写像列の収束

Convergence of mapping sequence

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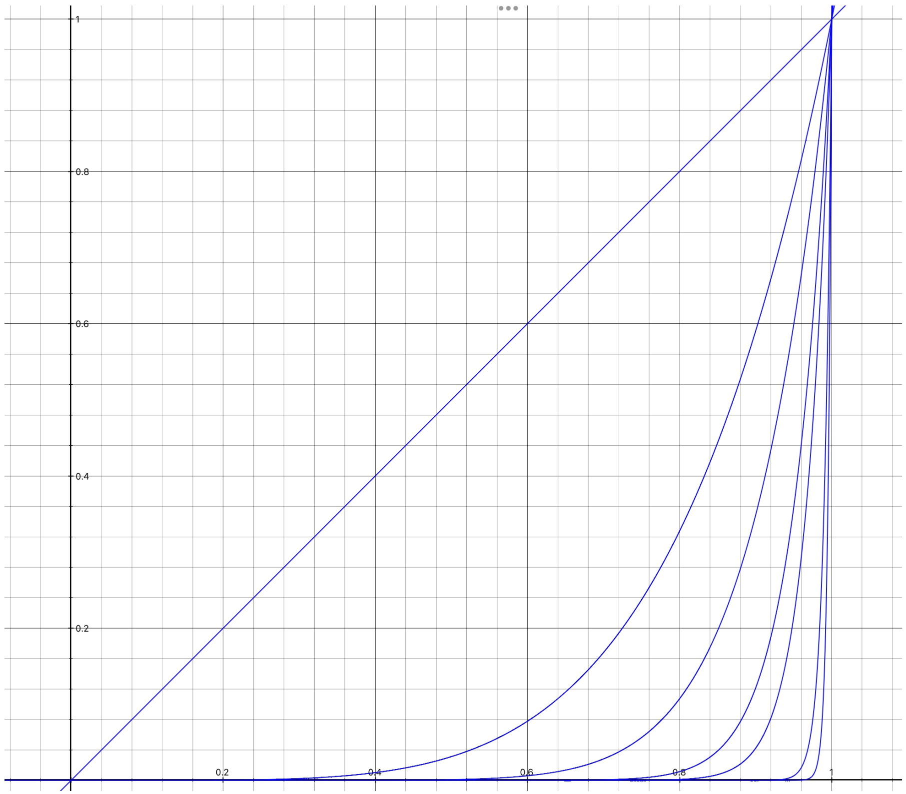
1. 収束する写像列 (Convergent mapping sequence)

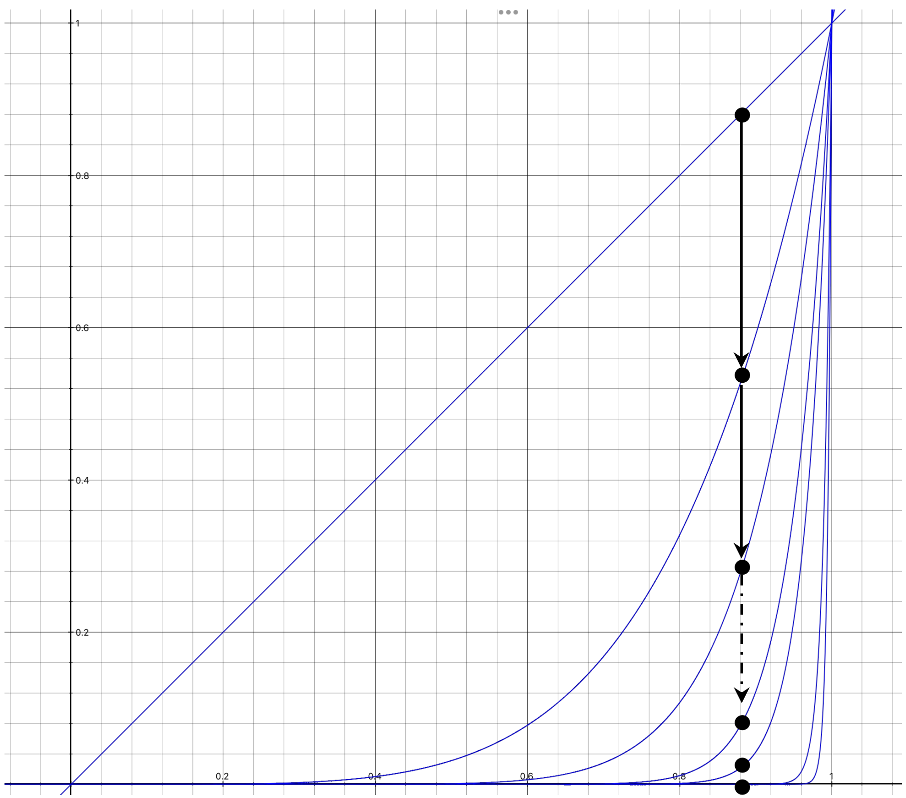


$$f_n(x) = x^n \quad (0 \leq x \leq 1)$$

⇩

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

2. 各点で収束 (Pointwise convergence)

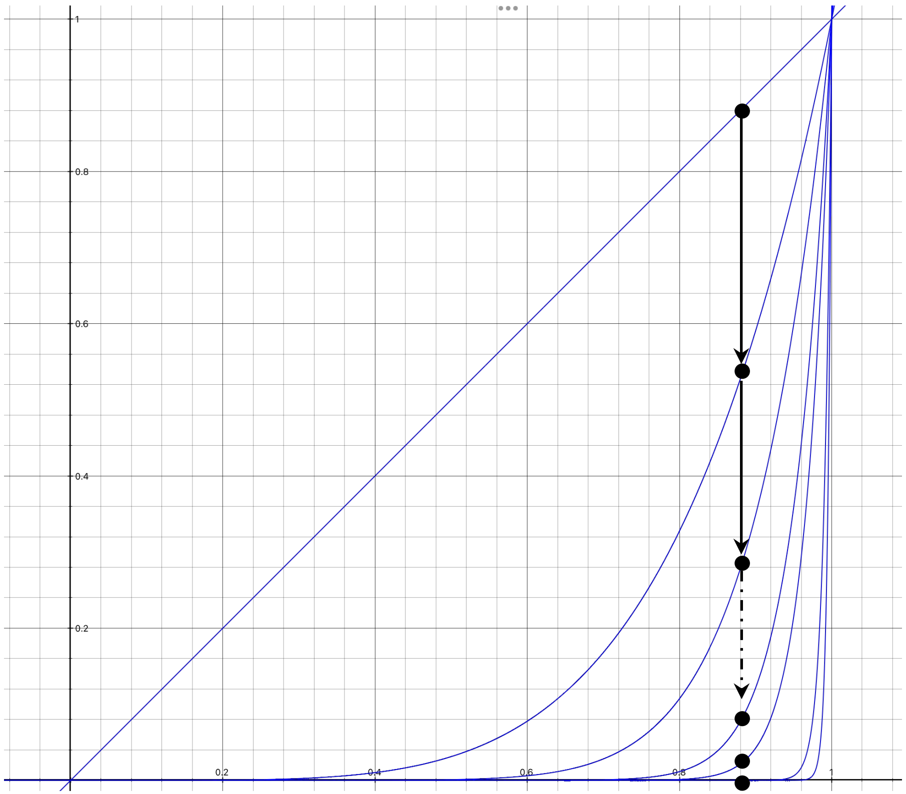


すべての点$x \in [0,1]$で、いかなる正数εについても
$|f_k(x) - f(x)| < \varepsilon$を満たす写像列$\{f_k\}_{n \le k}$が存在する

$$\forall x \in [0,1], \quad \forall \varepsilon \in \mathbb{R}_+, \quad \exists n \in \mathbb{N}, \quad \forall k \in \mathbb{N}:$$

$$n \le k \quad \Rightarrow \quad |f_k(x) - f(x)| < \varepsilon$$

3. 集合的な解釈 (Set theoretical interpretation)



各点で収束

$$\bigcup_{m\in\mathbb{N}}\left\{x\in[0,1]\,\middle|\,\limsup_{n\to+\infty}|f_n(x)-f(x)|\geq\frac{1}{m}\right\}=\emptyset$$

4. 述語と集合 (Representation of convergence)

$$\forall x \in S, \quad \forall \varepsilon \in \mathbb{R}_+, \quad \exists n \in \mathbb{N}, \quad \forall k \in \mathbb{N}: \qquad n \leq k \quad \Rightarrow \quad |f_k(x) - f(x)| < \varepsilon$$

$$\Leftrightarrow$$

$$\forall x \in S, \quad \forall m \in \mathbb{N}, \quad \exists n \in \mathbb{N}, \quad \forall k \in \mathbb{N}: \qquad n \leq k \quad \Rightarrow \quad |f_k(x) - f(x)| < \frac{1}{m}$$

$$\Leftrightarrow$$

$$\forall x \in S: \qquad \neg\left(\exists m \in \mathbb{N}, \quad \forall n \in \mathbb{N}, \quad \exists k \in \mathbb{N}: \quad n \leq k \quad \wedge \quad |f_k(x) - f(x)| \geq \frac{1}{m}\right)$$

$$\Leftrightarrow$$

$$\forall x \in S: \qquad x \notin \bigcup_{m \in \mathbb{N}} \left\{ x \in S \,\middle|\, \limsup_{n \to +\infty} |f_n(x) - f(x)| \geq \frac{1}{m} \right\}$$

$$\Leftrightarrow$$

$$\bigcup_{m \in \mathbb{N}} \left\{ x \in S \,\middle|\, \limsup_{n \to +\infty} |f_n(x) - f(x)| \geq \frac{1}{m} \right\} = \emptyset$$

5. 一様収束 (Uniform convergence)

$$\forall \varepsilon \in \mathbb{R}_+,\quad \exists n \in \mathbb{N},\quad \forall k \in \mathbb{N},\quad \forall x \in S:\qquad n \le k \quad \Rightarrow \quad |f_k(x) - f(x)| < \varepsilon$$

$$\Leftrightarrow$$

$$\forall m \in \mathbb{N},\quad \exists n \in \mathbb{N},\quad \forall k \in \mathbb{N},\quad \forall x \in S:\qquad n \le k \quad \Rightarrow \quad |f_k(x) - f(x)| < \frac{1}{m}$$

$$\Leftrightarrow$$

$$\neg\left(\exists m \in \mathbb{N},\quad \forall n \in \mathbb{N},\quad \exists k \in \mathbb{N},\quad \exists x \in S:\quad n \le k \quad \wedge \quad |f_k(x) - f(x)| \ge \frac{1}{m}\right)$$

$$\Leftrightarrow$$

$$\forall x \in S:\qquad x \notin \bigcup_{m \in \mathbb{N}} \limsup_{n \to +\infty} \left\{ x \in S \,\middle|\, |f_n(x) - f(x)| \ge \frac{1}{m} \right\}$$

$$\Leftrightarrow$$

$$\bigcup_{m \in \mathbb{N}} \limsup_{n \to +\infty} \left\{ x \in S \,\middle|\, |f_n(x) - f(x)| \ge \frac{1}{m} \right\} = \emptyset$$

6. 各点収束と一様収束 (Pointwise or uniform convergence)

各点収束 pointwise convergence

$$\bigcup_{m\in\mathbb{N}}\left\{x\in S \,\middle|\, \limsup_{n\to+\infty}|f_n(x)-f(x)|\geq\frac{1}{m}\right\}=\emptyset$$

一様収束 uniform convergence

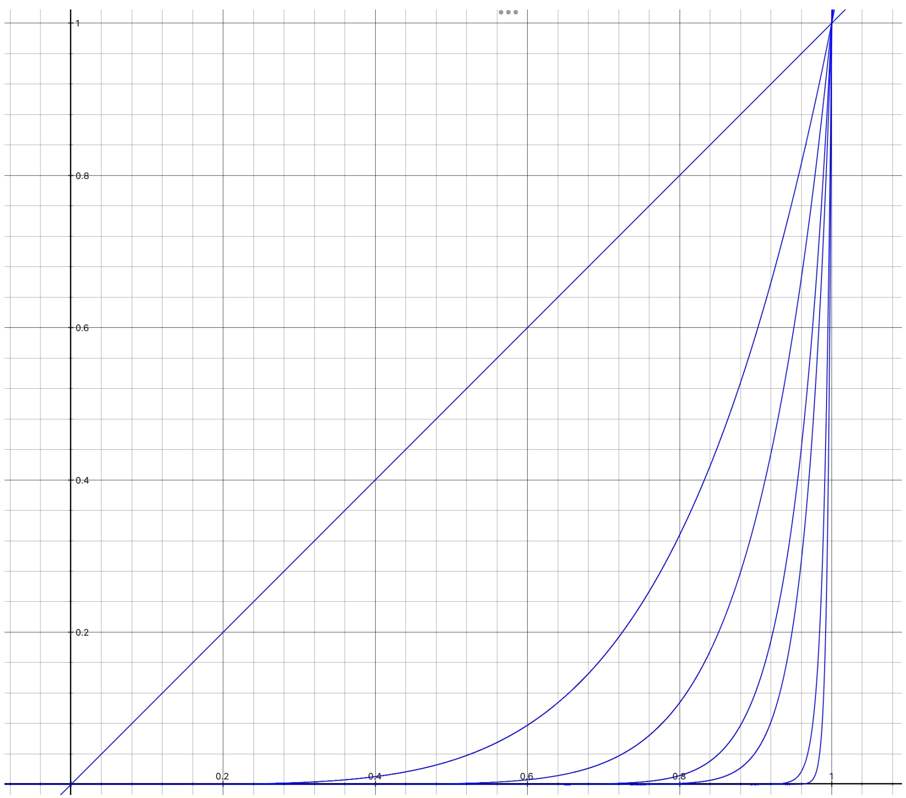
$$\bigcup_{m\in\mathbb{N}}\limsup_{n\to+\infty}\left\{x\in S \,\middle|\, |f_n(x)-f(x)|\geq\frac{1}{m}\right\}=\emptyset$$

一様収束 uniform convergence	$\Rightarrow$	各点収束 pointwise convergence

$$\because\quad \forall m\in\mathbb{N},\ \ \exists n\in\mathbb{N}:\quad n\leq k\quad\Rightarrow\quad |f_k(x)-f(x)|<\frac{1}{m}$$

$$\Rightarrow\quad \lim_{n\to+\infty}f_n(x)=f(x)$$

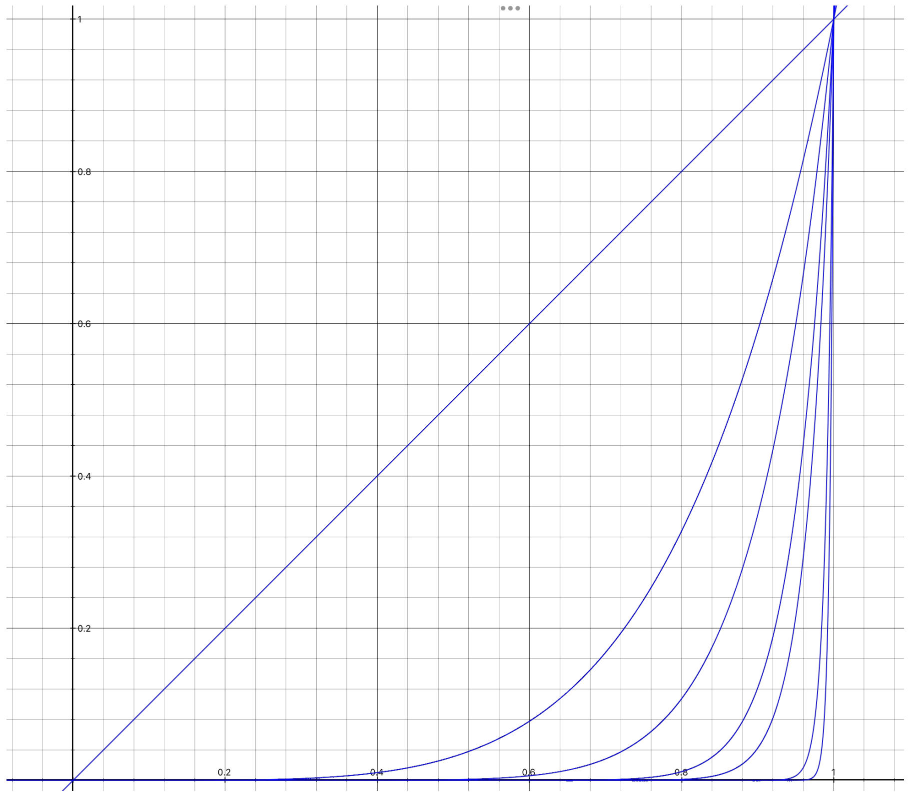
7. 各点収束 $\not\Rightarrow$ 一様収束 ("pointwise" does not imply "uniform")



$$f_n(x) = x^n \quad (0 \le x \le 1)$$

$$\because \quad \bigcup_{m\in\mathbb{N}} \limsup_{n\to+\infty} \left\{ x \in [0,1] \,\middle|\, |f_n(x) - f(x)| \ge \frac{1}{m} \right\} \neq \emptyset$$

7. 各点収束 $\not\Rightarrow$ 一様収束 (“pointwise” does not imply “uniform”)



$$f_n(x) = x^n \quad (0 \le x \le 1)$$

∵ 連続写像列が、写像fに一様連続すれば、fは連続

⇨ $$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$ 不連続

8. いたるところ収束 (convergence a.e.)

いたるところ各点収束(概収束) pointwise convergence a.e.

$$P\left(\bigcup_{m\in\mathbb{N}}\left\{x\in S \,\middle|\, \limsup_{n\to+\infty}|f_n(x)-f(x)|\geq\frac{1}{m}\right\}\right)=0$$

いたるところ一様収束 uniform convergence a.e.

$$P\left(\bigcup_{m\in\mathbb{N}}\limsup_{n\to+\infty}\left\{x\in S \,\middle|\, |f_n(x)-f(x)|\geq\frac{1}{m}\right\}\right)=0$$

いたるところ一様収束 uniform convergence a.e.	$\Rightarrow$	概収束 pointwise convergence a.e.

$\because$ 一様収束 uniform convergence ⇨ 各点収束 pointwise convergence

9. 測度収束 (convergence in measure)

$$\forall \varepsilon \in \mathbb{R}_+, \quad \forall n \in \mathbb{N}, \quad \exists k \in \mathbb{N}: \quad n \leq k \quad \wedge \quad P(\{x \in S \mid |f_k(x) - f(x)| \geq \varepsilon\}) = 0$$

$$\Leftrightarrow$$

$$\forall m \in \mathbb{N}, \quad \forall n \in \mathbb{N}, \quad \exists k \in \mathbb{N}: \quad n \leq k \quad \wedge \quad P\left(\left\{x \in S \,\middle|\, |f_k(x) - f(x)| \geq \frac{1}{m}\right\}\right) = 0$$

$$\Leftrightarrow$$

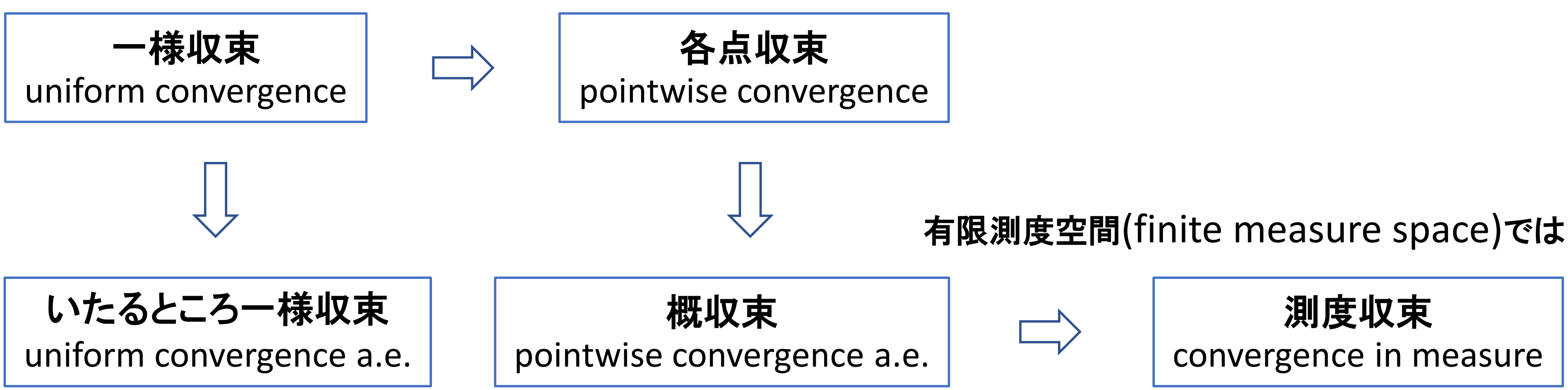
$$\forall m \in \mathbb{N}: \quad \limsup_{n \to +\infty} P\left(\left\{x \in S \,\middle|\, |f_n(x) - f(x)| \geq \frac{1}{m}\right\}\right) = 0$$

有限測度空間(finite measure space)**では**

概収束 pointwise convergence a.e.	$\Rightarrow$	**測度収束** convergence in measure

$\because$ **逆ファトゥの補題**(reverse Fatou's lemma)

10. 収束概念の関係 (Relationship of convergences)

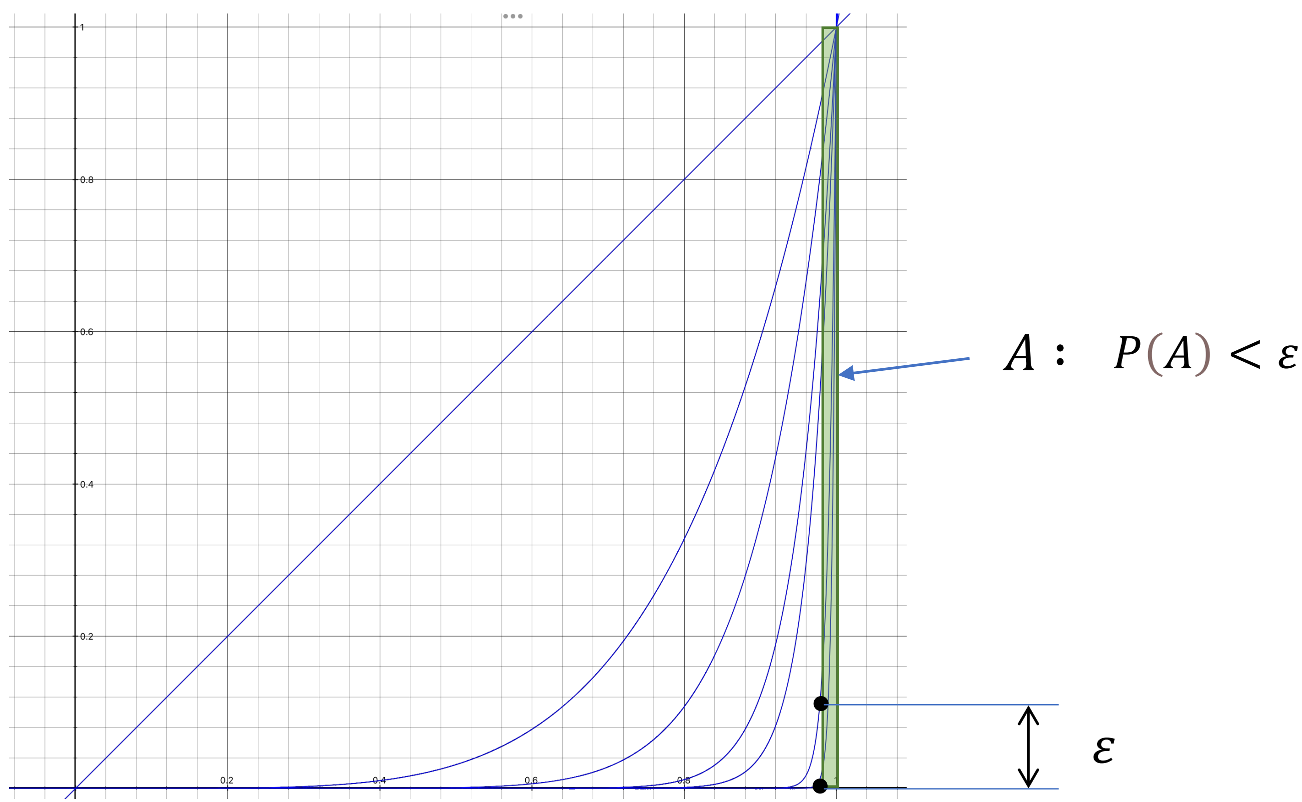


11. 概一様収束 (almost uniform convergence)

任意の正数 ε で、測度 ε より小さい除外集合 A の外で、一様収束

$$\forall \varepsilon \in \mathbb{R}_+, \quad \exists A \in \mathbb{A}:$$

$$P(A) < \varepsilon \qquad \wedge \qquad \bigcup_{m \in \mathbb{N}} \limsup_{n \to +\infty} \{x \in S - A \mid |f_n(x) - f(x)| \geq \varepsilon\} = \emptyset$$



12. 概一様収束 ⇒ 概収束 (“almost uniform” ⇒ “pointwise a.e.”)

概収束 pointwise convergence a.e.

$$P\left(\bigcup_{m\in\mathbb{N}}\left\{x\in S \,\middle|\, \limsup_{n\to+\infty}|f_n(x)-f(x)|\geq\frac{1}{m}\right\}\right)=0$$

概一様収束 almost uniform convergence

$$\forall k\in\mathbb{N},\quad \exists A\in\mathbb{A}:\qquad P(A)<\frac{1}{k}\qquad\wedge$$

$$\bigcup_{m\in\mathbb{N}}\limsup_{n\to+\infty}\left\{x\in S-A \,\middle|\, |f_n(x)-f(x)|\geq\frac{1}{m}\right\}=\emptyset$$

概一様収束 almost uniform convergence	⇒	**概収束** pointwise convergence a.e.

∵ **証明は次ページ** (Proof in next page)

13. 証明 (Proof)

$$\forall k \in \mathbb{N}: \quad P(A_k) < \frac{1}{k} \quad \wedge \quad \bigcup_{m \in \mathbb{N}} \limsup_{n \to +\infty} \left\{ x \in S - A_k \,\middle|\, |f_n(x) - f(x)| \geq \frac{1}{m} \right\} = \emptyset$$

① $S - A_k$: $\{f_n\}_{n \in \mathbb{N}}$ 一様収束 (uniformly convergent)

$$\Downarrow$$

各点収束 (pointwise convergent)

② A_k : $$P\left(\bigcap_{k \in \mathbb{N}} A_k\right) \leq \lim_{k \to +\infty} P(A_k) \leq \lim_{k \to +\infty} \frac{1}{k} = 0$$

$$x \in S - \bigcap_{k \in \mathbb{N}} A_k \iff x \notin \bigcap_{k \in \mathbb{N}} A_k \iff \exists k \in \mathbb{N}: x \in S - A_k$$

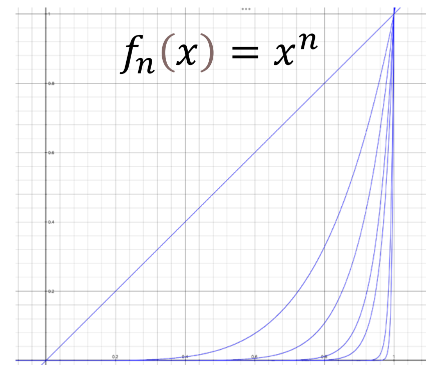
$$\Downarrow$$

$$\bigcup_{m \in \mathbb{N}} \left\{ x \in S - \bigcap_{k \in \mathbb{N}} A_k \,\middle|\, \limsup_{n \to +\infty} |f_n(x) - f(x)| \geq \frac{1}{m} \right\} = \emptyset$$

■

14. 収束概念の関係 (Relationship of convergences)

一様収束ではない例:



一様収束
uniform convergence

⇒

各点収束
pointwise convergence

⇓

いたるところ一様収束
uniform convergence a.e.

⇓

概収束
pointwise convergence a.e.

有限測度空間(finite measure space)では

⇒

測度収束
convergence in measure

概一様収束
almost uniform convergence

概一様収束ではない例:

$$\left\{1_{[n-1,n]}\right\}_{n\in\mathbb{N}}$$

15. エゴロフの定理 (Egorov theorem)

始域の測度空間が有限のならば（$P(S) < +\infty$）

概一様収束 almost uniform convergence	$\Leftrightarrow$	概収束 pointwise convergence a.e.

証明(Proof of “$\Leftarrow$”)

$$\exists N \subseteq S, \quad \forall x \in S - N: \quad P(N) = 0 \ \wedge \lim_{n \to +\infty} f_n(x) = f(x)$$

$$\Downarrow$$

$$\{A_{m,n}\}_{n \in \mathbb{N}} : \quad A_{m,n} = \left\{ x \in S - N \,\middle|\, |f_n(x) - f(x)| \geq \frac{1}{m} \right\}$$

$$\Rightarrow \quad \limsup_{n \to +\infty} A_{m,n} = \emptyset$$

$$\{B_{m,n}\}_{n \in \mathbb{N}} : \quad B_{m,n} = \bigcup_{n \le k} A_{m,k}$$

$$\Rightarrow \quad \lim_{n \to +\infty} P(B_{m,n}) = 0$$

$$\Downarrow$$

$$\forall n \in \mathbb{N}, \quad \forall \varepsilon \in \mathbb{R}_+, \quad \exists n_m \in \mathbb{N} : \qquad n \leq n_m \quad \wedge \quad P\left(B_{m,n_m}\right) < \frac{\varepsilon}{2^{m+1}}$$

$$\Downarrow$$

$$P\left(N \cup \bigcup_{m\in\mathbb{N}} B_{m,n_m}\right) \leq P(N) + \sum_{m\in\mathbb{N}} P\left(B_{m,n_m}\right) < \varepsilon$$

$$\Downarrow$$

$\forall a \in S - N :$

$$a \in \bigcup_{m\in\mathbb{N}} B_{m,n_m} \iff a \in \bigcup_{m\in\mathbb{N}} \left\{ x \in S - N \,\middle|\, \limsup_{n\to+\infty} |f_n(x) - f(x)| \geq \frac{1}{m} \right\}$$

$$\Downarrow$$

$\{f_n\}_{n\in\mathbb{N}}$ 概一様収束 (almost uniform convergence)

$$\because \bigcup_{k\in\mathbb{N}} \limsup_{n\to+\infty} \left\{ x \in S - \left(N \cup \bigcup_{m\in\mathbb{N}} B_{m,n_m}\right) \,\middle|\, |f_n(x) - f(x)| \geq \frac{1}{k} \right\} = \emptyset$$

■

16. 例 (Example)

$$f_n(x) = \sin^n(x) \qquad \left(0 \le x \le \frac{\pi}{2}\right)$$

$$\Downarrow$$

$$\bigcup_{m\in\mathbb{N}} \limsup_{n\to+\infty} \left\{x \in \left[0, \frac{\pi}{2} - \varepsilon\right] \,\middle|\, |\sin^n(x) - 0| \ge \frac{1}{m}\right\} = \emptyset$$

$\because$　エゴロフの定理 (Egorov theorem)

$$\Downarrow$$

$\{\sin^n\}_{n\in\mathbb{N}}$:　$f = 0$ に概一様収束 (almost uniform convergence)

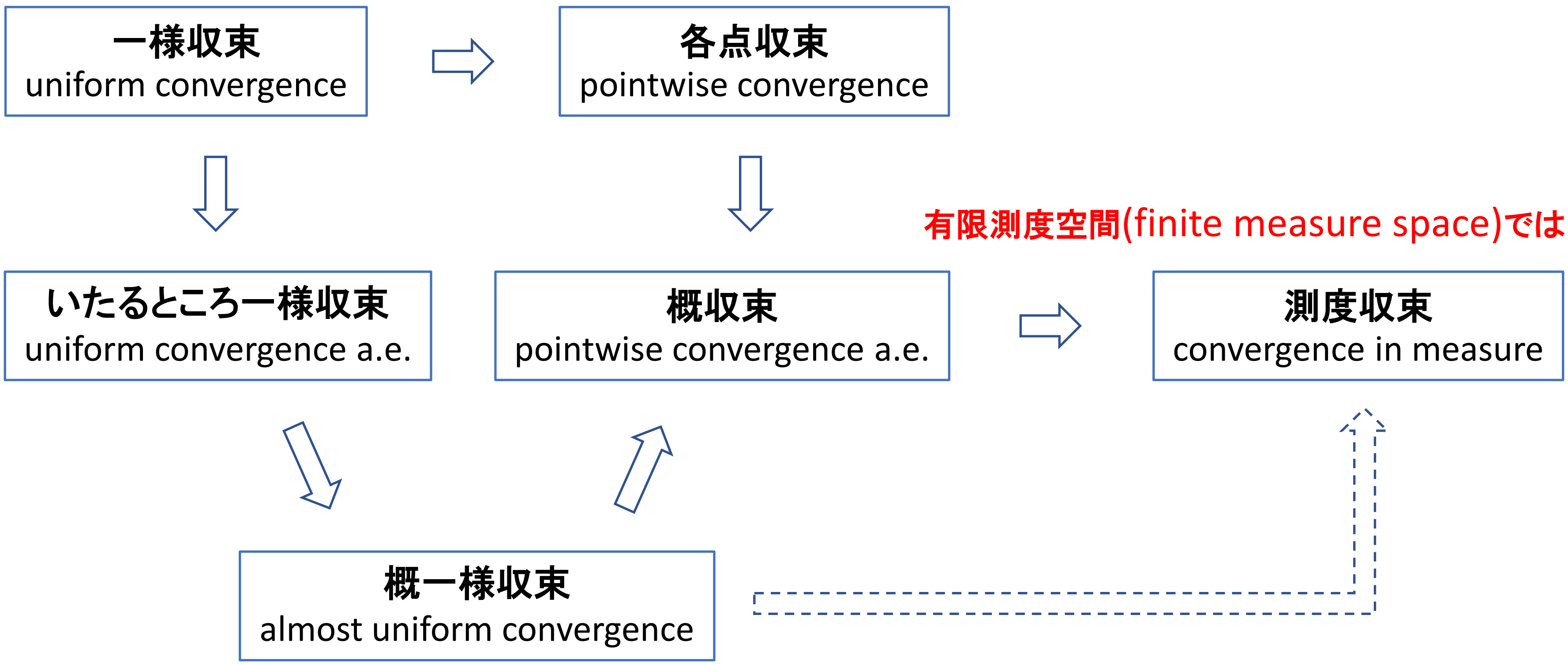
$$\Downarrow$$

ウォリスの積分 (Wallis's integral)

$$\int_0^{\frac{\pi}{2}} \lim_{n\to+\infty} \sin^n x \, dx = 0$$

■

17. 収束概念の関係 (Relationship of convergences)



<u>**概一様収束** almost uniform convergence</u>

$$\forall k \in \mathbb{N}, \quad \exists A \in \mathbb{A}: \quad P(A) < \frac{1}{k} \quad \wedge$$

$$\bigcup_{m\in\mathbb{N}} \limsup_{n\to+\infty} \left\{ x \in S - A \,\middle|\, |f_n(x) - f(x)| \geq \frac{1}{m} \right\} = \emptyset$$

<u>**測度収束** convergence in measure</u>

$$\forall m \in \mathbb{N}:$$

$$\limsup_{n\to+\infty} P\left(\left\{ x \in S \,\middle|\, |f_n(x) - f(x)| \geq \frac{1}{m} \right\}\right) = 0$$

18. 証明 (Proof)

$$\forall k \in \mathbb{N},\quad \exists A \in \mathbb{A}:\quad P(A) < \frac{1}{k}$$

$\wedge$

$$\bigcup_{m \in \mathbb{N}} \limsup_{n \to +\infty} \left\{ x \in S - A \,\middle|\, |f_n(x) - f(x)| \geq \frac{1}{m} \right\} = \emptyset$$

$$\forall m \in \mathbb{N},\quad \exists N \in \mathbb{N},\quad \forall n \in \mathbb{N},\quad x \in S - A:\quad N \leq n \quad \Rightarrow \quad |f_n(x) - f(x)| < \frac{1}{m}$$

$$\forall m \in \mathbb{N},\quad \exists N \in \mathbb{N},\quad \forall n \in \mathbb{N}:\quad N \leq n \quad \Rightarrow \quad P\left(\left\{ x \in S \,\middle|\, |f_n(x) - f(x)| \geq \frac{1}{m} \right\} \right) < \frac{1}{k}$$

$k \to +\infty$

$$\forall m \in \mathbb{N}:\quad \lim_{n \to +\infty} P\left(\left\{ x \in S \,\middle|\, |f_n(x) - f(x)| \geq \frac{1}{m} \right\} \right) = 0$$