ME 51500/15800 HW 7 SOLUTIONS P. GANATOS

1) From two position fixes $\mathbf{r}_1(t_1)$ and $\mathbf{r}_2(t_2)$, develop the approximate formula for the velocity vector $\mathbf{V}_1$ at time t_1

$$\mathbf{V}_1 = -[1/\tau - (\varepsilon_1/3)\tau]\,\mathbf{r}_1 + [1/\tau + (\varepsilon_2/6)\tau]\,\mathbf{r}_2$$

valid to third order in the time interval $\tau = t_2 - t_1$.

$$\bar{r} = \bar{a}_0 + t\bar{a}_1 + t^2\bar{a}_2 + t^3\bar{a}_3$$

$$V = \frac{d\bar{r}}{dt} = a_1 + 2t\bar{a}_2 + 3t^2\bar{a}_3$$

$$-\varepsilon\bar{r} = \frac{d^2\bar{r}}{dt^2} = 2\bar{a}_2 + 6t\bar{a}_3 \qquad \text{where } \varepsilon = \frac{\mu}{r^3}$$

$$\left.\begin{aligned}
\bar{r}_1 &= \bar{a}_0 \\
\bar{r}_2 &= \bar{a}_0 + \tau\bar{a}_1 + \tau^2\bar{a}_2 + \tau^3\bar{a}_3 \\
\bar{V}_1 &= \bar{a}_1 \\
-\varepsilon_1\bar{r}_1 &= 2\bar{a}_2 \\
-\varepsilon_2\bar{r}_2 &= 2\bar{a}_2 + 6\tau\bar{a}_3
\end{aligned}\right\} \quad \begin{array}{l}\text{5 equations} \\ \text{5 unknowns}: \bar{V}_1, \bar{a}_0, \ldots, \bar{a}_3\end{array}$$

Solve for $\bar{V}_1$

$$\boxed{\bar{V}_1 = -\left(\frac{1}{\tau} - \frac{\varepsilon_1}{3}\tau\right)\bar{r}_1 + \left(\frac{1}{\tau} + \frac{\varepsilon_2}{6}\tau\right)\bar{r}_2}$$

2) Two position fixes of a spacecraft in interplanetary space at times

$$t_1 = 0.010576712 \text{ year} \qquad t_2 = 0.021370777 \text{ year}$$

are found to be

$$\mathbf{r}(t_1) = \begin{bmatrix} 0.159321004 \\ 0.579266185 \\ 0.052359607 \end{bmatrix} \text{a.u.} \qquad \mathbf{r}(t_2) = \begin{bmatrix} 0.057594337 \\ 0.605750797 \\ 0.068345246 \end{bmatrix} \text{a.u.}$$

a) Determine the velocity vector at the location corresponding to the earlier time.

b) If a third position fix at time $t_3 = 0.005274926$ year is included in the data, where

$$\mathbf{r}(t_3) = \begin{bmatrix} 0.208200171 \\ 0.561804188 \\ 0.044088057 \end{bmatrix} \text{a.u.}$$

calculate the velocity vector at the same location as determined above. Use both equations (7.53) and (7.51) given in the notes.

c) Compare your results with the exact value given by

$$\mathbf{V}(0.010576712 \text{ year}) = \begin{bmatrix} -9.303603251 \\ 3.018641330 \\ 1.536362143 \end{bmatrix} \text{a.u./year}$$

a) $$T^2 = \frac{4\pi^2}{\mu} a^3$$

$$\mu_{SUN} = \frac{4\pi^2 a^3}{T^2} = \frac{4\pi^2 (1\,au)^3}{(1\,yr)^2} = 4\pi^2\ au^3/yr^2$$

$$\tau = t_2 - t_1 = 0.010794065\ yr$$

$$r_1 = 0.6030539145\ au \qquad \varepsilon_1 = \frac{\mu}{r_1^3} = 180.0077959\ yr^{-2}$$

$$r_2 = 0.6123089158\ au \qquad \varepsilon_2 = \frac{\mu}{r_2^3} = 171.9681373\ yr^{-2}$$

substitute numerical values into result of problem 1

$$\boxed{\bar{V}_1 = \begin{bmatrix} -9.303308992 \\ 3.016204370 \\ 1.536021596 \end{bmatrix} au/yr} \qquad \text{error} = \begin{bmatrix} -0.00316\% \\ -0.0807\% \\ -0.0222\% \end{bmatrix}$$

b) With additional position fix, taking

$t_1 = 0.005274926$ yr

$t_2 = 0.010576712$ yr

$t_3 = 0.021370777$ yr

Equation (7.53) in notes gives:

$$\boxed{\bar{V}_2 = \begin{bmatrix} -9.303478168 \\ 3.018664140 \\ 1.536349482 \end{bmatrix} \text{au/yr}} \qquad \text{error} = \begin{bmatrix} -0.00134\% \\ 0.000756\% \\ -0.000824\% \end{bmatrix}$$

Equation (7.51) in notes gives:

$$\boxed{\bar{V}_2 = \begin{bmatrix} -9.303601823 \\ 3.018636631 \\ 1.536361550 \end{bmatrix} \text{au/yr}} \qquad \text{error} = \begin{bmatrix} -0.0000153\% \\ -0.000156\% \\ -0.0000386\% \end{bmatrix}$$

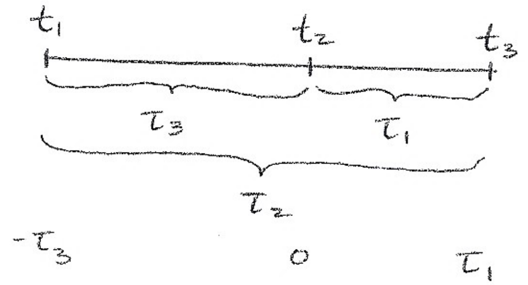
c) Percent error for each of the results is given above.

3) Using appropriate Taylor series expansions, show that

$$\left.\frac{d\mathbf{l}_\rho}{dt}\right|_{t_2} = -\frac{\tau_1}{\tau_2\tau_3}\mathbf{l}_{\rho 1} + \frac{\tau_1 - \tau_3}{\tau_1\tau_3}\mathbf{l}_{\rho 2} + \frac{\tau_3}{\tau_1\tau_2}\mathbf{l}_{\rho 3}$$

$$\left.\frac{d^2\mathbf{l}_\rho}{dt^2}\right|_{t_2} = \frac{2}{\tau_2\tau_3}\mathbf{l}_{\rho 1} - \frac{2}{\tau_1\tau_3}\mathbf{l}_{\rho 2} + \frac{2}{\tau_1\tau_2}\mathbf{l}_{\rho 3}$$

are valid to second order in the time intervals where the τ's are defined following equation (7.49c) in the notes. More accurate values for these derivatives can be obtained if more than three sets of observational data are available.



Taylor series expansions about t_2:

$$\hat{i}_{e_1} = \hat{i}_{e_2} + \left.\frac{d\hat{i}_e}{dt}\right|_{t_2}(-\tau_3) + \left.\frac{d^2\hat{i}_e}{dt^2}\right|_{t_2}\frac{(-\tau_3)^2}{2}$$

$$\hat{i}_{e_3} = \hat{i}_{e_2} + \left.\frac{d\hat{i}_e}{dt}\right|_{t_2}(\tau_1) + \left.\frac{d^2\hat{i}_e}{dt^2}\right|_{t_2}\frac{(\tau_1)^2}{2}$$

Solve simultaneously for $\left.\frac{d\hat{i}_e}{dt}\right|_{t_2}$ and $\left.\frac{d^2\hat{i}_e}{dt^2}\right|_{t_2}$

$$\left.\frac{d\hat{i}_e}{dt}\right|_{t_2} = \frac{-\tau_1^2\,\hat{i}_{e1} + (\tau_1^2 - \tau_3^2)\,\hat{i}_{e2} + \tau_3^2\,\hat{i}_{e3}}{\tau_1\tau_3(\tau_1+\tau_3)}$$

$$\left.\frac{d^2\hat{i}_e}{dt^2}\right|_{t_2} = \frac{2\tau_1\,\hat{i}_{e1} - 2(\tau_1+\tau_3)\,\hat{i}_{e2} + 2\tau_3\,\hat{i}_{e3}}{\tau_1\tau_3(\tau_1+\tau_3)}$$

Using $\tau_2 = \tau_1 + \tau_3$ get

$$\boxed{\begin{aligned} \left.\frac{d\hat{i}_\rho}{dt}\right|_{t_2} &= -\frac{\tau_1}{\tau_2\tau_3}\hat{i}_{\rho_1} + \frac{\tau_1-\tau_3}{\tau_1\tau_3}\hat{i}_{\rho_2} + \frac{\tau_3}{\tau_1\tau_2}\hat{i}_{\rho_3} \\ \left.\frac{d^2\hat{i}_\rho}{dt^2}\right|_{t_2} &= \frac{2}{\tau_2\tau_3}\hat{i}_{\rho_1} - \frac{2}{\tau_1\tau_3}\hat{i}_{\rho_2} + \frac{2}{\tau_1\tau_2}\hat{i}_{\rho_3} \end{aligned}}$$