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 Assignment: Final Exam
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1. Show that every quotient ring of a Noetherian ring is a Noetherian Ring. Is the same true for "every subring of..."?

Proof: Let R be a Noetherian ring with an ideal I . We want to show that R/I is also Noetherian. Let J be an ideal of R/I . Now by contracting J to R across the canonical map to the quotient φ , we know that $J^c = \varphi^{-1}(J)$ is an ideal of R . Since R is Noetherian, it is finitely generated. That is, there exist j_1, j_2, \dots, j_n such that $(j_1, j_2, \dots, j_n) = J^c$. Thus it must follow that $(\varphi(j_1), \varphi(j_2), \dots, \varphi(j_n)) = J$. Therefore J is finitely generated and hence Noetherian.

Proposition: Every subring of a Noetherian ring is a Noetherian ring.

Counterexample: Let k be a field. Then the polynomial ring over k with infinitely many variables $R = k[x_1, x_2, \dots]$ is a commutative domain, but R is not Noetherian because we can keep choosing more variables to extend the chain of ideals:

$$(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \subset \dots$$

Now by homework 9, we know that R has a field of fractions $R_{(0)}$. Fields only have two ideals, (0) and (1) , so $R_{(0)}$ is clearly Noetherian. We also know that $R \subset R_{(0)}$, so there exists a Noetherian ring which has a subring that is not Noetherian.

2. Let k be field. Prove that $k[x, y]$ is *not* a PID.

Proof: For the sake of contradiction, suppose $k[x, y]$ is a PID. Now consider the ideal (x, y) . There must exist some $i \in k[x, y]$ so that $(i) = (x, y)$. Hence as $x, y \in (i)$ it must follow that x and y are factors of i . However, x and y are both irreducible in $k[x, y]$, so i must be a unit. Therefore $(i) = (x, y) = (1)$, a contradiction. Therefore, $k[x, y]$ is not a PID.

3. Find a ring R which has a chain of prime ideals $P_1 \subsetneq P_2 \subsetneq P_3 \subsetneq P_4 \subsetneq R$.

Solution: Let $R = \mathbb{Z}[x, y, z]$ and let $p \in \mathbb{Z}$ be prime. Then we have the chain of prime ideals $(p) \subsetneq (p, x) \subsetneq (p, x, y) \subsetneq (p, x, y, z) \subsetneq \mathbb{Z}[x, y, z]$.

4. Prove that $\mathbb{Z}[i]$ is a Euclidean domain.

Proof: Define $d : \mathbb{Z}[i] - \{0\} \rightarrow \mathbb{N}$ by $d(a + bi) = a^2 + b^2$ for all $a + bi \in \mathbb{Z}[i] - \{0\}$. Let $(a + bi), (c + di) \in \mathbb{Z}[i] - \{0\}$. Then

$$\begin{aligned} d((a + bi)(c + di)) &= d(a + bi) \cdot d(c + di) \\ &= (a^2 + b^2) \cdot (c^2 + d^2). \end{aligned}$$

Thus as $a, b, c, d \in \mathbb{Z}$, it is clear that $d((a + bi)(c + di)) \geq d(a + bi)$.

5. Let $I \subseteq P \subsetneq R$, where I is an ideal and P is a prime ideal of R . Prove that the ring R_P/I^e (the quotient of the localization at P by the extension of the ideal I to that localization) and the ring $(R/I)_{P/I}$ (the localization of the quotient ring R/I at the prime ideal P/I) are isomorphic.
6. Let R be a ring and let $f \in R$ be not nilpotent. Let $S = \{f^n : n \in \mathbb{N}\}$. Prove that $R[S^{-1}]$ is isomorphic to $R[z]/(fz - 1)$.