2004-AL-P-MATH-1-Q08

8(a)

$$\begin{aligned} & \text{Let } s = \alpha - \beta \\ & X = \frac{1}{\alpha - \beta} (A - \beta I) \\ & = \frac{1}{s} (\begin{pmatrix} \alpha - k & \alpha - \beta - k \\ k & \beta + k \end{pmatrix} - \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}) \\ & = \frac{1}{s} \begin{pmatrix} \alpha - \beta - k & \alpha - \beta - k \\ k & k \end{pmatrix} \\ & = \frac{1}{s} \begin{pmatrix} s - k & s - k \\ k & k \end{pmatrix} \end{aligned}$$

$$\begin{split} Y &= \frac{1}{\beta - \alpha} (A - \alpha I) \\ &= -\frac{1}{s} (\begin{pmatrix} \alpha - k & \alpha - \beta - k \\ k & \beta + k \end{pmatrix} - \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}) \\ &= -\frac{1}{s} \begin{pmatrix} -k & \alpha - \beta - k \\ k & -\alpha + \beta + k \end{pmatrix} \\ &= -\frac{1}{s} \begin{pmatrix} -k & s - k \\ k & -s + k \end{pmatrix} \end{split}$$

Therefore,

$$XY = \frac{1}{s} \binom{s-k}{k} \frac{s-k}{k} (-1) \frac{1}{s} \binom{-k}{k} \frac{s-k}{-s+k}$$

$$= -\frac{1}{s^2} \binom{s-k}{k} \frac{s-k}{k} \binom{-k}{k} \frac{s-k}{-s+k}$$

$$= -\frac{1}{s^2} \binom{(-k)(s-k)+k(s-k)}{-k^2+k^2} \frac{(s-k)^2-(s-k)^2}{k(s-k)-k(s-k)}$$

$$= 0$$

$$YX = -rac{1}{s} inom{-k}{k} rac{s-k}{k-s+k} rac{1}{s} inom{s-k}{k} rac{s-k}{k}$$
 $= -rac{1}{s^2} inom{-k}{k} rac{s-k}{k-s+k} inom{s-k}{k} rac{s-k}{k}$

$$=-rac{1}{s^2} egin{pmatrix} (-k)(s-k)+k(s-k) & (-k)(s-k)+k(s-k) \ k(s-k)-k(s-k) & k(s-k)-k(s-k) \end{pmatrix}$$

= 0

$$\begin{split} X+Y&=\frac{1}{s}\begin{pmatrix}s-k&s-k\\k&k\end{pmatrix}-\frac{1}{s}\begin{pmatrix}-k&s-k\\k&-s+k\end{pmatrix}\\ &=\frac{1}{s}\begin{pmatrix}s&0\\0&s\end{pmatrix}\\ &=I \end{split}$$

$$\begin{split} X^2 &= \frac{1}{s^2} \begin{pmatrix} s-k & s-k \\ k & k \end{pmatrix} \begin{pmatrix} s-k & s-k \\ k & k \end{pmatrix} \\ &= \frac{1}{s^2} \begin{pmatrix} s(s-k) & s(s-k) \\ sk & sk \end{pmatrix} \\ &= \frac{1}{s} \begin{pmatrix} s-k & s-k \\ k & k \end{pmatrix} \\ &= X \end{split}$$

$$Y^{2} = \frac{1}{s^{2}} \begin{pmatrix} -k & s-k \\ k & -s+k \end{pmatrix} \begin{pmatrix} -k & s-k \\ k & -s+k \end{pmatrix}$$

$$= \frac{1}{s^{2}} \begin{pmatrix} sk & -s(s-k) \\ -sk & s(s-k) \end{pmatrix}$$

$$= \frac{1}{s} \begin{pmatrix} k & -(s-k) \\ -k & (s-k) \end{pmatrix}$$

$$= -\frac{1}{s} \begin{pmatrix} -k & s-k \\ k & -s+k \end{pmatrix}$$

$$= Y$$

8(b)

Let P(n) be the statement that $A^n = \alpha^n X + \beta^n Y$ for all positive integers n.

When n = 1,

$$\alpha^{n}X + \beta^{n}Y$$

$$= \frac{\alpha}{\alpha - \beta}(A - \beta I) + \frac{\beta}{\beta - \alpha}(A - \alpha I)$$

$$= \frac{\alpha}{\alpha - \beta}(A - \beta I) - \frac{\beta}{\alpha - \beta}(A - \alpha I)$$

$$= \frac{1}{\alpha - \beta}(\alpha A - \alpha \beta I - \beta A + \alpha \beta I)$$

$$= \frac{1}{\alpha - \beta} ((\alpha - \beta)A)$$

$$= \frac{1}{\alpha - \beta} (\alpha - \beta)A$$

Therefore, P(1) is true.

Assume that P(k) is true for some positive integer $k \ge 1$. Then,

$$\begin{split} &A^{k+1} = A^k A = (\alpha^k X + \beta^k Y)(\alpha X + \beta Y) \\ &= \alpha^{k+1} X^2 + \alpha^k \beta XY + \alpha \beta^k YX + \beta^{k+1} Y^2 \\ &= \alpha^{k+1} X + \alpha^k \beta 0 + \alpha \beta^k 0 + \beta^{k+1} Y \\ &= \alpha^{k+1} X + \beta^{k+1} Y \end{split}$$

Therefore, P(k+1) is true. By mathematical induction P(n) is true.

8(c)

Let
$$lpha=7$$
 , $eta=1$, $k=2$

$$egin{aligned} \left(egin{aligned} 5 & 4 \ 2 & 3 \end{aligned}
ight)^{2004} &= A^{2004} = lpha^{2004} X + eta^{2004} Y \ &= 7^{2004} (rac{1}{3}) \left(egin{aligned} 2 & 2 \ 1 & 1 \end{aligned}
ight) + (rac{1}{3}) \left(egin{aligned} 1 & -2 \ -1 & 2 \end{aligned}
ight) \ &= rac{1}{3} \left(egin{aligned} 2 \cdot 7^{2004} + 1 & 2 \cdot 7^{2004} - 2 \ 7^{2004} - 1 & 7^{2004} + 2 \end{aligned}
ight) \end{aligned}$$

8(d)

Let
$$B = \alpha^{-1}X + \beta^{-1}Y$$
. Then

$$AB = (\alpha X + \beta Y)(\alpha^{-1}X + \beta^{-1}Y)$$

$$= X^{2} + \frac{\alpha}{\beta}XY + \frac{\beta}{\alpha}YX + Y^{2}$$

$$= X + \frac{\alpha}{\beta}0 + \frac{\beta}{\alpha}0 + Y$$

$$= X + Y$$

$$= I$$

Also
$$BA = (\alpha^{-1}X + \beta^{-1}Y)(\alpha X + \beta Y)$$

 $= X^2 + \frac{\beta}{\alpha}XY + \frac{\alpha}{\beta}YX + Y^2$
 $= X + \frac{\beta}{\alpha}0 + \frac{\alpha}{\beta}0 + Y$
 $= X + Y$
 $= I$

Hence $A^{-1}=B=lpha^{-1}X+eta^{-1}Y$