1997-AL-P-MATH-1-Q07

7(a)

A is non-singular $\Rightarrow detA \neq 0$
Also,

$$A(A^{-1}-xI)=I-xA$$
$$\Rightarrow A(A^{-1}-xI)=x\,(x^{-1}I-A)$$
$$\Rightarrow A(A^{-1}-xI)=-x\,(A-x^{-1}I)$$
$$\Rightarrow det\;(\;A(A^{-1}-xI)\;)=det\;(-x\;(A-x^{-1}I)\;)$$
$$\Rightarrow (det\;A)\;(det\;(A^{-1}-xI))=(-x)^3\;det\;(A-x^{-1}I)$$ $\because$ A is 3x3
$$\Rightarrow (det\;A)\;(det\;(A^{-1}-xI))=-\,x^3\;det\;(A-x^{-1}I)$$
$$\Rightarrow det\;(A^{-1}-xI)=-\,\frac{x^3}{det\;A}\;det\;(A-x^{-1}I)$$

7(b)

$$A-xI=\begin{pmatrix}0&1&0\\0&0&1\\4&-17&8\end{pmatrix}-\begin{pmatrix}x&0&0\\0&x&0\\0&0&x\end{pmatrix}=\begin{pmatrix}-x&1&0\\0&-x&1\\4&-17&8-x\end{pmatrix}$$

$$\Rightarrow det(A-xI)=\begin{vmatrix}-x&1&0\\0&-x&1\\4&-17&8-x\end{vmatrix}$$

$$\Rightarrow det(A-4I)=\begin{vmatrix}-4&1&0\\0&-4&1\\4&-17&8-4\end{vmatrix}=\begin{vmatrix}-4&1&0\\0&-4&1\\4&-17&4\end{vmatrix}$$

$$\Rightarrow det(A-4I)=(-4)\begin{vmatrix}-4&1\\-17&4\end{vmatrix}-\begin{vmatrix}0&1\\4&4\end{vmatrix}=(-4)(1)-(-4)=0$$

Therefore, $x=4$ is a root of $det(A-xI)=0$

Also, $det(A-xI)=(-x)\begin{vmatrix}-x&1\\-17&8-x\end{vmatrix}-\begin{vmatrix}0&1\\4&8-x\end{vmatrix}$

$$\Rightarrow det(A-xI)=-x^3+8x^2-17x+4=-(x-4)(x^2-4x+1)=-(x-4)\;[x-(2+\sqrt{3})]\;[x-(2-\sqrt{3})]$$

Therefore, roots of $det(A-xI)$ are $4\;,2+\sqrt{3}$ and $2-\sqrt{3}$

Now, $detA=\begin{vmatrix}0&1&0\\0&0&1\\4&-11&8\end{vmatrix}=-4\begin{vmatrix}1&0\\0&1\end{vmatrix}=-4\neq 0$

Therefore, A is non-singular and $det\;(A^{-1}-xI)=-\,\dfrac{x^3}{det\;A}\;det\;(A-x^{-1}I)$

Hence, $det\;(A^{-1}-xI)=0$
$$\Rightarrow -\,\frac{x^3}{det\;A}\;det\;(A-x^{-1}I)=0$$

$\Rightarrow x^3 \; det\;(A - x^{-1}I) = 0$

$x = 0$ is NOT a solution because $det\;(A^{-1} - 0\;I) = det\;A^{-1} = \dfrac{1}{detA} \neq 0$

$\Rightarrow det\;(A - x^{-1}I) = 0$

$\Rightarrow x^{-1} = 4$ or $x^{-1} = 2 + \sqrt{3}$ or $x^{-1} = 2 - \sqrt{3}$

$\Rightarrow x = \dfrac{1}{4}$ or $x = \dfrac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}$ or $x = \dfrac{1}{2 - \sqrt{3}} = 2 + \sqrt{3}$