1999-AL-P-MATH-1-Q09

9(a)

Let P(n) be the statement that $(A+B)^n = A^n + B^n$ for any positive integer n

When n = 1,
L.H.S. = $(A+B)^1 = A+B$
R.H.S. = $A^1 + B^1 = A+B$
$\Rightarrow$ L.H.S. = R.H.S.
Therefore P(1) is true.

Assume P(k) is true for some integer k $\geq$ 1, then,
$(A+B)^{k+1} = (A+B)^k(A+B)$
$\Rightarrow (A+B)^{k+1} = (A^k+B^k)(A+B)$
$\Rightarrow (A+B)^{k+1} = A^{k+1} + A^kB + B^kA + B^{k+1}$
$\Rightarrow (A+B)^{k+1} = A^{k+1} + A^{k-1}AB + B^{k-1}BA + B^{k+1}$
$\Rightarrow (A+B)^{k+1} = A^{k+1} + A^{k-1}0 + B^{k-1}0 + B^{k+1}$
$\Rightarrow (A+B)^{k+1} = A^{k+1} + B^{k+1}$
Therefore, P(k+1) is true.

By mathematical induction, P(n) is true

9(b)

Note that :
$AB = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ 0 & 0 \end{pmatrix}$ and

$BA = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ap & bp \\ ar & br \end{pmatrix}$

if $AB = 0$ and $BA = 0$
$\Rightarrow \begin{pmatrix} ap+br & aq+bs \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} ap & bp \\ ar & br \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ aq+bs=0 and ap=0 and ar=0
$\Rightarrow$ aq+bs=0 and p=0 and r=0 ($\because a \neq 0$)

On the other hand, if aq+bs=0 and p=0 and r=0,

$\Rightarrow AB = \begin{pmatrix} ap+br & aq+bs \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a\cdot 0 + b\cdot 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and
$BA = \begin{pmatrix} ap & bp \\ ar & br \end{pmatrix} = \begin{pmatrix} a\cdot 0 & b\cdot 0 \\ a\cdot 0 & b\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow AB = BA = 0$

9(c)

Let a = x , $b = \frac{xy}{x-z}$, p = r = 0 , $q = -\frac{yz}{x-z}$ and s = z so that

$aq + bs = x \cdot (-\frac{yz}{x-z}) + \frac{xy}{x-z} \cdot z = 0$ and

$b + q = \frac{xy}{x-z} - \frac{yz}{x-z} = \frac{y(x-z)}{x-z} = y$

Now let

$D = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ and $E = \begin{pmatrix} 0 & q \\ 0 & s \end{pmatrix}$

$\Rightarrow D = \begin{pmatrix} x & \frac{xy}{x-z} \\ 0 & 0 \end{pmatrix}$ and $E = \begin{pmatrix} 0 & -\frac{yz}{x-z} \\ 0 & z \end{pmatrix}$

$\Rightarrow C = D + E$ and $DE = ED = 0$ (by 9(b))

9(d)

Let x=2, y=5, z=1, and also let

$C = \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 10 \\ 0 & 0 \end{pmatrix}$, $E = \begin{pmatrix} 0 & -5 \\ 0 & 1 \end{pmatrix}$ so that C = D + E and DE=ED=0

$$\Rightarrow \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^n = C^n = (D+E)^n$$

$$\Rightarrow \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^n = (D+E)^n = D^n + E^n \text{ (by 9(a))}$$

$$\Rightarrow \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 2 & 10 \\ 0 & 0 \end{pmatrix}^n + \begin{pmatrix} 0 & -5 \\ 0 & 1 \end{pmatrix}^n$$

$$\Rightarrow \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 5 \cdot 2^n \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -5 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 5 \cdot (2^n - 1) \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}^{99} = \begin{pmatrix} 2^{99} & 5 \cdot (2^{99} - 1) \\ 0 & 1 \end{pmatrix}$$