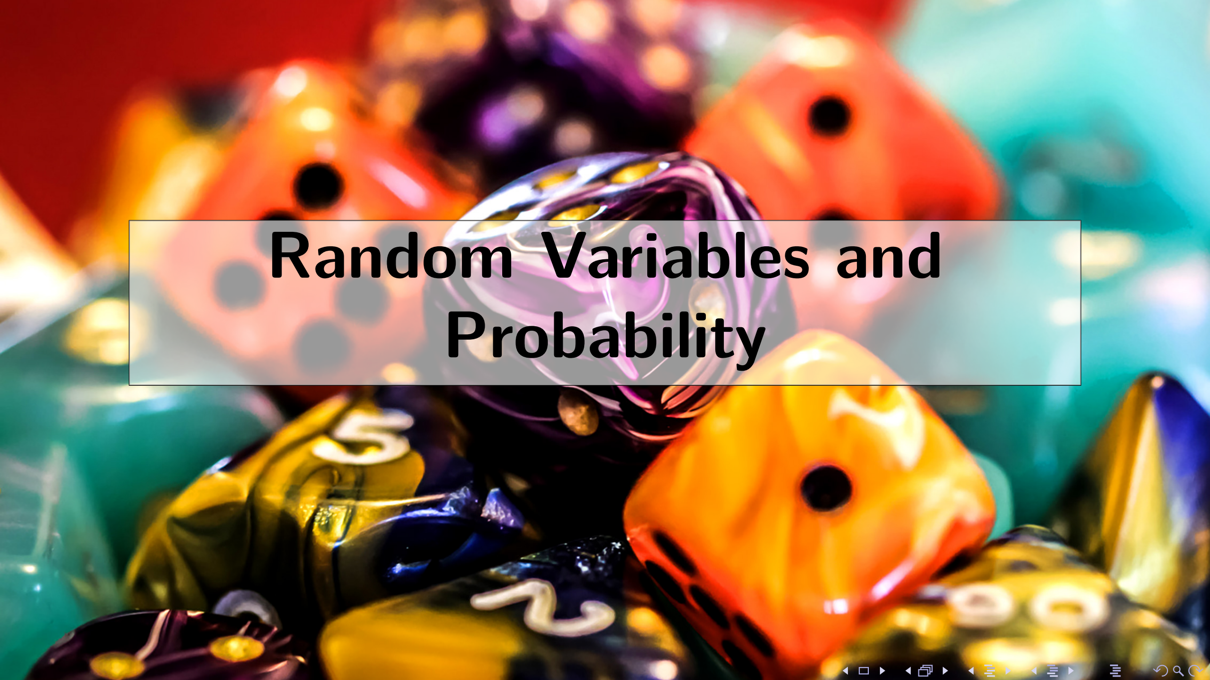
Random Variables and Probability



Today:

- Talk about a little bit of math;
- Introduce the ideas of random variables and probability;
- Discuss Bernoulli, Binomial, and Normal random variables;
- Compute these distributions in R.

Motivation – reasoning about voters...

- Cavalier Johnson: a political candidate running a campaign for Mayor of Milwaukee;
- Johnson's campaign claims that he will win the election – in this case that means getting more than 50% of votes from Milwaukee voters;
- How to assess the Johnson campaign claim? Conduct an experiment:
  - **Sample** n voters from Milwaukee;
  - Record the number of voters who will vote for Johnson.

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- **Sum** – shorthand to write adding up a bunch of stuff:

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  - We write the dependent variable as f, y, or p (but not always!), the independent variable as x, and the relationship between them as $f(x)$;
  - Example: $f(x) = 2x + 3$;
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- **Exponentiation** – we write $\exp\{x\}$ and this is shorthand for 2.718^x.

What are random variables?

- **Sample space** for an experiment or random trial = all possible outcomes or results of that experiment;

- **Random variable**:
  - A variable whose values depend on outcomes of a random phenomenon;
  - A variable whose values have a relative likelihood governed by a probability distribution;

- **Probability distribution**: a mathematical function that gives the probabilities that the different possible outcomes occur. Must satisfy three rules:
  1. Probabilities are > 0;
  2. If you add up the probabilities for all possible outcomes they must equal 1;
  3. To find the probability of two mutually exclusive events happening you add up their individual probabilities.

What are random variables: examples.

- The outcome of a FAIR coin toss:
  - Sample space is: $\{\text{Heads}, \text{Tails}\}$;
  - Probability distribution is: $\{0.5, 0.5\}$;

- The outcome of rolling a six sided dice:
  - Sample space is: $\{1, 2, 3, 4, 5, 6\}$;
  - Probability distribution is: $\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$;

- The number of heads you get if you toss a coin 10 times:
  - Sample space is: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$;
  - Probability distribution is?

- The number of voters who will vote for Johnson:
  - If N people live in Milwaukee the sample space is: $\{0, 1, 2, 3, \ldots, N\}$;
  - Probability distribution is?

Bernoulli Random Variables

- A **Bernoulli random variable** is used to model the outcome of an experiment that can be answered as a **boolean** – typical answers could be:
  - Yes or No;
  - Success or Failure;
  - True or False;
  - 1 or 0;
- Has a probability distribution with a single parameter: π, the probability of Yes/Success/True/1;
- Let's take the outcomes to be 0 and 1. Then we can write this as a function:

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$$p(k = 1, \pi) = \pi$$
$$p(k = 0, \pi) = 1 - \pi$$
$$\text{or: } p(k, \pi) = \pi^k (1 - \pi)^{1-k} \text{ for } k = 0 \text{ or } k = 1.$$

Binomial Random Variables

- A **binomial random variable** is used to model the sum of n Bernoulli experiments – how many successes after n tries?

- Has a probability distribution with three parameters:
  1. n, the number of tries;
  2. k, the number of successes;
  3. π, the probability of each success;

- If we had a single try we could have either $k = 1$ success or $k = 0$ successes:

$$p(k = 1, n = 1, \pi) = \pi$$
$$p(k = 0, n = 1, \pi) = 1 - \pi$$
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- Has a probability distribution with three parameters:
  1. n, the number of tries;
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  3. π, the probability of each success;
- If we had two tries we could have $k = 2, 1$ or 0 successes:

$$p(k = 2, n = 2, \pi) = \pi^2$$
$$p(k = 1, n = 2, \pi) = \pi(1 - \pi) + (1 - \pi)\pi$$
$$p(k = 0, n = 2, \pi) = (1 - \pi)^2$$
$$\text{or: } p(k, n = 2, \pi) \propto \pi^k (1 - \pi)^{2-k}.$$

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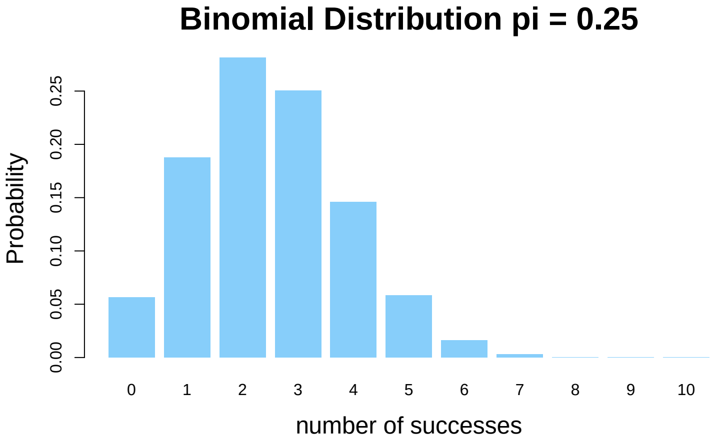

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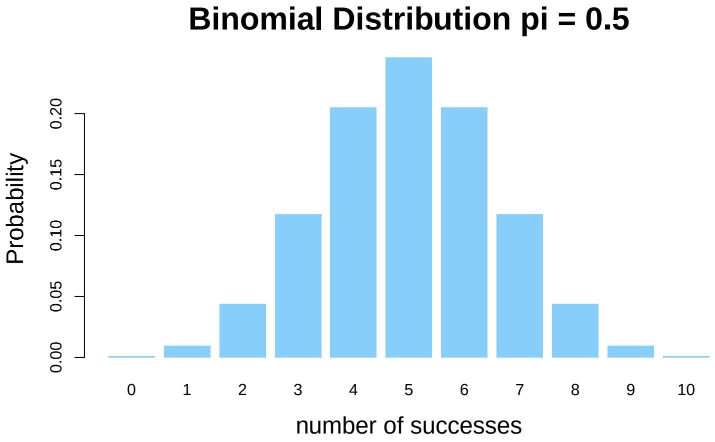

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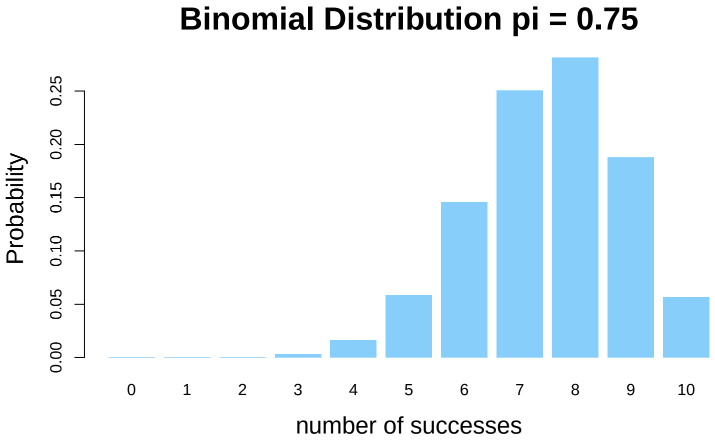

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- Has a probability distribution with two parameters: μ, the mean and σ the standard deviation:

$$N(x, \mu, \sigma) = \frac{1}{\sqrt{2 * 3.1416 * \sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\};$$

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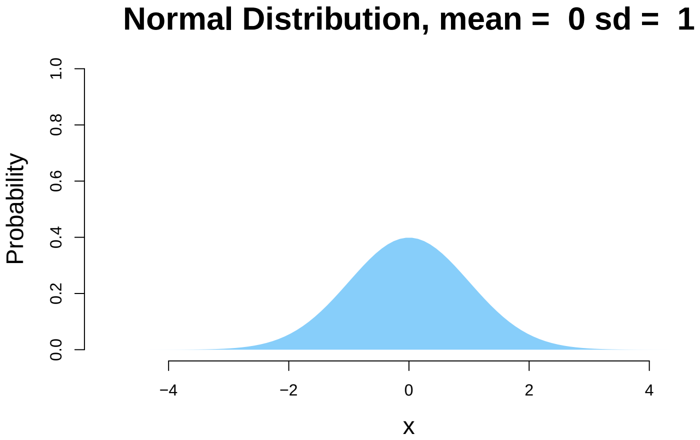
- When $\mu = 0$ and $\sigma = 1$ we call it the **standard normal**:

$$N(x, \mu = 0, \sigma = 1) = \frac{1}{\sqrt{2 * 3.1416}} \exp\left\{-\frac{x^2}{2}\right\}.$$

Normal – what happens as mean changes?



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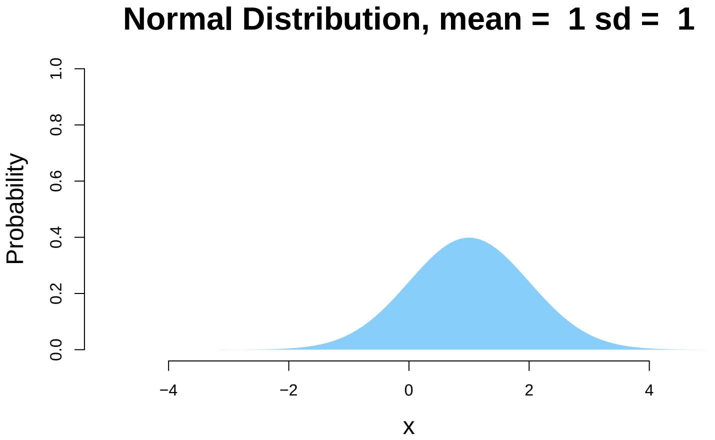

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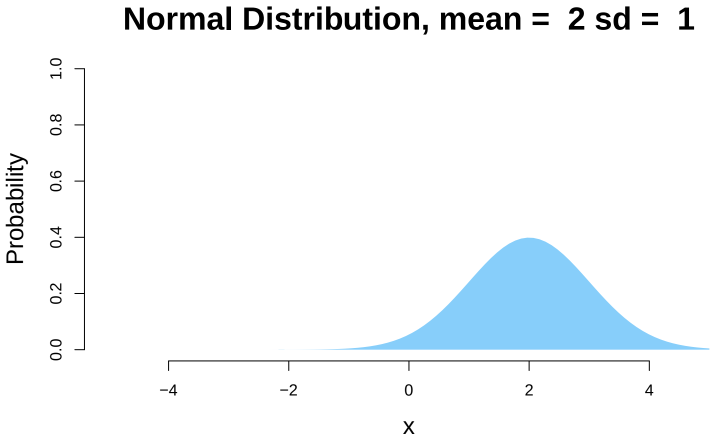

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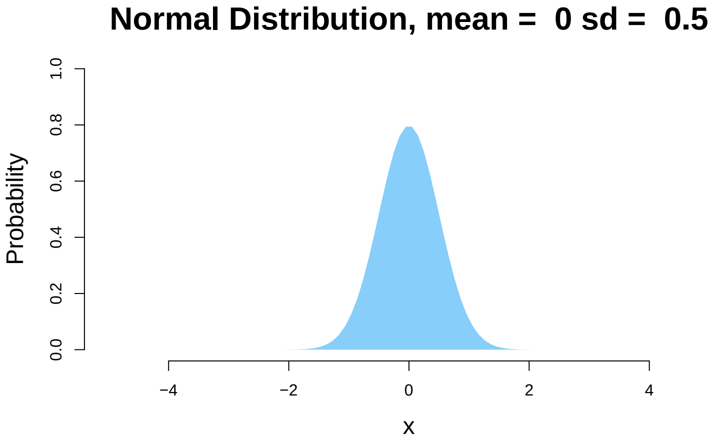

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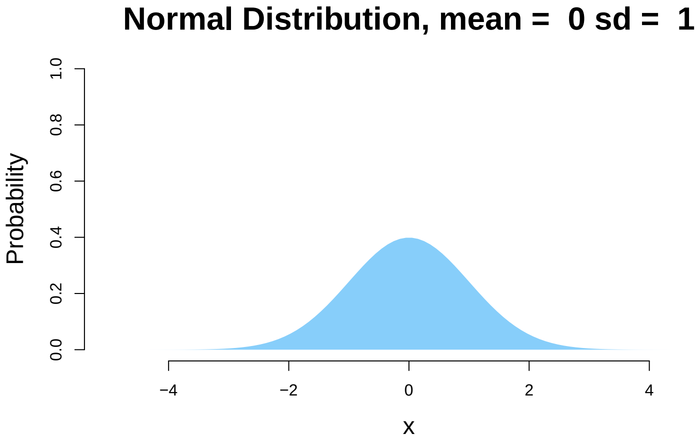

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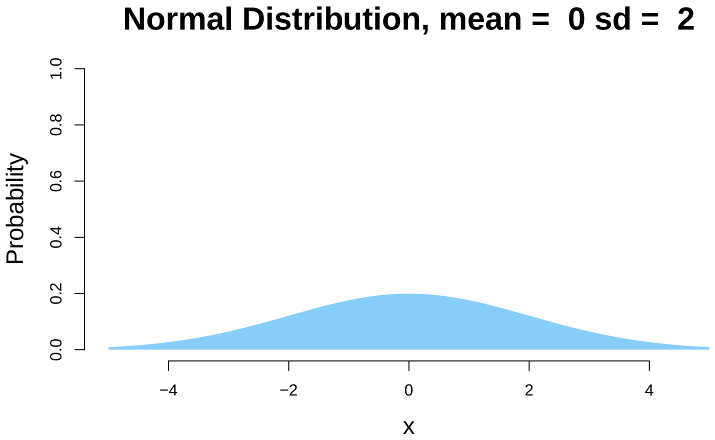

Motivation – reasoning about voters...

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- Johnson's campaign claims that he will win the election – in this case that means getting more than 50% of votes from Milwaukee voters;
- How to assess the Johnson campaign claim? Conduct an experiment:
  - **Sample** n voters from Milwaukee;
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If we see that k out of n voters say they will vote for Johnson, what is the probability his campaign's claim is correct?

Why should we care?

Probability is used for all hypothesis testing and statistical modeling.