Sampling a GPMP Trajectory

17) $\theta(J) = \tilde{\mu}(J) + \Lambda(J)(\theta_i - \tilde{\mu}_i) + \Psi(J)(\theta_{i+1} - \tilde{\mu}_{i+1}), \quad t_i < J < t_{i+1}$

$\theta(J)$ is the state θ_J at time J

$\theta(J): \mathbb{R} \to \mathbb{R}^{D \times V}$ (D, V), presented as $(D \times V)$

$\tilde{\mu}(J)$ is the expected state $\tilde{\mu}_J$ at time J

$\dot{\mu}(J): \mathbb{R} \to \mathbb{R}^{D \times V}$ (D, V) represented as $(D \times V)$

8) $\tilde{\mu}(t) = \tilde{\mu}_t = \Phi(t, t_0)\mu_0 + \int_{t_0}^{t} \Phi(t, s)\, u(s)\, ds$

Assuming the external force $u(s) = 0$, $\tilde{\mu}(t) = \Phi(t, t_0)\mu_0$

$\Phi(t, s)$ is the state transition matrix from t_s to t_f

We will set $D = 3$ and $V = 2$:

$$\theta_i = \begin{bmatrix} x & y \\ \dot{x} & \dot{y} \\ \ddot{x} & \ddot{y} \end{bmatrix} \quad (D, V) \to \quad \theta_i = [x \;\; y \;\; \dots \;\; \ddot{x} \;\; \ddot{y}] \quad (D \times V)$$

59) $$\Phi(t, s) = \begin{bmatrix} 1 & (t-s) & \frac{(t-s)^2}{2} \\ 0 & 1 & (t-s) \\ 0 & 0 & 1 \end{bmatrix} \quad (D, D)$$

$\Lambda(J, t_i)$ measures the effect of θ_i on θ_J (Alpha)

17) $\Lambda(J, t_i) = \Phi(J, t_i) - \Psi(J)\,\Phi(t_{i+1}, t_i)$ $\qquad t_{i+1} = t_i + 1$

$(D, D) \qquad (D, D) - (D, D) \;\; (D, D)$

$\Psi(t, t_i)$ measures the effect of θ_{i+1} on θ_J (Beta)

17) $\Psi(J, t_i) = Q_{i,J}\; \Phi(t_{i+1}, J)^T\; Q_{i,i+1}^{-1}$

$(D, D) \qquad (D, D) \quad (D, D) \quad (D, D)$

15) $Q_{a,b}$ is the process noise covariance between times t_a, t_b

$Q_{a,b} = \int_{t_a}^{t_b} \Phi(t_b, s)\, F(s)\, Q_c\, F(s)^T\, \Phi(t_b, s)^T\, ds$

$(D, D) \qquad (D, D) \;\; (D, 1) \;\; (1) \;\; (1, D) \quad (D, D)$

$F(s)$ chooses what $w(t)$ affects, since we want it on acceleration:

$$F(s) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Q_c is a scalar representing the "strength" of random movement

The integration in $Q_{a,b}$ is unfortunate

We can reduce this equation using our definitions:

$$Q_{a,b} = \int_{t_a}^{t_b} \begin{bmatrix} 1 & (t_b-s) & \frac{(t_b-s)^2}{2} \\ 0 & 1 & (t_b-s) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} Q_c \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ (t_b-s) & 1 & 0 \\ \frac{(t_b-s)^2}{2} & (t_b-s) & 1 \end{bmatrix} ds$$

$$= Q_c \int_{t_a}^{t_b} \begin{bmatrix} 1 & (t_b-s) & \frac{(t_b-s)^2}{2} \\ 0 & 1 & (t_b-s) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ (t_b-s) & 1 & 0 \\ \frac{(t_b-s)^2}{2} & (t_b-s) & 1 \end{bmatrix} ds$$

$$= Q_c \int_{t_a}^{t_b} \begin{bmatrix} 0 & 0 & \frac{(t_b-s)^2}{2} \\ 0 & 0 & (t_b-s) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ (t_b-s) & 1 & 0 \\ \frac{(t_b-s)^2}{2} & (t_b-s) & 1 \end{bmatrix} ds$$

$$= Q_c \int_{t_a}^{t_b} \begin{bmatrix} \frac{(t_b-s)^4}{4} & \frac{(t_b-s)^3}{2} & \frac{(t_b-s)^2}{2} \\ \frac{(t_b-s)^3}{2} & (t_b-s)^2 & (t_b-s) \\ \frac{(t_b-s)^2}{2} & (t_b-s) & 1 \end{bmatrix} ds$$

$u = t_b - s$

$du = -ds$

$$= Q_c \int_{0}^{t_b-t_a} \begin{bmatrix} \frac{u^4}{4} & \frac{u^3}{2} & \frac{u^2}{2} \\ \frac{u^3}{2} & u^2 & u \\ \frac{u^2}{2} & u & 1 \end{bmatrix} du$$

$$= Q_c \begin{bmatrix} \frac{u^5}{20} & \frac{u^4}{8} & \frac{u^3}{6} \\ \frac{u^4}{8} & \frac{u^3}{3} & \frac{u^2}{2} \\ \frac{u^3}{6} & \frac{u^2}{2} & u \end{bmatrix} \Bigg|_{0}^{t_b-t_a}$$

$$\boxed{= Q_c \begin{bmatrix} \frac{(t_b-t_a)^5}{20} & \frac{(t_b-t_a)^4}{8} & \frac{(t_b-t_a)^3}{6} \\ \frac{(t_b-t_a)^4}{8} & \frac{(t_b-t_a)^3}{3} & \frac{(t_b-t_a)^2}{2} \\ \frac{(t_b-t_a)^3}{6} & \frac{(t_b-t_a)^2}{2} & (t_b-t_a) \end{bmatrix}}$$