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Course: Math 672
Assignment: Homework 9
Due: 5/4/17

1. Let R be a commutative domain and let $S = R - \{0\}$. Prove that $R_{(0)} = R[S^{-1}]$ is a field. This field is called the *field of fractions* of R.

   Proof: Since R is a domain and $0 \notin S$, we know that S is a multiplicatively closed set. Thus $R_{(0)}$ is a ring. Now we will show that $R_{(0)}$ has multiplicative commutativity. Let $\frac{a}{b}, \frac{c}{d} \in R_{(0)}$. Now since

   $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

   and

   $$\frac{c}{d} \cdot \frac{a}{b} = \frac{ca}{db}$$

   we must show that there exists $s \in S$ so that $s(acdb - cabd) = 0$. Since R is commutative, we know that $acdb = cabd$. Thus we can choose $s = 1$ to satisfy $s(acdb - cabd) = 0$, so we now know that $\frac{ac}{bd} = \frac{ca}{db}$. Hence $R_{(0)}$ is commutative under multiplication.

   Now we will show that $R_{(0)}$ is closed under multiplicative inverse. Let $\frac{a}{b} \in R_{(0)}$ such that $\frac{a}{b} \neq 0$. Since $\frac{a}{b} \neq 0$, we know that $a \neq 0$. Hence we can choose $\frac{b}{a} \in R_{(0)}$ and we can see that

   $$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba}.$$

   Now we can choose $1 \in S$ and observe that since R is commutative, $1(ab \cdot 1 - 1 \cdot ba) = 0$, so

   $$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = 1.$$

   Thus $R_{(0)}$ is closed under multiplicative inverse.

   Therefore as $R_{(0)}$ is a ring that is commutative under multiplication and closed under multiplicative inverse, we can conclude that $R_{(0)}$ is a field.

2. As above, let $R \subseteq F$, where F is a (possibly different) field. Prove that the field of fractions of R is isomorphic to the smallest subfield of F which contains R.

   Proof: Observe that for any $b \in R - \{0\}$, $b^{-1} \in F$. Define $f : R_{(0)} \to F$ by $f(\frac{a}{b}) = ab^{-1}$ for all $\frac{a}{b} \in R_{(0)}$. We will first show that f is a ring homomorphism. Let $\frac{a}{b}, \frac{c}{d} \in R_{(0)}$. Then

$$\begin{aligned}
f\left(\frac{a}{b}+\frac{c}{d}\right) &= f\left(\frac{ad+bc}{bd}\right) \\
&= (ad+bc)(d^{-1}b^{-1}) \\
&= add^{-1}b^{-1} + bcd^{-1}b^{-1} \\
&= ab^{-1} + cd^{-1} \\
&= f\left(\frac{a}{b}\right) + f\left(\frac{c}{d}\right).
\end{aligned}$$

Also,

$$\begin{aligned}
f\left(\frac{a}{b}\cdot\frac{c}{d}\right) &= f\left(\frac{ac}{bd}\right) \\
&= acd^{-1}b^{-1} \\
&= ab^{-1}\cdot cd^{-1} \\
&= f\left(\frac{a}{b}\right)\cdot f\left(\frac{c}{d}\right).
\end{aligned}$$

Therefore we now know that f is a homomorphism.

Now we will show that f is injective. Let $\frac{a}{b}, \frac{c}{d} \in R_{(0)}$ so that $f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right)$. Then

$$\begin{aligned}
ab^{-1} &= cd^{-1} \\
ab^{-1}bd &= cd^{-1}bd \\
ad &= bc.
\end{aligned}$$

Thus $\frac{a}{b} = \frac{c}{d}$ since $ad - bc = 0$, so f is injective. Now it is clear that $f : R_{(0)} \to f(R_{(0)})$ is surjective, so f is an isomorphism. Therefore as $f(R_{(0)})$ is a field, and $f(R_{(0)}) \subseteq F$, we now know that $f(R_{(0)})$ is a subfield of F. Finally, we can conclude that $R_{(0)}$ is isomorphic to $f(R_{(0)})$, which is the smallest subfield of F which contains R.