2013-DSE-MATH-EP(M2)-Q13

13(a)(i)

$$MN = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$\Rightarrow tr(MN) = ae+bg+cf+dh$$

Also $NM = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$

$$\Rightarrow tr(NM) = ae+bg+cf+dh$$

Therefore $tr(MN) = tr(NM)$

13(a)(ii)

$$BAB^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\Rightarrow A = B^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} B$$

$$\Rightarrow tr(A) = tr(B^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} B)$$

$$\Rightarrow tr(A) = tr(BB^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix})$$

$$\Rightarrow tr(A) = tr(I \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix})$$

$$\Rightarrow tr(A) = tr(\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix})$$

$$\Rightarrow tr(A) = 1+3 = 4$$

13(a)(iii)

$$BAB^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\Rightarrow A = B^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} B$$

$$\Rightarrow |A| = \left| B^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} B \right|$$

$$\Rightarrow |A| = |B^{-1}| \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} |B|$$

$$\Rightarrow |A| = |B| \, |B^{-1}| \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix}$$

$\Rightarrow |A| = \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3$

13(b)(i)

$C\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{pmatrix} px+qy \\ rx+sy \end{pmatrix} = \begin{pmatrix} \lambda_1 x \\ \lambda_1 y \end{pmatrix}$

$\Rightarrow \begin{pmatrix} (p-\lambda_1)x+qy \\ rx+(s-\lambda_1)y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} (p-\lambda_1) & q \\ r & (s-\lambda_1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Since $\begin{pmatrix} x \\ y \end{pmatrix}$ is non-zero. Consider $\begin{pmatrix} kx \\ ky \end{pmatrix} \neq \begin{pmatrix} x \\ y \end{pmatrix}$ for any $k \neq 0$. Then

$\begin{pmatrix} (p-\lambda_1) & q \\ r & (s-\lambda_1) \end{pmatrix}\begin{pmatrix} kx \\ ky \end{pmatrix} = k\begin{pmatrix} (p-\lambda_1) & q \\ r & (s-\lambda_1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} (p-\lambda_1) & q \\ r & (s-\lambda_1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has infinitely many solutions.

Hence $\begin{vmatrix} (p-\lambda_1) & q \\ r & (s-\lambda_1) \end{vmatrix} = 0$

Similarly,
$C\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{pmatrix} (p-\lambda_2) & q \\ r & (s-\lambda_2) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which has infinitely many solutions $k\begin{pmatrix} x \\ y \end{pmatrix}$ for any $k \neq 0$

Hence $\begin{vmatrix} (p-\lambda_2) & q \\ r & (s-\lambda_2) \end{vmatrix} = 0$

13(b)(ii)

Now $\begin{vmatrix} (p-\lambda_1) & q \\ r & (s-\lambda_1) \end{vmatrix} = 0$ and $\begin{vmatrix} (p-\lambda_2) & q \\ r & (s-\lambda_2) \end{vmatrix} = 0$

$\Rightarrow (p-\lambda_1)(s-\lambda_1) - qr = 0$ and $(p-\lambda_2)(s-\lambda_2) - qr = 0$

$\Rightarrow \lambda_1^2 - (p+s)\lambda_1 + ps - qr = 0$ and $\lambda_2^2 - (p+s)\lambda_2 + ps - qr = 0$

$\Rightarrow \lambda_1^2 - tr(C)\cdot\lambda_1 + |C| = 0$ and $\lambda_2^2 - tr(C)\cdot\lambda_2 + |C| = 0$

$\Rightarrow \lambda_1$, λ_2 are roots of $\lambda^2 - tr(C)\cdot\lambda + |C| = 0$

13(c)

Now let C = A. Then
tr(C) = tr(A) = 4 and $|C| = |A| = 3$

The two values of λ are the roots of $\lambda^2 - tr(C) \cdot \lambda + |C| = 0$ or $\lambda^2 - 4\lambda + 3 = 0$

Now, $\lambda^2 - 4\lambda + 3 = 0$
$\Rightarrow (\lambda - 1)(\lambda - 3) = 0$
$\Rightarrow \lambda = 1$ or $\lambda = 3$