1995-AL-P-MATH-1-Q09

9(a)

Consider (S),

$$\Delta = \begin{vmatrix} 2 & 2 & -1 \\ h & -3 & -1 \\ -3 & h & 1 \end{vmatrix}$$

$$\begin{aligned} &= 2(-3)(1) + 2(-1)(-3) + (-1)h^2 - 2(-1)h - 2h + (-3)(-3) \\ &= -6 + 6 - h^2 + 2h - 2h + 9 \\ &= 9 - h^2 \end{aligned}$$

(1) when (S) has a unique solution
$\Rightarrow \Delta \neq 0$
$\Rightarrow 9 - h^2 \neq 0$
$\Rightarrow h^2 \neq 9$

(2) when $h^2 \neq 9$
$\Rightarrow 9 - h^2 \neq 0$
$\Rightarrow \Delta \neq 0$
$\Rightarrow$ (S) has a unique solution

From (1) and (2), (S) has a unique solution if and only if $h^2 \neq 9$

Consider,

$$\Delta_x = \begin{vmatrix} k & 2 & -1 \\ 0 & -3 & -1 \\ 0 & h & 1 \end{vmatrix} = k(-3 + h) = k(h - 3)$$

$$\Delta_y = \begin{vmatrix} 2 & k & -1 \\ h & 0 & -1 \\ -3 & 0 & 1 \end{vmatrix} = -k(h - 3)$$

$$\Delta_z = \begin{vmatrix} 2 & 2 & k \\ h & -3 & 0 \\ -3 & h & 0 \end{vmatrix} = k(h^2 - 9)$$

Then,

$$x = \frac{\Delta_x}{\Delta} = \frac{k(h-3)}{9-h^2} = -\frac{k}{h+3}$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-k(h-3)}{9-h^2} = \frac{k}{h+3}$$

$$z = \frac{\Delta_z}{\Delta} = \frac{k(h^2-9)}{9-h^2} = -k$$

9(b)(i)

When h = 3, (S) becomes:
2x + 2y - z = k
3x - 3y - z = 0

$\Rightarrow y = \dfrac{x+k}{5}$ and $z = \dfrac{3(4x-k)}{5}$ for any $k \in R$

$\Rightarrow$k can be any real number and solutions are $(t,\ \dfrac{t+k}{5},\ \dfrac{3(4t-k)}{5})$ for any $t \in R$

9(b)(ii)

When h = -3, (S) becomes:
2x + 2y - z = k
-3x - 3y - z = 0
-3x - 3y + z = 0

$\Rightarrow y = -x,\ z = 0,\ k = 0$
$\Rightarrow k = 0$ and solutions are $(t,\ -t,\ 0)$ for any $t \in R$

9(c)

The top 3 equations in (T) are actually (S) where $k = \dfrac{2}{3}$.
To ensure (T) has solutions , (S) must have solutions first of all.

Consider result 9(b)(ii), when h = -3, k must be 0.
Therefore, $h \neq -3$

Consider result 9(a), when $h \neq 3$

$$x = -\frac{k}{h+3} = -\frac{2}{3(h+3)}$$

$$y = \frac{k}{h+3} = \frac{2}{3(h+3)}$$

$$z = -k = -\frac{2}{3}$$

Put them to the last equation in (T),
L.H.S.

$$= -5x - 2y + 6z$$

$$= 5 \cdot \frac{2}{3(h+3)} - 2 \cdot \frac{2}{3(h+3)} - 6 \cdot \frac{2}{3}$$

$$= \frac{10 - 4 - 12h - 36}{3(h+3)}$$

$$= \frac{-12h - 30}{3(h+3)}$$

$$= \frac{-4h-10}{h+3}$$

To make L.H.S. = h, we must have:

$-4h-10 = h(h+3)$

$\Rightarrow h^2+7h+10=0$

$\Rightarrow (h+2)(h+5)=0$

$\Rightarrow h=-2$ or $h=-5$

Therefore, (T) has solution

when $h=-2,\ x=-\frac{2}{3},\ y=\frac{2}{3}, z=-\frac{2}{3}$

or

when $h=-5,\ x=\frac{1}{3},\ y=-\frac{1}{3}, z=-\frac{2}{3}$

From result 9(b)(i), when h = 3,

$x=t,\ y=\frac{t+k}{5},\ z=\frac{3(4t-k)}{5}$ for any $t \in R$

$\Rightarrow x=t,\ y=\frac{3t+2}{15},\ z=\frac{12t-2}{5}$ for any $t \in R$

Put them to the last equation in (T),

L.H.S.

$= -5x-2y+6z$

$= -5t-2\cdot\frac{3t+2}{15}+6\cdot\frac{12t-2}{5}$

$= \frac{-75t-6t-4+216t-36}{15}$

$= \frac{135t-40}{15} = 9t-\frac{8}{3}$

if $t=\frac{17}{27}$ then L.H.S. = 3 = h. resulting in (T) having a solution.

Hence, (T) has a solution when h = 3, $x=\frac{17}{27},\ y=\frac{7}{27},\ z=\frac{10}{9}$