2003-AL-P-MATH-1-Q07

7(a)(i)

(E) has a unique solution.
$\Rightarrow \Delta \neq 0$

$$\Rightarrow \begin{vmatrix} 1 & a & -1 \\ 2 & -1 & a \\ -1 & 2a^2 & a-3 \end{vmatrix} \neq 0$$

$\Rightarrow -2a^3 - 7a^2 + 5a + 4 \neq 0$
$\Rightarrow -(a+4)(2a+1)(a-1) \neq 0$
$\Rightarrow a \neq -4$ and $a \neq -\dfrac{1}{2}$ and $a \neq 1$

Also,
$$\Delta_x = \begin{vmatrix} 0 & a & -1 \\ -2a & -1 & a \\ 2a & 2a^2 & a-3 \end{vmatrix} = 2a(4a+1)(a-1) \text{ and}$$

$$\Delta_y = \begin{vmatrix} 1 & 0 & -1 \\ 2 & -2a & a \\ -1 & 2a & a-3 \end{vmatrix} = -4a(a-1) \text{ and}$$

$$\Delta_z = \begin{vmatrix} 1 & a & 0 \\ 2 & -1 & -2a \\ -1 & 2a^2 & 2a \end{vmatrix} = 2a(2a+1)(a-1)$$

When $\Delta \neq 0$
$\Rightarrow x = \dfrac{\Delta_x}{\Delta}$ and $y = \dfrac{\Delta_y}{\Delta}$ and $z = \dfrac{\Delta_z}{\Delta}$

$$\Rightarrow x = \frac{2a(4a+1)(a-1)}{-(a+4)(2a+1)(a-1)}, \; y = \frac{-4a(a-1)}{-(a+4)(2a+1)(a-1)}, \; z = \frac{2a(2a+1)(a-1)}{-(a+4)(2a+1)(a-1)}$$

$$\Rightarrow x = \frac{-2a(4a+1)}{(a+4)(2a+1)}, \; y = \frac{4a}{(a+4)(2a+1)}, \; z = \frac{-2a}{(a+4)}$$

8(a)(ii)(1)

When a = 1,

$$\text{(E)} \begin{cases} x + y - z = 0 \\ 2x - y + z = -2 \\ -x + 2y - 2z = 2 \end{cases}$$

$\Rightarrow \begin{cases} x+y-z=0 \\ 2x-y+z=-2 \end{cases}$

Consider the augmented matrix :
$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & -2 \end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & -2 \end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & \frac{2}{3} \end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & -1 & \frac{2}{3} \end{array}\right]$

$\Rightarrow$ The solutions are $x=-\dfrac{2}{3}$, $y=t+\dfrac{2}{3}$, $z=t$ where $t \in R$

8(a)(ii)(2)

When a = -4 (i.e. $\Delta=0$),

$\Delta_x = 2a(4a+1)(a-1) = -600 \neq 0$ and

$\Delta_y = -4a(a-1) = -80 \neq 0$ and

$\Delta_z = 2a(2a+1)(a-1) = -280 \neq 0$

$\Rightarrow$ (E) has NO solution.

8(b)

Using result in (a)(ii)(1),
$x=-\dfrac{2}{3}$, $y=t+\dfrac{2}{3}$, $z=t$ where $t \in R$

$24x^2+3y^2+2z$
=$24(-\dfrac{2}{3})^2+3(t+\dfrac{2}{3})^2+2t$ for some $t \in R$
=$3t^2+6t+12$ for some $t \in R$
=$3(t+1)^2+9$ for some $t \in R$
≤ 9

Therefore, the least value of $24x^2+3y^2+2z$ is 9 when $t=-1$, i.e. $x=-\dfrac{2}{3}$, $y=-\dfrac{1}{3}$, $z=-1$