Proceedings of the
46th IEEE Conference on Decision and Control
New Orleans, LA, USA, Dec. 12-14, 2007

A Model Predictive Control Scheme for Consensus in Multi-Agent Systems with Single-Integrator Dynamics and Input Constraints

Giancarlo Ferrari-Trecate⋆, Luca Galbusera°, Marco Pietro Enrico Marciandi*, Riccardo Scattolini°

⋆ Dipartimento di Informatica e Sistemistica - Università degli Studi di Pavia - Via Ferrata 1, 27100 Pavia, Italy
e-mail: giancarlo.ferrari@unipv.it

° Dipartimento di Elettronica e Informazione - Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy
e-mail: galbusera@elet.polimi.it, scattolini@elet.polimi.it

* CESI Ricerca S.p.a. - Via Rubattino 54, 20134 Milano, Italy
e-mail: mmarciandi@hotmail.com

Abstract— **In this paper we address the problem of driving a group of agents towards a consensus point when agents have a discrete-time single-integrator dynamics and the communication graph is undirected and time-varying. We propose a decentralized Model Predictive Control (MPC) scheme that takes into account constraints on the agent inputs and show that it guarantees consensus under mild assumptions. Since the global cost does not decrease monotonically, it cannot be used as a Lyapunov function for proving consensus. Rather, our proof exploits geometric properties of the optimal path followed by individual agents.**

I. Introduction

This paper deals with consensus problems for dynamically uncoupled agents described by a discrete-time single-integrator model and subject to possible time-varying interconnections. The objective is to define a decentralized control strategy guaranteeing that the states of all the agents converge to a common value, called consensus point, which generally speaking depends on the agents' states and on the communication network. The importance of the problem is due to many potential technical applications, *e.g.* in computer graphics, unmanned autonomous vehicles, sensor networks and, generally speaking, in the control of cooperating systems (see [1] and the references therein).

In recent years, many control laws capable of guaranteeing consensus for various models of the agents' dynamics and the communication network have been proposed, see *e.g.* [2], [3], [4], [5], [6], [7], [8]. Most of them do not exploit optimal control ideas and, with the exception of [2] and [6], do not account for input constraints, which in many cases have to be included in the problem formulation due to actuator limitations. In this paper we propose an innovative solution based on Model Predictive Control (MPC), which is a widely used approach in view of its ability to handle control and state constraints. Stabilizing MPC algorithms for decentralized and distributed systems have been proposed in [9], [10], [11], [12], [13] for dynamically coupled and uncoupled systems. However, to the best of our knowledge, no applications of MPC have been reported so far for the consensus problem with the notable exception of [14], where the properties of the proposed algorithm are confirmed by means of simulation experiments. On the contrary, the method described in this paper can be formally proven to guarantee (asymptotic) consensus under control constraints and for time-varying communication graphs. Notably, the proof of the main result does not rely on the standard arguments used in predictive control to guarantee closed-loop stability, see *e.g.* [15], [16], since the global cost to be minimized by the MPC algorithm is not monotonically decreasing and, as such, it cannot be used as a Lyapunov function. Rather, our proof exploits geometric properties of the optimal path followed by individual agents and relies on the general results described in [2] for analyzing consensus.

The paper is organized as follows: Section II is devoted to the definition of models for the agents' dynamics and the communication network. Section III summarizes some key results on convergence in multi-agent systems presented in [2] and used in this paper. Section IV describes the main result. Section V consists of two simulation examples and Section VI is devoted to conclusions. Finally Appendix A contains technical results used in the proof of convergence that is given in Appendix B.

II. Basic notions

We consider a set of n agents with discrete-time dynamics

$$x_i(k+1) = x_i(k) + u_i(k),\ i = 1, \dots, n \tag{1}$$

and initial condition $x_i(0) = \tilde{x}_i$, where $x_i(k) \in \mathbb{R}^d$ and $u_i(k) \in \mathbb{R}^d$ are the state and the control input, respectively, of agent i at time $k \in \mathbb{N}$. As an example, considering a group of autonomous vehicles moving in a d-dimensional geometric space, $x_i(k)$ and $u_i(k)$ can describe the position and control of agent i.

The communication network is modeled as a graph $G = (\mathcal{N}_G, \mathcal{E}_G)$, where $\mathcal{N}_G = \{1, \dots, n\}$ is the set of nodes indexing individual agents and $\mathcal{E}_G \subseteq \{(i,j) : i, j \in \mathcal{N}_G, j \neq i\}$ is the set of edges. We assume that G is *undirected*, which means that $(i,j) \in \mathcal{E}_G \Leftrightarrow (j,i) \in \mathcal{E}_G$. The *adjacency matrix* defined on G is the $n \times n$ matrix $A(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } \exists (j,i) \in \mathcal{E}_G \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

If $(j,i) \in \mathcal{E}_G$ we say that j is *neighbor* of i and j-th agent transmits instantaneously its state to the i-th agent. The set of neighbors to the node $i \in \mathcal{N}_G$ is $\mathcal{N}_i(G) = \{j \in \mathcal{N}_G : (j,i) \in \mathcal{E}_G\}$ and $|\mathcal{N}_i|$ is the valency of the i-th node. The *valency matrix* is $V(G) = \mathrm{diag}\{|\mathcal{N}_1|, \ldots, |\mathcal{N}_n|\}$. We introduce the matrices

$$\tilde{K}(G) = [V(G) + I_n]^{-1}[I_n + A(G)]$$

and

$$K(G) = \tilde{K}(G) \otimes I_d$$

where $\otimes$ denotes the Kronecker product and I_d is the identity matrix of order d. The matrix $\tilde{K}(G)$ is stochastic (*i.e.* it is square, nonnegative and its row sums are equal to 1) and $K(G)$ enjoys the same property (see [17]). Note that the entry (i,j) of $\tilde{K}(G)$ is non null if and only if $i = j$ or $(j,i) \in \mathcal{E}_G$.
The undirected graph G is *connected* if, $\forall (i,j) \in \mathcal{N}_G \times \mathcal{N}_G$, there exists a path from i to j. The creation and loss of communication links can be modeled by means of a time-dependent collection of graphs $\{G(k) = (\mathcal{N}_G, \mathcal{E}_G(k)), k \in \mathbb{N}\}$.

Definition 1 [17] *A collection of graphs* $\{G(1), \ldots, G(m)\}$ *is* jointly connected *if* $\cup_{i=1}^m G(i) = (\mathcal{N}_G, \cup_{i=1}^m \mathcal{E}_G(i))$ *is connected. The agents are* linked together *on the time interval* $l, (l+1), \ldots, m$, $l \le m \le +\infty$ *if the collection of graphs* $\{G(k), k = l, \ldots, m\}$ *is jointly connected.*

III. Preliminary results

For sake of completeness we review here results provided in [2] that will enable us to prove consensus under the MPC we will propose in the sequel. Assume that agents obey to the general closed-loop dynamics

$$x(k+1) = f(k, x(k)) \tag{3}$$

where $x(k) = [x_1(k)^T \cdots x_n(k)^T]^T$. Let Φ be the set of equilibria for (3).

Definition 2 [2] *System* (3) *is globally attractive w.r.t.* Φ *if for each* $\phi_1 \in \Phi$, $\forall c_1, c_2 > 0$ *and* $\forall k_0 \in \mathbb{N}$, $\exists T \ge 0$ *such that every solution* ζ *to* (3) *has the following property:*

$$|\zeta(k_0) - \phi_1| < c_1 \Rightarrow \exists \phi_2 \in \Phi : |\zeta(k) - \phi_2| < c_2, \forall k \ge k_0 + T.$$

Definition 3 *The multi-agent system* (3) *asymptotically reaches consensus if it is globally attractive w.r.t.* $\Phi = \{x \in \mathbb{R}^d : x_1 = x_2 = \cdots = x_n\}$.

The consensus results stated in [2] hinge on the following convexity assumption.

Assumption 1 *For every graph* $G(k)$, *agent* $i \in \mathcal{N}_G$ *and state* $x \in X^n, X \subseteq \mathbb{R}^d$, *there is a compact set* $e_i(G(k))(x) \subseteq X$ *such that:*

1) $f_i(x,k) \in e_i(G(k))(x), \forall k \in \mathbb{N}, \forall x \in X^n$;
2) $e_i(G(k))(x) = \{x_i\}$ *if* $x_i = x_j, \forall j \in \mathcal{N}_i(G(k))$;
3) *whenever the states of agent* i *and agents* $j \in \mathcal{N}_i(G(k))$ *are not all equal,* $e_i(G(k))(x) \in \mathrm{Ri}(\mathrm{Co}(\{x_i(k)\} \cup \{x_j(k), j \in \mathcal{N}_i(k)\}))$, *where* $\mathrm{Co}(A)$ *and* $\mathrm{Ri}(A)$ *denote the convex hull and the relative interior, respectively, of the set* A;
4) *the set-valued function* $e_i(G(k))(x) : X^n \mapsto 2^X$ *is continuous* (2^X *is the power set of* X).

We are now in a position to state the main Theorem on consensus we will use.

Theorem 1 [2] *Let* $\{G(k), k \in \mathbb{N}\}$, *be a collection of undirected graphs and assume that* f *in* (3) *verifies Assumption 1. Then, the system* (3) *asymptotically reaches consensus if and only if,* $\forall k_0 \in \mathbb{N}$, *all agents are linked together across the interval* $[k_0, +\infty)$.

IV. Problem formulation and solution

In this section we propose a distributed control strategy for consensus based on an MPC scheme defined on a finite horizon of length $N \ge 1$. We associate to the i-th agent, whose dynamics is described by (1), the input vector $U_i(k) = [u_i^T(k) \cdots u_i^T(k+N-1)]^T$ and the cost

$$J_i(x(k), U_i(k)) = J_i^x(x(k), U_i(k)) + J_i^u(U_i(k)) \tag{4}$$

with

$$J_i^x(x(k), U_i(k)) = q_i \sum_{j=1}^{N} \|x_i(k+j) - z_i(k)\|^2 \tag{5}$$

$$J_i^u(U_i(k)) = r_i \sum_{j=0}^{N-1} \|u_i(k+j)\|^2 \tag{6}$$

where $q_i, r_i > 0$ are weights, $x(k) = [x_1^T(k) \cdots x_n^T(k)]^T$ is the state of the multi-agent system at the beginning of the prediction horizon, $z_i(k) \doteq K_i(G(k))x(k)$ and $K_i(G(k))$ is the i-th block of the matrix $K(G(k))$, partitioned as $K(G(k)) = [K_1^T(G(k)) \cdots K_n^T(G(k))]^T$, with $K_i(G(k)) \in \mathbb{R}^{d \times dn}$, $i = 1, ..., n$. Notice that $z_i(k)$ is just the barycenter of $\{x_i(k)\} \cup \{x_j(k), j \in \mathcal{N}_i(k)\}$. Cost (4) is decentralized, because the term $K_i(G(k))x(k)$ depends on the state variables of neighbors to the i-th agent at time k.
Consider the following finite-horizon control problem for agent $i \in \mathcal{N}_G$:

$$\min_{U_i(k)} J_i(x(k), U_i(k)) \tag{7}$$

subject to the following constraints:

(a) the agent dynamics (1);
(b) the input constraint $\|u_i(k+j)\| \le u_{i,max}$, $u_{i,max} > 0, \forall i \in \mathcal{N}_G, \forall j \in \{0, \ldots, (N-1)\}$.

Optimal inputs will be denoted with $U_i^o(x(k)) = [u_i^{oT}(k) \cdots u_i^{oT}(k+N-1)]^T$ and we will investigate the consensus properties provided by the Receding-Horizon control law

$$u_i^{RH}(k) = \kappa_i^{RH}(x(k)),\ \kappa_i^{RH}(x(k)) = u_i^o(k) \tag{8}$$

Note that problem (7) is always feasible since $U_i(x(k)) = 0$ is a feasible input.

Theorem 2 *Consider the closed-loop multi-agent system given by*

$$x(k+1) = f(x(k)) \tag{9}$$

where

$$f(x(k)) = \begin{bmatrix} f_1(x(k)) \\ \vdots \\ f_n(x(k)) \end{bmatrix},\ f_i(x(k)) = x_i(k) + \kappa_i^{RH}(x_i(k))$$

Then, the map f fulfills Assumption 1.

The proof of this Theorem can be found in Appendix B and is based on the geometrical properties of optimal state trajectories that are provided in Appendix A.

We are now in a position to state the main result of the paper.

Theorem 3 *Let $G(k), k \in \mathbb{N}$ be a sequence of undirected graphs sharing the node set $\mathcal{N}_G = \{1, \dots, n\}$. Then the closed-loop multi-agent system given by* (1) *and* (8) *asymptotically reaches consensus if and only if, $\forall k_0 \in \mathbb{N}$, all agents are linked together across the interval $[k_0, +\infty)$.*

Proof: In view of Theorem 2, the proof is a straightforward application of Theorem 1. ■

We highlight that the application of the Receding-Horizon technique to the finite-horizon optimal control problem (7) produces a dependency of the consensus point from both the instantaneous communication network and the agents' states.

V. Simulation results

Example 1 *We consider a set of $n = 5$ agents moving in a bidimensional space, with initial states $x_1(0) = [-30\ 30]^T$, $x_2(0) = [-25\ \ 35]^T$, $x_3(0) = [65\ \ -75]^T$, $x_4(0) = [70\ -68]^T$, $x_5(0) = [100\ -25]^T$. We consider a prediction horizon $N = 3$. The weights in the cost function* (4) *are $q_i = 1, i = 1, \dots, 5$, $r_1 = 100$, $r_i = 1, i = 2, \dots, 5$. The communication network is described by the time-invariant undirected graph represented in Fig. 1, that corresponds to the following matrix:*

$$\tilde{K}(G) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{10}$$

The input constraints (b) *are given by $u_{i,max} = 100, i = 1, \dots, 5$. The simulation in Fig. 2 shows asymptotic convergence towards a consensus point. Since $r_1 \gg r_i, i = 2, 3, 4, 5$, agent 1 behaves approximatively as a leader, in the sense that its state influences the movement of the neighbors while its movement is almost uninfluenced by the others' states. In this case, input constraints are never active.*

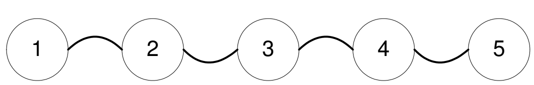

Fig. 1. Communication network.

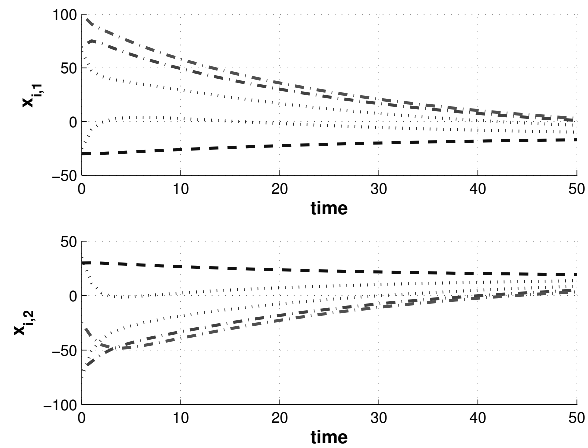

Fig. 2. Example 1: Evolution of agents' position. Dashed line: path followed by agent 1.

It is important to notice that in this example both the cost functions $J_1(x(k), U_1(k))$ and $J(x(k), U(k)) = \sum_{i=1}^{5} J_i(x(k), U_i(k))$ are not monotonically decreasing over time (see Fig. 3). This implies that the global cost function $J(x(k), U(k))$ cannot be interpreted as a Lyapunov function as it is usually done to prove the stability of MPC algorithms, see e.g. *[15], [16]. This justifies the necessity of an alternative technique to prove convergency, as the one we propose, based on geometrical concepts.*

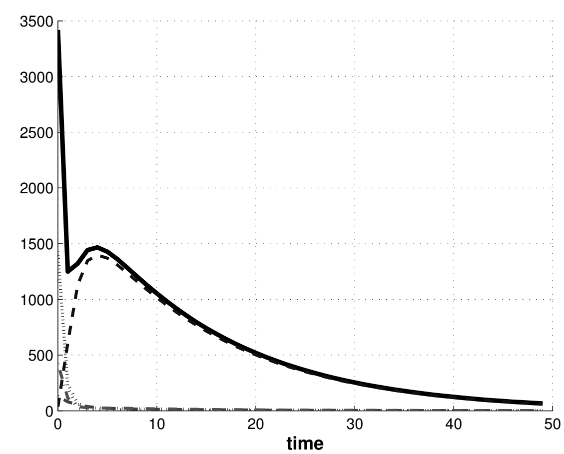

Fig. 3. Example 1: Cost of individual agents and global cost $J(x(k), U(k)) = \sum_{i=1}^{5} J_i(x(k), U_i(k))$ (continuous line).

Example 2 *We use the same setting of Example 1, but now we suppose $r_i = 1, i = 1, \dots, 5$ and $u_{i,max} = 5$. We also assume that the communication network is time-varying, while it respects the connection condition required in Theorem*

3 for consensus. More in details, we generated a random sequence of undirected graphs such that all agents are linked together across the interval $[k_0, +\infty), \forall k_0 \in \mathbb{N}$. In this case, input constraints are active (see Fig. 5). Nevertheless, consensus is asymptotically achieved, in accordance with Theorem 3.

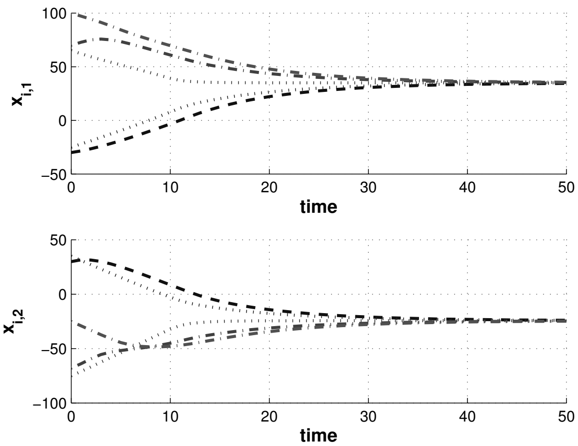

Fig. 4. Example 2: Evolution of agents' position.

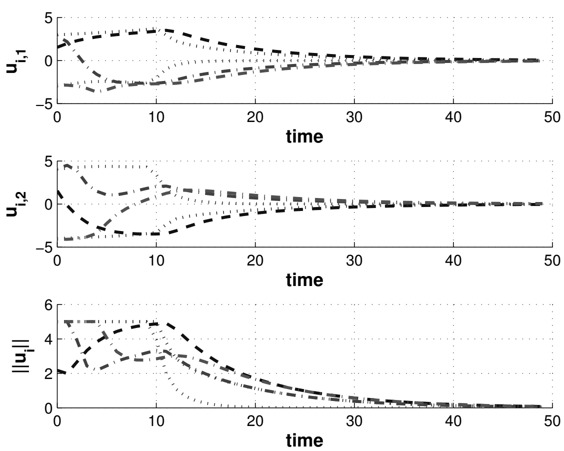

Fig. 5. Example 2: Control inputs of individual agents.

VI. CONCLUSIONS

We have proposed an MPC control scheme able to guarantee consensus in a multi-agent system where individual dynamics are described by a single-integrator model. The proof of consensus, which holds under suitable assumptions on the communication graph, relies on a convergence result by Moreau ([2]), which is applicable because of the particular properties of optimal state trajectories. Future extensions of this work will concern alternative MPC schemes to control agents with double-integrator dynamics, as well as an evaluation of the effect of communication delays and/or uncertainties on the performances of the proposed control schemes.

APPENDIX A
PROPERTIES OF DISCRETE-TIME PATHS

Given the points $P_1, P_2 \in \mathbb{R}^d$, let $\overline{P_1P_2}$ be the segment joining them and denote with $|\overline{P_1P_2}|$ the segment length. The straight line passing through P_1 and P_2 will be denoted with $r_{P_1P_2}$. An N-path is an ordered sequence of N points $T = \{P_1, P_2, \ldots, P_N\} \subset \mathbb{R}^d$.

Definition 4 *Two N-paths $T_A = \{A_1, \ldots, A_N\} \subset \mathbb{R}^d$, $T_B = \{B_1, \ldots, B_N\} \in \mathbb{R}^d$ are* equivalent *with respect to a point $O \in \mathbb{R}^d$ if $|\overline{A_jO}| = |\overline{B_jO}|,\ j = 1, \ldots, N$ and we write $T_A \overset{\circ}{\sim} T_B$.*

Note that $\overset{\circ}{\sim}$ is an equivalence relation.

Definition 5 *An N-path $T = \{A_1, \ldots, A_N\} \subset \mathbb{R}^d$ is straight if $A_j \in r_{A_1A_N}, \forall j \in \{1, \ldots, N\}$.*

Note that, given an N-path T_A and $O \in \mathbb{R}^d$, there always exists a straight N-path T_B verifying $T_A \overset{\circ}{\sim} T_B$. This property is depicted in Fig. 6.

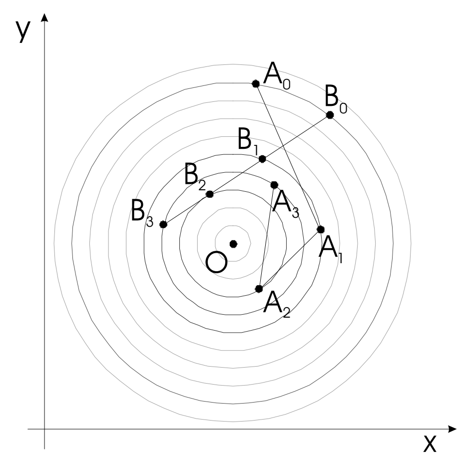

Fig. 6. Two paths T_A and T_B in $\mathbb{R}^2$ equivalent with respect to O.

For $P, Q \in \mathbb{R}^d$ let s_{PQ} be a the straight half-line starting from P and passing through Q.

Lemma 1 *For a given $O \in \mathbb{R}^d$, let $T_A = \{A_1, \ldots, A_N\} \subset \mathbb{R}^d$ and $T_C = \{C_1, \ldots, C_N\} \subset \mathbb{R}^d$ be two N-paths such that $T_C \overset{\circ}{\sim} T_A$, and $C_j \in s_{OC_1},\ j = 1, \ldots, N$. Then, it holds $|A_jA_{j+1}| \geq |C_jC_{j+1}|,\ j = 1, \ldots, (N-1)$.*

Proof: Since $C_j \in s_{OC_1}$ we have two possibilities, for $j = 1, \ldots, (N-1)$:

- if $|\overline{C_{j+1}O}| \leq |\overline{C_jO}|$ one has

$$\begin{aligned} |\overline{A_jO}| &\leq |\overline{A_{j+1}O}| + |\overline{A_jA_{j+1}}| \\ |\overline{C_jO}| &= |\overline{C_{j+1}O}| + |\overline{C_jC_{j+1}}| \end{aligned}$$

- if $|\overline{C_{j+1}O}| > |\overline{C_jO}|$ one has

$$\begin{aligned} |\overline{A_{j+1}O}| &\leq |\overline{A_jO}| + |\overline{A_jA_{j+1}}| \\ |\overline{C_{j+1}O}| &= |\overline{C_jO}| + |\overline{C_jC_{j+1}}| \end{aligned}$$

and the result follows immediately from the definition of equivalent trajectories. ∎

We highlight that under the assumptions of the previous Lemma, one has

$$\sum_{j=1}^{N-1} |\overline{A_j A_{j+1}}|^2 \geq \sum_{j=1}^{N-1} |\overline{C_j C_{j+1}}|^2$$

Moreover, this inequality is strict if $\exists j : A_j \notin s_{OA_1}$.

Definition 6 *An N-path is* non-increasing with respect to (w.r.t.) $O \in \mathbb{R}^d$ *if* $|\overline{A_{j+1}O}| \leq |\overline{A_jO}|, \forall j \in \{1, \ldots, (N-1)\}$.

Note that if T_A is non-increasing w.r.t. O and $T_B \overset{\circ}{\sim} T_A$ then also T_B enjoys the same property.

Definition 7 *The N-path $T_A = \{A_1, \ldots, A_N\}$ is pointing towards $O \in \mathbb{R}^d$ if it is non-increasing w.r.t. O and $A_j \in \overline{A_1O}, \forall j \in 1, \ldots, N$.*

Note that if T_A points towards O one has $|\overline{A_1 A_N}| = \sum_{j=1}^{N-1} |A_j A_{j+1}|$.

Theorem 4 *Let $T_A = \{A_1, \ldots, A_N\} \in \mathbb{R}^d$ be an N-path. Given $O \in \mathbb{R}^d$, there always exists an N-path $T_B = \{B_1, \ldots, B_N\} \in \mathbb{R}^d$ with $B_1 = A_1$, pointing towards O and that verifies the following inequalities:*

$$|\overline{B_jO}| = |\overline{A_jO}|, \; j = 1, \ldots, N \tag{11}$$

$$|\overline{B_j B_{j+1}}| \leq |\overline{A_j A_{j+1}}|, \; j = 1, \ldots, (N-1) \tag{12}$$

Proof: From Lemma 1 there exists an N-path $T_C = \{C_1, \ldots, C_N\} \subset \mathbb{R}^d : T_C \overset{\circ}{\sim} T_A$, $C_1 = A_1$ and verifying $C_j \in s_{OC_1}, j = 1, \ldots, N$. We set $B_1 = C_1 = A_1$ and build up T_B in a recursive fashion.

Assume that (11) and (12) hold for all $j \in \{1, \ldots, \bar{j}\}, \bar{j} < N$.

- If $B_{\bar{j}} = C_{\bar{j}}$ and $|\overline{C_{\bar{j}+1}O}| \leq |\overline{C_{\bar{j}}O}|$ we set $B_{\bar{j}+1} = C_{\bar{j}+1}$ and inequalities (11), (12) are verified for $j = \bar{j}+1$ since $|\overline{B_{\bar{j}+1}O}| = |\overline{C_{\bar{j}+1}O}|$ and $|\overline{B_{\bar{j}}B_{\bar{j}+1}}| = |\overline{C_{\bar{j}}C_{\bar{j}+1}}|$.
- If $B_{\bar{j}} = C_{\bar{j}}$ and $|\overline{C_{\bar{j}+1}O}| > |\overline{C_{\bar{j}}O}|$, it results $|\overline{B_{\bar{j}+1}O}| < |\overline{C_{\bar{j}+1}O}|$ for any choice of $B_{\bar{j}+1}$ such that $|\overline{B_{\bar{j}}B_{\bar{j}+1}}| \leq |\overline{C_{\bar{j}}C_{\bar{j}+1}}|$, having imposed the following conditions: $|\overline{B_{\bar{j}+1}O}| \leq |\overline{B_{\bar{j}}O}|$ and $B_{\bar{j}+1} \in \overline{A_1O}$.
- If $B_{\bar{j}} \neq C_{\bar{j}}$ and $|\overline{C_{\bar{j}+1}O}| \leq |\overline{B_{\bar{j}}O}|$ we set $B_{\bar{j}+1} = C_{\bar{j}+1}$; in this way conditions (11) and (12) are verified, because $|\overline{B_{\bar{j}+1}O}| = |\overline{C_{\bar{j}+1}O}|$ and $|\overline{B_{\bar{j}}B_{\bar{j}+1}}| < |\overline{C_{\bar{j}}C_{\bar{j}+1}}|$. The latter inequality, in particular, follows from the fact that necessarily $|\overline{C_{\bar{j}}O}| > |\overline{B_{\bar{j}}O}|$ and from the system of equations
$$\begin{aligned} |\overline{C_{\bar{j}}O}| &= |\overline{C_{\bar{j}+1}O}| + |\overline{C_{\bar{j}}C_{\bar{j}+1}}| \\ |\overline{B_{\bar{j}}O}| &= |\overline{B_{\bar{j}+1}O}| + |\overline{B_{\bar{j}}B_{\bar{j}+1}}| \end{aligned}$$
- If $B_{\bar{j}} \neq C_{\bar{j}}$ and $|\overline{C_{\bar{j}+1}O}| > |\overline{B_{\bar{j}}O}|$, one has $|\overline{B_{\bar{j}+1}O}| < |\overline{C_{\bar{j}+1}O}|$ for any choice of $B_{\bar{j}+1}$ such that $|\overline{B_{\bar{j}}B_{\bar{j}+1}}| \leq |\overline{C_{\bar{j}}C_{\bar{j}+1}}|$. This follows from the conditions $|\overline{B_{\bar{j}+1}O}| \leq |\overline{B_{\bar{j}}O}|$ and $B_{\bar{j}+1} \in \overline{A_1O}$.

For the auxiliary trajectory T_C, recalling that $T_C \overset{\circ}{\sim} T_A$, one has $|\overline{C_jO}| = |\overline{A_jO}|, j = 1, ..., N$. Furthermore, from Lemma 1, the inequality $|\overline{C_j C_{j+1}}| \leq |\overline{A_j A_{j+1}}|, j = 1, \ldots, (N-1)$ follows.

Thus in all cases it is possible to construct an N-path T_B pointing towards O and verifying (11) and (12). ∎

If the N-paths T_A and T_B fulfill the assumptions of Theorem 4, one has

$$\sum_{j=1}^{N} |\overline{B_jO}|^2 \leq \sum_{j=1}^{N} |\overline{A_jO}|^2 \tag{13}$$

Moreover we have the following result:

Corollary 1 *Under the assumptions of Theorem 4, $T_B = T_A$ if and only if T_A is pointing towards O.*

Proof: The implication $\Rightarrow$ is trivial. Assume that T_A points towards O. Then under the assumption of Theorem 4, one has $\sum_{j=1}^{N-1} |\overline{B_j B_{j+1}}|^2 \leq \sum_{j=1}^{N-1} |\overline{A_j A_{j+1}}|^2$. The inequality is strict only if $\exists j : A_j \notin s_{OA_1}$. However this case cannot happen if T_A is pointing towards O. Therefore

$$|\overline{B_j B_{j+1}}| = |\overline{A_j A_{j+1}}|, \forall j = 1, \ldots, (N-1)$$

and from the fact that $A_1 = B_1$ the thesis follows. ∎

Appendix B
Technical proof of Theorem 2

Proof: With the notations used in Assumption 1, we set $X = \mathbb{R}^d$ and $e_i(G(k))(x) = \{f_i(x)\}$. Then point 1 of Assumption 1 is trivially verified. For point 2, when $x_i(k) = x_j(k), \forall j \in \mathcal{N}_i(G(k))$, then $z_i(k) = x_i(k)$. From the definition of the cost J_i, it is immediate to verify that $J_i(x(k), 0) = 0$ and hence the optimal inputs $U_i^o(x(k))$ are zero. This implies that $f_i(x) = x_i$, and point 2 is verified.

The most difficult part of the proof consists in verifying point 3. If there exists $j \in \mathcal{N}_i(G(k))$ such that $x_i(k) \neq x_j(k)$ one has that $x_i(k) \neq z_i(k)$.

Let $X_i(x(k), U_i(k)) = [x_i^T(k) \cdots x_i^T(k+N)]^T$ be the vector collecting the states of the i-th agent when the input sequence $U_i(k)$ is applied, with $u_i(k+j), j = 0, \ldots, (N-1)$ fulfilling the constraint (b).

As a first step we show that $X_i^o(x(k), U_i^o(k)) = [x_i^{oT}(k) \cdots x_i^{oT}(k+N)]^T$ is a path pointing towards $z_i(k)$. Note that, because of the single-integrator dynamics (1), expression J_i^u can be rewritten as:

$$J_i^u(U_i(k)) = r_i \sum_{j=0}^{N-1} \|x_i(k+j+1) - x_i(k+j)\|^2$$

Theorem 4 applied with $O = z_i(k)$ shows that, given the generic N-path $X_i(k) = [x_i^T(k) \cdots x_i^T(k+N)]^T$, there is an N-path $\hat{X}_i(k) = [\hat{x}_i^T(k) \cdots \hat{x}_i^T(k+N)]^T$ with $\hat{x}_i(k) = x_i(k)$ pointing towards $z_i(k)$ and such that, $\forall j = 0, \ldots, (N-1)$, both the following inequalities hold:

$$\|\hat{x}_i(k+j+1) - z_i(k)\| \leq \|x_i(k+j+1) - z_i(k)\| \tag{14}$$

$$\|\hat{x}_i(k+j+1)-\hat{x}_i(k+j)\| \leq \|x_i(k+j+1)-x_i(k+j)\| \quad (15)$$

Inequality (15) implies that the input vector $\hat{U}_i(k) = [\hat{u}_i^T(k)\cdots\hat{u}_i^T(k+N-1)]^T$, producing $\hat{X}_i(x(k))$, fulfills the constraint (b). Moreover (14) implies that $J_i^x(x(k),\hat{U}_i(k)) \leq J_i^x(x(k),U_i(k))$ and (15) that $J_i^u(\hat{U}_i(k)) \leq J_i^u(U_i(k))$. From Corollary 1, one concludes that the optimal control vector is $U_i^o(k)=\hat{U}_i(k)$; correspondingly, the optimal state path for agent i is $X_i^o(x(k),U_i^o(k)) = \hat{X}_i(x(k),\hat{U}_i(k))$ and therefore it points towards $z_i(k)$.

As a second step, we show that $x_i(k) \neq z_i(k)$ implies that $u_i^o(k) \neq 0$. This will be proved by first showing that $U_i^o \neq 0$. By contradiction assume that $U_i^o(x(k)) = 0$ is the optimal input sequence. Then $J_i(x(k),U_i^o(k)) = Nq_i\|x_i(k)-z_i(k)\|^2$. Consider the input sequence $\bar{U}_i(k) = [(\alpha(x_i(k)-z_i(k)))^T\ 0^T\cdots 0^T]^T,\ 0<\alpha\leq 1$, where α is such that $\|\alpha(x_i(k)-z_i(k))\|^2 < u_{i,max}$, and let $\bar{X}(k) = [\bar{x}(k)\cdots\bar{x}(k+N)]$ be the corresponding state path. One has

$$\begin{aligned} J_i(x(k),\bar{U}_i(k)) =& Nq_i(1-\alpha)^2\|x_i(k)-z_i(k)\|^2 \\ &+ r_i^2\alpha^2\|x_i(k)-z_i(k)\|^2 \end{aligned} \quad (16)$$

For $\alpha < \frac{2Nq_i}{Nq_i+r_i}$ one has $J(x(k),\bar{U}_i(k)) < J(x(k),\bar{U}_i^o(k))$ and therefore $U_i^o(k) = 0$ cannot be the optimal input sequence.

Now we prove that $u_i^o(k)\neq 0$. By contradiction, assume that the optimal input vector is $U_i^o(k) = [0^T\ \ u_i^{oT}(k+1)\ \cdots\ u_i^{oT}(k+N-1)]^T$, with $u_i^{oT}(k+1)\neq 0$. Now consider the input $\bar{U}_i(k) = [u_i^{oT}(k+1)\ \cdots\ u_i^{oT}(k+N-1)\ 0^T]^T$. We observe that all input samples in $\bar{U}_i(k)$ verify the constraint (b), as those of $U_i^o(k)$ do. Then,

$$\begin{aligned} J(x(k),&\bar{U}_i(k)) - J(x(k),U_i^o(k)) \\ &= q_i\left\|x_i(k)+\sum_{j=0}^{N-1}u_i^o(k+j)-z_i(k)\right\|^2 \\ &\quad - q_i\|x_i(k)-z_i(k)\|^2 < 0 \end{aligned} \quad (17)$$

Inequality (17) follows from the fact that some input $u_i^o(k+j), j=0,\ldots,(N-1)$ is non null and every input makes the state follow a path that points towards $z_i(k)$.

The property of stochasticity of the matrix $K(G(k))$, together with the assumptions that $\mathcal{N}_i(G)\neq\emptyset$ and $x_i(k)\neq z_i(k)$ implies that $z_i(k)\in \mathrm{Ri}(\mathrm{Co}(\{x_i(k)\}\cup\{x_j(k), j\in\mathcal{N}_i(k)\}))$. When also $x_i(k)\in\mathrm{Ri}(\mathrm{Co}(\{x_i(k)\}\cup\{x_j(k), j\in\mathcal{N}_i(k)\}))$ it is trivial to conclude that the optimal trajectory belongs to $\mathrm{Ri}(\mathrm{Co}(\{x_i(k)\}\cup\{x_j(k), j\in\mathcal{N}_i(k)\}))$. When $x_i(k)\in cl(\mathrm{Co}(\{x_i(k)\}\cup\{x_j(k), j\in\mathcal{N}_i(k)\}))$, by applying the *line segment principle* [18] the same conclusion follows because the optimal trajectory is totally included in the segment connecting $x_i(k)$ and $z_i(k)$.

Finally, also point 4 in Assumption 1 is verified. In fact, since we have proven that the optimal state trajectory $X_i^o(x_i(k),U_i^o(k))$ is pointing towards $z_i(k)$, it is possible to write: $x_i(k+1) = x_i(k)+\beta(k)L(k)(z_i(k)-x_i(k)), \forall k\in\mathbb{N}$, where $0\leq L(k)\leq 1$ (specifically, $L(k)=0$ only when $z_i(k)=x_i(k)$) and

$$\beta(k) = \begin{cases} 1 & \text{if } L(k)\|z_i(k)-x_i(k)\|\leq u_{i,max} \\ \frac{u_{i,max}}{L(k)\|z_i(k)-x_i(k)\|} & \text{otherwise} \end{cases}$$

It is then evident that the function $f_i(x)$ is continuous, and therefore $e_i(G(k))(x)$ is also continuous.

In conclusion, Assumption 1 holds and the proof is complete. ■

ACKNOWLEDGMENTS

The work has been partially done in the framework of the HYCON Network of Excellence, contract number FP6-IST-511368, and it has also been supported by MIUR Project "Advanced methodologies for hybrid systems control".

REFERENCES

[1] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: Algorithms and theory," *IEEE Trans. on Autom. Control*, vol. 51, no. 3, pp. 401–420, 2006.

[2] L. Moreau, "Stability of multi-agent systems with time-dependent communication links," *IEEE Trans. Automat. Control*, vol. 50, no. 2, pp. 198–182, 2005.

[3] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans on Autom. Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[4] H. Tanner, A. Jadbabaie, and G. Pappas, "Stable flocking of mobile agents, part I : Fixed topology," in *Proceedings of the 42^{nd} IEEE Conference on Decision and Control*, 2003, pp. 2010–2015.

[5] ——, "Stable flocking of mobile agents, part II : Dynamic topology," in *Proceedings of the 42^{nd} IEEE Conference on Decision and Control*, 2003, pp. 2016–2021.

[6] J. Cortes, S. Martinez, and F. Bullo, "Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions," *IEEE Trans. on Autom. Control*, vol. 51, no. 8, pp. 1289–1298, 2006.

[7] G. Ferrari-Trecate, A. Buffa, and M. Gati, "Analysis of coordination in multi-agent systems through partial difference equations," *IEEE Trans. on Autom. Control*, vol. 51, no. 6, pp. 1058–1063, 2006.

[8] D. Bauso, L. Giarré, and R. Pesenti, "Non-linear protocols for optimal distributed consensus in networks of dynamic agents," *System and Control Letters*, vol. 55, no. 11, pp. 918–928, 2006.

[9] E. Camponogara, D. Jia, B. Krough, and S. Talukdar, "Distributed model predictive control," *IEEE Control Systems Magazine*, pp. 44–52, 2002.

[10] W. Dunbar and R. Murray, "Distributed receding horizon control for multi-vehicle formation stabilization," *Automatica*, vol. 42, pp. 549–558, 2006.

[11] L. Magni and R. Scattolini, "Stabilizing decentralized model predictive control of nonlinear systems," *Automatica*, vol. 42, pp. 1231–1236, 2006.

[12] T. Keviczki, F. Borrelli, and G. Balas, "Decentralized receding horizon control of large scale dynamically decoupled systems," *Automatica*, vol. 42, pp. 2105–2115, 2006.

[13] D. Raimondo, L. Magni, and R. Scattolini, "Decentralized MPC of nonlinear systems: an input-to-state stability approach," *Int. Journal of Robust and Nonlinear Control*, 2007, in press.

[14] B. Johansson, A. Speranzon, M. Johansson, and K. Johansson, "Distributed model predictive consensus," in *17th International Symposium on Mathematical Theory of Networks and Systems*, July 2006.

[15] D. Mayne, J. Rawlings, C. Rao, and P. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789–814, 2000.

[16] G. De Nicolao, L. Magni, and R. Scattolini, "Stability and robustness of nonlinear receding-horizon control," in *Nonlinear Model Predictive Control*, F. Allgöwer and A. Zheng, Eds. Progress in Systems and Control Theory, Birkhauser Verlag, 2000, pp. 3–22.

[17] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. on Automatic Control*, vol. 48, no. 6, pp. 988 – 1001, 2003.

[18] D. Bertsekas, *Convex Analysis and Optimization*. Athena Scientific, 2003.