Intuition about Vector Calculus

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Contents

1 Flux 2

2 Divergence incl. Divergence Theorem 4

3 Line Integral of scalar-valued functions 5

4 Introduction to a line integral of a vector field 6

5 Line integrals as circulation (Macroscopic) 8

6 The curl (Microscopic) 9

7 Gradient Theorem 10

8 Green's Theorem 11

9 Stoke's Theorem 12

10 Gauss Theorem 13

The content is mainly a summary of different websites(mainly http://mathinsight.org/ and http://betterexplained.com). Sometimes sentences are cited one-to-one, but I have refrained from making citations and a bibliography because this paper's goal is not to be published but to use privately.

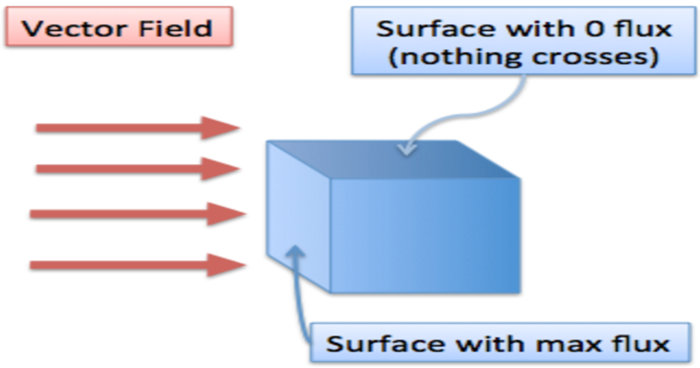
1 Flux

- Flux is the amount of "something"(a force, an electric field, potatoes...) passing through a surface.
- The **total flux** depends on:
  - **Strength of the field**: The source of the flux has a huge impact on the total flux. Doubling it's strenght will double the amount of potatoes you shoot through your surface.
  - **Size of the surface it passes through**: The bigger the surface, the more fits through.
  - **Their orientation**: When the surface faces the source(they are orthogonal), all of the potatoes can pass the surface. As the surface titls away from the field, less potatoes can enter the surface an the flux decreases.
- **Vector Field**: This is the source of the flux that exerts some force on the surface.
- **Surface**: This is the boundary the flus passes through. It can be a sphere, a plane or the top of a bucket. Notice that the top of a bucket is empty, but that we can measure the flux passing through this empty region.
- **Timing**: We measure the flux at a specific time. Because the flux can change, we have to freeze time and look at it at this moment.
- We say that we have *positive flux* when it leaves the closed surface and *negative flux* when it enters a closed surface.
- Now let's get to the math and derive a formula. We want to know how much of a vector field is passing through our surface, taking the magnitude, orientation, and size into account.

  Total flux = Field Strength · Surface Size · Orientation

  Because the vector field is not the same everywhere we will look at tiny little surface pieces a time and add them up.

  Total flux = (Field Strength · dS · Orientation) for every dS) = Integral(Field Strength · Orientation · dS)



We see that the flux is parallel with the top of the surface. But the left surface gets maximal flux as it is directed orthogonal to it.

- Mathematically we denote a surface with its normal vector which sticks out of the surface orthogonally. This means the normal vector of the left surface is parallel to the flux.

- As we are trying to measure how much the strenght of the vector field passes through the surface at different angles, you could rephrase it as how does the angle between the source and the surface affect the strenght of the flux on the surface. How could we mathematically denote that?
  Well that's easy, we use the dot product. It gives us a number between 0 and 1 that tells us what percentage of the field is passing through the surface.

- If we take together both aspects we get following new formula. Total flux = Integral(Vector Field Strength dot normal Vector)dS
  $= \int_S \overrightarrow{F} \cdot \overrightarrow{n} dS$

2 Divergence incl. Divergence Theorem

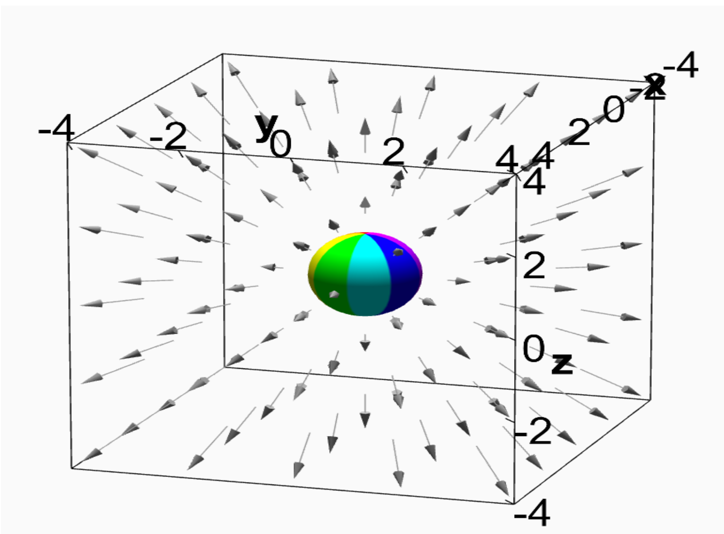
- Divergence is nothing else than "flux density" meaning the net flux per unit volume. It tells you how much flux is entering or leaving a specific point. You can also think about it as flux expansion(positive divergence) or flux contraction(negative divergence).

- Imagine you could talk to a specific point. If he saw flux entering he would tell you that ther is negative divergence. If he saw flux leaving he would say there is positive divergence. Also the bigger the flux densitivy(positive or negative), the stronger the flux source or sink. If the divergence is zero, it menas that there is no net flux change.

- Mathematically speaking:
  $Divergence = \frac{Flux}{Volume}$

  So if you imagine that you had a tiny little cube(in the space R^3) you'd have to calculate the flux change in all directions to get divergence:

  Total flux change = (field change in X direction) + (field change in Y direction) + (field change in Z direction)
  $\leftrightarrow Divergence = \lim_{Vol \to 0} \frac{Flux}{Vol}$
  $\leftrightarrow Divergence = \bigtriangledown(gradient) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$



Divergent vector field with embedded sphere in three-dimensional space. The vector field is expanding thus the divergence is positive.

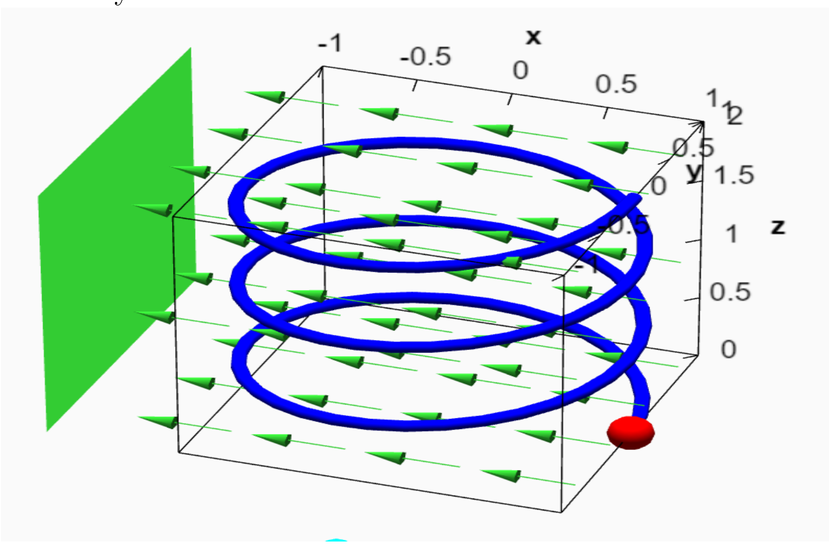
3 Line Integral of scalar-valued functions

- The idea is similar to the one of an Integral over a two dimensioal Surface. As analogy you can imagine to calculate the mass of a wire from its density.
- Assume you have a wire whose density is not constant over its length. We have a function $C(t)$ which describes a certain point on the wire. How can you calcuate it's mass from its density?
- We can segment the wire into lots of smaller segments and calculate their mass according to the specific densities. The length of the i-th segment is defined as: $||c(t_i) - c(t_i - 1)||$.
  The densitiy for a specific point is defined as: $f(c(t_i))$
- The mass of a segment is just the Linesegment $\cdot$ its density:
  $f(c(t_i)) \cdot ||c(t_i) - c(t_i - 1)||$
  To calculate the total mass of the whole wire you just have to sum up all the masses of the segments: $\sum_i^n f(c(t_i)) \cdot ||c(t_i) - c(t_i - 1)||$
- To turn this into an integral we define: $\Delta t_i = t_i - t_{i-1}$. Now multiply and divide each term by Δt_i and obtain the more complicated looking expression:
  $\sum_i^n f(c(t_i)) \cdot ||c(t_i) - c(t_{i-1})|| = \sum_i^n ||\frac{c(t_{i-1}+\Delta t_i) - c(t_{i-1})}{\Delta t_i}|| \cdot \Delta t_{i-1}$
  If you are confused by the counter just realize, that $c(t_{i-1} + \Delta t_i) - c(t_{i-1} = c(t_i) - c(t_i - 1)$
- You realize, that the term in the expression $||.||$ is actually the definition of the derivate off $c(t)$. The counter is nothing else than a function c(t) applied to an interval and the denominator is that exact interval. So if you apply it to a two-dimensional function it's nothing else than $\frac{\Delta y}{\Delta x}$ which defines the secant.

  For comparison here is the definition of the definition of a Secant: $\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$
- If we let $\Delta t_i \to 0$ and $n \to \infty$ the above Riemann sum converges to the integral: $\int_a^b f(c(t)) \cdot ||c'(t)||dt$ which is often denoted as $\int_c f ds$.

4 Introduction to a line integral of a vector field

- The only change to the chapter above is that now we want to integrate a vector valued curve function along a curve. They are usually represented by a vector field.



A very easy interpretation is to imagine the amount of work that a force field does on a particle as it moves along a curve.

- Imagine you have a Bead that is fixed to move along a helix defined as $c(t)$. The green rectangle is a magnet inducing a magnetic field $F(x, y, z)$(illustrated by the green arrows). The line integral of a vector field can be thought of, as work on the bead.

- When the bead changes its position along the function, the force exorted by the magnetic field, and thus also the work, changes.

Work is defined as $F \cdot s$ (force in direction of movement). For example, when the movement is 90 degrees from the direction of the force, the magnetic field does no work at all. So how can we get the direction of the bead?

We just use its direction given by the velocity $c'(t)$. We denote the unit vector in the direction of the movement as $T(t) = \frac{c'(t)||}{c'(t)||}$. If you think of it, this absolutely makes sense as the derivate of $c(t)$ defines teh tangent vector to the path.

- The component of force in direction of the movement is simply the force of the magnetic field at a specific point on the function $c(t)$ along the tangent of $c(t)$: $F(c(t)) \cdot T(t)$.(for future reference we will use$F \cdot T$)

  Per definition we know that if we take the product of force and distance we get work. So $F \cdot T$ actually denotes the work per unit length along the helix.
  To get the total work we need to ake the line integral of this scalar-valued function.
  Work $= \int_C F \cdot T ds$ (where C is the path along the function$c(t)$)

- To derive a final formula we go back to the first chapter where we looked at scalar-valued line integrals and take the end-formula that we derived.
  $\int_C f ds = \int_a^b f(c(t)) \cdot ||c'(t)||dt$
  If we replace f with our new equation for work $F \cdot T$ we get:

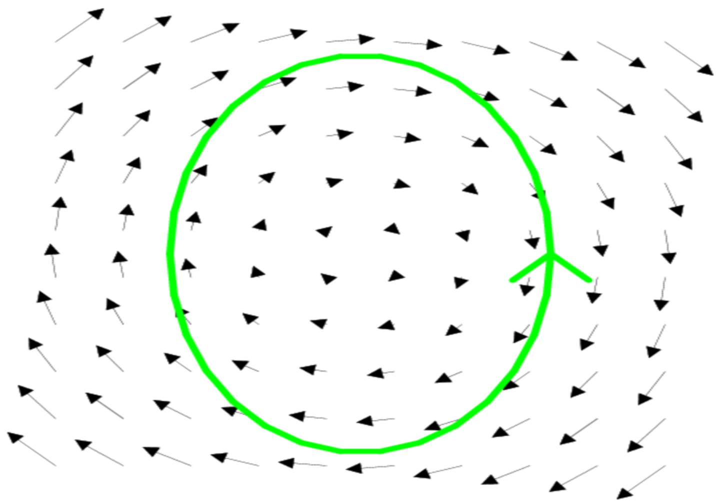
  $\int_C F \cdot T ds = \int_a^b F(c(t)) \cdot T(t) \cdot ||c'(t)||dt$

  We remember that we defined $T(t) = \frac{c'(t)}{||c'(t)}$. If we replace $T(t)$ we get:

  $\int_a^b F \cdot T ds = \int_a^b F(c(t)) \cdot \frac{c'(t)}{||c(t)||} \cdot ||c'(t)||dt$
  $= \int_a^b = F(c(t)) \cdot c'(t)dt = \int_C F ds.$
  (because we usually write $T ds$ as ds.)

5 Line integrals as circulation (Macroscopic)

- In the previous chapter we learned how a vector field F over an oriented curve C "adds up" the component of the vector field that is tangential to the curve. In simple words, this defines how much the vector field is aligned with the curve. But what do we do if the curve is closed?
- The line inegral then indicates how much the vector fields tends to circulate around the curve C.
  $\int_C F \cdot ds = \oint_C F \cdot ds$ (other notation) = circulation of F around C
- The circulation can be positive or negative according to the "circling"-direction of the vector field around the curve C. It is positive if it circles with the Curve and negative if it circles counter-clockwise to the Curve.



In this example the circulation of the vector field circulates clockwise in the opposite direction of the Curve.

6 The curl (Microscopic)

7 Gradient Theorem

8 Green's Theorem

9 Stoke's Theorem

10 Gauss Theorem