PP-DSE-MATH-EP(M2)-Q11

11(a)

$$A^2 = \begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (\alpha+\beta)^2 - \alpha\beta & -\alpha\beta(\alpha+\beta) \\ \alpha+\beta & -\alpha\beta \end{pmatrix}$$

$$= \begin{pmatrix} (\alpha+\beta)^2 & -\alpha\beta(\alpha+\beta) \\ \alpha+\beta & 0 \end{pmatrix} - \begin{pmatrix} \alpha\beta & 0 \\ 0 & \alpha\beta \end{pmatrix}$$

$$= (\alpha+\beta) \begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix} - \alpha\beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= (\alpha+\beta)A - \alpha\beta I$$

11(b)

$(A-\alpha I)^2$
$= A^2 - 2\alpha A + \alpha^2 I$
$= (\alpha+\beta)A - \alpha\beta I - 2\alpha A + \alpha^2 I$
$= (\alpha+\beta-2\alpha)A + (\alpha^2 - \alpha\beta)I$
$= (\beta-\alpha)A + \alpha(\alpha-\beta)I$
$= (\beta-\alpha)(A-\alpha I)$

Also $(A-\beta I)^2$
$= A^2 - 2\beta A + \beta^2 I$
$= (\alpha+\beta)A - \alpha\beta I - 2\beta A + \beta^2 I$
$= (\alpha+\beta-2\beta)A + (\beta^2 - \alpha\beta)I$
$= (\alpha-\beta)A + \beta(\beta-\alpha)I$
$= (\alpha-\beta)(A-\beta I)$

11(c)(i)

$A = X + Y$

$$\Rightarrow \begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix} = s\begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix} - s\alpha \begin{pmatrix} 1 & -0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix} - t\beta \begin{pmatrix} 1 & -0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (\alpha+\beta)(s+t) - s\alpha - t\beta & -\alpha\beta(s+t) \\ s+t & -s\alpha - t\beta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} s\beta + t\alpha & -\alpha\beta(s+t) \\ s+t & -s\alpha - t\beta \end{pmatrix}$$

$\Rightarrow \alpha+\beta = t\alpha + s\beta$ and $s\alpha + t\beta = 0$

$\Rightarrow s = \dfrac{\beta}{\beta-\alpha}$ and $t = -\dfrac{\alpha}{\beta-\alpha} = \dfrac{\alpha}{\alpha-\beta}$

11(c)(ii)

Let P(n) be the statement that $X^n = \frac{\beta^n}{\beta-\alpha}(A-\alpha I)$ and $Y^n = \frac{\alpha^n}{\alpha-\beta}(A-\beta I)$ for any positive n.

When n = 1,
$X^1 = X = s(A-\alpha I) = \frac{\beta}{\beta-\alpha}(A-\alpha I)$ and
$Y^1 = Y = t(A-\beta I) = \frac{\alpha}{\alpha-\beta}(A-\beta I)$
Therefore, P(1) is true.

Assume that P(k) is true for some positive integer k ≥ 1. Then
$X^{k+1} = X^k X = \frac{\beta^k}{\beta-\alpha}(A-\alpha I)\cdot\frac{\beta}{\beta-\alpha}(A-\alpha I)$
$= \frac{\beta^{k+1}}{(\beta-\alpha)^2}(A-\alpha I)^2$
$= \frac{\beta^{k+1}}{(\beta-\alpha)^2}(A^2-2\alpha A+\alpha^2 I)$
$= \frac{\beta^{k+1}}{(\beta-\alpha)^2}((\alpha+\beta)A-\alpha\beta I-2\alpha A+\alpha^2 I)$
$= \frac{\beta^{k+1}}{(\beta-\alpha)^2}((\beta-\alpha)A-\alpha(\beta-\alpha)I)$
$= \frac{\beta^{k+1}}{\beta-\alpha}(A-\alpha I)$

Also, $Y^{k+1} = Y^k Y = \frac{\alpha^k}{\alpha-\beta}(A-\beta I)\cdot\frac{\alpha}{\alpha-\beta}(A-\beta I)$
$= \frac{\alpha^{k+1}}{(\alpha-\beta)^2}(A-\beta I)^2$
$= \frac{\alpha^{k+1}}{(\alpha-\beta)^2}(A^2-2\beta A+\beta^2 I)$
$= \frac{\alpha^{k+1}}{(\alpha-\beta)^2}((\alpha+\beta)A-\alpha\beta I-2\beta A+\beta^2 I)$
$= \frac{\alpha^{k+1}}{(\alpha-\beta)^2}((\alpha-\beta)A-\beta(\alpha-\beta)I)$
$= \frac{\alpha^{k+1}}{\alpha-\beta}(A-\beta I)$
Therefore, P(k+1) is true. By mathematical induction, P(n) is true.

11(c)(iii)

Consider,
$XY = s(A-\alpha I)\cdot t(A-\beta I)$
$= st(A^2-(\alpha+\beta)A+\alpha\beta I)$
$= st((\alpha+\beta)A-\alpha\beta I-(\alpha+\beta)A+\alpha\beta I)$
$= st(0) = 0$

$YX = t(A-\beta I)\cdot s(A-\alpha I)$
$= st(A^2-(\alpha+\beta)A+\alpha\beta I)$
$= st((\alpha+\beta)A-\alpha\beta I-(\alpha+\beta)A+\alpha\beta I)$
$= st(0) = 0$

Therefore $XY = YX = 0$

$\Rightarrow (X+Y)^n = X^n + K^n$ for any positive integer n.

Now, $A = X + Y$

$\Rightarrow A^n = (X+Y)^n$

$\Rightarrow A^n = X^n + Y^n$

$\Rightarrow A^n = \dfrac{\beta^n}{\beta - \alpha}(A - \alpha I) + \dfrac{\alpha^n}{\alpha - \beta}(A - \beta I)$

$\Rightarrow A^n = \dfrac{\beta^n - \alpha^n}{\beta - \alpha} A + \dfrac{\alpha^n \beta - \alpha \beta^n}{\beta - \alpha} I$