Mathematics HL yr1
Unit 3: Optimization (derivative apps)

Topics

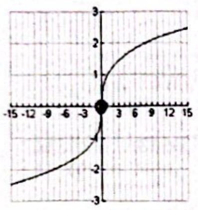
Extrema of a function
MVT for Derivatives
1^{st} and 2^{nd} derivative tests
Optimization
Limits & Curve Sketching

* Ignore this Calender *

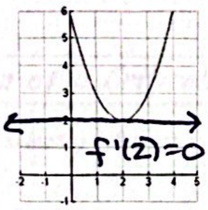
			Sept 14	15
			Extrema MVT	1^{st} & 2^{nd} Derivative Tests
18	19	20	21	22
Optimization	Limits/Curve Sketching	ICA1	ICA2	Quiz
25	26	27	28	29
	Unit 3 Test			

Definition of a Critical Number

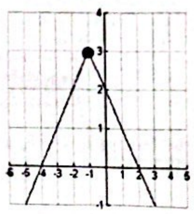
Critical Number: A defined point on $f(x)$ where $f'(x)$ is undefined or equals zero. $f'(x)=0$



$f(x)=\sqrt[3]{x}$



$f(x)=(x-2)^2+2$



$f(x)=|x+1|+3$

$f(x)=x^{1/3}$

$f'(x)=\frac{1}{3}x^{-\frac{2}{3}}$

$=\frac{1}{3x^{2/3}}$

$f'(x)\neq 0$

Critical number of $x=0$.

$f'(x)=2(x-2)\cdot 1$

$=2(x-2)$

$2(x-2)=0$

Critical number @ $x=2$.

Critical number at $x=-1$

$f(x)=\pm(x+1)+3$

$f(x)=\begin{cases}(x+1)+3, & x<-1\\ -(x+1)+3, & x\geq -1\end{cases}$

$f(x)=\begin{cases}x+4, & x<-1\\ -x+2, & x\geq -1\end{cases}$

$f'(x)=\begin{cases}1, & x<-1\\ -1, & x\geq -1\end{cases}$

$f'(x)\neq -1$ (und),

$\therefore$ critical point.

Absolute & Relative Extrema

$[-3,5]$

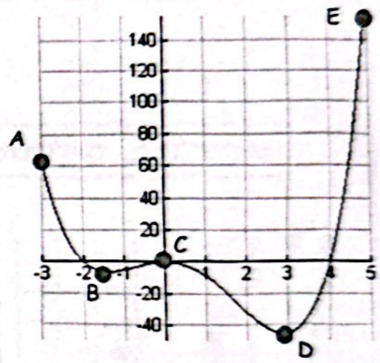
Absolute Minimum: D

Absolute Maximum: E

$(-3,5)$

Relative Minimum: B, D

Relative Maximum: C



* Absolute extrema can be endpoints or relative extrema (critical numbers)

* "Candidates" for absolute extrema on a closed interval

"Candidates Test"

Guidelines for Finding Absolute Extrema on a Closed Interval

Candidates

1. Find the critical numbers of f in (a, b).
2. Evaluate f at each critical number in (a, b)
3. Evaluate f at each endpoint in $[a, b]$
4. The least value is the minimum. The greatest is the maximum.

Find the absolute extrema of $f(x) = x^3 - 3x^2 - x + 3$ on the interval $[-1.5, 2.5]$
"abs min & abs max"

$f'(x) = 3x^2 - 6x - 1 = 0$

$x = \dfrac{6 \pm \sqrt{36 - 4(3)(-1)}}{2(3)}$

$= \dfrac{6 \pm \sqrt{48}}{6}$

$= \dfrac{6 \pm 4\sqrt{3}}{6} = \dfrac{3 \pm 2\sqrt{3}}{3} \approx \underbrace{-.15 \text{ and } 1.15}_{\text{w/i the domain}}$

X	f(x)
-1.5	-5.625
$\frac{3-2\sqrt{3}}{3}$	3.079
$\frac{3+2\sqrt{3}}{3}$	-3.079
2.5	-2.625

Absolute min of -5.625 @ $x = -1.5$.

Absolute max of 3.079 @ $x = \frac{3-2\sqrt{3}}{3}$.

Find the absolute extrema of $f(x) = 4\cos\left(\frac{1}{2}x\right) + 1$ over the interval $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$.

$f'(x) = -4\sin\left(\frac{1}{2}x\right) \cdot \frac{1}{2}$

$= -2\sin\left(\frac{1}{2}x\right) = 0$

$\sin\left(\frac{1}{2}x\right) = 0$

$\frac{1}{2}x = \pi k$

$x = 2\pi k$

$x = \ldots, -2\pi, 0, \underbrace{2\pi}_{\text{w/i the domain}}, 4\pi, \ldots$

$x = 2\pi$ on $[\pi/2, 5\pi/2]$

X	f(x)
$\frac{\pi}{2}$	$4\cos\left(\frac{\pi}{4}\right) + 1 = 4\left(\frac{\sqrt{2}}{2}\right) + 1 = 2\sqrt{2} + 1$
2π	$4\cos(\pi) + 1 = 4(-1) + 1 = -3$
$\frac{5\pi}{2}$	$4\cos\left(\frac{5\pi}{4}\right) + 1 = 4\left(-\frac{\sqrt{2}}{2}\right) + 1 = -2\sqrt{2} + 1 \approx -1.8$

Absolute min of -3 @ $x = 2\pi$.

Absolute max of $2\sqrt{2} + 1$ @ $x = \frac{\pi}{2}$.

Find the absolute extrema of $f(x) = x^3 - \underset{\frac{11}{2}}{\cancel{5}}x^2 - 4x + 20$ on $1 \le x \le 6$.

$f'(x) = 3x^2 - 11x - 4$

$= (3x+1)(x-4) = 0$

$x = -\frac{1}{3}, 4$

$-\frac{1}{3}$ is not on the interval of $1 \le x \le 6$, so it can not be a candidate.

4: w/i the domain

X	f(x)
1	11.5
4	-20
6	14

Absolute min of -20 @ $x = 4$.

Absolute max of 14 @ $x = 6$.

Mean Value Theorem

If f is continuous on [a, b] & differentiable on (a, b) then there exists a number c in (a, b) such that

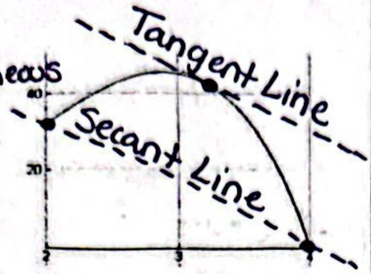
$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(c)}$$

Average Rate of Change — Instantaneous Rate of Change

The "mean" in the MVT refers to the mean rate change of f in the interval [a, b].

"Average"

Result is parallel slopes



1) Find the values guaranteed by the MVT. AROC = IROC

$f(x) = x^3 + 1, [0,2]$

$$m_{sec} = \frac{f(2)-f(0)}{2-0} = \frac{(2^3+1)-(0^3+1)}{2} = \frac{9-1}{2} = 4$$

$m_{tan} = f'(x) = 3x^2$

$4 = 3x^2$

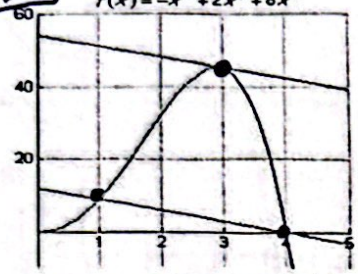
$x = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$

$x = \pm\frac{2}{\sqrt{3}}$ is guarenteed by MVT.

2) Find all values of c in the interval [1,4] such that $f'(c) = \frac{f(4)-f(1)}{4-1}$

CALC

$f(x) = -x^4 + 2x^3 + 8x^2$



$\frac{f(4)-f(1)}{4-1} = \frac{0-9}{3} = -3$

$-3 = -4x^3 + 6x^2 + 16x$

$0 = -4x^3 + 6x^2 + 6x + 3$

* use CALC for roots.

The are three solutions, but only x=2.944 is w/i the interval, ∴ it is the only value guarenteed by MVT.

Rolle's Theorem

Critical Points

-specific case of the MVT where the avg rate of change is zero

Let f be cont. on the closed interval [a, b] and diff. on the open interval (a, b). If

f(a) = f(b) → secant line is horizontal

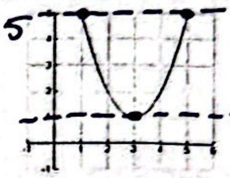
Then there is at least one number c in (a, b) such that $f'(x) = 0$.

$f(1) = 5$, $f(5) = 5$

$f(1) = f(5)$

and

$f'(3) = 0$



x=3 is guarenteed by Rolle's Thm.

3) Can MVT be applied?

$f(x) = \frac{x^2-16}{x+2}, [-3,6]$

No, b/c f(x) is NOT cont. on [-3,6].

4) Can MVT be applied?

$f(x) = x^{\frac{1}{3}}, [-3,3]$

No, b/c f(x) has a vertical tangent line at x=0.

5) Can Rolle's be applied?

$f(x) = \frac{x^2}{x^2-1}, [-3,3]$

No, b/c f(x) has VA's at x=±1.

6) Can Rolle's be applied?

$f(x) = \frac{x^2}{x^2+1}, [-3,3]$

Yes.

7) Let $f(x) = x^4 - 6x^2$. Find all values of c in the interval (-3,3) guaranteed by Rolle's Thm. $f'(x) = 0$

$f(-3) = (-3)^4 - 6(-3)^2 = 81 - 54 = 27$ ✓

$f(3) = 3^4 - 6(3)^2 = 81 - 54 = 27$

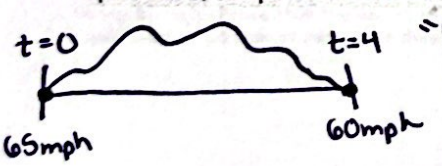
$f'(x) = 4x^3 - 12x$

$= 4x(x^2-3)$

$= 4x(x-\sqrt{3})(x+\sqrt{3})$

$x = 0, \pm\sqrt{3}$ are guarenteed by Rolle's Thm.

8) Two stationary patrol cars equipped with radar are 5 miles apart on a highway. As a car passes the 1st patrol car its speed is clocked at 65 mph and 4 minutes later the 2nd patrol car clocks 60 mph. Prove that the car must have exceeded the speed limit of 65 mph at some time during the 4 minutes.



"Average Rate of Change"

$$\text{Average Velocity} = \frac{\Delta\text{ distance}}{\Delta\text{ time}} = \frac{5\text{ miles}}{4\text{ min}}$$

Need mph

$$\frac{5\text{ miles}}{\cancel{4}_{1}\text{ min}} \cdot \frac{\cancel{60}^{15}\text{ min}}{1\text{ hour}} = 75\text{ mph}$$

The car went 75mph at least once.

Increasing/Decreasing Intervals & The First Derivative

If $f'(2) > 0$ then f(x) is inc. at x=2.

If $f'(x) > 0$ on [-3,5] then f(x) is in on [-3,5].

If $f''(-5) < 0$ then f(x) is con. down at x=-5.

If $f''(x) < 0$ on $\left[\frac{\pi}{2}, 3\pi\right]$ then f(x) is con. down on $\left[\frac{\pi}{2}, 3\pi\right]$.

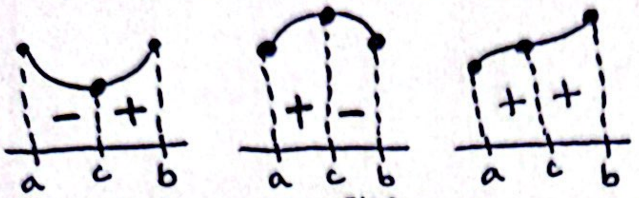
First Derivative Test

Let c be a critical number of a function f that is cont. on an open interval I containing c. If f is diff. on the interval, except possibly at c, then f(c) can be classified as follows.

If f'(x) = 0 or is undefined & changes from...

pos. to neg. at c, then f has a rel. max. at (c, f(c)).

neg. to pos. at c, then f has a rel. min. at (c, f(c)).



$f'<0$ $f'>0$ — Rel. max.

$f'>0$ $f'<0$ — Rel. min.

Neither rel. min. or rel. max.

10) Given $f(x) = x^3 - 6x^2 + 15$...

a) Find the critical numbers of f (if any).
b) Find the open interval(s) on which the function is increasing or decreasing.
c) Apply the 1st Derivative Test to identify all relative extrema.

a) $f'(x) = 3x^2 - 12x$

f'(x) = 0 | f'(x) is und

$3x^2 - 12x = 0$ | ∅

$3x(x-4) = 0$

$x = 0, 4$

f' −− +− ++

\+ − +

0 4

f inc. dec. inc.

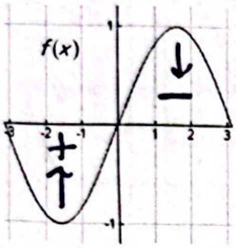
max min

$f(0) = 15$

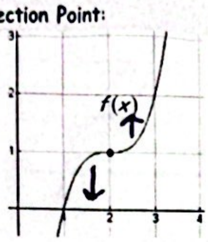
$f(4) = -17$

Relative maximum at (0,15) b/c f'(x) changes from pos. to neg. Relative minimum at (4,−17) b/c f'(x) changes from neg. to pos.

Concavity & Inflection Points



Inflection Point:



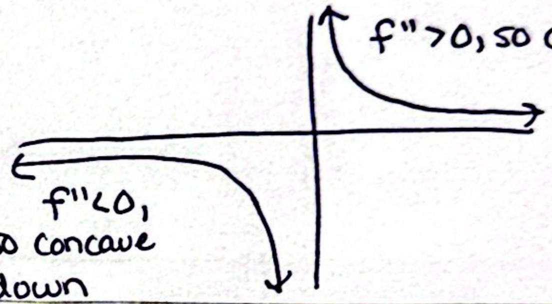
Slope is inc., so f(x) is con. up | Slope is dec., so f(x) is con. down

When a function changes concavity, a point of inflection occurs.

Concavity of f(x)

If $f''>0$ on [a, b] then the graph of f is con. up

If $f''<0$ on [a, b] then the graph of f is con. down.



f''<0, so concave down

Finding Points of Inflection

A sign change of f" occurs when f"(c) = 0 or f"(c) is undefined. This results in an inflection pt. at x = c on the graph of f(x) as long as f(c) is defined.

Similar to critical point for f'(x)

12) Let $f(x) = x^4 - 6x^2$. Find the points of inflection.

$f'(x) = 4x^3 - 12x$

$f''(x) = 12x^2 - 12$

f''(x) = 0 | f''(x) is und

$12(x^2 - 1) = 0$ | ∅

$12(x-1)(x+1) = 0$

$x = \pm 1$

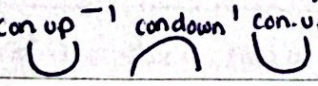
Points of inflection at $x = \pm 1$ b/c f''(x) changes signs at $x = \pm 1$.

f'' ++ +− ++

\+ − +

−1 1

f con up con down con. up



Second Derivative Test

uses only critical points

- another way of determining types of relative extrema other than the 1st derivative test

Given f'(c) = 0 or is undefined...

If $f''(c) > 0$ then f(c) is a rel. min.

If $f''(c) < 0$ then f(c) is a rel. max.

If $f''(c) = 0$, inconclusive → use 1st der. test (#10)

13) Determine the type of relative extrema on f(x) given the following info.

$f(-2) = 10$	$f'(-2) = 0$	$f''(x) = x^2 - 6x$
$f(5) = -4$	$f'(5) = 0$	
$f(7) = 3$	$f'(7) = -5$	

$f''(-2) = (-2)^2 - 6(-2) = 4 + 12 > 0$, ∴ rel. min.

$f''(5) = 5^2 - 6(5) = 25 - 30 < 0$, ∴ rel. max.

f(7) is neither b/c $f'(7) \neq 0$.

Ex: Given the function $f(x) = x^3 - 11x^2 - 16x + 1$, find all relative extrema using the 2nd derivative test.

$f'(x) = 3x^2 - 22x - 16$ $f''(x) = 6x - 22$

f'(x) = 0

$3x^2 - 22x - 16 = 0$

$(3x+2)(x-8) = 0$

$x = -2/3, 8$

$f''(8) = 6(8) - 22 > 0$ (min)

$f(8) = 8^3 - 11(8)^2 - 16(8) + 1 = -319$

$\boxed{\text{Rel. min @ } (8, -319).}$

$f(-2/3) = (-2/3)^3 - 11(-2/3)^2 - 16(-2/3) + 1 = 175/27$

$f''(-2/3) = 6(-2/3) - 22 < 0$ (max)

$\boxed{\text{Rel. max @ } (-2/3, 175/27).}$

Remember from Precal?

Optimization Notes

A manufacturer wants to design an open box having a square base and a surface area of 108 sq inches. What dimensions will produce a box with maximum volume?

no top

x & h

We are maximizing volume

Bottom $= x^2$

Side $= x \cdot h$

4 sides $= 4xh$

Restriction

Primary Eq

$V = x^2 h$ (max)

Secondary Eq

$SA = x^2 + 4xh = 108$

* Easier to solve for h

$4xh = 108 - x^2$

$h = \frac{108}{4x} - \frac{x^2}{4x}$

$h = \frac{27}{x} - \frac{1}{4}x$

$V = x^2\left(\frac{27}{x} - \frac{1}{4}x\right)$

$V = 27x - \frac{1}{4}x^3$

$\frac{dV}{dt} = 27 - \frac{3}{4}x^2$

$= -\frac{3}{4}(x^2 - 36)$

$= -\frac{3}{4}(x+6)(x-6)$

when $x = 6$, $h = \frac{27}{6} - \frac{1}{4}(6) = \frac{9}{2} - \frac{3}{2} = \frac{6}{2} = 3$

Volume is maximized when $x=6$ b/c dV/dt changes from pos. to neg. @ $x=6$.

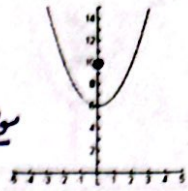
$\boxed{x = 6\text{ in}, h = 3\text{ in}}$

$-\quad+\quad-$

-6 min, 6 max

Which points on the graph of $y = 6 + x^2$ are closest to the point $(0, 10)$?

minimizing distance b/w (x,y) and $(0,10)$



Primary Eq

$D = \sqrt{(x-0)^2 + (y-10)^2}$ (min)

Secondary Eq

$y = 6 + x^2$

$D = \sqrt{x^2 + (6 + x^2 - 10)^2}$

$= \sqrt{x^2 + (x^2-4)^2}$

let $f(x) = x^2 + (x^2-4)^2$

* minimizing f minimizes D

$f'(x) = 2x + 2(x^2-4) \cdot 2x$

$= 2x(1 + 2x^2 - 8)$

$= 2x(2x^2 - 7)$

Distance is minimized when $x = \pm\sqrt{7/2}$ b/c $f'(x)$ changes from $-$ to $+$ @ $x = \pm\sqrt{7/2}$

$y = 6 + (\pm\sqrt{7/2})^2 = 6 + \frac{7}{2} = 9.5$

$\boxed{\left(\pm\sqrt{\tfrac{7}{2}}, 9.5\right)}$

$-\quad+\quad-\quad+$

$-\sqrt{7/2}\quad 0\quad \sqrt{7/2}$

Find 2 positive numbers that satisfy the given requirements.

x and y

1) The product is 185 and the sum is a minimum.

Prim Eq

$S = x + y$ (min)

$S = \frac{185}{y} + y$

$S = 185y^{-1} + y$

$\frac{dS}{dt} = -185y^{-2} + 1 = -\frac{185}{y^2} + \frac{y^2}{y^2} = \frac{y^2 - 185}{y^2}$

Sec Eq

$P = xy = 185$

$x = \frac{185}{y}$

Sum is minimized when $y = \sqrt{185}$ b/c $\frac{dS}{dt}$ changes from $-$ to $+$ @ $y = \sqrt{185}$. At $y = \sqrt{185}$, $x = \frac{185}{\sqrt{185}}$.

$+\quad-\quad-\quad+$

$-\sqrt{185}\quad 0\quad \sqrt{185}$

2) The sum of the first number squared and the second number is 54 and the product is a maximum.

Prim Eq

$P = xy$ (max)

$P = x(54 - x^2)$

$= 54x - x^3$

$\frac{dP}{dx} = 54 - 3x^2$

$= -3(x^2 - 18)$

Sec Eq

$x^2 + y = 54$

$y = 54 - x^2$

Product is maximized when $x = \sqrt{18}$ bc $\frac{dP}{dx}$ changes from $+$ to $-$ @ $x = \sqrt{18}$. At $x = \sqrt{18}$, $y = 36$.

$-\quad+\quad-$

$-\sqrt{18}\quad \sqrt{18}$

3) The sum of the first number and twice the second number is 108 and the product is a maximum.

Prim Eq

$P = xy$ (max)

$P = (108 - 2y)y$

$= 108y - 2y^2$

$\frac{dP}{dy} = 108 - 4y$

$= -4(y - 27)$

$+\quad-$

27

Sec Eq

$x + 2y = 108$

$x = 108 - 2y$

Product is maximized when $y = 27$ b/c $\frac{dP}{dy}$ changes from $+$ to $-$ @ $y = 27$. At $y = 27$, $x = 54$.

4) The second number is the reciprocal of the first number and the sum is a minimum.

Prim Eq

$S = x + y$ (min)

$S = x + \frac{1}{x}$

$= x + x^{-1}$

$\frac{dS}{dt} = 1 - x^{-2}$

$= 1 - \frac{1}{x^2}$

$= \frac{x^2}{x^2} - \frac{1}{x^2}$

$= \frac{x^2 - 1}{x^2} = \frac{(x+1)(x-1)}{x^2}$

Sec Eq

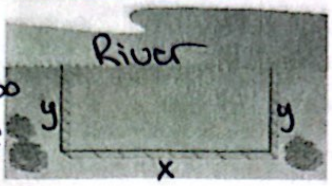
$y = \frac{1}{x}$

Sum is minimized when $x = 1$ b/c $\frac{dS}{dt}$ changes from neg. to pos. at $x = 1$. At $x = 1$, $y = 1$.

$+\quad-\quad+$

$-1\quad 0\quad 1$

Minimum Length A farmer plans to fence a rectangular pasture adjacent to a river (see figure). The pasture must contain 245,000 square meters in order to provide enough grass for the herd. No fencing is needed along the river. What dimensions will require the least amount of fencing?



Prim Eq — Sec. Eq

$F = 2y + x$ (min) — $A = xy = 245{,}000$

$F = 2y + \frac{245{,}000}{y}$ — $x = \frac{245{,}000}{y}$

$= 2y + 245{,}000y^{-1}$

$\frac{dF}{dy} = 2 - 245{,}000y^{-2}$

$= \frac{2y^2}{y^2} - \frac{245{,}000}{y^2}$

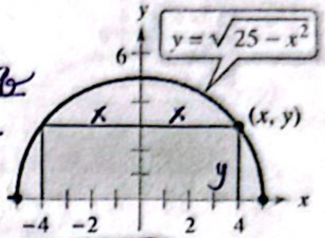
$= \frac{2(y^2 - 122{,}500)}{y^2}$

$= \frac{2(y+350)(y-350)}{y^2}$

$+ \quad - \quad - \quad +$; $-350 \quad 0 \quad 350$

The dimensions will require the least amount of fence when $y = 350$ b/c dF/dy changes from neg. to pos. at $y = 350$. At $y = 350$, $x = 700$.

A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$. What length and width should the rectangle have so that its area is a maximum?



Prim Eq — Sec Eq

$y = 2xy$ — $y = \sqrt{25 - x^2}$

$= (25 - x^2)^{1/2}$

$f'g + g'f$

$y = \underbrace{2x}_{f}\underbrace{(25 - x^2)^{1/2}}_{g}$

$\frac{dy}{dx} = 2(25 - x^2)^{1/2} + 2x\left(\frac{1}{2}\right)(25 - x^2)^{-\frac{1}{2}}(-2x)$

$= 2\sqrt{25 - x^2} - \frac{2x^2}{\sqrt{25 - x^2}}$

$= \frac{2(25 - x^2) - 2x^2}{\sqrt{25 - x^2}}$

$= \frac{50 - 2x^2 - 2x^2}{\sqrt{25 - x^2}}$

$= \frac{50 - 4x^2}{\sqrt{25 - x^2}}$

$= \frac{-4\left(x^2 - \frac{25}{2}\right)}{\sqrt{25 - x^2}}$

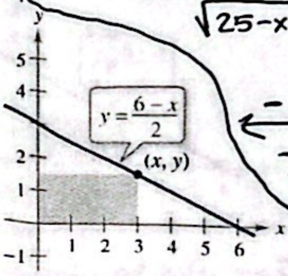
$- \quad + \quad -$; $-\frac{5}{\sqrt{2}} \quad \frac{5}{\sqrt{2}}$

Area is maximized when $x = \frac{5}{\sqrt{2}}$ b/c $\frac{dA}{dx}$ changes from + to − at $x = \frac{5}{\sqrt{2}}$. At $x = \frac{5}{\sqrt{2}}$, $y = \frac{5}{\sqrt{2}}$, ∴

length $= \frac{10}{\sqrt{2}}$

width $= \frac{5}{\sqrt{2}}$

A rectangle is bounded by the x- and y-axes and the graph of $y = (6 - x)/2$ (see figure). What length and width should the rectangle have so that its area is a maximum?



Prim Eq — Sec Eq

$A = xy$ (max) — $y = \frac{6 - x}{2}$

$A = x\left(\frac{6 - x}{2}\right)$

$= \frac{1}{2}(6x - x^2)$

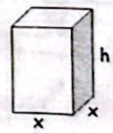
$\frac{dA}{dx} = \frac{1}{2}(6 - 2x) = 3 - x = -(x - 3)$

$+ \quad -$; 3

Area is maximized when $x = 3$ b/c dA/dx changes from + to − at $x = 3$. At $x = 3$, $y = \frac{6-3}{2} = \frac{3}{2}$.

3 by $3/2$.

A box with a square base and no top has volume 8 m^3. The material for the base costs \$8.00 per square meter and the material for the sides costs \$6.00 per square meter. Find the dimensions of the box that will minimize the cost.



Primary Eq — Secondary Eq

$SA = x^2 + 4xh$ — $V = x^2h = 8$

$\boxed{C = \text{Cost}}$ $C = 8x^2 + 24xh$ — $h = \frac{8}{x^2}$

$C = 8x^2 + 24x\left(\frac{8}{x^2}\right)$

$= 8x^2 + 192x^{-1}$

$\frac{dC}{dx} = 16x - 192x^{-2} = 16x - \frac{192}{x^2} = \frac{16x^3}{x^2} - \frac{192}{x^2} = \frac{16x^3 - 192}{x^2} = \frac{16(x^3 - 12)}{x^2}$

$- \quad - \quad +$; $0 \quad \sqrt[3]{12}$

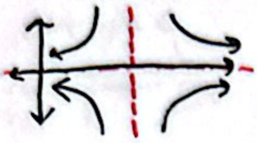
The cost, C, is minimized when $x = \sqrt[3]{12}$ m b/c dC/dx changes from neg. to pos. when $x = \sqrt[3]{12}$. When $x = \sqrt[3]{12}$, $h = \frac{8}{12^{2/3}}$ m.

Limits of ∞ and at ∞

limits at ∞	limits of ∞
$\lim\limits_{x\to\infty} f(x) = c$	$\lim\limits_{x\to c} f(x) = \infty$

Find the limit of each of the following.

1. $\lim\limits_{x\to\infty}\left(\dfrac{1}{x^2+5}\right) = 0$



EBA: y=0

"Bottom Heavy"

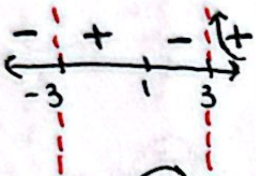
2. $\lim\limits_{x\to-\infty}\left(\dfrac{1}{x^2+5}\right) = 0$

3. $\lim\limits_{x\to-\infty}\left(\dfrac{10x^3+1}{11x^3-x+5}\right) = \dfrac{10}{11}$

EBA: $y=\dfrac{10}{11}$

"Equal Weight"

4. $\lim\limits_{x\to-3^+}\left(\dfrac{x-1}{x^2-9}\right) = \lim\limits_{x\to-3^+}\dfrac{x-1}{(x+3)(x-3)} = \infty$



5. $\lim\limits_{x\to\infty}\left(\dfrac{\sin x}{x}+1\right)$

oscillates b/w [-1,1]

$= \lim\limits_{x\to\infty}\left(\dfrac{c}{x}+1\right) = 1$

EBA: y=0

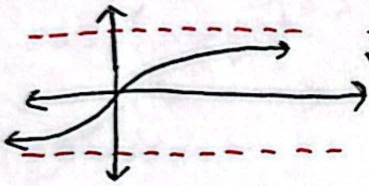
"Bottom Heavy"

6. $\lim\limits_{x\to1^+}\left(\dfrac{x-1}{x^2-9}\right) = 0$

7. $\lim\limits_{x\to-\infty}\dfrac{5x}{\sqrt{4x^2+3}} = -\dfrac{5}{2}$

EBA

$\dfrac{5x}{\sqrt{4x^2}} = \pm\dfrac{5}{2}$



8. $\lim\limits_{x\to0}\left(\dfrac{\sin x}{x}+1\right) = \lim\limits_{x\to0} 2 = 2$

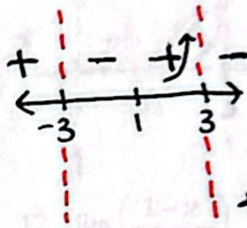
$1 + 1$

Special Trig Limit

$\lim\limits_{x\to0}\dfrac{\sin x}{x} = 1$

9. $\lim\limits_{x\to3^-}\left(\dfrac{1-x}{x^2-9}\right) = \lim\limits_{x\to3^-}\left(\dfrac{-(x-1)}{(x-3)(x+3)}\right)$

$=\infty$



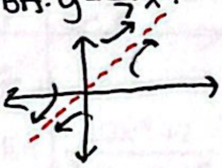
+compare to #4

10. $\lim\limits_{x\to\infty}\left(\dfrac{4x^3+1}{7x^3-x+5}\right) = \dfrac{4}{7}$

EBA: $y=\dfrac{4}{7}$

11. $\lim\limits_{x\to\infty}\left(\dfrac{10x^4+1}{7x^3-x+5}\right) = \infty$

EBA: $y=\dfrac{10}{7}x+\cdots$



12. $\lim\limits_{x\to\infty}\left(\dfrac{1-x}{x^2-9}\right) = 0$

EBA: y=0

13. $\lim\limits_{x\to-\infty}\dfrac{-5x}{\sqrt{4x^2+3}} = \dfrac{5}{2}$

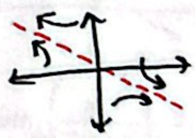
see #7

14. $\lim\limits_{x\to-\infty}\left(\dfrac{10x^4+1}{7x^3-x+5}\right) = -\infty$

see #11

15. $\lim\limits_{x\to-\infty}\left(\dfrac{-10x^4+1}{7x^3-x+5}\right) = \infty$

EBA: $y=-\dfrac{10}{7}x+\cdots$



[-1,1]

16. $\lim\limits_{x\to-\infty}\left(\dfrac{\cos x}{x}+1\right)$

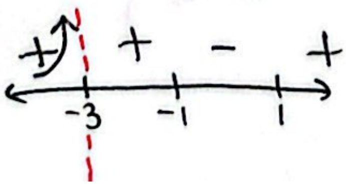
$= \lim\limits_{x\to-\infty}\left(\dfrac{c}{x}+1\right) = 1$

EBA: y=0

"Bottom Heavy"

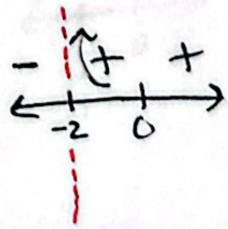
17. $\lim\limits_{x\to-3^-}\left(\dfrac{x^2-1}{x^2+6x+9}\right)$

$= \lim\limits_{x\to-3^-}\left(\dfrac{(x-1)(x+1)}{(x+3)^2}\right) = \infty$



18. $\lim\limits_{x\to-2^+}\left(\dfrac{x^2}{2x+4}\right) = \lim\limits_{x\to-2^+}\left(\dfrac{x^2}{2(x+2)}\right)$

$=\infty$



Quotient = EBA

$$\begin{array}{r} x+3 \\ x^2+0x-1 \overline{\big) x^3+3x^2+0x+5} \\ -(x^3+0x^2-x) \\ \hline 3x^2+x+5 \\ -(3x^2+0x-3) \\ \hline R:\ x+2 \end{array}$$

Curve Sketching

1. Identify any asymptotes on the graph of $f(x)=\dfrac{x^3+3x^2+5}{x^2-1}$.

$(x+1)(x-1)$

$\dfrac{x^3}{x^2}=x$

$\dfrac{3x^2}{x^2}=3$

VA: $x=\pm 1$

EBA: $y=x+3$

2. Describe the EBA for each of the following functions.

a) $f(x)=\dfrac{3}{x^2-x+1}$

EBA: $y=0$

b) $f(x)=\dfrac{7x}{3x+1}$

EBA: $y=\frac{7}{3}$

c) $f(x)=\dfrac{x^3+3x^2+5}{x^2-1}$

EBA: $y=x+3$

see #1

Find any intercepts and asymptotes on f(x). Write all statements describing the behavior of the graph of the function $f(x)$.

Ex1: $f(x)=\dfrac{2x}{x-3}$ $\quad f'(x)=\dfrac{-6}{(x-3)^2}$ $\quad f''(x)=\dfrac{12}{(x-3)^3}$

No POI b/c $x=3$ is a VA

VA: $x=3$

EBA: $y=2$

Intercept: $(0,0)$

f': $-$ | $-$ at 3

f decreases on $(-\infty,3)\cup(3,\infty)$ b/c $f'(x)<0$.

No extrema.

f'': $-$ | $+$ at 3

f is concave down on $(-\infty,3)$ b/c $f''(x)<0$.

f is concave up on $(3,\infty)$ b/c $f''(x)>0$.

Ex2: $f(x)=(x^2+9)(x-6)$ $\quad f'(x)=3(x-1)(x-3)$ $\quad f''(x)=6(x-2)$

x-int: $(6,0)$

y-int: $(0,-54)$

x	f(x)
1	-50
3	-54

f': $+$ | $-$ | $+$ at 1, 3

f(x) increases on $(-\infty,1)\cup(3,\infty)$ b/c $f'(x)>0$.

f(x) decreasing on $(1,3)$ b/c $f'(x)<0$.

$y=-50$ is a rel. max b/c $f'(x)$ changes from + to − at $x=1$.

$y=-54$ is a rel. min b/c $f'(x)$ changes from − to + at $x=3$.

f'': $-$ | $+$ at 2

f(x) is concave down on $(-\infty,2)$ b/c $f''(x)<0$.

f(x) is concave up on $(2,\infty)$ b/c $f''(x)>0$.

$(2,-52)$ is a POI b/c $f''(x)$ changes signs at $x=2$.