2007-AL-P-MATH-1-Q05

5(a)

$M^2 = \lambda M + \mu I$
$\Rightarrow P^{-1}Q^2P = \lambda P^{-1}QP + \mu I$
$\Rightarrow P^{-1}Q^2P = \lambda P^{-1}QP + \mu P^{-1}IP$
$\Rightarrow P^{-1}Q^2P = P^{-1}(\lambda Q)P + P^{-1}(\mu I)P$
$\Rightarrow P^{-1}Q^2P = P^{-1}(\lambda Q + \mu I)P$
$\Rightarrow Q^2 = \lambda Q + \mu I$

$$\Rightarrow \begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{pmatrix} = \begin{pmatrix} \lambda\alpha + \mu & 0 \\ 0 & \lambda\beta + \mu \end{pmatrix}$$

$\Rightarrow \alpha^2 = \lambda\alpha + \mu$ and $\beta^2 = \lambda\beta + \mu$
$\Rightarrow \lambda = \alpha + \beta$ and $\mu = -\alpha\beta$

5(b)

$det\ M = det(P^{-1}QP)$
$= (det\ P^{-1})(det\ Q)(det\ P)$
$= (det\ Q)(det\ P^{-1})(det\ P)$
$= det\ Q$
$= \alpha\beta$

Now, $det\ (M^2 + \alpha\beta I)$
$= det\ (\lambda M + \mu I + \alpha\beta I)$
$= det\ ((\alpha + \beta)M - \alpha\beta I + \alpha\beta I)$
$= det\ ((\alpha + \beta)M)$
$= (\alpha + \beta)^2 det\ M$ ($\because M$ is 2x2)
$= (\alpha + \beta)^2 \alpha\beta$