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Course: Math 672
Assignment: Homework 8
Due: 4/27/17

1. Let $S = \mathbb{Z} - (p)$. Prove that $\mathbb{Z}[S^{-1}]$ has exactly three ideals. Describe them completely. Tell any containments among them.

   Proof: We know that (0) and (1) are ideals for any ring. Thus we need to show that $\mathbb{Z}[S^{-1}]$ has exactly one more ideal which is not equal to (0) or (1). Let $p \in \mathbb{Z}$ be prime. Consider the ideal $\left(\frac{p}{1}\right)$. We can easily see that $\frac{p}{1} \notin (0)$ and $\frac{1}{1} \notin \left(\frac{p}{1}\right)$. Hence $\left(\frac{p}{1}\right)$ is an ideal of $\mathbb{Z}[S^{-1}]$, and $(0) \neq \left(\frac{p}{1}\right) \neq (1)$. For the sake of contradiction, suppose there exists an ideal I of $\mathbb{Z}[S^{-1}]$ such that I is not equal to (0), $\left(\frac{p}{1}\right)$, or (1). Let $\frac{a}{b} \in I$. Then we know that $a \in \mathbb{Z}$ and $b \in \mathbb{Z} - (p)$. Notice $\frac{b}{1} \in \mathbb{Z}[S^{-}1]$. Thus as I is an ideal, we know that $\frac{b}{1} \cdot \frac{a}{b} = \frac{a}{1} \in I$. Now since $I \neq (0)$, it follows that $a \neq 0$. Therefore $\frac{1}{a} \in \mathbb{Z}[S^{-1}]$. Hence $\frac{a}{1} \cdot \frac{1}{a} = \frac{1}{1}$ and since I is an ideal, we now know that $\frac{1}{1} \in \mathbb{Z}[S^{-1}]$. However this means that $I = (1)$, a contradiction. Therefore we can conclude that there are exactly three ideals of $\mathbb{Z}[S^{-1}]$, with the following order of containment: $(0) \subset (\frac{p}{1}) \subset (1)$.

2. Let R be a domain and let S be any multiplicatively closed subset of R. Prove that the canonical map $R \to R[S^{-1}]$ is injective.

   Proof: Let $\varphi : R \to R[S^{-1}]$ be the canonical map defined by $\varphi(r) = \frac{r}{1}$ for all $r \in R$. Let $a, b \in R$ so that $\varphi(a) = \varphi(b)$. We want to show that $a = b$. By the definition of the canonical map φ, we have $\varphi(a) = \frac{a}{1}$ and $\varphi(b) = \frac{b}{1}$. Hence we have $\frac{a}{1} = \frac{b}{1}$, so there exists $s \in S$ such that $s(a \cdot 1 - 1 \cdot b) = 0$. Since R is a domain, we know that either $s = 0$ or $a - b = 0$. However, $0 \notin S$, so it is clear that $s \neq 0$. Hence $a - b = 0$ which further implies that $a = b$, which is what we needed to show. Therefore the canonical map φ is injective.