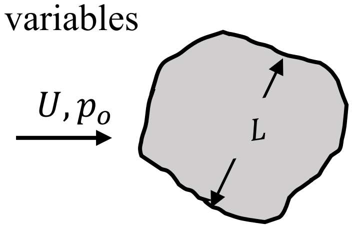
IV. Scaling & Hydrodynamic Parameter

The Navier-Stokes Equations presented in II and III have very few exact solutions. To obtain applicable solutions for other more complex problems, approximations and assumptions have to be introduced that can simplify the equation and provide analytical or numerical solutions.
To this end, the Navier-Stokes equations can be written in dimensionless form by introducing characteristic scales.

For example, a flow over a solid body, or a flow through a pipe has a typical geometric dimension associated with this flow. Define this characteristic dimension as the length scale L.
Similarly, the free stream velocity, or average fluid velocity can de defined as the characteristic velocity, U.

Other characteristic quantities can include the ambient pressure p_o, time scale τ, and the gravitational constant g_o.

Using the characteristic dimensional variables, dimensionless variables can be defined as follows:



$$u_i^* = \frac{u_i}{U} \qquad x_i^* = \frac{x_i}{L}$$
$$P^* = \frac{p}{P_0} \qquad t^* = \frac{t}{\tau}$$

The time scale can be defined as $\tau = L/U$, but other definitions are possible.

If selected properly, the dimensionless variables are assumed to be $O(1)$. This means that all the dimensionless quantities have numerical values that cannot be much different than unity.

Using the definition of the dimensionless variables, both dependent and independent, the Navier Stokes equations can be written in dimensionless form

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = g_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$\frac{U}{\tau}\left(\frac{\partial u_i}{\partial t}\right)^* + \frac{U^2}{L}\left(u_j\frac{\partial u_i}{\partial x_j}\right)^* = g_i - \frac{P_o}{\rho L}\left(\frac{\partial p}{\partial x_i}\right)^* + \frac{\nu U}{L^2}\left(\frac{\partial^2 u_i}{\partial x_j \partial x_j}\right)^*$$

Multiplying the equation by $\frac{L}{U^2}$ gives

$$\frac{L}{U\tau}\left(\frac{\partial u_i}{\partial t}\right)^* + \left(u_j\frac{\partial u_i}{\partial x_j}\right)^* = g_i^*\frac{g_0 L}{U^2} - \frac{P_o}{\rho U^2}\left(\frac{\partial p}{\partial x_i}\right)^* + \frac{\nu}{UL}\left(\frac{\partial^2 u_i}{\partial x_j \partial x_j}\right)^*$$

where $g_i^* = g_i/g_o$

Define the following dimensionless parameters

Strouhal Number	$S_t = \frac{L}{U\tau}$	(IV-1)
Froude Number	$F_r = \frac{U^2}{g_0 L}$	(IV-2)
Euler Number	$E = \frac{P_o}{\rho U^2}$	(IV-3)
Reynolds Number	$Re = \frac{UL}{\nu}$	(IV-4)

The meaning of these parameters is as follows:

Reynolds number (Osborne Reynolds, 1842-1912): the ratio between inertial forces and viscous forces $\rho U^2/(\frac{\mu U}{L})$

Froude number (William Froude, 1810-1879): the ratio inertial forces to gravitational forces $\rho U^2/\rho g L$

Euler number (Leonhard Euler, 1707-1783): the ratio between the force due to pressure and inertial forces,

Strouhal number (Vincent Strouhal, 1850-1922): represents fluid oscillations or the rate of propagation of vorticity.

With these definitions the dimensionless Navier Stokes equations become

$$S_t \left(\frac{\partial u_i}{\partial t}\right)^* + \left(u_j \frac{\partial u_i}{\partial x_j}\right)^* = g_i^* \frac{1}{F_r} - E \left(\frac{\partial p}{\partial x_i}\right)^* + \frac{1}{Re} \left(\frac{\partial^2 u_i}{\partial x_j \partial x_j}\right)^* \quad \text{(IV-5)}$$

All the dimensionless quantities in this equation are $O(1)$, but the parameters can have different magnitudes which can be used to simplify the Navier Stokes equations. For example:

<u>Large Reynolds Number</u>.

$Re \gg 1$, with $S_t \sim 1$ and $E \sim 1$, the viscous terms in (IV-5) can be ignored and the flow can behave as inviscid flow provided it is not turbulent,

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = g_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i}$$

This equation can be integrated into the familiar Bernoulli Equation:

$$\frac{u_i u_i}{2} + \frac{p}{\rho} + gh = Constant$$

where h is the elevation

<u>Small Reynolds Number</u>.

$Re \ll 1$, and $S_t Re \sim 1$, $S_t E \sim 1$.

The non-linear term can be neglected, and the Navier Stokes equations become

$$\frac{\partial u_i}{\partial t} = g_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad \text{(IV-6)}$$

If the $S_t \sim 1$, the transient term can also be ignored, and the equation is reduced to

$$0 = g_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad \text{(IV-7)}$$

which represents very slow-moving steady flow called *Stokes Flow* or *Creeping Motion*.

Other parameters are generated from boundary conditions, or compressibility.

Weber Number: $$W = U\sqrt{\frac{L\rho}{\gamma}} \tag{VI-8}$$

represents the square root of the ratio between inertial forces and surface tension per volume γ/L^2.

Mach Number: $$M = \frac{U}{c} \tag{VI-9}$$

Represents the ratio between the fluid speed and the speed of sound c. The speed of sound can be written in the form

$$c = \sqrt{B/\rho}$$

where B is the *Bulk Modulus*, which measures the resistance of the substance to compression, and is defined as follows

$$B = \rho\frac{dp}{d\rho}$$

Generally, when $M < 0.3$, the fluid (gas) can be treated as incompressible.