Orthogonal Time Frequency Space (OTFS)

CSE870

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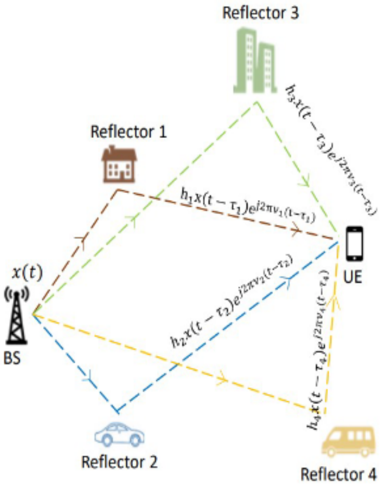
Table of Contents

Motivation: Delay-only model

System Model:

- Transmitted signal x
- Received signal y
- Paths parameterized by gain h and delay τ

$$y(t) = \sum_i h_i x(t - \tau_i)$$



[1]

[1]Figure 9: https://arxiv.org/pdf/2302.08696.pdf

Motivation: Delay-only model

Paths with distinct delays can be non-ambiguously localized in time domain using pulse

$$x(t) = \delta(t)e^{\mathrm{i}[2\pi f_0 t+\phi]}$$

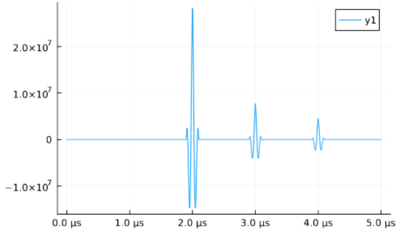
$$y(t) = \sum_i h_i\delta(t-\tau_i)e^{\mathrm{i}\phi}$$

Channel parameters can be recovered from received signal, so channel is predictable.

```
# nascent delta
η(t,ϵ)=max(1-abs(ustrip(t))/ϵ, 0)/ϵ
# signal
x(t) = η(t,1e-7) * exp(2im*π*fz*t + ϕ)

# parameters
τ = [2μs 2μs 3μs 4μs;]
h = [1/√2 1/√3 1/√8 1/√24;]

y(t) = sum(h .* x.(t.-τ), dims=2)
plot_waveform(y(t), t)
```



[1]

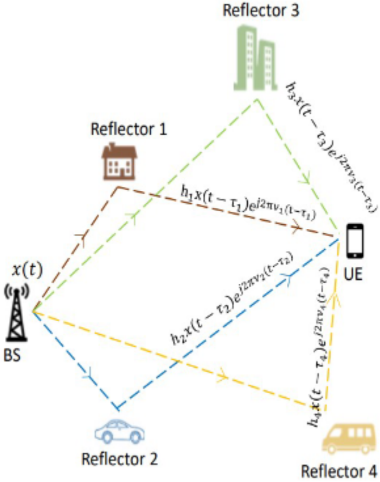
[1] https://github.com/JeremyArsenault/cse910/blob/main/OTFS/OFTS_presentation.ipynb

Motivation: Doppler-only model

System Model:

- Transmitted signal x
- Received signal y
- Paths parameterized by gain h and doppler ν

$$y(t) = \sum_i h_i x(v_i t)$$



[1]

[1] Figure 9: https://arxiv.org/pdf/2302.08696.pdf

Motivation: Doppler-only model

Paths with distinct dopplers can be non-ambiguously localized in frequency domain using sinusoid

$$X(f) = \delta(f_0 - f)e^{\mathrm{i}[\phi]}$$

$$x(t) = e^{\mathrm{i}[2\pi f_0 t + \phi]}$$

$$Y(f) = \sum_i h_i \delta(f_0 \nu_i - f)e^{\mathrm{i}[\phi]}$$

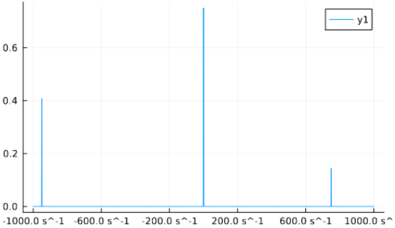
$$y(t) = \sum_i h_i e^{\mathrm{i}[2\pi f_0 \nu_i t + \phi]}$$

Channel parameters can be recovered from received signal, so channel is predictable.

```
# signal
X(f) = η(f,1) * exp(1im*ϕ)

# model parameters
ν = [0Hz -950Hz 0Hz 750Hz;]
h = [1/√2 1/√3 1/√8 1/√24;]

Y(f) = sum(h .* X.(ν.-f), dims=2)
plot_waveform(Y(f), f)
```



[1]

[1] https://github.com/JeremyArsenault/cse910/blob/main/OTFS/OFTS_presentation.ipynb

Motivation: Orthogonality

Signals which are orthogonal along different paths have predictable channel interactions.

TD pulse is orthogonal in delay only.

FD pulse is orthogonal in doppler only.

Can we design a signal that is orthogonal in delay and doppler?

Zak-OTFS

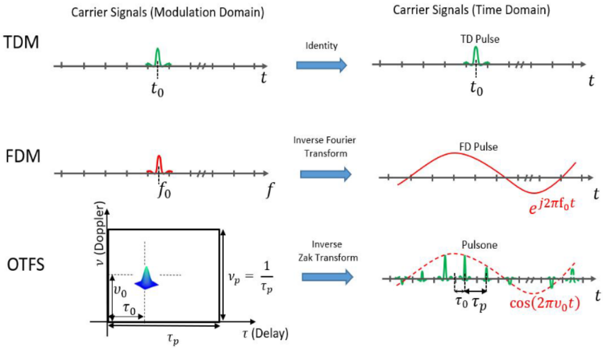


Fig. 2. Information carriers for TDM/FDM/OTFS in modulation domain and time domain. Traditional TDM and FDM carriers are narrow pulses in TD and FD but spread in FD and TD respectively, manifesting the fundamental obstruction for TF localization. In contrast, the OTFS carrier is a quasi-periodic pulse in the DD domain, viewed as “effectively” localized jointly in time and frequency.

[1]

[1] Figure 2: https://arxiv.org/pdf/2302.08696.pdf

Zak-OTFS: Zak Transform

Zak Transform (for delay period τ_p and doppler period $\nu_p = \frac{1}{\tau_p}$):

$$\mathcal{Z}_t(x(t)) = \sqrt{\tau_p} \sum_{k=-\infty}^{\infty} x(\tau + k\tau_p) e^{-\mathrm{i}[2\pi\nu k\tau_p]}$$

Inverse time Zak Transform:

$$\mathcal{Z}_t^{-1}(x_{dd}(\tau,\nu)) = \sqrt{\tau_p} \int_0^{\nu_p} x_{dd}(t,\nu)\, d\nu$$

Inverse frequency Zak Transform:

$$\mathcal{Z}_f^{-1}(x_{dd}(\tau,\nu)) = \sqrt{\nu_p} \int_0^{\tau_p} x_{dd}(\tau,f) e^{-\mathrm{i}[2\pi f\tau]}\, d\tau$$

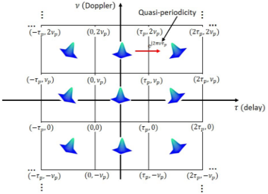


Fig. 3. A DD domain pulse is localized only within the fundamental DD domain period $\mathcal{D}_0$, as it repeats infinitely many times in a quasi-periodic fashion.

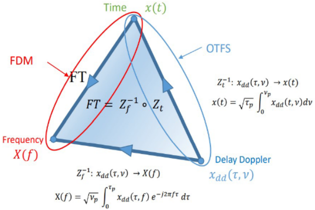


Fig. 8. Three different basic signal realizations, TD, FD and DD domains. Signal representations in these domains are related through transforms. The well known Fourier transform is in fact a composition of the Zak transform $\mathcal{Z}_t$ and the inverse Zak transform $\mathcal{Z}_f^{-1}$ from DD domain to FD.

[1]

[1] Figure 3,8: https://arxiv.org/pdf/2302.08696.pdf

Zak-OTFS: DD pulse

On $[0, \tau_p) \times [0, \nu_p)$, we want $x_{dd}^{\tau_0,\nu_0,\phi}(\tau, \nu) = \delta(\tau - \tau_0)\delta(\nu - \nu_0)e^{\mathrm{i}[\phi]}$

By $\mathcal{Z}_t$ for $n, m \in \mathbb{Z}$ (quasi-periodicity)

$$x_{dd}(\tau + n\tau_p, \nu + m\nu_p) = x_{dd}(\tau, \nu)e^{\mathrm{i}[2\pi\nu n\tau_p]}$$

Therefore

$$x_{dd}^{\tau_0,\nu_0,\phi}(\tau, \nu) = \delta(\sin\frac{\pi}{\tau_p}(\tau - \tau_0))\delta(\sin\frac{\pi}{\nu_p}(\nu - \nu_0))e^{\mathrm{i}[2\pi\nu_0(\tau-\tau_0)+\phi]}$$

$$x^{\tau_0,\nu_0,\phi}(t) = \mathcal{Z}_t^{-1}(x_{dd}^{\tau_0,\nu_0,\phi}(\tau, \nu)) = \sqrt{\tau_p}\delta(\sin\frac{\pi}{\tau_p}(t - \tau_0))e^{\mathrm{i}[2\pi\nu_0(t-\tau_0)+\phi]}$$

Zak-OTFS: Path orthogonality of DD pulse

System Model:

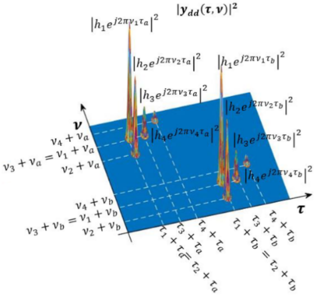
$$y(t) = \sum_i h_i x^{\tau_0,\nu_0,\phi}(\nu_i t + \tau_i)$$

Received Signal in DD (on $[0, \tau_p) \times [0, \nu_p)$):

$$\mathcal{Z}_t(y(t)) = \mathcal{Z}_t(\sum_i h_i x^{\tau_0,\nu_0,\phi}(\nu_i t + \tau_i))$$

$$= \sum_i h_i \mathcal{Z}_t(x^{\tau_0,\nu_0,\phi}(\nu_i t + \tau_i))$$

$$= \sum_i h_i x_{dd}^{(\tau_0-\tau_i),\nu_0,\phi}(\tau, \nu_i \nu) = \sum_i h_i x_{dd}^{\frac{\tau_0-\tau_i}{\nu_i},\nu_0\nu_i,\phi}(\tau, \nu)$$



[1]

Last equality is provided $\frac{\tau_0-\tau_1}{\tau_p} \geqslant \nu_i - 1$
(constellation shrink due to doppler does not cause ambiguities on co-domain)

[1] Figure 10: https://arxiv.org/pdf/2302.08696.pdf

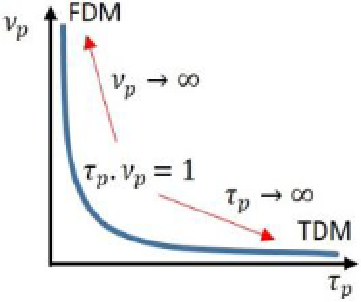
Zak-OTFS: OTFS as interpolation between TD and FD

OTFS pulsone is both TD and FD pulse

$$\tau_p \to \infty \quad \nu_p \to 0 \quad x(t) \to \delta(t)$$

$$\tau_p \to 0 \quad \nu_p \to \infty \quad x(t) \to e^{it}$$

Parameters $\tau_p, \ \nu_p$ should be chosen based on system delay and doppler spread



[1]

[1] Figure 4: https://arxiv.org/pdf/2302.08696.pdf

Implications for Radar Sensing

Looking back to 1953, only 5 years after Claude Shannon created information theory, Philip Woodward described how to think of radar in information theoretic terms. He suggested that we view the radar scene as an unknown operator parametrized by delay and Doppler, and that we view radar waveforms as questions that we ask the operator. [1]

[1] https://arxiv.org/pdf/2302.08696.pdf

Implications for Radar Sensing: Predictability

The radar ambiguity function is essentially the likelihood function of different environmental parameters.

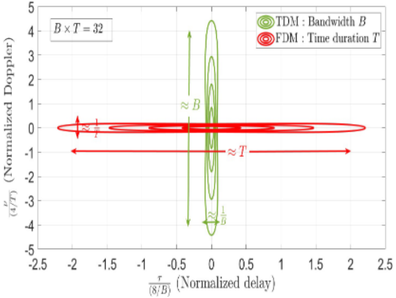


Fig. 19. Squared magnitude $|A_{s,s}(\tau,\nu)|^2$ of the ambiguity functions for TDM and FDM. The TDM carrier waveform is not able to separate targets in Doppler, and the FDM carrier waveform is not able to separate targets in delay.

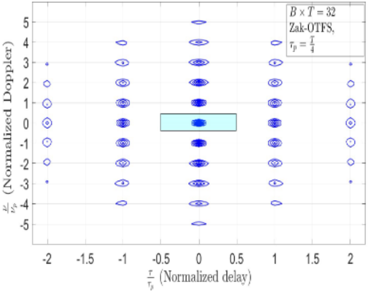


Fig. 21. Plot of the ambiguity function (squared magnitude) for the Zak-OTFS carrier waveform. Simultaneous delay and Doppler resolvability can be achieved. Unambiguous target estimation is achievable in the crystalline regime.

[1]

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[1]Figure 19,21: https://arxiv.org/pdf/2302.08705.pdf

Implications for Radar Sensing: Non-fading

When the received signal power depends on the parameters of the transmitted signal in ways that are difficult to predict, the signal is fading.

TD pulse with doppler-spread paths:

$$y(t_0) = e^{\phi} \sum h_i e^{\mathrm{i}[2\pi \nu_i t_0]}$$

FD pulse with delay-spread paths:

$$Y(f_0) = e^{\phi} \sum h_i e^{-\mathrm{i}[2\pi \tau_i f_0]}$$

DD pulsone with doubly-spread paths:

$$y_{dd}(\tau_i, \nu_i) = e^{\mathrm{i}[2\pi \nu_0 (\tau_i - \tau_0) + \phi]}$$

OTFS is non-fading

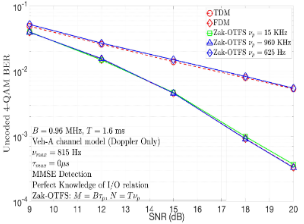

Fig. 9. BER performance of Zak-OTFS, TDM, and FDM, on a Doppler-only Veh-A channel. The I/O relation for Zak-OTFS in the crystalline regime (ν_p = 15 KHz) is non-fading, hence BER performance matches that of FDM. Time selective fading degrades BER performance of TDM.

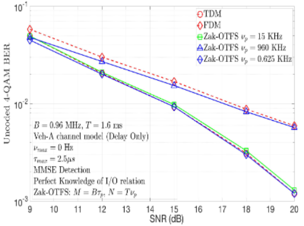

Fig. 8. BER performance of Zak-OTFS, TDM, and FDM on a delay-only Veh-A channel. The I/O relation for Zak-OTFS in the crystalline regime (ν_p = 15 KHz) is non-fading, hence BER performance matches that of TDM. Frequency selective fading degrades the BER performance of FDM.

[1] Figure 8,9: https://arxiv.org/pdf/2302.08705.pdf

Discussion: Key points

- Path estimation in radar sensing using DD modulation is less ambiguous than with TD, FD modulation in doubly-spread environment (predictability)
- Channel estimation in DD modulation is theoretically superior to TD, FD modulation in doubly-spread environment (non-fading)
- Performance of DD modulation in Zak-OTFS depends on crystallization condition (can't have simultaneously large delay spread and large doppler spread)