# 2015-DSE-MATH-EP(M2)-Q11

## 11(a)(i)

$$M^2$$
$$= \begin{pmatrix} \lambda & 1 \\ \lambda-\mu+1 & \mu \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ \lambda-\mu+1 & \mu \end{pmatrix}$$
$$= \begin{pmatrix} \lambda^2+\lambda-\mu+1 & \lambda+\mu \\ (\lambda-\mu+1)(\lambda+\mu) & \mu^2+\lambda-\mu+1 \end{pmatrix}$$

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$$AB$$
$$= \frac{1}{\lambda-\mu+2}(I-\mu I+M)\cdot\frac{1}{\lambda-\mu+2}(I+\lambda I-M)$$
$$= \frac{1}{(\lambda-\mu+2)^2}(I-\mu I+M)(I+\lambda I-M)$$
$$= \frac{1}{(\lambda-\mu+2)^2}(\ (1-\mu)(1+\lambda)I+(\lambda+\mu)M-M^2\ )$$
$$= \frac{1}{(\lambda-\mu+2)^2}[\ (1-\mu)(1+\lambda)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(\lambda+\mu)\begin{pmatrix} \lambda & 1 \\ \lambda-\mu+1 & \mu \end{pmatrix}-\begin{pmatrix} \lambda^2+\lambda-\mu+1 & \lambda+\mu \\ (\lambda-\mu+1)(\lambda+\mu) & \mu^2+\lambda-\mu+1 \end{pmatrix}\ ]$$
$$= \frac{1}{(\lambda-\mu+2)^2}[\ \begin{pmatrix} (1-\mu)(1+\lambda) & 0 \\ 0 & (1-\mu)(1+\lambda) \end{pmatrix}+\begin{pmatrix} (\lambda+\mu)\lambda & (\lambda+\mu) \\ (\lambda+\mu)(\lambda-\mu+1) & \mu(\lambda+\mu) \end{pmatrix}-\begin{pmatrix} \lambda^2+\lambda-\mu+1 & \lambda+\mu \\ (\lambda-\mu+1)(\lambda+\mu) & \mu^2+\lambda-\mu+1 \end{pmatrix}\ ]$$
$$= \frac{1}{(\lambda-\mu+2)^2}[\ \begin{pmatrix} 1-\mu+\lambda+\lambda^2 & \lambda+\mu \\ (\lambda+\mu)(\lambda-\mu+1) & 1-\mu+\lambda+\mu^2 \end{pmatrix}-\begin{pmatrix} \lambda^2+\lambda-\mu+1 & \lambda+\mu \\ (\lambda-\mu+1)(\lambda+\mu) & \mu^2+\lambda-\mu+1 \end{pmatrix}\ ]$$
$$= 0$$

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$$
\begin{aligned}
&BA \\
&= \frac{1}{\lambda-\mu+2}(I+\lambda I-M)\cdot\frac{1}{\lambda-\mu+2}(I-\mu I+M) \\
&= \frac{1}{(\lambda-\mu+2)^2}(I+\lambda I-M)(I-\mu I+M) \\
&= \frac{1}{(\lambda-\mu+2)^2}(\,(1-\mu)(1+\lambda)I+(\lambda+\mu)M-M^2\,) \\
&= 0
\end{aligned}
$$

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$$
\begin{aligned}
&A+B \\
&= \frac{1}{\lambda-\mu+2}(I-\mu I+M)+\frac{1}{\lambda-\mu+2}(I+\lambda I-M) \\
&= \frac{1}{\lambda-\mu+2}(I-\mu I+M+I+\lambda I-M) \\
&= \frac{1}{\lambda-\mu+2}(2-\mu+\lambda)I \\
&= I
\end{aligned}
$$

# 11(a)(ii)

$A+B=I$
$\Rightarrow (A+B)A=A$ and $(A+B)B=B$
$\Rightarrow A^2+BA=A$ and $AB+B^2=B$
$\Rightarrow A^2+0=A$ and $0+B^2=B$
$\Rightarrow A^2=A$ and $B^2=B$

# 11(a)(iii)

Let P(n) be the statement that $M^n=(\lambda+1)^nA+(\mu-1)^nB$ for all positive integers n.

When n = 1,
$(\lambda+1)^1 A + (\mu-1)^1 B$

$= (\lambda+1)A + (\mu-1)B$

$= (\lambda+1)\dfrac{1}{\lambda-\mu+2}(I-\mu I+M) + (\mu-1)\dfrac{1}{\lambda-\mu+2}(I+\lambda I-M)$

$= \dfrac{1}{\lambda-\mu+2}[\,(\lambda+1)(I-\mu I+M) + (\mu-1)(I+\lambda I-M)\,]$

$= \dfrac{1}{\lambda-\mu+2}[\,(\lambda+1)(1-\mu)I + (\lambda+1)M + (\mu-1)(1+\lambda)I - (\mu-1)M)\,]$

$= \dfrac{1}{\lambda-\mu+2}(\lambda-\mu+2)M$

$= M$
Therefore P(1) is true.

Assume P(k) is true for some positive integer $k \geq 1$. Then,
$M^{k+1} = M^k \cdot M$
$= [\,(\lambda+1)^k A + (\mu-1)^k B\,][\,(\lambda+1)A + (\mu-1)B\,]$
$= (\lambda+1)^{k+1}A^2 + (\lambda+1)^k(\mu-1)AB + (\mu-1)^k(\lambda+1)BA + (\mu-1)^{k+1}B^2$
$= (\lambda+1)^{k+1}A^2 + (\lambda+1)^k(\mu-1)(0) + (\mu-1)^k(\lambda+1)(0) + (\mu-1)^{k+1}B^2$
$= (\lambda+1)^{k+1}A^2 + (\mu-1)^{k+1}B^2$
$= (\lambda+1)^{k+1}A + (\mu-1)^{k+1}B$
Therefore P(k+1) is true. By mathematical induction P(n) is true.

# 11(b)

Let $\lambda = 2$ , $\mu = 3$ such that $\mu - \lambda = 1 \neq 2$. Then,
$M = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ , $A = M - 2I = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = 3I - M = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

Therefore,
$\begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix}^{315}$

$$= (2\begin{pmatrix}2 & 1\\ 0 & 3\end{pmatrix})^{315}$$

$$= 2^{315}M^{315}$$

$$= 2^{315}(3^{315}A + 2^{315}B)$$

$$= 2^{315}(\ (3^{315} - 2^{315})A + 2^{315}A + 3^{315}B\ )$$

$$= 2^{315}(\ (3^{315} - 2^{315})A + 2^{315}(A + B)\ )$$

$$= 2^{315}(\ (3^{315} - 2^{315})A + 2^{315}I\ )$$

$$= 2^{315}\begin{pmatrix}2^{315} & 3^{315} - 2^{315}\\ 0 & 3^{315}\end{pmatrix}$$