2011-AL-P-MATH-1-Q08

8(a)(i)

$ab > 0$
$\Rightarrow a \neq 0$ and $b \neq 0$
$\Rightarrow$Also, a, b are either both positive or both negative
$\Rightarrow a + b > 0$ (both positive) or $a + b < 0$ (both negative)

$det\ P = a + b$
$\Rightarrow det\ P > 0$ or $det\ P < 0$
$\Rightarrow det\ P \neq 0$
$\Rightarrow P$ is non-singular.

8(a)(ii)

$$P^{-1}AP = \frac{1}{a+b}\begin{pmatrix}1 & 1\\ -b & a\end{pmatrix}\begin{pmatrix}4-b & a\\ b & 4-a\end{pmatrix}\begin{pmatrix}a & -1\\ b & 1\end{pmatrix}$$

$$= \frac{1}{a+b}\begin{pmatrix}4 & 4\\ b(a+b-4) & a(4-a-b)\end{pmatrix}\begin{pmatrix}a & -1\\ b & 1\end{pmatrix}$$

$$= \frac{1}{a+b}\begin{pmatrix}4a+4b & 0\\ 0 & (a+b)(4-a-b)\end{pmatrix}$$

$$= \begin{pmatrix}4 & 0\\ 0 & 4-a-b\end{pmatrix}$$

8(a)(iii)

$$P^{-1}AP = \begin{pmatrix}4 & 0\\ 0 & 4-a-b\end{pmatrix}$$

$$\Rightarrow A = P\begin{pmatrix}4 & 0\\ 0 & 4-a-b\end{pmatrix}P^{-1}$$

$$\Rightarrow A^n = P\begin{pmatrix}4 & 0\\ 0 & 4-a-b\end{pmatrix}^n P^{-1}$$

$$\Rightarrow A^n = P\begin{pmatrix}4^n & 0\\ 0 & (4-a-b)^n\end{pmatrix}P^{-1}$$

$\Rightarrow d_1 = 4^n$ and $d_2 = (4-a-b)^n$

8(b)

Let a=4, b=1, then

$$A = \begin{pmatrix}4-1 & 4\\ 1 & 4-4\end{pmatrix} = \begin{pmatrix}3 & 4\\ 1 & 0\end{pmatrix} = B$$

Also, let

$D = \begin{pmatrix} 4 & 0 \\ 0 & 4-4-1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$ so that $B^k = PD^kP^{-1}$

where $P = \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}$ and $P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix}$ for any positive integer k

$$\begin{aligned}
&B + B^3 + B^5 + \cdots + B^{2n-1} \\
&= \sum_{k=1}^{n} B^{2k-1} \\
&= \sum_{k=1}^{n} (PD^{2k-1}P^{-1}) \\
&= P(\sum_{k=1}^{n} D^{2k-1})P^{-1}
\end{aligned}$$

$$= P\begin{pmatrix} \sum_{k=1}^{n} 4^{2k-1} & 0 \\ 0 & \sum_{k=1}^{n} (-1)^{2k-1} \end{pmatrix} P^{-1}$$

$$= P\begin{pmatrix} \sum_{k=1}^{n} 4 \cdot 4^{2k-2} & 0 \\ 0 & \sum_{k=1}^{n} (-1)^{2k-1} \end{pmatrix} P^{-1}$$

$$= P\begin{pmatrix} \sum_{k=1}^{n} 4 \cdot (4^2)^{k-1} & 0 \\ 0 & -\sum_{k=1}^{n} (-1)^{2k} \end{pmatrix} P^{-1}$$

$$= P\begin{pmatrix} \sum_{k=1}^{n} 4 \cdot (4^2)^{k-1} & 0 \\ 0 & -\sum_{k=1}^{n} ((-1)^2)^{k} \end{pmatrix} P^{-1}$$

$$= P\begin{pmatrix} 4\sum_{k=1}^{n} (4^2)^{k-1} & 0 \\ 0 & -\sum_{k=1}^{n} (1)^{k} \end{pmatrix} P^{-1}$$

$$= P\begin{pmatrix} 4\sum_{k=1}^{n} (4^2)^{k-1} & 0 \\ 0 & -\sum_{k=1}^{n} 1 \end{pmatrix} P^{-1}$$

$$= P\begin{pmatrix} 4\sum_{k=1}^{n} (4^2)^{k-1} & 0 \\ 0 & -n \end{pmatrix} P^{-1}$$

Consider $A + Ar + Ar^2 + Ar^3 + \cdots + Ar^{n-1} = A\dfrac{r^n - 1}{r - 1}$

Let $A = 4$ and $r = 4^2$, then

$$4\sum_{k=1}^{n} (4^2)^{k-1} = 4\frac{(4^2)^n - 1}{4^2 - 1} = \frac{4^{2n+1} - 4}{15}$$

$$= P\begin{pmatrix} \frac{4^{2n+1}-4}{15} & 0 \\ 0 & -n \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{4^{2n+1}-4}{15} & 0 \\ 0 & -n \end{pmatrix}\frac{1}{5}\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix}$$

$$= \frac{1}{75}\begin{pmatrix}4 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}4^{2n+1}-4 & 0\\ 0 & -15n\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 4\end{pmatrix}$$

$$= \frac{1}{75}\begin{pmatrix}4(4^{2n+1}-4) & 15n\\ 4^{2n+1}-4 & -15n\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 4\end{pmatrix}$$

$$= \frac{1}{75}\begin{pmatrix}4^{2n+2}-16 & 15n\\ 4^{2n+1}-4 & -15n\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 4\end{pmatrix}$$

$$= \frac{1}{75}\begin{pmatrix}4^{2n+2}-15n-16 & 4^{2n+2}+60n-16\\ 4^{2n+1}+15n-4 & 4^{2n+1}-60n-4\end{pmatrix}$$