1998-AL-P-MATH-1-Q08

8(a)

(E) has a unique solution
$\Rightarrow \Delta \neq 0$

$$\Rightarrow \begin{vmatrix} a & 1 & b \\ 1 & a & b \\ 1 & 1 & ab \end{vmatrix} \neq 0$$

$\Rightarrow a(a^2b - b) - ab + b + b - ab \neq 0$

$\Rightarrow a^3b - 3ab + 2b \neq 0$

$\Rightarrow b(a^3 - 3a + 2) \neq 0$

$\Rightarrow b(a+2)(a^2 - 2a + 1) \neq 0$

$\Rightarrow b(a+2)(a-1)^2 \neq 0$

$\Rightarrow b \neq 0$ and $a \neq -2$ and $a \neq 1$

On the other hand, when $b \neq 0$ and $a \neq -2$ and $a \neq 1$

$\Rightarrow \Delta \neq 0$

$\Rightarrow$ (E) has unique solution.

Hence (E) has a unique solution if and only if $b \neq 0$ and $a \neq -2$ and $a \neq 1$

When $b \neq 0$ and $a \neq -2$ and $a \neq 1$

$$\Delta_x = \begin{vmatrix} 1 & 1 & b \\ 1 & a & b \\ b & 1 & ab \end{vmatrix}$$

$= a^2b - b - ab + b + b^2 - ab^2$
$= a^2b - ab + b^2 - ab^2$
$= b(a^2 - a + b - ab)$
$= b(a-1)(a-b)$

$$\Delta_y = \begin{vmatrix} a & 1 & b \\ 1 & 1 & b \\ 1 & b & ab \end{vmatrix}$$

$= a(ab - b^2) - ab + b^2$
$= a^2b - ab^2 - ab + b^2$
$= a^2b - ab + b^2 - ab^2$
$= ab(a-1) - b^2(a-1)$
$= b(a-1)(a-b)$

$$\Delta_z = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & b \end{vmatrix}$$

$= a(ab - 1) - b + 1 + 1 - a$
$= a^2b - a - b + 2 - a$
$= a^2b - 2a - b + 2$
$= a^2b - b - 2a + 2$
$= b(a^2 - 1) - 2(a - 1)$
$= (a-1)(b(a+1) - 2)$
$= (a-1)(ab + b - 2)$

Then, $x = \dfrac{\Delta_x}{\Delta}$ and $y = \dfrac{\Delta_y}{\Delta}$ and $z = \dfrac{\Delta_z}{\Delta}$

$\Rightarrow x = \dfrac{b(a-1)(a-b)}{b(a+2)(a-1)^2}$ and $y = \dfrac{b(a-1)(a-b)}{b(a+2)(a-1)^2}$ and $z = \dfrac{(a-1)(ab+b-2)}{b(a+2)(a-1)^2}$

$\Rightarrow x = \dfrac{a-b}{(a+2)(a-1)}$ and $y = \dfrac{a-b}{(a+2)(a-1)}$ and $z = \dfrac{ab+b-2}{b(a+2)(a-1)}$

8(b)(i)

When a = -2,
$\Delta_x = \Delta_y = b(a-1)(a-b)$ and $\Delta_z = (a-1)(ab+b-2)$
$\Rightarrow \Delta_x = \Delta_y = 3b(b+2)$ and $\Delta_z = 3(b+2)$

(E) is consistent and $\Delta = 0$
$\Rightarrow \Delta_x = \Delta_y = \Delta_z = 0$
$\Rightarrow 3b(b+2) = 0$ and $3(b+2) = 0$
$\Rightarrow (b = 0$ or $b = -2)$ and $b = -2$
$\Rightarrow b = -2$

Then (E) $\begin{cases} -2x + y - 2z = 1 \\ x - 2y - 2z = 1 \\ x + y + 4z = -2 \end{cases}$

$\Rightarrow$ (E) $\begin{cases} x - 2y - 2z = 1 \\ x + y + 4z = -2 \end{cases}$

Consider the augmented matrix :

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 1 & 1 & 4 & -2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 0 & 3 & 6 & -3 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$\Rightarrow$ Solutions are $x = y = -2t - 1$, $z = t \in R$

8(b)(ii)

When a = 1,
$\Delta_x = \Delta_y = b(a-1)(a-b)$ and $\Delta_z = (a-1)(ab+b-2)$
$\Rightarrow \Delta_x = \Delta_y = \Delta_z = 0$ for any $b \in R$

Then (E) $\begin{cases} x + y + bz = 1 \\ x + y + bz = b \end{cases}$

(E) is consistent $\Rightarrow b = 1$

Solutions are $x = 1 - m - n, y = m \in R$, $z = n \in R$

8(c)

When b = 0,
$\Delta_x = \Delta_y = b(a-1)(a-b)$ and $\Delta_z = (a-1)(ab+b-2)$
$\Rightarrow \Delta_x = \Delta_y = 0$ and $\Delta_z = 2(1-a)$

When $a \neq 1$, (E) has NO solution because $\Delta = 0$ but $\Delta_z \neq 0$.
When a=1, according to result in 8(b)(ii), (E) has NO solution because $b = 0 \neq 1$.

Hence, (E) is NOT consistent for b=0.