SP-DSE-MATH-EP(M2)-Q11

11(a)(i)

$$P^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$P^3 = P^2P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

$$\begin{aligned} &\Rightarrow P^3 - 2P^2 - P + I \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} - 2\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &\Rightarrow P^3 - 2P^2 - P + I \\ &= \begin{pmatrix} 1-2-0+1 & 2-2-0+0 & 3-2-1+0 \\ 2-2-0+0 & 4-4-1+1 & 5-4-1+0 \\ 3-2-1+0 & 5-4-1+0 & 6-6-1+1 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

11(a)(ii)

$$\begin{aligned} &P^3 - 2P^2 - P + I = 0 \\ &\Rightarrow I = -P^3 + 2P^2 + P \\ &\Rightarrow P^{-1} = -P^2 + 2P + I \\ &\Rightarrow P^{-1} = -P^2 + 2P + I \end{aligned}$$

$$= -\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1+0+1 & -1+0+0 & -1+2+0 \\ -1+0+0 & -2+2+1 & -2+2+0 \\ -1+2+0 & -2+2+0 & -3+2+1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

11(b)(i)

$$P^{-1}AP = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D$$

11(b)(ii)

$det\ D = 2 \neq 0 \Rightarrow$ D is non-singular.
$det\ A = 2 \neq 0 \Rightarrow$ A is non-singular.

11(b)(iii)

$$D^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^T = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Also
$D = P^{-1}AP$
$\Rightarrow D^{-1} = (P^{-1}AP)^{-1}$
$\Rightarrow D^{-1} = (AP)^{-1}(P^{-1})^{-1}$
$\Rightarrow D^{-1} = P^{-1}A^{-1}P$
$\Rightarrow A^{-1} = PD^{-1}P^{-1}$
$\Rightarrow (A^{-1})^{100} = (PD^{-1}P^{-1})^{100}$
$\Rightarrow (A^{-1})^{100} = P(D^{-1})^{100}P^{-1}$

$$\Rightarrow (A^{-1})^{100} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}^{100} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2^{100}} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \frac{1}{2^{100}} \\ 0 & 1 & \frac{1}{2^{100}} \\ 1 & 1 & \frac{1}{2^{100}} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2^{100}} & 0 & 0 \\ \frac{1}{2^{100}} - 1 & 1 & 0 \\ \frac{1}{2^{100}} - 1 & 0 & 1 \end{pmatrix}$$