1999-AL-P-MATH-1-Q08

8(a)

(E) has a unique solution
$\Rightarrow \Delta \neq 0$

$\Rightarrow \begin{vmatrix} 1 & \lambda & 1 \\ 3 & -1 & \lambda+2 \\ 1 & -1 & 1 \end{vmatrix} \neq 0$
$\Rightarrow \lambda(\lambda+2)+1+(\lambda+2)-3-1-3\lambda \neq 0$
$\Rightarrow \lambda^2-1 \neq 0$
$\Rightarrow (\lambda+1)(\lambda-1) \neq 0$
$\Rightarrow \lambda \neq -1$ and $\lambda \neq 1$

On the other hand,
$\lambda \neq -1$ and $\lambda \neq 1$
$\Rightarrow \Delta \neq 0$
$\Rightarrow$ (E) has a unique solution.

Hence, (E) has a unique solution if and only if $\lambda \neq \pm 1$

8(b)(i)

$\Delta_x = \begin{vmatrix} \lambda & \lambda & 1 \\ 7 & -1 & \lambda+2 \\ 3 & -1 & 1 \end{vmatrix}$ and $\Delta_y = \begin{vmatrix} 1 & \lambda & 1 \\ 3 & 7 & \lambda+2 \\ 1 & 3 & 1 \end{vmatrix}$ and $\Delta_z = \begin{vmatrix} 1 & \lambda & \lambda \\ 3 & -1 & 7 \\ 1 & -1 & 3 \end{vmatrix}$

$\Rightarrow \Delta_x = 4(\lambda+1)(\lambda-1)$ and $\Delta_y = (\lambda-1)(\lambda-3)$ and $\Delta_z = 4(1-\lambda)$

$\Rightarrow x = \dfrac{\Delta_x}{\Delta}$ and $y = \dfrac{\Delta_y}{\Delta}$ and $z = \dfrac{\Delta_z}{\Delta}$

$\Rightarrow x = \dfrac{4(\lambda+1)(\lambda-1)}{(\lambda+1)(\lambda-1)}$ and $y = \dfrac{(\lambda-1)(\lambda-3)}{(\lambda+1)(\lambda-1)}$ and $z = \dfrac{4(1-\lambda)}{(\lambda+1)(\lambda-1)}$

$\Rightarrow x = 4$ and $y = \dfrac{\lambda-3}{\lambda+1}$ and $z = \dfrac{-4}{\lambda+1}$

8(b)(ii)

When $\lambda = -1$
$\Delta = (\lambda+1)(\lambda-1)$ and $\Delta_x = 4(\lambda+1)(\lambda-1)$ and $\Delta_y = (\lambda-1)(\lambda-3)$ and $\Delta_z = 4(1-\lambda)$
$\Rightarrow \Delta = \Delta_x = 0$ and $\Delta_y = \Delta_z = 8 \neq 0$
$\Rightarrow$ (E) has NO solution.

8(b)(iii)

When $\lambda = 1$, consider the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & -1 & 3 & 7 \\ 1 & -1 & 1 & 3 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & 0 & 4 \\ 0 & -2 & 0 & 2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

$\Rightarrow$ The solutions are $x = 2 - t$, $y = -1$, $z = t \in R$

8(c)(iii)

The solution must fulfill (E) for $\lambda = 1$ and $ax + by + cz = d$

$\Rightarrow a(2-t) + b(-1) + ct = d$ for some $t \in R$
$\Rightarrow a(2-t) - b + ct = d$ for some $t \in R$
$\Rightarrow (c-a)t + 2a - b - d = 0$ for some $t \in R$

Therefore, there are 2 possible ways for the system to be consistent

1. a=c and 2a-b-d=0
   or
2. $a \neq c$ and a, b, c, d can be any number so that $t = \dfrac{2a-b-d}{a-c}$