

# Homework6

March 9, 2017

## 1 Homework 6: $Z[\sqrt{2}]$

In this homework you'll analyze the ring  $Z[\sqrt{2}]$ . We define:

$$Z[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Z\}$$

1. Prove that this subset of  $\mathbb{R}$  is a subring by applying the subring test.
2. Construct a surjective map from  $Z[x]$  to  $Z[\sqrt{2}]$ , calculate its kernel, and conclude that  $Z[\sqrt{2}]$  is isomorphic to some (be specific) quotient of the polynomial ring.
3. Define a conjugation map:  $\overline{a + b\sqrt{2}} = a - b\sqrt{2}$ . Prove that it is a ring isomorphism.
4. Define a function on  $Z[\sqrt{2}]$  by  $\phi(a + bi) = a^2 - 2b^2$ .
  - Prove that  $x\bar{x} = \phi(x)$ .
  - Prove that  $\phi$  is multiplicative in this sense:  $\phi(xy) = \phi(x)\phi(y)$
  - Prove that if  $\phi(x) = \pm 1$ , then  $\frac{\bar{x}}{\phi(x)}$  is an inverse in the ring.
5. Define a function on  $Z[\sqrt{2}]$  by  $v(x) = |\phi(x)|$ .
  - Calculate (with proof) the minimum possible value of  $v(x)$  for  $x \in Z[\sqrt{2}]$ .
  - Prove that  $v(x)$  satisfies the first axiom of a Euclidean domain.

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In [27]: class zrtelement:
    def __init__(self, a, b):
        self.a = a
        self.b = b

    def __add__(self, other):
        return zrtelement(self.a+other.a, self.b+other.b)

    def __sub__(self, other):
        return zrtelement(self.a-other.a, self.b-other.b)

    def __mul__(self, other):
        a, b, c, d = self.a, self.b, other.a, other.b
        return zrtelement(a*c+2*b*d, b*c+a*d)

    def v(self):
        return abs(self.a**2 - 2*self.b**2)
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def __str__(self):
    if self.b == 0:
        return str(self.a)
    op = "+" if (self.b>0) else "-"
    return "("+str(self.a)+op+str(abs(self.b))+" $\sqrt{2}$ ")

def conj(self):
    return zrtelement(self.a,-self.b)

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In [29]: x = zrtelement(6, 7)
        y = zrtelement(10,-5)
        print(x,y, x+y, x-y, x*y, x.v(), y.v(), (x*y).v(), x.conj(), y.conj(), (x*
        z = zrtelement(7,5)
        w = zrtelement(-7,5)
        print(z, w, z*w)

(6+7 $\sqrt{2}$ ) (10-5 $\sqrt{2}$ ) (16+2 $\sqrt{2}$ ) (-4+12 $\sqrt{2}$ ) (-10+40 $\sqrt{2}$ ) 62 50 3100 (6-7 $\sqrt{2}$ ) (10+5 $\sqrt{2}$ ) (-10-
(7+5 $\sqrt{2}$ ) (-7+5 $\sqrt{2}$ ) 1

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In [ ]: !jupyter nbconvert --to pdf Homework6.ipynb

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In [ ]:

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