SP-DSE-MATH-EP(M2)-Q10

10(a)

Let P(n) be the statement that
$A^n = \begin{pmatrix} cos\ n\theta & -sin\ n\theta \\ sin\ n\theta & cos\ n\theta \end{pmatrix}$ for all positive integers n.

When n = 1,
$A^1 = A = \begin{pmatrix} cos\ \theta & -sin\ \theta \\ sin\ \theta & cos\ \theta \end{pmatrix} = \begin{pmatrix} cos\ 1\theta & -sin\ 1\theta \\ sin\ 1\theta & cos\ 1\theta \end{pmatrix}$
Therefore, P(1) is true.

Assume P(k) is true for some positive integer $k \geq 1$, then
$A^{k+1} = A^k A$

$= \begin{pmatrix} cos\ k\theta & -sin\ k\theta \\ sin\ k\theta & cos\ k\theta \end{pmatrix} \begin{pmatrix} cos\ \theta & -sin\ \theta \\ sin\ \theta & cos\ \theta \end{pmatrix}$

$= \begin{pmatrix} cos\ k\theta\ cos\ \theta - sin\ k\theta\ sin\ \theta & -cos\ k\theta\ sin\ \theta - sin\ k\theta\ cos\ \theta \\ sin\ k\theta\ cos\ \theta + cos\ k\theta\ sin\ \theta & -sin\ k\theta\ sin\ \theta + cos\ k\theta\ cos\ \theta \end{pmatrix}$

$= \begin{pmatrix} cos\ (k\theta + \theta) & -sin\ (\theta + k\theta) \\ sin\ (k\theta + \theta) & cos\ (k\theta + \theta) \end{pmatrix}$

$= \begin{pmatrix} cos\ (k+1)\theta & -sin\ (k+1)\theta \\ sin\ (k+1)\theta & cos\ (k+1)\theta \end{pmatrix}$

Therefore, P(k+1) is true. By mathematical induction P(n) is true.

10(b)

$sin\ 3\theta + sin\ 2\theta + sin\theta = 0$
$\Rightarrow 2sin(\frac{3\theta + \theta}{2})cos(\frac{3\theta - \theta}{2}) + sin2\theta = 0$
$\Rightarrow 2sin\ 2\theta\ cos\theta + sin2\theta = 0$
$\Rightarrow sin\ 2\theta\ (2\ cos\theta + 1) = 0$
$\Rightarrow sin\ 2\theta = 0$ or $cos\ \theta = -\frac{1}{2}$
$\Rightarrow \theta = \frac{\pi}{2}$ or $\theta = \frac{2\pi}{3}$

10(c)

$A^3 + A^2 + A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$\Rightarrow \begin{pmatrix} cos\ 3\theta & -sin\ 3\theta \\ sin\ 3\theta & cos\ 3\theta \end{pmatrix} + \begin{pmatrix} cos\ 2\theta & -sin\ 2\theta \\ sin\ 2\theta & cos\ 2\theta \end{pmatrix} + \begin{pmatrix} cos\ \theta & -sin\ \theta \\ sin\ \theta & cos\ \theta \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$\Rightarrow \begin{pmatrix} cos\ 3\theta + cos\ 2\theta + cos\ \theta & -sin\ 3\theta - sin\ 2\theta - sin\ \theta \\ sin\ 3\theta + sin\ 2\theta + sin\ \theta & cos\ 3\theta + cos\ 2\theta + cos\ \theta \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$\Rightarrow cos\ 3\theta + cos\ 2\theta + cos\ \theta = a$ and $sin\ 3\theta + sin\ 2\theta + sin\ \theta = 0$

$\Rightarrow cos\ 3\theta + cos\ 2\theta + cos\ \theta = a$ and ($\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{2\pi}{3}$)

$\Rightarrow a = cos\ 3\dfrac{\pi}{2} + cos\ 2\dfrac{\pi}{2} + cos\ \dfrac{\pi}{2}$ or $cos\ 3\dfrac{2\pi}{3} + cos\ 2\dfrac{2\pi}{3} + cos\ \dfrac{2\pi}{3}$

$\Rightarrow a = 0 - 1 + 0$ or $= 1 - \frac{1}{2} - \frac{1}{2}$

$\Rightarrow a = -1$ or 0