1997-AL-P-MATH-1-Q08

8(a)

(S) has infinitely many solutions
$\Rightarrow \Delta = 0$

$$\Rightarrow \begin{vmatrix} a+1 & 2 & -2 \\ 1 & a & 2 \\ 3 & -1 & a-7 \end{vmatrix} = 0$$

$\Rightarrow (a+1)[a(a-7)+2] - 2(a-7-6) - 2(-1-3a) = 0$
$\Rightarrow (a+1)(a^2-7a+2) - 2a + 26 + 2 + 6a = 0$
$\Rightarrow a^3 - 7a^2 + 2a + a^2 - 7a + 2 + 4a + 28 = 0$
$\Rightarrow a^3 - 6a^2 - a + 30 = 0$
$\Rightarrow (a+2)(a^2-8a+15) = 0$
$\Rightarrow (a+2)(a-3)(a-5) = 0$
$\Rightarrow a = -2$ or $a = 3$ or $a = 5$

When a=-2,
(S) : $\begin{cases} x - 2y + 2z = 0 \\ 3x - y - 9z = 0 \end{cases}$

Consider augmented matrix,
$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 3 & -1 & -9 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 5 & -15 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$\Rightarrow$ Solutions are $x = 4t$, $y = 3t$, $z = t \in R$

When a=3,
(S) : $\begin{cases} 4x - 2y + 2z = 0 \\ x + 3y + 2z = 0 \end{cases}$

Consider augmented matrix,
$$\Rightarrow \left[\begin{array}{ccc|c} 4 & -2 & 2 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & -14 & -6 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 7 & 3 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & \frac{3}{7} & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{7} & 0 \\ 0 & 1 & \frac{3}{7} & 0 \end{array}\right]$$

$\Rightarrow$ Solutions are $x = -\dfrac{5}{7}t,\ y = -\dfrac{3}{7}t,\ z = t \in R$

When a=5,

(S) : $\begin{cases} 6x + 2y - 2z = 0 \\ x + 5y + 2z = 0 \\ 3x - y - 2z = 0 \end{cases}$

Consider augmented matrix,

$$\left[\begin{array}{ccc|c} 6 & 2 & -2 & 0 \\ 1 & 5 & 2 & 0 \\ 3 & -1 & -2 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 3 & -1 & -2 & 0 \\ 6 & 2 & -2 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 3 & -1 & -2 & 0 \\ 3 & 1 & -1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 3 & -1 & -2 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 0 & -16 & -8 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right]$$

$\Rightarrow$ Solutions are $x = \dfrac{1}{2}t,\ y = -\dfrac{1}{2}t,\ z = t \in R$

8(b)

The smallest value of a is -2.

(T) : $\begin{cases} -x+2y-2z=6 \\ x-2y+2z=5b-1 \\ 3x-y-9z=1-b \end{cases}$

(T) is consistent.

$\Rightarrow \Delta_x = 0$ and $\Delta_y = 0$ and $\Delta_z = 0$

$$\Rightarrow \begin{vmatrix} 6 & 2 & -2 \\ 5b-1 & -2 & 2 \\ 1-b & -1 & -9 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} -1 & 6 & -2 \\ 1 & 5b-1 & 2 \\ 3 & 1-b & -9 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} -1 & 2 & 6 \\ 1 & -2 & 5b-1 \\ 3 & -1 & 1-b \end{vmatrix} = 0$$

$\Rightarrow 108+4(1-b)+2(5b-1)+12+18(5b-1)-4(1-b)=0$ and
$9(5b-1)+36-2(1-b)+2(1-b)+54+6(5b-1)=0$ and
$2(1-b)+6(5b-1)-6-(5b-1)-2(1-b)+36=0$

$\Rightarrow 120+4-4b+10b-2+90b-18-4+4b=0$ and $45b-9+90-2+2b+2-2b+30b-6=0$ and
$2-2b+30b-6-5b+1-2+2b+30=0$

$\Rightarrow 100+100b=0$ and $75b+75=0$ and $25b+25=0$

$\Rightarrow b=-1$

When a = -2 , b = -1,

(T) : $\begin{cases} -x+2y-2z=6 \\ x-2y+2z=-6 \\ 3x-y-9z=2 \end{cases}$

$\Rightarrow$ (T) : $\begin{cases} x-2y+2z=-6 \\ 3x-y-9z=2 \end{cases}$

Consider augmented matrix,

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & -6 \\ 3 & -1 & -9 & 2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -6 \\ 0 & 5 & -15 & 20 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 2 & -6 \\ 0 & 1 & -3 & 4 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -4 & 2 \\ 0 & 1 & -3 & 4 \end{array}\right]$$

$\Rightarrow$ Solutions are $x=2+4t,\ y=4+3t,\ z=t\in R$

8(c)

Using result in (b), the solution of the top 3 equations are :
$x=2+4t,\ y=4+3t,\ z=\sqrt{t}$ where t is any non-negative real number

To satisfy the fourth equation ,

$3x - 4y - z = -11$

$\Rightarrow 3(2 + 4t) - 4(4 + 3t) - \sqrt{t} = -11$

$\Rightarrow 6 + 12t - 16 - 12t - \sqrt{t} = -11$

$\Rightarrow \sqrt{t} = 1$

$\Rightarrow t = 1$

$\Rightarrow x = 6,\ y = 7,\ z = 1$