2014-DSE-MATH-EP(M2)-Q12

12(a)(i)

$A^{-1} = \frac{1}{1+p}\begin{pmatrix} 1 & 1 \\ -p & 1 \end{pmatrix}^T = \frac{1}{1+p}\begin{pmatrix} 1 & -p \\ 1 & 1 \end{pmatrix}$

12(a)(ii)

$A^{-1}MA$

$= \frac{1}{1+p}\begin{pmatrix} 1 & -p \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k-1 & k \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & p \\ -1 & 1 \end{pmatrix}$

$= \frac{1}{1+p}\begin{pmatrix} k-1-p & k \\ k & k \end{pmatrix}\begin{pmatrix} 1 & p \\ -1 & 1 \end{pmatrix}$

$= \frac{1}{1+p}\begin{pmatrix} -1-p & pk-p^2-p+k \\ 0 & kp+k \end{pmatrix}$

$= \frac{1}{1+p}\begin{pmatrix} -(1+p) & (1+p)(k-p) \\ 0 & k(p+1) \end{pmatrix}$

$= \begin{pmatrix} -1 & k-p \\ 0 & k \end{pmatrix}$

12(a)(iii)

p = k

$\Rightarrow A = \begin{pmatrix} 1 & k \\ -1 & 1 \end{pmatrix}$, $A^{-1} = \frac{1}{1+k}\begin{pmatrix} 1 & -k \\ 1 & 1 \end{pmatrix}$ and $A^{-1}MA = \begin{pmatrix} -1 & 0 \\ 0 & k \end{pmatrix}$

Then $M = A\begin{pmatrix} -1 & 0 \\ 0 & k \end{pmatrix}A^{-1}$

$\Rightarrow M^n = A\begin{pmatrix} -1 & 0 \\ 0 & k \end{pmatrix}^n A^{-1}$

$\Rightarrow M^n = A\begin{pmatrix} (-1)^n & 0 \\ 0 & k^n \end{pmatrix}A^{-1}$

$\Rightarrow M^n = \frac{1}{1+k}\begin{pmatrix} 1 & k \\ -1 & 1 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & k^n \end{pmatrix}\begin{pmatrix} 1 & -k \\ 1 & 1 \end{pmatrix}$

$\Rightarrow M^n = \frac{1}{1+k}\begin{pmatrix} (-1)^n & k^{n+1} \\ (-1)^{n+1} & k^n \end{pmatrix}\begin{pmatrix} 1 & -k \\ 1 & 1 \end{pmatrix}$

$\Rightarrow M^n = \frac{1}{1+k}\begin{pmatrix} (-1)^n + k^{n+1} & (-1)^{n+1}k + k^{n+1} \\ (-1)^{n+1} + k^n & (-1)^n k + k^n \end{pmatrix}$

12(b)

Let k=2, then $M = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

Now $\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix}$ for $n \geq 3$

$\Rightarrow \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = M^{n-2} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$ for $n \geq 3$

$\Rightarrow \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \dfrac{1}{1+k} \begin{pmatrix} (-1)^{n-2} + k^{n-1} & (-1)^{n-1}k + k^{n-1} \\ (-1)^{n-1} + k^{n-2} & (-1)^{n-2}k + k^{n-2} \end{pmatrix} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$ for $n \geq 3$

$\Rightarrow x_n = \dfrac{1}{1+k}[\,((-1)^{n-2} + k^{n-1})x_2 + ((-1)^{n-1}k + k^{n-1})x_1\,]$ for $n \geq 3$

$\Rightarrow x_n = \dfrac{1}{3}[\,((-1)^{n-2} + 2^{n-1})(1) + ((-1)^{n-1}2 + 2^{n-1})(0)\,]$ for $n \geq 3$

$\Rightarrow x_n = \dfrac{1}{3}((-1)^{n-2} + 2^{n-1})$ for $n \geq 3$