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AMS 131: Quiz 7

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All parts of this problem are unrelated (i.e., the assumptions in part (x) apply only to part (x)). All expectations and variances are assumed to exist and to be finite.

(a) You're working with two random variables X and Y, which may be dependent and for which $V(X) = V(Y)$. Show that the random variables $W = (X+Y)$ and $Z = (X-Y)$ are uncorrelated. *Hint:* Nothing fancy — just simplify the covariance of W and Z, using properties of covariance discussed in class and discussion section.

$$Cov(W, Z) = Cov(X+Y, X-Y) = E[WZ] - E[W]E[Z]$$
$$WZ = (X+Y)(X-Y) = X^2 - Y^2$$
$$E[WZ] = E[X^2 - Y^2] = E[X^2] - E[Y^2]$$
$$E[W] = E[X] + E[Y]$$
$$E[Z] = E[X] - E[Y]$$
$$E[W]E[Z] = E[X^2] - E[Y^2]$$
$$Cov(W, Z) = E[X^2] - E[Y^2] - E[X^2] + E[Y^2]$$
$$Cov(W, Z) = E[X^2] - E[Y^2] - E[X^2] + E[Y^2] = 0$$
$$V(X) - V(Y) = 0$$

(b) You're working with two random variables X and Y that are negatively correlated. Which is bigger — $V(X+Y)$ or $V(X-Y)$ — or are they equal? Show your calculations.

$$V(X+Y) = V(X) + V(Y) + Z * Cov(X, Y)$$

$$V(X-Y) = V(X) + V(Y) - Z * Cov(X,Y)$$

Since: Cov(X,Y), V(X), and V(Y) are all less than 0; $V(X-Y) > V(X+Y)$. Therefore, V(X-Y) is bigger.

(c) You're working with two random variables X and Y such that $V(X) = 9, V(Y) = 4$, and $\rho(X,Y) = -\frac{1}{6}$. Compute $V(X+Y)$ and $V(X-Y)$ (show your calculations).

$$\begin{aligned}
sd(X) &= \sqrt{9} = 3 \\
sd(Y) &= \sqrt{4} = 2 \\
Cov(X,Y) = \rho(X,Y) * sd(X)sd(Y) &= -\frac{1}{6} * 3 * 2 = -1 \\
V(X+Y) = V(X) + V(Y) + 2Cov(X,Y) &= 9 + 4 - 2 = 11 \\
V(X-Y) = V(X) + V(Y) - 2Cov(X,Y) &= 9 + 4 + 2 = 15
\end{aligned}$$

(d) You and your research assistant (RA) are working with two random variables X and Y, and your RA has computed the following values: $E(X) = 3, E(Y) = 2, E(X^2) = 10, E(Y^2) = 29$, and $E(XY) = 0$. Show that there must be something wrong in this computation. *Hint:* Consider the bounds on variances and correlations.

$$\begin{gathered}
Var(X) = E[X^2] - E[X]^2 = 10 - 3^2 = 10 - 9 = 1 \\
sd(X) = \sqrt{1} = 1 \\
var(Y) = E[Y^2] - E[Y]^2 = 29 - 2^2 = 29 - 4 = 25 \\
sd(Y) = \sqrt{25} = 5
\end{gathered}$$

Since $E[XY] = 0, Cov(X,Y) = E[XY] - E[x]E[Y] = 0 - 3 * 2 = -6$

$$\rho = \frac{Cov(X,Y)}{sd(X)sd(Y)} = \frac{-6}{5} = -1.2$$

Since ρ must be between -1 and 1, something must be wrong.