# 1993-AL-P-MATH-1-Q03

# 1(a)

Consider augmented matrix of the system:

$$\left[\begin{array}{ccc|c} a & b & c & 1 \\ b & c & a & 1 \\ c & a & b & 1 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ a & b & c & 1 \\ b & c & a & 1 \\ c & a & b & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & b-a & c-a & 1-3a \\ 0 & c-b & a-b & 1-3b \\ 0 & a-c & b-c & 1-3c \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & b-a & c-a & 1-3a \\ 0 & c-b & a-b & 1-3b \\ 0 & b-c & b-a & 2-3a-3c \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & b-a & c-a & 1-3a \\ 0 & c-b & a-b & 1-3b \\ 0 & 0 & 0 & 3-3a-3b-3c \end{array}\right]$$

$\Rightarrow 3-3a-3b-3c=0$
$\Rightarrow a+b+c=1$

# 1(b)

Consider the augmented matrix and the fact that a+b+c=0

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & b-a & c-a & 1-3a \\ 0 & c-b & a-b & 1-3b \\ 0 & 0 & 0 & 3-3a-3b-3c \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & b-a & c-a & 1-3a \\ 0 & c-b & a-b & 1-3b \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & b-a & c-a & 1-3a \\ 0 & c-b & a-b & 1-3b \end{array}\right]$$

When the system (*) has a unique solution,

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & c-b & a-b \end{vmatrix} \neq 0$$

$\Rightarrow (b-a)(a-b)-(c-a)(c-b) \neq 0$
$\Rightarrow -a^2-b^2+2ab-(c^2-bc-ac+ab) \neq 0$
$\Rightarrow -a^2-b^2+2ab-c^2+bc+ac-ab \neq 0$
$\Rightarrow -a^2-b^2-c^2+bc+ac+ab \neq 0$

$$\Rightarrow -\frac{a^2}{2}-\frac{a^2}{2}-\frac{b^2}{2}-\frac{b^2}{2}-\frac{c^2}{2}-\frac{c^2}{2}+bc+ac+ab \neq 0$$

$$\Rightarrow -\frac{1}{2}(a^2+a^2+b^2+b^2+c^2+c^2-2bc-2ac-2ab) \neq 0$$

$\Rightarrow a^2-2ab+b^2+a^2-2ac+c^2+b^2-2bc+c^2 \neq 0$
$\Rightarrow (a-b)^2+(a-c)^2+(b-c)^2 \neq 0$
$\Rightarrow a \neq b$ or $a \neq c$ or $b \neq c$
$\Rightarrow$ a , b and c are NOT all equal

On the other hand, when a , b and c are NOT all equal
$\Rightarrow a \neq b$ or $a \neq c$ or $b \neq c$
$\Rightarrow (a-b)^2+(a-c)^2+(b-c)^2 \neq 0$
$\Rightarrow -a^2-b^2-c^2+bc+ac+ab \neq 0$
$\Rightarrow (b-a)(a-b)-(c-a)(c-b) \neq 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & c-b & a-b \end{vmatrix} \neq 0$$

$\Rightarrow$ The system (*) has a unique solution.

Hence the system (*) has a unique solution if and only if a , b and c are NOT all equal.

# 1(c)

if a = b = c,
$\Rightarrow$ 3a = 1 (due to result (a) a+b+c=1)
$\Rightarrow$ a = b = c = $\frac{1}{3}$

The system (*) has only one equation x + y + z = 3. Therefore, the solution is :
x = m which is any real number,
y = n which is any real number and
z = 3 - m - n