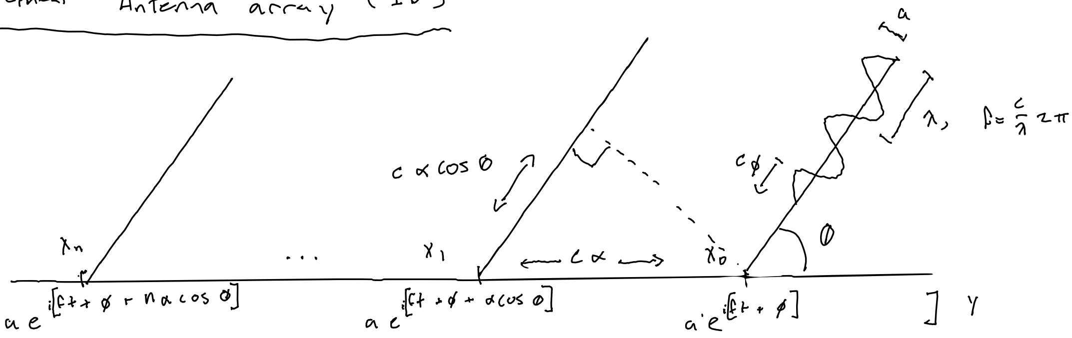
HW 1: MUSIC Algorithm

Linear Antenna array (1D)



gives model $Y = a e^{i[ft+\phi]} [1 \quad e^{if\alpha\cos\theta} \quad \dots \quad e^{ifn\alpha\cos\theta}]^T + \mathcal{N}_\sigma$

$$= X A(\theta) + \mathcal{N}_\sigma$$

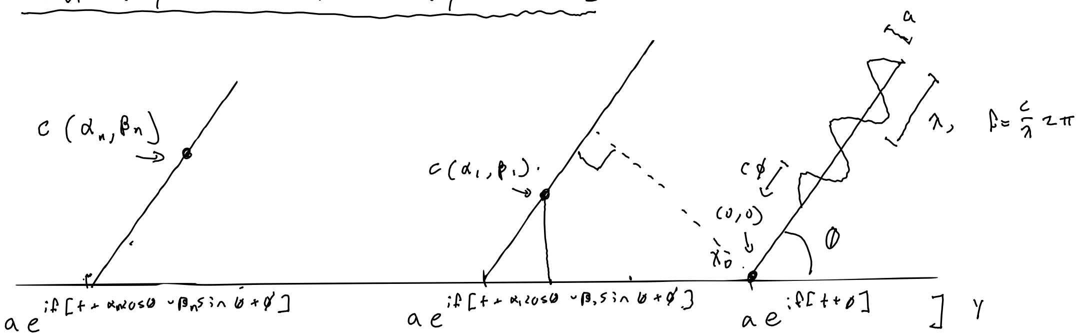
for one signal, and model $\vec{Y} = [A(\theta_1) \dots A(\theta_d)]\vec{x} + \vec{n}_\sigma = A\vec{x} + \vec{n}_\sigma$ for d signals.

Note: $\vec{y}, \vec{n} \in \mathbb{C}^{n\times 1}$, $A \in \mathbb{C}^{n\times d}$, $\vec{x} \in \mathbb{C}^{d\times 1}$

The autocorrelation matrix $R = \vec{y}^T \vec{y} = \vec{x}^T A^T A \vec{x} + I\sigma^2$ is full rank, and has signal subspace dim $= d' \leq d$. Assume the vectors R_n corresponding to the n-d smallest eigenvalues span the noise subspace (1).

The MUSIC algorithm approximates the MLE of θ (2) by solving $\arg\max_{\theta} (A(\theta)^H R_n R_n^H A(\theta))^{-1}$

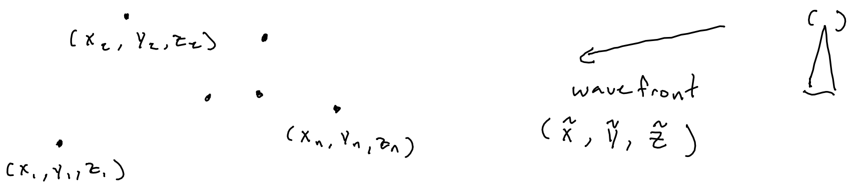
Arbitrary antenna array (2D)



Redefine $A(\theta) = [e^{if[\alpha_n \cos\theta - \beta_n \sin\theta]}]_n$ and leave the rest of the MUSIC algorithm unchanged. This works so long as antenna spacing $[\alpha]_n$ is chosen s.t.

$$A(\theta_1) = A(\theta_2) \iff \theta_1 = \theta_2 \qquad \text{(Nyquist works)}$$

Arbitrary antenna array (3D)



For this DOA problem, consider a projection of each axis onto zero (e.g. $y = 0$) then use the arbitrary 2D antenna array MUSIC algorithm to approx the ratio of the non-zero dimensions (e.g. $\tilde{x}/\tilde{z} = \tan(\theta_{y=0})$). Repeating for each axis gives a simple system of 3 eqs and 3 unknowns, and is easily solvable for $(\tilde{x}, \tilde{y}, \tilde{z})$.

Errata

(1) Noise / Signal Subspaces and Eigendecomposition

$$\frac{1}{N}\sum_n^N A(\theta_1)_n \overline{A(\theta_2)_n} \approx \frac{1}{N}\int_0^N e^{if(\theta_1-\theta_2)x}\,dx \to \delta_{\theta_1}^{\theta_2} \text{ as } N\to\infty,$$

Then $(Ax)(Ax)^H$ is a diagonal matrix and by uniqueness of eigenvalues/vectors, with small noise the largest d eigenvectors will be $= A(\theta_d)$, and have eigenvalues equal to the amplitudes of each signal.

Therefore the small eigenvalues correspond to the noise subspace.

Another way to show this is that for any orthogonal basis, the noise component decomposes the same way, so that

$$E\left[\left\|\sum_j^d A(\theta_j)e_i + \sum_j^N \mathcal{N} c_j e_i\right\|^2\right] = \sum_j^d \|A(\theta_j)e_i\|^2 + \sigma^2 \geq \sigma^2$$

and in expectation signal components are largest. Since we have noise and are sampling in a finite frame, however, the eigendecomposition will not be "perfect" and $|\{\lambda_i > \sigma\}| \geq d$, and you will have to use a subspace of the noise subspace.

(2) MLE vs. Subspace Methods.

A maximum likelihood estimate seems like the most straightforward, and is indeed the best, way of estimating the DOA. The problem is that this is a high dimensional, non-convex optimization problem.

Convexity: $$LE = \underset{x,\theta,\sigma}{\arg\min} \prod^{S,N} \left(\| Y_{s,n} - \sum^{d} A_{\theta_d,n} x_d \|^2 - \sigma^2 \right)$$

consider an example with $S=1$ sample, $N=2$ recievers.

$$LLE = \log\left((Y_1 - X_1 - X_2)\overline{(Y_1 - X_1 - X_2)} - \sigma^2 \right)$$
$$+ \log\left((Y_2 - X_1 e^{i\theta} - X_2 e^{i\theta})\overline{(Y_2 - X_1 e^{i\theta} - X_2 e^{i\theta})} - \sigma^2 \right)$$

$$\frac{\partial LLE}{\partial \theta} = \frac{2\cos\theta\,(Y_2^r (X_1^r + X_2^r) - Y_2^i (X_1^i + X_2^i))}{Y_2 \overline{Y_2} + 2\cos\theta\,(Y_2^r (X_1^r + X_2^r) - Y_2^i (X_1^i + X_2^i)) + (X_1 + X_2)\overline{(X_1 + X_2)} - \sigma^2} \approx \frac{\cos\theta\,\alpha}{\cos\theta\,\alpha + \beta}$$

$$\frac{\partial^2 LLE}{\partial \theta^2} = \frac{-2\sin\theta\cos\theta\,\alpha^2 - \sin\theta\,\alpha\beta}{(\cos\theta\,\alpha + \beta)^2}$$ $\leftarrow$ is not strictly positive $\forall$ possible α, β, so is nonconvex.

Dimensionality: $$\underbrace{d}_{\theta} + \underbrace{2d}_{x \text{ (amplitude + phase)}} + \underbrace{1}_{\sigma}$$

Because the MLE problem is computationally hard to solve, we use heuristic subspace methods either instead of, or to give a good initial estimate for the MLE problem.

③ $A(\theta) = \sum_i^d a_i(\theta)\, e_i + \sum_i^{N-d} a_{d+i}(\theta)\, e_{d+i}$ $\qquad \sum_i^N a_i(\theta)^2 = 1$

$\| R_s A(\theta) \|^2 = \sum_i^d a_i(\theta)^2 \lambda_i^2$

$\| R_n A(\theta) \|^2 = \sigma^2 \sum_i^{N-d} a_{d+i}^2(\theta) = \sigma^2 \left(1 - \sum_i^N a_i(\theta)^2\right)$

from this, you can conclude that signal subspace and noise subspace methods are equivalent when $\lambda_1 = \dots = \lambda_d$. This can be an artificial condition (e.g. ignore eigenvals) or natural (e.g. if $d = 1$).

When $\lambda_i \neq \lambda_j$, $i, j \leq d$ why is noise subspace method preferred?

My best idea is that it may be easier to choose "good" noise eigenvectors than "good" signal ones, especially if $N \gg d$. In this case, while all signal eigenvectors will have some noise component in them, it should be possible to identify a subset of the noise eigenvectors with no signal component.