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Course: Math 672
Assignment: Final Exam
Due: 5/19/17

1. Show that every quotient ring of a Noetherian ring is a Noetherian Ring. Is the same true for "every subring of..."?

   Proof: Let R be a Noetherian ring with an ideal I. We want to show that R/I is also Noetherian. Recall that $R/I = \{r + I | r \in R\}$. Since R is Noetherian, we know that I is finitely generated. That is, $I = (i_1, i_2, ..., i_n)$ for some $i_1, i_2, ..., i_n \in R, n \in \mathbb{N}$. Now consider the canonical map to the quotient $\varphi : R \to R/I$ defined by $\varphi(r) = r + I$ for all $r \in R$. Now we can extend I to R/I through φ to get $I^e = (\varphi(I))$. Note that $\varphi(I) = \{i+I|i \in I\}$, so $\varphi(I) = \{I\}$.

2. Let k be field. Prove that $k[x, y]$ is *not* a PID.

3. Find a ring R which has a chain of prime ideals $P_1 \subsetneq P_2 \subsetneq P_3 \subsetneq P_4 \subsetneq R$.

4. Prove that $\mathbb{Z}[i]$ is a Euclidean domain.

   Proof: Define $d : \mathbb{Z}[i] - \{0\} \to \mathbb{N}$ by $d(a + bi) = a^2 + b^2$ for all $a + bi \in \mathbb{Z}[i] - \{0\}$. Let $(a + bi), (c + di) \in \mathbb{Z}[i] - \{0\}$. Then

$$\begin{aligned} d((a + bi)(c + di)) &= (ac - bd)^2 + (ad + bc)^2 \\ &= a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2 \\ &= a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2. \end{aligned}$$

   Thus as $d(a + bi) = a^2 + b^2$, and $a, b, c, d \in \mathbb{Z}$, it is clear that

$$d((a + bi)(c + di)) \geq d(a + bi).$$

5. Let $I \subseteq P \subsetneq R$, where I is an ideal and P is a prime ideal of R. Prove that the ring R_P/I^e (the quotient of the localization at P by the extension of the ideal I to that localization) and the ring $(R/I)_{P/I}$ (the localization of the quotient ring R/I at the prime ideal P/I) are isomorphic.

6. Let R be a ring and let $f \in R$ be not nilpotent. Let $S = \{f^n : n \in \mathbb{N}\}$. Prove that $R[S^{-1}]$ is isomorphic to $R[z]/(fz - 1)$.