1994-AL-P-MATH-1-Q02

$$\Delta = \begin{vmatrix} 4-\lambda & 3 & 1 \\ 3 & -4-\lambda & 7 \\ 1 & 7 & -6-\lambda \end{vmatrix}$$

$$\begin{aligned}
&= (4-\lambda)(-4-\lambda)(-6-\lambda) + 21 + 21 - 49(4-\lambda) - 9(-6-\lambda) - (-4-\lambda) \\
&= (4-\lambda)(4+\lambda)(6+\lambda) + 42 - 49(4-\lambda) + 9(6+\lambda) + (4+\lambda) \\
&= (16-\lambda^2)(6+\lambda) + 42 - 196 + 49\lambda + 54 + 9\lambda + 4 + \lambda \\
&= (-\lambda^3 - 6\lambda^2 + 16\lambda + 96) - 96 + 59\lambda \\
&= -\lambda^3 - 6\lambda^2 + 75\lambda \\
&= -\lambda(\lambda^2 + 6\lambda - 75) \\
&= -\lambda(\lambda + 3 + 2\sqrt{21})(\lambda + 3 - 2\sqrt{21})
\end{aligned}$$

(*) has nontrivial solutions

$\Rightarrow \Delta = 0$

$\Rightarrow -\lambda(\lambda + 3 + 2\sqrt{21})(\lambda + 3 - 2\sqrt{21}) = 0$

$\Rightarrow \lambda = 0$ or $\lambda = -3 - 2\sqrt{21}$ or $\lambda = -3 + 2\sqrt{21}$

$\Rightarrow \lambda = 0$ ($\because \lambda$ is an integer)

When $\lambda = 0$, we have x + 7y - 6z = 0 and 3x - 4 + 7z = 0,

Consider the augmented matrix :

$$\left[\begin{array}{ccc|c} 1 & 7 & -6 & 0 \\ 3 & -4 & 7 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 7 & -6 & 0 \\ 0 & -25 & 25 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 7 & -6 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

Therefore, the solution is,

y = t which is any real number

z = t

x = 6z - 7y = -t