1994-AL-P-MATH-1-Q09

9(a)(i)

If (I) has a unique solution
⇒ This unique solution is the trivial solution (x, y, z) = (0, 0, 0)
⇒ The trivial solution (x, y, z) = (0, 0, 0) is the unique solution of (I)

Also, (II) is a particular case of (I) where z=1
⇒ (II) has NO solution otherwise (I) has a non-trivial solution where z=1.

9(a)(ii)

(1) When (u, v) is a solution of (II)

$$\Rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow t \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ for any real number t}$$

$$\Rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} ut \\ vt \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ for any real number t}$$

⇒ (ut, vt, t) are solutions of (I) for any real number t.

(2) When (ut, vt, t) are solutions of (I) for any real number t,

$$\Rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} ut \\ vt \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow t \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ (}\because\text{ t can be any real number otherwise t must be 0 only)}$$

⇒ (u, v) is a solution of (II)

From results (1) and (2), (u, v) is a solution of (II) if and only if (ut, vt, t) are solutions of (I) for all real number t.

9(a)(iii)

Let (u, v, w) be a non-trivial solution of (I).

$\Rightarrow$ w must be 0 otherwise $(\frac{u}{w}, \frac{v}{w}, 1)$ is a solution of (I) and (II) has a solution $(\frac{u}{w}, \frac{v}{w})$

(I) becomes :
$a_{11}x + a_{21}y = 0 \quad --- L1$
$a_{21}x + a_{22}y = 0 \quad --- L2$
$a_{31}x + a_{32}y = 0 \quad --- L3$

They are equations of 3 lines which must all represent the same single line otherwise they intercept at only (0,0) and does NOT have non-trivial solutions.

Therefore, the solutions of (I) is actually the solutions of any L1 or L2 or L3.

9(b)(i)

For (III),
$$\Delta = \begin{vmatrix} -(3+k) & 1 & -1 \\ -7 & 5-k & -z \\ -6 & 6 & k-2 \end{vmatrix}$$

$$\begin{aligned}
&= -(3+k)(5-k)(k-2) + 1(-1)(-6) + (-1)(-7)(6) + (3+k)(-1)(6) - (1)(-7)(k-2) + (5-k)(-6) \\
&= (k+3)(k-5)(k-2) + 6 + 42 - 6(k+3) + 7(k-2) + 6(k-5) \\
&= (k+3)(k^2 - 7k + 10) + 48 - 6k - 18 + 7k - 14 + 6k - 30 \\
&= (k^3 - 7k^2 + 10k + 3k^2 - 21k + 30) - 14 + 7k \\
&= k^3 - 4k^2 - 4k + 16 \\
&= k^2(k-4) - 4(k-4) \\
&= (k^2 - 4)(k-4) \\
&= (k+2)(k-2)(k-4)
\end{aligned}$$

When (III) has non-trivial solutions, $\Delta = 0$
$\Rightarrow (k+2)(k-2)(k-4) = 0$
$\Rightarrow k = -2$ or $k = 2$ or $k = 4$

When k = -2, (III) becomes:
-x + y - z =0
-7x + 7y - z = 0
-6x + 6y - 4z = 0

Consider the augmented Matrix
$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ -7 & 7 & -1 & 0 \\ -6 & 6 & -4 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -7 & 7 & -1 & 0 \\ -6 & 6 & -4 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$\Rightarrow x = y$ and $z = 0$

$\Rightarrow$ the solution is $(t, t, 0)$ for any $t \in R$

When k = 2, (III) becomes:
-5x + y - z =0
-7x + 3y - z = 0
-6x + 6y = 0

$\Rightarrow x = y$ and $x = -\frac{z}{4}$

$\Rightarrow$ the solution is $(t,\ t,\ -4t)$ for any $t \in R$

When k = 4, (III) becomes:
-7x + y - z = 0
-3x + 3y + z = 0

$\Rightarrow y = \frac{5}{2}x$ and $z = -\frac{9}{2}x$

$\Rightarrow$ the solution is $(t,\ \frac{5}{2}t,\ -\frac{9}{2}t)$ for any $t \in R$

9(b)(ii)

Note that (IV) is a particular case of (III) where z=1.

When (IV) is consistent,
$\Rightarrow$ the solutions of (IV) are also solutions of (III) where z=1
$\Rightarrow$ (k=2 when $t = -\frac{1}{4}$) or (k=4 when $t = -\frac{2}{9}$)
$\Rightarrow$ (k=2 and solution is $x = -\frac{1}{4},\ y = -\frac{1}{4}$) or (k=4 and solution is $x = -\frac{2}{9},\ y = -\frac{5}{9}$)

9(b)(iii)

According to results in 9(a)(i),

When k = -2, the solution is $(t, t, 0)$ for any $t \in R$
When k = 2, the solution is $(t,\ t,\ -4t)$ for any $t \in R$
When k = 4, the solution is $(t,\ \frac{5}{2}t,\ -\frac{9}{2}t)$ for any $t \in R$