2013-AL-P-MATH-1-Q11

11(a)(i)

$$A\begin{pmatrix}1\\-2\end{pmatrix}=\lambda\begin{pmatrix}1\\-2\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}a & -2\\-2 & a+3\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}\lambda\\-2\lambda\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}a+4\\-2a-8\end{pmatrix}=\begin{pmatrix}\lambda\\-2\lambda\end{pmatrix}$$

$\Rightarrow \lambda=a+4$

11(a)(ii)

$$A\begin{pmatrix}b\\1\end{pmatrix}=\mu\begin{pmatrix}b\\1\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}a & -2\\-2 & a+3\end{pmatrix}\begin{pmatrix}b\\1\end{pmatrix}=\begin{pmatrix}\mu b\\\mu\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}ab-2\\-2b+a+3\end{pmatrix}=\begin{pmatrix}\mu b\\\mu\end{pmatrix}$$

$\Rightarrow ab-2=\mu b$ and $-2b+a+3=\mu$
$\Rightarrow ab-2=(-2b+a+3)b$
$\Rightarrow ab-2=-2b^2+ab+3b$
$\Rightarrow 2b^2-3b-2=0$
$\Rightarrow (2b+1)(b-2)=0$
$\Rightarrow b=-\frac{1}{2}$(rejected since b>0) or 2
$\Rightarrow b=2$

Therefore $\mu=-2b+a+3=a-1$

11(a)(iii)(1)

$$M^TM=\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}$$

$$=\begin{pmatrix}5 & 0\\0 & 5\end{pmatrix}=5I$$

11(a)(iii)(2)

Let P(n) be the statement that $A^n=\frac{1}{5}MD^nM^T$ for any $n\in N$ where $D=\begin{pmatrix}\lambda & 0\\0 & \mu\end{pmatrix}$

When n = 1,
$\frac{1}{5}MD^1M^T$

$$= \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} \lambda & 2\mu \\ -2\lambda & \mu \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} \lambda + 4\mu & -2\lambda + 2\mu \\ -2\lambda + 2\mu & 4\lambda + \mu \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} a + 4 + 4(a-1) & -2(a+4) + 2(a-1) \\ -2(a+4) + 2(a-1) & 4(a+4) + a - 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 5a & -10 \\ -10 & 5a + 15 \end{pmatrix}$$

$$= \begin{pmatrix} a & -2 \\ -2 & a + 3 \end{pmatrix}$$

$$= A$$

Therefore P(1) is true.

Assume that P(k) is true for some natural number $k \geq 1$. Then,

$$A^{k+1}$$
$$= A^k A$$
$$= \frac{1}{5}MD^kM^T \cdot \frac{1}{5}MDM^T$$
$$= \frac{1}{25}MD^k(M^TM)DM^T$$
$$= \frac{1}{25}MD^k(5I)DM^T$$
$$= \frac{1}{5}MD^kDM^T$$
$$= \frac{1}{5}MD^{k+1}M^T.$$

Therefore P(k+1) is true and by mathematical induction P(n) is true.

11(b)(i)

Let a=3, then $\lambda = 7$, $\mu = 2$ and $A = \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}$. Then

$$(x \quad y)\begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}^{2013}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow (x \quad y)A^{2013}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow (x \quad y)\frac{1}{5}MD^{2013}M^T\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \frac{1}{5}(x \quad y)\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 7^{2013} & 0 \\ 0 & 2^{2013} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \frac{1}{5}(x \quad y)\begin{pmatrix} 7^{2013} & -2^{2014} \\ 2 \cdot 7^{2013} & 2^{2013} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \frac{1}{5}(x \quad y)\begin{pmatrix} 7^{2013} + 2^{2015} & 2 \cdot 7^{2013} - 2^{2014} \\ 2 \cdot 7^{2013} - 2^{2014} & 4 \cdot 7^{2013} + 2^{2013} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix}(7^{2013}+2^{2015})x+(2\cdot 7^{2013}-2^{2014})y & (2\cdot 7^{2013}-2^{2014})x+(4\cdot 7^{2013}+2^{2013})y\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0$$

$$\Rightarrow (7^{2013}+2^{2015})x^2+(2\cdot 7^{2013}-2^{2014})xy+(2\cdot 7^{2013}-2^{2014})xy+(4\cdot 7^{2013}+2^{2013})y^2=0$$

$$\Rightarrow (7^{2013}+2^{2015})x^2+2(7^{2013}-2^{2013})xy+(4\cdot 7^{2013}+2^{2013})y^2=0$$

$$\Rightarrow (7^{2013}+4\cdot 2^{2013})x^2+2(7^{2013}-2^{2013})xy+(4\cdot 7^{2013}+2^{2013})y^2=0$$

$$\Rightarrow (7^{2013}-2^{2013}+6\cdot 2^{2013})x^2+2(7^{2013}-2^{2013})xy+(7^{2013}-2^{2013}+3\cdot 7^{2013}+2\cdot 2^{2013})y^2=0$$

$$\Rightarrow (7^{2013}-2^{2013})(x^2+2xy+y^2)+6\cdot 2^{2013}x^2+(3\cdot 7^{2013}+2\cdot 2^{2013})y^2=0$$

$$\Rightarrow (7^{2013}-2^{2013})(x+y)^2+6\cdot 2^{2013}x^2+(3\cdot 7^{2013}+2\cdot 2^{2013})y^2=0$$

$\Rightarrow (x+y)^2=0$ or $x^2=0$ or $y^2=0$

$\Rightarrow x=0$ or $y=0$

11(b)(ii)

Let a=1, then $\lambda=5$, $\mu=0$ and $A=\begin{pmatrix}1 & -2\\-2 & 4\end{pmatrix}$. Then

$$\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1 & -2\\-2 & 4\end{pmatrix}^{2013}\begin{pmatrix}x\\y\end{pmatrix}=0$$

$$\Rightarrow \frac{1}{5}\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\begin{pmatrix}5^{2013} & 0\\0 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0$$

$$\Rightarrow \frac{1}{5}\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}5^{2013} & 0\\2\cdot 5^{2013} & 0\end{pmatrix}\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0$$

$$\Rightarrow \frac{1}{5}\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}5^{2013} & 2\cdot 5^{2013}\\2\cdot 5^{2013} & 4\cdot 5^{2013}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0$$

$$\Rightarrow \begin{pmatrix}5^{2013}x+2\cdot 5^{2013}y & 2\cdot 5^{2013}x+4\cdot 5^{2013}y\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0$$

$$\Rightarrow 5^{2013}x^2+2\cdot 5^{2013}xy+2\cdot 5^{2013}xy+4\cdot 5^{2013}y^2=0$$

$$\Rightarrow x^2+4xy+4y^2=0$$

$$\Rightarrow (x+2y)^2=0$$

$$\Rightarrow x+2y=0$$

Therefore, it is NOT necessary that both x and y equal 0.