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Course: Math 460
Assignment: Weekly #12
Due: 4/24/17

12.1 To avoid paying a toll of \$2 on a direct road, Mr. Fred Cheaply first goes west 10 miles, then south 5 miles, then 30 miles west and finally 35 miles north. Freds old Hyundai gets 30 miles to the gallon and he can still buy gas for \$1.659 per gallon.

(a) How far out of his way is Fred going to avoid paying the \$2 toll?

Solution: Observe that Fred is travelling north-west. Thus we must calculate the net distance traveled by Fred in the north and west directions:

$$d_{north} = 35 - 5 = 30$$

$$d_{west} = 10 + 30 = 40$$

Now using Pythagorean's Theorem, we can calculate the distance of the direct road:

$$d_{direct} = \sqrt{30^2 + 40^2} = 50$$

Therefore by avoiding the \$2 toll, Fred travels a total distance of $10+5+30+35 = 80$ miles. Hence Fred is going $80 - 50 = 30$ miles out of his way in order to avoid the \$2 toll.

(b) On the basis of gas mileage his Hyundai gets and what he pays per gallon of gas, is Fred saving any money (and if so how much)?

Solution: If Fred had taken the direct road he would have used $50/30 = 5/3$ gallons of gas. Thus the total cost of using the direct road would be:

$$cost_{direct} = (5/3) \cdot \$1.659 + \$2 = \$4.765.$$

Now by not taking the direct road, Fred uses $80/30 = 8/3$ gallons of gas. Thus the cost of not taking the direct road is

$$cost_{indirect} = (8/3) \cdot \$1.659 = \$4.424.$$

Therefore Fred saves $\$4.765 - \$4.424 = \$0.34$ by not taking the direct road.

12.2 Given that a, b and c are odd integers, prove that the equation

$$ax^2 + bx + c = 0$$

cannot have a rational root (i.e. a root of the form $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Hint: Consider $b^2 - 4ac$ mod 8.

Proof: Let a, b, and c be odd integers so that $ax^2 + bx + c = 0$. Recall that a quadratic equation has rational roots if and only if there exists $d \in \mathbb{Z}$ so that $d^2 = b^2 - 4ac$. Now notice that as a, b, and c are odd, b^2 must be odd and $4ac$ must be even. Hence $b^2 - 4ac$ is odd, so d must also be odd.

Claim: For any odd integer n, $n^2 \equiv 1 \pmod 8$

Proof of Claim: Let n be an odd integer. We need to show that $8 \mid n^2 - 1$. Since n is odd, there exists $k \in \mathbb{Z}$ so that $n = 2k + 1$. Thus

$$\begin{aligned} n^2 - 1 &= (2k+1)^2 - 1 \\ &= 4k^2 + 4k + 1 - 1 \\ &= 4(k^2 + k). \end{aligned}$$

Now notice that regardless of whether k is even or odd, $k^2 + k$ must be even. Hence there exists $j \in \mathbb{Z}$ so that $k^2 + k = 2j$, so

$$n^2 - 1 = 4(k^2 + k) = 4(2j) = 8j.$$

Therefore we can conclude that $8 \mid n^2 - 1$, so $n^2 \equiv 1 \pmod 8$.

Now we know that $d^2 \equiv 1 \pmod 8$, and also that $b^2 \equiv 1 \pmod 8$. Next we will show that $-4ac \equiv 4 \pmod 8$. Thus we will show that $8 \mid -4ac - 4$. Notice that ac is odd, so $ac + 1$ must be even. That is, there exists some $k \in \mathbb{Z}$ so that $ac + 1 = 2k$. Thus:

$$\begin{aligned} -4ac - 4 &= -4(ac + 1) \\ &= -4(2k) \\ &= 8 \cdot -k \end{aligned}$$

so we can conclude that $-4ac \equiv 4 \pmod 8$. Therefore $b^2 + (-4ac) \equiv 1 + 4 \pmod 8$, so $b^2 - 4ac \not\equiv 1 \pmod 8$. Hence $b^2 - 4ac$ is not a square of some integer, so the equation

$$ax^2 + bx + c = 0$$

cannot have a rational root.