2016-DSE-MATH-EP(M2)-Q08

8(a)(i)

$$A^2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

8(a)(ii)

Let $X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then,

$$X^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$$

Therefore, for all positive integer $n \geq 3$
$X^n = X^2 X^{n-2} = (0)X^{n-2} = 0$

Now, for all positive integer $n \geq 3$, A^n
$= (I + X)^n$
$= I + nX + \binom{n}{2}X^2 + \sum_{r=3}^{n}\binom{n}{r}X^r$
$= I + nX + \binom{n}{2}(0) + \sum_{r=3}^{n}\binom{n}{r}(0)$
$= I + nX$

$= \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$

Consider also the definition of A (case n=1) and result in (i) (case n=2),
$A^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ for all positive integer n.

8(a)(iii)

$(A^{-1})^n$
$= (A^n)^{-1}$

$= \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$

8(b)(i)

Let P(n) be the statement that $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ for all positive integers n.
When n=1,
$L.H.S = \sum_{k=0}^{1-1} 2^k = \sum_{k=0}^{0} 2^k = 2^0 = 1 = 2 - 1 = 2^1 - 1 = R.H.S.$
Therefore, P(1) is true.

Assume P(r) is true for positive integer $r \geq 1$. Then
$\sum_{k=0}^{(r+1)-1} 2^k$

$= \sum_{k=0}^{r} 2^k$

$= 2^r + \sum_{k=0}^{r-1} 2^k$

$= 2^r + 2^r - 1$

$= 2 \cdot 2^r - 1$

$= \cdot 2^{r+1} - 1$
Therefore, P(r+1) is also true and by mathematical induction, P(n) is true.

Hence, $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ for all positive integers n

8(b)(ii)

Let $Y = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

Let P(n) be the statement that $Y^n = \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{n-1} 2^k & 2^n \end{pmatrix}$ for all positive integers n.

When n=1,

R.H.S. = $\begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{1-1} 2^k & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2^0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = Y = Y^1$ = L.H.S.

Therefore, P(1) is true.

Assume that P(r) is true for some positive integer $r \geq 1$. Then,

Y^{r+1}

$= Y^r Y$

$= \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{r-1} 2^k & 2^r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{r-1} 2^k + 2^r & 2^{r+1} \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{r} 2^k & 2^{r+1} \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{(r+1)-1} 2^k & 2^{r+1} \end{pmatrix}$

Therefore, P(r+1) is true and by mathematical induction P(n) is true.

Hence,

$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{n-1} 2^k & 2^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{pmatrix}$