Fourier Transform

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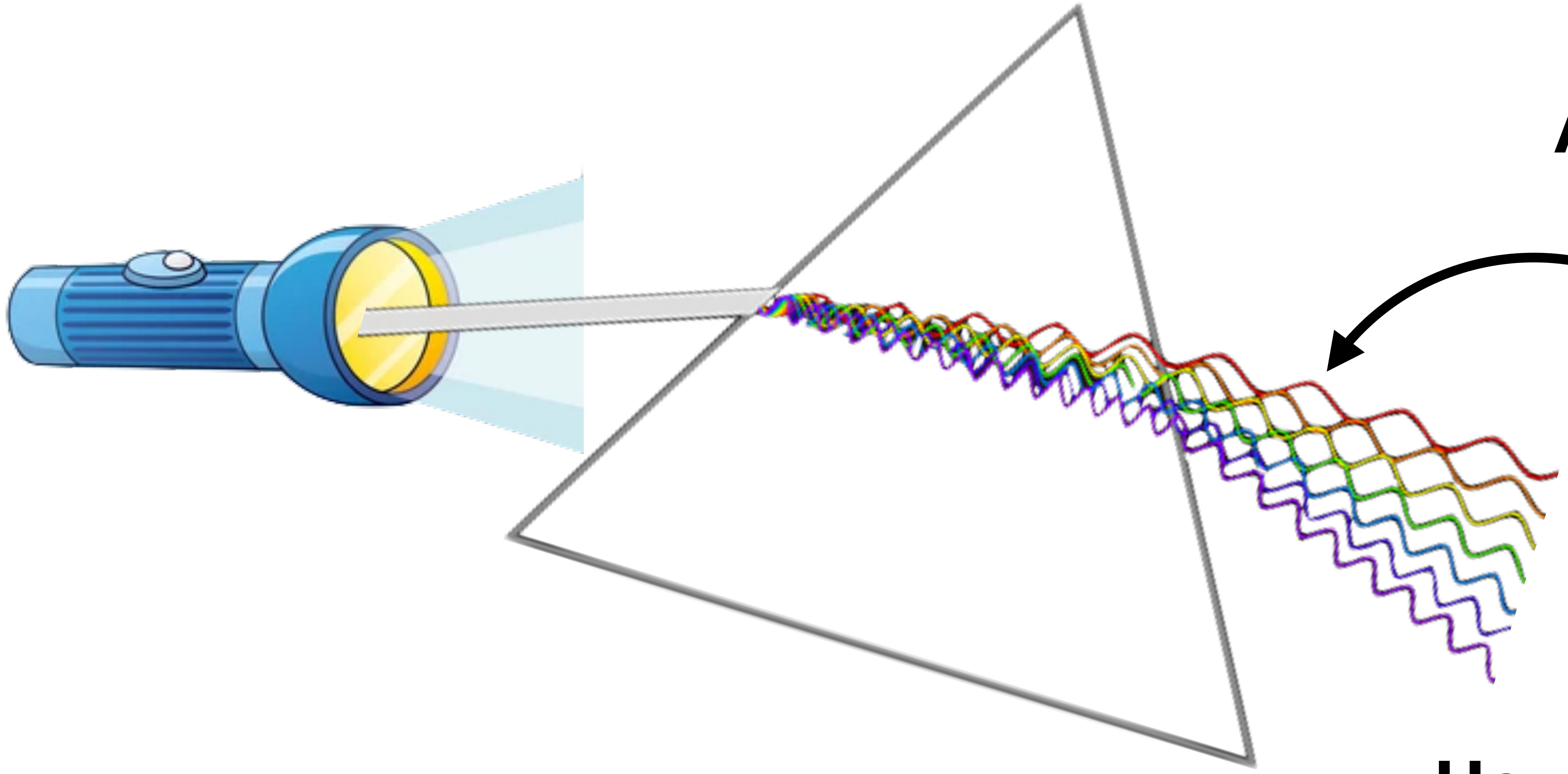
7th December 2022

Intuition of the Fourier transform



A light beam is composed of several waves of different wavelength (that is, colors)

How to get information on the different components from a measurement of the whole beam?

Formal definition

If we have a Lebesgue integrable function:

$$f \colon \mathbb{R} \to \mathbb{C}$$

Its Fourier transform is a function:

$$\mathscr{F}\{f\} \colon \mathbb{R} \to \mathbb{C}$$

$\mathscr{F}\{f\}(\xi) = |\mathscr{F}\{f\}(\xi)| \,.\, \mathrm{e}^{i\phi_f(\xi)}$

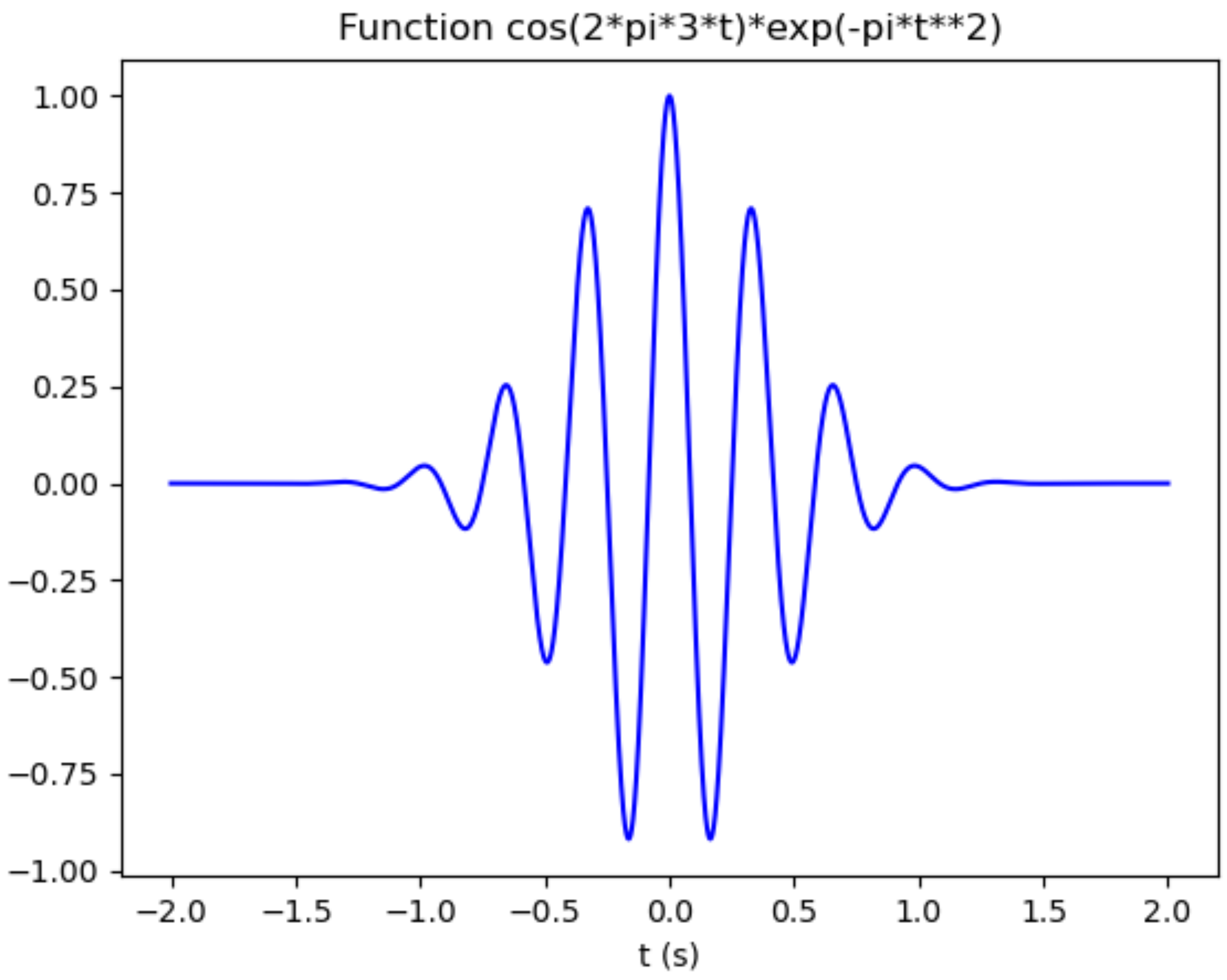
$|\mathscr{F}\{f\}(\xi)|$**: magnitude**

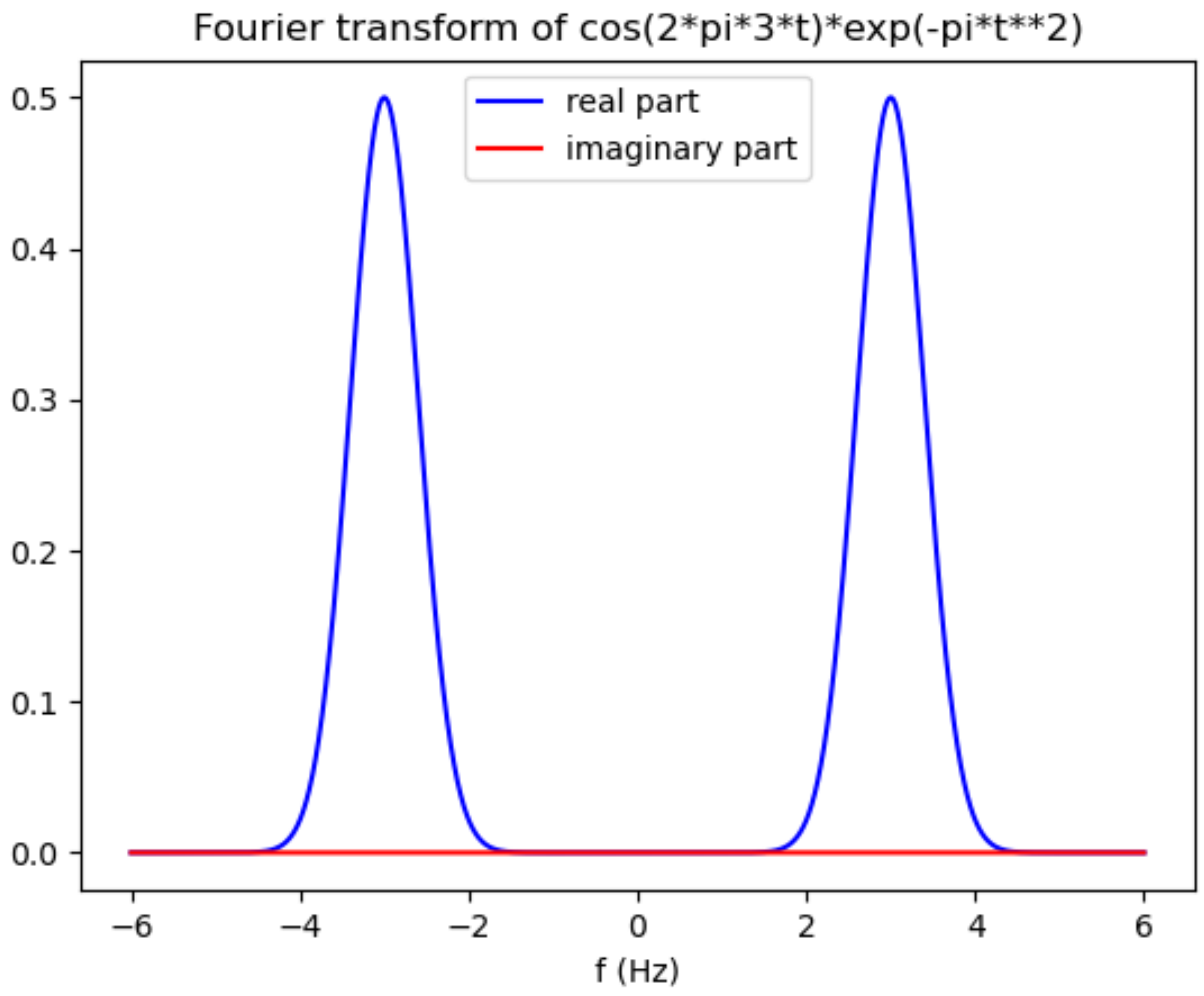
$\phi_f(\xi)$**: phase**

Defined by the integrals:

$$\mathscr{F}\{f\}(\xi) = \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-i2\pi\xi x}dx$$

Example 1: one principal wavelength





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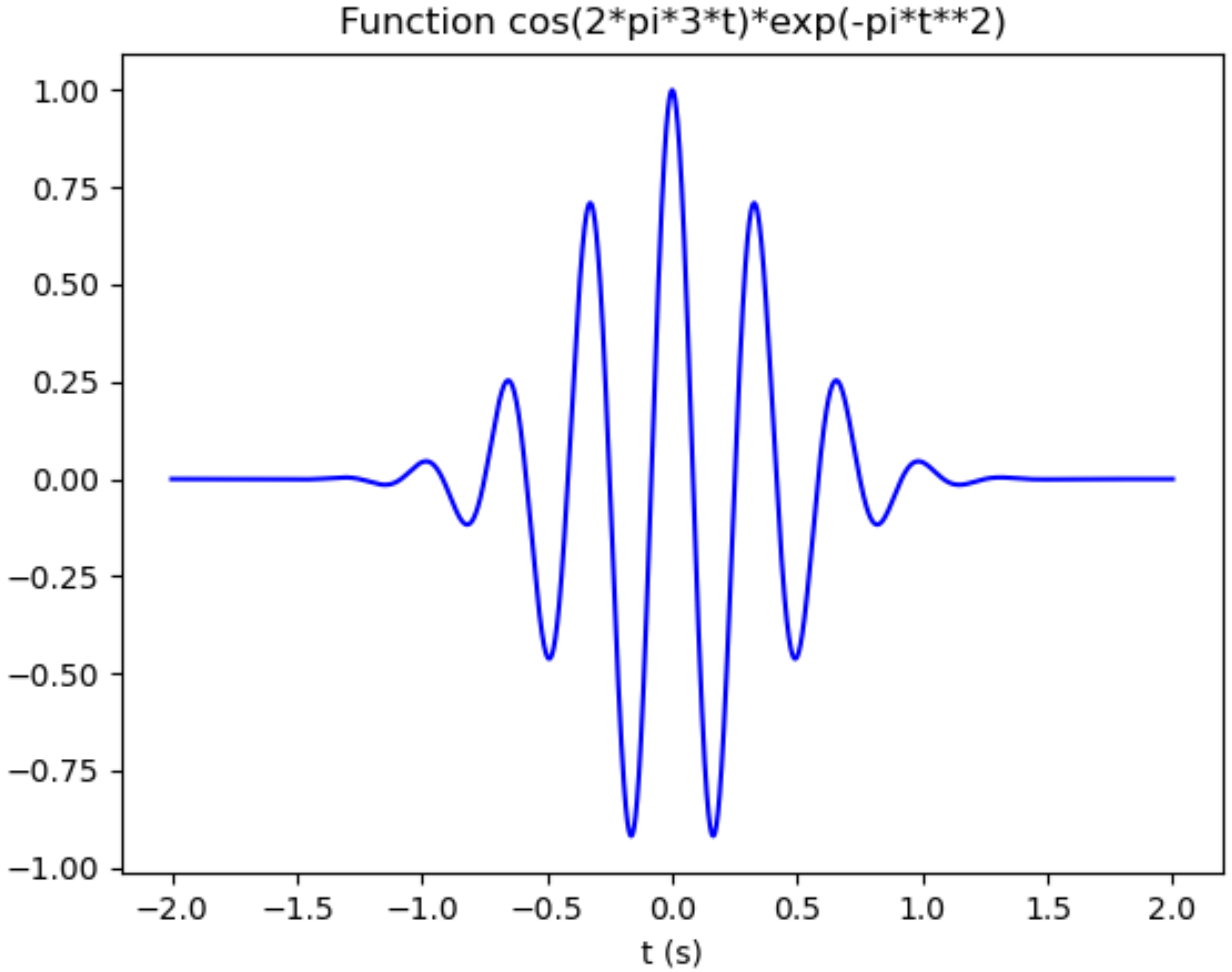
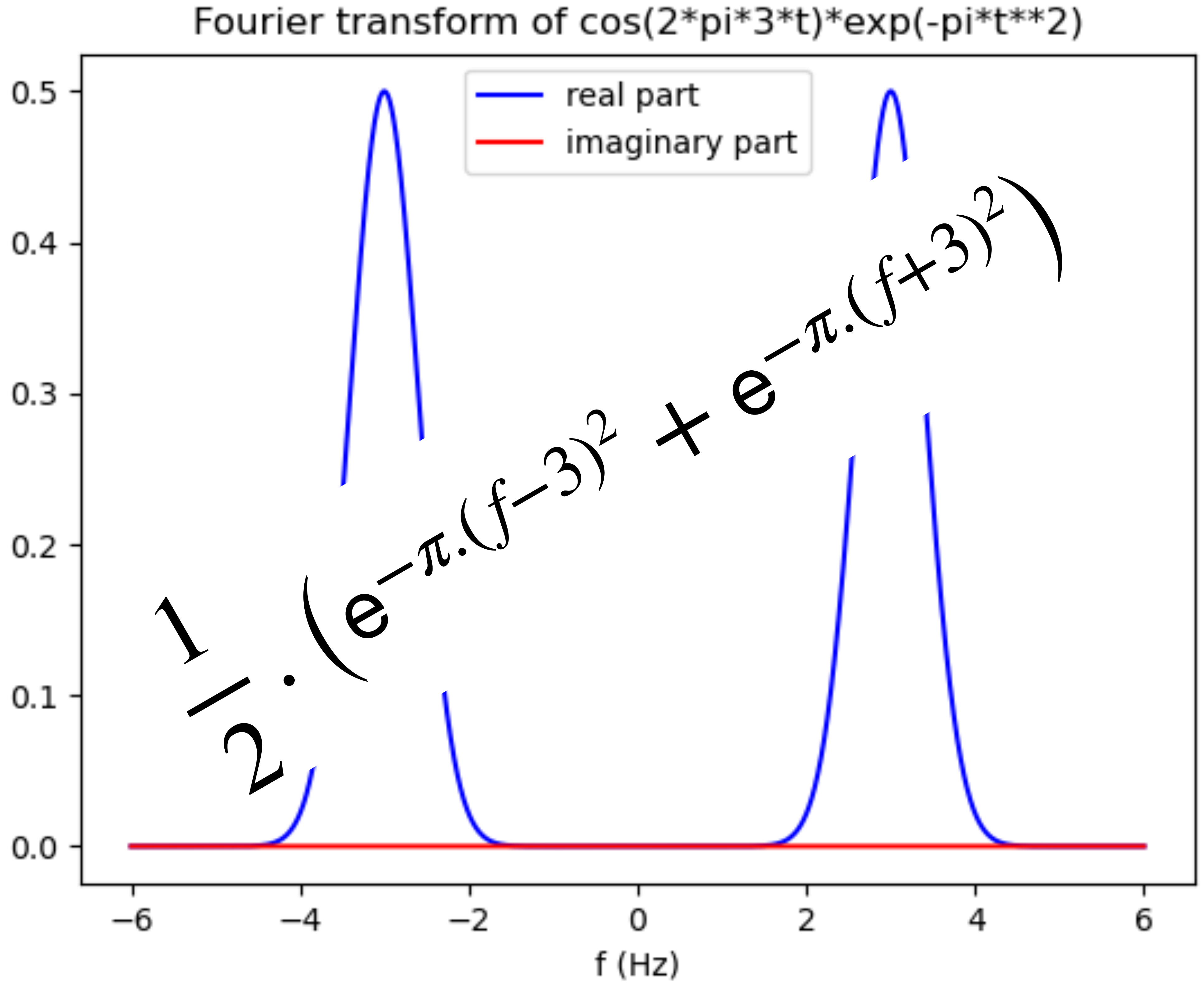
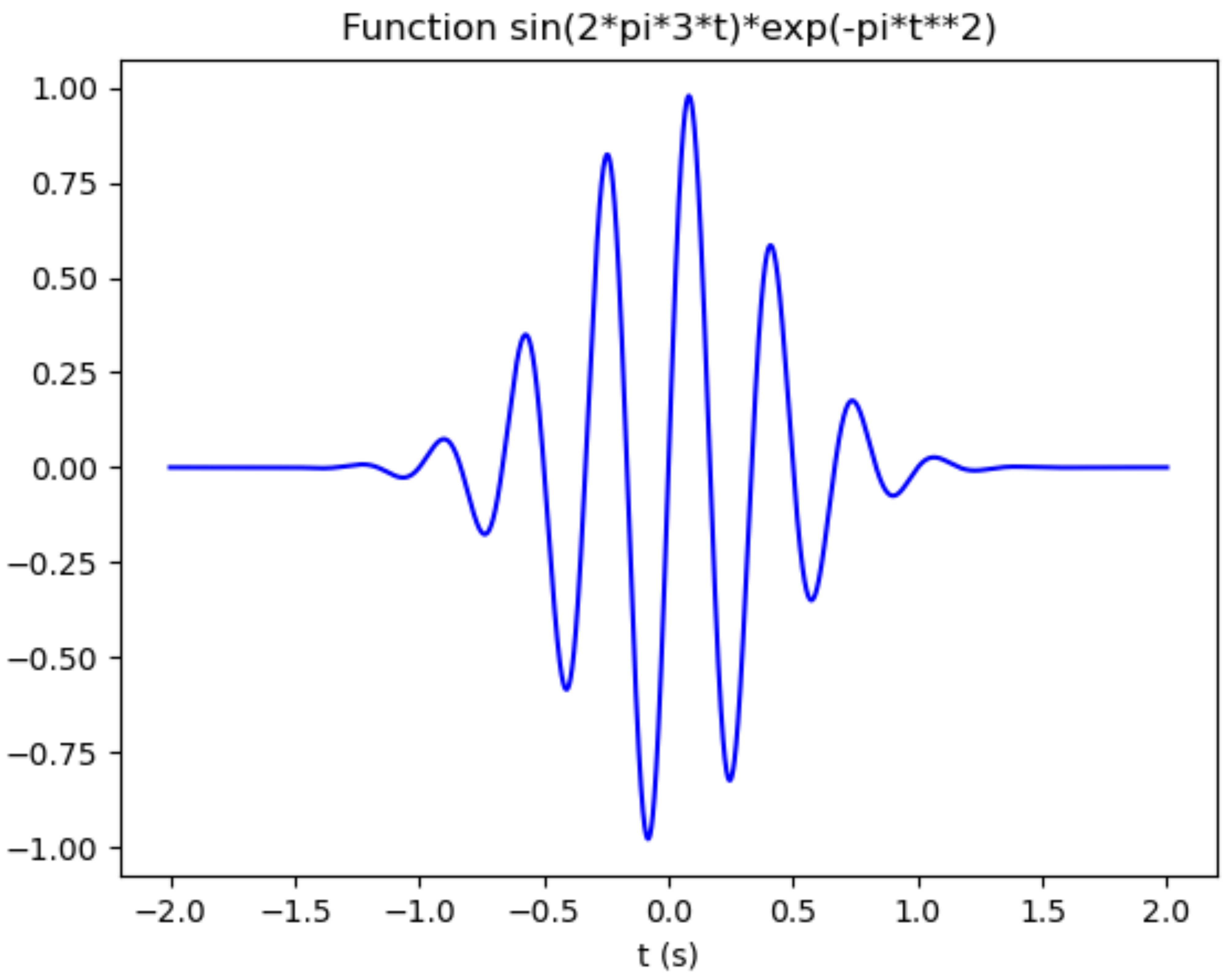
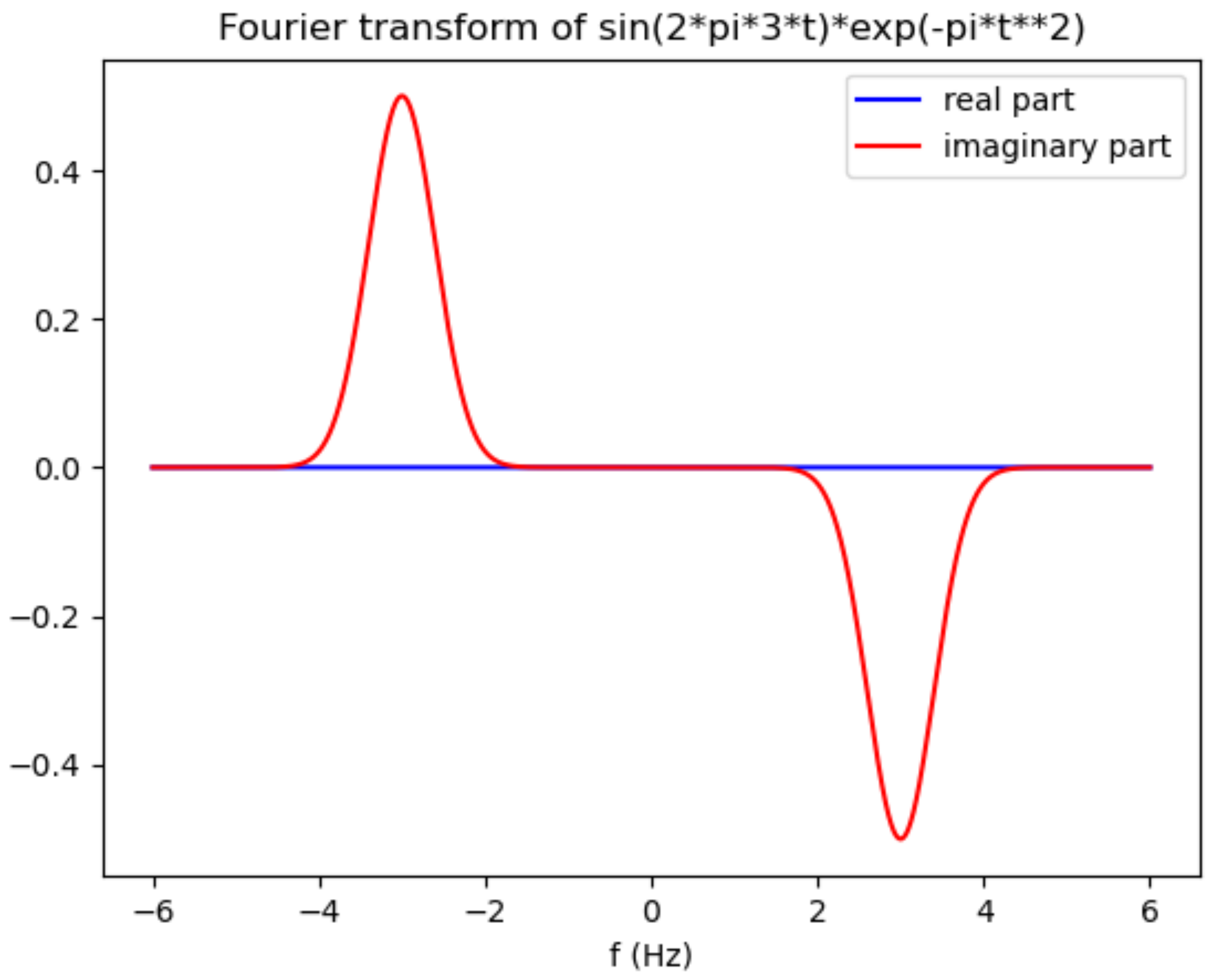




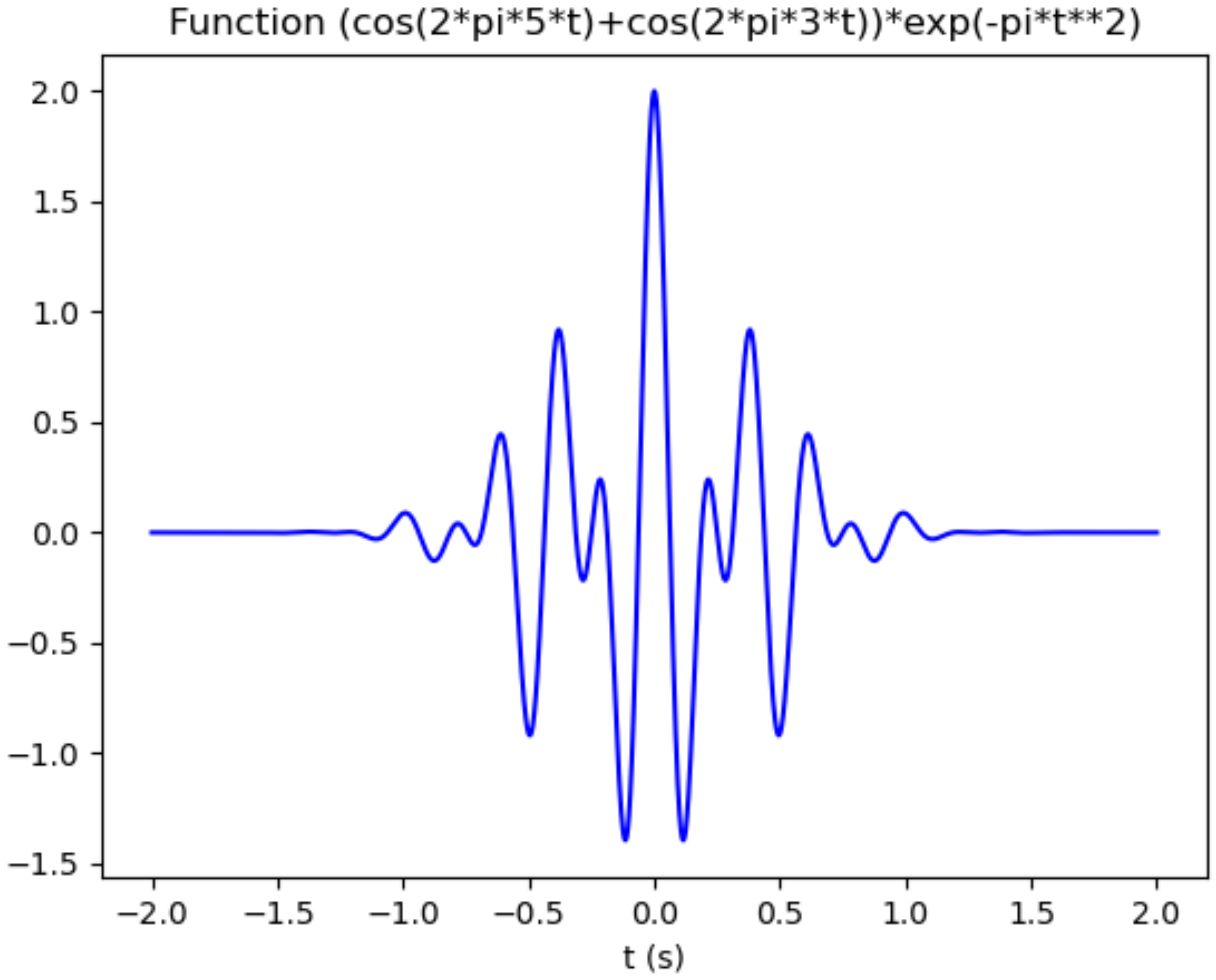
$$\frac{1}{2}.\left(e^{-\pi.(f-3)^2} + e^{-\pi.(f+3)^2}\right)$$

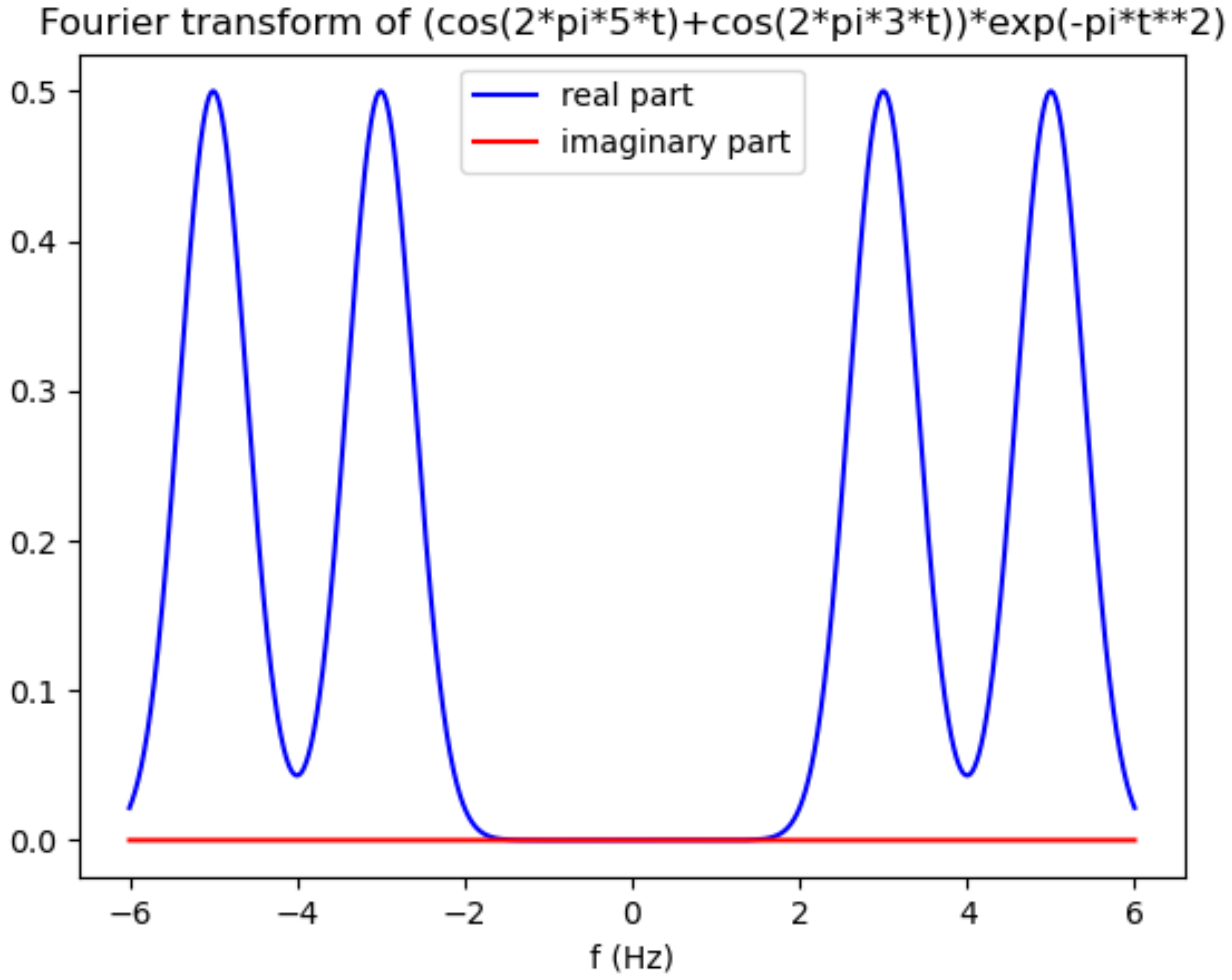
Example 1: one principal wavelength



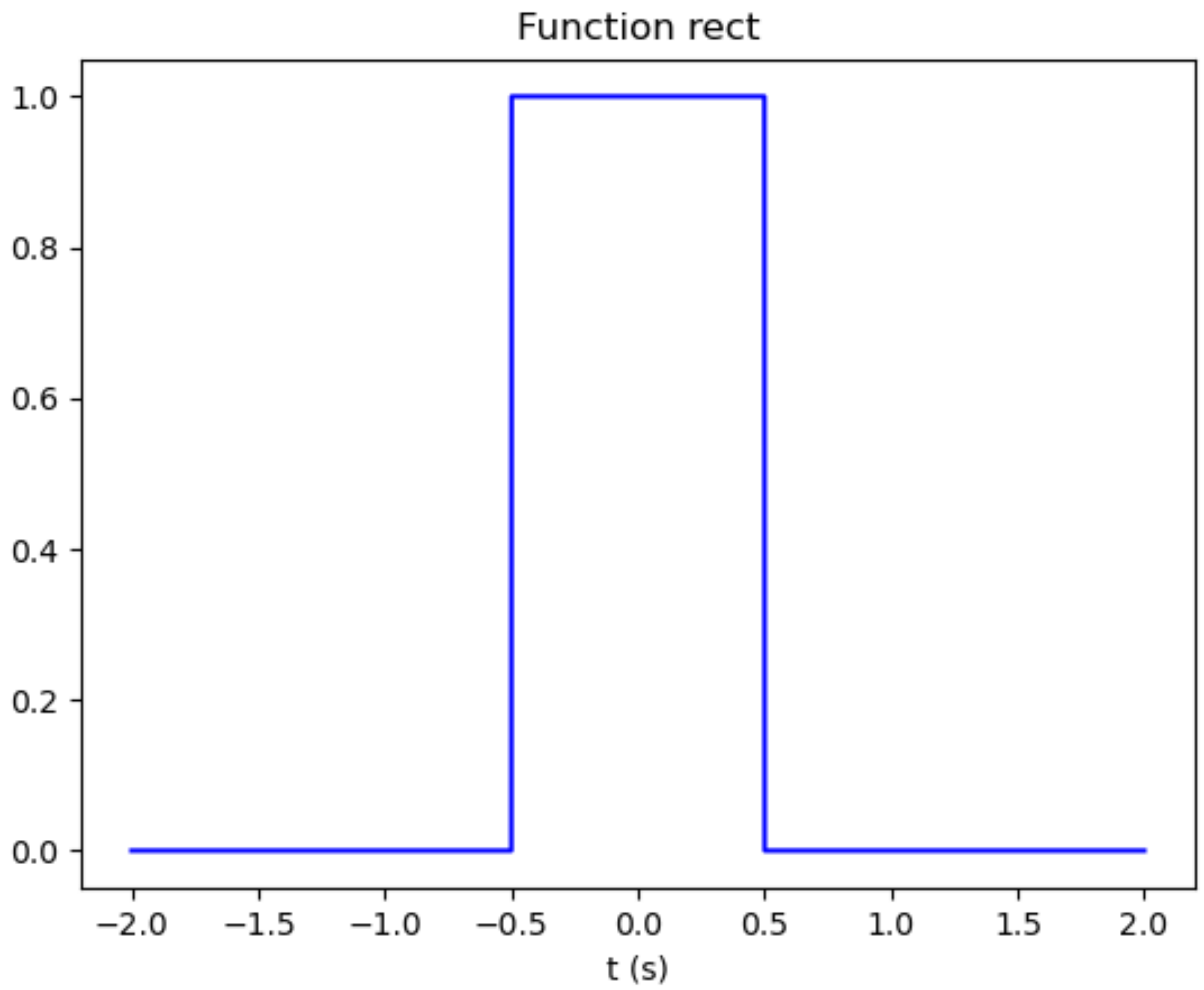


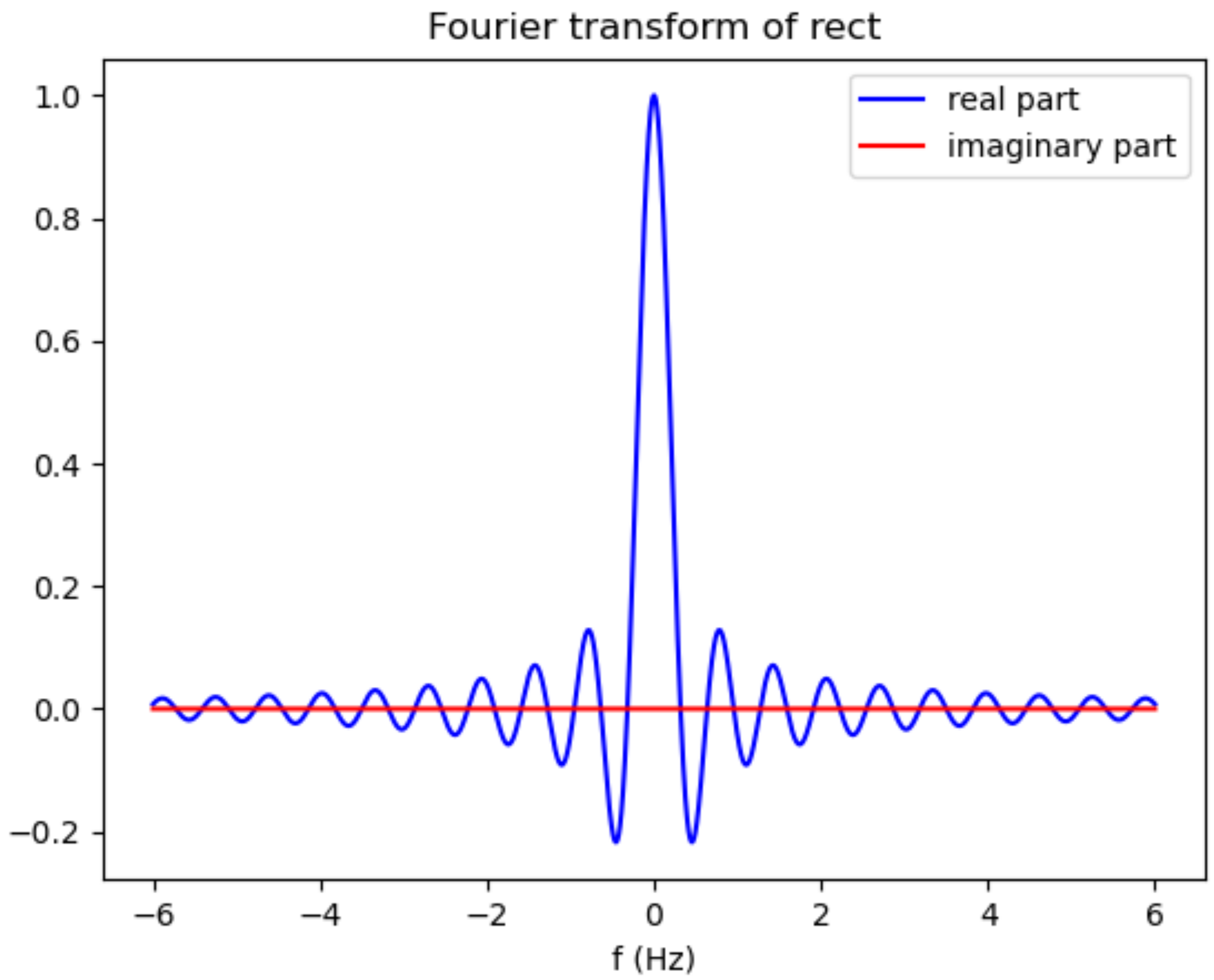
Example 2: two in-phase principal wavelengths





Example 4: rectangle signal





Basic properties: Linearity

If we have:

$$f, g : \mathbb{R} \to \mathbb{C} \text{ and } a, b \in \mathbb{C}$$

Then:

$$\mathscr{F}\{a \,.\, f + b \,.\, g\} = a \,.\, \mathscr{F}\{f\} + b \,.\, \mathscr{F}\{g\}$$

Proof: Linearity of integrals.

Basic properties: Time shift

If we have:

$$g(t) = f(t - t_0)$$

Then:

$$\mathscr{F}\{g\}(\xi) = \mathrm{e}^{-2\pi i \xi t_0} . \mathscr{F}\{f\}(\xi)$$

That is: same magnitude, but different (drifted) phase

Proof: If $u = t - t_0$**,**

$$\mathscr{F}\{g\}(\xi) = \int_{-\infty}^{+\infty} g(t)\mathrm{e}^{-i2\pi\xi t}dt = \int_{-\infty}^{+\infty} f(u)\mathrm{e}^{-i2\pi\xi(u+t_0)}du = \mathrm{e}^{-2\pi i\xi t_0} . \mathscr{F}\{f\}(\xi)$$

Basic properties: Time drift

If we have:

$$g(t) = f(at) \textbf{ for } a \neq 0$$

Then:

$$\mathscr{F}\{g\}(\xi) = \frac{1}{|a|} . \mathscr{F}\{f\}\left(\frac{\xi}{a}\right)$$

Proof: If $u = at$**,**

$$\mathscr{F}\{g\}(\xi) = \int_{-\infty}^{+\infty} g(t)\mathrm{e}^{-i2\pi\xi t}dt = \int_{-\infty}^{+\infty} \frac{1}{|a|} f(u)\mathrm{e}^{-i2\pi\xi\frac{u}{a}}du = \frac{1}{|a|} . \mathscr{F}\{f\}\left(\frac{\xi}{a}\right)$$

Basic properties: conjugate

If we have:

$$g = \bar{f}$$

Then:

$$\mathcal{F}\{g\}(\xi) = \overline{\mathcal{F}\{f\}(-\xi)}$$

Proof: If $u = -t$,

$$\mathcal{F}\{g\}(\xi) = \int_{-\infty}^{+\infty} \overline{f(t)}\mathrm{e}^{-i2\pi\xi t}dt = \overline{\int_{-\infty}^{+\infty} f(t)\mathrm{e}^{i2\pi\xi t}dt} = \overline{\int_{-\infty}^{+\infty} f(-u)\mathrm{e}^{-i2\pi\xi u}du} = \overline{\mathcal{F}\{f\}(-\xi)}$$

Basic properties: consequences

If $f\colon\ \mathbb{R} \to \mathbb{R}$ (that is, purely real) then $\mathscr{F}\{f\}(-\xi) = \overline{\mathscr{F}\{f\}(\xi)}$ (that is, Hermitian).

If $f\colon\ \mathbb{R} \to \mathbb{R}$ is even, then $\mathscr{F}\{f\} : \mathbb{R} \to \mathbb{R}$ (that is, purely real)

If $f\colon\ \mathbb{R} \to \mathbb{R}$ is odd, then $\mathscr{F}\{f\}$ is purely imaginary

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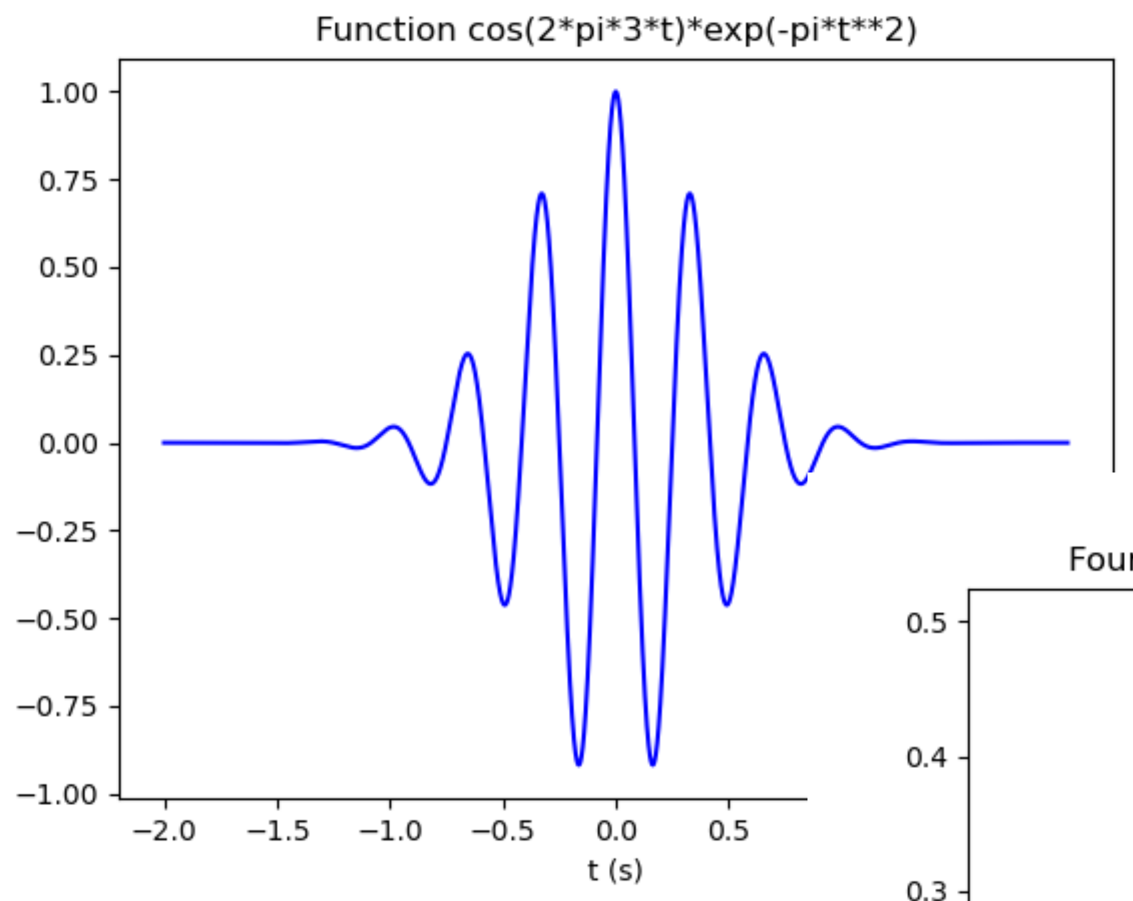

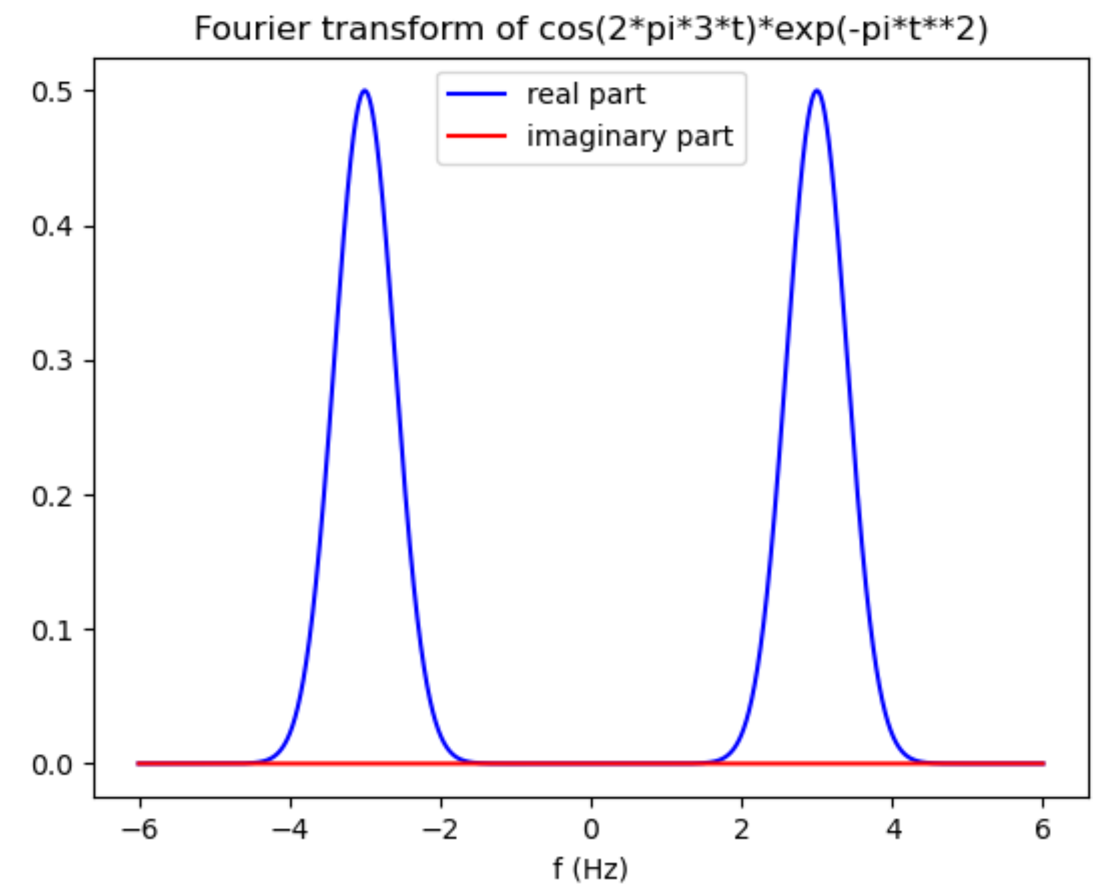

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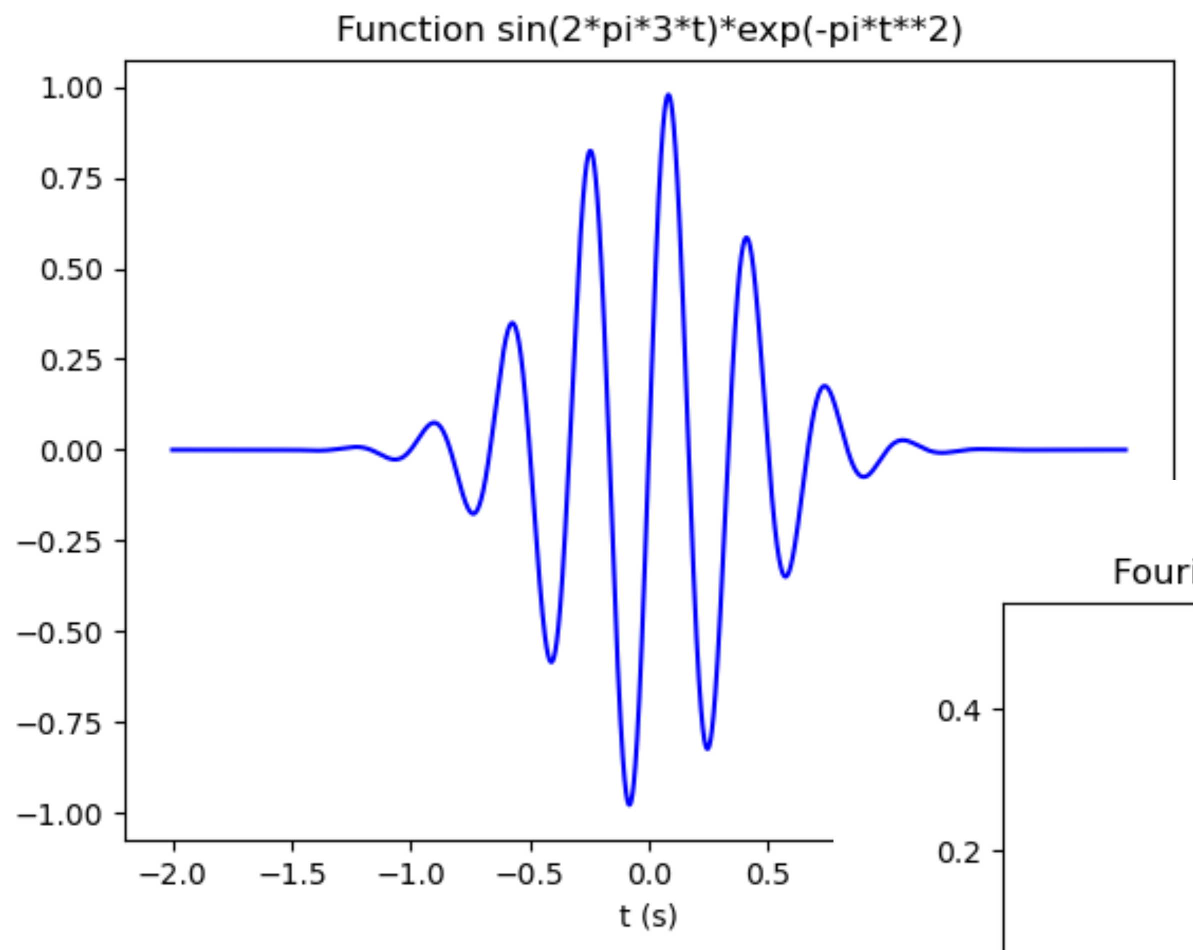

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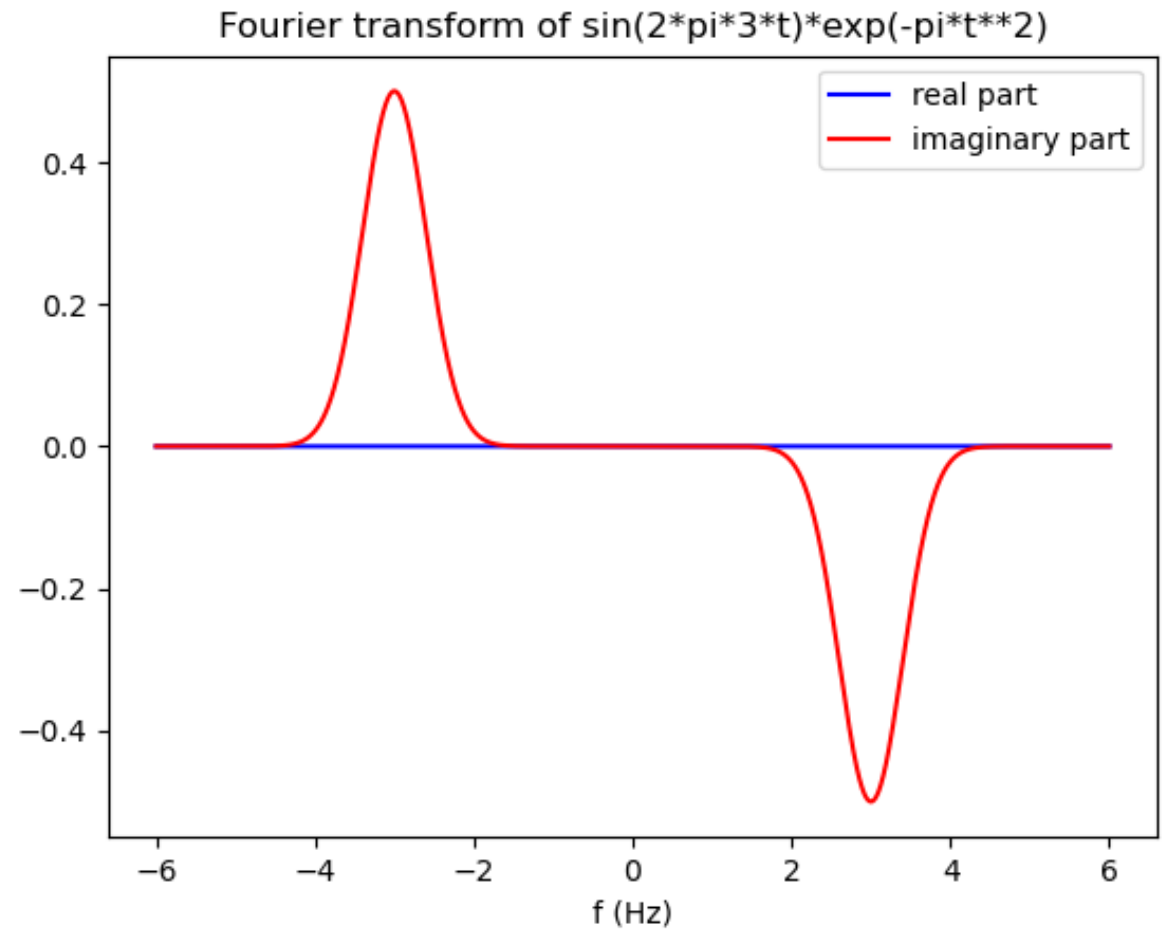

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Inversion theorem

Assume that f and $\mathscr{F}\{f\}$ are Lebesgue integrable.

Then, for almost every t:

$$f(t) = \mathscr{F}^{-1}\{\mathscr{F}\{f\}\}(t) \text{ where } \mathscr{F}^{-1}\{g\}(t) = \int_{-\infty}^{+\infty} g(\xi)e^{2\pi i\xi t}d\xi$$

Furthermore, if f is continuous, then the equality is for every t.

Proof of inversion theorem

Idea of the proof when f is continuous.

We want to prove $f(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(s)e^{-2\pi i\xi s}e^{2\pi i\xi t}dsd\xi$

Trick: consider $I_\epsilon(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(s)e^{-2\pi i\xi s}e^{2\pi i\xi t}e^{-2\pi^2\epsilon^2\xi^2}dsd\xi$

$$I_\epsilon(t) = \int_{-\infty}^{+\infty} \mathscr{F}\{f\}(\xi)e^{2\pi i\xi t}e^{-2\pi^2\epsilon^2\xi^2}d\xi \xrightarrow{\epsilon\to 0} \int_{-\infty}^{+\infty} \mathscr{F}\{f\}(\xi)e^{2\pi i\xi t}d\xi$$

$$I_\epsilon(t) = \int_{-\infty}^{+\infty} f(s)g_\epsilon(t-s)ds \xrightarrow{\epsilon\to 0} f(t)$$

Proof of inversion theorem

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Warning: we cannot use Fubini directly!

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Integration over s

Trick: consider $$I_\epsilon(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(s)e^{-2\pi i\xi s}e^{2\pi i\xi t}e^{-2\pi^2\epsilon^2\xi^2}dsd\xi$$

$$I_\epsilon(t) = \int_{-\infty}^{+\infty} \mathcal{F}\{f\}(\xi)e^{2\pi i\xi t}e^{-2\pi^2\epsilon^2\xi^2}d\xi \xrightarrow{\epsilon\to 0} \int_{-\infty}^{+\infty} \mathcal{F}\{f\}(\xi)e^{2\pi i\xi t}d\xi$$

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Dominated convergence

$$I_\epsilon(t) = \int_{-\infty}^{+\infty} \mathcal{F}\{f\}(\xi)e^{2\pi i\xi t}e^{-2\pi^2\epsilon^2\xi^2}d\xi \xrightarrow{\epsilon\to 0} \int_{-\infty}^{+\infty} \mathcal{F}\{f\}(\xi)e^{2\pi i\xi t}d\xi$$

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Integration over ξ

$$\ldots\}(\xi)e^{2\pi i\xi t}e^{-2\pi^2\epsilon^2\xi^2}d\xi \xrightarrow{\epsilon\to 0} \int_{-\infty}^{+\infty} \mathscr{F}\{f\}(\xi)e^{2\pi i\xi t}d\xi$$

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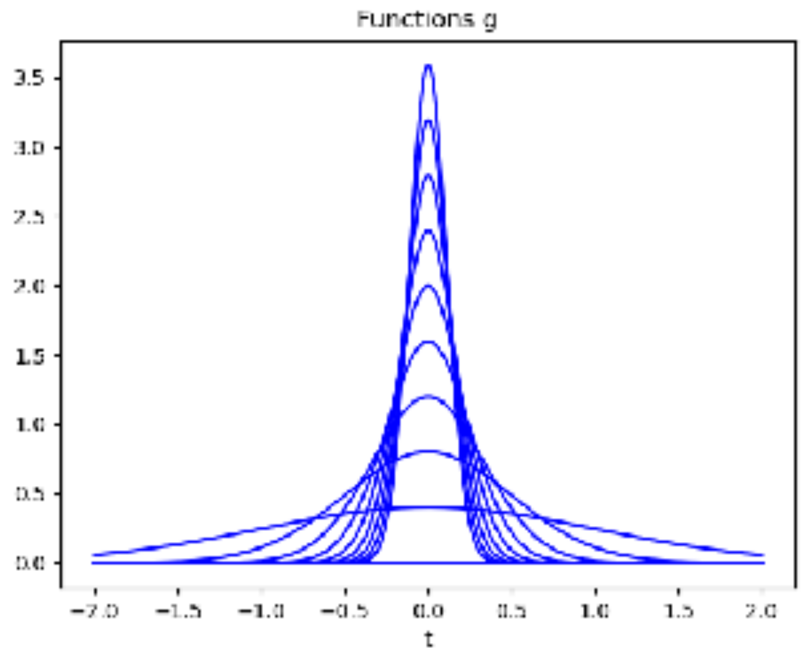
Using that f is continuous



Inversion theorem

Assume that f and $\mathscr{F}\{f\}$ are Lebesgue integrable.

Then, for almost every t:

$$f(t) = \int_{-\infty}^{+\infty} \mathscr{F}\{f\}(\xi) e^{2\pi i \xi t} d\xi$$

Furthermore, if f is continuous, then the equality is for every t.

Inversion theorem

Assume that f and $\boxed{\mathscr{F}\{f\} \text{ are Lebesgue integrable.}}$ **Inconvenient…**

Then, for almost every t:

$$f(t) = \int_{-\infty}^{+\infty} \mathscr{F}\{f\}(\xi)e^{2\pi i\xi t}d\xi$$

Furthermore, if f is continuous, then the equality is for every t.

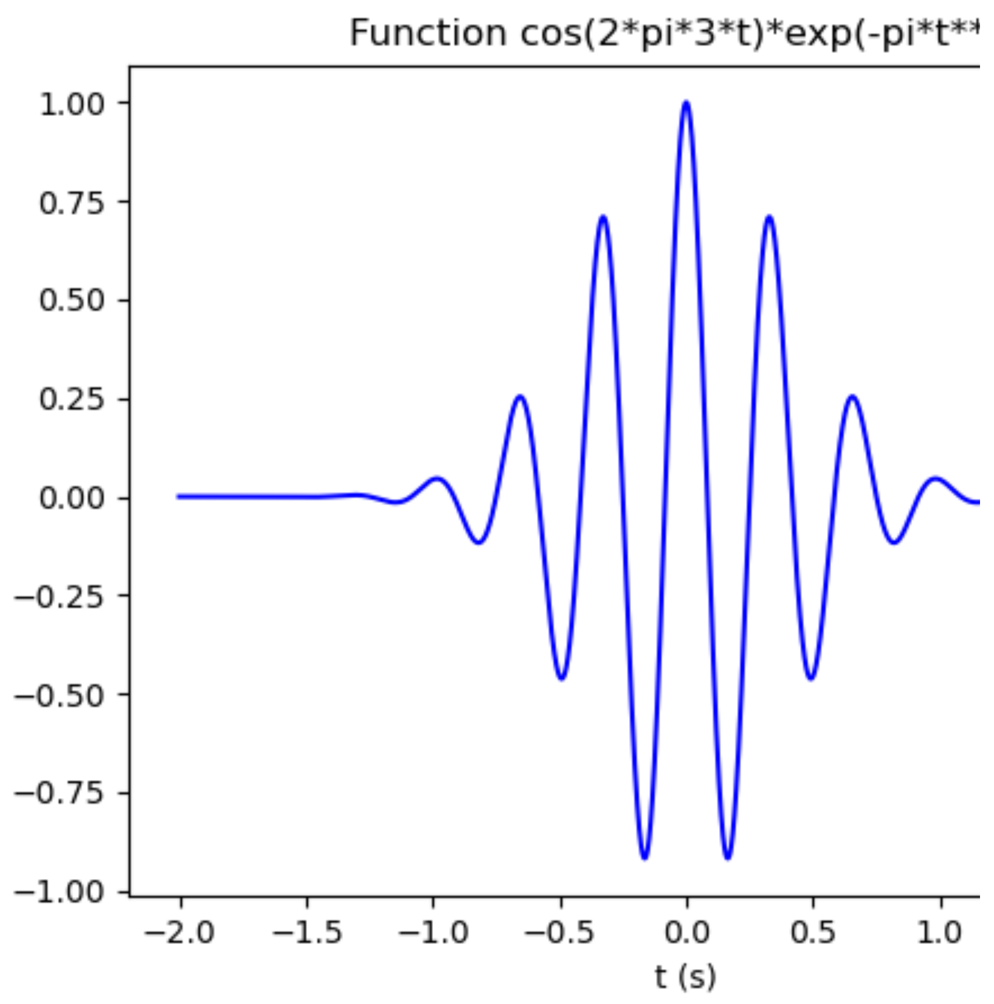
Schwarz space

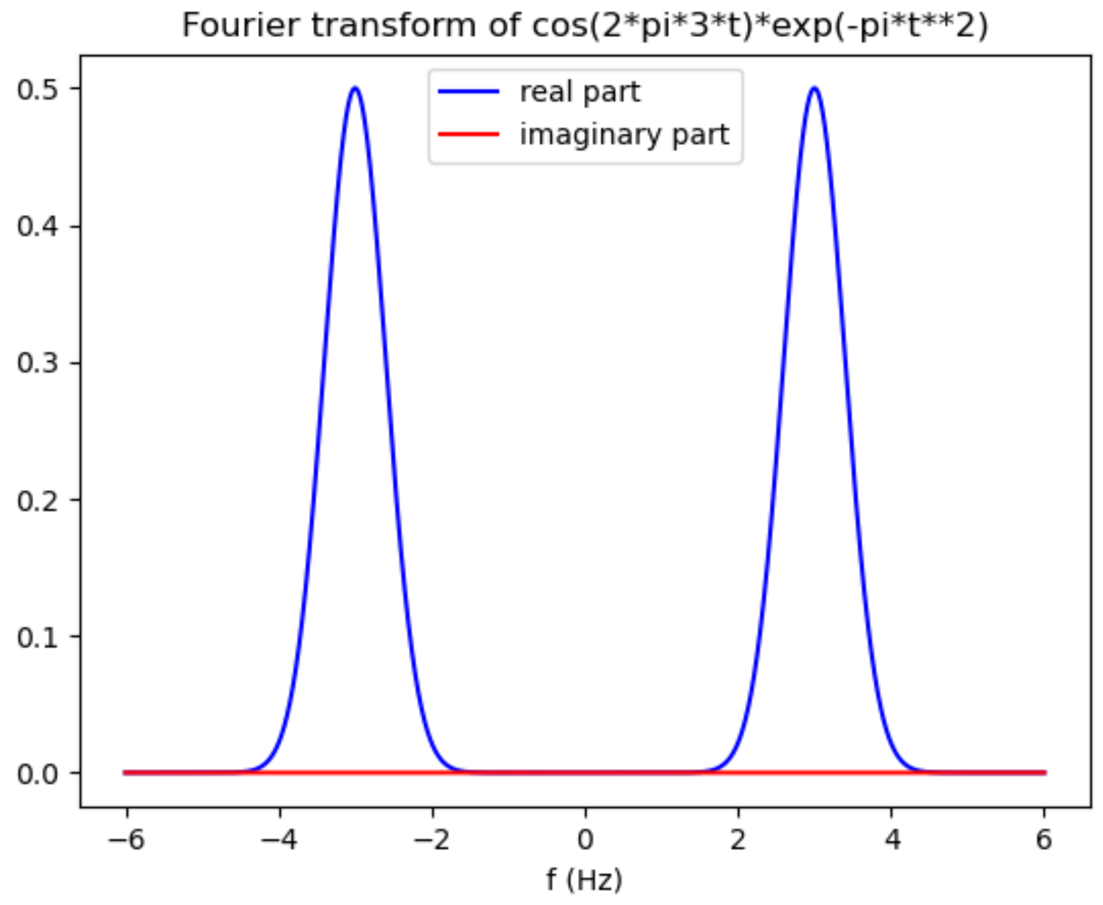
A function $f\colon\ \mathbb{R} \to \mathbb{C}$ is Schwartz if it is smooth (C^∞) and for all $n, m \in \mathbb{N}$:

$$\sup |x^n f^{(m)}(x)| < +\infty$$

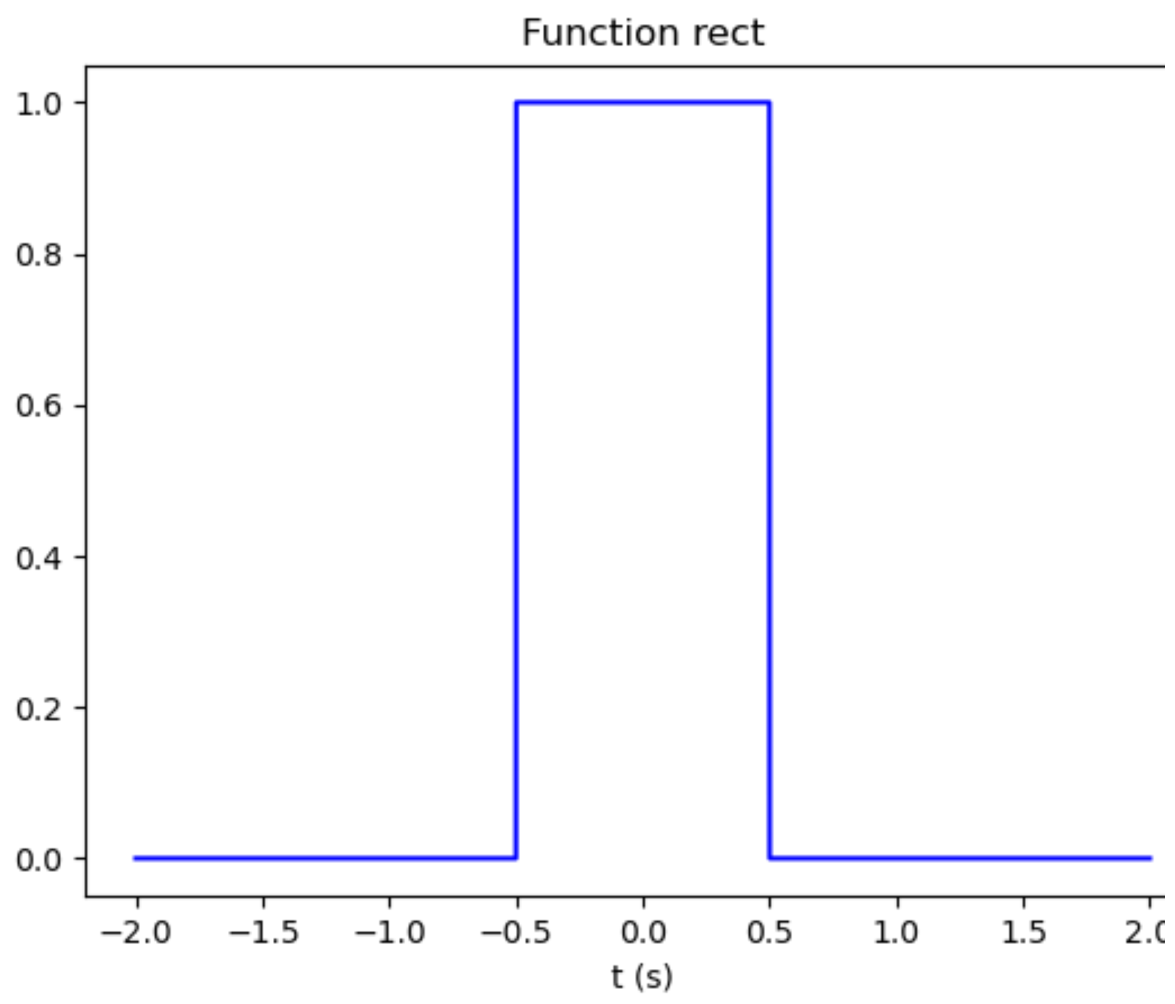
That is, f and all its derivatives collapse to 0 quickly.

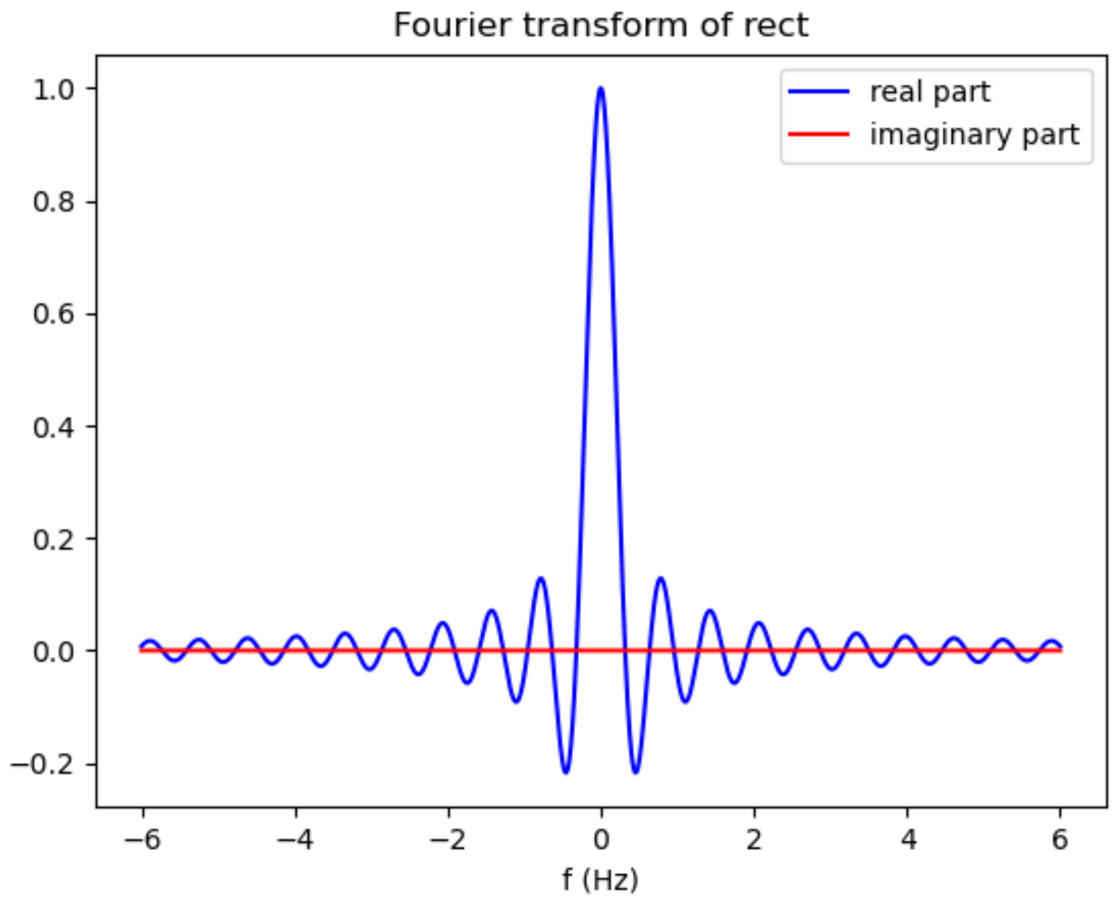
Examples





Non-example





Properties of Schwartz functions

- If f is Schwartz then all its derivative are Schwartz.
- If f is Schwartz then f is Lebesgue integrable.
- If f is Schwartz then $\mathscr{F}\{f\}$ and $\mathscr{F}^{-1}\{f\}$ are Schwartz.
- $\mathscr{F}$ is an isomorphism on Schwartz functions, with inverse $\mathscr{F}^{-1}$.

Fourier transform for L^2 functions

Everything I said can be extended to L^2 functions by density, by extending the Fourier transform and its inverse:

$$\mathcal{F}\{f\}(\xi) = \lim_{R \to +\infty} \int_{-R}^{+R} f(x) \mathrm{e}^{-i2\pi\xi x} dx$$

$$\mathcal{F}^{-1}\{f\}(\xi) = \lim_{R \to +\infty} \int_{-R}^{+R} f(x) \mathrm{e}^{i2\pi\xi x} dx$$

Basic properties: derivation

If $f\colon\ \mathbb{R} \to \mathbb{C}$ is Schwartz, then

$$\mathcal{F}\{f^{(n)}\}(\xi) = (2\pi i\xi)^n \mathcal{F}\{f\}(\xi)$$

Proof: If f is Schwartz, then all its derivative $f^{(n)}$ are Schwartz.
By Fourier inversion,

$$f^{(n)}(t) = \frac{d^n}{dt^n}\int_{-\infty}^{+\infty} \mathcal{F}\{f\}(\xi)e^{2\pi i\xi t}d\xi = \int_{-\infty}^{+\infty} \frac{\partial^n}{\partial t^n}\mathcal{F}\{f\}(\xi)e^{2\pi i\xi t}d\xi = \int_{-\infty}^{+\infty} (2\pi i\xi)^n \mathcal{F}\{f\}(\xi)e^{2\pi i\xi t}d\xi$$

By Fourier inversion again, we obtain the result.

Basic properties: derivation

If $f\colon\ \mathbb{R} \to \mathbb{C}$ is Schwartz, then

$$\mathcal{F}\{f^{(n)}\}(\xi) = (2\pi i \xi)^n \mathcal{F}\{f\}(\xi)$$

Differentiation under the integral

Possible because $\mathcal{F}\{f\}$ is Schwartz

Proof: If f … are Schwartz.

By Fourier inversion,

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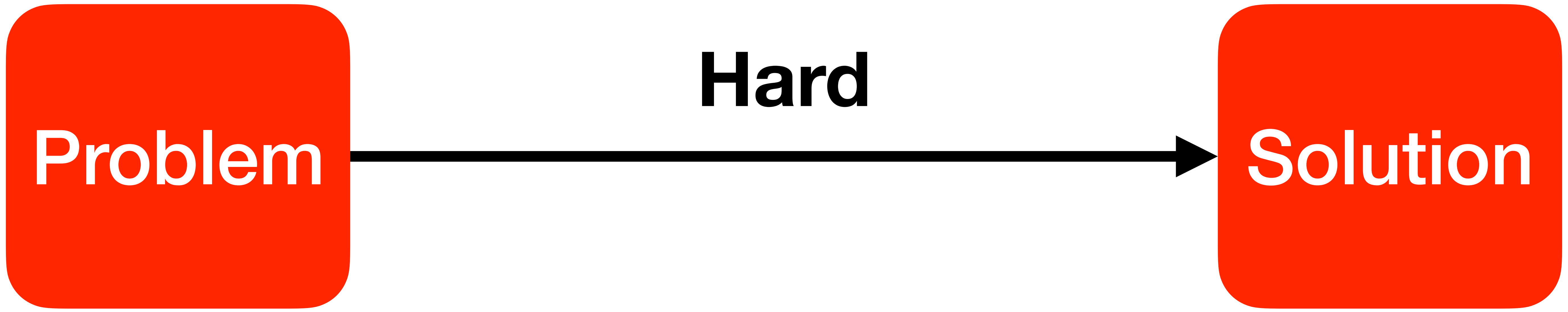
Fourier analysis



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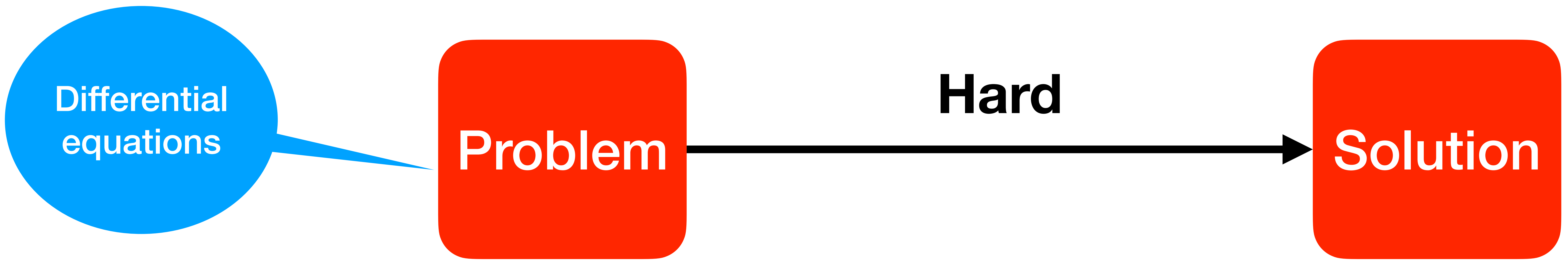

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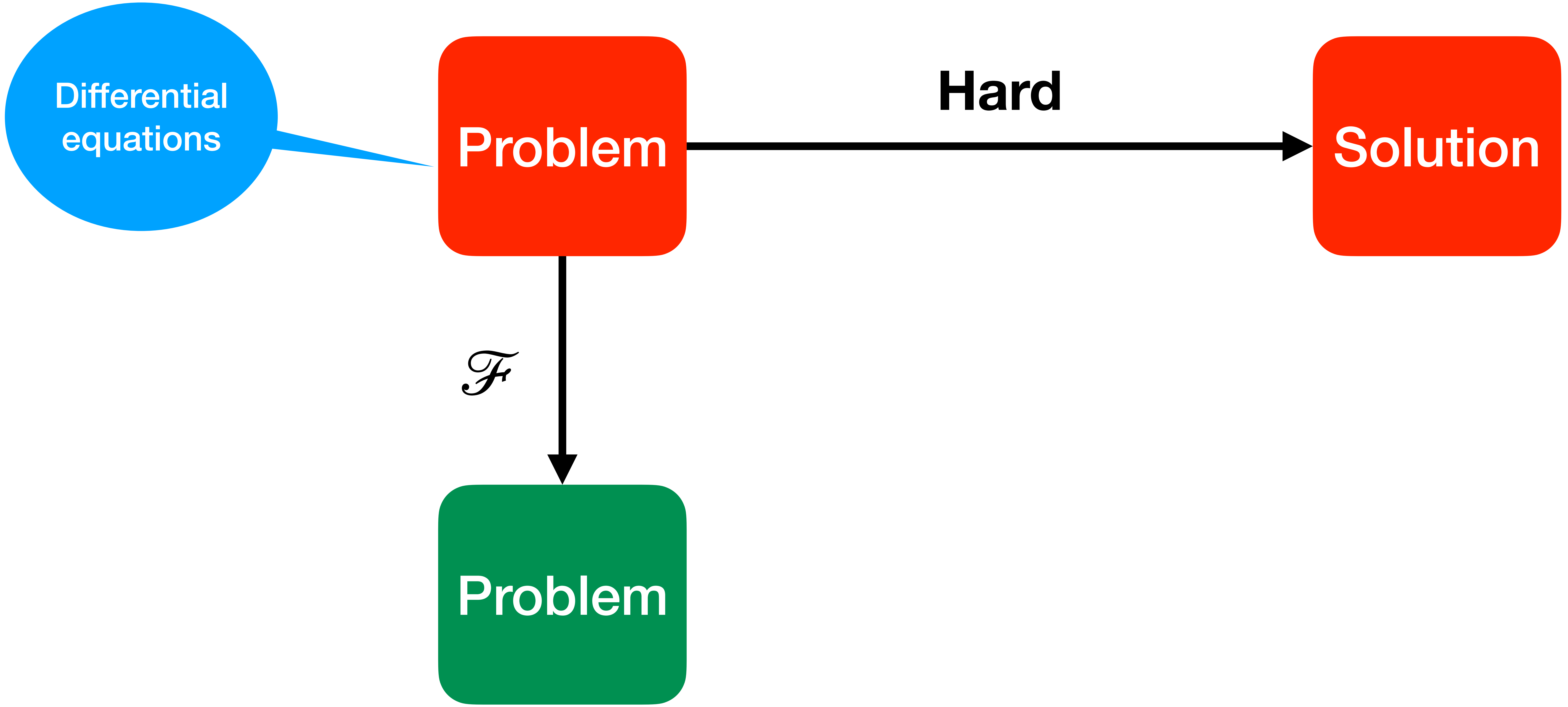

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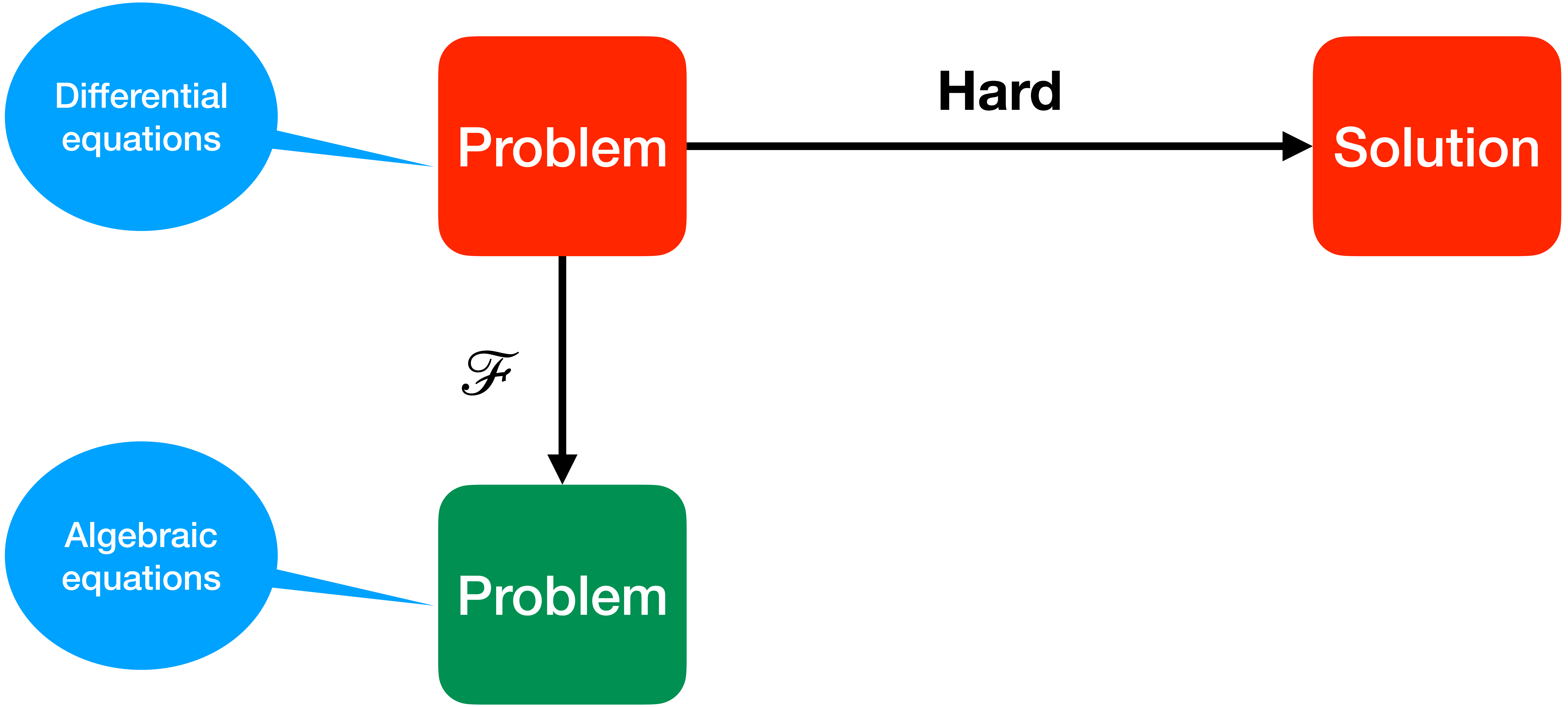

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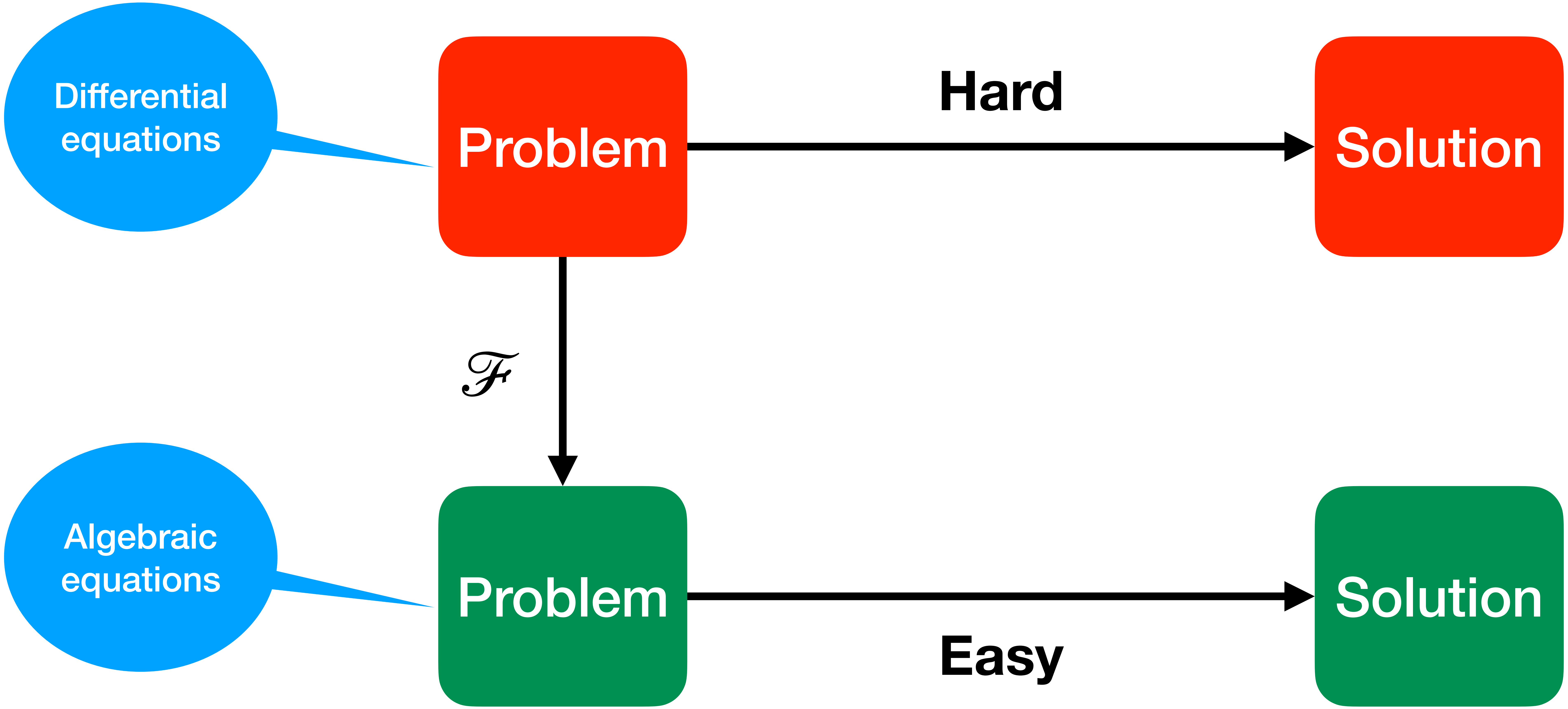
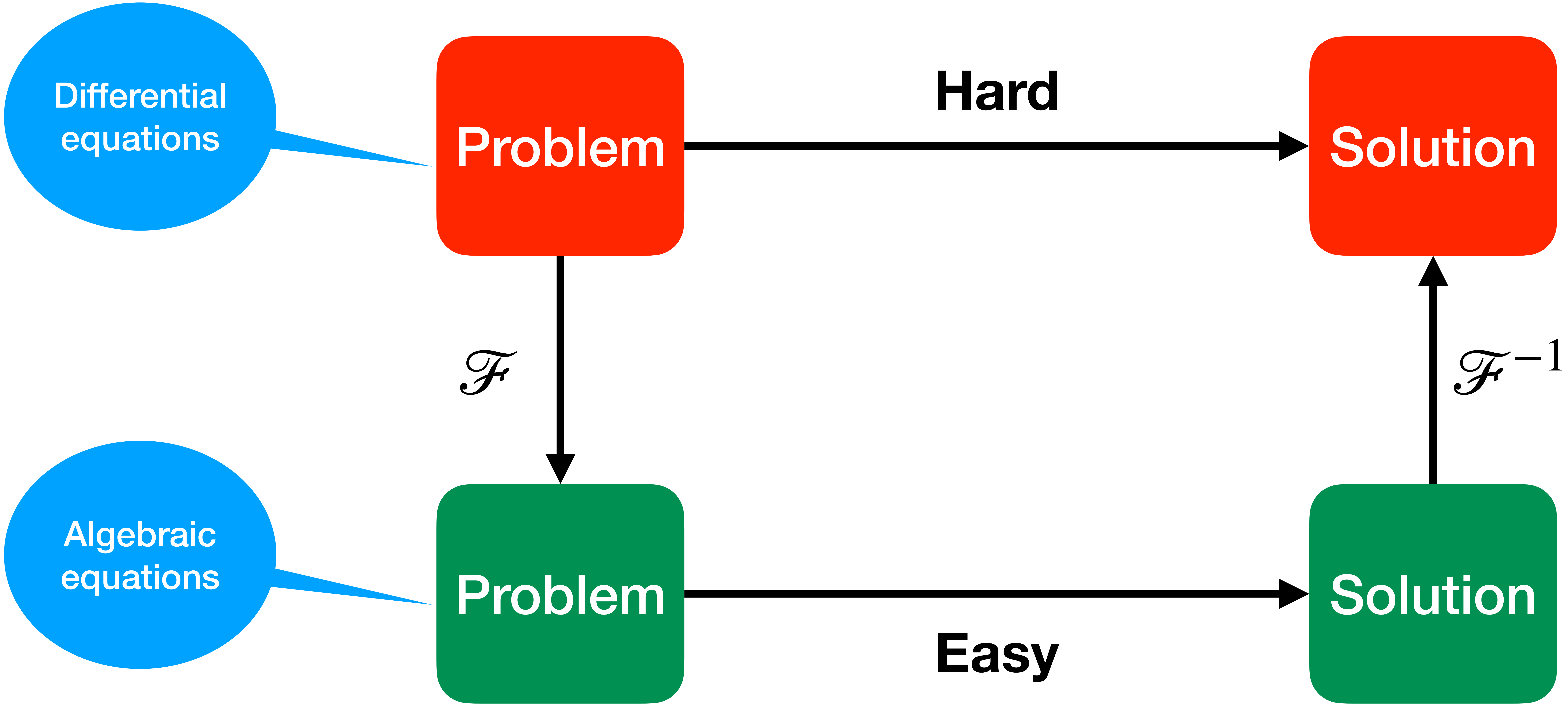


Fourier analysis



Convolution

When $f, g : \mathbb{R} \to \mathbb{C}$ are Lebesgue integrable, the function

$$f \otimes g(t) = \int_{\mathbb{R}} f(x)g(t-x)dx$$

Is defined almost everywhere and is Lebesgue integrable (by Fubini-Tonelli).

When f and g are Schwartz, then the convolution is defined everywhere and is Schwartz.

When f is L^1 and g is L^2, then the convolution is L^2 (by Young's inequality)

Example: smoothening

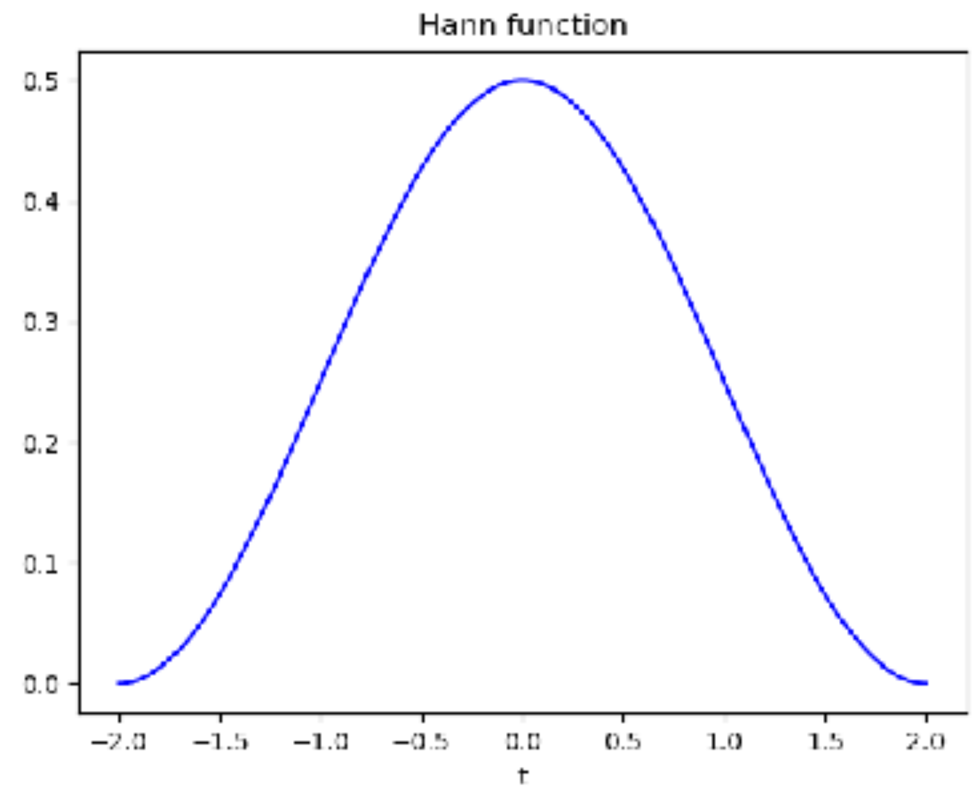
Imagine that f is a noisy signal and g is a window function (that is, with compact support and integral 1). Then $f \otimes g(t)$ can be thought as averaging the values of f around t, the average being weighted by g.

Discrete view: $g(0) = 0.5$, $g(1) = g(-1) = 0.25$, then

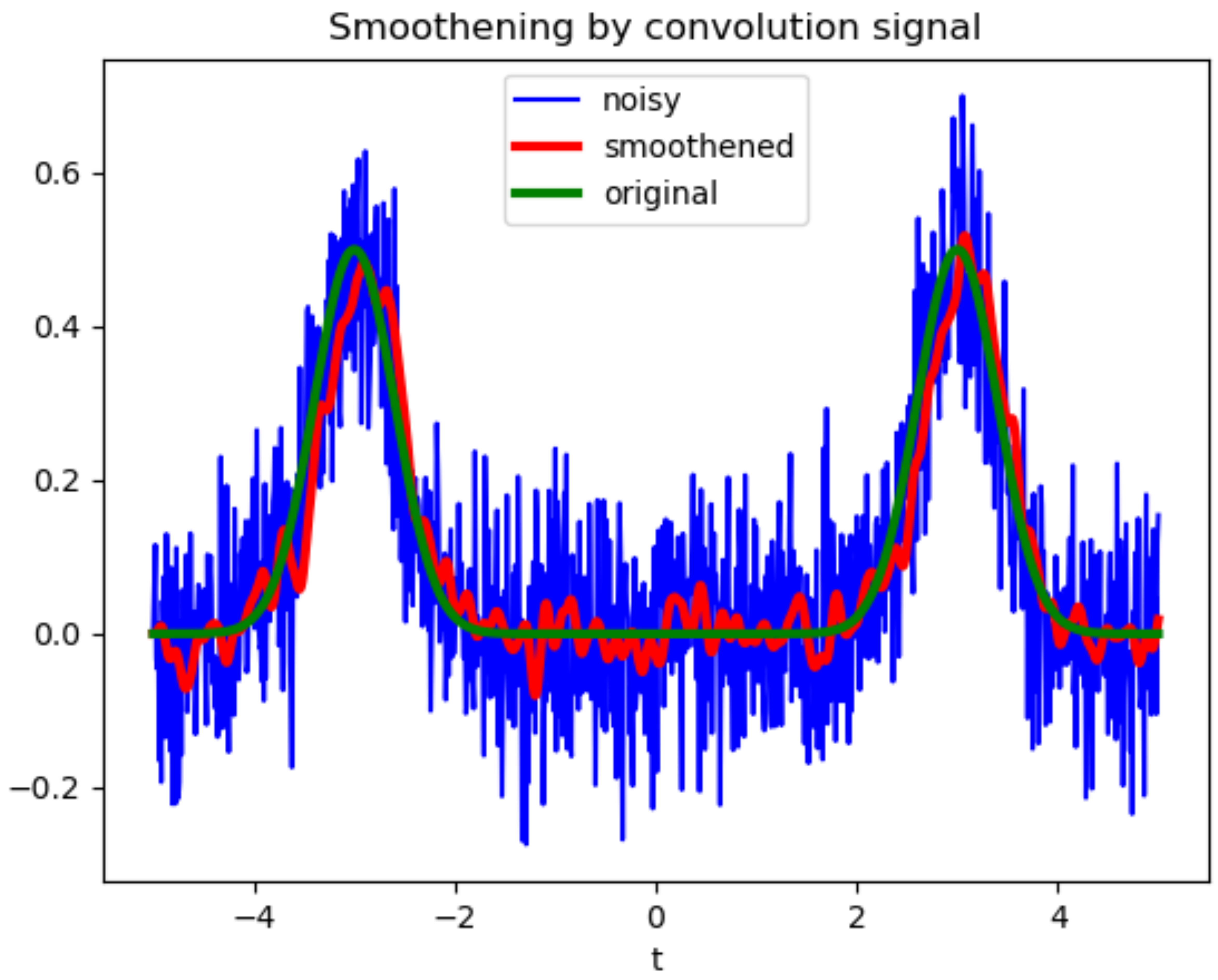
$$f \otimes g(t) = 0.25f(t-1) + 0.5f(t) + 0.25f(t+1)$$

A typical example of window function:

Hann function $\dfrac{2}{L}\cos^2\left(\dfrac{\pi t}{L}\right)$ on $\left[-\dfrac{L}{2}, \dfrac{L}{2}\right]$



Example of smoothening



Convolution theorem

When all the objects are well defined almost everywhere

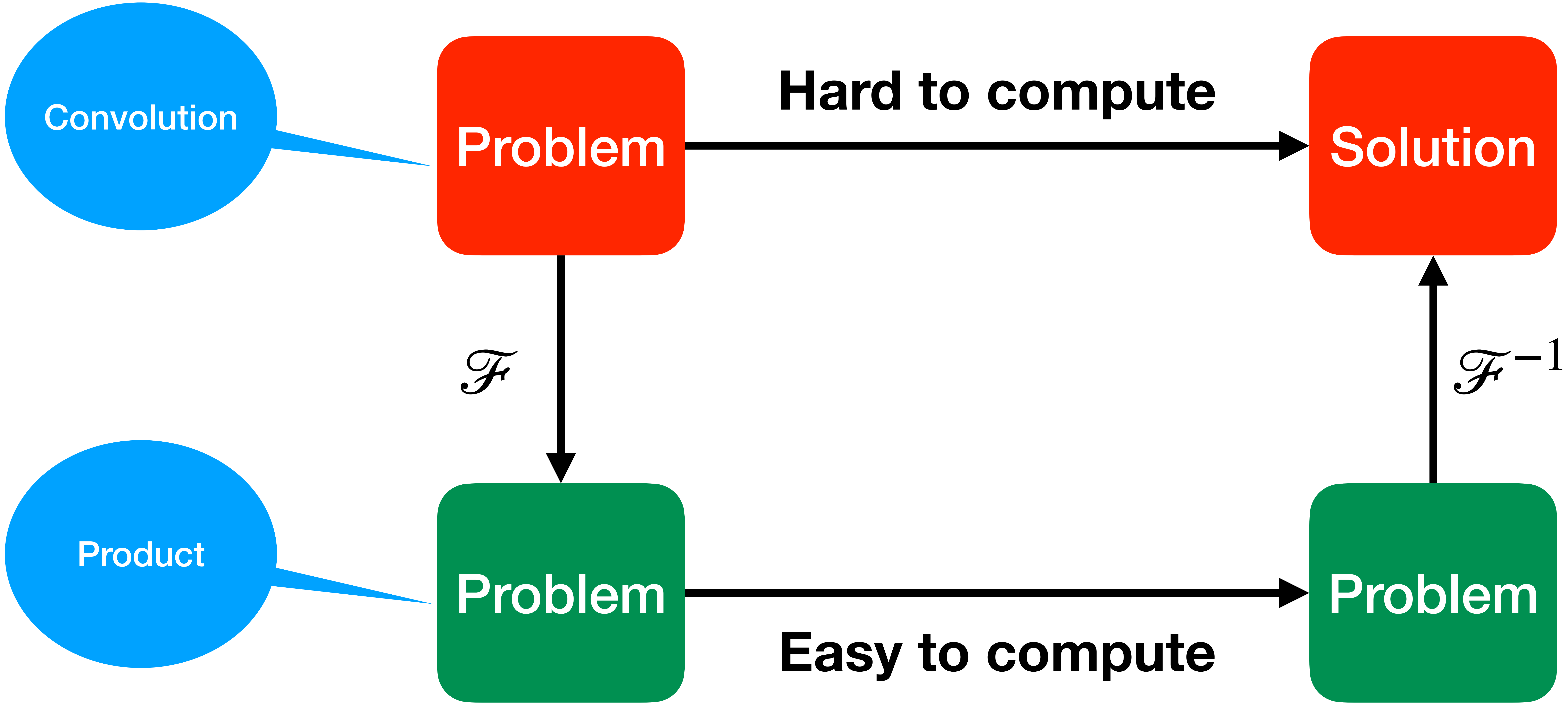
$$\mathscr{F}\{f \otimes g\} = \mathscr{F}\{f\} \, . \, \mathscr{F}\{g\}$$

Proof: In the case when f and g are L^1, by Fubini.

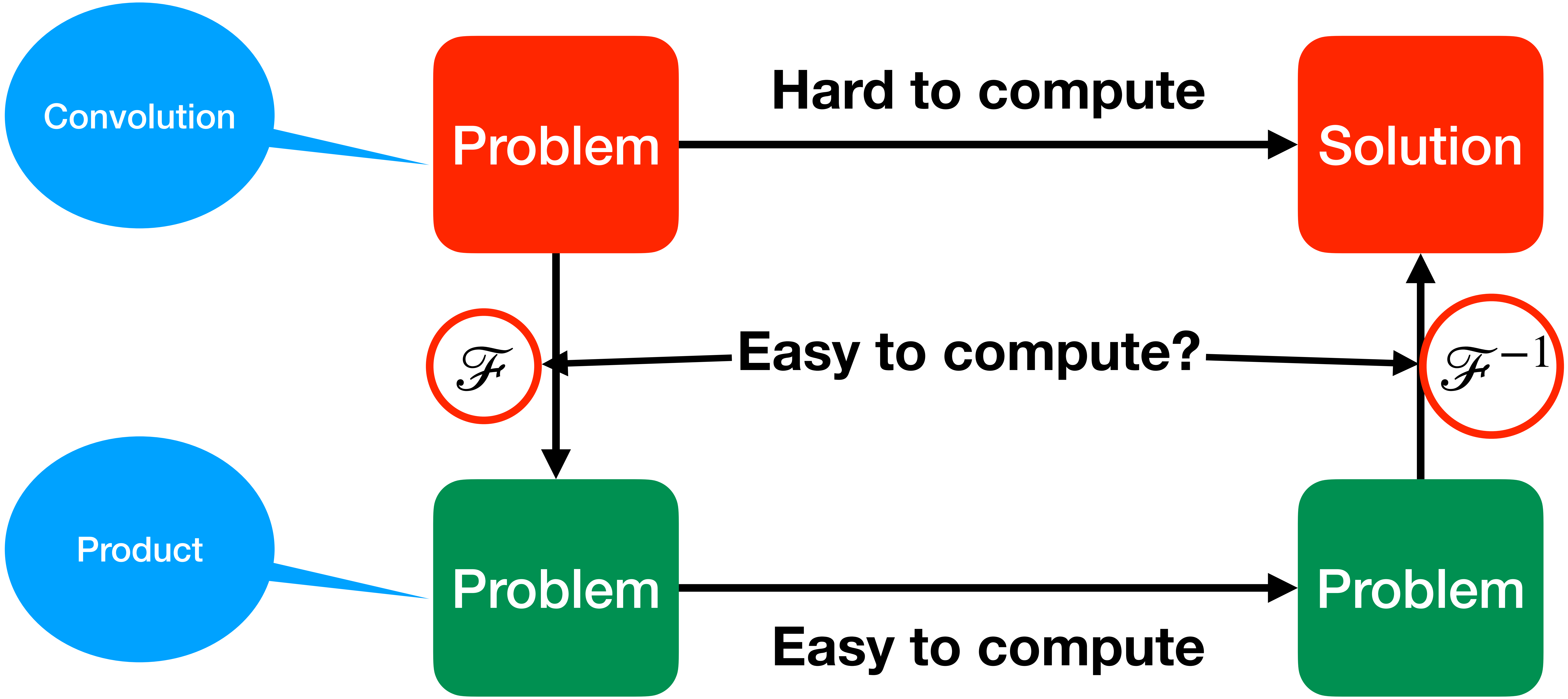
If furthermore the inversion theorem applies, almost everywhere:

$$f \otimes g = \mathscr{F}^{-1}\{\mathscr{F}\{f\} \, . \, \mathscr{F}\{g\}\}$$

Fourier analysis



Fourier analysis



Computation: Fast Fourier Transform

Assume that you can measure N samples of f:

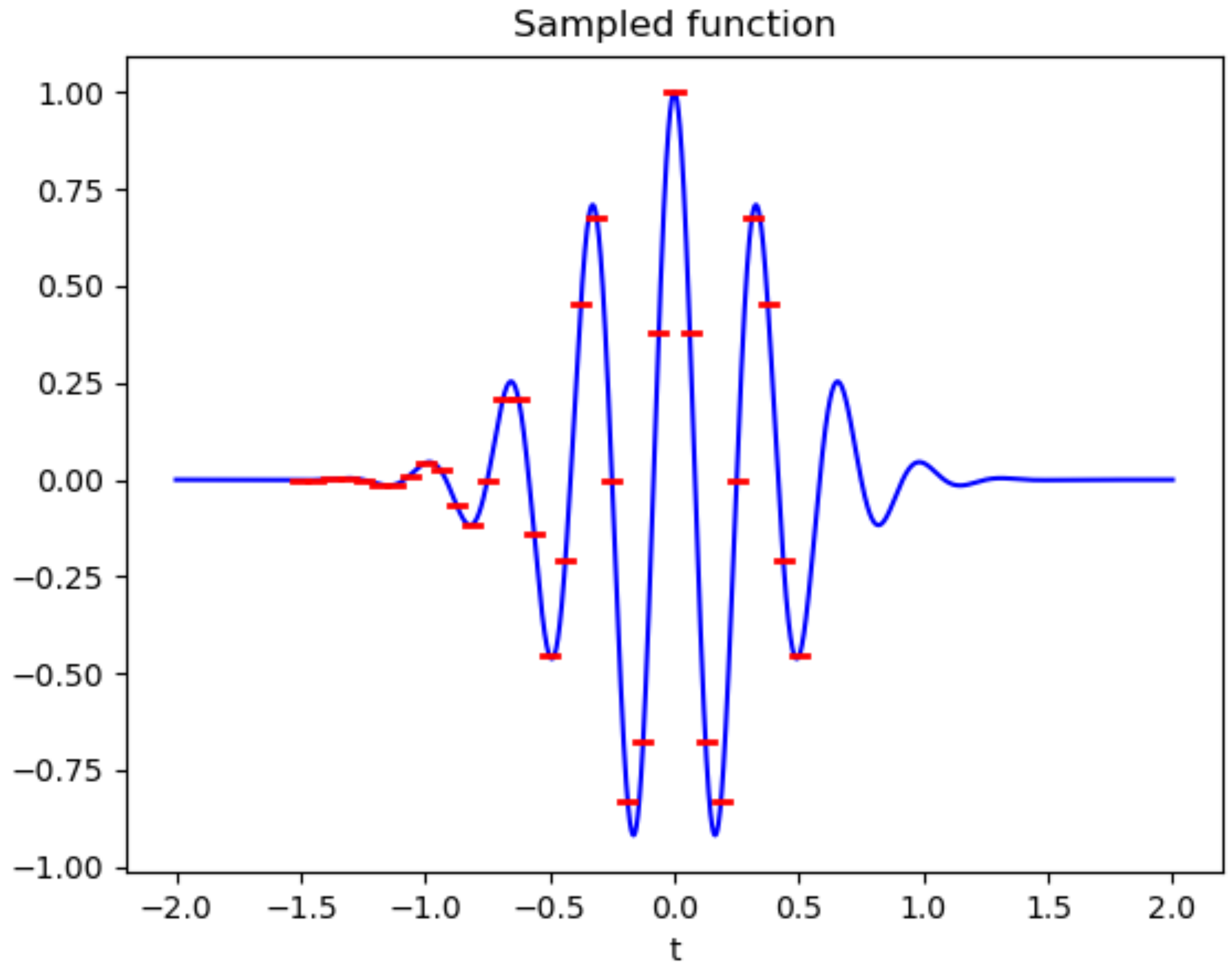
$$f(t_0), f(t_0 + \Delta_t), \ldots, f(t + \Delta_t(N-1))$$

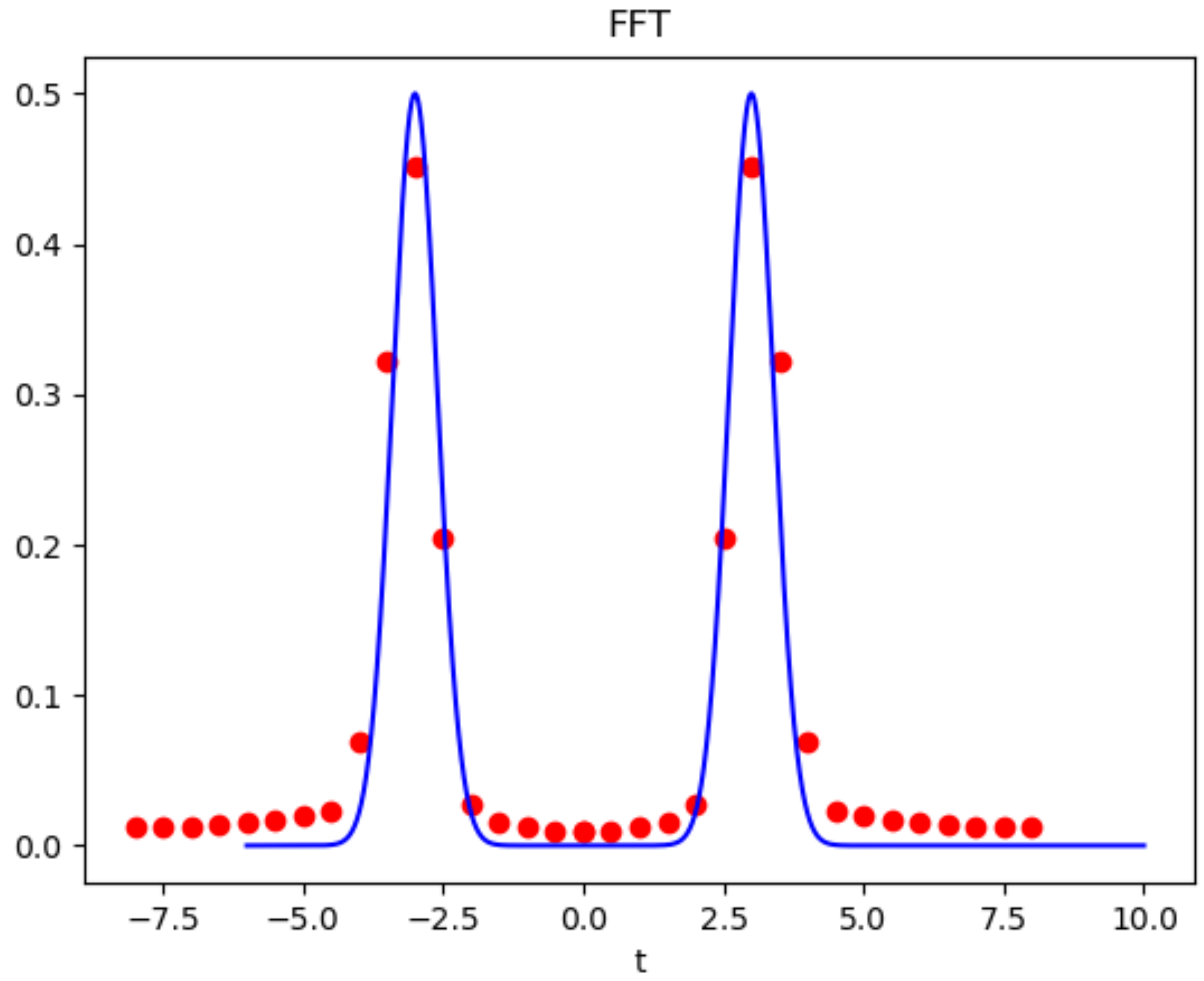
We can approximate the Fourier transform by:

$$\mathscr{F}\{f\}(\xi) \simeq \int_{t_0}^{t_0+N\Delta_t} f(t)e^{-2\pi i\xi t}dt \simeq \sum_{k=0}^{N-1} \Delta_t f(t_0 + k\Delta_t)e^{-2\pi i\xi(t_0+N\Delta_t)}$$

A priori, to compute the approximation of N values of $\mathscr{F}\{f\}$, N^2 operations are needed, but efficient algorithms are in $O(N \ln N)$.

Computation, in pictures





Computation, in pictures