# 2001-AL-P-MATH-1-Q09

## 9(a)

(S) has an unique solution.
$\Rightarrow \Delta \neq 0$

$$\Rightarrow \begin{vmatrix} 1 & \lambda & 1 \\ \lambda & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix} \neq 0$$

$\Rightarrow -3 - \lambda(2\lambda - 3) + \lambda + 3 \neq 0$
$\Rightarrow 2\lambda(2 - \lambda) \neq 0$
$\Rightarrow \lambda \neq 0$ and $\lambda \neq 2$

On the other hand,
$\lambda \neq 0$ and $\lambda \neq 2$
$\Rightarrow \Delta \neq 0$
$\Rightarrow$ (S) has a unique solution.

Hence, (S) has a unique solution if and only if $\lambda \neq 0$ and $\lambda \neq 2$

## 9(b)(i)

$\Delta_x = \begin{vmatrix} k & \lambda & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 2 \end{vmatrix}$ and $\Delta_y = \begin{vmatrix} 1 & k & 1 \\ \lambda & 1 & 1 \\ 3 & -1 & 2 \end{vmatrix}$ and $\Delta_z = \begin{vmatrix} 1 & \lambda & k \\ \lambda & -1 & 1 \\ 3 & 1 & -1 \end{vmatrix}$

$\Rightarrow \Delta_x = -3(k + \lambda)$ and $\Delta_y = 3k - 2k\lambda - \lambda$ and $\Delta_z = (k + \lambda)(\lambda + 3)$

When $\lambda \neq 0$ and $\lambda \neq 2$ (i.e. $\Delta \neq 0$)
$\Rightarrow x = \dfrac{\Delta_x}{\Delta}$ and $y = \dfrac{\Delta_y}{\Delta}$ and $z = \dfrac{\Delta_z}{\Delta}$

$\Rightarrow x = \dfrac{-3(k + \lambda)}{2\lambda(2 - \lambda)}$ and $y = \dfrac{3k - 2k\lambda - \lambda}{2\lambda(2 - \lambda)}$ and $z = \dfrac{(k + \lambda)(\lambda + 3)}{2\lambda(2 - \lambda)}$

## 9(b)(ii)

When $\lambda = 0$ (i.e. $\Delta = 0$), $\Delta_x = -3k$ and $\Delta_y = 3k$ and $\Delta_z = 3k$

When $k \neq 0$, $\Delta_x \neq 0$ and $\Delta_y \neq 0$ and $\Delta_z \neq 0 \Rightarrow$ (S) is NOT consistent.

When k = 0,
(S) $\begin{cases} x + z = 0 \\ -y + z = 1 \\ 3x + y + 2z = -1 \end{cases}$

Consider the augmented matrix,
$$\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & -1 & 1 & 1\\3 & 1 & 2 & -1\end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & -1 & 1 & 1\\0 & 1 & -1 & -1\end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & -1 & -1\end{array}\right]$$

$\Rightarrow$ The solutions are $x = -t$, $y = t - 1$, $z = t \in R$.

# 9(b)(iii)

When $\lambda = 2$ (i.e. $\Delta = 0$), $\Delta_x = -3(k+2)$ and $\Delta_y = -k-2$ and $\Delta_z = 5(k+2)$

When $k \neq -2$, $\Delta_x \neq 0$ and $\Delta_y \neq 0$ and $\Delta_z \neq 0 \Rightarrow$ (S) is NOT consistent.

When k = -2,
(S) $\begin{cases} x + 2y + z = -2 \\ 2x - y + z = 1 \\ 3x + y + 2z = -1 \end{cases}$

Consider the augmented matrix,
$$\left[\begin{array}{ccc|c}1 & 2 & 1 & -2\\2 & -1 & 1 & 1\\3 & 1 & 2 & -1\end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c}1 & 2 & 1 & -2\\0 & -5 & -1 & 5\\0 & -5 & -1 & 5\end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c}1 & 2 & 1 & -2\\0 & -5 & -1 & 5\end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c}1 & 2 & 1 & -2\\0 & 1 & \frac{1}{5} & -1\end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c}1 & 0 & \frac{3}{5} & 0\\0 & 1 & \frac{1}{5} & -1\end{array}\right]$$

$\Rightarrow$ The solutions are $x = -\frac{3t}{5}$, $y = -1 - \frac{t}{5}$, $z = t \in R$.

# 9(c)

According to 9(b)(ii), the solution $x = -t$, $y = t - 1$, $z = t \in R$. satisfies $(x-p)^2 + y^2 + z^2 = 1$
$\Rightarrow (-t-p)^2 + (t-1)^2 + t^2 = 1$
$\Rightarrow t^2 + 2pt + p^2 + t^2 - 2t + 1 + t^2 = 1$

$\Rightarrow 3t^2 + 2pt + p^2 - 2t = 0$
$\Rightarrow 3t^2 + 2(p-1)t + p^2 = 0$
$\Rightarrow 4(p-1)^2 - 12p^2 \geq 0$ ($\because$ some solutions of t exist)
$\Rightarrow (p-1)^2 - 3p^2 \geq 0$
$\Rightarrow (p - 1 + \sqrt{3}p)(p - 1 - \sqrt{3}p) \geq 0$
$\Rightarrow [(\sqrt{3}+1)p - 1]\ [(1-\sqrt{3})p - 1] \geq 0$
$\Rightarrow [(\sqrt{3}+1)p - 1]\ [-(\sqrt{3}-1)p - 1] \geq 0$
$\Rightarrow [(\sqrt{3}+1)p - 1]\ [(\sqrt{3}-1)p + 1] \leq 0$

$\Rightarrow (\sqrt{3}+1)p - 1 \geq 0$ and $(\sqrt{3}-1)p + 1 \leq 0$, or
$(\sqrt{3}+1)p - 1 \leq 0$ and $(\sqrt{3}-1)p + 1 \geq 0$

$\Rightarrow p \geq \dfrac{1}{\sqrt{3}+1}$ and $p \leq \dfrac{-1}{\sqrt{3}-1}$, or
$p \leq \dfrac{1}{\sqrt{3}+1}$ and $p \geq \dfrac{-1}{\sqrt{3}-1}$

$\Rightarrow p \leq \dfrac{1}{\sqrt{3}+1}$ and $p \geq \dfrac{-1}{\sqrt{3}-1}$

$\Rightarrow p \leq \dfrac{\sqrt{3}-1}{2}$ and $p \geq \dfrac{-\sqrt{3}-1}{2}$