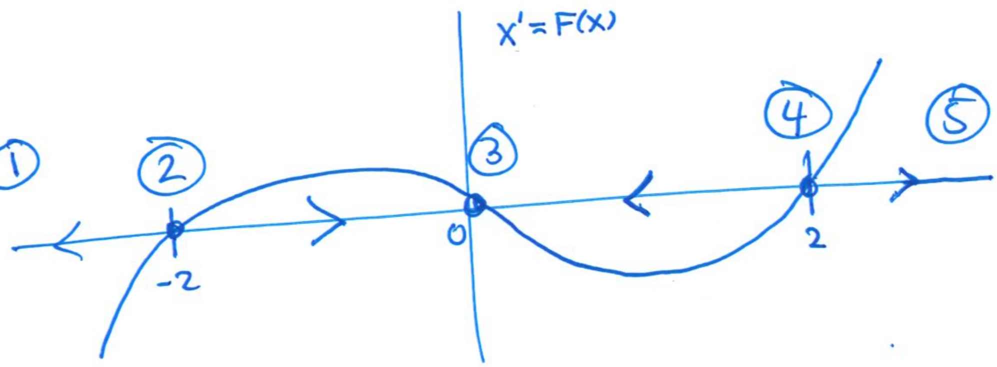
16 $X' = X^3 - 4X =: F(X).$

$= x \cdot (x^2 - 4)$

$= x \cdot (x-2) \cdot (x+2).$

Equilibria pts: $0 = x \cdot (x-2) \cdot (x+2)$

$\Longrightarrow \bar{x}_1 = -2, \ \bar{x}_2 = 0, \ \bar{x}_3 = 2.$



1 $x(0) < -2 \Longrightarrow \lim_{t\to\infty} x(t) = -\infty.$

2 $x(0) = -2 \Longrightarrow \lim_{t\to\infty} x(t) = -2.$

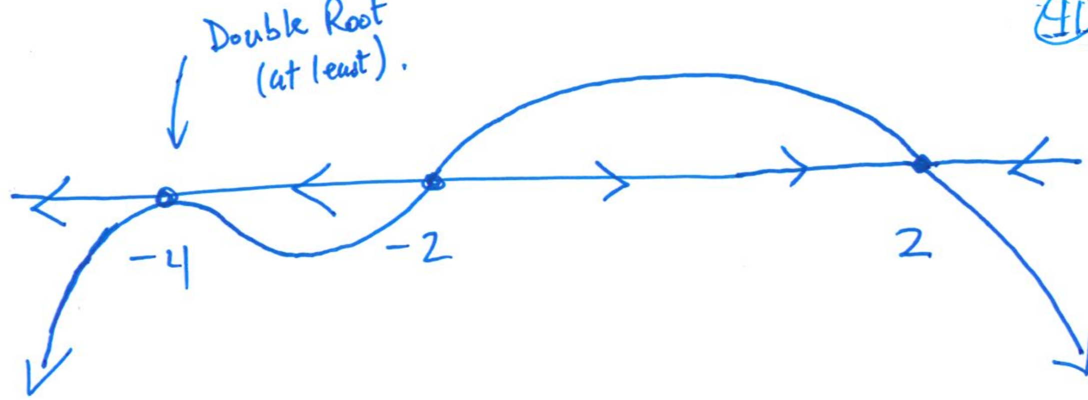
3 $-2 < x(0) < 2 \Longrightarrow \lim_{t\to\infty} x(t) = 0.$

4 $x(0) = 2 \Longrightarrow \lim_{t\to\infty} x(t) = 2.$

5 $x(0) > 2 \Longrightarrow \lim_{t\to\infty} x(t) = +\infty.$

Long-term behavior based on initial condition.

18



The following Autonomous DE

$$X' = -(X+4)^2 \cdot (x+2) \cdot (x-2).$$

has the above qualitative phase line.

<u>Note</u>: there are many sol'ns

$$\text{e.g., } X' = -(x+4)^{2n}(x+2)(x-2),$$

where n is an integer, are also correct answers. Can you find others?

Section 2.4: Existence & Uniqueness of Sol'ns

Initial Value Problem (IVP)

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

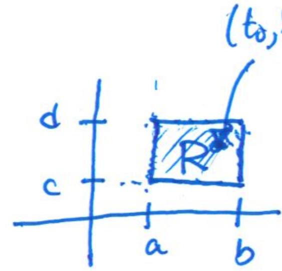
Two Questions:

(1) When do sol'ns exist?
(Existence)

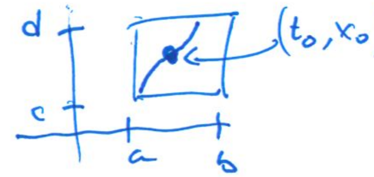
(2) When are they unique?
(uniqueness)

<u>Thm 2.1</u>: Suppose $f(t,x)$ is cont's on $R := \{(t,x) \in \mathbb{R}^2 \mid a \le t \le b,\ c \le x \le d\}$.

$$= [a,b] \times [c,d]$$



with $(t_0, x_0) \in R$, then there <u>exists</u> a sol'n to the IVP with $x(t_0) = x_0$.



If $\frac{\partial f}{\partial x}$ is cont's on R, then the sol'n is <u>unique</u>.

Ex | Example 2.4.2

IVP $\begin{cases} x' = x(1-x) =: F(x) \\ x(t_0) = x_0 . \end{cases}$

(1) Existence: $F(x) = x(1-x) = x - x^2$,

is a quadratic in x, which is continuous for all x.

$\Longrightarrow$ Sol'n exists for all initial conditions, $x(t_0) = x_0$.

(2) Uniqueness: $\frac{\partial F(x)}{\partial x} = F'(x) = 1-2x$ is continuous for all $x \Longrightarrow$ Sol'n is unique.

(1) + (2) $\Longrightarrow$ Sol'n exists & is unique

Ex | Example 2.4.4

$$\text{IVP} \begin{cases} x' = x^{1/2} \\ x(0) = 0. \end{cases}$$

Find 2 sol'ns to the IVP.

① $x(t) = 0$ for all t.

Indeed, $x'(t) = 0 = 0^{1/2} = (x(t))^{1/2}$,

and $x(0) = 0$.

② use separation of variables

$$\frac{dx}{dt} = x^{1/2} \Rightarrow \frac{dx}{x^{1/2}} = dt$$

$$\Rightarrow \int x^{-1/2}\,dx = \int dt.$$

$$\Rightarrow 2x^{1/2} = t + C.$$

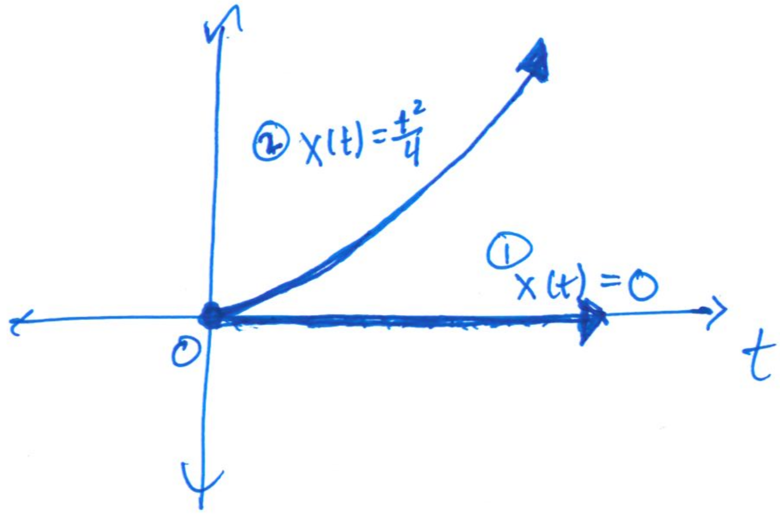
$$\Rightarrow x(t) = \left(\frac{t+C}{2}\right)^2$$

$$0 = x(0) = \left(\frac{0+C}{2}\right)^2$$

$$\Rightarrow C = 0 \Rightarrow \boxed{x(t) = \frac{t^2}{4}}$$

...

Graphs of sol'ns ① & ②: $\begin{cases} x' = x^{1/2} = F(x) \\ x(0) = 0 \end{cases}$



Here, we see 2 sol'ns that emanate from I.C. $x(0) = 0$.

Note: $\frac{\partial F(x)}{\partial x} = F'(x) = \frac{1}{2x^{1/2}}$

is discontinuous at $x = 0$.

$\Longrightarrow$ Uniqueness is not guaranteed by the E&U Thm.

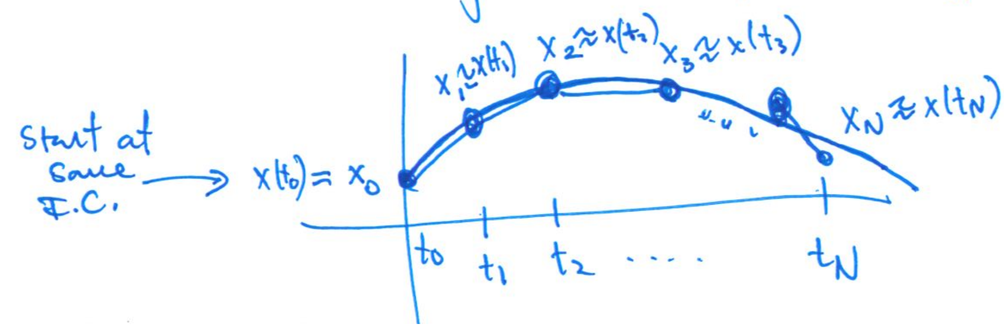
2.6 Numerical Methods

We want numerically approximate sol'ns to IVPs:

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

Def'n: A numerial method for solving an IVP is a set of discrete pts: $(t_0,x_0), (t_1,x_1), \ldots, (t_N,x_N)$

So that: $x_j = x(t_j)$ for $j = 0,1,2,\ldots,N,$



Euler's Method

Based on Linearization (OR tangent line approximation)

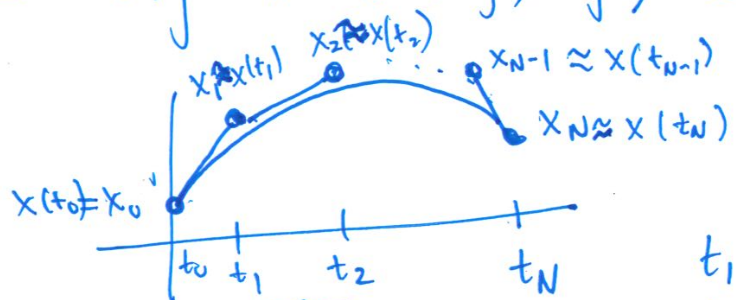
$$X(t_{j+1}) = X(t_j + \Delta t)$$

$$\approx X(t_j) + X'(t_j)\cdot\Delta t$$

$$= X(t_j) + f(t_j, X(t_j))\cdot\Delta t.$$

Euler's Method: $(j+1)$ in terms of j; explicit method.

$$X_{j+1} = X_j + f(t_j, X_j)\cdot\Delta t.$$



Uniform Mesh: $\Delta t = \dfrac{t_N - t_0}{N} \Rightarrow$

$t_1 = t_0 + \Delta t$

$t_2 = t_1 + \Delta t = t_0 + 2\Delta t$

$\vdots$

$t_N = t_0 + N\cdot\Delta t.$

$\boxed{\text{Ex}}$ Example 2.6.1 (page 61)

$$\begin{cases} x' = t - x \\ x(0) = 1. \end{cases}$$

Use Euler's Method on $[0, 2]$.

Exact Sol'n: $x(t) = t - 1 + 2e^{-t}$.

Compare to: increasing N

$N = 4,\ N = 8,\ N = 10, \dots$

Look at code in Matlab

Section 3.2: 2^{nd} order linear DEs w/ constant coefficients.

Consider:

$$a x''(t) + b x'(t) + c x(t) = 0.$$

where $a \neq 0$, b, and c are constant coefficients.

Guess: $g(t) = e^{rt}$, $g'(t) = re^{rt}$, $g''(t) = r^2 e^{rt}$.

$$\Rightarrow a r^2 e^{rt} + b r e^{rt} + c e^{rt} \overset{\text{Want}}{=} 0$$

$$\Rightarrow \underbrace{(ar^2 + br + c)}_{=0} \underbrace{e^{rt}}_{>0} = 0$$

Quadratic Eqn (in r): characteristic Polynomial.

Quadratic Formula: $$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Recall HW1 Question:

$x(t) = e^{rt}$

$$4r^2 + br + 16 = 0$$

Homogeneous 2^{nd} order linear DE w/ constant coefficients.

$$4x'' + bx' + 16x = 0$$

3 cases:

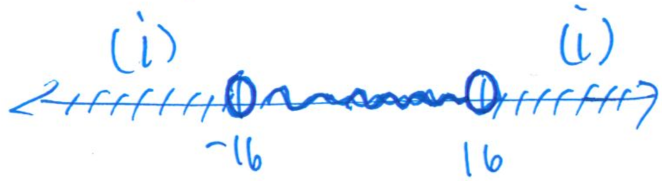
$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 16^2}}{8}$$

(i) two real roots, $r_1 \neq r_2$: $b^2 - 16^2 > 0$

$\Rightarrow \boxed{b > 16 \text{ OR } b < -16}$

(ii) one repeated real root: $\boxed{b = \pm 16}$

(iii) Complex Conjugates: $b^2 - 16^2 < 0 \Leftrightarrow \boxed{-16 < b < 16}$



Generally,

$$ar^2 + br + c = 0.$$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

<u>3 cases</u> : Based on Discriminant, $\underline{K = b^2 - 4ac}$

<u>General Sol'n Form</u>

$K>0$ (1) Two real Roots: $r_1 \neq r_2$. $\quad x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$K=0$ (2) One repeated real root: $r = r_1 = r_2$. $\quad x(t) = C_1 e^{rt} + C_2 t e^{rt}$

$K<0$ (3) Complex conjugates: $r_{1,2} = \alpha \pm \beta i$. $\quad x(t) = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)$

Ex

2 real roots (1) $x'' - 4x' + 3x = 0.$

Char. Equ:

$x = e^{rt}$
$x' = re^{rt}$
$x'' = r^2 e^{rt}$

$r^2 - 4r + 3 = 0.$

$(r-3)(r-1) = 0.$

$r_{1,2} = 3, 1.$

$\Rightarrow \quad x_1(t) = e^{3t}, \quad x_2(t) = e^{t}$

$\Rightarrow$ General Sol'n: $\boxed{x(t) = C_1 e^{3t} + C_2 e^{t}}$

1 repeated real root (2) Example 3.2.1 (page 40)

$x'' + 4x' + 4x = 0$

Char. Equ: $r^2 + 4r + 4 = 0.$

$(r+2)^2 = 0.$

$\Rightarrow r = r_1 = r_2 = -2.$

$\Rightarrow$ Gen. Sol'n: $\boxed{x(t) = C_1 e^{-2t} + C_2 t e^{-2t}}$

Complex conjugates (3) Example 3.2.3 (page 92)

$$x'' + 2x' + 5x = 0.$$

Characteristic Eqn: $r^2 + 2r + 5 = 0.$

$$r_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4(5)}}{2} = \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

$\Rightarrow$ General Sol'n:

$$x(t) = e^{-t}\left(C_1\cos(2t) + C_2\sin(2t)\right).$$

2nd order Homogeneous Linear DE's w/ constant coefficients:

$$ax'' + bx' + cx = 0 \quad (a \neq 0).$$

$$\Longrightarrow ar^2 + br + c = 0 \quad \text{(associated Char. Eqn)}$$

$$\Longrightarrow r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(Quadratic Formula)}.$$

3 cases based on $K := b^2 - 4ac$

Table 3.1 General Sol'ns of $ax'' + bx' + cx = 0$.

$K = b^2 - 4ac$	Characteristic Equation: $ar^2 + br + c = 0$	Form of General Sol'n
$K > 0$	2 Distinct Real Roots $r_1 \neq r_2$	$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
$K = 0$	Single repeated Real Root r	$x(t) = C_1 e^{rt} + C_2 t e^{rt}$
$K < 0$	Complex Conjugates: $r_{1,2} = \alpha \pm \beta i$	$x(t) = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)$

Section 3.1 General 2^{nd} order Linear DEs.

Warm up:

Exercise 3.1: #14. Show that

$x_1(t) = e^{-t}$, $x_2(t) = e^{-2t}$.

are both sol'ns.

to $x'' + 3x' + 2x = 0$

Characteristic Equ: $r^2 + 3r + 2 = 0$

$\Rightarrow (r+2)(r+1) = 0$

$\Rightarrow r_1 = -2, \ r_2 = -1$.

General Sol'n: $x(t) = c_1 e^{-2t} + c_2 e^{-t}$.

c_1, c_2 are constants.

First, Consider

$$a(t)x'' + b(t)x' + c(t)x = 0.$$

$$\xrightarrow{a(t)\neq 0} \quad x'' + \left(\frac{b(t)}{a(t)}\right)x' + \left(\frac{c(t)}{a(t)}\right)x = 0.$$

$\Longrightarrow$ General 2nd order linear DEs have the form:

$$x'' + p(t)x' + q(t)x = f(t)$$

w/ <u>associated homogeneous DE</u>

$$x'' + p(t)x' + q(t)x = 0$$

Exercise 3.1 : #14 Find particular solutions.

(1) $x'' + 3x' + 2x = \underline{5}$

$f_1(t) = \text{constant}.$

$\Rightarrow$ Good Guess : $x_p(t) = A$ (constant)

$x_p'(t) = 0$

$x_p''(t) = 0.$

Substitute :

$$x_p'' + 3x_p' + 2x_p \overset{\text{Want}}{=} 5$$

$$\Rightarrow 0 + 3\cdot 0 + 2\cdot A = 5 \Rightarrow 2A = 5$$

$$\Rightarrow A = 5/2.$$

$$\Rightarrow \boxed{x_{p,1}(t) = 5/2.}$$

particular sol'n.

---

(2) $x'' + 3x' + 2x = \underline{t}$

$f_2(t) = \text{linear in } t.$

Guess : $x_p(t) = \underline{A + Bt}$ (general line)

$x_p'(t) = 0 + B = B.$

$x_p''(t) = 0$

Substitute : $x_p'' + 3x_p' + 2x_p \overset{\text{Want}}{=} t$

$$\Rightarrow 0 + 3B + 2(A + Bt) = t.$$

$$\Rightarrow 3B + 2A + 2Bt = t.$$

$$\Rightarrow 3B + 2A = 0 \quad , \quad 2B = 1 \Rightarrow \boxed{B = 1/2}.$$

$$3/2 + 2A = 0 \Rightarrow \boxed{A = -\frac{3}{4}}$$

$$\boxed{x_{p,2}(t) = -\frac{3}{4} + \frac{1}{2}t}$$

(3) $x'' + 3x' + 2x = \underbrace{5 + t}$.

Note: $f_1(t) + f_2(t)$

Guess: $\underline{x_p(t)} = A + Bt$.

$x_p'(t) = 0 + B$

$x_p''(t) = 0$.

Substitute: $x_p'' + 3x_p' + 2x_p =$

$0 + 3B + 2(A+Bt) \overset{\text{Want}}{=} 5+t$

$\Rightarrow$ ① $3B + 2A = 5$, ② $2B = 1$.

$\Rightarrow \boxed{B = 1/2}$

$\frac{3}{2} + 2A = 5 \Rightarrow 2A = \frac{7}{2}$

$\Rightarrow \boxed{A = 7/4}$

$$\boxed{x_{p,3}(t) = \frac{7}{4} + \frac{1}{2}t}$$

Note: $x_{p,1} + x_{p,2}$

$$= \frac{5}{2} + \frac{-3}{4} + \frac{1}{2}t$$

$$= \frac{7}{4} + \frac{1}{2}t = x_{p,3}.$$

$$\Longrightarrow \quad x_{p,3} = x_{p,1} + x_{p,2}$$

Question: Does this hold generally?! ...

True for Linear DEs.

Let's formalize!

Let $x_{p,1}$, $x_{p,2}$ be particular sol'ns of:

(1) $ax'' + bx' + cx = f_1(t)$

(2) $ax'' + bx' + cx = f_2(t)$, respectively.

Consider:

$$ax'' + bx' + cx = f_1(t) + f_2(t)$$

Then: $x_p(t) = x_{p,1} + x_{p,2}$.

Also, Suppose $x_{p,1}(t)$ solves

$$ax'' + bx' + cx = f_1(t)$$

Then, $x_p(t) = d \cdot x_{p,1}$ solves

$$ax'' + bx' + cx = d \cdot f_1(t)$$

Proof: Substitute: $a x_p'' + b x_p' + c x_p$

$$= a(d x_{p,1})'' + b(d x_{p,1})' + c(d x_{p,1})$$

$$= a \cdot d \cdot x_{p,1}'' + b \cdot d \cdot x_{p,1}' + c \cdot d \cdot x_{p,1}$$

$$= d \cdot (a x_{p,1}'' + b x_{p,1}' + c x_{p,1})$$

$$= \underline{d \cdot f_1(t)}$$

$\Longrightarrow$ $a x_p'' + b x_p' + c x_p = d \cdot f_1(t)$, where $x_p = d \cdot x_{p,1}$

The Principle of Linear Superposition:

Consider:

$$(\star) \quad ax'' + bx' + cx = d_1 f_1(t) + d_2 f_2(t).$$

Let $x_p(t) = d_1 x_{p,1}(t) + d_2 x_{p,2}(t)$, where

$$(1) \quad a \cdot x_{p,1}'' + b \cdot x_{p,1}' + c\, x_{p,1} = f_1(t)$$

$$(2) \quad a \cdot x_{p,2}'' + b \cdot x_{p,2}' + c \cdot x_{p,2} = f_2(t).$$

Then $x_p = d_1 x_{p,1} + d_2 x_{p,2}$ solves $(\star)$.

Continue: See In-Class Worksheet on

"Method" of Undetermined Coefficients

In-class Worksheet: Method of Undetermined Coefficients

Table 1: **Method of Undetermined Coefficients.** Adapted from Table 3.2, page 104 of DE Textbook.

Given Functional Form, $f(t)$	**Particular Solution* ("Guess"), x_p**
Exponential, $a\,e^{b\,t}$	$Ae^{b\,t}$
Poylnomial (degree n), $a_0 + a_1\,t + \dots a_n\,t^n$	$A_0 + A_1\,t + \dots A_n\,t^n$
Periodic, $a\,\sin(\omega\,t) + b\,\cos(\omega\,t)$	$A\,\sin(\omega\,t) + B\,\cos(\omega\,t)$

**If x_p is a solution of the associated homogeneous equation, then use $t^k\,x_p$ where k is the smallest positive integer such that $t^k\,x_p$ is not a homogeneous solution.*

Problem 1. *Find general solutions to the following differential equations:*

1. $x'' + 5x' + 4x = 10$

2. $x'' + 5x' + 4x = 3\,t$

3. $x'' + 5x' + 4x = 20 + 18\,t$

4. $x'' + 5x' + 4x = e^{-3\,t}$

5. $x'' + 5x' + 4x = 20 + 18\,t + 5\,e^{-3\,t}$

6. $x'' + 5x' + 4x = \cos(4\,t)$

7. $x'' + 5x' + 4x = e^{-t}$

How to find general sol'n to:

$$a \cdot x'' + bx' + cx = f(t)\,?$$

Thm 3.1: Suppose $x_h(t)$ is general sol'n to $x'' + p(t)x' + q(t)x = 0$ and $x_p(t)$ a particular sol'n to

$$x'' + p(t)x' + q(t)x = f(t).$$

Then, the sum: $x(t) = x_h(t) + x_p(t)$ is general sol'n to

$$x'' + p(t)x' + q(t)x = f(t).$$

Ex Extending Exercise 3.1: #14.

Find General Sol'ns to:

(1) $x'' + 3x' + 2x = 5$

$$\left.\begin{aligned} x_1(t) &= e^{-t} \\ x_2(t) &= e^{-2t} \end{aligned}\right\} \quad \begin{aligned} X_h(t) &= C_1 X_1(t) + C_2 X_2(t) \\ &= C_1 e^{-t} + C_2 e^{-2t} \end{aligned}$$

Recall: $X_{p,1} = 5/2$.

$$\Longrightarrow \quad X(t) = \overbrace{C_1 e^{-t} + C_2 e^{-2t}}^{X_h(t)} + \overbrace{5/2}^{X_p(t).}$$

(2) $x'' + 3x' + 2x = t$

Recall: $X_{p,2} = -\frac{3}{4} + \frac{1}{2}t$.

$$\Longrightarrow \boxed{X(t) = C_1 e^{-t} + C_2 e^{-2t} - \frac{3}{4} + \frac{1}{2}t.}$$

(3) $x'' + 3x' + 2x = 5 + t$

Recall: $x_{p,3} = \frac{7}{4} + \frac{1}{2}t.$

$$\Longrightarrow \boxed{x(t) = C_1 e^{-t} + C_2 e^{-2t} + \frac{7}{4} + \frac{1}{2}t.}$$

Section 3.2: Existence & Uniqueness Thm for 2^{nd} order linear DEs.

Consider the 2^{nd} order linear DE:

$$x'' + p(t)\,x' + q(t)\,x = f(t).$$

with initial conditions: $x(t_0) = x_0$, $x'(t_0) = v_0$. If p, q, and f are continuous on an interval (t_1, t_2) containing t_0, then there exists a unique sol'n that is continuous on (t_1, t_2) (based on the initial conditions).

$\boxed{\text{Ex}}$ Example 3.1.2

Find the largest interval (around $t_0 = 0$) for which Thm 3.2 guarantees existence & Uniqueness.

(a) $$x'' + \underbrace{2}_{\substack{p(t)\\ \text{cts}}} x' + \underbrace{6}_{\substack{q(t).\\ \text{cts}.}} x = \underbrace{e^t \sin(t)}_{\substack{f(t) \text{ continuous}\\ \text{for all } t.}};$$

(b) $$x'' + \underbrace{\tan(t)}_{p(t)} x' + \underbrace{\frac{1}{(1-t)}}_{q(t)} x = \underbrace{0}_{\substack{f(t)\\ \text{cts. all } t.}}.$$

cts $(-\pi/2, \pi/2)$

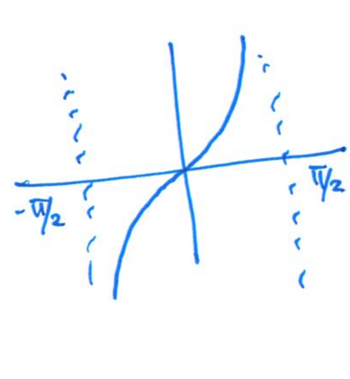
cts $t \neq 1$.

Intersection!

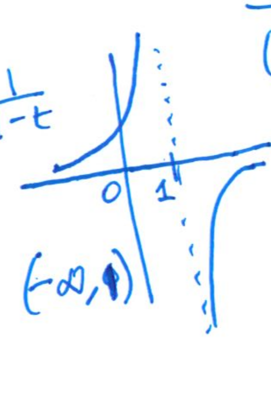
$(-\frac{\pi}{2}, \pi/2) \cap (-\infty, 1)$

$= \underline{\underline{(-\pi/2, 1)}}$

$\tan(t)$



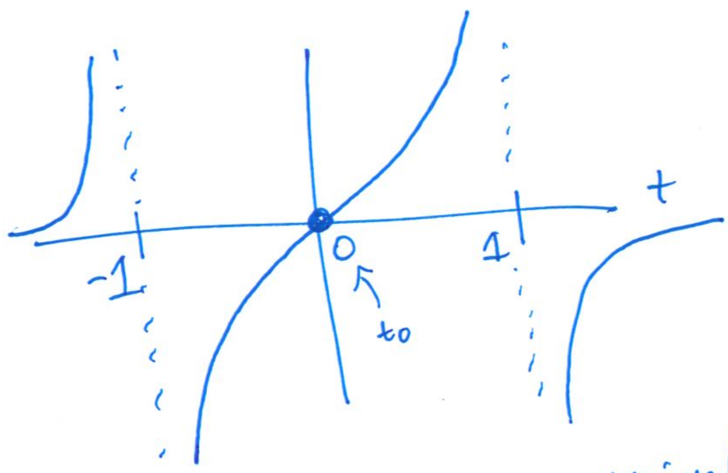
$\frac{1}{1-t}$



(c) $(1-t^2)x'' + 2tx' + 6x = 0.$

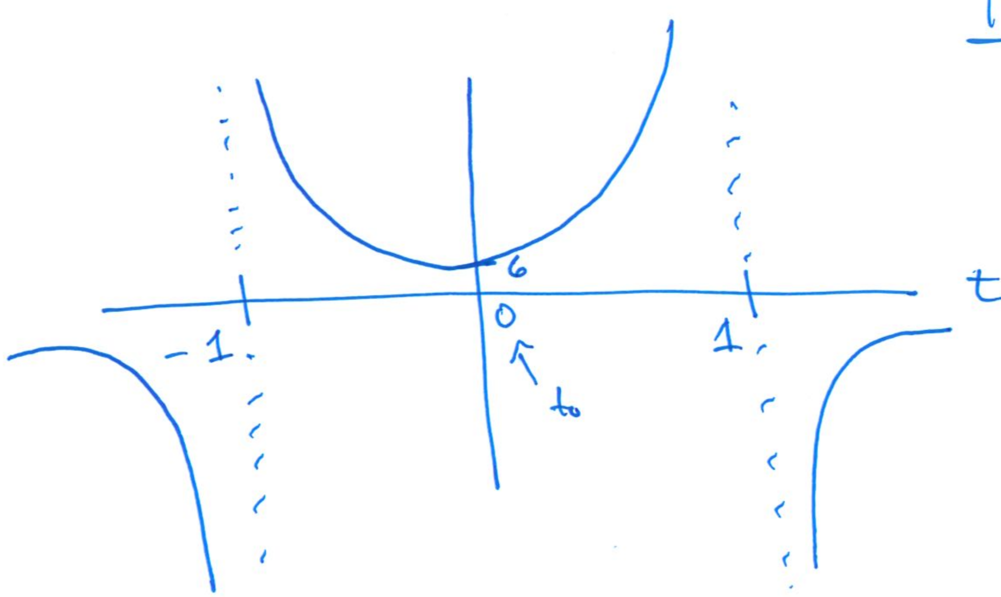
$$\Rightarrow \quad x'' + \left(\frac{2t}{1-t^2}\right)x' + \left(\frac{6}{1-t^2}\right)x = 0.$$

$p(t) = \dfrac{2t}{1-t^2}$



maximal interval containing $t_0=0$ where both $p(t)$ & $q(t)$ are cont's is $\boxed{(-1,1)}$

$q(t) = \dfrac{6}{1-t^2}$



<u>Def'n:</u> $x_1(t), x_2(t)$ form a fundamental set of sol'ns to $x'' + p(t)x' + q(t)x = 0$, if both $x_1(t)$ & $x_2(t)$ are sol'ns

<u>and</u> $\det\begin{pmatrix} x_1 & x_2 \\ x_1' & x_2' \end{pmatrix}$

$$= x_1 \cdot x_2' - x_1' x_2 \neq 0$$

<u>Thm:</u> Suppose $x_1(t), x_2(t)$ form a fundamental set of sol'ns to $x'' + p(t)x' + q(t)x = 0$

Then $x(t) = c_1 x_1(t) + c_2 x_2(t)$ (with $c_1, c_2 \neq 0$ constants) is the <u>general</u> sol'n.

Ex

Go Back to:

Exercise 3.1 : #14

$x_1(t) = e^{-t}$ and $x_2(t)$ are sol'ns to

$$x'' + 3x' + 2x = 0.$$

Show that $x_1(t)$ & $x_2(t)$ form a fundamental set of sol'ns:

Wronskian!

$$\left.\begin{matrix} x_1(t) = e^{-t}, & x_2(t) = e^{-2t} \\ x_1'(t) = -e^{-t}, & x_2'(t) = -2e^{-2t} \end{matrix}\right\} \det\begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} = \det\begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix}$$

$$= -2e^{-t}\cdot e^{-2t} - (-e^{-t})\cdot e^{-2t}$$

$$= -2e^{-3t} + e^{-3t}$$

$$= -e^{-3t} \neq 0$$

for all t.

Section 3.3: The Spring-mass system

Consider an object with mass, m, suspended from a spring

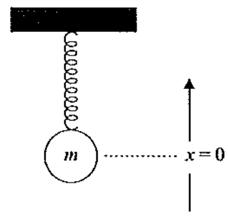


Figure 3.5. A simple spring-mass system

By **Newton's Second Law of Motion**, (the mass) $\times$ (the acceleration) of an object is equal to the sum of the forces acting on it:

$$m\,x''(t) = F_s + F_d + f(t)$$

Hooke's Law for a linear spring gives that $F_s = -k\,x(t)$, where k is the spring constant. Also, the **damping force** can be assumed to be, $F_d = -d\,x'(t)$, where b is the damping constant. The following differential equation can be obtained for the position of the object:

$$m\,x''(t) + b\,x'(t) + k\,x(t) = f(t)$$

The associated homogeneous equation corresponds to the **unforced spring-mass system**:

$$m\,x''(t) + b\,x'(t) + k\,x(t) = 0\,,$$

which has characteristic equation given by

$$m\,r^2 + b\,r + k = 0\,.$$

Table 2: **Unforced Spring-mass System.** Here, the external force is equal to zero, i.e., $f(t) = 0$.

Discriminant, $b^2 - 4\,m\,k$	**Roots of Characteristic Equation**	**General Solution Form**
$b^2 - 4\,m\,k > 0$ (Overdamped)	Two real roots, $r_1 \neq r_2$	$x(t) = C_1 e^{r_1\,t} + C_2 e^{r_2\,t}$
$b^2 - 4\,m\,k = 0$ (Critically Damped)	A single real root, r	$x(t) = C_1 e^{r\,t} + C_2 t\, e^{r\,t}$
$b^2 - 4\,m\,k < 0$ (Underdamped)	Complex conjugate roots, $r_{1,2} = \alpha \pm \beta\, i$	$A\, e^{\alpha\, t}\, \sin(\beta\, t) + B^{\,\alpha\, t}\, \cos(\beta\, t)$

Problem 2. *Example 3.3.1.* *A 1 kg object is suspended on a spring with spring constant $k = 4N/m$. Initially, the object is lifted up $1\,m$ (from equilibrium) and let go (from rest). In general, the damping constant depends on the media in which the spring is immersed (e.g., higher viscosity media correspond to larger damping constants). For each of the given damping constants, write an initial value problem and characterize the behavior of the spring-mass system.*

1. $b = 5\,N\,s/m$

2. $b = 4\,N\,s/m$

3. $b = 2\,N\,s/m$

Section 3.3: the spring-mass System

Agenda: based on worksheet.

(1) Derive spring-mass system from force ~~[illegible]~~ Diagram.

(2) No Damping: $b = 0$ (page 91)

$\Rightarrow$ Simple Harmonic Motion

$$Mx'' + Kx = 0 \Rightarrow x(t) = A\cos(\omega t) + B\sin(\omega t)$$
$$\omega = \sqrt{\tfrac{k}{m}}.$$

Rewrite as: $x(t) = R\sin(\omega t + \phi)$

System parameters
$$\begin{cases} \omega = \sqrt{\frac{K}{m}} & \text{(natural frequency)} \\ T = 2\pi/\omega & \text{(period)} \\ R = \sqrt{A^2+B^2} & \text{(amplitude)} \end{cases}$$
$$\phi = \tan^{-1}(A/B) \quad \text{(phase)}$$

(π added to ϕ if $B<0$)

3. Relate Roots of the Characteristic Eqn to the dynamics.

4. write as a system of 1st order DEs:

$$x'' + 6x' + 4x = 0.$$

Let $\underbrace{v}_{\text{velocity}} = \underbrace{x'}_{\substack{\text{derivative}\\ \text{of}\\ \text{position}}}. \Rightarrow v' = x'' = -6\underbrace{x'}_{v} - 4x$

$\Rightarrow v' = -4x - 6v.$

$$\Longrightarrow \begin{cases} \begin{pmatrix} x' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ -4x - 6v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -6 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \end{cases}$$

A.11 Arithmetic of Complex Numbers

Consider the quadratic eqn:

$$r^2 + 1 = 0. \quad \text{want "sol'ns"}$$

$$\Leftrightarrow \quad r^2 = -1$$

$$\Leftrightarrow \quad r = \pm i, \text{ where}$$

$i = \sqrt{-1}$ so that

$$i^2 = -1 \quad \Leftrightarrow \quad i^2 + 1 = 0$$

<u>has a sol'n</u>.

Def'n: A complex number has the form:

$$z = a + bi,$$

where a, b are real numbers

"Real Part" $\mathrm{Re}(z) = a$

"Imaginary Part" $\mathrm{Im}(z) = b$.

Def'n: Complex conjugate of $z = a+bi$ is $\bar{z} = a - bi$.

a α a

Complex Arithmetic: $z = a+bi$, $w = c+di$

- **Addition**: just sum real parts & complex parts separately.

$$z+w = (a+bi) + (c+di)$$
$$= (a+c) + (bi+di)$$
$$= (a+c) + (b+d)i$$

- **Subtraction**:

$$z-w = (a+bi) - (c+di)$$
$$= (a-c) + (b-d)i$$

$\boxed{\text{Ex}}$ $z = 2+3i$, $w = -1+i$.

$z+w = (2-1) + (3+1)i = \boxed{1+4i}$ $\quad z-w = (2-(-1)) + (3-1)i = \boxed{3+2i}$

- **Multiplication!**

$$z \cdot w = (a+bi)\cdot(c+di)$$

$$= ac + adi + bci + bdi^2$$

$$= (ac - bd) + (ad + bc)i.$$

Ex $z = 2+3i$, $w = -1+i$.

$$z \cdot w = (2+3i)\cdot(-1+i)$$

$$= -2 + 2i - 3i + 3\overset{-1}{\boxed{i^2}}$$

$$= -2 - i - 3$$

$$= \boxed{-5 - i}.$$

- **Division:**

Aside: Recall rationalizing.

$$\frac{1}{1+\sqrt{2}}\frac{(1-\sqrt{2})}{(1-\sqrt{2})} = \frac{1-\sqrt{2}}{1+2} = \frac{1-\sqrt{2}}{3}.$$

similar here ...

$\boxed{\text{Ex}}$

$$\frac{z}{w} = \frac{2+3i}{-1+i}\cdot\frac{(-1-i)}{(-1-i)}$$

$$= \frac{-2-2i-3i-3i^2}{2}$$

$$= \frac{1}{2} - \frac{5}{2}i.$$

Note: $\frac{z}{w} = \frac{z}{w}\cdot\frac{\bar{w}}{\bar{w}} = \frac{z\cdot\bar{w}}{|\bar{w}|^2} = \frac{z\cdot\bar{w}}{|w|^2}$.

Note: $z = a+bi$, $\bar{z} = a-bi$.

$Re(z) = a$, $Im(z) = b$.

(i) $z + \bar{z} = a + \cancel{b}i + a - \cancel{b}i = 2a$

$\Longrightarrow Re(z) = \dfrac{z+\bar{z}}{2}$.

(ii) $z - \bar{z} = a+bi - (a-bi) = 2bi$

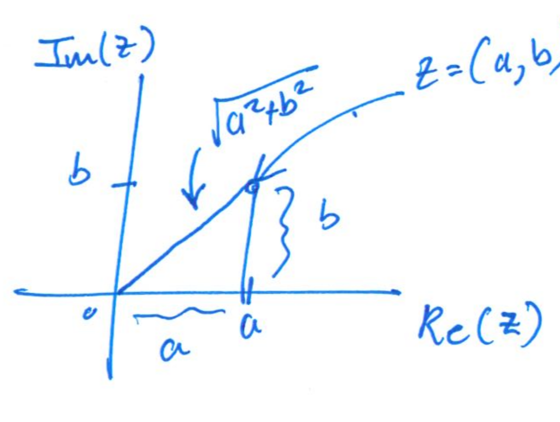
$\Longrightarrow Im(z) = \dfrac{z-\bar{z}}{2i}$.

(iii) $z \cdot \bar{z} = (a+bi)\cdot(a-bi)$

$= a^2 + abi - abi - b^2 \boxed{i^2}^{-1}$

$= a^2 + b^2$.

$= |z|^2$



$|z| = \sqrt{a^2 + b^2}$

Euler's Identity !! :)

$z = x + iy$.

$$e^z = e^{x+iy} = e^x \cdot \underbrace{e^{iy}}.$$

<u>Euler's Identity</u>:

$$e^{iy} = \cos(y) + i\sin(y).$$

<u>Ex 1</u> Recall "T-shirt Formula":

$$e^{i\pi} = \underbrace{\cos(\pi)}_{-1} + i\underbrace{\sin(\pi)}_{0}$$

$$\Rightarrow \boxed{e^{i\pi} = -1}$$

Proof of Euler's Identity (if there's time)

real #

$$\text{Let } f(y) = \underbrace{e^{iy}}_{\text{complex \#}}$$

$$= \underbrace{g(y)}_{\text{Real \#}} + \underbrace{h(y)}_{\text{Real \#.}}\, i \, .$$

Differentiate either side:

LHS: $f'(y) = i \cdot e^{iy}$.

$$= i \cdot \left(g(y) + h(y) \cdot i \right)$$

$$= i \cdot g(y) + h(y) \underbrace{\boxed{i^2}}_{-1}$$

$$= -h(y) + g(y)\, i \, .$$

RHS: $g'(y) + h'(y)\, i$

LHS = RHS

Equate Real vs. complex parts:

$$\begin{cases} g'(y) = -h(y) \\ h'(y) = +g(y). \end{cases}$$

We Require : $f(0) = g(0) + h(0)\,i$

$$\Rightarrow 1 = e^0 = f(0) = g(0) + h(0)\,i$$

$$\Rightarrow g(0) = 1, \quad h(0) = 0.$$

IVP
$$\begin{cases} g'(y) = -h(y) \\ h'(y) = +g(y) \end{cases} \text{ 2D system of } 1^{st} \text{ order Eqns.}$$

1 $g(0) = 1,\ h(0) = 0$

$\Rightarrow g'(0) = h(0) = 0.$ 2 initial conditions

$\Rightarrow$ 2 $g'(0) = 0.$

We have $g''(y) = -h'(y) = -g(y)$

$$\Longrightarrow g'' + g = 0.$$

Characteristic Eqn : $r^2 + 1 \Rightarrow r_{1,2} = \pm i \Rightarrow g(y) = A\cos(y) + B\sin(y)$

$$1 = g(0) = A\cos(0) + B\sin(0).$$

$$= A.$$

$$\Longrightarrow \underline{A = 1}.$$

$$g'(y) = -A\sin(y) + B\cos(y).$$

$$0 = g'(0) = -A \cancel{\sin(0)}^{\,0} + B\underbrace{\cos(0)}_{1}$$

$$= B.$$

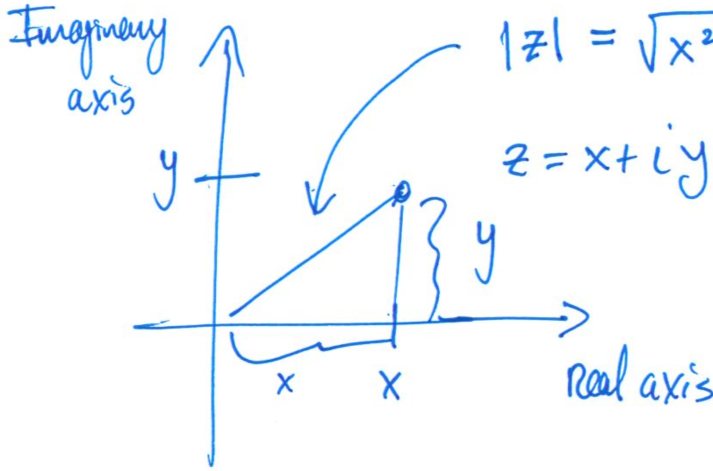
$$\Longrightarrow \underline{B = 0}.$$

$$\Longrightarrow g(y) = \cos(y)$$

$$\underline{h(y) = g'(y) = \sin(y).}$$

$$\Longrightarrow f(y) = \boxed{e^{iy} = \cos(y) + i\sin(y)}$$

A.2 | Geometry of Complex Numbers.



$|z| = \sqrt{x^2+y^2} = \sqrt{z \cdot \bar{z}} = \sqrt{\bar{z} \cdot z}$.

$z = x + iy$

Polar Coordinates: $z = x + iy$

$(x, y) \longmapsto (r, \theta)$

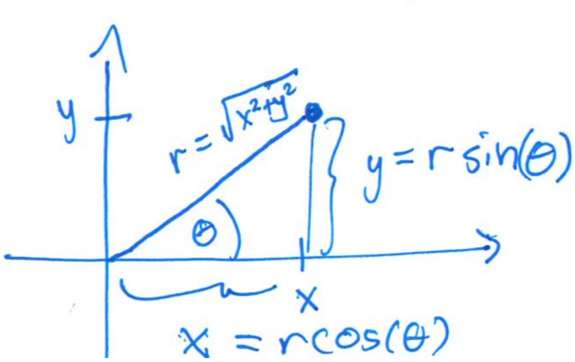
$x = r\cos(\theta)$, $y = r\sin(\theta)$.

$$x^2 + y^2 = r^2\cos^2(\theta) + r^2\sin^2(\theta)$$

$$= r^2 \big(\underbrace{\cos^2(\theta) + \sin^2(\theta)}_{1} \big)$$

$$= r^2 \implies r = \sqrt{x^2+y^2}$$

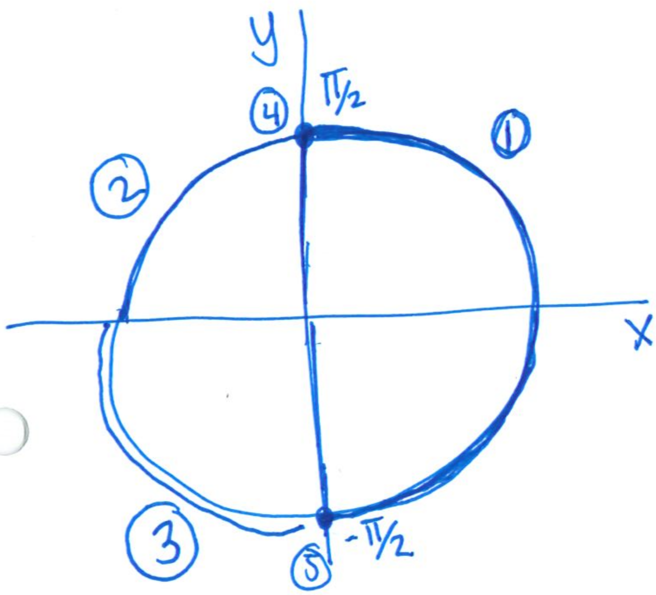
$$= |z|.$$



(modulus of z)

$\theta = \underline{\text{Arctan}}(y, x) =$

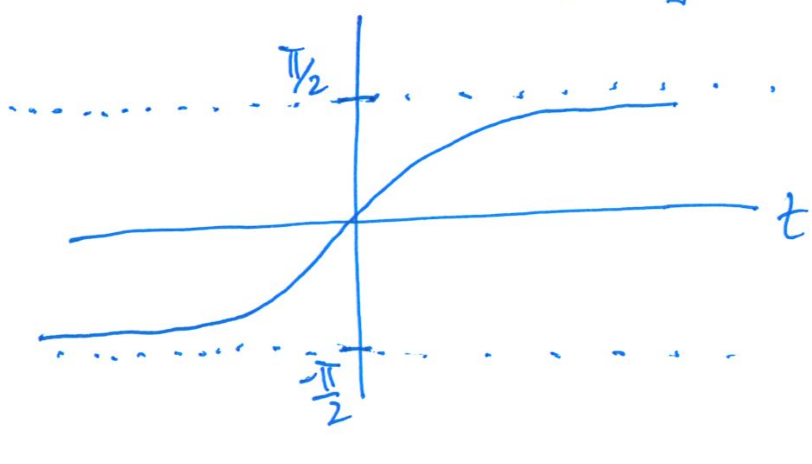
Returns Polar Angle.



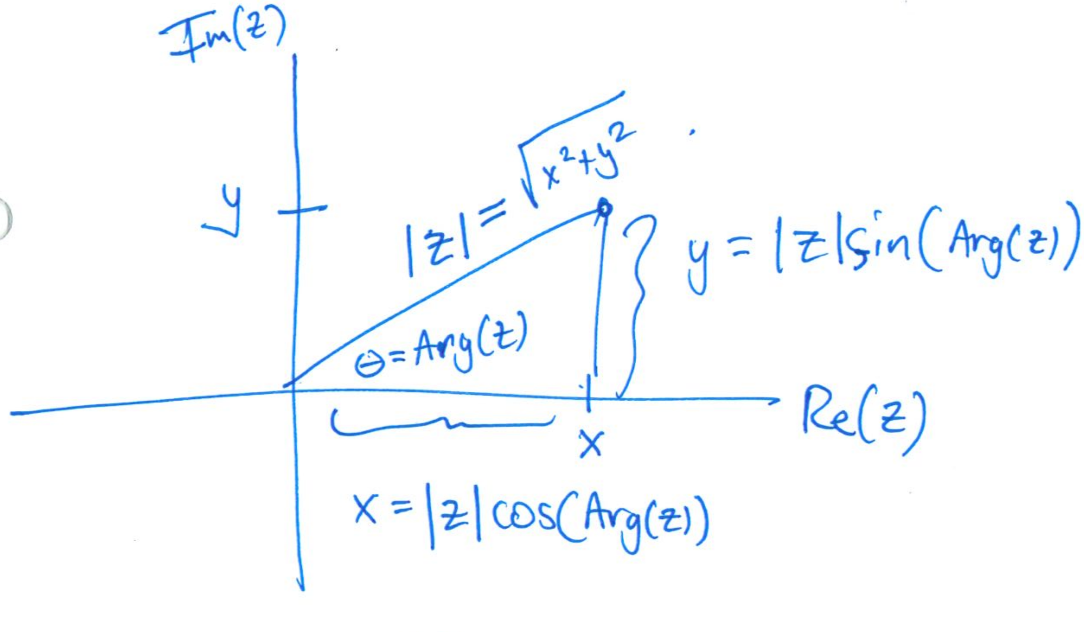
$$\begin{cases} \arctan\left(\frac{y}{x}\right), & x > 0 \quad (1) \\ \arctan\left(\frac{y}{x}\right) + \pi, & x < 0,\ y \geq 0 \quad (2) \\ \arctan\left(\frac{y}{x}\right) - \pi, & x < 0,\ y < 0 \quad (3) \\ \pi/2, & x = 0,\ y > 0 \quad (4) \\ -\pi/2, & x = 0,\ y < 0 \quad (5) \end{cases}$$

(undefined for $x = y = 0$)

$\arctan(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Given $z = x + iy$, we call $\theta = \text{Arctan}(y, x)$ the argument of z (denoted $\text{Arg}(z)$) and call $r = \sqrt{x^2+y^2}$ the modulus of z (denoted $|z|$).



Ex | 142 (page 271)

Let $z = 1 + i$. Find $|z|$, $\text{Arg}(z)$.

$$|z| = \sqrt{1^2 + 1^2} = \boxed{\sqrt{2}}$$

$$\text{Arg}(z) = \text{Arctan}\underbrace{(1, 1)}_{\text{Region } ①} = \arctan(1) = \boxed{\pi/4}$$

Polar Form:

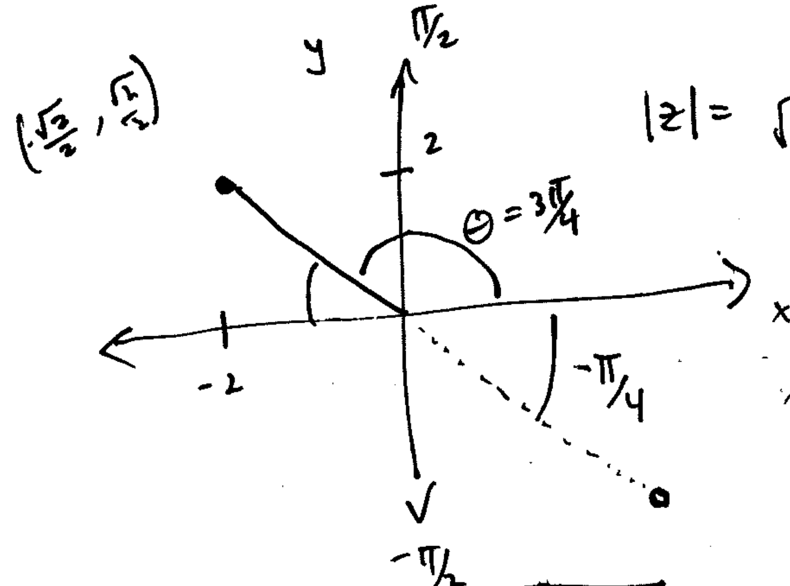
Euler's Identity.

$$z = re^{i\theta} = \underbrace{r\cos\theta}_{x} + i\underbrace{r\sin\theta}_{y}.$$

$r = |z|$, $\theta = \arg(z)$

$$= |z|\, e^{i\arg(z)}$$

Ex. $z = -2 + 2i.$ $\left.\begin{array}{l} x = -2 < 0 \\ y = 2 > 0. \end{array}\right\}$



$|z| = \sqrt{4+4} = \sqrt{8}$

$= 2\sqrt{2}.$

$\mathrm{Arg}(z) = -\pi/4 + \pi = 3\pi/4$

$$\boxed{z = 2\sqrt{2}\, e^{i\arg(z)}}$$

Geometry of Multiplication:

$|z_1| = r_1$, $\arg(z_1) = \theta_1$.

$z_1 = r_1 e^{i\theta_1}$

$|z_2| = r_2$, $\arg(z_2) = \theta_2$.

$z_2 = r_2 e^{i\theta_2}$.

Multiply $z_1 \cdot z_2$

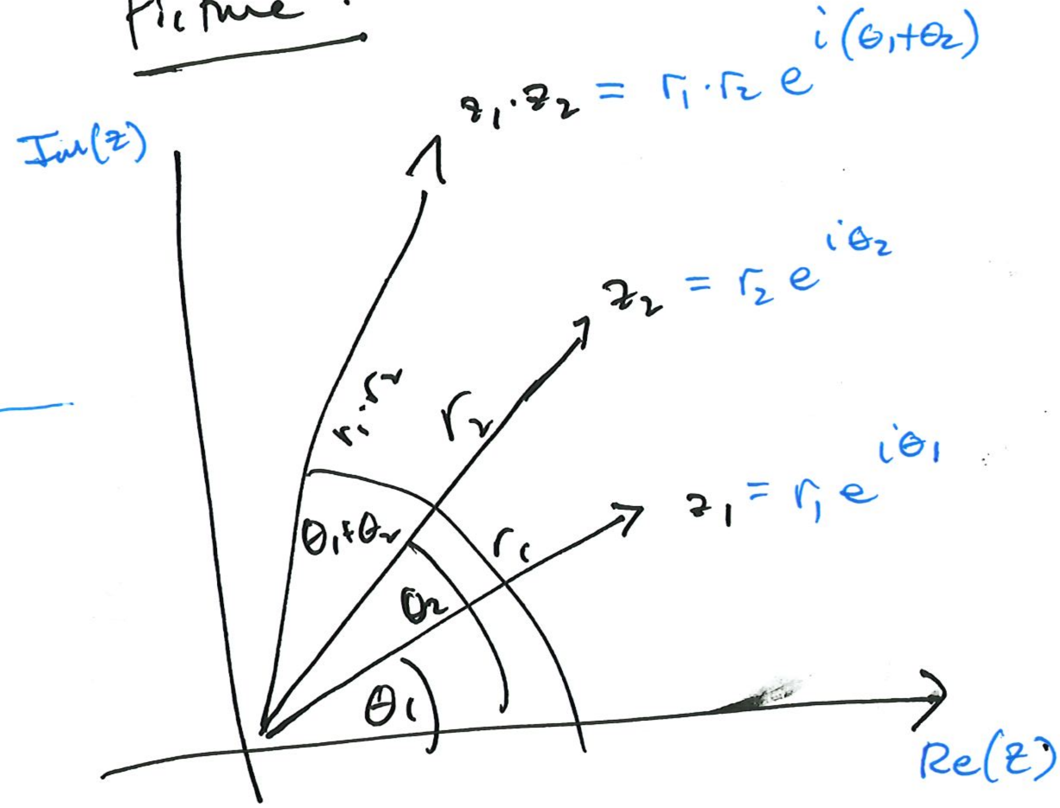
$= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}$

$= r_1 \cdot r_2 \cdot e^{i\theta_1} \cdot e^{i\theta_2}$

$= \underbrace{r_1 \cdot r_2}_{\text{moduluses multiply.}} \cdot e^{i\underbrace{(\theta_1 + \theta_2)}}$

angles add

Picture:



As a consequence, $n \geq 0$ integer.

$$\begin{cases} |z^n| = |z|^n \\ \arg(z^n) = n \cdot \arg(z). \end{cases}$$

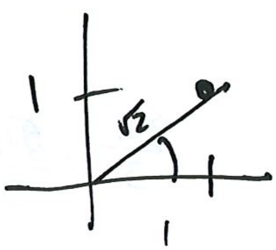
$z = re^{i\theta}$

Proof $|z| = r$, $\arg(z) = \theta$.

$$z^n = \left(re^{i\theta}\right)^n = r^n \cdot e^{i\theta n}$$

Example

$z = 1 + i$. Compute z^6.



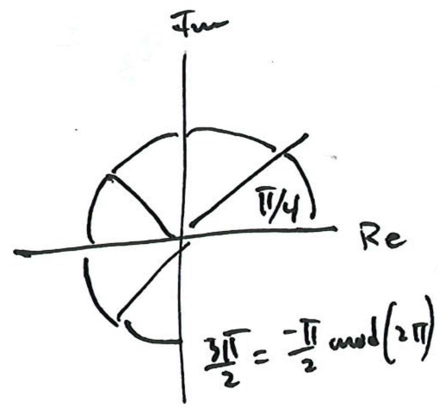
$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$.

$\theta = \arg(z) = \arctan(1) = \pi/4$.

$z = \sqrt{2}\, e^{i\pi/4}$.

$z^6 = (\sqrt{2})^6 \cdot \left(e^{i\pi/4}\right)^6$

$= 2^3 \cdot e^{i 3\pi/2}$



$= 2^3 \cdot e^{-\frac{\pi}{2} i}$.

$= 8 \cdot (0 + -1i)$

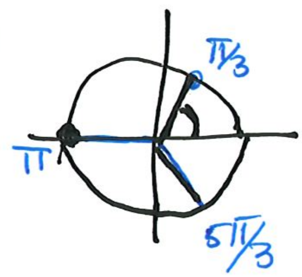
$= -8i$

Ex. Find Sol'ns to.

$$z^3 + 1 = 0.$$

$$z^3 = -1 = e^{i\pi + 2\pi i k} \qquad k = \text{integer}$$

$$= \underbrace{\cos(\pi)}_{-1} + i\underbrace{\sin(\pi)}_{0}$$



Two More Sol'ns:

$$\Rightarrow z = \left(e^{i\pi + 2\pi i k}\right)^{1/3}$$

$$= e^{i\pi/3 + \frac{2\pi}{3} k i}$$

Case 5: $k = 0$: $z = e^{i\pi/3} = \cos(\pi/3) + i\sin(\pi/3)$

$= \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$k = 1$: $z = e^{i\pi} = \cos(\pi) + i\sin(\pi)$

$= -1$

$k = 2$: $z = e^{i5\pi/3} = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

1.1 Intro to linear Eqns.

<u>Def'n</u>: A linear eqn can be written in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = c.$$

where x_i are variables (unknowns) and a_i, c are (known) constant coefficients.

<u>Def'n</u> A <u>sol'n</u> to a linear eqn is a set of values x_i so that all eqns are satisfied.

Ex] Determine which Eqns are Linear:

(1) $2x + 3y - 7z = 29$ (linear)

(2) $2xy + z = 1$ (non linear)

(3) $\frac{1}{x} + \sqrt{y} + 24z = 3$ (non linear)

(4) $\sqrt{7}\, r + \pi s + \frac{3}{5} t = \cos(45^\circ)$

(linear).

Ex] Solve the system of linear Equations.

$$x + y = -1$$
$$2x - 3y = 8$$

3 Approaches

Approach 1: Substitution

$$x + y = -1$$

$$\Rightarrow \quad x = -1 - y \xrightarrow{\text{substitute}}$$

$$2x - 3y = 8$$

$$2(-1-y) - 3y = 8$$

$$-2 - 2y - 3y = 8.$$

$$\Rightarrow \quad -5y = 10$$

$$\Rightarrow \quad \boxed{y = -2}$$

$$x = -1 - (-2)$$

$$= -1 + 2 = 1$$

Back substitute.

$$\boxed{x = 1, \ y = -2} \quad \text{Soln.}$$

Approach 2:

$$2(x+y) = -2$$

$$-\ (2x - 3y) = -8$$

$$5y = -10$$

$$\Rightarrow \boxed{y = -2}$$

Back Substitute: $x + y = -1 \Rightarrow x + (-2) = -1.$

$$\Rightarrow \boxed{x = 1}$$

Approach 3 : Using Augmented Matrix to solve linear systems via Row Operations.

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & -3 & 8 \end{array}\right) \Longleftrightarrow \begin{cases} x + y = -1 \\ 2x - 3y = 8 \end{cases}$$

Row Operation

$$\xrightarrow[\text{Row Being Replaced } R_2]{-R_2 + 2R_1} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 5 & -10 \end{array}\right) \Longleftrightarrow \begin{cases} x + y = -1 \\ 5y = -10 \end{cases}$$

$$\xrightarrow[R_2]{\frac{1}{5}R_2} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & -2 \end{array}\right) \Longleftrightarrow \begin{cases} x + y = -1 \\ \boxed{y = -2} \end{cases}$$

Row Echelon Form

Back sub:

$x - 2 = -1$

$\Rightarrow x = 1.$

OR, Keep Going

$$\xrightarrow[R_1]{R_1 - R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right) \Longleftrightarrow \boxed{\begin{array}{l} x = 1 \\ y = -2 \end{array}}$$

Reduced Row Echelon Form

2.1 Matrix Addition & Scalar Multiplication

① When are two matrices equal?

② Add Matrices, multiply by constant.

A matrices.

m-rows, n-columns

$$\underset{(m \times n)}{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix}$$

Ex. 1 $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 8 & 11 \end{pmatrix}$

$$3\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$$

Matrix Addition:

$$A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & & \vdots \\ a_{m1}+b_{m1} & \cdots & \ddots & \vdots \\ & & & a_{mn}+b_{mn} \end{pmatrix}$$

k - scalar

$$k \cdot A = \begin{pmatrix} k \cdot a_{11} & \cdots & k \cdot a_{1n} \\ \vdots & \ddots & \vdots \\ k\, a_{m1} & \cdots & k \cdot a_{mn} \end{pmatrix}$$

Properties of matrix addition & scalar Mult.

A, B, C $(m \times n)$. k - scalar.

$$A+B = B+A$$

$$(A+B)+C = A+(B+C)$$

$$k(A+B) = kA+kB$$

$$kA = Ak$$

$$0 \cdot A = 0.$$

2 **Ex** $A = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$, $b = \begin{pmatrix} -1 \\ 8 \end{pmatrix}$

Find x, y so that:

$$x \begin{pmatrix} 1 \\ 2 \end{pmatrix} + y \begin{pmatrix} -1 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \end{pmatrix}.$$

1 **Ex** 13 Page 55 $A = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 7 \\ 3 & -4 \end{pmatrix}$

$$3A + 2X = -B$$

$$\Rightarrow \quad 2X = -B - 3A.$$

$$\Rightarrow \quad X = -\tfrac{1}{2}B - \tfrac{3}{2}A.$$

$$= \begin{pmatrix} -1/2 & -7/2 \\ -\frac{3}{2} & 2 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} & \frac{21}{2} \\ 9/2 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -14 \\ -6 & -4 \end{pmatrix}.$$

When Can we multiply two matrices or vectors?

Row Vector: $(1 \times n)$ $(u_1, \ldots, u_n)$

Column Vector: $(m \times 1)$ $\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$

Row Vector By Column Vector:
(inner product)

$$\underset{1\times n}{(v_1 \cdots v_n)} \cdot \underset{n \times 1}{\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} (v_1 \cdots v_n) = \begin{pmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ \vdots & & \ddots & \\ u_n v_1 & \cdots & & u_n v_n \end{pmatrix} = \begin{pmatrix} u_1 \vec{v} \\ u_2 \cdot \vec{v} \\ \vdots \\ u_n \vec{v} \end{pmatrix}$$

Ex.

$$\underset{(1\times 3)}{\vec{u}} = (-2, 4, 3), \quad \underset{(3\times 1)}{\vec{v}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$(1\times 3)\cdot(3\times 1)$ Inner Product.

$$\vec{u}\cdot\vec{v} = (-2, 4, 3)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -2 + 8 + 9 = 15$$

$(3\times 1)\cdot(1\times 3)$

$$\vec{v}\cdot\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\cdot(-2, 4, 3)$$

$$= \begin{pmatrix} -2 & 4 & 3 \\ -4 & 8 & 6 \\ -6 & 12 & 9 \end{pmatrix}$$ Outer Product.

$$\vec{a}_i \quad (1 \times n)$$

$$\overset{(m \times \overset{r}{\cancel{n}})}{A} = \begin{pmatrix} \leftarrow \vec{a}_1 \rightarrow \\ \leftarrow \vec{a}_2 \rightarrow \\ \vdots \\ \leftarrow \vec{a}_m \rightarrow \end{pmatrix} \qquad \overset{(\overset{r}{\cancel{m}} \times \overset{n}{\cancel{m}})}{B} = \cancel{\begin{pmatrix} \leftarrow \vec{b}_1 \rightarrow \\ \leftarrow \vec{b}_2 \rightarrow \\ \vdots \\ \leftarrow \vec{b}_m \rightarrow \end{pmatrix}}$$

$$\begin{pmatrix} \uparrow & & \uparrow \\ \vec{b}_1 & \cdots & b_n \\ \downarrow & & \downarrow \end{pmatrix}$$

$$\underbrace{(m \times r)(r \times n)}_{\text{match}}$$

$$(A \cdot B) = \begin{pmatrix} \vec{a}_1 \vec{b}_1 & \vec{a}_1 \vec{b}_2 & \cdots & \vec{a}_1 \vec{b}_n \\ \vdots & \ddots & & \vdots \\ \vec{a}_m \vec{b}_1 & \cdots & \cdots & \vec{a}_m \vec{b}_n \end{pmatrix}$$

$(m \times n)$

Result

2.2 Matrix Multiplication

$$(2\times 2)\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\cdot\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}(2\times 2)$$

$$= \begin{pmatrix} 1\cdot 1 + 2\cdot 2 & 1\cdot(-1) + 2\cdot(2) \\ 3\cdot 1 + 4\cdot 2 & -(3) + 4\cdot 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 3 \\ 11 & 5 \end{pmatrix}$$

Can we do? $\underset{(2\times 2)}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}\underset{(2\times 3)}{\begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & -1 \end{pmatrix}}$

How about $\overset{(2\times 3)}{\begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & -1 \end{pmatrix}}\overset{(2\times 2)}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}$?

What's different about Matrix Mult. from regular (Scalar) multi?

Consider: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Matrix Mult

$$A \cdot B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 10 \end{pmatrix}$$

$\neq$ $\neq$

$$B \cdot A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 3 & 4 \end{pmatrix}$$

$\Rightarrow$ ① Generally, $AB \neq BA$.

Question: Does $AB = AC \Rightarrow B = C$?

Not Necessarily !!

Can you find an example where:

$$AB = AC$$

But, $B \neq C$.

(Picks out 1st Row)

Consider: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$AC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

But, $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = C$.

$\Rightarrow$ 2 Generally, $AB = AC \not\Rightarrow B = C$.

Properties of Matrix Multiplication

Let A, B, C matrices, & k a scalar.

(1) $A(BC) = (AB)C$ (Assoc.)

(2) $A(B+C) = AB + AC$ (Dist.)

$(B+C)A = BA + CA$

(3) $k(AB) = (kA)B = A(kB)$

(4) $AI = IA = A$.

Warm-up Exercise

$$A = \begin{pmatrix} 1 & -1 \\ 5 & 2 \\ -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 6 & 7 \end{pmatrix}$$

Calculate:

$$\underset{(3\times 2)\cdot(2\times 3)}{A \cdot B} = \underset{(3\times 3)}{\begin{pmatrix} -1 & -5 & -6 \\ 9 & 17 & 19 \\ 4 & 16 & 19 \end{pmatrix}}$$

$$\underset{(2\times 3)\cdot(3\times 2)}{B \cdot A} = \underset{(2\times 2)}{\begin{pmatrix} 4 & 4 \\ 18 & 31 \end{pmatrix}}$$

Example $AB \neq BA$

$\Rightarrow$ Diff Dimension!

~~E~~Exercise :

Start

1 Consider the following systems :

concept
consistent vs. inconsistent

(a) $x_1 + \cancel{y}_2 = 1$

$2\cancel{x}_1 + 2\cancel{y}_2 = 2$

Consistent,
infinitely many sol'ns

Concept
no sol'n, one sol'n, infinitely many sol'ns.

(b) $x_1 + x_2 = 0$

$2x_1 + 2x_2 = 4$

inconsistent
$\Rightarrow$
no sol'n

(c) $x_1 + \cancel{y}_2 = 2$

$x_1 - x_2 = 0$

Consistent,
one sol'n.

(i) Draw the lines & describe the sets of sol'ns.

Concepts: Rank + Pivots

(2) Write corresponding augmented matrices.

(108)

(a)
$$\left(\begin{array}{cc:c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right)$$

(Rank 1)

~~# pivots~~

$$\longrightarrow \left(\begin{array}{cc:c} \boxed{1} & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

(b)
$$\left(\begin{array}{cc:c} 1 & 1 & 0 \\ 2 & 2 & 4 \end{array}\right)$$

(Rank 1)

$$\begin{array}{cc:c} \boxed{1} & 1 & 0 \\ 0 & 0 & 4 \end{array}$$

Rank 2

(c)
$$\left(\begin{array}{cc:c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right)$$

(Rank 2)

Obtain Reduced Row Echelon Form.

3 Write in the form: $A\vec{x} = \vec{b}$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(a) $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$; $\vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\det(A) = 0$

(b) $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$.

$\det(A) = 0$

(c) $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$\det(A) = -2 \neq 0.$

Square Matrix has Full Rank

Concept

$\Longleftrightarrow$

columns linearly independent

$\Longleftrightarrow$

A^{-1} (invertible)

$\Longleftrightarrow$

$\det(A) \neq 0$.

4 Obtain Reduced Row Echelon form:

$$\left(\begin{array}{cc:cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1]{R_2 - R_1} \left(\begin{array}{cc:cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right).$$

$$\xrightarrow[R_2]{-\frac{1}{2} R_2} \left(\begin{array}{cc:cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

$$\xrightarrow[R_1]{R_1 - R_2} \left(\begin{array}{cc:cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

$B := A^{-1}$ Let B

5 Compute

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Show $AB = BA = I$.

$$AB=BA=I \iff B = A^{-1}.$$ def'n:

Formula: Check $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A) = ad-bc$

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

6

Find Sol'n $\vec{x}$:

$$A\vec{x} = \vec{b}$$

$$\Longrightarrow \vec{x} = A^{-1}\vec{b}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2.8 Linear Independence.

Concept Check: True/False.

① If $\vec{v}$ & $\vec{w}$ are linearly dependent, then $\vec{v}$ is a scalar multiple of $\vec{w}$. True

② If $\vec{v}_2 = 2\vec{v}_1 - 4\vec{v}_3$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent. True

③ 3 vectors in $\mathbb{R}^3$ must be linearly independent. False

④ 3 vectors in $\mathbb{R}^2$ can be linearly independent. False

Def'n:

A linear combination of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ has the form

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n ,$$

where $c_1, c_2, \ldots, c_n$ are constants.

~~Sometime~~ Summation Notation:

$$\sum_{j=1}^{n} c_j \vec{v}_j$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Write $\vec{v}_3$ as a linear combination of $\vec{v}_2, \vec{v}_3$.

$$\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \overset{c_1}{\boxed{-\tfrac{1}{2}}} \overset{\vec{v}_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} + \overset{c_2}{\boxed{-\tfrac{1}{2}}} \overset{\vec{v}_2}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

How can we do this systematically?

Write:
$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$A\vec{c} = \vec{b} \qquad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

$$\Rightarrow \vec{c} = A^{-1}\vec{b}, \quad A^{-1} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \vec{c} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}. \checkmark$$

(Finally!)

Def'n: $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are linearly **dependent** if

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where at least one $c_k \neq 0$.

Consider:

$$\Longleftrightarrow \underbrace{\begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{V}_1 & \vec{V}_2 & \cdots & \vec{V}_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{\vec{c}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\vec{0}}$$

$\Longleftrightarrow \quad A \cdot \vec{c} = \vec{0}$

① A^{-1} (invertible),

$\Longleftrightarrow$

② $\det(A) \neq 0$.

$\Longleftrightarrow$

③ columns linearly independent

$\Longleftrightarrow$

④ the only sol'n to $A\vec{c} = 0$ is $\vec{c} = 0$

~ Example 69 page 131

Consider $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

$\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ k \end{pmatrix}$

(a) Find k s.t. $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent.

$(k = 1)$

(b) Find c_1, c_2, c_3 s.t. $A\vec{c} = \vec{0}$

$\Leftrightarrow \quad c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$.

Recall: $\boxed{\text{Rank}(A) = 2}$

$(c_3 = 1)$, $c_1 = -3$, $c_2 = 1$. works.

We say $\vec{c}$ is a sol'n to the homogeneous eqn !!

<u>Recall</u>: the rank of a matrix is # of linearly independent collumns (or rows, equivalently)

Determinant of 2x2:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Computing Determinants of nxn matrices via co-factor expansion...

Def: Matrix Minor, Cofactor

A $(n \times n)$: $\underline{i, j \text{ minor}}$ of A

denoted A_{ij} is the det.

of $(n-1) \times (n-1)$ matrix formed

by deleting i^{th} row & j^{th}

column of A.

$\Longrightarrow$ i, j cofactor of A is

$$C_{ij} = (-1)^{i+j} A_{ij}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Find $C_{1,3}$

$$C_{1,3} = (-1)^{1+3} \cdot A_{1,3}$$

$$= \det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$= 32 - 35$$

$$= -3.$$

Determinant Via Cofactor Expansion:

The determinant of the matrix A $(n \times n)$ is the cofactor expansion along a chosen row or column,

e.g., along 1st row.

$$\det(A) = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$$

E.g. compute the

$$\det\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Determinant & Elementary Row Operations:

A $(n \times n)$, B formed after the described one row operation on A:

1. ~~Adding~~ a scalar multiple of one row to another:

   $\det(B) = \det(A)$. (unchanged)

2. Multiplying one row of A by a scalar k;

   $\det(B) = k \cdot \det(A)$ (multiplied by k)

3. Interchanging two rows:

   $\det(B) = -\det(A)$. (negated)

~ Ex. 89 Page 164

Write $A = \begin{pmatrix} 2 & 4 & -2 \\ -1 & -2 & 5 \\ 3 & 2 & 1 \end{pmatrix}$

as upper triangular using row operations. Find it's determinant by tracking the det. the row operations.

Determinant Properties

A, B $(n \times n)$, $k \in \mathbb{R}$.

① $\det(kA) = k^n \det(A)$.

② $\det(A^T) = \det(A)$. A^T = transpose of A.

③ $\det(AB) = \det(A) \cdot \det(B)$.

④ A invertible $\Rightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$.

⑤ $A^{-1} \Leftrightarrow \det(A) \neq 0$.

Problem 1 a)

Quiz 7

126

$$\begin{pmatrix} 3 & 2 & 4 \\ 1 & 5 & -2 \\ 1 & -1 & 8 \end{pmatrix}$$

Det(A) =

21 · 10 · 3

$$\xrightarrow[R_2]{3R_2-R_1} \begin{pmatrix} 3 & 2 & 4 \\ 0 & 13 & -10 \\ 1 & -1 & 8 \end{pmatrix}$$

$$\xrightarrow[R_3]{3R_3-R_1} \begin{pmatrix} 3 & 2 & 4 \\ 0 & \boxed{\begin{matrix} 13 & -10 \\ -5 & 20 \end{matrix}} \\ 0 & \end{pmatrix}$$

$$\xrightarrow[R_3]{\frac{5}{13}R_2+R_3} \begin{pmatrix} 3 & 2 & 4 \\ 0 & 13 & -10 \\ 0 & 0 & \boxed{\frac{-5}{13}\cdot 10 + \frac{2\cdot 13\cdot 10}{13}} \end{pmatrix}$$

$$= \frac{21\cdot 10}{13}$$

Ex

$$B = \begin{pmatrix} 1 & -2 & 4 & 7 \\ 1 & -2 & 3 & 14 \\ 1 & -2 & 4 & 3 \\ 2 & 2 & 2 & 2 \end{pmatrix}$$

$\det(B) = \det(B^T)$.

$$B^T = \begin{pmatrix} 1 & 1 & 1 & 2 \\ -2 & -2 & -2 & 2 \\ 4 & 3 & 4 & 2 \\ 7 & 14 & 3 & 2 \end{pmatrix}$$

$$\xrightarrow[R_2]{2R_1 + R_2} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 6 \\ 4 & 3 & 4 & 2 \\ 7 & 14 & 3 & 2 \end{pmatrix} \quad \underline{\det(B^T)} = \det(B)$$

① $R_2 \leftrightarrow R_3$

② $R_3 \leftrightarrow R_4$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 \\ 4 & 3 & 4 & 2 \\ 7 & 14 & 3 & 2 \\ \boxed{0 \quad 0 \quad 0 \quad 6} \end{pmatrix} \quad (-1)\cdot(-1)\det(B) = \det(B).$$

cofactor expansion along last row.

keep going...

4.1) Eigenvalues & Eigenvectors

Ex

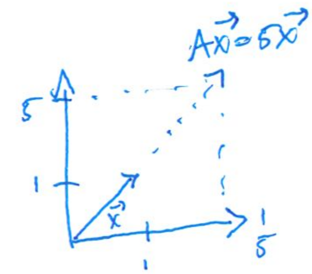
$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

$$= 5\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \underset{\text{eigenvalue}}{5}\,\underset{\text{eigenvector}}{\vec{x}} \Longrightarrow A\vec{x} = 5\vec{x}$$



$\Longrightarrow$

"eigen" = German for "characteristic" or "proper"

Def'n: $A\ (n\times n)$, $\vec{x} \neq 0$, $\underbrace{\lambda \in \mathbb{R}}_{\text{scalar}}$.

* If

$$A\vec{x} = \lambda \vec{x}.$$

then $\vec{x}$ is an eigenvector of A
and λ is the corresponding eigenvalue of A.

$\Rightarrow$ Solve by ~~…~~ rewriting a Homogeneous eqn:

$$A\vec{x} = \lambda\vec{x}$$

$$\Rightarrow A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$\Rightarrow \underbrace{(A - \lambda I)}_{B}\vec{x} = \vec{0}$$

$$\Rightarrow B\vec{x} = \vec{0}. \overset{\vec{x}\neq 0}{\iff} \det(B) = 0$$

Example 95, page 184

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

Find the eig'values eig'vectors s.t.

$$\det(A - \lambda I) = 0 .$$

Recall spring-mass-damper system. 2^{nd} order linear eqn.

$$mx'' + bx' + kx = 0.$$

Let $y = x'$. $\Rightarrow$ $y' = x'' = -\frac{b}{m}x' - \frac{k}{m}x$

$\Rightarrow$ $= -\frac{b}{m}y - \frac{k}{m}x.$

Write as system of 1^{st} order eqns
$$\begin{cases} x' = y \\ y' = -\frac{k}{m}x - \frac{b}{m}y \end{cases}$$

Let $\vec{X} = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}$

$$\Longrightarrow \vec{X}' = \begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\vec{X}' = A\vec{X}$$

Recall the characteristic eqn:

$$mr^2 + br + k = 0.$$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

Find the eigenvalues of

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \qquad A\vec{x} = \lambda\vec{x}.$$

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{b}{m} - \lambda \end{pmatrix}$$

$$= \lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} \overset{\text{want}}{=} 0.$$

Ex. 195, page 181 (Continued)

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$ ... Find e'values/e'vectors.

For a ~~general~~ A ($n \times n$), w/ e'values $\lambda_1, \lambda_2, \ldots, \lambda_n$?

1. What are the e'values of A^T?
2. What are the e'value/e'vectors of A^{-1}?
3. How does $\text{Trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}$ relate to the e'values: $\lambda_1 + \lambda_2 + \cdots + \lambda_n$?
4. How does the $\det(A)$ relate to the e'values?

Keep on Board...

$$④ \Longrightarrow \left(A^{-1} \Rightarrow \lambda_i \neq 0\right).$$

Ex 103, page 198

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

Find e'values / e'vectors.

$\det(A-\lambda I) = \lambda(\lambda-3) \implies \lambda = 0, 3.$

$\lambda = 3$

$$\vec{x} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$\lambda = 0$

$$\vec{x} = x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

A is invertible $\iff$ $(\det(A) \neq 0 \iff)$ $\lambda \neq 0$.

(because $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$)

Ex. 106 / page 204

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}.$$

$\lambda = 2+3i$. What must the other e'value be?

$\Longrightarrow$ A ($n \times n$) matrix w/ real entries.

If λ is complex, e'value ($\vec{x}$)

then $\bar{\lambda}$ is also e'values ($\bar{\vec{x}}$

Proof:

$$A\vec{x} = \lambda\vec{x}$$

$$\Rightarrow \underbrace{\bar{A}}\,\bar{\vec{x}} = \overline{\lambda\vec{x}} = \bar{\lambda}\cdot\bar{\vec{x}}$$

(has real entries) $A = \bar{A}$

$$\Rightarrow A\bar{\vec{x}} = \bar{\lambda}\,\bar{\vec{x}} \Rightarrow$$ $\bar{\lambda}$ e'value w/ $\bar{\vec{x}}$ e'vector.

Defective E'values

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)^2 = 0$$

$$\Rightarrow \lambda = 1 \quad \begin{pmatrix} \text{Algebraic} \\ \text{mult} = 2 \end{pmatrix}$$

$$\vec{X} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ldots$$

($\lambda = 1$ is called defective).

because no 2 linearly independent vectors.