2020-DSE-MATH-EP(M2)-Q08

8(a)

$PM = MQ$ and $|M| = 1$

$\Rightarrow \begin{pmatrix} -5 & -2 \\ 15 & 6 \end{pmatrix} \begin{pmatrix} 1 & a \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & a \\ b & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $-ab + c = 1$

$\Rightarrow \begin{pmatrix} -2b-5 & -5a-2c \\ 6b+15 & 15a+6c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$ and $-ab + c = 1$

$\Rightarrow a = -\dfrac{22}{25}$, $b = -3$ and $c = \dfrac{11}{5}$

8(b)(i)

$M = \begin{pmatrix} 1 & a \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & -\frac{22}{25} \\ -3 & \frac{11}{5} \end{pmatrix} = \dfrac{1}{25} \begin{pmatrix} 25 & -22 \\ -75 & 55 \end{pmatrix}$

$M^{-1} = -\dfrac{1}{11} \begin{pmatrix} 55 & 22 \\ 75 & 25 \end{pmatrix}$

$M^{-1}RM$

$= -\dfrac{1}{11} \begin{pmatrix} 55 & 22 \\ 75 & 25 \end{pmatrix} \begin{pmatrix} 6 & 2 \\ -15 & -5 \end{pmatrix} \cdot \dfrac{1}{25} \begin{pmatrix} 25 & -22 \\ -75 & 55 \end{pmatrix}$

$= -\dfrac{1}{275} \begin{pmatrix} 55 & 22 \\ 75 & 25 \end{pmatrix} \begin{pmatrix} 6 & 2 \\ -15 & -5 \end{pmatrix} \begin{pmatrix} 25 & -22 \\ -75 & 55 \end{pmatrix}$

$= -\dfrac{1}{275} \begin{pmatrix} 0 & 0 \\ 75 & 25 \end{pmatrix} \begin{pmatrix} 25 & -22 \\ -75 & 55 \end{pmatrix}$

$= -\dfrac{1}{275} \begin{pmatrix} 0 & 0 \\ 0 & -275 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

8(b)(ii)

Now $PM = MQ$ and $M^{-1}RM = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow P = MQM^{-1}$ and $R = M \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} M^{-1}$

$\Rightarrow P = M \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M^{-1}$ and $R = M \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} M^{-1}$

$\Rightarrow \alpha P = M \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} M^{-1}$ and $\beta R = M \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} M^{-1}$

$$\Rightarrow \alpha P + \beta R = M \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} M^{-1}$$

$$\Rightarrow (\alpha P + \beta R)^{99} = M \begin{pmatrix} \alpha^{99} & 0 \\ 0 & \beta^{99} \end{pmatrix} M^{-1}$$

$$\Rightarrow (\alpha P + \beta R)^{99} = M \begin{pmatrix} \alpha^{99} & 0 \\ 0 & 0 \end{pmatrix} M^{-1} + M \begin{pmatrix} 0 & 0 \\ 0 & \beta^{99} \end{pmatrix} M^{-1}$$

$$\Rightarrow (\alpha P + \beta R)^{99} = \alpha^{99} M \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M^{-1} + \beta^{99} M \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} M^{-1}$$

$$\Rightarrow (\alpha P + \beta R)^{99} = \alpha^{99} P + \beta^{99} R$$