

## 2002-AL-P-MATH-1-Q08

### 8(a)(i)

(S) has a unique solution.

$$\Rightarrow \Delta \neq 0$$

$$\Rightarrow \begin{vmatrix} a & -2 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & a \end{vmatrix} \neq 0$$

$$\Rightarrow 1 - a^2 \neq 0$$

$$\Rightarrow a^2 \neq 1$$

On the other hand,  $a^2 \neq 1$

$$\Rightarrow \Delta \neq 0$$

$\Rightarrow$  (S) has a unique solution.

Hence, (S) has a unique solution if and only if  $a^2 \neq 1$ .

Moreover,

$$\Delta_x = \begin{vmatrix} 0 & -2 & 1 \\ b & -1 & 2 \\ b & 1 & a \end{vmatrix} \text{ and } \Delta_y = \begin{vmatrix} a & 0 & 1 \\ 1 & b & 2 \\ 0 & b & a \end{vmatrix} \text{ and } \Delta_z = \begin{vmatrix} a & -2 & 0 \\ 1 & -1 & b \\ 0 & 1 & b \end{vmatrix}$$

$$\Rightarrow \Delta_x = 2b(a-1) \text{ and } \Delta_y = b(a-1)^2 \text{ and } \Delta_z = -2b(a-1)$$

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When  $a^2 \neq 0$  (i.e.  $\Delta \neq 0$ )

$$\Rightarrow x = \frac{\Delta_x}{\Delta} \text{ and } y = \frac{\Delta_y}{\Delta} \text{ and } z = \frac{\Delta_z}{\Delta}$$

$$\Rightarrow x = \frac{2b(a-1)}{1-a^2} \text{ and } y = \frac{b(a-1)^2}{1-a^2} \text{ and } z = \frac{-2b(a-1)}{1-a^2}$$

$$\Rightarrow x = \frac{-2b}{1+a} \text{ and } y = \frac{b(1-a)}{1+a} \text{ and } z = \frac{2b}{1+a}$$

### 8(a)(ii)(1)

When  $a = 1$ ,

$$(S) \begin{cases} x - 2y + z = 0 \\ x - y + 2z = b \\ y + z = b \end{cases}$$

$$\Rightarrow \begin{cases} x - 2y + z = 0 \\ y + z = b \end{cases}$$

Consider the augmented matrix :

$$\left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & b \end{array} \right]$$

$$\Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 3 & 2b \\ 0 & 1 & 1 & b \end{array} \right]$$

$\Rightarrow$  The solutions are  $x = 2b - 3t$ ,  $y = b - t$ ,  $z = t$  where  $b, t \in R$

## 8(a)(ii)(2)

When  $a = -1$ ,

$$\Delta = 0, \Delta_x = -4b \text{ and } \Delta_y = 4b \text{ and } \Delta_z = 4b$$

For (S) to be consistent,

$$\Delta_x = \Delta_y = \Delta_z = 0$$

$$\Rightarrow b = 0$$

$$(S) \begin{cases} -x - 2y + z = 0 \\ x - y + 2z = 0 \\ y - z = 0 \end{cases}$$

$$\Rightarrow (S) \begin{cases} x - y + 2z = 0 \\ y - z = 0 \end{cases}$$

Consider the augmented matrix :

$$\left[ \begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right]$$

$$\Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right]$$

$\Rightarrow$  The solutions are  $x = -t$ ,  $y = t$ ,  $z = t \in R$

## 8(b)

(T) is a combination of (S) where  $b = -1$  and equation  $5x - 2y + z = a$

When  $a^2 \neq 1$ ,

According to 8(a)(i), the solution of in 8(a)(i) must fulfill the equation for (T) being consistent.

Put the solution in 8(a)(i)  $x = \frac{2}{1+a}$  and  $y = \frac{a-1}{1+a}$  and  $z = \frac{-2}{1+a}$  into the equation.

$$5x - 2y + z = a$$

$$\Rightarrow \frac{10}{1+a} - \frac{2a-2}{1+a} + \frac{-2}{1+a} = a$$

$$\Rightarrow \frac{10-2a}{1+a} = a$$

$$\Rightarrow 10 - 2a = a + a^2$$

$$\Rightarrow a^2 + 3a - 10 = 0$$

$$\Rightarrow (a + 5)(a - 2) = 0$$

$$\Rightarrow a = -5 \text{ or } a = 2$$

When  $a = -5$ , (T) is consistent where

$$x = -\frac{1}{2} \text{ and } y = \frac{3}{2} \text{ and } z = \frac{1}{2}$$

When  $a = 2$ , (T) is consistent where

$$x = \frac{2}{3} \text{ and } y = \frac{1}{3} \text{ and } z = -\frac{2}{3}$$


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When  $a = -1$ ,

According to 8(a)(ii)(2), (S) is NOT consistent when  $b \neq 0$  and now  $b = -1 \neq 0$ .

Therefore, when  $a = -1$ , (T) is NOT consistent.

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When  $a = 1$ ,

According to 8(a)(i), the solution of in 8(a)(i) must fulfill the equation for (T) being consistent.

Put the solution in 8(a)(i)  $x = -2 - 3t$ ,  $y = -1 - t$ ,  $z = t$  into the equation.

$$5x - 2y + z = a$$

$$\Rightarrow -10 - 15t + 2 + 2t + t = 1$$

$$\Rightarrow -12t = 9$$

$$\Rightarrow t = -\frac{3}{4}$$

Therefore, when  $a = 1$ , (T) is consistent where

$$x = \frac{1}{4}, y = -\frac{1}{4}, z = -\frac{3}{4}$$