2017-DSE-MATH-EP(M2)-Q12

12(a)

Let P(n) be the statement that $A^n = 3^n I + 3^{n-1} n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for all positive integers n.

When n = 1,
R.H.S.
$= 3^n I + 3^{n-1} n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$= 3I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$
$= A = A^1 = L.H.S.$
Therefore, P(1) is true.

Assume P(k) is true for some positive integer $k \geq 1$. Then
A^{k+1}
$= A^k A$
$= [\ 3^k I + 3^{k-1} k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}]\ A$

$= 3^k A + 3^{k-1} k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A$

$= \ 3^k \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} + 3^{k-1} k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

$= 3^k \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} + 3^{k-1} k \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$

$= 3^k \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + 3^k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 3^{k-1} k \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$

$= 3^{k+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3^k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 3^k k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$= 3^{k+1} I + 3^k \begin{pmatrix} 0 & k+1 \\ 0 & 0 \end{pmatrix}$

$= 3^{k+1} I + 3^k (k+1) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$= 3^{k+1} I + 3^{(k+1)-1} (k+1) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
Therefore, P(k+1) is true and by mathematical induction P(n) is true.

12(b)(i)

$P^{-1}BP$
$= P^{-1}\begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$

$= P^{-1}\begin{pmatrix} -3 & -1 \\ 6 & -1 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} -3 & -1 \\ 6 & -1 \end{pmatrix}$

$= \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

$= A$

12(b)(ii)

$P^{-1}BP = A$
$\Rightarrow B = PAP^{-1}$
$\Rightarrow B^n = PA^nP^{-1}$
$\Rightarrow B^n = P\ [\ 3^nI + 3^{n-1}n\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ ]\ P^{-1}$

$\Rightarrow B^n = PA^nP^{-1}$

$\Rightarrow B^n = 3^nPIP^{-1} + 3^{n-1}nP\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ P^{-1}$

$\Rightarrow B^n = 3^nI + 3^{n-1}n\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$

$\Rightarrow B^n = 3^nI + 3^{n-1}n\begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}$

$\Rightarrow B^n = 3^nI + 3^{n-1}n\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$

12(b)(iii)

For all positive integers m, $A^m - B^m$

$= 3^mI + 3^{m-1}m\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - 3^mI - 3^{m-1}m\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$

$= 3^{m-1}m\ [\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\ ]$

$= 3^{m-1}m\begin{pmatrix} -2 & 0 \\ 4 & 2 \end{pmatrix}$

Therefore,
$|\,A^m - B^m\,|$

$= 3^{m-1}m\begin{vmatrix} -2 & 0 \\ 4 & 2 \end{vmatrix}$

$= -4 \cdot (3^{m-1}m)^2$

$$= -4 \cdot 3^{2m-2} m^2$$

$$= -3^{2m-2}(4m^2) < 0 < 4m^2$$

Therefore, $|A^m - B^m| \neq 4m^2$