Name: Jeremy Florence
Course: Math 460
Assignment: GMK #48 Presentation
Due: 4/12/17

48. Let G be a group with identity e and finite order n. Which of the following conditions is sufficient for G to be abelian?

I. $n = 6$

Counterexample: Let $G = S_3 = \{(), (1,2), (2,3), (1,3), (1,2,3), (1,3,2)\}$, the symmetric group on a set of three elements. It is clear that $|S_3| = 6$. Now consider the elements $(1,2)$ and $(2,3)$ in S_3. Notice that $(1,2) \cdot (2,3) = (1,2,3)$ but $(2,3) \cdot (1,2) = (1,3,2)$. Hence $(1,2) \cdot (2,3) \neq (2,3) \cdot (1,2)$, so S_3 is not abelian. Thus there exists a group G with order n=6 where G is not abelian. Therefore $|G| = 6$ is not a sufficient condition for G to be abelian

II. $n = 15$

Definition: Let G be a group and let p be a prime.

(1) A group of order p^k for some $k \geq 0$ is a called a *p-group*. Subgroups of G which are p-groups are called *p-subgroups*.

(2) If G is a group of order $p^k m$, where $p \nmid m$, then a subgroup of order p^k is called a *Sylow p-subgroup* of G.

(3) The number of Sylow p-subgroups of G will be denoted by n_p.

Sylow's Theorem: Let G be a group of order $p^k m$, where p is a prime not dividing m. Then the following are true:

(1) Sylow p-subgroups of G exist.

(2) If P is a Sylow p-subgroup of G and Q is any p-subgroup of G, then there exists $g \in G$ such that Q is a subgroup of gPg^{-1}, i.e., Q is contained in some conjugate of P. In particular, any two Sylow p-subgroups of G are conjugate in G.

(3) $n_p \equiv 1(\text{mod } p)$, and n_p divides m

Proof: Let G be a group with order $n = 15$. Since $n = 15 = 3 \cdot 5$, and both 3 and 5 are prime, we know that Sylow 3-subgroups of G and Sylow-5 subgroups of G exist by part (1) of Sylow's Theorem. Now by part (3) of Sylow's Theorem, we know that $n_3 \equiv 1 (\text{mod } 3)$ and $n_3 \mid 5$. Thus as 5 is prime it must follow that $n_3 = 1$. Following the same reasoning, we can see that $n_5 = 1$. Let P be the Sylow 3-subgroup in G and let Q be the Sylow 5-subgroup in G. Recall that $P \cap Q$ is a subgroup of Q and P. Thus by Lagrange's Theorem, we know that $|P \cap Q|$ divides 3 and 5, so it follows that $|P \cap Q| = 1$. That is, $P \cap Q = \{e\}$. Now consider the elements $a \in P$ and $b \in Q$. Now notice that $ab \notin P$ and $ab \notin Q$, and furthermore that $|\langle ab \rangle|$ must divide 15 by Lagrange's Theorem. Hence $|\langle ab \rangle|$ must be 1, 3, 5, or 15. Obviously $|\langle ab \rangle| \neq 1$ because this would imply $ab = e$, and further that $ab \in P \cap Q$, a contradiction. Also, $|\langle ab \rangle|$ cannot be 3, as this would imply that $\langle ab \rangle$ is a Sylow 3-subgroup of G, and moreover as $n_3 = 1$, $\langle ab \rangle = P$, which contradicts $ab \notin P$. For the same reasoning, we can conclude that $|\langle ab \rangle| \neq 5$. Therefore $|\langle ab \rangle| = 15$, which means that ab generates the entire group G. Hence G is cyclic and therefore also abelian.

III. n is a prime number

Proof: Let G be a group with prime order p. Then $|G| > 1$, so let $g \in G$ such that $g \neq e$. Then it follows that $|\langle g \rangle| > 1$. Additionally, we know by Lagrange's Theorem that $|\langle g \rangle|$ divides p. Hence either $|\langle g \rangle| = 1$ or $|\langle g \rangle| = p$. Thus as we already know $|\langle g \rangle| > 1$, it must follow that $|\langle g \rangle| = p$. Therefore $\langle g \rangle = G$, and g generates G. Hence G is cyclic and therefore abelian.

IV. $(ab)^2 = a^2b^2$ for all $a, b \in G$

Proof: Let G be a group such that $(ab)^2 = a^2b^2$ for all $a, b \in G$. Let $a, b \in G$. Then it follows that:

$$\begin{aligned}
(ab)^2 &= a^2b^2 \\
(ab)(ab)b^{-1} &= a^2b^2b^{-1} \\
aba &= a^2b \\
a^{-1}aba &= a^{-1}a^2b \\
ba &= ab.
\end{aligned}$$

Therefore as $ab = ba$, we can conclude that G is abelian.

Answer: D (II, III, and IV only)