1996-AL-P-MATH-1-Q09

9(a)

Consider augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & 1 & 2 & 4 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 3 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

$\Rightarrow x = -5t + 5, y = 3t - 1, z = t$ for any $t \in R$

9(b)

xy + yz + zx = 2
$\Rightarrow (-5t+5)(3t-1) + (3t-1)t + t(-5t+5) = 2$
$\Rightarrow -15t^2 + 20t - 5 + 3t^2 - t - 5t^2 + 5t = 2$
$\Rightarrow -17t^2 + 24t - 7 = 0$
$\Rightarrow 17t^2 - 24t + 7 = 0$
$\Rightarrow (17t - 7)(t - 1) = 0$
$\Rightarrow t = \dfrac{7}{17}$ or $t = 1$

$\Rightarrow x = \dfrac{50}{17},\ y = \dfrac{4}{17},\ z = \dfrac{7}{17}$
or
$x = 0,\ y = 2,\ z = 1$

9(c)

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ a & 1 & 1 \end{vmatrix} = 1 + 4a - 1 - 2 - 2 + a = 5a - 4$$

$$\Delta_x = \begin{vmatrix} 3 & 2 & -1 \\ 4 & 1 & 2 \\ \lambda & 1 & 1 \end{vmatrix} = 3 + 4\lambda - 4 - 6 - 8 + \lambda = 5\lambda - 15$$

$$\Delta_y = \begin{vmatrix} 1 & 3 & -1 \\ 1 & 4 & 2 \\ a & \lambda & 1 \end{vmatrix} = 4 + 6a - \lambda - 2\lambda - 3 + 4a = 10a - 3\lambda + 1$$

$$\Delta_z = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 4 \\ a & 1 & \lambda \end{vmatrix} = \lambda + 8a + 3 - 4 - 2\lambda - 3a = 5a - \lambda - 1$$

When the system is solvable,

Case 1 : it has unique solution, then

$\Delta \neq 0$ and λ is any real number

$\Rightarrow 5a - 4 \neq 0$ and λ is any real number

$\Rightarrow a \neq \dfrac{4}{5}$ and λ is any real number

Case 2 : it has infinitely many solutions, then

$\Delta = 0$ and $\Delta_x = 0$ and $\Delta_y = 0$ and $\Delta_z = 0$

$\Rightarrow a = \dfrac{4}{5}$ and $5\lambda - 15 = 0$ and $10a - 3\lambda + 1 = 0$ and $5a - \lambda - 1 = 0$

$\Rightarrow a = \dfrac{4}{5}$ and $\lambda = 3$

9(d)

Case 1 : $a \neq \dfrac{4}{5}$ and λ is any real number and $x = \dfrac{50}{17},\ y = \dfrac{4}{17},\ z = \dfrac{7}{17}$

$\Rightarrow ax + y + z = \lambda$ and $a \neq \dfrac{4}{5}$

$\Rightarrow a \cdot \dfrac{50}{17} + \dfrac{4}{17} + \dfrac{7}{17} = \lambda$ and $a \neq \dfrac{4}{5}$

$\Rightarrow a \cdot \dfrac{50}{17} = \lambda - \dfrac{4}{17} - \dfrac{7}{17}$ and $a \neq \dfrac{4}{5}$

$\Rightarrow a = \dfrac{17\lambda - 11}{50}$ and $a \neq \dfrac{4}{5}$

$\Rightarrow a = \dfrac{17\lambda - 11}{50}$ and $\dfrac{17\lambda - 11}{50} \neq \dfrac{4}{5}$

$\Rightarrow a = \dfrac{17\lambda - 11}{50}$ and $17\lambda - 11 \neq 40$

$\Rightarrow a = \dfrac{17\lambda - 11}{50}$ and $\lambda \neq \dfrac{51}{17} = 3$

Therefore, $a = \dfrac{17\lambda - 11}{50}$ and $\lambda \neq 3$ and $x = \dfrac{50}{17},\ y = \dfrac{4}{17},\ z = \dfrac{7}{17}$ fulfill ALL the four equations.

Case 2 : $a \neq \dfrac{4}{5}$ and λ is any real number and $x = 0,\ y = 2,\ z = 1$

$\Rightarrow ax + y + z = \lambda$ and $a \neq \dfrac{4}{5}$

$\Rightarrow a(0) + 2 + 1 = \lambda$ and $a \neq \dfrac{4}{5}$

$\Rightarrow \lambda = 3$ and $a \neq \dfrac{4}{5}$

Therefore $a \neq \dfrac{4}{5}$ and $\lambda = 3$ and $x = 0,\ y = 2,\ z = 1$ fulfil ALL the four equations.

Case 3 : $a=\frac{4}{5}$ and $\lambda=3$ and $x=\frac{50}{17},\ y=\frac{4}{17},\ z=\frac{7}{17}$

Then $ax+y+z$

$=\frac{4}{5}\cdot\frac{50}{17}+\frac{4}{17}+\frac{7}{17}$

$=\frac{40+4+7}{17}$

$=\frac{51}{17}$

$=3=\lambda$

Therefore, $a=\frac{4}{5}$ and $\lambda=3$ and $x=\frac{50}{17},\ y=\frac{4}{17},\ z=\frac{7}{17}$ fulfil ALL the four equations

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Case 4 : $a=\frac{4}{5}$ and $\lambda=3$ and $x=0,\ y=2,\ z=1$

Then $ax+y+z$

$=\frac{4}{5}\cdot(0)+2+1$

$=3=\lambda$

Therefore, $a=\frac{4}{5}$ and $\lambda=3$ and $x=0,\ y=2,\ z=1$ fulfil ALL the four equations

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Hence,
when $\lambda\neq 3$, $a=\frac{17\lambda-11}{50}$ (case 1)
when $\lambda=3,\ a$ can be any real numbers. (cases 2,3,4)