1995-AL-P-MATH-1-Q01

1995-AL-P MATH 1 #01(a)

Let $P(n)$ be the statement that $A^n = \begin{pmatrix} a^n & \frac{a^n-b^n}{a-b} \\ 0 & b^n \end{pmatrix}$ for all positive integers n

For n = 1,
L.H.S = $A = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$
R.H.S = $\begin{pmatrix} a^1 & \frac{a^1-b^1}{a-b} \\ 0 & b^1 \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$
$\Rightarrow$ L.H.S = R.H.S.
$\Rightarrow P(1)$ is true

Assume that $P(k)$ is true for some integer $k \geq 1$, then
$A^{k+1} = A^k A$

$$\Rightarrow A^{k+1} = \begin{pmatrix} a^k & \frac{a^k-b^k}{a-b} \\ 0 & b^k \end{pmatrix} \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} a^{k+1} & a^k + \frac{a^k-b^k}{a-b} \cdot b \\ 0 & b^{k+1} \end{pmatrix}$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} a^{k+1} & \frac{a^k(a-b)+a^k \cdot b-b^{k+1}}{a-b} \\ 0 & b^{k+1} \end{pmatrix}$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} a^{k+1} & \frac{a^{k+1}-a^k \cdot b+a^k \cdot b-b^{k+1}}{a-b} \\ 0 & b^{k+1} \end{pmatrix}$$

$$\Rightarrow A^{k+1} = \begin{pmatrix} a^{k+1} & \frac{a^{k+1}-b^{k+1}}{a-b} \\ 0 & b^{k+1} \end{pmatrix}$$

$\Rightarrow P(k+1)$ is true
Therefore, $P(n)$ is true by mathematical induction.

1995-AL-P MATH 1 #01(b)

Let $a = \frac{1}{2}$, and $b = \frac{3}{2}$

Then $A = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{3}{2} \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = 2A$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{95} = (2A)^{95}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{95} = 2^{95} \cdot A^{95}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{95} = 2^{95} \cdot \begin{pmatrix} (\frac{1}{2})^{95} & \frac{(\frac{1}{2})^{95} - (\frac{3}{2})^{95}}{\frac{1}{2} - \frac{3}{2}} \\ 0 & (\frac{3}{2})^{95} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{95} = 2^{95} \cdot \begin{pmatrix} \frac{1}{2^{95}} & \frac{3^{95}-1}{2^{95}} \\ 0 & (\frac{3}{2})^{95} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{95} = \begin{pmatrix} 1 & 3^{95} - 1 \\ 0 & 3^{95} \end{pmatrix}$$