2015-DSE-MATH-EP(M2)-Q11

11(a)(i)

$$
\begin{aligned}
&M^2\\
&=\begin{pmatrix}\lambda & 1\\ \lambda-\mu+1 & \mu\end{pmatrix}\begin{pmatrix}\lambda & 1\\ \lambda-\mu+1 & \mu\end{pmatrix}\\
&=\begin{pmatrix}\lambda^2+\lambda-\mu+1 & \lambda+\mu\\ (\lambda-\mu+1)(\lambda+\mu) & \mu^2+\lambda-\mu+1\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&AB\\
&=\frac{1}{\lambda-\mu+2}(I-\mu I+M)\cdot\frac{1}{\lambda-\mu+2}(I+\lambda I-M)\\
&=\frac{1}{(\lambda-\mu+2)^2}(I-\mu I+M)(I+\lambda I-M)\\
&=\frac{1}{(\lambda-\mu+2)^2}(\,(1-\mu)(1+\lambda)I+(\lambda+\mu)M-M^2\,)\\
&=\frac{1}{(\lambda-\mu+2)^2}[\,(1-\mu)(1+\lambda)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+(\lambda+\mu)\begin{pmatrix}\lambda & 1\\ \lambda-\mu+1 & \mu\end{pmatrix}-\begin{pmatrix}\lambda^2+\lambda-\mu+1 & \lambda+\mu\\ (\lambda-\mu+1)(\lambda+\mu) & \mu^2+\lambda-\mu+1\end{pmatrix}\,]\\
&=\frac{1}{(\lambda-\mu+2)^2}[\,\begin{pmatrix}(1-\mu)(1+\lambda) & 0\\ 0 & (1-\mu)(1+\lambda)\end{pmatrix}+\begin{pmatrix}(\lambda+\mu)\lambda & (\lambda+\mu)\\ (\lambda+\mu)(\lambda-\mu+1) & \mu(\lambda+\mu)\end{pmatrix}-\begin{pmatrix}\lambda^2+\lambda-\mu+1 \\ (\lambda-\mu+1)(\lambda+\mu\end{pmatrix}\\
&=\frac{1}{(\lambda-\mu+2)^2}[\,\begin{pmatrix}1-\mu+\lambda+\lambda^2 & \lambda+\mu\\ (\lambda+\mu)(\lambda-\mu+1) & 1-\mu+\lambda+\mu^2\end{pmatrix}-\begin{pmatrix}\lambda^2+\lambda-\mu+1 & \lambda+\mu\\ (\lambda-\mu+1)(\lambda+\mu) & \mu^2+\lambda-\mu+1\end{pmatrix}\,]\\
&=0
\end{aligned}
$$

$$
\begin{aligned}
&BA\\
&=\frac{1}{\lambda-\mu+2}(I+\lambda I-M)\cdot\frac{1}{\lambda-\mu+2}(I-\mu I+M)\\
&=\frac{1}{(\lambda-\mu+2)^2}(I+\lambda I-M)(I-\mu I+M)\\
&=\frac{1}{(\lambda-\mu+2)^2}(\,(1-\mu)(1+\lambda)I+(\lambda+\mu)M-M^2\,)\\
&=0
\end{aligned}
$$

$$
\begin{aligned}
&A+B\\
&=\frac{1}{\lambda-\mu+2}(I-\mu I+M)+\frac{1}{\lambda-\mu+2}(I+\lambda I-M)
\end{aligned}
$$

$$= \frac{1}{\lambda - \mu + 2}(I - \mu I + M + I + \lambda I - M)$$

$$= \frac{1}{\lambda - \mu + 2}(2 - \mu + \lambda)I$$

$$= I$$

11(a)(ii)

$A + B = I$
$\Rightarrow (A+B)A = A$ and $(A+B)B = B$
$\Rightarrow A^2 + BA = A$ and $AB + B^2 = B$
$\Rightarrow A^2 + 0 = A$ and $0 + B^2 = B$
$\Rightarrow A^2 = A$ and $B^2 = B$

11(a)(iii)

Let P(n) be the statement that $M^n = (\lambda+1)^n A + (\mu - 1)^n B$ for all positive integers n.

When n = 1,
$(\lambda+1)^1 A + (\mu-1)^1 B$

$$= (\lambda+1)A + (\mu-1)B$$

$$= (\lambda+1)\frac{1}{\lambda-\mu+2}(I - \mu I + M) + (\mu-1)\frac{1}{\lambda-\mu+2}(I + \lambda I - M)$$

$$= \frac{1}{\lambda-\mu+2}[\,(\lambda+1)(I-\mu I+M) + (\mu-1)(I+\lambda I-M)\,]$$

$$= \frac{1}{\lambda-\mu+2}[\,(\lambda+1)(1-\mu)I + (\lambda+1)M + (\mu-1)(1+\lambda)I - (\mu-1)M)\,]$$

$$= \frac{1}{\lambda-\mu+2}(\lambda-\mu+2)M$$

$$= M$$

Therefore P(1) is true.

Assume P(k) is true for some positive integer $k \geq 1$. Then,
$M^{k+1} = M^k \cdot M$
$= [\,(\lambda+1)^k A + (\mu-1)^k B\,][\,(\lambda+1)A + (\mu-1)B\,]$
$= (\lambda+1)^{k+1}A^2 + (\lambda+1)^k(\mu-1)AB + (\mu-1)^k(\lambda+1)BA + (\mu-1)^{k+1}B^2$
$= (\lambda+1)^{k+1}A^2 + (\lambda+1)^k(\mu-1)(0) + (\mu-1)^k(\lambda+1)(0) + (\mu-1)^{k+1}B^2$
$= (\lambda+1)^{k+1}A^2 + (\mu-1)^{k+1}B^2$
$= (\lambda+1)^{k+1}A + (\mu-1)^{k+1}B$
Therefore P(k+1) is true. By mathematical induction P(n) is true.

11(b)

Let $\lambda = 2$, $\mu = 3$ such that $\mu - \lambda = 1 \neq 2$. Then,
$M = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$, $A = M - 2I = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = 3I - M = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

Therefore,

$$\begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix}^{315}$$

$$= (2 \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix})^{315}$$

$$= 2^{315} M^{315}$$

$$= 2^{315}(3^{315} A + 2^{315} B)$$

$$= 2^{315}(\ (3^{315} - 2^{315})A + 2^{315} A + 3^{315} B\ )$$

$$= 2^{315}(\ (3^{315} - 2^{315})A + 2^{315}(A + B)\ )$$

$$= 2^{315}(\ (3^{315} - 2^{315})A + 2^{315} I\ )$$

$$= 2^{315} \begin{pmatrix} 2^{315} & 3^{315} - 2^{315} \\ 0 & 3^{315} \end{pmatrix}$$