2021-DSE-MATH-EP(M2)-Q11

11(a)

Note that $P = \begin{pmatrix} sin\theta & cos\theta \\ -cos\theta & sin\theta \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} sin\theta & -cos\theta \\ cos\theta & sin\theta \end{pmatrix}$

$$PAP^{-1}$$
$$= \begin{pmatrix} sin\theta & cos\theta \\ -cos\theta & sin\theta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} sin\theta & -cos\theta \\ cos\theta & sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \alpha sin\theta + \beta cos\theta & \beta sin\theta - \alpha cos\theta \\ -\alpha cos\theta + \beta sin\theta & -\beta cos\theta - \alpha sin\theta \end{pmatrix} \begin{pmatrix} sin\theta & -cos\theta \\ cos\theta & sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \alpha sin^2\theta + \beta sin\theta cos\theta + \beta sin\theta cos\theta - \alpha cos^2\theta & -\alpha sin\theta cos\theta - \beta cos^2\theta + \beta sin^2\theta - \alpha sin\theta cos\theta \\ -\alpha sin\theta cos\theta + \beta sin^2\theta - \beta cos^2\theta - \alpha sin\theta cos\theta & \alpha cos^2\theta - \beta sin\theta cos\theta - \beta sin\theta cos\theta - \alpha sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(sin^2\theta - cos^2\theta) + \beta(sin\theta cos\theta + sin\theta cos\theta) & -\alpha(sin\theta cos\theta + sin\theta cos\theta) - \beta(cos^2\theta - sin^2\theta) \\ -\alpha(sin\theta cos\theta + sin\theta cos\theta) + \beta(sin^2\theta - cos^2\theta) & \alpha(cos^2\theta - sin^2\theta) - \beta(sin\theta cos\theta + sin\theta cos\theta) \end{pmatrix}$$

$$= \begin{pmatrix} -\alpha(cos^2\theta - sin^2\theta) + \beta(2sin\theta cos\theta) & -\alpha(2sin\theta cos\theta) - \beta(cos^2\theta - sin^2\theta) \\ -\alpha(2sin\theta cos\theta) - \beta(cos^2\theta - sin^2\theta) & \alpha(cos^2\theta - sin^2\theta) - \beta(2sin\theta cos\theta) \end{pmatrix}$$

$$= \begin{pmatrix} -\alpha\ cos2\theta + \beta\ sin2\theta & -\alpha\ sin2\theta - \beta\ cos2\theta \\ -\alpha\ sin2\theta - \beta\ cos2\theta & \alpha\ cos2\theta - \beta\ sin2\theta \end{pmatrix}$$

$$= \begin{pmatrix} -\alpha\ cos2\theta + \beta\ sin2\theta & -\beta\ cos2\theta - \alpha\ sin2\theta \\ -\beta\ cos2\theta - \alpha\ sin2\theta & \alpha\ cos2\theta - \beta\ sin2\theta \end{pmatrix}$$

11(b)(i)

Let $\alpha = 1$, $\beta = \sqrt{3}$, then $A = B$.

$$PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

$$\Rightarrow PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\alpha\ cos2\theta + \beta\ sin2\theta & -\beta\ cos2\theta - \alpha\ sin2\theta \\ -\beta\ cos2\theta - \alpha\ sin2\theta & \alpha\ cos2\theta - \beta\ sin2\theta \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

$$\Rightarrow -\beta\ cos2\theta - \alpha\ sin2\theta = 0$$
$$\Rightarrow \alpha\ sin2\theta = -\beta\ cos2\theta$$

$$\Rightarrow tan2\theta = -\frac{\beta}{\alpha} = -\sqrt{3}$$

$$\Rightarrow 2\theta = 2\pi - \frac{\pi}{3}\ (\because \pi < 2\theta < 2\pi)$$

$$\Rightarrow \theta = \frac{5\pi}{6}$$

11(b)(ii)

$\lambda = -\alpha\, cos2\theta + \beta\, sin2\theta$ and $\mu = \alpha\, cos2\theta - \beta\, sin2\theta$

$\Rightarrow \lambda = -cos(\frac{5\pi}{3}) + \sqrt{3}\, sin(\frac{5\pi}{3})$ and $\mu = cos(\frac{5\pi}{3}) - \sqrt{3}\, sin(\frac{5\pi}{3})$

$\Rightarrow \lambda = -\frac{1}{2} + \sqrt{3}\,(-\frac{\sqrt{3}}{2})$ and $\mu = \frac{1}{2} - \sqrt{3}\,(-\frac{\sqrt{3}}{2})$

$\Rightarrow \lambda = -2$ and $\mu = 2$

Also $P = \begin{pmatrix} sin\theta & cos\theta \\ -cos\theta & sin\theta \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} sin\theta & -cos\theta \\ cos\theta & sin\theta \end{pmatrix}$

$\Rightarrow P = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$ and $P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$

Therefore,

$B = P^{-1}\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}P$

$\Rightarrow B^n = P^{-1}\begin{pmatrix} (-2)^n & 0 \\ 0 & 2^n \end{pmatrix}P$

$= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} (-2)^n & 0 \\ 0 & 2^n \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} (-2)^n & 0 \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

$= \frac{2^n}{4}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

$= 2^{n-2}\begin{pmatrix} (-1)^n & \sqrt{3} \\ -\sqrt{3}(-1)^n & 1 \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

$= 2^{n-2}\begin{pmatrix} (-1)^n + 3 & -\sqrt{3}(-1)^n + \sqrt{3} \\ -\sqrt{3}(-1)^n + \sqrt{3} & 3(-1)^n + 1 \end{pmatrix}$

$= 2^{n-2}\begin{pmatrix} (-1)^n + 3 & \sqrt{3}(-1)^{n+1} + \sqrt{3} \\ \sqrt{3}(-1)^{n+1} + \sqrt{3} & 3(-1)^n + 1 \end{pmatrix}$

11(b)(iii)

$B^{555} = 2^{555-2}\begin{pmatrix} (-1)^{555} + 3 & \sqrt{3}(-1)^{555+1} + \sqrt{3} \\ \sqrt{3}(-1)^{555+1} + \sqrt{3} & 3(-1)^{555} + 1 \end{pmatrix}$

$B^{555} = 2^{553}\begin{pmatrix} 2 & 2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix}$

$B^{555} = 2^{554}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

$\Rightarrow (B^{555})^{-1} = -2^{-556}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$

$$\Rightarrow (B^{-1})^{555} = -2^{-556}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

$$\Rightarrow (B^{-1})^{555} = 2^{-556}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$