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Course: Math 460
Assignment: Weekly #10
Due: 4/10/17

For each positive number n, let $a_n = \sum_{k=n}^{2n} \frac{1}{k}$.

10.1 Compute a_n for $n = 1$, 2, 3, and 4. give your answers in rational form (i.e. $\frac{a}{b}$).

Solution:

<u>$n = 1$:</u>

$$a_1 = \sum_{k=1}^{2} \frac{1}{k} = 1 + \frac{1}{2} = \frac{3}{2}$$

<u>$n = 2$:</u>

$$a_2 = \sum_{k=2}^{4} \frac{1}{k} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$$

<u>$n = 3$:</u>

$$a_3 = \sum_{k=3}^{6} \frac{1}{k} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{19}{20}$$

<u>$n = 4$:</u>

$$a_4 = \sum_{k=4}^{8} \frac{1}{k} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{743}{840}$$

10.2 Prove that the sequence $a_1, a_2, a_3, ...$ is convergent.

Proof: We want to show that $\{a_n\}$ is convergent. Hence it is sufficient to show that $\{a_n\}$ is bounded and monotone.

First we will show that $\{a_n\}$ is monotonic. Specifically, we will show that $\{a_n\}$ is monotonic decreasing. Let $n \in \mathbb{N}$. We need to show that $a_{n+1} \leq a_n$. Thus it is sufficient to show that $0 \leq a_n - a_{n+1}$. Now notice that

$$a_n = \sum_{k=n}^{2n} \frac{1}{k}$$

and

$$a_{n+1} = \sum_{k=n+1}^{2(n+1)} \frac{1}{k}.$$

Thus

$$\begin{aligned}
a_n - a_{n+1} &= \sum_{k=n}^{2n} \frac{1}{k} - \sum_{k=n+1}^{2(n+1)} \frac{1}{k} \\
&= (\frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n}) - (\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}) \\
&= \frac{1}{n} - \frac{1}{2n+1} - \frac{1}{2n+2} \\
&= \frac{1}{n} - \frac{2n+2-2n+1}{(2n+1)(2n+2)} \\
&= \frac{1}{n} - \frac{1}{(2n+1)(2n+2)} \\
&= \frac{(2n+1)(2n+2)-n}{n(2n+1)(2n+2)}.
\end{aligned}$$

Now observe that $(2n+1)(2n+2) > n$, so it follows that

$$a_n - a_{n+1} = \frac{(2n+1)(2n+2)-n}{n(2n+1)(2n+2)} > 0.$$

Hence $a_{n+1} \leq a_n$, and therefore $\{a_n\}$ is monotonic decreasing

We will now show that $\{a_n\}$ is bounded. Thus we need to show that there exists $M \in \mathbb{R}^+$ so that $|a_n| \leq M$ for each $n \in \mathbb{N}$. Choose $M = \frac{3}{2}$, and let $n \in \mathbb{N}$. Now notice that as $a_1 = M$, and $\{a_n\}$ is monotonic decreasing, $a_n \leq M$. Additionally, since a_n is a sum of positive real numbers, it follows that $-M \leq 0 \leq a_n$. Therefore as $-M \leq a_n \leq M$, we can conclude that $|a_n| \leq M$, so $\{a_n\}$ is bounded.

Therefore as we have shown that $\{a_n\}$ is monotonic and bounded, we can conclude that a_n is convergent.