Prof. David Draper
Department of Applied Mathematics and Statistics
University of California, Santa Cruz

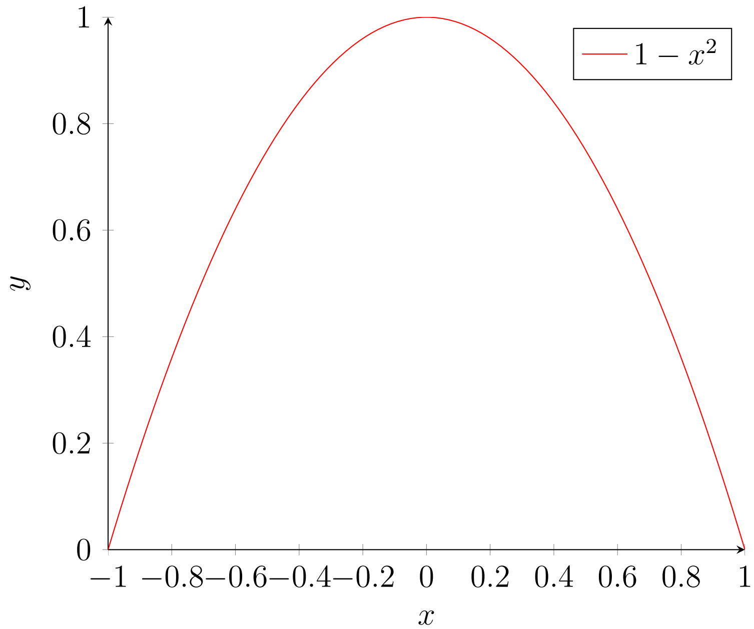
AMS 131: Quiz 5

Name:
Jeremy Lafond

You're working on a problem involving two continuous random variables X and Y, and you figure out that their joint PDF has the following form:

$$f_{X,Y}(x,y) = \left\{ \begin{array}{cc} c\,x^2 & \text{for } 0 \leq y \leq 1 - x^2 \\ 0 & \text{otherwise} \end{array} \right\} . \tag{1}$$

(a) Sketch the support S of this bivariate distribution.



(b) Compute the normalizing constant c.

Recall that the below must be true:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} cx^2, \mathrm{d}x\, \mathrm{d}y = 1$$

Our bounds are $0 \leq y \leq 1 - x^2$ and $-1 \leq x \leq 1$ respectively.

Thus:

$$\int_{-1}^{1} \int_{0}^{1-x^2} cx^2, \mathrm{d}y\, \mathrm{d}x = c * \frac{4}{15}$$

and our normalizing constant must equal $15/4$

(c) It can be shown that the marginal PDFs of X and Y with this joint PDF are

$$f_X(x) = \left\{ \begin{array}{cc} \frac{15}{4}x^2(1-x^2) & \text{for } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{array} \right\} \tag{2}$$

and

$$f_Y(y) = \left\{ \begin{array}{cc} \frac{5}{2}(1-y)^{\frac{3}{2}} & \text{for } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{array} \right\}. \tag{3}$$

Verify that both of these marginals are correct.

$$f_{\underline{\overline{x}}}(x) = \int_S f_{\underline{\overline{xy}}}(x,y)\mathrm{d}y = \int_0^{1-x^2} \frac{15}{4}x^2\mathrm{d}y = \frac{15}{4}x^2(1-x^2)$$

and

$$f_{\underline{\overline{y}}}(y) = \int_S f_{\underline{\overline{xy}}}(x,y)\mathrm{d}x = \int_0^{\sqrt{1-y}} \frac{15}{4}x^2\mathrm{d}x = \frac{15}{12}(\sqrt{1-y})^3 = \frac{5}{4}(1-y)^{\frac{3}{2}}$$

I believe $f_{\underline{\overline{y}}}(y)$ has an incorrect leading constant as the integral becomes $\frac{x^3}{3} * \frac{15}{4} = x^3 * \frac{15}{12}$ which shows the constant reduces to $\frac{5}{4}$ not $\frac{5}{2}$.

(d) Are X and Y independent in this joint distribution? Explain briefly.

Recall the joint PDF:

$$f_{X,Y}(x,y) = \left\{ \begin{array}{cc} c\,x^2 & \text{for } 0 \leq y \leq 1-x^2 \\ 0 & \text{otherwise} \end{array} \right\} \tag{4}$$

where c $= \frac{15}{4}$

and the marginals:

$$f_X(x) = \left\{ \begin{array}{cc} \frac{15}{4}x^2(1-x^2) & \text{for } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{array} \right\} \tag{5}$$

and

$$f_Y(y) = \left\{ \begin{array}{cc} \frac{5}{2}(1-y)^{\frac{3}{2}} & \text{for } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{array} \right\}. \tag{6}$$

In order for X and Y to be independent the following equation must hold true:

$$f_{\underline{\overline{y}}}(y) * f_{\underline{\overline{x}}}(x) = f_{\underline{\overline{x,y}}}(x,y)$$

however,

$$f_{\underline{\overline{y}}}(y) * f_{\underline{\overline{x}}}(x) = \frac{15}{4}x^2(1-x^2) * \frac{5}{4}(1-y)^{\frac{3}{2}} \neq f_{\underline{\overline{x,y}}}(x,y)$$

Therefore, because the product of the marginals clearly does not equal the joint PDF, X and Y are most certainly **not independent** in this joint distribution.