2004-AL-P-MATH-1-Q08

8(a)

Let $s = \alpha - \beta$

$$X = \frac{1}{\alpha - \beta}(A - \beta I)$$

$$= \frac{1}{s}(\begin{pmatrix} \alpha - k & \alpha - \beta - k \\ k & \beta + k \end{pmatrix} - \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix})$$

$$= \frac{1}{s}\begin{pmatrix} \alpha - \beta - k & \alpha - \beta - k \\ k & k \end{pmatrix}$$

$$= \frac{1}{s}\begin{pmatrix} s - k & s - k \\ k & k \end{pmatrix}$$

$$Y = \frac{1}{\beta - \alpha}(A - \alpha I)$$

$$= -\frac{1}{s}(\begin{pmatrix} \alpha - k & \alpha - \beta - k \\ k & \beta + k \end{pmatrix} - \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix})$$

$$= -\frac{1}{s}\begin{pmatrix} -k & \alpha - \beta - k \\ k & -\alpha + \beta + k \end{pmatrix}$$

$$= -\frac{1}{s}\begin{pmatrix} -k & s - k \\ k & -s + k \end{pmatrix}$$

Therefore,

$$XY = \frac{1}{s}\begin{pmatrix} s - k & s - k \\ k & k \end{pmatrix}(-1)\frac{1}{s}\begin{pmatrix} -k & s - k \\ k & -s + k \end{pmatrix}$$

$$= -\frac{1}{s^2}\begin{pmatrix} s - k & s - k \\ k & k \end{pmatrix}\begin{pmatrix} -k & s - k \\ k & -s + k \end{pmatrix}$$

$$= -\frac{1}{s^2}\begin{pmatrix} (-k)(s-k) + k(s-k) & (s-k)^2 - (s-k)^2 \\ -k^2 + k^2 & k(s-k) - k(s-k) \end{pmatrix}$$

$$= 0$$

$$YX = -\frac{1}{s}\begin{pmatrix} -k & s - k \\ k & -s + k \end{pmatrix}\frac{1}{s}\begin{pmatrix} s - k & s - k \\ k & k \end{pmatrix}$$

$$= -\frac{1}{s^2}\begin{pmatrix} -k & s - k \\ k & -s + k \end{pmatrix}\begin{pmatrix} s - k & s - k \\ k & k \end{pmatrix}$$

$$= -\frac{1}{s^2}\begin{pmatrix} (-k)(s-k)+k(s-k) & (-k)(s-k)+k(s-k) \\ k(s-k)-k(s-k) & k(s-k)-k(s-k) \end{pmatrix}$$

$$= 0$$

$$X+Y = \frac{1}{s}\begin{pmatrix} s-k & s-k \\ k & k \end{pmatrix} - \frac{1}{s}\begin{pmatrix} -k & s-k \\ k & -s+k \end{pmatrix}$$

$$= \frac{1}{s}\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$

$$= I$$

$$X^2 = \frac{1}{s^2}\begin{pmatrix} s-k & s-k \\ k & k \end{pmatrix}\begin{pmatrix} s-k & s-k \\ k & k \end{pmatrix}$$

$$= \frac{1}{s^2}\begin{pmatrix} s(s-k) & s(s-k) \\ sk & sk \end{pmatrix}$$

$$= \frac{1}{s}\begin{pmatrix} s-k & s-k \\ k & k \end{pmatrix}$$

$$= X$$

$$Y^2 = \frac{1}{s^2}\begin{pmatrix} -k & s-k \\ k & -s+k \end{pmatrix}\begin{pmatrix} -k & s-k \\ k & -s+k \end{pmatrix}$$

$$= \frac{1}{s^2}\begin{pmatrix} sk & -s(s-k) \\ -sk & s(s-k) \end{pmatrix}$$

$$= \frac{1}{s}\begin{pmatrix} k & -(s-k) \\ -k & (s-k) \end{pmatrix}$$

$$= -\frac{1}{s}\begin{pmatrix} -k & s-k \\ k & -s+k) \end{pmatrix}$$

$$= Y$$

8(b)

Let P(n) be the statement that $A^n = \alpha^n X + \beta^n Y$ for all positive integers n.

When n = 1,

$$\alpha^n X + \beta^n Y$$
$$= \frac{\alpha}{\alpha-\beta}(A-\beta I) + \frac{\beta}{\beta-\alpha}(A-\alpha I)$$
$$= \frac{\alpha}{\alpha-\beta}(A-\beta I) - \frac{\beta}{\alpha-\beta}(A-\alpha I)$$
$$= \frac{1}{\alpha-\beta}(\alpha A - \alpha\beta I - \beta A + \alpha\beta I)$$

$= \frac{1}{\alpha - \beta}((\alpha - \beta)A)$
$= A$
Therefore, P(1) is true.

Assume that P(k) is true for some positive integer k ≥ 1. Then,
$A^{k+1} = A^k A = (\alpha^k X + \beta^k Y)(\alpha X + \beta Y)$
$= \alpha^{k+1} X^2 + \alpha^k \beta XY + \alpha \beta^k YX + \beta^{k+1} Y^2$
$= \alpha^{k+1} X + \alpha^k \beta 0 + \alpha \beta^k 0 + \beta^{k+1} Y$
$= \alpha^{k+1} X + \beta^{k+1} Y$
Therefore, P(k+1) is true. By mathematical induction P(n) is true.

8(c)

Let $\alpha = 7$, $\beta = 1$, $k = 2$

$$\begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}^{2004} = A^{2004} = \alpha^{2004} X + \beta^{2004} Y$$

$$= 7^{2004} (\frac{1}{3}) \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} + (\frac{1}{3}) \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 2 \cdot 7^{2004} + 1 & 2 \cdot 7^{2004} - 2 \\ 7^{2004} - 1 & 7^{2004} + 2 \end{pmatrix}$$

8(d)

Let $B = \alpha^{-1} X + \beta^{-1} Y$. Then

$AB = (\alpha X + \beta Y)(\alpha^{-1} X + \beta^{-1} Y)$
$= X^2 + \frac{\alpha}{\beta} XY + \frac{\beta}{\alpha} YX + Y^2$
$= X + \frac{\alpha}{\beta} 0 + \frac{\beta}{\alpha} 0 + Y$
$= X + Y$
$= I$

Also $BA = (\alpha^{-1} X + \beta^{-1} Y)(\alpha X + \beta Y)$
$= X^2 + \frac{\beta}{\alpha} XY + \frac{\alpha}{\beta} YX + Y^2$
$= X + \frac{\beta}{\alpha} 0 + \frac{\alpha}{\beta} 0 + Y$
$= X + Y$
$= I$

Hence $A^{-1} = B = \alpha^{-1}X + \beta^{-1}Y$