Variable Interactions

Today:

- Define variable interactions in a linear regression context.
- Understand how to interpret the coefficients and p-values for variable interactions.

What are **variable interactions**?

- So far, we've used linear regression to:
  1. Model a linear relationship between dependent variable y and independent variable x:
  $$y = \beta_0 + \beta_1 x + \varepsilon;$$
  2. Model a linear relationship between dependent variable y and many independent variables $x_1, x_2, x_3, \ldots$:
  $$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots + \varepsilon;$$
  3. Model a NON-linear relationship between dependent variable y and many independent variables x:
  $$y = \beta_0 + \beta_1 x_1 + \beta_2 x_1^2 + \ldots + \varepsilon;$$

What are **variable interactions**?

- So far, we've used linear regression to:
  1. Model a linear relationship between dependent variable y and independent variable x:

  $$y = \beta_0 + \beta_1 x + \varepsilon;$$

  2. Model a linear relationship between dependent variable y and many independent variables $x_1, x_2, x_3, \ldots$:

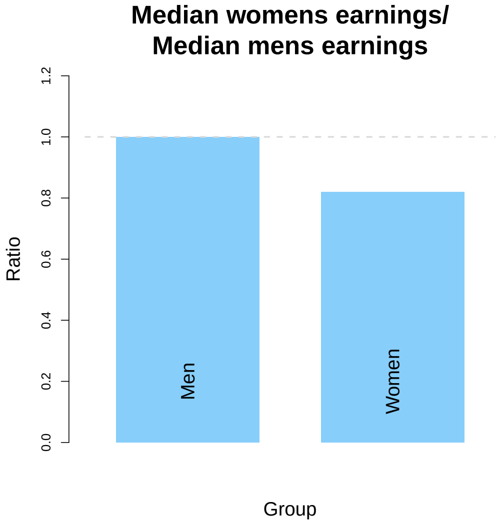
  $$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots + \varepsilon;$$

  3. Model a NON-linear relationship between dependent variable y and many independent variables x:

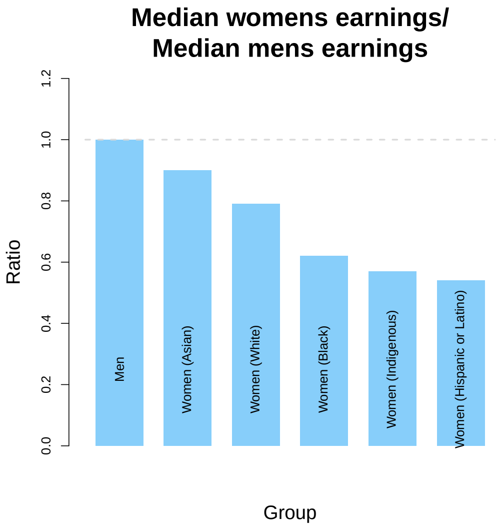
  $$y = \beta_0 + \beta_1 x_1 + \beta_2 x_1^2 + \ldots + \varepsilon;$$

- But what if the effect of one independent variable depends on another independent variable?!

What are **variable interactions**?



What are **variable interactions**?



What are **variable interactions**?

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \varepsilon$$

- y is the **dependent variable** – this is part of the data set;
- The x's are **independent variables** that explain y – also part of the data set;
- The β's are called **coefficient effects** that control how the x's affect y – they are **learned** from the data;
- The ε is a **noise** term.

What are **variable interactions**?

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \underbrace{\beta_{12} x_1 x_2}_{\text{interaction}} + \ldots + \varepsilon$$

- y is the **dependent variable** – this is part of the data set;
- The x's are **independent variables** that explain y – also part of the data set;
- The β's are called **coefficient effects** that control how the x's affect y – they are **learned** from the data;
- The ε is a **noise** term.

Interpreting variable interactions: coefficients

What is the effect of a unit increase in x_1 on y?

$$y_1 = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$

Interpreting variable interactions: coefficients

What is the effect of a unit increase in x_1 on y?

$$y_1 = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$

for a unit increase in x_1: $y_2 = \beta_0 + \beta_1 (x_1 + 1) + \beta_2 x_2 + \beta_{12} (x_1 + 1) x_2$

Interpreting variable interactions: coefficients

What is the effect of a unit increase in x_1 on y?

$$
\begin{aligned}
y_1 &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 \\
\text{for a unit increase in } x_1\text{:}\quad y_2 &= \beta_0 + \beta_1 (x_1 + 1) + \beta_2 x_2 + \beta_{12} (x_1 + 1) x_2 \\
&= \beta_0 + \beta_1 x_1 + \beta_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{12} x_2
\end{aligned}
$$

Interpreting variable interactions: coefficients

What is the effect of a unit increase in x_1 on y?

$$\begin{aligned}
y_1 &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 \\
\text{for a unit increase in } x_1\text{: } y_2 &= \beta_0 + \beta_1 (x_1 + 1) + \beta_2 x_2 + \beta_{12} (x_1 + 1) x_2 \\
&= \beta_0 + \beta_1 x_1 + \beta_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{12} x_2 \\
\text{so change in } y \text{ is: } y_2 - y_1 &=
\end{aligned}$$

Interpreting variable interactions: coefficients

What is the effect of a unit increase in x_1 on y?

$$
\begin{aligned}
y_1 &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 \\
\text{for a unit increase in } x_1\text{:}\quad y_2 &= \beta_0 + \beta_1 (x_1 + 1) + \beta_2 x_2 + \beta_{12} (x_1 + 1) x_2 \\
&= \beta_0 + \beta_1 x_1 + \beta_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{12} x_2 \\
\text{so change in } y \text{ is:}\quad y_2 - y_1 &= \underbrace{\beta_0 + \beta_1 x_1 + \beta_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{12} x_2}_{y_2} \\
&\qquad - (\underbrace{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2}_{y_1})
\end{aligned}
$$

Interpreting variable interactions: coefficients

What is the effect of a unit increase in x_1 on y?

$$
\begin{aligned}
y_1 &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 \\
\text{for a unit increase in } x_1\text{: } y_2 &= \beta_0 + \beta_1 (x_1 + 1) + \beta_2 x_2 + \beta_{12} (x_1 + 1) x_2 \\
&= \beta_0 + \beta_1 x_1 + \beta_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{12} x_2 \\
\text{so change in } y \text{ is: } y_2 - y_1 &= \underbrace{\beta_0 + \beta_1 x_1 + \beta_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{12} x_2}_{y_2} \\
&\qquad \underbrace{- \beta_0 - \beta_1 x_1 - \beta_2 x_2 - \beta_{12} x_1 x_2}_{y_1} \\
&= \beta_1 + \beta_{12} x_2.
\end{aligned}
$$

What does this actually look like?

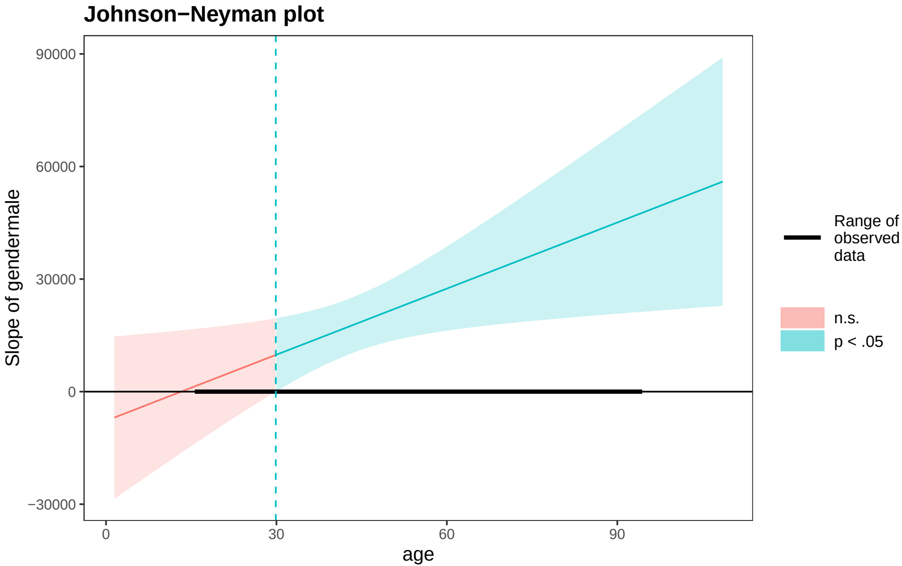
```
Coefficients:
                Estimate Std. Error t value          Pr(>|t|)
(Intercept)    -26945.95    8232.57  -3.273           0.00111 **
age               540.39     122.27   4.420 0.000011286396345 ***
hrs_work         1061.82     149.48   7.103 0.000000000002758 ***
gendermale      19484.80    3688.59   5.282 0.000000165718705 ***
time_to_work       93.06      80.10   1.162           0.24567
edugrad         44734.13    6140.20   7.285 0.000000000000789 ***
eduhs or lower -18519.50    4077.02  -4.542 0.000006446559114 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

What does this actually look like? Gender and age...

Example: the effect of gender on income depends on age:

```
Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)    -25275.9     9645.6  -2.620  0.00894 **
age               274.9      186.4   1.475  0.14056
hrs_work         1223.2      145.9   8.386  < 2e-16 ***
gendermale      -7770.5    11368.2  -0.684  0.49446
age:gendermale    587.2      250.1   2.347  0.01914 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Effect of gender on income is: $\beta_{\text{gender}} + \beta_{\text{age}} * \text{age}$

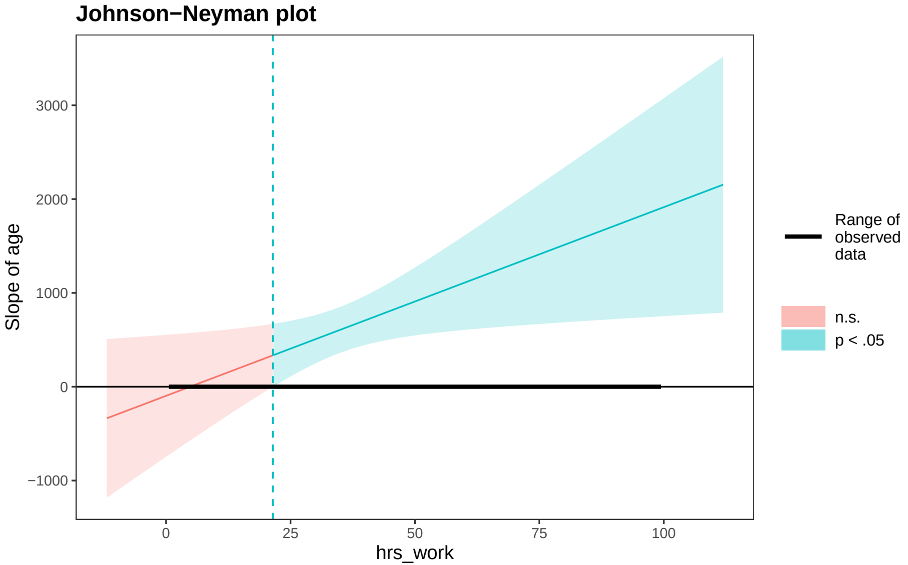


What does this actually look like? Age and hours worked...

Example: the effect of age on income depends on hours worked:

```
Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)   -10784.380  14674.170  -0.735   0.4626
age              -96.882    331.164  -0.293   0.7699
hrs_work         399.850    393.857   1.015   0.3103
gendermale     16923.305   3783.253   4.473 8.77e-06 ***
age:hrs_work      20.107      8.851   2.272   0.0234 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Effect of age on income is: $\beta_{\text{age}} + \beta_{\text{hrs\_work}} * \text{hrs\_work}$



Interpreting variable interactions: p-values

Fair warning – when you use interaction terms you can no longer just interpret p-values to determine whether to reject the null of no effect.

Why should we care?

Linear regression can deal with complicated relationships between independent variables – with careful design this can make it as powerful as modern machine learning techniques.