Prof. David Draper
Department of Applied Mathematics and Statistics
University of California, Santa Cruz

AMS 131: Quiz 6

Name:
Jeremy Lafond

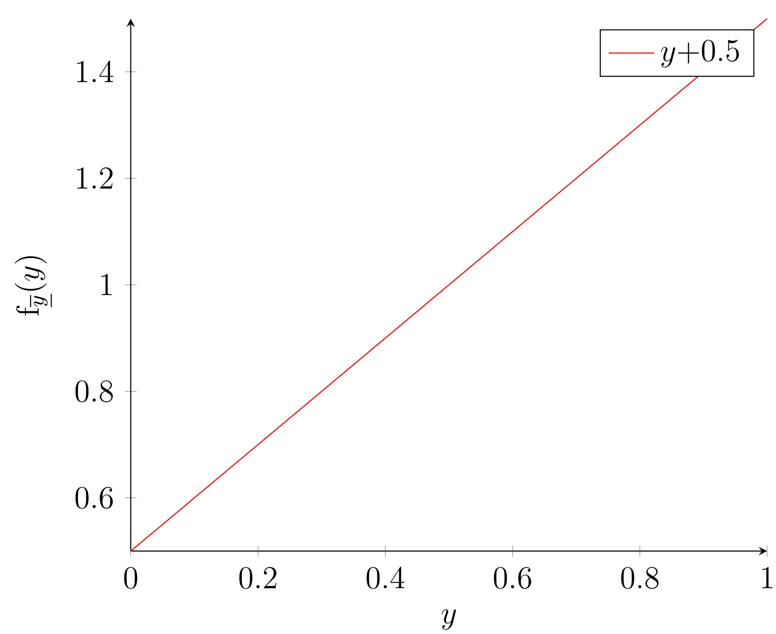
(Note that part (e) of this question is on the second page.)

In a problem you're working on, you need to simulate random draws from the following PDF for the continuous random variable Y:

$$f_Y(y) = \left\{ \begin{array}{cc} \frac{1}{2}(2\,y+1) & \text{for } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{array} \right\} . \tag{1}$$

(a) Sketch the PDF in equation (1) for y in the interesting range $[0, 1]$.

Due to the behavior of tikz in LaTeX, the graph at the origin seems slightly misleading. The $f_{\underline{y}}(y)$ axis is actually starting at 0.5 to indicate the $f_{\underline{y}}(y)$ intercept.



(b) Work out the CDF $F_Y(y)$ for Y, specifying its values for all $-\infty < y < +\infty$, and sketch it in the interesting range $0 \leq y \leq 1$.

Recall:

$$F_{\underline{y}}(y) = \int_{-\infty}^{y} f_{\underline{y}}(t)\mathrm{d}t$$

Thus:

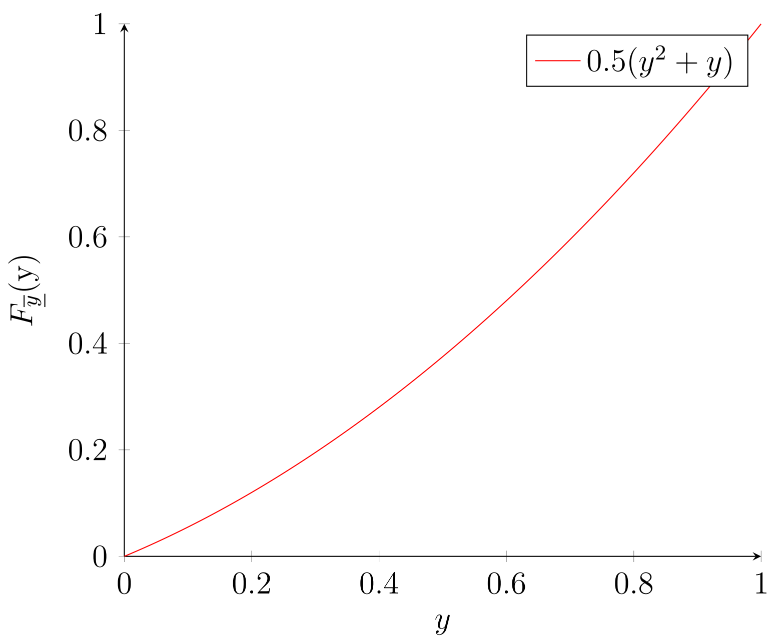
$$F_{\underline{y}}(y) = \int_{-\infty}^{y} f_{\underline{y}}(t)\mathrm{d}t = \int_{0}^{y} t + 0.5\mathrm{d}t = 0.5(y^2 + y)$$

for $0 \leq y \leq 1$.

Therefore:

$$F_{\underline{y}}(y) = \left\{ \begin{array}{cc} 0 & \text{for } y \leq 0 \\ \frac{1}{2}(y^2 + y) & \text{for } 0 \leq y \leq 1 \\ 1 & y \geq 1 \end{array} \right\}. \tag{2}$$

As seen sketched in the interesting range below:



(c) Work out the inverse CDF (quantile function) $F_Y^{-1}(p)$ for Y, specifying its values for all $0 < p < 1$, and sketch it for p in that range.

Recall:

$F_{\underline{y}}^{-1}(p)$ for $0 \leq y \leq 1 = y \leftrightarrow$ p $=$ $\mathrm{F}_{\underline{y}}(y)$

Thus we solve for y in terms of p:

$0.5(y^2 + y) \rightarrow .5p = y^2 + y = -0.5 \pm 0.5(\sqrt{8p+1}) = y$

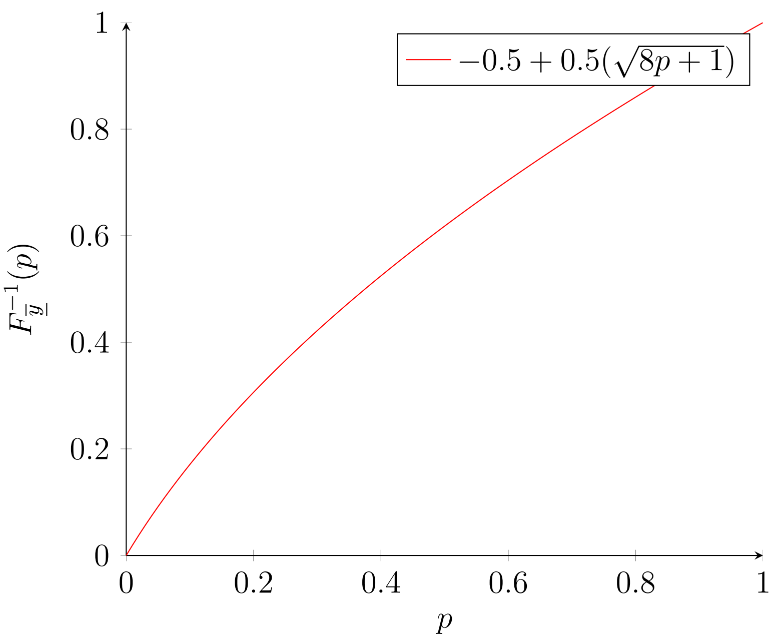
However, we can see that $-0.5 - 0.5(\sqrt{8p+1})$ does not map to the region we are concerned with.

Therefore:

$$F_{\underline{y}}^{-1}(p) = y = \begin{cases} -0.5 + 0.5(\sqrt{8p+1}) \text{for } 0 < p < 1 \end{cases}$$

(3)

As seen sketched in the interesting range below:



Using the result presented in Discussion Section 6 that demonstrates how to employ the quantile function of a random variable Y to generate random draws from its PDF $f_Y(y)$, and building on your result in part (c), explicitly specify how you can generate IID random draws from the PDF in equation (1).

Given a PDF: $f_{\underline{y}}$, if you calculate the CDF: $F_{\underline{y}}$,

and then the Inverse CDF in terms of p: $F_Y^{-1}(p)$.

With this information you can then generate a bunch of uniform draws and use them each as inputs in $F_Y^{-1}(p)$. The outputs from this process turn out to be random IID draws from the original PDF.

Once you have your random sample in part (d), briefly explain how you could graphically check whether it really *is* a sample from the PDF in equation (1).

You could super impose the original PDF as a line graph across the histogram of the sample of all F_Y^{-1}(u) outputs in which each u is a uniform draw. In theory, the more uniform samples that are generated within a sample, the closer the histogram becomes to approximating the curve. This resembles a sort of probabilistically random Reimann sum that improves in accuracy and positioning with its increasing number of rectangles. This makes sense based on the theorems we have learned so far and how they pertain to the basic principles of calculus.