

# 2017-DSE-MATH-EP(M2)-Q12

## 12(a)

Let  $P(n)$  be the statement that  $A^n = 3^n I + 3^{n-1} n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  for all positive integers  $n$ .

When  $n = 1$ ,

R.H.S.

$$= 3^n I + 3^{n-1} n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= 3I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

$$= A = A^1 = L.H.S.$$

Therefore,  $P(1)$  is true.

Assume  $P(k)$  is true for some positive integer  $k \geq 1$ . Then

$$A^{k+1}$$

$$= A^k A$$

$$= [3^k I + 3^{k-1} k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}] A$$

$$= 3^k A + 3^{k-1} k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A$$

$$= 3^k \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} + 3^{k-1} k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

$$= 3^k \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} + 3^{k-1} k \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$$

$$= 3^k \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + 3^k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 3^{k-1} k \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$$

$$= 3^{k+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3^k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 3^k k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= 3^{k+1} I + 3^k \begin{pmatrix} 0 & k+1 \\ 0 & 0 \end{pmatrix}$$

$$= 3^{k+1} I + 3^k (k+1) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= 3^{k+1} I + 3^{(k+1)-1} (k+1) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Therefore,  $P(k+1)$  is true and by mathematical induction  $P(n)$  is true.

## 12(b)(i)

$$\begin{aligned}
& P^{-1}BP \\
&= P^{-1} \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} \\
&= P^{-1} \begin{pmatrix} -3 & -1 \\ 6 & -1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 6 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \\
&= A
\end{aligned}$$

## 12(b)(ii)

$$\begin{aligned}
P^{-1}BP &= A \\
\Rightarrow B &= PAP^{-1} \\
\Rightarrow B^n &= PA^nP^{-1} \\
\Rightarrow B^n &= P \left[ 3^n I + 3^{n-1}n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] P^{-1} \\
\Rightarrow B^n &= PA^nP^{-1} \\
\Rightarrow B^n &= 3^n PIP^{-1} + 3^{n-1}nP \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} P^{-1} \\
\Rightarrow B^n &= 3^n I + 3^{n-1}n \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \\
\Rightarrow B^n &= 3^n I + 3^{n-1}n \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \\
\Rightarrow B^n &= 3^n I + 3^{n-1}n \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}
\end{aligned}$$

## 12(b)(iii)

For all positive integers m,  $A^m - B^m$

$$\begin{aligned}
&= 3^m I + 3^{m-1}m \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - 3^m I - 3^{m-1}m \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \\
&= 3^{m-1}m \left[ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \right] \\
&= 3^{m-1}m \begin{pmatrix} -2 & 0 \\ 4 & 2 \end{pmatrix}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& |A^m - B^m| \\
&= 3^{m-1}m \begin{vmatrix} -2 & 0 \\ 4 & 2 \end{vmatrix} \\
&= -4 \cdot 3^{m-1} < 0 < 4m^2
\end{aligned}$$

Therefore,  $|A^m - B^m| \neq 4m^2$