

# Homework 4 & 5

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CSE598 - Analysis of Algorithms

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**Problem 1.** Show how to multiply the complex numbers  $a + bj$  and  $c + dj$  using only three real multiplications...

Given:

$$\begin{aligned} R + jI &= (a + jb) \cdot (c + jd) \\ &= (ac - bd) + j(bc + ad) \end{aligned}$$

A common pattern in Digital Signal Processing is to break out the operations into 3 constants.

$$\begin{aligned} k_1 &= a(c + d) &= ac + ad \\ k_2 &= d(a + b) &= ad + bd \\ k_3 &= c(b - a) &= bc - ac \end{aligned}$$

From these we can define the Real part as

$$\begin{aligned} \Re &= k_1 - k_2 \\ &= ac + ad - (ad + bd) \\ &= ac + ad - ad - bd \\ &= ac - bd \end{aligned}$$

The Imaginary Part as

$$\begin{aligned} \Im &= k_1 + k_3 \\ &= ac + ad + bc - ac \\ &= ab + bc \end{aligned}$$

In Python we can clearly see this uses only 3 multiplications.

```

1 | def complex_mul(a, b, c, d):
2 |     k1 = a(c + d)
3 |     k2 = d(a + b)
4 |     k3 = c(b - a)
5 |     return (k1 - k2) + (k1 - k3)*j

```

**Problem 2.** Consider in the closest pair of points problem, splitting the points along  $\frac{n}{4}$  and  $\frac{3n}{4}$ . Is the complexity still  $O(n \lg n)$ ? Yes

Proof:

Consider the base case  $n \geq 4$ :

$$T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{3n}{4}\right) + n$$

Substitute  $n \lg n$

$$\begin{aligned}
 T(n) &= T\left(\frac{n}{4}\right) + T\left(\frac{3n}{4}\right) + n \\
 T(n) &\leq c \frac{n}{4} \log_2 \frac{n}{4} + c \frac{3n}{4} \log_2 \frac{3n}{4} + cn \\
 &\leq c \left( \frac{n}{4} \log_2 n - \frac{n}{4} \log_2 4 \right) + c \left( \frac{3n}{4} \log_2 3n - \frac{3n}{4} \log_2 4 \right) + cn \\
 &\leq c \left( \frac{n}{4} \log_2 n - \frac{2n}{4} \right) + c \left( \frac{3n}{4} \log_2 3n - \frac{3n \cdot 2}{4} \right) + cn \\
 &\leq c \left( \frac{n}{4} \log_2 n - \frac{1}{2}n \right) + c \left( \frac{3n}{4} \log_2 3 + \frac{3n}{4} \log_2 n - \frac{3n}{2} \right) + cn \\
 &\leq c \frac{n}{4} \log_2 n - \frac{1}{2}cn + \frac{3nc}{4} \log_2 3 - cn + cn \\
 &\leq c \frac{n}{4} \log_2 n + \frac{3cn}{4} \log_2 3 \\
 &\leq cn \log_2 3
 \end{aligned}$$

Intuitively this makes sense because we are still dividing the space into some log relatable ratio for each subproblem.

Consider dividing the space into  $\sqrt{n}$  and  $n - \sqrt{n}$ . This problem will not remain at  $n \log_2 n$  since the division of the space is not an even recurrence. Instead there is a linear adjustment in the form of  $n - \dots$

$$\begin{aligned}
 T(n) &= T(n^{\frac{1}{2}}) + T(n - n^{\frac{1}{2}}) + nc \\
 &\leq n^{\frac{1}{2}} \log_2 n^{\frac{1}{2}} + \left( n - n^{\frac{1}{2}} \right) \log_2 \left( n - n^{\frac{1}{2}} \right) c + cn \dots
 \end{aligned}$$

Eventually we get to where we cannot resolve the logs with the roots. So let's try a greater

complexity:  $n^2$

$$\begin{aligned} &\leq \left(n^{\frac{1}{2}}\right)^2 + \left(n - n^{\frac{1}{2}}\right)^2 + nc \\ &\leq n + n^2 - n + nc \\ &\leq n^2 + n(1 - 1 + 1)c \\ &\leq n^2 + nc \\ &\leq \lim_{n \rightarrow \infty} (n^2 + nc) \\ &\leq cn^2 \end{aligned}$$

Thus by adding an irregular division to the sub problems we increase the complexity to recombine those subproblems.

**Problem 5.1.** Find the median of two databases. Use at most  $O(\log_2 n)$  queries.

To achieve this complexity target we must do at most a recurrence of

$$T(n) = T\left(\frac{n}{2}\right) + c \quad (1)$$

With a Base case of  $T(1) = 1$ , since a median of a single element is itself  $\therefore$

$$T(n) = T\left(\frac{n}{2}\right) + O(1) \quad (2)$$

Intuitively, this represents an algorithm which divides the input into 2 subproblems, considers one side and spends a constant amount of time at each step. Such an algorithm may be implemented in Python:

```
1 import numpy
2
3 def median( ns ):
4     l = len(ns)
5     if l % 2 == 0:
6         return (ns[l/2] + ns[(l/2)-1])/2.0
7     else:
8         return ns[l/2]
9
10
11 def find_median(a, b):
12     l = len(a)
13     if l == 1:
14         return (median(a) + median(b)) / 2
15     if l == 2:
16         return ( max(a[0], b[0]) + min(a[1], b[1]) ) / 2
17
18     ma = median(a)
19     mb = median(b)
20
```

```

21     if ma == mb:
22         return ma
23
24     if ma < mb:
25         if l % 2 == 0:
26             return find_median(a[l/2:], b[:l/2])
27         else:
28             return find_median(a[l/2:], b[:l/2+1])
29     else:
30         if l % 2 == 0:
31             return find_median(a[l/2-1:], b[l/2-1:])
32         else:
33             return find_median(a[:l/2+1], b[l/2:])
34
35 import random
36 for i in range(10):
37     a = [random.randint(0,10) for r in xrange(10)]
38     b = [random.randint(0,10) for r in xrange(10)]
39     c = a + b
40
41     a.sort()
42     b.sort()
43     c.sort()
44     print a, b, c
45     if median(c) == find_median(a,b):
46         print "PASS"
47     else:
48         print "FAIL"

```

This algorithm leverages subranges (called splices in Python). Lines 12 - 16 describe the various end conditions and base cases. Once complete, one must start the merging of each subrange. If the median of list a is less than the median of list b, then the median of the entire set must lie in the right half of list a, or the left half of list b.

Conversely, if list b's median is greater, then the median must lie in the left half of a.

**Problem 5.5 (graduate).** Design an algorithm which finds the visible lines in  $O(n \lg n)$  time.

The brute force of this problem, would be to compare the intersections of all lines, sorting the y values, and dropping the invisible lines. This however results in  $\Omega(n^2)$ .

The algorithm must then divide the input in such a way that it doesn't have to enumerate *all* intersections.

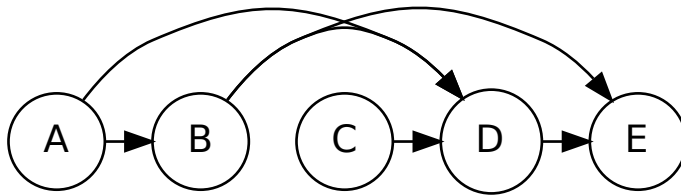
Consider:

1. Sort the list of lines by their slope.
2. Add line to the visible list.

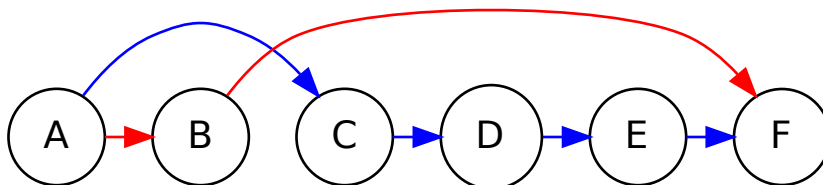
3. Starting at the beginning of the list, find all intersections with this line to the lines to its right in the list.
4. Take the line whose intersection is greatest. Add this line to the visible list.
5. Now repeat with this line, find all intersections with lines to its right.

This algorithm sorts in  $O(n \lg n)$  using something like Merge Sort. Then at each step it only considers lines to its right in the sorted slope list. Each line with a greater y value gets added to the visible list, and becomes the next line to consider, never considering lines behind us, or already in the visible list. In this way we beat  $n^2$  time.

**Problem 6.3.** Given an ordered graph  $G$ , find the length of the longest path that begins at A and ends at E.



The given algorithm is a greedy algorithm and always chooses the nearest node hoping to get to the end as late as possible. However in the example below, the blue path is the longest path, with length 4, while the red path is length 2. The greedy algorithm does not see beyond node 2, and incorrectly follows the red path.



Dynamic programming can help here by “carefully” evaluating all possible paths. First a few constraints. The graph must be acyclic,, and have a topological sort, which by definition it does.

Next we use a 5 step process.

1. Define sub problems.
2. Guess a metric to minimize/maximize.
3. Relate the sub problems (recurrence)
4. Solve the recurrence, and memoize the algorithm.
5. Recombine the subproblems (verify the original problem gets solved.)

1. Define sub problems. Consider the incoming edges to the final node. The longest path must be one of those edges, so use dynamic programming to carefully try them all.

2. Guess the last edge coming into the final node  $v$ . Next recursively find the longest path to  $u$  and add the longer path to  $v$ .

3. Relate sub problems. Define a function  $\delta(s, v)$  which is the length of the edge from the path from  $s$  to  $v$ .

$$\delta(s, v) = \max\{\delta(s, u) + (u \rightarrow v).weight \mid \forall \text{ Edges } = u\} \quad (3)$$

Time per subproblem is  $\delta(s, v) = indegree(v)$ . Total time

$$\sum_{v \in G} indegree(v) = \theta(E + V) \quad (4)$$

**Problem 6.5.** Consider the string `ilovemywife`.

Consider all the suffixes of this string.

define a recurrence  $Q(s, f)$  where  $s$  is the starting index, and  $f$  is the closed finish index.  $Q(s, f)$  returns the quality for that string. We want to maximize the quality of the sentence. So let's “carefully” try all suffixes.

1. Subproblems. suffixes of words `words[i:]` (in python notation).  
# number of subproblems is  $n$  in the number of letters
2. Guess where to start the second word.  
# number of choices  $(n - 1)$  since the first letter must be at the first word.
3. Recurrence

$$word(i) = \max\{word(j) + Q(i, j) \mid j \text{ in range}(i+1, n + 1)\} \quad (5)$$

4. Total running time is time / subproblem \* number of problems =  $\theta(n^2)$ .

5. Original subproblem is the maximum quality starting from the first character i.e.  $word(0)$ .