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Course: Math 672
Assignment: Final Exam
Due: 5/19/17

1. Show that every quotient ring of a Noetherian ring is a Noetherian Ring. Is the same true for "every subring of..."?

   Proof: Let R be a Noetherian ring with an ideal I. We want to show that R/I is also Noetherian. Let J be and ideal of R/I. Now by contracting J to R across the canonical map to the quotient φ, we know that $J^c = \varphi^{-1}(J)$ is an ideal of R. Since R is Noetherian, it is finitely generated. That is, there exist $j_1, j_2, ..., j_n$ such that $(j_1, j_2, ..., j_n) = J^c$. Thus it must follow that $(\varphi(j_1), \varphi(j_2), ..., \varphi(j_n)) = J$. Therefore J is finitely generated and hence Noetherian.

   Proposition: Every subring of a Noetherian ring is a Noetherian ring.

   Counterexample: Let k be a field. Then the polynomial ring over k with infinitely many variables $R = k[x_1, x_2, ...]$ is a commutative domain, but R is not Noetherian because we can keep choosing more variables to extend the chain of ideals:

   $$(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \subset ...$$

   Now by homework 9, we know that R has a field of fractions $R_{(0)}$. Fields only have two ideals, (0) and (1), so $R_{(0)}$ is clearly Noetherian. We also know that $R \subset R_{(0)}$, so there exists a Noetherian ring which has a subring that is not Noetherian.

2. Let k be field. Prove that $k[x, y]$ is *not* a PID.

   Proof: For the sake of contradiction, suppose $k[x, y]$ is a PID. Now consider the ideal (x, y). There must exist some $i \in k[x, y]$ so that $(i) = (x, y)$. Hence as $x, y \in (i)$ it must follow that x and y are factors of i. However, x and y are both irreducible in $k[x, y]$, so i must be a unit. Therefore $(i) = (x, y) = (1)$, a contradiction. Therefore, $k[x, y]$ is not a PID.

3. Find a ring R which has a chain of prime ideals $P_1 \subsetneq P_2 \subsetneq P_3 \subsetneq P_4 \subsetneq R$.

   Solution: Let $R = \mathbb{Z}[x, y, z]$ and let $p \in \mathbb{Z}$ be prime. Then we have the chain of prime ideals $(p) \subsetneq (p, x) \subsetneq (p, x, y) \subsetneq (p, x, y, z) \subsetneq \mathbb{Z}[x, y, z]$.

4. Prove that $\mathbb{Z}[i]$ is a Euclidean domain.

   Proof: Define $d : \mathbb{Z}[i] - \{0\} \to \mathbb{N}$ by $d(a+bi) = a^2 + b^2$ for all $a + bi \in \mathbb{Z}[i] - \{0\}$. Let $(a+bi), (c+di) \in \mathbb{Z}[i] - \{0\}$. Now since d is the modulus function for Gaussian integers, we know it is multiplicative. Hence

$$\begin{aligned} d((a+bi)(c+di)) &= d(a+bi) \cdot d(c+di) \\ &= (a^2+b^2) \cdot (c^2+d^2). \end{aligned}$$

   Thus as $a, b, c, d \in \mathbb{Z}$, it is clear that $d((a+bi)(c+di)) \geq d(a+bi)$.

5. Let $I \subseteq P \subsetneq R$, where I is an ideal and P is a prime ideal of R. Prove that the ring R_P/I^e (the quotient of the localization at P by the extension of the ideal I to that localization) and the ring $(R/I)_{P/I}$ (the localization of the quotient ring R/I at the prime ideal P/I) are isomorphic.

6. Let R be a ring and let $f \in R$ be not nilpotent. Let $S = \{f^n : n \in \mathbb{N}\}$. Prove that $R[S^{-1}]$ is isomorphic to $R[z]/(fz-1)$.