2000-AL-P-MATH-1-Q08

8(a)

(S) has an unique solution.
$\Rightarrow \Delta \neq 0$

$$\Rightarrow \begin{vmatrix} 1 & -1 & -1 \\ 2 & \lambda & -2 \\ 1 & 2\lambda+3 & \lambda^2 \end{vmatrix} \neq 0$$

$\Rightarrow \lambda^3 + 2(2\lambda+3) + 2\lambda^2 + 2 - 2(2\lambda+3) + \lambda \neq 0$
$\Rightarrow \lambda^3 + 2\lambda^2 + \lambda + 2 \neq 0$
$\Rightarrow (\lambda^2+1)(\lambda+2) \neq 0$
$\Rightarrow \lambda + 2 \neq 0$
$\Rightarrow \lambda \neq -2$

On the other hand,
$\lambda \neq -2$
$\Rightarrow \Delta = (\lambda^2+1)(\lambda+2) \neq 0$
$\Rightarrow$ (S) has an unique solution.

Hence, (S) has an unique solution if and only if $\lambda \neq -2$.

When $\lambda = -1$, then

$\Delta = (\lambda^2+1)(\lambda+2)$ and $\Delta_x = \begin{vmatrix} a & -1 & -1 \\ b & \lambda & -2 \\ c & 2\lambda+3 & \lambda^2 \end{vmatrix}$ and $\Delta_y = \begin{vmatrix} 1 & a & -1 \\ 2 & b & -2 \\ 1 & c+3 & \lambda^2 \end{vmatrix}$ and $\Delta_z = \begin{vmatrix} 1 & -1 & a \\ 2 & \lambda & b \\ 1 & 2\lambda+3 & c \end{vmatrix}$

$\Rightarrow \Delta = 2$ and $\Delta_x = \begin{vmatrix} a & -1 & -1 \\ b & -1 & -2 \\ c & 1 & 1 \end{vmatrix}$ and $\Delta_y = \begin{vmatrix} 1 & a & -1 \\ 2 & b & -2 \\ 1 & c+3 & 1 \end{vmatrix}$ and $\Delta_z = \begin{vmatrix} 1 & -1 & a \\ 2 & -1 & b \\ 1 & 1 & c \end{vmatrix}$

$\Rightarrow \Delta = 2$ and $\Delta_x = a + c$ and $\Delta_y = -4a + 2b$ and $\Delta_z = 3a - 2b + c$

$\Rightarrow x = \dfrac{\Delta_x}{\Delta}$ and $y = \dfrac{\Delta_y}{\Delta}$ and $z = \dfrac{\Delta_z}{\Delta}$

$\Rightarrow x = \dfrac{a+c}{2}$ and $y = \dfrac{-4a+2b}{2}$ and $z = \dfrac{3a-2b+c}{2}$

$$\Rightarrow x=\frac{a+c}{2} \text{ and } y=-2a+b \text{ and } z=\frac{3a-2b+c}{2}$$

8(b)(i)

When $\lambda=-2$ (i.e. $\Delta=0$) and (S) has infinitely many solutions

$\Rightarrow \Delta_x=\Delta_y=\Delta_z=0$

$$\Rightarrow \begin{vmatrix} a & -1 & -1 \\ b & \lambda & -2 \\ c & 2\lambda+3 & \lambda^2 \end{vmatrix}=0 \text{ and } \begin{vmatrix} 1 & a & -1 \\ 2 & b & -2 \\ 1 & c+3 & \lambda^2 \end{vmatrix}=0 \text{ and } \begin{vmatrix} 1 & -1 & a \\ 2 & \lambda & b \\ 1 & 2\lambda+3 & c \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} a & -1 & -1 \\ b & -2 & -2 \\ c & -1 & 4 \end{vmatrix}=0 \text{ and } \begin{vmatrix} 1 & a & -1 \\ 2 & b & -2 \\ 1 & c+3 & 4 \end{vmatrix}=0 \text{ and } \begin{vmatrix} 1 & -1 & a \\ 2 & -2 & b \\ 1 & -1 & c \end{vmatrix}=0$$

$\Rightarrow -10a+5b=0$ and $-10a+5b=0$ and $0=0$

$\Rightarrow -10a+5b=0$

$\Rightarrow b=2a$

8(b)(ii)

When $\lambda=-2$, a=-1, b=-2, c=3 (c = -3 is wrong)

$$\text{(S) } \begin{cases} x-y-z=-1 \\ 2x-2y-2z=-2 \\ x-y+4z=3 \end{cases}$$

$$\Rightarrow \text{(S) } \begin{cases} x-y-z=-1 \\ x-y+4z=3 \end{cases}$$

$$\Rightarrow \text{(S) } \begin{cases} x-y-z=-1 \\ 5z=4 \end{cases}$$

$$\Rightarrow \text{(S) } \begin{cases} x-y=-\dfrac{1}{5} \\ z=\dfrac{4}{5} \end{cases}$$

$\Rightarrow$ The solutions are $x=t-\frac{1}{5}$, $y=t\in R$, $z=\frac{4}{5}$

8(c)

$$\text{(T) } \begin{cases} x-y-z+3\mu-5=0 \\ 2x-2y-2z+2\mu-2=0 \\ x-y+4z-\mu-1=0 \end{cases}$$

$$\Rightarrow \text{(T) } \begin{cases} x-y-z=5-3\mu \\ 2x-2y-2z=2-2\mu \\ x-y+4z=\mu+1 \end{cases}$$

(T) is equivalent to (S) where $\lambda=-2$, $a=5-3\mu$, $b=2-2\mu$ and $c=\mu+1$

According to (b)(i), (T) is consistent when

$b = 2a$

$\Rightarrow 2 - 2\mu = 2(5 - 3\mu)$

$\Rightarrow \mu = 2$

$\Rightarrow a = 5 - 3\mu$, $b = 2 - 2\mu$ and $c = \mu + 1$

$\Rightarrow a = -1$, $b = -2$ and $c = 3$

According (b)(ii), the solutions are $x = t - \dfrac{1}{5}$, $y = t \in R$, $z = \dfrac{4}{5}$

On the other hand, when $\mu \neq 2$, (T) is inconsistent.