2000-AL-P-MATH-1-Q01

2000-AL-P MATH 1 #01

Let P(n) be the statement that $M^{2n} = \begin{pmatrix} 1 & 0 & 0 \\ n[\lambda(1+b)+\mu a] & 1 & 0 \\ n[\lambda c+\mu(1-b)] & 0 & 1 \end{pmatrix}$ for all positive integers n.

When n = 1,

$$M^2 = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & b & a \\ \mu & c & -b \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \lambda & b & a \\ \mu & c & -b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda+b\lambda+a\mu & b^2+ac & ab-ab \\ \mu+c\lambda-b\mu & bc-bc & ac+b^2 \end{pmatrix}$$

$$\Rightarrow M^2 = \begin{pmatrix} 1 & 0 & 0 \\ \lambda(1+b)+\mu a & 1 & 0 \\ \lambda c+\mu(1-b) & 0 & 1 \end{pmatrix} \ (\because b^2+ac=1)$$

Therefore P(1) is true.

Assume that $P(k)$ is true for some positive integer $k \geq 1$. Then

$$M^{2(k+1)} = M^{2k+2} = M^{2k}M^2 = \begin{pmatrix} 1 & 0 & 0 \\ k[\lambda(1+b)+\mu a] & 1 & 0 \\ k[\lambda c+\mu(1-b)] & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \lambda(1+b)+\mu a & 1 & 0 \\ \lambda c+\mu(1-b) & 0 & 1 \end{pmatrix}$$

$$\Rightarrow M^{2(k+1)} = \begin{pmatrix} 1 & 0 & 0 \\ k[\lambda(1+b)+\mu a]+\lambda(1+b)+\mu a & 1 & 0 \\ k[\lambda c+\mu(1-b)]+\lambda c+\mu(1-b) & 1 & 0 \end{pmatrix}$$

$$\Rightarrow M^{2(k+1)} = \begin{pmatrix} 1 & 0 & 0 \\ (k+1)[\lambda(1+b)+\mu a] & 1 & 0 \\ (k+1)[\lambda c+\mu(1-b)] & 1 & 0 \end{pmatrix}$$

Therefore P(k+1) is true. By mathematical induction, P(n) is true for positive integers n.

Let $\lambda=-2,\ \mu=1,\ a=2,\ b=3,\ c=-4,\ n=1000$. Then

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & 2 \\ 1 & -4 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1000[-2(1+3)+2] & 1 & 0 \\ 1000[(-2)(-4)+1-3] & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1000(-6) & 1 & 0 \\ 1000(6) & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -6000 & 1 & 0 \\ 6000 & 0 & 1 \end{pmatrix}$$