2000-AL-P-MATH-1-Q08

8(a)

(S) has an unique solution.
$\Rightarrow \Delta \neq 0$

$$\Rightarrow \begin{vmatrix} 1 & -1 & -1 \\ 2 & \lambda & -2 \\ 1 & 2\lambda+3 & \lambda^2 \end{vmatrix} \neq 0$$

$\Rightarrow \lambda^3 + 2(2\lambda+3) + 2\lambda^2 + 2 - 2(2\lambda+3) + \lambda \neq 0$
$\Rightarrow \lambda^3 + 2\lambda^2 + \lambda + 2 \neq 0$
$\Rightarrow (\lambda^2+1)(\lambda+2) \neq 0$
$\Rightarrow \lambda + 2 \neq 0$
$\Rightarrow \lambda \neq -2$

On the other hand,
$\lambda \neq -2$
$\Rightarrow \Delta = (\lambda^2+1)(\lambda+2) \neq 0$
$\Rightarrow$ (S) has an unique solution.

Hence, (S) has an unique solution if and only if $\lambda \neq -2$.

When $\lambda = -1$, then

$$\Delta = (\lambda^2+1)(\lambda+2) \text{ and } \Delta_x = \begin{vmatrix} a & -1 & -1 \\ b & \lambda & -2 \\ c & 2\lambda+3 & \lambda^2 \end{vmatrix} \text{ and } \Delta_y = \begin{vmatrix} 1 & a & -1 \\ 2 & b & -2 \\ 1 & c+3 & \lambda^2 \end{vmatrix} \text{ and } \Delta_z = \begin{vmatrix} 1 & -1 & a \\ 2 & \lambda & b \\ 1 & 2\lambda+3 & c \end{vmatrix}$$

$$\Rightarrow \Delta = 2 \text{ and } \Delta_x = \begin{vmatrix} a & -1 & -1 \\ b & -1 & -2 \\ c & 1 & 1 \end{vmatrix} \text{ and } \Delta_y = \begin{vmatrix} 1 & a & -1 \\ 2 & b & -2 \\ 1 & c+3 & 1 \end{vmatrix} \text{ and } \Delta_z = \begin{vmatrix} 1 & -1 & a \\ 2 & -1 & b \\ 1 & 1 & c \end{vmatrix}$$

$\Rightarrow \Delta = 2$ and $\Delta_x = a + c$ and $\Delta_y = -4a + 2b$ and $\Delta_z = 3a - 2b + c$

$$\Rightarrow x = \frac{\Delta_x}{\Delta} \text{ and } y = \frac{\Delta_y}{\Delta} \text{ and } z = \frac{\Delta_z}{\Delta}$$

$$\Rightarrow x = \frac{a+c}{2} \text{ and } y = \frac{-4a+2b}{2} \text{ and } z = \frac{3a-2b+c}{2}$$

$$\Rightarrow x = \frac{a+c}{2} \text{ and } y = -2a + b \text{ and } z = \frac{3a-2b+c}{2}$$

8(b)(i)

When $\lambda = -2$ (i.e. $\Delta = 0$) and (S) has infinitely many solutions
$\Rightarrow \Delta_x = \Delta_y = \Delta_z = 0$

$$\Rightarrow \begin{vmatrix} a & -1 & -1 \\ b & \lambda & -2 \\ c & 2\lambda+3 & \lambda^2 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} 1 & a & -1 \\ 2 & b & -2 \\ 1 & c+3 & \lambda^2 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} 1 & -1 & a \\ 2 & \lambda & b \\ 1 & 2\lambda+3 & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & -1 & -1 \\ b & -2 & -2 \\ c & -1 & 4 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} 1 & a & -1 \\ 2 & b & -2 \\ 1 & c+3 & 4 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} 1 & -1 & a \\ 2 & -2 & b \\ 1 & -1 & c \end{vmatrix} = 0$$

$\Rightarrow -10a + 5b = 0$ and $-10a + 5b = 0$ and $0 = 0$
$\Rightarrow -10a + 5b = 0$
$\Rightarrow b = 2a$

8(b)(ii)

When $\lambda = -2$, a=-1, b=-2, c=3 (c = -1 is wrong)

(S) $\begin{cases} x - y - z = -1 \\ 2x - 2y - 2z = -2 \\ x - y + 4z = 3 \end{cases}$

$\Rightarrow$ (S) $\begin{cases} x - y - z = -1 \\ x - y + 4z = 3 \end{cases}$

$\Rightarrow$ (S) $\begin{cases} x - y - z = -1 \\ 5z = 4 \end{cases}$

$\Rightarrow$ (S) $\begin{cases} x - y = -\dfrac{1}{5} \\ z = \dfrac{4}{5} \end{cases}$

$\Rightarrow$ The solutions are $x = t - \dfrac{1}{5}$, $y = t \in R$, $z = \dfrac{4}{5}$

8(c)

(T) $\begin{cases} x - y - z + 3\mu - 5 = 0 \\ 2x - 2y - 2z + 2\mu - 2 = 0 \\ x - y + 4z - \mu - 1 = 0 \end{cases}$

$\Rightarrow$ (T) $\begin{cases} x - y - z = 5 - 3\mu \\ 2x - 2y - 2z = 2 - 2\mu \\ x - y + 4z = \mu + 1 \end{cases}$

(T) is equivalent to (S) where $\lambda = -2$, $a = 5 - 3\mu$, $b = 2 - 2\mu$ and $c = \mu + 1$

According to (b)(i), (T) is consistent when
$b = 2a$
$\Rightarrow 2 - 2\mu = 2(5 - 3\mu)$
$\Rightarrow \mu = 2$
$\Rightarrow a = 5 - 3\mu$, $b = 2 - 2\mu$ and $c = \mu + 1$
$\Rightarrow a = -1$, $b = -2$ and $c = 3$
According (b)(ii), the solutions are $x = t - \dfrac{1}{5}$, $y = t \in R$, $z = \dfrac{4}{5}$

On the other hand, when $\mu \neq 2$, (T) is inconsistent.