Unit 1: Limits & Derivatives

Monday	Tuesday	Wednesday	Thursday	Friday
August 12	13	14	15	16
STUDENT HOLIDAY TEACHER WORK DAY	Introductions Introducing Derivative	Differentiability Intro to Limits	Limits (evaluating techniques) Definition of the Derivative	Alternate Form of Derivative Concavity & Inflection Points
19	20	21	22	23
f, f', f'' Power Rule Junior/Staff Pictures	ICA1 Junior/Staff Pictures	ICA2	ICA3	Quiz
26	27	28	29	30
Review	Unit 1 Test	Notes: Product Rule, Quotient Rule, Chain Rule, Implicit Differentiation, Related Rates		

Warm-up Problems

1. Write a function representing a line with slope of 10 passing through the point (7, 23).

Point-Slope Form

$y - y_1 = m(x - x_1)$

$y_1 = 23$

$x_1 = 7$

$m = 10$

$\boxed{y - 23 = 10(x - 7)}$

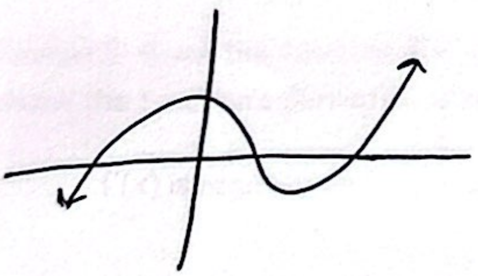
$y = m(x - h) + k$

line passes through (h, k)

$(7, 23)$

$y = 10(x - 7) + 23$

2. Given the non-linear function f(x) is tangent to a line with a slope of 3 at the ordered pair (5,2). Use the tangent line to predict the output for f(x) at x = 6.



$y = 3(x - 5) + 2$

$y(6) = 3(6 - 5) + 2$

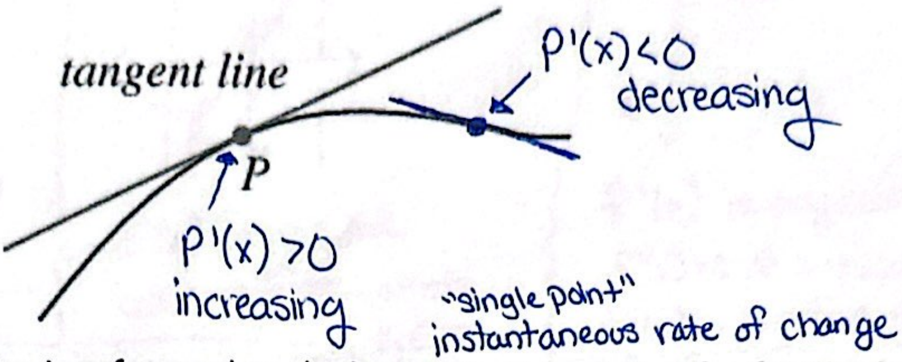
$= 3(1) + 2$

$= 3 + 2$

$= 5$

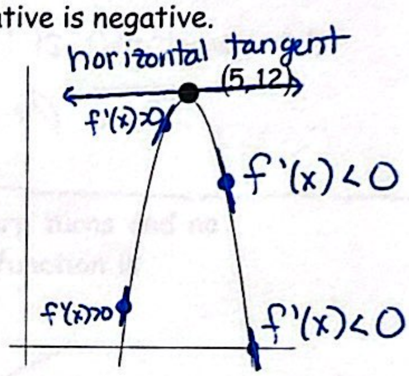
Prior to calculus, we have primarily discussed rates of change in relation to a linear function, where the rate of change (slope) is constant (always the same). The value of m, from y = mx + b.

A curve has different rates of change at its points, where the direction of the function at a particular point is described by the direction (slope) of a tangent line



The slope of this tangent line is referenced as the Derivative at point P. If a function is increasing at some point, the derivative is positive at that point. If a function is decreasing at some point, the derivative is negative at that point. The derivative function for f(x) is notated as f'(x).

Example1: Given the function $f(x) = -3(x-5)^2 + 12$, identify where the function's derivative is positive, where the function's derivative is zero, and where the function's derivative is negative.

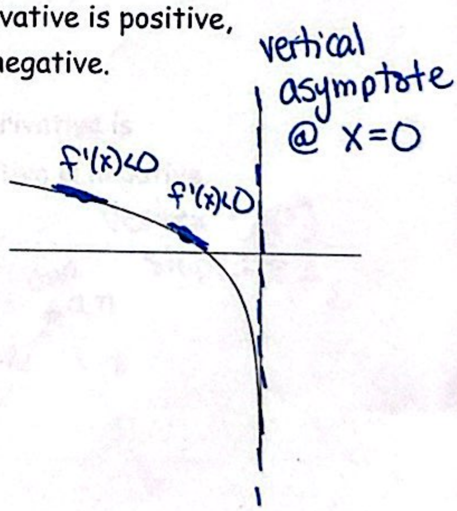


f'(x) is positive on $(-\infty,5)$

f'(x) is zero at x = 5. That is, $f'(5) = 0$ (horizontal tangent line)

f'(x) is negative on $(5,\infty)$

Example2: Given the function $f(x) = \log_3(-x)$, identify where the function's derivative is positive, where the function's derivative is zero, and where the function's derivative is negative.



f'(x) is negative on $(-\infty,0)$, which is the entire domain of f(x)

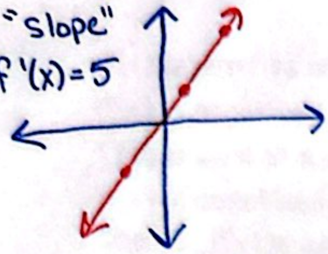
$\mathbb{R}$ is the set of all reals — rational & irrational numbers (René Descartes) "Father of Modern Phil."

$\mathbb{Z} = \{\dots -4, -3, -2, -1, 0, 1, 2, \dots\}$ integers (German: Zahlen ← numbers)

For each function, identify where the function's derivative is positive, where the function's derivative is zero, and where the function's derivative is negative.

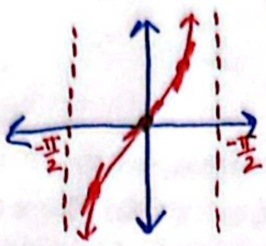
1. $f(x) = \underline{5}x$

"slope" $f'(x) = 5$



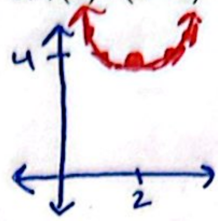
$f'(x)$ is positive on $(-\infty, \infty)$ or $\mathbb{R}$

2. $f(x) = \tan x$ * Note: tangent is periodic



$f'(x)$ is positive on $x \neq \frac{\pi}{2} + \pi k$, where $k \in \mathbb{Z}$

3. $f(x) = (x-2)^2 + 4$

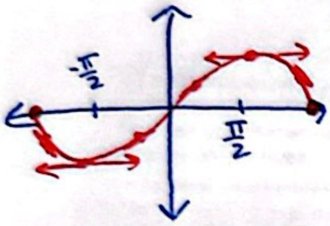


$f'(x)$ is negative of $(-\infty, 2)$

$f'(x) = 0$ @ $x = 2$, or $f'(2) = 0$

$f'(x)$ is positive of $(2, \infty)$

4. $f(x) = \sin x, -\pi \le x \le \pi$ Note: sine is periodic, but we have a domain restriction

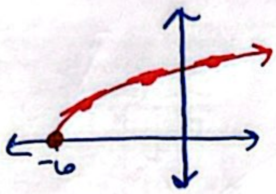


$f'(x)$ is negative of $(-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$

$f'(x) = 0$ @ $x = -\frac{\pi}{2}$ & $x = \frac{\pi}{2}$

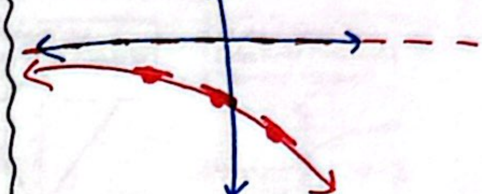
$f'(x)$ is positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$

5. $f(x) = \sqrt{x+6}$



$f'(x)$ is positive on $(-6, \infty)$

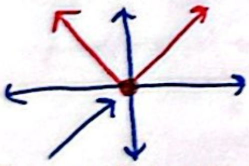
6. $f(x) = -5^x$



$f'(x)$ is negative on $(-\infty, \infty)$ or $\mathbb{R}$

The derivative will exist at a point as long as the curve is smooth with no sharp turns and no vertical tangent lines. If the derivative exists at a point it can be said that function is <u>differentiable</u> at that point.

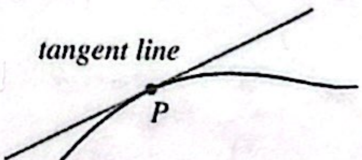
Sharp turns commonly occur on absolute functions.



$g(x) = |x| = \begin{cases} x, & x > 0;\ f'(x) > 0 \\ -x, & x \le 0;\ f'(x) < 0 \end{cases}$

"nondifferentiable"

* when the "equal to" notation is placed does not matter



Example1: Given the function $f(x) = \underline{-2}|x-3| + 7$, identify where the function's derivative is positive, where the function's derivative is zero, and where the function's derivative is negative. Also identify any defined domain values where f(x) is not differentiable.

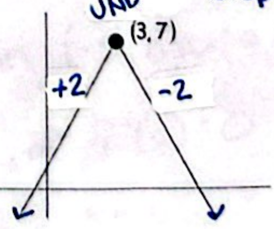
Vertex: (3,7)
Slope: ±2

$f'(x)$ is positive on $(-\infty, 3)$; $f'(x) = 2$ on $(-\infty, 3)$

$f'(x)$ is negative on $(3, \infty)$; $f'(x) = -2$ on $(3, \infty)$

$f'(x)$ is undefined at $x = 3$. (<u>sharp turn</u>)

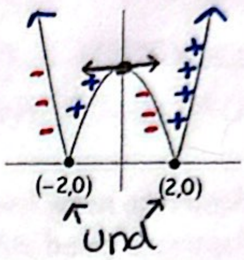
That is, $f(x)$ is <u>not differentiable</u> at $x = 3$



Note: $|f(x)| = |y|$ reflects the entire function about the x-axis
→ ALL output values are positive

Example2: Given the function $f(x) = |x^2 - 4|$, identify where the function's derivative is positive, where the function's derivative is zero, and where the function's derivative is negative. Also identify any defined domain values where f(x) is not differentiable.

f'(x) is positive on $(-2,0)\cup(2,\infty)$
f'(x) is negative on $(-\infty,-2)\cup(0,2)$
f'(x) is zero at x = 0. That is, $f'(0) = 0$ (horizontal tangent line)
f'(x) is undefined at x = -2 and x = 2. (sharp turn)
That is, f(x) is not differentiable at x = -2 and x = 2.



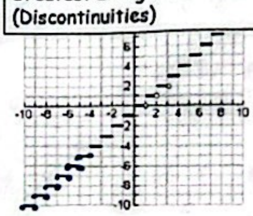
Differentiability & Continuity

A function is differentiable if the derivative exists for all x values.

Differentiability implies continuity.
Continuity does not imply differentiability.

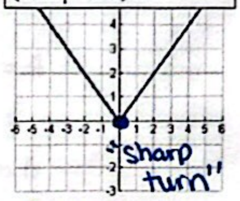
Examples of points that are not differentiable:

Any integer value from the Greatest Integer Function (Discontinuities)



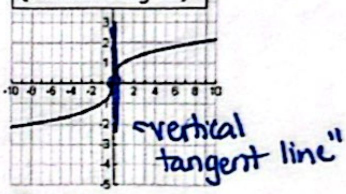
Discontinuous AND Nondifferentiable

At x = 0 from the Absolute Value Function (Sharp Turn)



Continuous, but Nondifferentiable

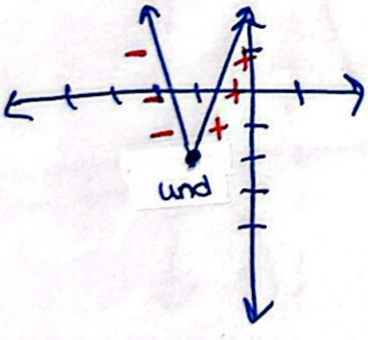
At x = 0 from the Cube Root Function (Vertical Tangent)



Continuous, but Nondifferentiable

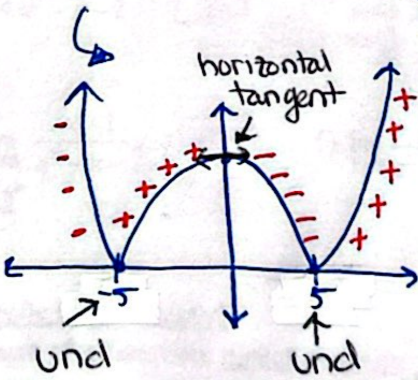
For each function, identify where the function's derivative is positive, where the function's derivative is zero, and where the function's derivative is negative. Also identify any defined domain values where f(x) is not differentiable.

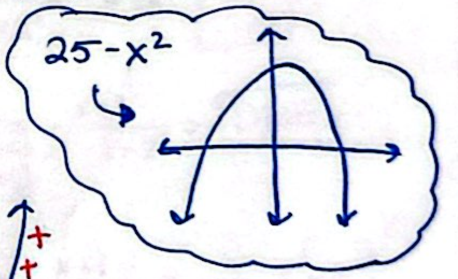
7. $f(x) = 5|x+1| - 2$



f'(x) is negative on $(-\infty,-1)$
f'(x) is positive on $(-1,\infty)$
f'(x) is undefined @ $x=-1$
(not differentiable)
or $f'(-1)$ is undefined

8. $f(x) = |25 - x^2|$





f'(x) is negative on $(-\infty,-5)\cup(0,5)$
f'(x) is positive on $(-5,0)\cup(5,\infty)$
$f'(x) = 0$ @ $x=0$, or $f'(0)=0$
f'(x) is undefined @ $x=\pm 5$
(not differentiable)

Limits $\longrightarrow$ y-value approached when approaching a given x-value (c)

"Left Sided"
$\lim_{x\to c^-} f(x) =$
y-value approached when x approaches c from the left

"Right Sided"
$\lim_{x\to c^+} f(x) =$
y-value approached when x approaches c from the left

$\lim_{x\to c} f(x) =$
y-value approached when x approaches c from both sides

Limits Graphically

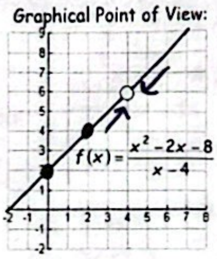
Graphically, the existence of a limit will occur if the same y-value is being approached when approaching the x-value from both the left and the right. If the function is continuous at the x-value being approached, then the limit equals the corresponding y-value. When approaching a removable discontinuity (as shown below for the 3rd limit) the limit will equal the y-value for the removable discontinuity.

$\lim_{x\to c} f(x) = \frac{5}{0}$ DNE

$\frac{x^2-2x-8}{x-4} = \frac{\cancel{(x-4)}(x+2)}{\cancel{x-4}}$

$f(x) = \lim_{x\to 4}\frac{x^2-2x-8}{x-4} = \frac{0}{0}$

Graphical Point of View:



Limit examples:

$\lim_{x\to 0}\frac{x^2-2x-8}{x-4} = 2$

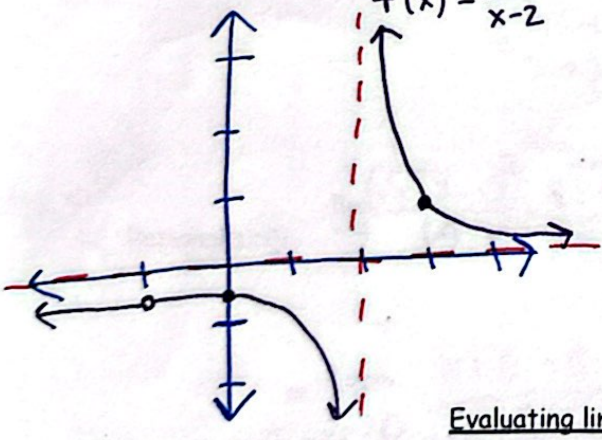
$\lim_{x\to 2}\frac{x^2-2x-8}{x-4} = 4$

$\lim_{x\to 4}\frac{x^2-2x-8}{x-4} = 6$

9. Sketch a graph of $f(x) = \frac{\cancel{x+1}}{x^2-x-2}$. Then evaluate the limits shown below.

$\cancel{(x+1)}(x-2)$

$f(x) = \frac{1}{x-2}$



a) $\lim_{x\to -1}\frac{x+1}{x^2-x-2} = \frac{1}{-1-2} = \boxed{-\frac{1}{3}}$

b) $\lim_{x\to 0}\frac{x+1}{x^2-x-2} = \boxed{-\frac{1}{2}}$

c) $\lim_{x\to 2^-}\frac{x+1}{x^2-x-2} = \boxed{-\infty \text{ or DNE}}$

d) $\lim_{x\to 2^+}\frac{x+1}{x^2-x-2} = \boxed{\infty \text{ or DNE}}$

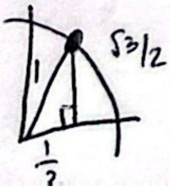
e) $\lim_{x=2}\frac{x+1}{x^2-x-2} = \boxed{\text{DNE}}$

f) $\lim_{x\to 3}\frac{x+1}{x^2-x-2} = \frac{1}{3-2} = \boxed{1}$

Evaluating limits analytically (algebraically)

When evaluating a limit at a continuous point on the graph of a function, direct substitution is a method for evaluating analytically.

10. Evaluate each limit.



a) $\lim_{x\to\frac{\pi}{3}}\sin x = \sin\frac{\pi}{3}$
$= \frac{\sqrt{3}}{2}$

b) $\lim_{x\to 2}\frac{2}{x-3} = \frac{2}{2-3}$
$= -2$

c) $\lim_{x\to 2.5}[x] = [2.5]$
$= 2$

Evaluating Techniques for Limits w/ Discontinuities (removable)

If you divide and get $\frac{0}{0}$ ← indeterminate form

1) Dividing Common Factor:

Simplify the rational expression by dividing out common factors within numerator & denominator.

2) Simplifying Complex Fraction:

Simplify the complex fraction by multiplying numerator & denominator by "fraction-within-fraction" denominators

3) Rationalizing Technique: expression containing radical

Evaluate by <u>rationalizing numerator</u>:
multiply numerator & denominator by conjugate expression

For all 3 techniques, use direct substitution after simplifying.

Evaluate each limit using the appropriate technique.

Tech #2

11. Simplifying Complex Fraction: $\lim_{x \to 3} \frac{\left(\frac{1}{x+1} - \frac{1}{4}\right)}{(x-3)} \cdot \frac{4(x+1)}{4(x+1)}$ → weight of $\frac{1}{1}$ LCD

* using direct sub results in the denominator being 0

$$= \lim_{x \to 3} \frac{4 - (x+1)}{4(x-3)(x+1)}$$

$$= \lim_{x \to 3} \frac{-x+3}{4(x-3)(x+1)}$$

$$= \lim_{x \to 3} \frac{-1}{4(x+1)} = \frac{-1}{4(4)} = \boxed{-\frac{1}{16}}$$

Tech #3

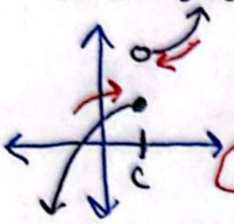
12. Rationalizing: $\lim_{x \to 0} \frac{(\sqrt{x+3} - \sqrt{3})}{(x)} \cdot \frac{\sqrt{x+3} + \sqrt{3}}{\sqrt{x+3} + \sqrt{3}}$ → weight of $\frac{1}{1}$

$$= \lim_{x \to 0} \frac{x+3-3}{x(\sqrt{x+3} + \sqrt{3})}$$

$$= \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+3} + \sqrt{3})} = \frac{1}{\sqrt{3} + \sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}}}$$

Limits that Fail to Exist

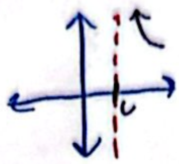
a) Differing Right & Left Behavior



$\lim_{x \to c} f(x)$ $\boxed{DNE}$

left and right limits do not agree

b) Unbounded Behavior

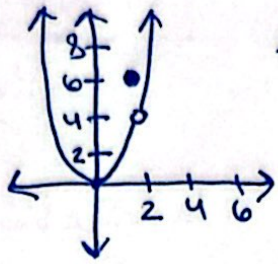


$\lim_{x \to c} f(x)$ $\boxed{DNE}$

c) Oscillating Behavior

$\lim_{x \to \infty} f(x)$ $\boxed{DNE}$

11. Graph $f(x) = \begin{cases} \dfrac{x^2(x-2)}{x-2}, & x \neq 2 \\ 6, & x = 2 \end{cases}$



$f(x) = x^2, \; x \neq 2$

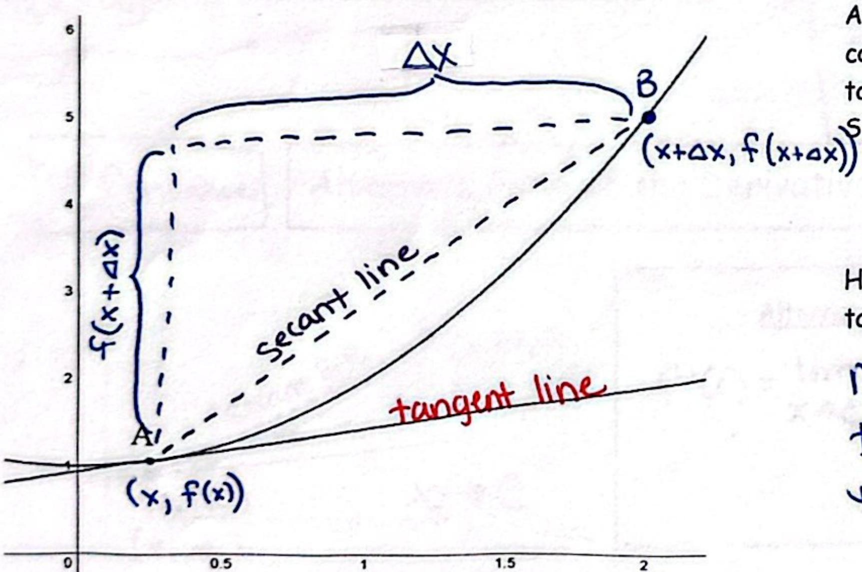
and evaluate the following limits.

a) $\lim_{x \to 0} f(x) = 0$ b) $\lim_{x \to 1} f(x) = 1$ c) $\lim_{x \to 2} f(x) = 4$ d) $f(2) = 6$

* limits are about what value we are "approaching"

$\boxed{1^{st}\text{ Derivative}}$ Definition of the Derivative

Finding the tangent line slope (the derivative)



"2 points"

A secant line slope could be calculated to approximate the tangent line slope.

Secant line slope = $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

How can we improve the tangent line slope at point A?

Move point B closer to point A. As we do this, what happens to Δx?

$\Delta x \to 0$

secant → average rate of change (b/w 2 points)

tangent → instantaneous rate of change (at 1 point)

Definition of the Derivative (Limit Process or 1st Principles)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = m_{tan}$$

f prime of x equals the limit as Δx approaches 0...

12. Given $f(x) = x^2$

a) Find the derivative function of f(x) using 1st principles. (Limit process)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

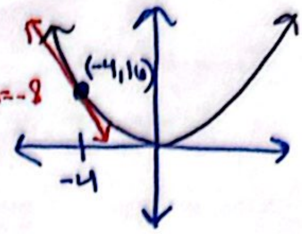
$$\boxed{f'(x) = 2x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x+\Delta x)}{\Delta x} = 2x + \Delta x$$

$$= 2x$$

b) Find the derivative of f(x) at x = -4

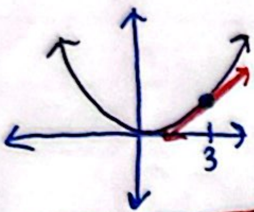


$f(-4) = (-4)^2 = 16$

$f'(-4) = 2(-4) = \boxed{-8}$

Slope is -8 and x=-4.

c) Find the slope of the line tangent to f(x) at x = 3.



$f'(3) = 2(3) = \boxed{6}$

We can use $f'(x)$ to find the slope at any point of $f(x)$.

d) Write the equation of the tangent line. at $x=3$

Point-Slope Form: *need 3 things

$(y - y_1) = m(x - x_1)$

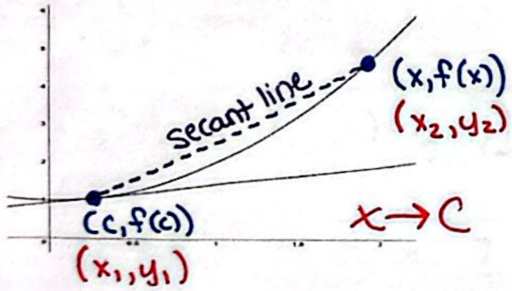
$f(3) = 9 \rightarrow (3, 9)$

$f'(3) = 6 \rightarrow m = 6$

$$\boxed{\begin{gathered} y - 9 = 6(x-3) \\ \text{or} \\ y = 6(x-3) + 9 \end{gathered}}$$

1st Derivative

Alternate Form of the Derivative → *Allows you to find a derivative at a single point*



Alternate Form of the Derivative (at $x=c$)

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = m_{sec}$$

13. Given $f(x) = x^3$ find the derivative of f(x) at $x = 2$ ($c=2$) using the alternate form of the derivative.

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{x^3 - (2)^2}{x-2}$$

difference of cubes SOAP

$$= \lim_{x \to 2} \frac{x^3 - 8}{x - 2}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{x-2}}$$

$$= \lim_{x \to 2} (x^2 + 2x + 4) = 4 + 4 + 4 = \boxed{12}$$

$f'(2) = 12$

Limits: Properties

Properties: If b, c are real #'s ⇒

$$\left.\begin{array}{l} 1)\ \lim_{x\to c} b = b \\ \\ 2)\ \lim_{x\to c} x = c \\ \\ 3)\ \lim_{x\to c} x^n = c^n \end{array}\right\}\text{ Think about the graphs}$$

If we know two limits ($\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = K$)

1) $\lim_{x\to c} [b \cdot f(x)] = b \cdot L$

2) $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$

3) $\lim_{x\to c} [f(x) \cdot g(x)] = L \cdot K$

4) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, $K \neq 0$

5) $\lim_{x\to c} [f(x)]^n = L^n$

Ex) If $\lim_{x\to 2} f(x) = 7$ & $\lim_{x\to 2} g(x) = 4$, find

a) $\lim_{x\to 2} [f(x) - g(x)] = 7 - 4 = 3$

b) $\lim_{x\to 2} \frac{f(x)}{g(x)} = \frac{7}{4}$

Limits of ∞

VA: The line $x = c$ is a vertical asymptote of $f(x)$ if,
$\lim_{x\to c^+} f(x)$ or $\lim_{x\to c^-} f(x)$ is ∞ or −∞.

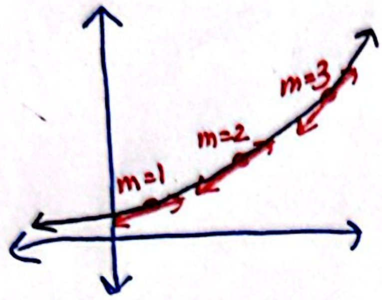
2nd Derivative

f double prime

Concavity of f(x)

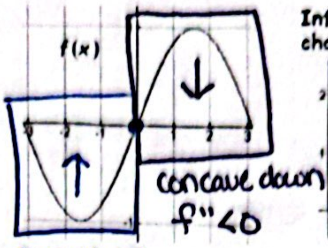
If $f''>0$ on [a, b] then $f(x)$ is concave up on. $a<x<b$

If $f''<0$ on [a, b] then $f(x)$ is concave down on $a<x<b$



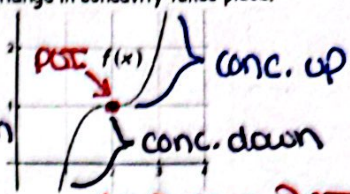
f is conc. up
f' is Increasing
f'' is positive

Concavity & Inflection Points



(0,0) is a point of inflection (POI)

Inflection Point: Point on the graph where a change in concavity takes place.



(2,1) is a POI
$f''>0$ when $x>2$,
$\therefore f(x)$ is conc. up.
$f''<0$ when $x<2$,
$\therefore f(x)$ is conc. down

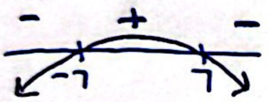
Given each f"(x) function,

a) sketch a sign pattern of f"(x)
b) Identify the domain in which f(x) is concave up.
c) Determine the x-values in which f(x) has inflection points.

14. $f''(x)=49-x^2$

$f''(x)=(7+x)(7-x)$

a)



b) f is conc. up on $(-7,7)$ b/c $f''>0$ on that interval.

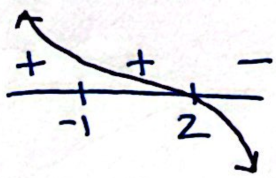
c) f has inflection points at $x=\pm 7$ b/c f" changes signs at these points.

15. $f''(x)=-x^3+3x+2$ (hint: f" has a zero of x = -1)

$$\begin{array}{r|rrrr} -1 & -1 & 0 & 3 & 2 \\ & & 1 & -1 & -2 \\ \hline & -1 & 1 & 2 & 0 \end{array}$$

$f(x)=(x+1)(-x^2+x+2)$
$=-(x+1)(x^2-x-2)$
$=-(x+1)(x-2)(x+1)$
$=-(x+1)^2(x-2)$

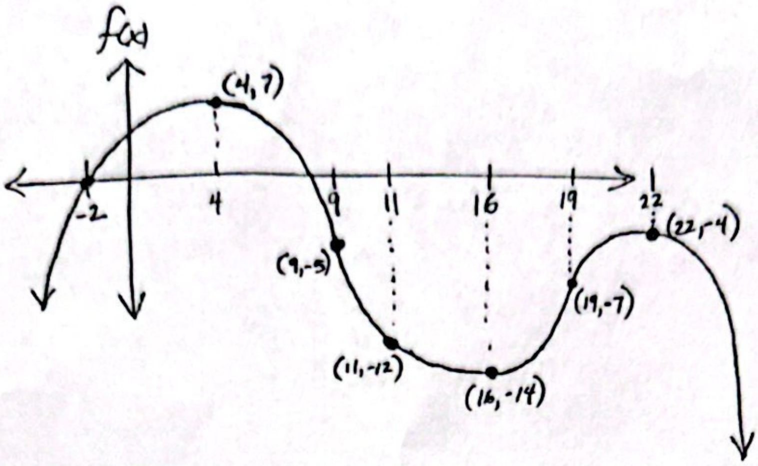
a)



b) f is conc. up on $(-\infty,-1)\cup(-1,2)$ b/c $f''>0$ over these intervals.

c) f has a inflection points at $x=2$ b/c f" changes signs at $x=2$.

Relating f(x), f'(x), and f''(x)



16. Use the graph displayed above to evaluate the following. If the exact value cannot be determined, answer with positive or negative.

"POI"

a) $\cancel{f(2)}$
$f(-2) = 0$

b) $f'(9)$ is a neg. value

c) $f''(19) = 0$

d) $\lim_{x \to 4} f(x) = 7$

conditions for continuity

e) $\lim_{\Delta x \to 0} \frac{f(19+\Delta x) - f(19)}{\Delta x} = f'(19)$ is a pos. value

f) $\lim_{x \to \infty} f(x) = -\infty$ or DNE

g) $f(4) = 7$

h) $f'(-2)$ is a pos. value

i) $f''(16)$ is a pos. value

j) $\lim_{\Delta x \to 0} \frac{f(16+\Delta x) - f(16)}{\Delta x} = f'(16) = 0$

k) $\lim_{x \to 22} f(x) = -4$

l) $f(19) = -7$

m) $f''(11)$ is a pos. value

n) $\lim_{h \to 4} \frac{f(11+\cancel{\Delta x}\, h) - f(11)}{\cancel{\Delta x}\, h} = f'(11)$ is a neg. value

0) $f'(19)$ is a pos. value

p) $\lim_{x \to -\infty} f(x) = -\infty$ or DNE

q) $\frac{f(19) - f(9)}{19 - 9} = \frac{-7-(-5)}{10}$
$= -\frac{2}{10}$
$= -\frac{1}{5}$

r) $\frac{f(22) - f(11)}{22 - 11}$
$= \frac{-4-(-12)}{11}$
$= \frac{8}{11}$

17. Sketch sign patterns for f'(x) and f''(x).

f'

+	4	−	16	+	22	−

extrema

f''

−	9	+	19	−

Points of Inflections

* Leibniz: German polymath who invented calculus (last universal genius).

Derivative Notations
for $y = f(x)$

2nd: $\frac{d^2y}{dx^2}$

1st Derivative (notations) : $f'(x)$ $\frac{dy}{dx}$ $\frac{d}{dx}[f(x)]$ ~~X~~ $y'(x)$

3rd: $\frac{d^3y}{dx^3}$

Leibniz Form

derivative of y w/respect to x

$y'(x) \neq y'(t)$

Power Rule for Differentiation

$$f(x) = x^n \Leftrightarrow f'(x) = nx^{n-1} \quad \text{or} \quad \frac{d}{dx}(x^n) = nx^{n-1}$$

n = constant

$y = x^n$

Use the power rule to differentiate each function.

18. $f(x) = 2x^7 + 3x^5$

$f'(x) = 7(2)x^{7-1} + 5(3)x^{5-1}$

$\boxed{f'(x) = 14x^6 + 15x^4}$

19. $f(x) = \frac{6}{x} + \sqrt{x}$

$f(x) = 6x^{-1} + x^{1/2}$

$f'(x) = -6x^{-2} + \frac{1}{2}x^{-\frac{1}{2}}$

$\boxed{f'(x) = -\frac{6}{x^2} + \frac{1}{2\sqrt{x}}}$

20. $f(x) = \frac{2}{\sqrt[4]{x^3}} + 8x + 12$

$f(x) = 2x^{-\frac{3}{4}} + 8x + 12$ → derivative of a constant is 0.

$f'(x) = (-\frac{3}{4})2x^{-\frac{7}{4}} + 8 + 0$

$\boxed{f'(x) = -\frac{3}{2x^{7/4}} + 8}$

21. $f(x) = \sqrt[3]{e}$

$\boxed{f'(x) = 0}$

Use the power rule to find $f''(x)$.

22. $f(x) = 5x^4 + 3x^2 - 3x + 1$

$f'(x) = 20x^3 + 6x - 3$

$\boxed{f''(x) = 60x^2 + 6}$

23. $f(x) = \frac{1}{x^3} + \sqrt[3]{x^{13}}$

$f(x) = x^{-3} + x^{13/3}$

$f'(x) = -3x^{-4} + \frac{13}{3}x^{10/3}$

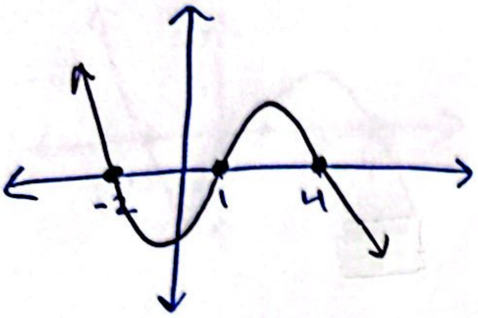
$f''(x) = 12x^{-5} + \frac{130}{9}x^{7/3}$

$\boxed{f''(x) = \frac{12}{x^5} + \frac{130}{9}x^{7/3}}$

24. Given $f(x)=(x-4)(x+2)(1-x) = (x-4)(x+2)(-(x-1))$

a) Sketch a graph of f(x).

$$\begin{array}{cccc} + & - & + & - \\ & -2 & 1 & 4 \end{array}$$



b) Expand f(x).

$f(x) = (x^2-2x-8)(1-x)$

$f(x) = x^2-2x-8-x^3+2x^2+8x$

$f(x) = -x^3+3x^2+6x-8$

*Standard Form

c) Find f'(x) and sketch a sign pattern.

$f'(x) = -3x^2+6x+6$

$= -3(x^2-2x-2)$

f'(x)

$$\begin{array}{ccccc} - & & + & & - \\ & 1-\sqrt{3} & & 1+\sqrt{3} & \end{array}$$

$x = \frac{2 \pm \sqrt{4-4(1)(-2)}}{2(1)}$

$= \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$

d) Find f"(x) and sketch a sign pattern.

$f''(x) = -6x+6$

$f''(x) = -6(x-1)$

f"(x)

$$\begin{array}{ccc} + & & - \\ & 1 & \end{array}$$

e) Identify any intervals in which f(x) decreases.

*Use $f'(x)$. When is $f'(x)<0$?

f decreased on $(-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, \infty)$

b/c $f'<0$ over these intervals.

f) Identify any intervals in which f(x) is concave up.

*Use $f''(x)$. When is $f''(x)>0$?

f is concave up on $(-\infty, 1)$

b/c $f''>0$ over this inverval.

g) Identify the x-values for any inflection points on the graph of f(x).

*Use $f''(x)$. When is $f''(x)=0$?

f has a POI at $x=1$

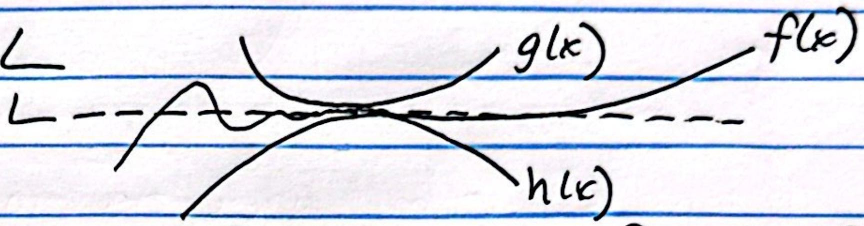
b/c f" changes signs at $x=1$.

Limits: Squeeze Theorem

Ex) $\lim_{x\to 0} \frac{\sin \pi x}{x} = ?$

The Squeeze Thm: If $f(x)$ is b/w $g(x)$ and $h(x)$ on some interval containing c and $\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x)$, then,

$\lim_{x\to c} f(x) = L$

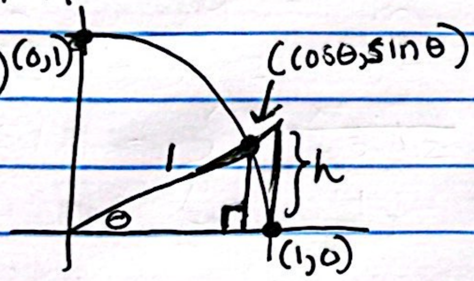


Ex) Given $4 + x^2 \le f(x) \le 7 - 2x$, find $\lim_{x\to 1} f(x)$,

$\lim_{x\to 1} (4 + x^2) = 4 + 1^2 = 5$

and

$\lim_{x\to 1} (7 - 2x) = 7 - 2(1) = 5$, $\therefore \lim_{x\to 1} f(x) = 5$

Ex) 

(small) (big)

Area of $\Delta \le$ Area of sector $\le$ Area of Δ

$\frac{1}{2}bh \qquad \frac{\theta}{2\pi}\cdot\pi r^2 \qquad \frac{1}{2}bh$

$\frac{1}{2}(1)(\sin\theta) \le \frac{1}{2}\theta(1)^2 \le \frac{1}{2}(1)\tan\theta$

$\frac{1}{2}\sin\theta \le \frac{1}{2}\theta \le \frac{1}{2}\tan\theta$

$\sin\theta \le \theta \le \tan\theta$

$\sin\theta \le \theta \le \frac{\sin\theta}{\cos\theta}$

$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$ $\quad \frac{1}{\cos(0)} = \frac{1}{1} = 1$

$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$

↳ reciprocate $\leftrightarrow$ $1 \ge \frac{\sin\theta}{\theta} \ge \cos\theta$,

$\therefore \boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$

Can also use squeeze thm to prove $\boxed{\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0}$

Ex) $\lim_{x \to 3^+} \frac{5+x}{3-x} \approx \frac{5+3.0001}{3-3.0001} \approx \frac{8}{\text{small neg \#}} \approx -\infty$

5x) Find any vertical asymptotes for $f(x) = \frac{\sin(x+2)}{x+2}$

$\lim_{x \to 2^-} \frac{\sin(x+2)}{x+2}$

$u = x+2$, $\lim_{u \to 0} \frac{\sin u}{u} = 1$ (by squeeze thm)

$x \to -2$, $u \to 0$

Limits at ∞

$\lim_{x \to \infty} \frac{c}{x} = 0$, Ex) $\lim_{x \to \infty} \frac{x^2+3x-1}{x^2+5}$

$= \lim_{x \to \infty} \frac{x^2/x^2 + 3x/x^2 - 1/x^2}{x^2/x^2 + 5/x^2}$

$= \lim_{x \to \infty} \frac{1+0-0}{1+0} = 1$

Ex) $\lim_{x \to \infty} \frac{3x^2+2x-1}{x^3+3x}$ Bottom Heavy

$= \lim_{x \to \infty} \frac{3x^2/x^3 + 2x/x^3 - 1/x^3}{x^3/x^3 + 3x/x^3}$

$= \lim_{x \to \infty} \frac{0+0-0}{1+0} = 0$

Ex) $\lim_{x \to -\infty} \frac{3x^2+2x-1}{x+3} = \lim_{x \to -\infty} \frac{3x^2/x + 2x/x - 1/x}{x/x + 3/x}$

$= \lim_{x \to -\infty} (3x+2) = -\infty$

$$\begin{array}{r|rrr} -3 & 3 & 2 & -1 \\ & & -9 & 21 \\ \hline & 3 & -7 & 20 \end{array}$$

EBA: $3x-7$