2004-AL-P-MATH-1-Q07

7(a)(i)

(E) has a unique solution.
$\Rightarrow \Delta \neq 0$

$$\Rightarrow \begin{vmatrix} 1 & a-2 & a \\ 1 & 2 & 4 \\ a & -1 & 3 \end{vmatrix} \neq 0$$

$\Rightarrow 2a^2 - 12a + 16 \neq 0$
$\Rightarrow 2(a-2)(a-4) \neq 0$
$\Rightarrow a \neq 2$ and $a \neq 4$

On the other hand, when $a \neq 2$ and $a \neq 4$
$\Rightarrow \Delta \neq 0$
$\Rightarrow$ (S) has an unique solution.

Hence, (E) has a unique solution if and only if $a \neq 2$ and $a \neq 4$.

Moreover,
$\Delta_x = \begin{vmatrix} 1 & a-2 & a \\ 1 & 2 & 4 \\ b & -1 & 3 \end{vmatrix}$ and $\Delta_y = \begin{vmatrix} 1 & 1 & a \\ 1 & 1 & 4 \\ a & b & 3 \end{vmatrix}$ and $\Delta_z = \begin{vmatrix} 1 & a-2 & 1 \\ 1 & 2 & 1 \\ a & -1 & b \end{vmatrix}$

$\Rightarrow \Delta_x = 2(b-2)(a-4)$ and $\Delta_y = (b-a)(a-4)$ and $\Delta_z = (a-b)(a-4)$

When $a \neq 2$ and $a \neq 4$ (i.e. $\Delta \neq 0$)
$\Rightarrow x = \dfrac{\Delta_x}{\Delta}$ and $y = \dfrac{\Delta_y}{\Delta}$ and $z = \dfrac{\Delta_z}{\Delta}$

$\Rightarrow x = \dfrac{2(b-2)(a-4)}{2(a-2)(a-4)}$ and $y = \dfrac{(b-a)(a-4)}{2(a-2)(a-4)}$ and $z = \dfrac{(a-b)(a-4)}{2(a-2)(a-4)}$

$\Rightarrow x = \dfrac{b-2}{a-2}$ and $y = \dfrac{b-a}{2(a-2)}$ and $z = \dfrac{a-b}{2(a-2)}$

7(a)(ii)(1)

When a = 2 (i.e. $\Delta = 0$),

$\Delta_x = 2(b-2)(a-4)$ and $\Delta_y = (b-a)(a-4)$ and $\Delta_z = (a-b)(a-4)$
$\Rightarrow \Delta_x = -4(b-2)$ and $\Delta_y = -2(b-2)$ and $\Delta_z = 2(b-2)$

When (E) is consistent, $\Delta_x = \Delta_y = \Delta_z = 0$
$\Rightarrow -4(b-2) = -2(b-2) = 2(b-2) = 0$
$\Rightarrow b = 2$

In this case, (E) $\begin{cases} x + 2z = 1 \\ x + 2y + 4z = 1 \\ 2x - y + 3z = 2 \end{cases}$

Consider the augmented matrix :
$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 1 & 2 & 4 & 1 \\ 2 & -1 & 3 & 2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$\Rightarrow$ The solutions are $x = 1 - 2t$, $y = -t$, $z = t$ where $t \in R$

7(a)(ii)(2)

When a = 4,

(E) $\begin{cases} x + 2y + 4z = 1 \\ x + 2y + 4z = 1 \\ 4x - y + 3z = b \end{cases}$

$\Rightarrow$ (E) $\begin{cases} x + 2y + 4z = 1 \\ 4x - y + 3z = b \end{cases}$

Consider the augmented matrix :
$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 4 & -1 & 3 & b \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & -9 & -13 & b-4 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & 1 & \frac{13}{9} & \frac{4-b}{9} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{10}{9} & \frac{1+2b}{9} \\ 0 & 1 & \frac{13}{9} & \frac{4-b}{9} \end{array}\right]$$

$\Rightarrow$ The solutions are $x = \dfrac{2b - 10t + 1}{9}$, $y = \dfrac{4 - b - 13t}{9}$, $z = t$ where $b,\ t \in R$

7(b)

Using result in (a)(ii)(1), $x = 1 - 2t$, $y = -t$, $z = t$ where $t \in R$
Since x, y, z satisfy $k(x^2 - 3) > yz$

$\Rightarrow k[\,(1-2t)^2 - 3\,] > -t^2$
$\Rightarrow (4k+1)t^2 - 4kt - 2k > 0$
$\Rightarrow 4k + 1 > 0$ and $(-4k)^2 - 4(4k+1)(-2k) < 0$
$\Rightarrow k > -\dfrac{1}{4}$ and $48k^2 + 8k < 0$
$\Rightarrow k > -\dfrac{1}{4}$ and $k(6k+1) < 0$
$\Rightarrow k > -\dfrac{1}{4}$ and { ($k < 0$ and $6k + 1 > 0$) or ($k > 0$ and $6k + 1 < 0$) }
$\Rightarrow k > -\dfrac{1}{4}$ and { ($k < 0$ and $k > -\dfrac{1}{6}$) or ($k > 0$ and $k < -\dfrac{1}{6}$) rejected }
$\Rightarrow k > -\dfrac{1}{4}$ and $k < 0$ and $k > -\dfrac{1}{6}$
$\Rightarrow k > -\dfrac{1}{6}$ and $k < 0$