2002-AL-P-MATH-1-Q08

8(a)(i)

(S) has a unique solution.
$\Rightarrow \Delta \neq 0$

$$\Rightarrow \begin{vmatrix} a & -2 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & a \end{vmatrix} \neq 0$$

$\Rightarrow 1 - a^2 \neq 0$
$\Rightarrow a^2 \neq 1$

On the other hand, $a^2 \neq 1$
$\Rightarrow \Delta \neq 0$
$\Rightarrow$ (S) has an unique solution.

Hence, (S) has a unique solution if and only if $a^2 \neq 1$.

Moreover,
$$\Delta_x = \begin{vmatrix} 0 & -2 & 1 \\ b & -1 & 2 \\ b & 1 & a \end{vmatrix} \text{ and } \Delta_y = \begin{vmatrix} a & 0 & 1 \\ 1 & b & 2 \\ 0 & b & a \end{vmatrix} \text{ and } \Delta_z = \begin{vmatrix} a & -2 & 0 \\ 1 & -1 & b \\ 0 & 1 & b \end{vmatrix}$$

$\Rightarrow \Delta_x = 2b(a-1)$ and $\Delta_y = b(a-1)^2$ and $\Delta_z = -2b(a-1)$

When $a^2 \neq 1$ (i.e. $\Delta \neq 0$)
$$\Rightarrow x = \frac{\Delta_x}{\Delta} \text{ and } y = \frac{\Delta_y}{\Delta} \text{ and } z = \frac{\Delta_z}{\Delta}$$

$$\Rightarrow x = \frac{2b(a-1)}{1-a^2} \text{ and } y = \frac{b(a-1)^2}{1-a^2} \text{ and } z = \frac{-2b(a-1)}{1-a^2}$$

$$\Rightarrow x = \frac{-2b}{1+a} \text{ and } y = \frac{b(1-a)}{1+a} \text{ and } z = \frac{2b}{1+a}$$

8(a)(ii)(1)

When a = 1,

$$\text{(S) } \begin{cases} x - 2y + z = 0 \\ x - y + 2z = b \\ y + z = b \end{cases}$$

$\Rightarrow \begin{cases} x-2y+z=0 \\ y+z=b \end{cases}$

Consider the augmented matrix :

$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & b \end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2b \\ 0 & 1 & 1 & b \end{array}\right]$

$\Rightarrow$ The solutions are $x = 2b - 3t$, $y = b - t$, $z = t$ where $b, t \in R$

8(a)(ii)(2)

When a = -1,

$\Delta = 0$, $\Delta_x = -4b$ and $\Delta_y = 4b$ and $\Delta_z = 4b$

For (S) to be consistent,

$\Delta_x = \Delta_y = \Delta_x = 0$

$\Rightarrow b = 0$

(S) $\begin{cases} -x-2y+z=0 \\ x-y+2z=0 \\ y-z=0 \end{cases}$

$\Rightarrow$ (S) $\begin{cases} x-y+2z=0 \\ y-z=0 \end{cases}$

Consider the augmented matrix :

$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$

$\Rightarrow$ The solutions are $x = -t$, $y = t$, $z = t \in R$

8(b)

(T) is a combination of (S) where b = -1 and equation $5x - 2y + z = a$

When $a^2 \neq 1$,

According to 8(a)(i), the solution of in 8(a)(i) must fulfill the equation for (T) being consistent.

Put the solution in 8(a)(i) $x = \dfrac{2}{1+a}$ and $y = \dfrac{a-1}{1+a}$ and $z = \dfrac{-2}{1+a}$ into the equation.

$5x - 2y + z = a$

$\Rightarrow \dfrac{10}{1+a} - \dfrac{2a-2}{1+a} + \dfrac{-2}{1+a} = a$

$\Rightarrow \dfrac{10-2a}{1+a} = a$

$\Rightarrow 10 - 2a = a + a^2$

$\Rightarrow a^2 + 3a - 10 = 0$
$\Rightarrow (a+5)(a-2) = 0$
$\Rightarrow a = -5$ or $a = 2$

When $a = -5$, (T) is consistent where
$x = -\dfrac{1}{2}$ and $y = \dfrac{3}{2}$ and $z = \dfrac{1}{2}$

When $a = 2$, (T) is consistent where
$x = \dfrac{2}{3}$ and $y = \dfrac{1}{3}$ and $z = -\dfrac{2}{3}$

When a = -1,
According to 8(a)(ii)(2), (S) is NOT consistent when $b \neq 0$ and now $b = -1 \neq 0$.
Therefore, when a = -1, (T) is NOT consistent.

When a = 1,
According to 8(a)(i), the solution of in 8(a)(i) must fulfill the equation for (T) being consistent.
Put the solution in 8(a)(i) $x = -2 - 3t$, $y = -1 - t$, $z = t$ into the equation.

$5x - 2y + z = a$
$\Rightarrow -10 - 15t + 2 + 2t + t = 1$
$\Rightarrow -12t = 9$
$\Rightarrow t = -\dfrac{3}{4}$

Therefore, when a = 1, (T) is consistent where
$x = \dfrac{1}{4}$, $y = -\dfrac{1}{4}$, $z = -\dfrac{3}{4}$