Next generation matrix calculation

1 Model

Let S_i represent the proportion of susceptible individuals in age group i and I_i represent the proportion of infected individuals in age group $i = 1, \ldots, n$. Then, the model is given by:

$$\frac{dI_{s,i}}{dt} = \beta_a \sigma_{s,i} S_i \left(\sum_{j=1}^{N} C_{i,j} I_{a,j} \right) + \beta_s \sigma_{s,i} S_i \left(\sum_{j=1}^{N} C_{i,j} I_{s,j} \right) - \delta_s I_{s,i} \tag{1}$$

$$\frac{dI_{a,i}}{dt} = \beta_a \sigma_{a,i} S_i \left(\sum_{j=1}^{N} C_{i,j} I_{a,j} \right) + \beta_s \sigma_{a,i} S_i \left(\sum_{j=1}^{N} C_{i,j} I_{s,j} \right) - \delta_a I_{s,i} \tag{2}$$

2 NGM

In this case, we have (using rough notations):

$$\mathcal{F} = \begin{pmatrix} \beta_a \sigma_{s,i} S_i \left(\sum_{j=1}^{N} C_{i,j} I_{a,j} \right) + \beta_s \sigma_{s,i} S_i \left(\sum_{j=1}^{N} C_{i,j} I_{s,j} \right) \\ \beta_a \sigma_{a,i} S_i \left(\sum_{j=1}^{N} C_{i,j} I_{a,j} \right) + \beta_s \sigma_{a,i} S_i \left(\sum_{j=1}^{N} C_{i,j} I_{s,j} \right) \end{pmatrix} \tag{3}$$

and

$$\mathcal{V} = \begin{pmatrix} \gamma_s \\ \gamma_a \end{pmatrix} \tag{4}$$

Then, we can write F and V using a block matrix:

$$F = \begin{pmatrix} F_{s,s,i,j} & F_{s,a,i,j} \\ F_{a,s,i,j} & F_{a,a,i,j} \end{pmatrix} \tag{5}$$

where $F_{x,y,i,j}$ represents transmission from infection type y to x from age group j to i. Then, we have

$$F_{x,y,i,j} = \beta_y \sigma_{x,i} C_{i,j}. \tag{6}$$

Likewise,

$$V = \begin{pmatrix} V_{s,s,i,j} & V_{s,a,i,j} \\ V_{a,s,i,j} & V_{a,a,i,j} \end{pmatrix} \tag{7}$$

In this case, $V_{s,a,i,j}$ and $V_{a,s,i,j}$ are zero matrices and $V_{s,s,i,j}$ and $V_{a,a,i,j}$ are diagonal matrices whose entries are δ_s and δ_a, respectively. Then, we have

$$V^{-1} = \begin{pmatrix} 1/\delta_s & 0 \\ 0 & 1/\delta_a \end{pmatrix} \tag{8}$$

a $2n \times 2n$ diagonal matrix.

Then, the next generation matrix is given by:

$$FV^{-1} = \begin{pmatrix} \mathcal{R}_s \sigma_{s,i} C_{i,j} & \mathcal{R}_a \sigma_{s,i} C_{i,j} \\ \mathcal{R}_s \sigma_{a,i} C_{i,j} & \mathcal{R}_a \sigma_{a,i} C_{i,j} \end{pmatrix} \tag{9}$$

If we want to calculate the exp growth rate, we take the largest eigenvalue of the matrix $F - V$.