2019-DSE-MATH-EP(M2)-Q11

11(a)

$M^2 = aM + bI$
$\Rightarrow \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix} \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix} = a \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix} + b \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -3 & -28 \\ 4 & 29 \end{pmatrix} = \begin{pmatrix} 2a+b & 7a \\ -a & -6a+b \end{pmatrix}$

$\Rightarrow a = -4$ and $b = 5$

11(b)

Let P(n) be the statement that $6M^n = (1-(-5)^n)M + (5+(-5)^n)I$ for all positive integers n.

When n = 1,
R.H.S. = $(1-(-5)^n)M + (5+(-5)^n)I$
= $(1-(-5))M + (5+(-5))I$
= $6M - 4I$
= $6M$
= L.H.S.
Therefore P(1) is true.

Assume P(k) is true for some positive integer $k \geq 1$. Then,
$6M^{k+1}$
$= 6M^kM$
$= [(1-(-5)^k)M + (5+(-5)^k)I]M$
$= (1-(-5)^k)M^2 + (5+(-5)^k)M$
$= (1-(-5)^k)(-4M+5I) + (5+(-5)^k)M$
$= [\,(5+(-5)^k) - 4(1-(-5)^k)\,]M + 5(1-(-5)^k)I$
$= [\,5+(-5)^k - 4 + 4(-5)^k)\,]M + 5(1-(-5)^k)I$
$= [\,1+5(-5)^k)\,]M + 5(1-(-5)^k)I$
$= [\,1-(-5)(-5)^k)\,]M + (5-5(-5)^k)I$
$= [\,1-(-5)^{k+1})\,]M + [\,5+(-5)(-5)^k\,]I$
$= (1-(-5)^{k+1}))M + (5+(-5)^{k+1})I$
Therefore P(k+1) is true and by mathematical indication P(n) is true.

11(c)

Note that $M^{-1} = -\frac{1}{5}\begin{pmatrix} -6 & -7 \\ 1 & 2 \end{pmatrix}$ and $M^{-2} = \frac{1}{25}\begin{pmatrix} -6 & -7 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -6 & -7 \\ 1 & 2 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 29 & 28 \\ -4 & -3 \end{pmatrix}$

Consider matrices A and B such that
$M^{-1} = A - \frac{1}{5}B$ and $M^{-2} = A + \frac{1}{25}B$

$$\Rightarrow -\frac{1}{5}\begin{pmatrix}-6 & -7\\ 1 & 2\end{pmatrix} = A - \frac{1}{5}B \text{ and } \frac{1}{25}\begin{pmatrix}29 & 28\\ -4 & -3\end{pmatrix} = A + \frac{1}{25}B$$

$$\Rightarrow 5A - B = \begin{pmatrix}6 & 7\\ -1 & -2\end{pmatrix} \text{ and } 25A + B = \begin{pmatrix}29 & 28\\ -4 & -3\end{pmatrix}$$

$$\Rightarrow A = \frac{1}{6}\begin{pmatrix}7 & 7\\ -1 & -1\end{pmatrix} \text{ and } B = \frac{1}{6}\begin{pmatrix}-1 & -7\\ 1 & 7\end{pmatrix}$$

Also,

$$AM = \frac{1}{6}\begin{pmatrix}7 & 7\\ -1 & -1\end{pmatrix}\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix} = \frac{1}{6}\begin{pmatrix}7 & 7\\ -1 & -1\end{pmatrix} = A$$

$$MA = \begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}\cdot\frac{1}{6}\begin{pmatrix}7 & 7\\ -1 & -1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}\begin{pmatrix}7 & 7\\ -1 & -1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}7 & 7\\ -1 & -1\end{pmatrix} = A$$

$$BM = \frac{1}{6}\begin{pmatrix}-1 & -7\\ 1 & 7\end{pmatrix}\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix} = \frac{1}{6}\begin{pmatrix}5 & 35\\ -5 & -35\end{pmatrix} = -\frac{5}{6}\begin{pmatrix}-1 & -7\\ 1 & 7\end{pmatrix} = -5B$$

$$MB = \begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}\cdot\frac{1}{6}\begin{pmatrix}-1 & -7\\ 1 & 7\end{pmatrix} = \frac{1}{6}\begin{pmatrix}2 & 7\\ -1 & -6\end{pmatrix}\begin{pmatrix}-1 & -7\\ 1 & 7\end{pmatrix} = \frac{1}{6}\begin{pmatrix}5 & 35\\ -5 & -35\end{pmatrix} = -5B$$

$$A + B = \frac{1}{6}\begin{pmatrix}7 & 7\\ -1 & -1\end{pmatrix} + \frac{1}{6}\begin{pmatrix}-1 & -7\\ 1 & 7\end{pmatrix} = I$$

Now let $X = A + \frac{1}{(-5)^n}B$ for all positive integers n. Then,

M^nX

$$= \frac{1}{6}[\,(1-(-5)^n)M + (5+(-5)^n)I\,](A + \frac{1}{(-5)^n}B)$$

$$= \frac{1}{6}[\,(1-(-5)^n)MA + (5+(-5)^n)A + \frac{1-(-5)^n}{(-5)^n}MB + \frac{5+(-5)^n}{(-5)^n}B\,]$$

$$= \frac{1}{6}[\,(1-(-5)^n)A + (5+(-5)^n)A + \frac{1-(-5)^n}{(-5)^n}(-5)B + \frac{5+(-5)^n}{(-5)^n}B\,]$$

$$= \frac{1}{6}[\,6A + \frac{5(-5)^n - 5}{(-5)^n}B + \frac{5+(-5)^n}{(-5)^n}B\,]$$

$$= \frac{1}{6}[\,6A + \frac{6(-5)^n}{(-5)^n}B\,]$$

$$= A + B = I$$

XM^n

$$= (A + \frac{1}{(-5)^n}B)\cdot\frac{1}{6}[\,(1-(-5)^n)M + (5+(-5)^n)I\,]$$

$$= \frac{1}{6}(A + \frac{1}{(-5)^n}B)[\,(1-(-5)^n)M + (5+(-5)^n)I\,]$$

$$= \frac{1}{6}[\,(1-(-5)^n)AM + (5+(-5)^n)A + \frac{1-(-5)^n}{(-5)^n}BM + \frac{5+(-5)^n}{(-5)^n}B]$$

$$= \frac{1}{6}[\,(1-(-5)^n)A + (5+(-5)^n)A + \frac{1-(-5)^n}{(-5)^n}(-5)B + \frac{5+(-5)^n}{(-5)^n}B]$$

$$= \frac{1}{6}[\,6A + \frac{5(-5)^n - 5}{(-5)^n}B + \frac{5+(-5)^n}{(-5)^n}B]$$

$$= \frac{1}{6}[\, 6A + \frac{6(-5)^n}{(-5)^n} B \,]$$

$$= A + B = I$$

Therefore $(M^n)^{-1} = X = A + \frac{1}{(-5)^n} B$