

2013-AL-P-MATH-1-Q11

11(a)(i)

$$A \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & -2 \\ -2 & a+3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \lambda \\ -2\lambda \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a+4 \\ -2a-8 \end{pmatrix} = \begin{pmatrix} \lambda \\ -2\lambda \end{pmatrix}$$

$$\Rightarrow \lambda = a + 4$$

11(a)(ii)

$$A \begin{pmatrix} b \\ 1 \end{pmatrix} = \mu \begin{pmatrix} b \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & -2 \\ -2 & a+3 \end{pmatrix} \begin{pmatrix} b \\ 1 \end{pmatrix} = \begin{pmatrix} \mu b \\ \mu \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} ab-2 \\ -2b+a+3 \end{pmatrix} = \begin{pmatrix} \mu b \\ \mu \end{pmatrix}$$

$$\Rightarrow ab-2 = \mu b \text{ and } -2b+a+3 = \mu$$

$$\Rightarrow ab-2 = (-2b+a+3)b$$

$$\Rightarrow ab-2 = -2b^2+ab+3b$$

$$\Rightarrow 2b^2-3b-2 = 0$$

$$\Rightarrow (2b+1)(b-2) = 0$$

$$\Rightarrow b = -\frac{1}{2} \text{ (rejected since } b > 0) \text{ or } 2$$

$$\Rightarrow b = 2$$

$$\text{Therefore } \mu = -2b + a + 3 = a - 1$$

11(a)(iii)(1)

$$M^T M = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = 5I$$

11(a)(iii)(2)

Let P(n) be the statement that $A^n = \frac{1}{5} M D^n M^T$ for any $n \in N$ where $D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$

When $n = 1$,

$$\frac{1}{5} M D^1 M^T$$

$$\begin{aligned}
&= \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} \lambda & 2\mu \\ -2\lambda & \mu \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} \lambda + 4\mu & -2\lambda + 2\mu \\ -2\lambda + 2\mu & 4\lambda + \mu \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} a + 4 + 4(a - 1) & -2(a + 4) + 2(a - 1) \\ -2(a + 4) + 2(a - 1) & 4(a + 4) + a - 1 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} 5a & -10 \\ -10 & 5a + 15 \end{pmatrix} \\
&= \begin{pmatrix} a & -2 \\ -2 & a + 3 \end{pmatrix} \\
&= A
\end{aligned}$$

Therefore P(1) is true.

Assume that P(k) is true for some natural number $k \geq 1$. Then,

$$\begin{aligned}
&A^{k+1} \\
&= A^k A \\
&= \frac{1}{5} M D^k M^T \cdot \frac{1}{5} M D M^T \\
&= \frac{1}{25} M D^k (M^T M) D M^T \\
&= \frac{1}{25} M D^k (5I) D M^T \\
&= \frac{1}{5} M D^k D M^T \\
&= \frac{1}{5} M D^{k+1} M^T.
\end{aligned}$$

Therefore P(k+1) is true and by mathematical induction P(n) is true.

11(b)(i)

Let $a=3$, then $\lambda = 7$, $\mu = 2$ and $A = \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}$. Then

$$\begin{aligned}
&\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}^{2013} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow \begin{pmatrix} x & y \end{pmatrix} A^{2013} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow \begin{pmatrix} x & y \end{pmatrix} \frac{1}{5} M D^{2013} M^T \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow \frac{1}{5} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 7^{2013} & 0 \\ 0 & 2^{2013} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow \frac{1}{5} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 7^{2013} & -2^{2014} \\ 2 \cdot 7^{2013} & 2^{2013} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow \frac{1}{5} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 7^{2013} + 2^{2015} & 2 \cdot 7^{2013} - 2^{2014} \\ 2 \cdot 7^{2013} - 2^{2014} & 4 \cdot 7^{2013} + 2^{2013} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow ((7^{2013} + 2^{2015})x + (2 \cdot 7^{2013} - 2^{2014})y \quad (2 \cdot 7^{2013} - 2^{2014})x + (4 \cdot 7^{2013} + 2^{2013})y) \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow (7^{2013} + 2^{2015})x^2 + (2 \cdot 7^{2013} - 2^{2014})xy + (2 \cdot 7^{2013} - 2^{2014})xy + (4 \cdot 7^{2013} + 2^{2013})y^2 = 0 \\
&\Rightarrow (7^{2013} + 2^{2015})x^2 + 2(7^{2013} - 2^{2013})xy + (4 \cdot 7^{2013} + 2^{2013})y^2 = 0 \\
&\Rightarrow (7^{2013} + 4 \cdot 2^{2013})x^2 + 2(7^{2013} - 2^{2013})xy + (4 \cdot 7^{2013} + 2^{2013})y^2 = 0 \\
&\Rightarrow (7^{2013} - 2^{2013} + 5 \cdot 2^{2013})x^2 + 2(7^{2013} - 2^{2013})xy + (7^{2013} - 2^{2013} + 3 \cdot 7^{2013} + 2 \cdot 2^{2013})y^2 = 0 \\
&\Rightarrow (7^{2013} - 2^{2013})(x^2 + 2xy + y^2) + 5 \cdot 2^{2013}x^2 + (3 \cdot 7^{2013} + 2 \cdot 2^{2013})y^2 = 0 \\
&\Rightarrow (7^{2013} - 2^{2013})(x + y)^2 + 5 \cdot 2^{2013}x^2 + (3 \cdot 7^{2013} + 2 \cdot 2^{2013})y^2 = 0 \\
&\Rightarrow (x + y)^2 = 0 \text{ or } x^2 = 0 \text{ or } y^2 = 0 \\
&\Rightarrow x = 0 \text{ or } y = 0
\end{aligned}$$

11(b)(ii)

Let $a=1$, then $\lambda = 5$, $\mu = 0$ and $A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$. Then

$$\begin{aligned}
&(x \quad y) \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}^{2013} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow \frac{1}{5} (x \quad y) \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5^{2013} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow \frac{1}{5} (x \quad y) \begin{pmatrix} 5^{2013} & 0 \\ 2 \cdot 5^{2013} & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow \frac{1}{5} (x \quad y) \begin{pmatrix} 5^{2013} & 2 \cdot 5^{2013} \\ 2 \cdot 5^{2013} & 4 \cdot 5^{2013} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow (5^{2013}x + 2 \cdot 5^{2013}y \quad 2 \cdot 5^{2013}x + 4 \cdot 5^{2013}y) \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow 5^{2013}x^2 + 2 \cdot 5^{2013}xy + 2 \cdot 5^{2013}xy + 4 \cdot 5^{2013}y^2 = 0 \\
&\Rightarrow x^2 + 4xy + 4y^2 = 0 \\
&\Rightarrow (x + 2y)^2 = 0 \\
&\Rightarrow x + 2y = 0
\end{aligned}$$

Therefore, it is NOT necessary that both x and y equal 0.