

2016-DSE-MATH-EP(M2)-Q08

8(a)(i)

$$A^2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

8(a)(ii)

Let $X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then,

$$X^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$$

Therefore, for all positive integer $n \geq 3$

$$X^n = X^2 X^{n-2} = (0) X^{n-2} = 0$$

Now, for all positive integer $n \geq 3$, A^n

$$= (I + X)^n$$

$$= I + nX + \binom{n}{2} X^2 + \sum_{r=3}^n \binom{n}{r} X^r$$

$$= I + nX + \binom{n}{2} (0) + \sum_{r=3}^n \binom{n}{r} (0)$$

$$= I + nX$$

$$= \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

Consider also the definition of A (case $n=1$) and result in (i) (case $n=2$),

$$A^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \text{ for all positive integer } n.$$

8(a)(iii)

$$\begin{aligned}
& (A^{-1})^n \\
&= (A^n)^{-1} \\
&= \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}
\end{aligned}$$

8(b)(i)

Let $P(n)$ be the statement that $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ for all positive integers n .

When $n=1$,

$$L.H.S = \sum_{k=0}^{1-1} 2^k = \sum_{k=0}^0 2^k = 2^0 = 1 = 2 - 1 = 2^1 - 1 = R.H.S.$$

Therefore, $P(1)$ is true.

Assume $P(r)$ is true for positive integer $r \geq 1$. Then

$$\begin{aligned}
& \sum_{k=0}^{(r+1)-1} 2^k \\
&= \sum_{k=0}^r 2^k \\
&= 2^r + \sum_{k=0}^{r-1} 2^k \\
&= 2^r + 2^r - 1 \\
&= 2 \cdot 2^r - 1 \\
&= 2^{r+1} - 1
\end{aligned}$$

Therefore, $P(r+1)$ is also true and by mathematical induction, $P(n)$ is true.

Hence, $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ for all positive integers n

8(b)(ii)

$$\text{Let } Y = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

Let $P(n)$ be the statement that $Y^n = \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{n-1} 2^k & 2^n \end{pmatrix}$ for all positive integers n .

When $n=1$,

$$\text{R.H.S.} = \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{1-1} 2^k & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2^0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = Y = Y^1 = \text{L.H.S.}$$

Therefore, $P(1)$ is true.

Assume that $P(r)$ is true for some positive integer $r \geq 1$. Then,

$$\begin{aligned} Y^{r+1} &= Y^r Y \\ &= \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{r-1} 2^k & 2^r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{r-1} 2^k + 2^r & 2^{r+1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^r 2^k & 2^{r+1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{(r+1)-1} 2^k & 2^{r+1} \end{pmatrix} \end{aligned}$$

Therefore, $P(r+1)$ is true and by mathematical induction $P(n)$ is true.

Hence,

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ \sum_{k=0}^{n-1} 2^k & 2^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2^n - 1 & 2^n \end{pmatrix}$$