2003-AL-P-MATH-1-Q08

8(a)

$$det\begin{pmatrix}-2-\alpha & \sqrt{3}\\ \sqrt{3} & -\alpha\end{pmatrix}=0$$

$\Rightarrow \alpha(\alpha+2)-3=0$
$\Rightarrow \alpha^2+2\alpha-3=0$
$\Rightarrow (\alpha+3)(\alpha-1)=0$
$\Rightarrow \alpha=-3$ or $\alpha=1$

8(b)

Now $\alpha_1=-3$ and $\alpha_2=1$

$$\begin{pmatrix}-2-\alpha_1 & \sqrt{3}\\ \sqrt{3} & -\alpha_1\end{pmatrix}\begin{pmatrix}cos\theta_1\\ sin\theta_1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & 3\end{pmatrix}\begin{pmatrix}cos\theta_1\\ sin\theta_1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$\Rightarrow cos\theta_1+\sqrt{3}sin\theta_1=0$

$\Rightarrow tan\theta_1=-\dfrac{1}{\sqrt{3}}$
$\Rightarrow \theta_1=\dfrac{5\pi}{6}$

Also,
$$\begin{pmatrix}-2-\alpha_2 & \sqrt{3}\\ \sqrt{3} & -\alpha_2\end{pmatrix}\begin{pmatrix}cos\theta_2\\ sin\theta_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}-3 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}cos\theta_2\\ sin\theta_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$\Rightarrow \sqrt{3}cos\theta_2-sin\theta_2=0$
$\Rightarrow tan\theta_2=\sqrt{3}$
$\Rightarrow \theta_2=\dfrac{\pi}{3}$

$$P=\begin{pmatrix}cos\theta_1 & cos\theta_2\\ sin\theta_1 & sin\theta_2\end{pmatrix}=\begin{pmatrix}cos\frac{5\pi}{6} & cos\frac{\pi}{3}\\ sin\frac{5\pi}{6} & sin\frac{\pi}{3}\end{pmatrix}=\begin{pmatrix}\frac{-\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-\sqrt{3} & 1\\ 1 & \sqrt{3}\end{pmatrix}$$

$$\Rightarrow P^2=\frac{1}{2}\begin{pmatrix}-\sqrt{3} & 1\\ 1 & \sqrt{3}\end{pmatrix}\frac{1}{2}\begin{pmatrix}-\sqrt{3} & 1\\ 1 & \sqrt{3}\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = I$$

Since $P^2 = I$
$\Rightarrow PP = I$
$\Rightarrow P^{-1} = P$

Therefore,

- For n = 2k where k is a positive integer,
  $P^n = P^{2k} = (P^2)^k = I^k = I$
- For n = 2k-1 where k is a positive integer,
  $P^n = P^{2k-1} = P^{2k}P^{-1} = IP = P$

$$P^{-1}\begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}P$$

$$= P\begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}P$$

$$= \frac{1}{2}\begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 3\sqrt{3} & -3 \\ 1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -12 & 0 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$$

8(c)

$$\begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}^n$$

$$= (P\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}P^{-1})^n$$

$$= P\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}^n P^{-1}$$

$$= P\begin{pmatrix} (-3)^n & 0 \\ 0 & 1 \end{pmatrix}P$$

$$= \frac{1}{4} \begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} (-3)^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} (-\sqrt{3})(-3)^n & 1 \\ (-3)^n & \sqrt{3} \end{pmatrix} \begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 3(-3)^n + 1 & (-\sqrt{3})(-3)^n + \sqrt{3} \\ (-\sqrt{3})(-3)^n + \sqrt{3} & (-3)^n + 3 \end{pmatrix}$$