2012-DSE-MATH-EP(M2)-Q11

11(a)

$$\begin{vmatrix} 1-x & 4 \\ 2 & 3-x \end{vmatrix} = 0$$

$\Rightarrow (1-x)(3-x) - 8 = 0$
$\Rightarrow x^2 - 4x - 5 = 0$
$\Rightarrow (x-5)(x+1) = 0$
$\Rightarrow x = -1$ or $x = 5$

11(b)(i)

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = x_1 \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a+4b \\ 2a+3b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2a+4b \\ 2a+4b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow a = -2t$, $b = t$ for some real number t.

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} = x_2 \begin{pmatrix} c \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} = \begin{pmatrix} 5c \\ 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c+4 \\ 2c+3 \end{pmatrix} = \begin{pmatrix} 5c \\ 5 \end{pmatrix}$$

$\Rightarrow c = 1$

Also, $|P| = 1$

$$\Rightarrow \begin{vmatrix} a & c \\ b & 1 \end{vmatrix} = 1$$

$$\Rightarrow \begin{vmatrix} -2t & 1 \\ t & 1 \end{vmatrix} = 1$$ for some real number t

$\Rightarrow -3t = 1$
$\Rightarrow t = -\frac{1}{3}$

Therefore, $P = \begin{pmatrix} \frac{2}{3} & 1 \\ -\frac{1}{3} & 1 \end{pmatrix}$

11(b)(ii)

$$P^{-1}\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}P$$

$$=\begin{pmatrix}1 & -1\\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}\begin{pmatrix}\frac{2}{3} & 1\\ -\frac{1}{3} & 1\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 1\\ \frac{5}{3} & \frac{10}{3}\end{pmatrix}\begin{pmatrix}\frac{2}{3} & 1\\ -\frac{1}{3} & 1\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix}$$

11(b)(iii)

$$P^{-1}\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}P=\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}=P\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix}P^{-1}$$

$$\Rightarrow\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}^{12}=P\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix}^{12}P^{-1}$$

$$\Rightarrow\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}^{12}=\begin{pmatrix}\frac{2}{3} & 1\\ -\frac{1}{3} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 5^{12}\end{pmatrix}\begin{pmatrix}1 & -1\\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}^{12}=\begin{pmatrix}\frac{2}{3} & 5^{12}\\ -\frac{1}{3} & 5^{12}\end{pmatrix}\begin{pmatrix}1 & -1\\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}^{12}=\frac{1}{3}\begin{pmatrix}5^{12}+2 & 2\cdot 5^{12}-2\\ 5^{12}-1 & 2\cdot 5^{12}+1\end{pmatrix}$$