EL2700 - Assignment 3: Linear Quadratic and Gaussian Regulators

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1 Part1: LQR Implementation

1.1 Q1: Infinite Horizon Stability

This first part brings into question system stability and the role of the state cost matrix Q. If the cost matrix Q is **positive semi-definite**, it does not imply stability. For example, in the case of $Q = 0$ and $U = 0$, where the states can't be affected by the input control. In this theoretical situation, the 0 value could not effect the eigenvalue outside the unit circle. However, if a **positive definite** matrix Q is used in the design of a reachable LQR with input, the system will be stabilizable since all states can be affected by the control input. The designer can find a proper control input to move the unstable poles back to left half plane in case of continuous system and inside unity circle in case of discrete time system.

In our particular system, it can be seen that it is inherently unstable by analyzing the eigenvalues of A. Figure 1 below shows the eigenvalues ('E'). Since all eigenvalues of the state matrix are 1, this correlates to being on the instability boundary of the discrete unit circle; meaning that all states will never converge to 0. With that being said, if A and B are both reachable, then there exists a control sequence that allows the target states to be reached. Below these values in the same figure are the corresponding eigenvectors ('V') multiplied by B. This is demonstrating the PH test for reachability. The PBH test claims that the discrete linear system is unreachable if and only if $V \neq 0$ and $V^T B = 0$. Since the printed values show that the result of this is full rank, we can conclude that the system is reachable.

```
E: [1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]
V: [[ 7.72199122e-20  1.15038317e-18 -9.43562824e-20  0.00000000e+00
   0.00000000e+00  0.00000000e+00]
 [ 2.30093711e-17 -1.38842006e-18  1.90306926e-18  0.00000000e+00
   0.00000000e+00  0.00000000e+00]
 [-1.77922338e-18  2.00378677e-18  2.29738926e-17  0.00000000e+00
   0.00000000e+00  0.00000000e+00]
 [ 4.83502401e-04 -5.49360266e-04  8.53603541e-05  0.00000000e+00
   0.00000000e+00  0.00000000e+00]
 [-1.04025760e-02  6.70423501e-04 -3.39586182e-04  0.00000000e+00
   0.00000000e+00  0.00000000e+00]
 [ 1.49661920e-03  9.45622259e-03 -1.11961544e-02  0.00000000e+00
   0.00000000e+00  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  1.44319869e-15
   0.00000000e+00  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   1.55602386e-15  0.00000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  1.36405840e-15]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00 -6.17354684e-01
   2.34961511e-04  2.36856475e-03]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00 -4.35719882e-03
  -6.63204043e-01 -5.62144784e-02]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00 -4.99563225e-02
   6.39446485e-02 -5.83021624e-01]]
```

Figure 1: Eigenvalues ('E') and corresponding eigenvectors multiplied by B ('V')

1.2 Q2: Effect of State Cost Q and Input Cost R

- **Original**

  To compare the effects of the different Q and R gains, we first examine the nominal system response with both initialized as 1 for all states. The system is shown to be stable with no overshoot, but max forces, torques, and Euler angle deviations are surpassed according to constraints mentioned in assignment outline.

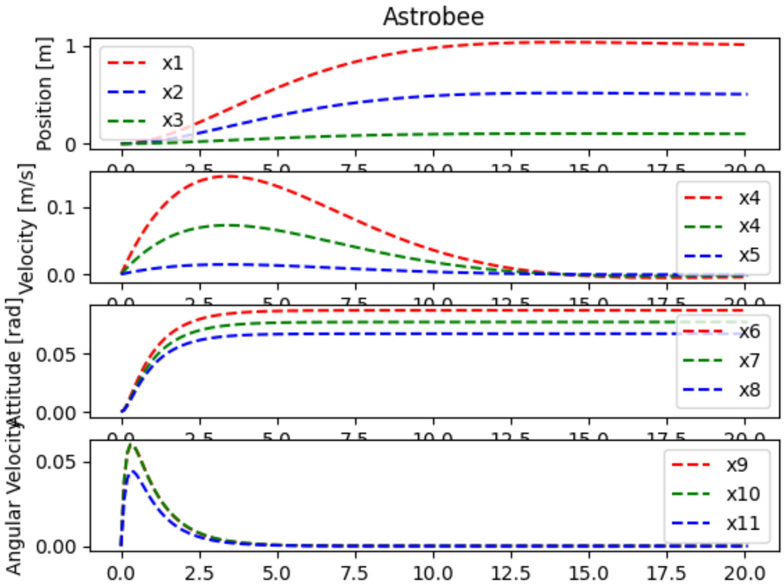
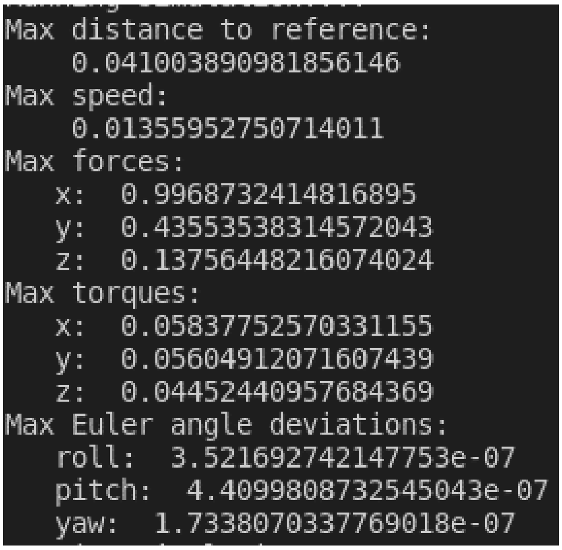


Figure 2: Graph plotting system response with no change to Q and R



```
Max distance to reference:
    0.041003890981856146
Max speed:
    0.01355952750714011
Max forces:
   x:  0.99687324148168895
   y:  0.43553538314572043
   z:  0.13756448216074024
Max torques:
   x:  0.05837752570331155
   y:  0.05604912071607439
   z:  0.04452440957684369
Max Euler angle deviations:
   roll:  3.521692742147753e-07
   pitch:  4.409980873254504 3e-07
   yaw:  1.7338070337769018e-07
```

Figure 3: Max system variables with no change to Q and R

- **Multiply R by 10**

By multiplying the R matrix by 10, we increase the cost of the control inputs. In other words, control inputs are more expensive when minimizing the objective function. From the simulation, we can see that the max forces and max torques in all axies are much smaller than the original, which also further results in a longer time to the reference position (after 12 seconds).

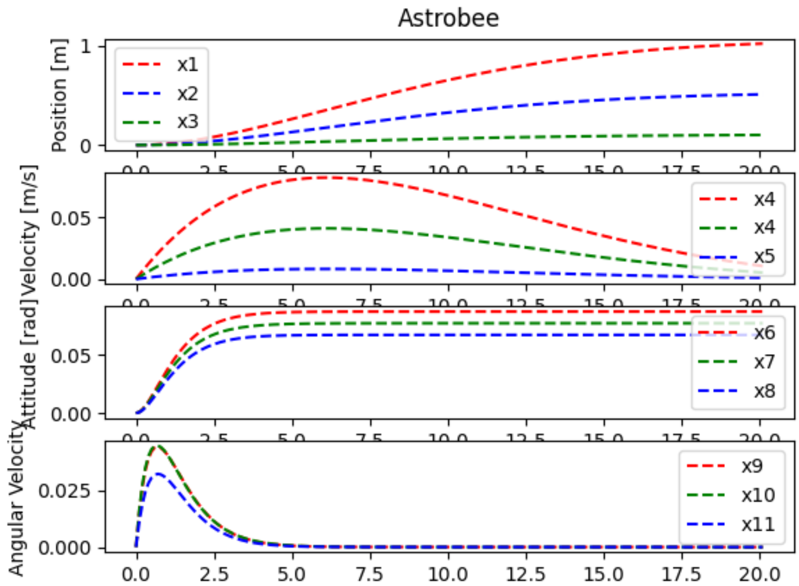


Figure 4: Graph plotting system response with R*10

```
Max distance to reference:
    0.2455667293555134
Max speed:
    0.06067999474351313
Max forces:
   x:  0.31855664025538916
   y:  0.13917786393871945
   z:  0.04395953009070111
Max torques:
   x:  0.023070435191139018
   y:  0.022482012258091114
   z:  0.017392632804820462
Max Euler angle deviations:
   roll:  4.799934065491396e-08
   pitch:  1.2970672601253508e-07
   yaw:  1.9038587525943562e-08
```

Figure 5: Max system variables with R*10

- **Add 100 to the velocity components of the diagonal of Q, Q[3:6] and Q[9:]**

By adding 100 to the velocity components in the Q (state cost) matrix, we increase the constraints of the state velocities. In other words, any variance from the refence his higher . From the figure below, we can see that the magnitude of velocities is less and are lengthened out further. This is because the system intial state and reference state are 0, so the objective function is attempting to minimize this variance as much as possible, thus keeping velocity slower. This has a direct impact on the time postion takes to reach steady state as well.

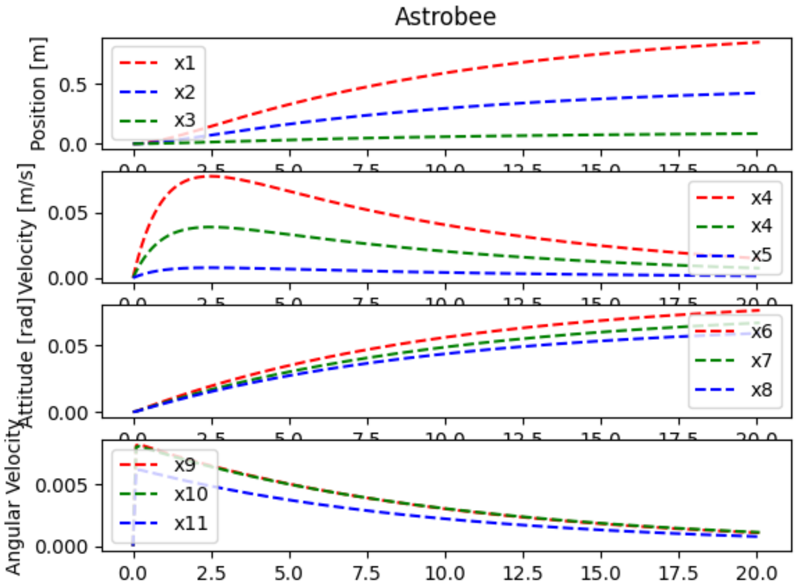


Figure 6: Graph plotting system response with 100 added to velocity of Q

```
Max distance to reference:
    0.37046687908572556
Max speed:
    0.03703172158488006
Max forces:
   x:  0.9638674954770575
   y:  0.4211151242459592
   z:  0.13300982248233884
Max torques:
   x:  0.012525060983108377
   y:  0.011474067754808186
   z:  0.009923579100960967
Max Euler angle deviations:
   roll:  0.025260710662083713
   pitch:  0.023219190807121474
   yaw:  0.018961144727154183
```

Figure 7: Max system variables with 100 added to velocity of Q

- **Revert the velocity components to their initial value of 1 and add 100 to the position and attitude components of the diagonal of Q, Q[0:3] and Q[6:9]**

By adding 100 to the position and attitude components in Q matrix, we increase the constraints of the state positions and attitudes. The system response becomes significantly faster since the cost of X not being 0 becomes so great. In addition, the addition of 100 on these state variables makes the difference between Q and R significantly greater than the velocity case because the positional values of matrix A are far greater than that of velocity. Another side effect of this is the slight overshoot seen because the control inputs gains are so large with the smaller R.

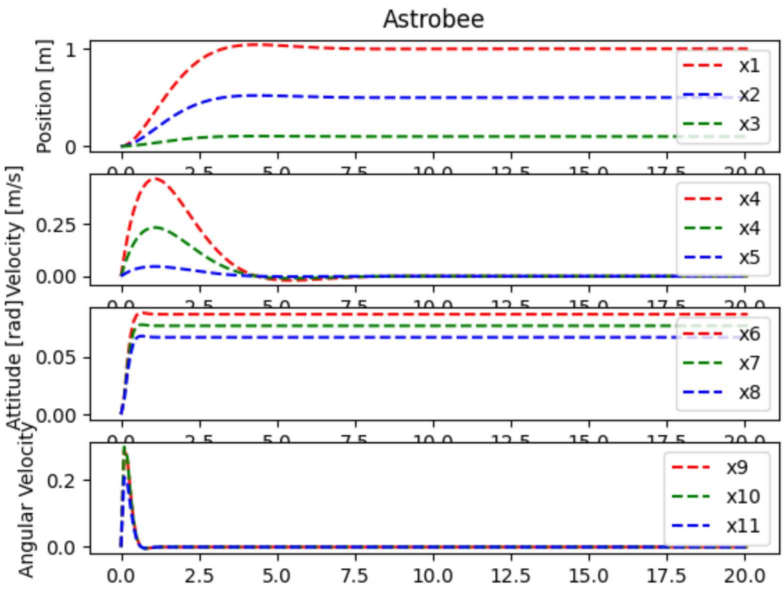


Figure 8: Graph plotting system response with 100 added to position and attitude of Q

```
Max distance to reference:
    8.653885832587684e-05
Max speed:
    0.0001605962370685525
Max forces:
   x:  9.538314000126835
   y:  4.167303393994692
   z:  1.316248818318604
Max torques:
   x:  0.4416216558944475
   y:  0.42396595127806064
   z:  0.3362052192152465
Max Euler angle deviations:
   roll:  0.0
   pitch:  0.0
   yaw:  1.3877787807814457e-17
```

Figure 9: Max system variables with 100 added to position and attitude of Q

- **Increase all elements of Q by 100**

  Increasing both cost matrices Q and R to 100 results in a system response that is still quite fast (compared to the previous position case) without the downfalls of over shoot as seen earlier. This is because control now does cost a bit more than the previous example, so control input gain is a little less and therefore slows the system down just enough to prevent surpassing the target reference. This can all be seen in figure 10 below.

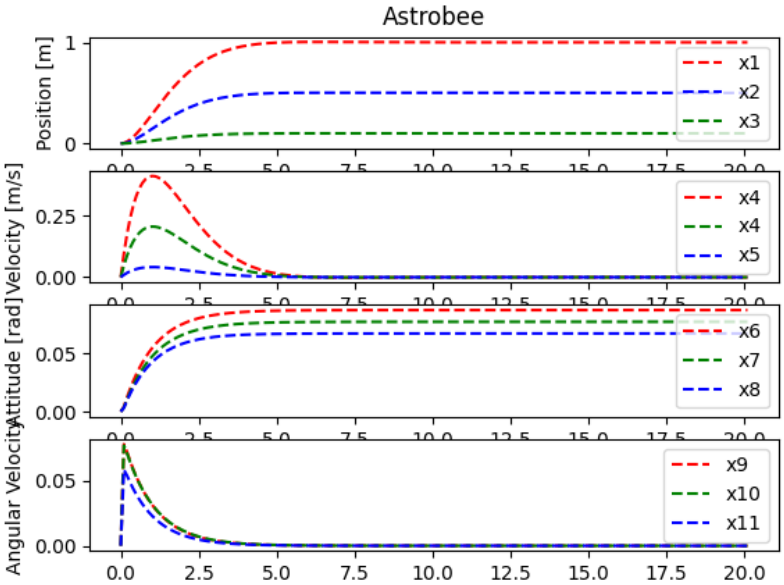


Figure 10: Graph plotting system response with 100 added to all elements of Q

```
Max distance to reference:
    1.4989482816505207e-05
Max speed:
    1.126283835967986e-05
Max forces:
   x:  9.379991539021344
   y:  4.098132078235428
   z:  1.294400957958666
Max torques:
   x:  0.12032417631311008
   y:  0.1103554590824718
   z:  0.09522807043629235
Max Euler angle deviations:
   roll:  3.626088593139398e-07
   pitch:  4.464564921508041e-07
   yaw:  1.8271773603861785e-07
```

Figure 11: Max system variables with 100 added to all elements of Q

1.3 Q3: Tune the LQR controller

To find a controller that fits the constraints, we tested multiple values, but ultimately settled on multiplying the cost of attitude components in matrix Q by 2, force components in matrix R by 1.5, and torque in matrix R by 7. All other state costs in Q were held at 1. These summarized values and their resulting max system state response values can be seen in figure 12 below. The corresponding plot is also depicted in figure 13.

```
Max distance to reference:
    0.04231874891757645
Max speed:
    0.02385368374182879
Max forces:
   x:  0.8158608002807639
   y:  0.3564507817620244
   z:  0.11258549616498624
Max torques:
   x:  0.03761511409608861
   y:  0.0366051933535468
   z:  0.02838206799036402
Max Euler angle deviations:
   roll:  2.0525997568299204e-11
   pitch:  5.5877288906991396e-11
   yaw:  1.584206099636276e-10
```

Figure 12: Max states of system response after 12 seconds with chosen Q and R

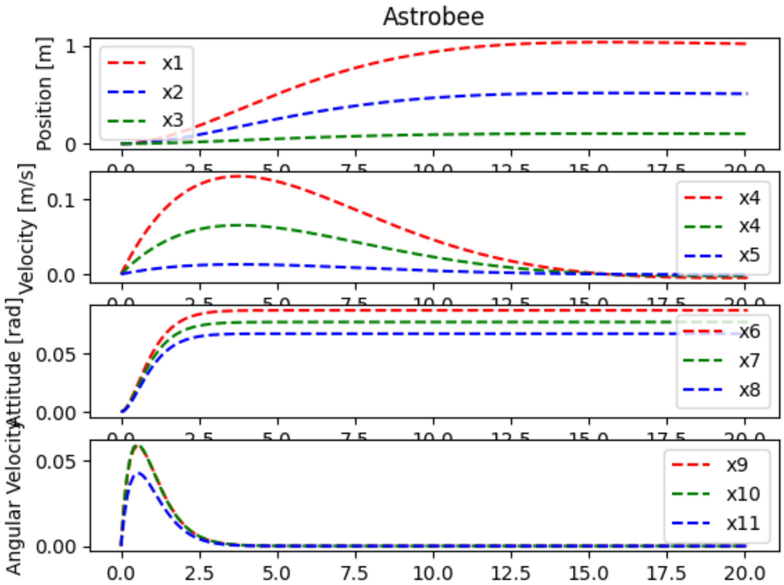


Figure 13: State response plot with chosen Q and R

With these Q and R values, our system now adheres to all constraints...

- Max force = 0.8157 N ($< .85 \rightarrow PASS$)
- Max Torque = 0.0376 Nm ($< .04 \rightarrow PASS$)
- Max Distance to reference = .0423 m ($< .06 \rightarrow PASS$)
- Speed of Astrobee = .0239 m/s ($< .03 \rightarrow PASS$)
- Max Euler Angle = 1.5842×10^{-10} rad ($< 10^{-7} \rightarrow PASS$)
- No overshoot seen...PASS

2 Part2: LQG Design

2.1 Q4: Effect of Q and R on Measurement Noise

With a noisy reading of the position state, the derived estimated velocity output is thus very noisy as well. Figure 14 shows this noisy velocity output when the random generated measurement is increased to +/- 0.5 (from .005). The velocity estimate than in turn effects the position as well.

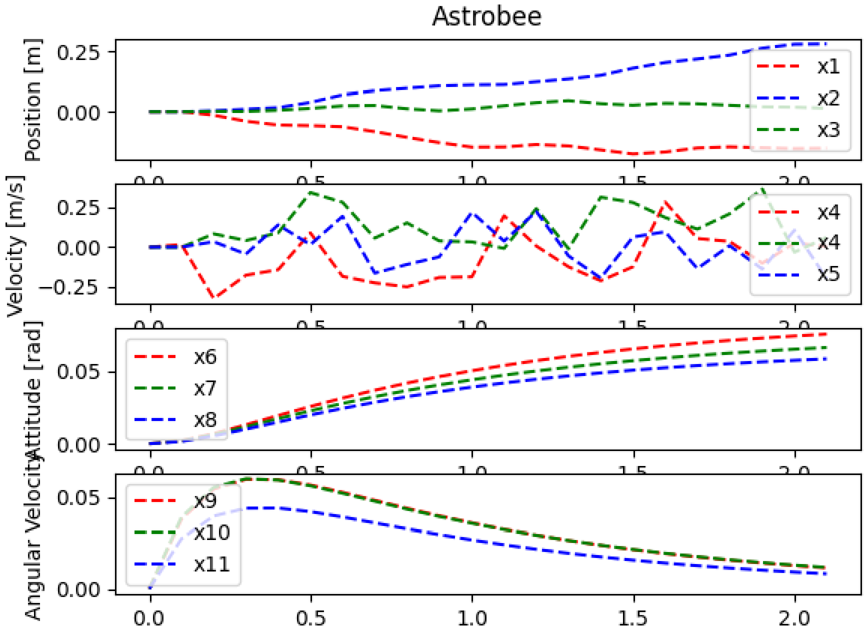


Figure 14: State output/estimate with measurement noise multiplied by 1000

Intuition would suggest that increasing R would minimize the effect of the measurement noise seen since this correlates to the Σ_v^{-1} of the Kalman filter objective function. This equation in full can be seen below. By increasing the cost associated to this noise, the optimal feedback gain will be calculated to combat this.

$$J_t = (x_0 - \bar{x}_0)\Sigma_0^{-1}(x_0 - \bar{x}) + \sum_{t=0}^{T-1} w_t^T \Sigma_w^{-1} w_t + v_t^T \Sigma_v^{-1} v_t \tag{2.1}$$

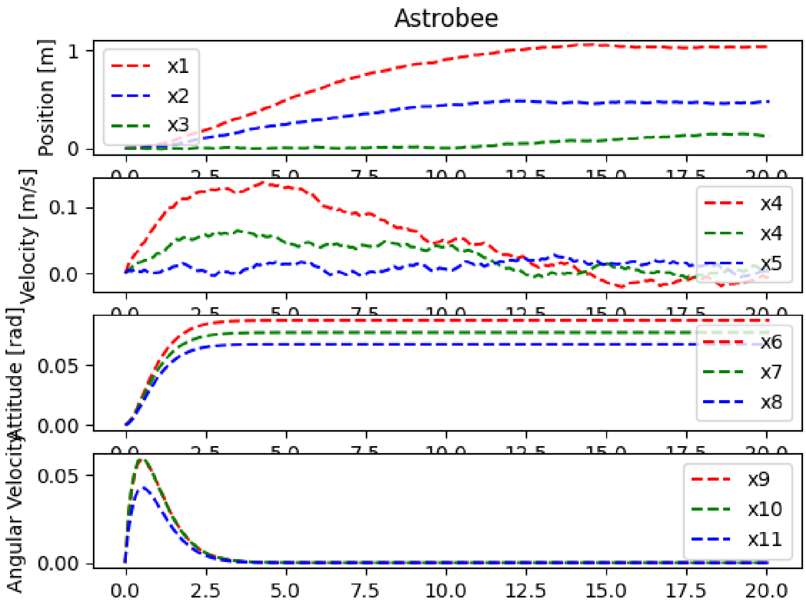


Figure 15: State output/estimate with Q set to 5 and R left at 1

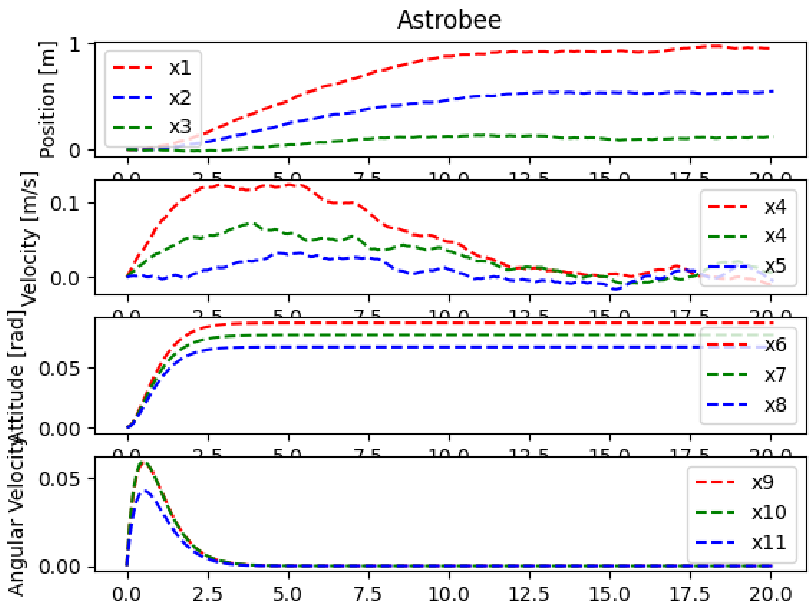


Figure 16: State output/estimate with Q left at 1 and R set to 5

Although it may be hard to notice, it does seem that our intuition was correct. It can be seen from figure 16 that the noise is here after applying a cost of R than in the previous figure (with higher cost of Q). This is best seen further on the horizon as velocity states reach there reference. Perhaps the reason why it is not even more obvious is due to the large amount of noise added; it could be too much to fully filter out regardless.

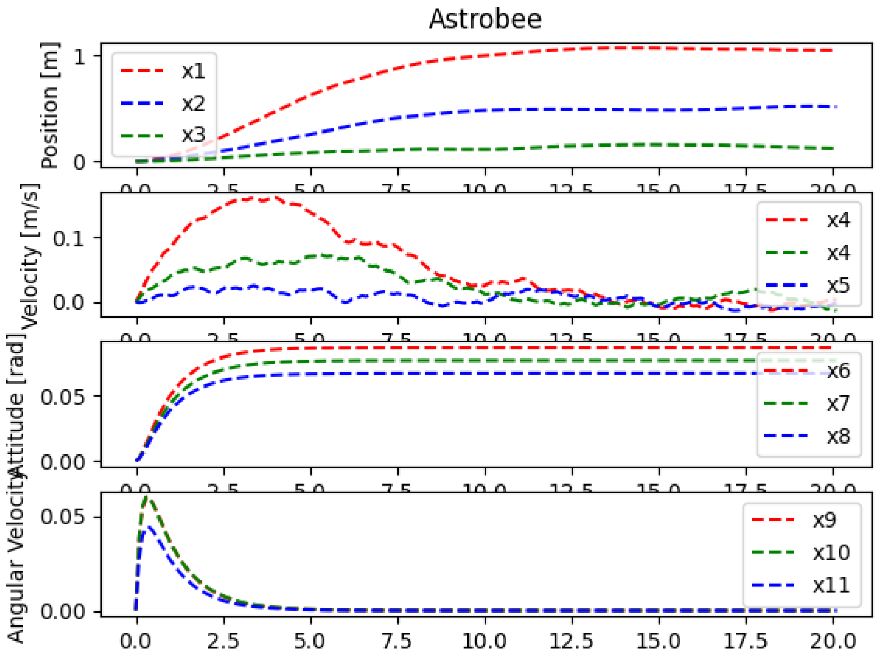


Figure 17: State output/estimate with Q and R both set to 5

With both Q and R matrices set to 5, the result is nearly the same. This is too be expected since we don't actually have any process noise in our system as of yet.

2.2 Q5: Effect of Q and R on Positional Process Noise

In question 5, we add in process noise into the system. Figure 18 now shows that not only the measurement of the velocity is noisy but also has some serious issue with tracking references. The process noise randomly generated and its magnitude is multiplied by 2 to +/- 0.01 (from .005).

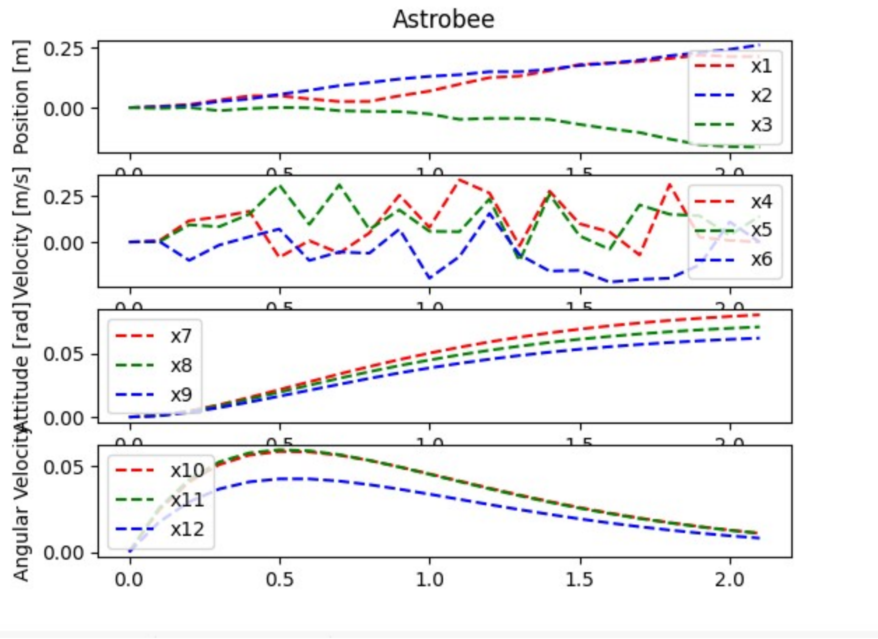


Figure 18: State output/estimate with process noise multiplied by 2

Intuition suggests that we should then increase Q to minimize the effect of the process noise since this correlates to the Σ_w^{-1} of the Kalman filter objective function.

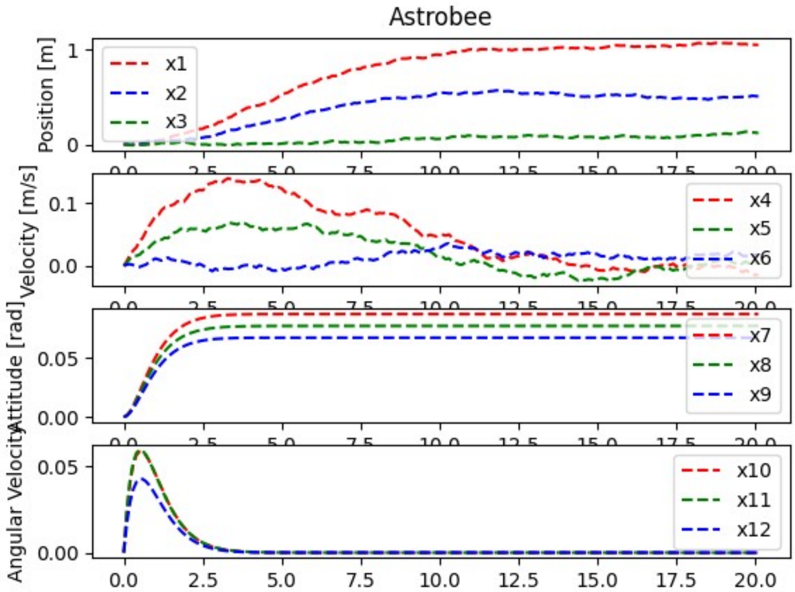


Figure 19: State output/estimate with Q set to 10 and R set to 10

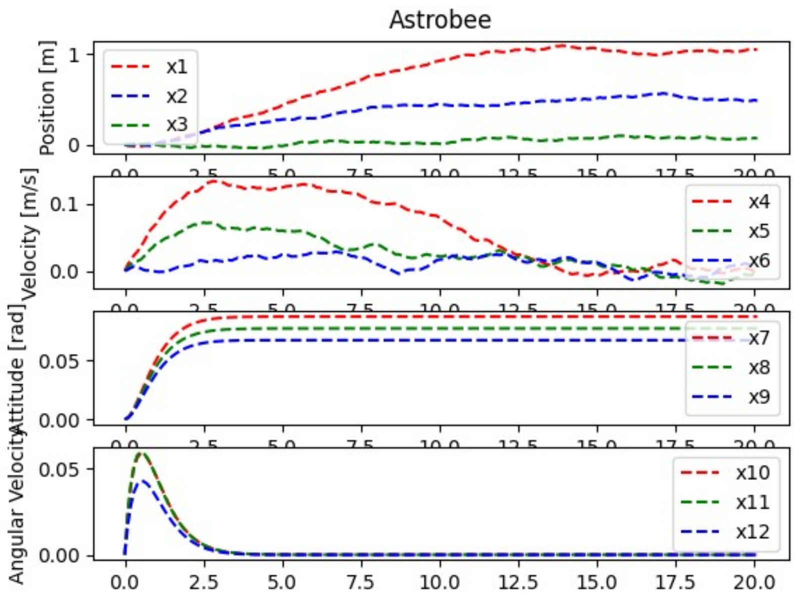


Figure 20: State output/estimate with Q set to 10 and R set to 20

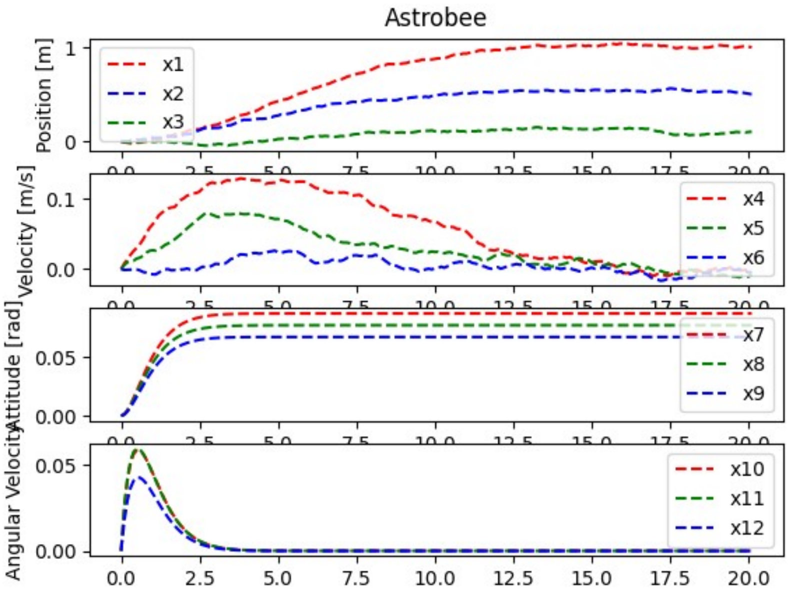


Figure 21: State output/estimate with Q set to 20 and R set to 20

As shown in Figure 19 - 21, we can see that the results fits our intuition that increasing the penalty of Q results in better rejecting the process noise.