ME 57200 Aerodynamic Design

Lecture #3: Dimensional Analysis

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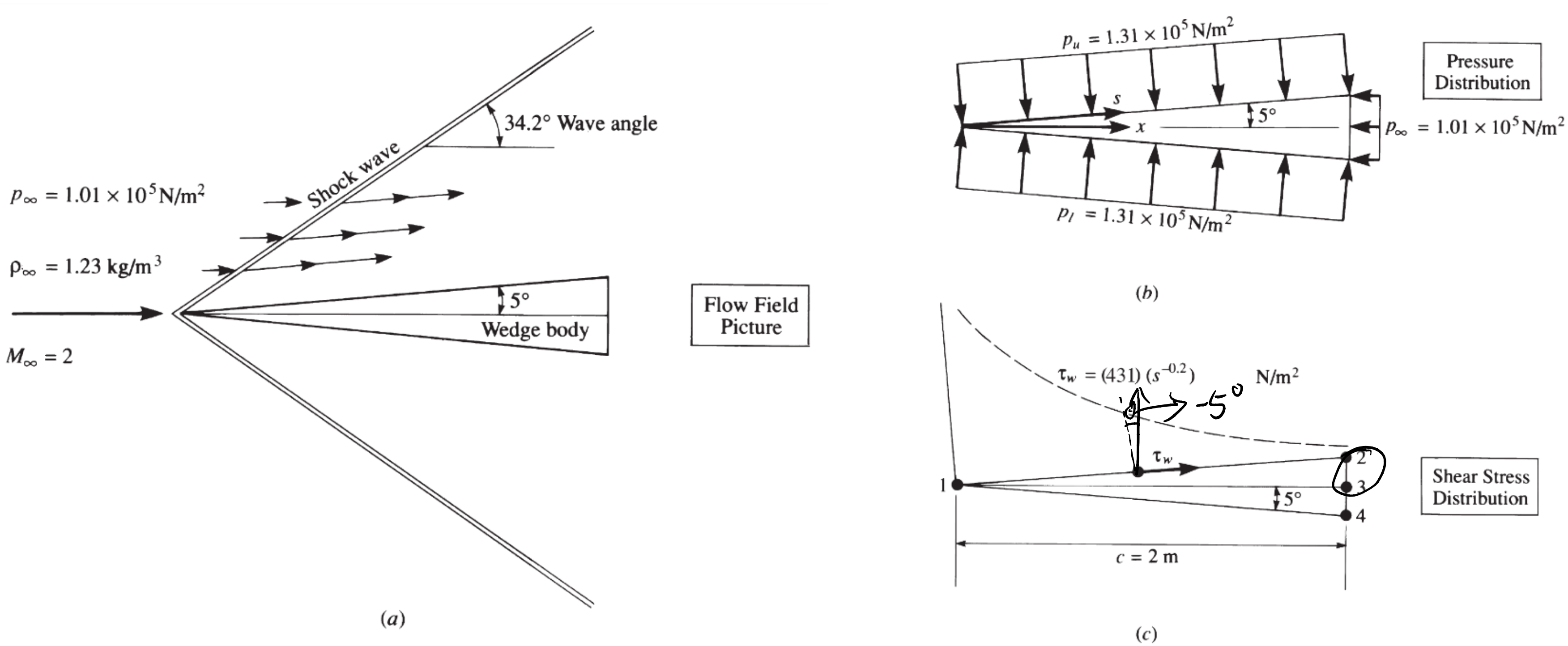
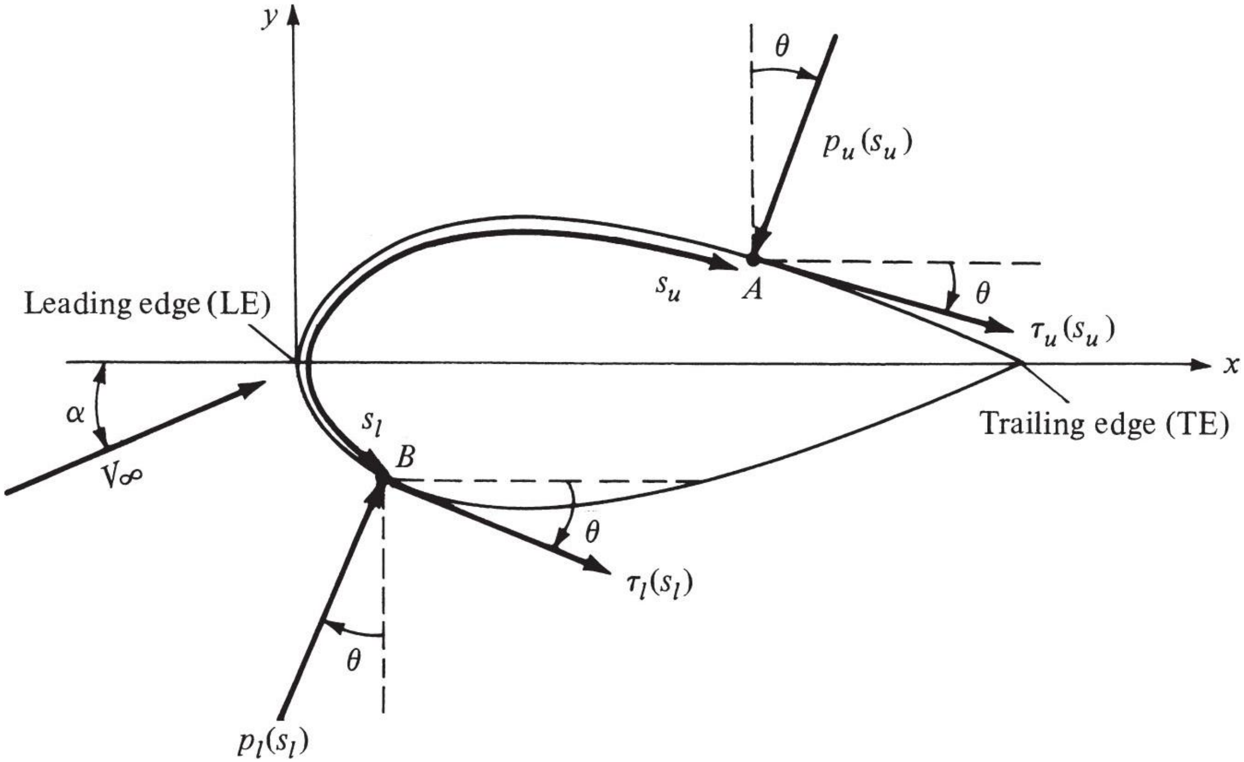
In-class Practice

Consider the supersonic flow over a 5° half-angle wedge at zero angle of attack, as sketched in **Figure 1.23*a***. The freestream Mach number ahead of the wedge is 2.0, and the freestream pressure and density are 1.01×10^5 N/m^2 and 1.23 kg/m^3, respectively (this corresponds to standard sea level conditions). The pressures on the upper and lower surfaces of the wedge are constant with distance s and equal to each other, namely, $p_u = p_l = 1.31 \times 10^5$ N/m^2, as shown in **Figure 1.23*b***. The pressure exerted on the base of the wedge is equal to p_∞. As seen in **Figure 1.23*c***, the shear stress varies over both the upper and lower surfaces as $\tau_w = 431 s^{-0.2}$. The chord length, c, of the wedge is 2 m. Calculate the drag coefficient for the wedge.



Type of Forces Acting on Airfoil

- How to calculate the aerodynamic forces



In-class Practice

$$D' = \int_{LE}^{TE} \left(-p_u \sin\theta + \tau_u \cos\theta\right) ds_u + \int_{LE}^{TE} \left(p_l \sin\theta + \tau_l \cos\theta\right) ds_l$$

$$\int_{LE}^{TE} -p_u \sin\theta ds_u = \int_{s_1}^{s_2} -(1.31\times10^5)\sin(-5^\circ) ds_u + \int_{s_2}^{s_3} -(1.01\times10^5)\sin 90^\circ ds_u$$

$$= 1.142\times10^4 (s_2 - s_1) - 1.01\times10^5 (s_3 - s_2)$$

$$= 1.142\times10^4 \left(\frac{c}{\cos 5^\circ}\right) - 1.01\times10^5 (c)(\tan 5^\circ)$$

$$= 1.142\times10^4 (2.008) - 1.01\times10^5 (0.175) = 5260\text{N}$$

$$\int_{s_1}^{s_2} ds_a$$

$$\int_{LE}^{TE} p_l \sin\theta ds_l = \int_{s_1}^{s_4} (1.31\times10^5)\sin(5^\circ) ds_l + \int_{s_4}^{s_3} (1.01\times10^5)\sin(-90^\circ) ds_l$$

$$= 1.142\times10^4 (s_4 - s_1) + 1.01\times10^5 (-1)(s_3 - s_4)$$

$$= 1.142\times10^4 \left(\frac{c}{\cos 5^\circ}\right) - 1.01\times10^5 (c)(\tan 5^\circ)$$

$$= 2.293\times10^4 - 1.767\times10^4 = 5260\text{N}$$

In-class Practice

$$D' = \int_{\text{LE}}^{\text{TE}} \left(-p_u \sin\theta + \tau_u \cos\theta\right) ds_u + \int_{\text{LE}}^{\text{TE}} \left(p_l \sin\theta + \tau_l \cos\theta\right) ds_l$$

$$\int_{\text{LE}}^{\text{TE}} \tau_u \cos\theta ds_u = \int_{s_1}^{s_2} 431 s^{-0.2} \cos(-5^\circ) ds_u$$

$$= 429 \left(\frac{s_2^{0.8} - s_1^{0.8}}{0.8} \right)$$

$$= 429 \left(\frac{c}{\cos 5^\circ} \right)^{0.8} \frac{1}{0.8} = 936.5\text{N}$$

$$\int_{\text{LE}}^{\text{TE}} \tau_l \cos\theta ds_l = \int_{s_1}^{s_4} 431 s^{-0.2} \cos(-5^\circ) ds_l$$

$$= 429 \left(\frac{s_4^{0.8} - s_1^{0.8}}{0.8} \right)$$

$$= 429 \left(\frac{c}{\cos 5^\circ} \right)^{0.8} \frac{1}{0.8} = 936.5\text{N}$$

In-class Practice

$$D' = \int_{LE}^{TE} (-p_u \sin\theta + \tau_u \cos\theta)\, ds_u + \int_{LE}^{TE} (p_l \sin\theta + \tau_l \cos\theta)\, ds_l$$

$$D' = \underbrace{1.052 \times 10^4}_{\text{pressure drag}} + \underbrace{0.1873 \times 10^4}_{\text{skin friction drag}} = \boxed{1.24 \times 10^4 \text{ N}}$$

The drag coefficient is obtained as follows. The velocity of the freestream is twice the sonic speed, which is given by

$q_\infty = \frac{1}{2} \rho_\infty V_\infty^2$

$$a_\infty = \sqrt{\gamma R T_\infty} = \sqrt{(1.4)(287)(288)} = 340.2 \text{m/s}$$

$M_\infty = \frac{V_\infty}{a_\infty} = 2$

$$V_\infty = 2(340.2) = 680.4 \text{ m/s.}$$

$$q_\infty = \tfrac{1}{2}\rho_\infty V_\infty^2 = (0.5)(1.23)(680.4)^2 = 2.847 \times 10^5 \text{N/m}^2$$

$$S = c(1) = 2.0\text{m}^2$$

$$c_d = \frac{D'}{q_\infty S} = \frac{1.24 \times 10^4}{(2.847 \times 10^5)(2)} = \boxed{0.022}$$

In-class Practice

An alternate solution?

Aerodynamic Coefficients

Lift coefficient: $$C_L \equiv \frac{L}{q_\infty S}$$

Drag coefficient: $$C_D \equiv \frac{D}{q_\infty S}$$

Normal force coefficient: $$C_N \equiv \frac{N}{q_\infty S}$$

Axial force coefficient: $$C_A \equiv \frac{A}{q_\infty S}$$

Aerodynamic Coefficients

Moment coefficient: $$C_M \equiv \frac{M}{q_\infty S l}$$

Aerodynamic Coefficients

- The symbols in capital letters (i.e., C_L, C_D, C_A , C_N , C_M) denote the force and moment coefficients for a complete three-dimensional body such as an airplane or a finite wing.

- For a two-dimensional body, the forces and moments are per unit span. The aerodynamic coefficients are denoted by lowercase letters:

$$c_l \equiv \frac{L'}{q_\infty c} \quad c_d \equiv \frac{D'}{q_\infty c} \quad c_m \equiv \frac{M'}{q_\infty c^2}$$

where the reference area $S = c(1) = c$.

Aerodynamic Coefficients

- Two additional dimensionless quantities:

Pressure coefficient: $$C_p \equiv \frac{p - p_\infty}{q_\infty}$$

Skin friction coefficient: $$c_f \equiv \frac{\tau}{q_\infty}$$

In-class Practice

An alternate solution?

1. Calculate the pressure coefficients and skin friction coefficients.

2. Then integrate the pressure coefficients and skin friction coefficients to obtain the drag coefficient.

Dimensions/Units

The F-22 Raptor

- Maximum thrust is in the 70,000 lbf (312 kN) class.
- *Is the engine performance of F-22 good or bad?*

Dimensions/Units

Person 1

Weight: 40 kg

Age: 8

Height: 105 cm

Average Weight at 8: 25.6 kg

BMI: 30.0 (Obesity)

Person 2

Weight: 50 kg

Age: 25

Height: 180 cm

Average Weight at 25: 73.8 kg

BMI: 15.4 (Underweight)

Dimensions/Units

The F-22 Raptor

- Maximum thrust is in the 70,000 lbf (312 kN) class.
- *Is the engine performance of F-22 good or bad?*

Dual Pratt & Whitney F119-PW-100 augmented turbofan engines

Dimensions/Units

Type	Takeoff weight/(10^3kg)	Maximum thrust of engine/(10^3kg)	Takeoff thrust-weight ratio
Su-27	22.5	2×7.7	0.68
Mig-29	15.24	2×5	0.67
F-15	20	2×6.7	0.67
F-16C	10.8	1×6.7	0.62
Su-30	25.7	2×7.7	0.60
X-47A	2.5	1×1.45	0.58
X-45A	5.5	1×3	0.55
X-45B	9.7	1×4.8	0.49
A-10	20	2×4.18	0.42
X-47B	20.9	1×6.7	0.32
X-45C	16.6	1×4.8	0.29
B-2	152.6	4×8.6	0.23

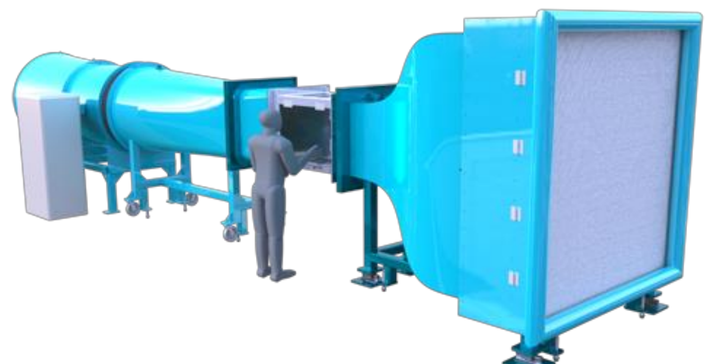
The F-22's thrust-to-weight ratio at typical combat weight is nearly at unity in maximum military power and 1.25 in full afterburner.

Dimensions/Units

F-111B Take-off

Full-Scale Wind Tunnel Test at NASA Ames Research Center

F-111B Model



Laboratory Wind Tunnel Facility

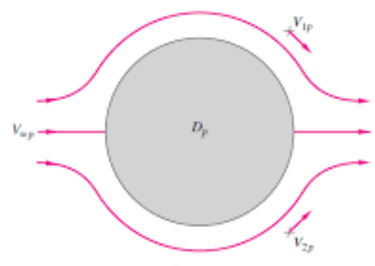
Dimensional Analysis and Similitude

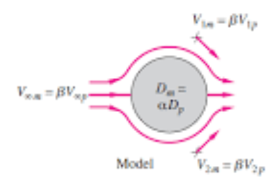
- **Geometric similarity: The model have the same shape as the prototype:**

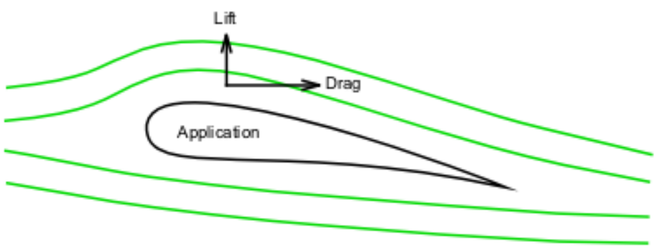
$$\frac{l}{L} = \frac{w}{W} = \frac{h}{H} = \cdots = constant$$

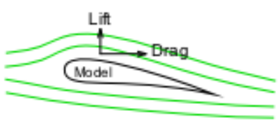
Dimensional Analysis and Similitude

- ***Kinematic similarity: Condition where the velocity ratio is a constant between all corresponding points in the flow field.***
  - *The streamline pattern around the model is the same as that around the prototype*





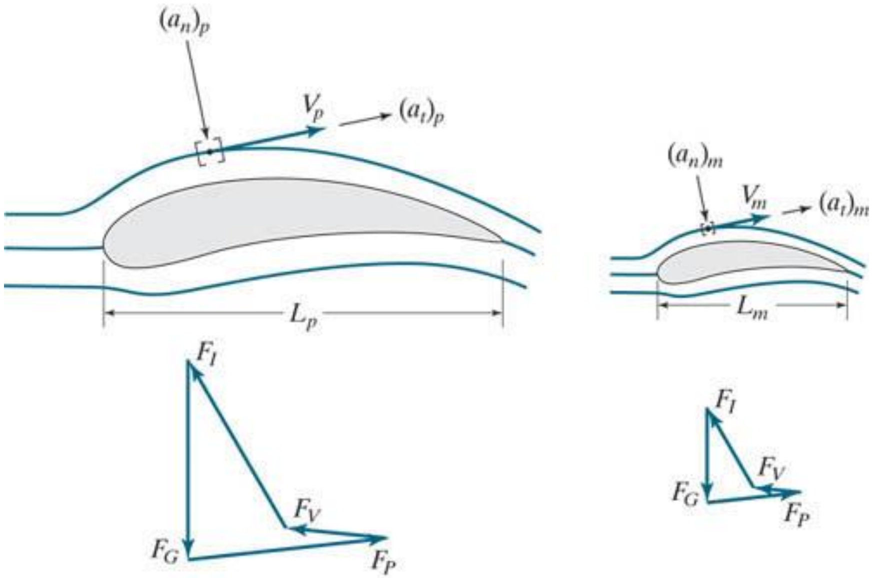




$$\frac{v_1}{V_1} = \frac{v_2}{V_2} = \frac{v_3}{V_3} = \cdots = constant$$

Dimensional Analysis and Similitude

- ***Dynamic similarity: Forces which act on corresponding masses in the model flow and prototype flow are in the same ratio through out the entire flow.***



(*a*) Prototype (*b*) Model

$$\frac{f_1}{F_1} = \frac{f_2}{F_2} = \frac{f_3}{F_3} = \cdots = constant$$

Dimensional Analysis and Similitude

- ***Dimensional analysis: a dimensionless quantity is a quantity to which no physical dimension is assigned.***

⇩

- ***Similitude: the study of predicting prototype conditions from model observations.***

Dimensions/Units

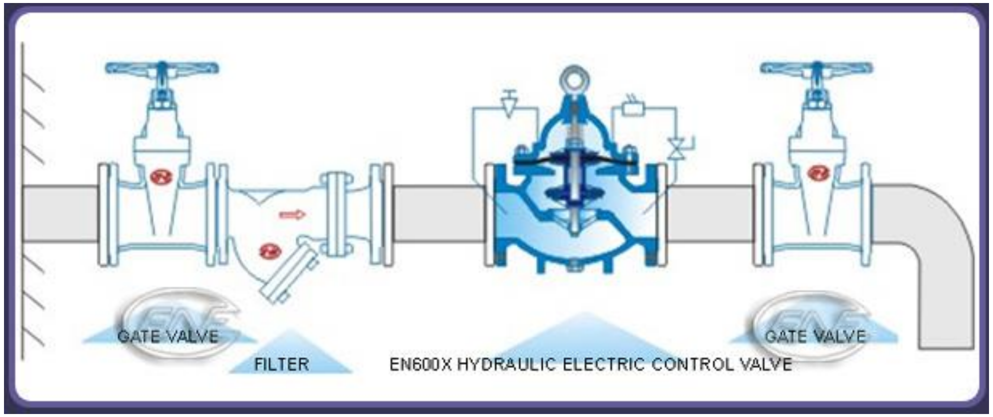
Primary dimensions and their associated primary SI and English units

Dimension	Symbol	SI Unit	English Unit
Mass	m	Kg (kilogrom)	lbm (pound-mass)
Length	L	m (meter)	ft (foot)
Time	t	s (second)	s (second)
Temperature	T	K (kelvin)	R (rankine)
Electric current	I	A (ampere)	A (ampere)
Amount of light	C	Cd (candela)	Cd (candela)
Amount of Matter	N	mol (mole)	mol (mole)

All nonprimary dimensions can be formed by some combination of the seven primary dimensions: e.g., Force ($F = m\ L/t^2$)

Dimensional Analysis

Example: Pressure drop in pipe flow



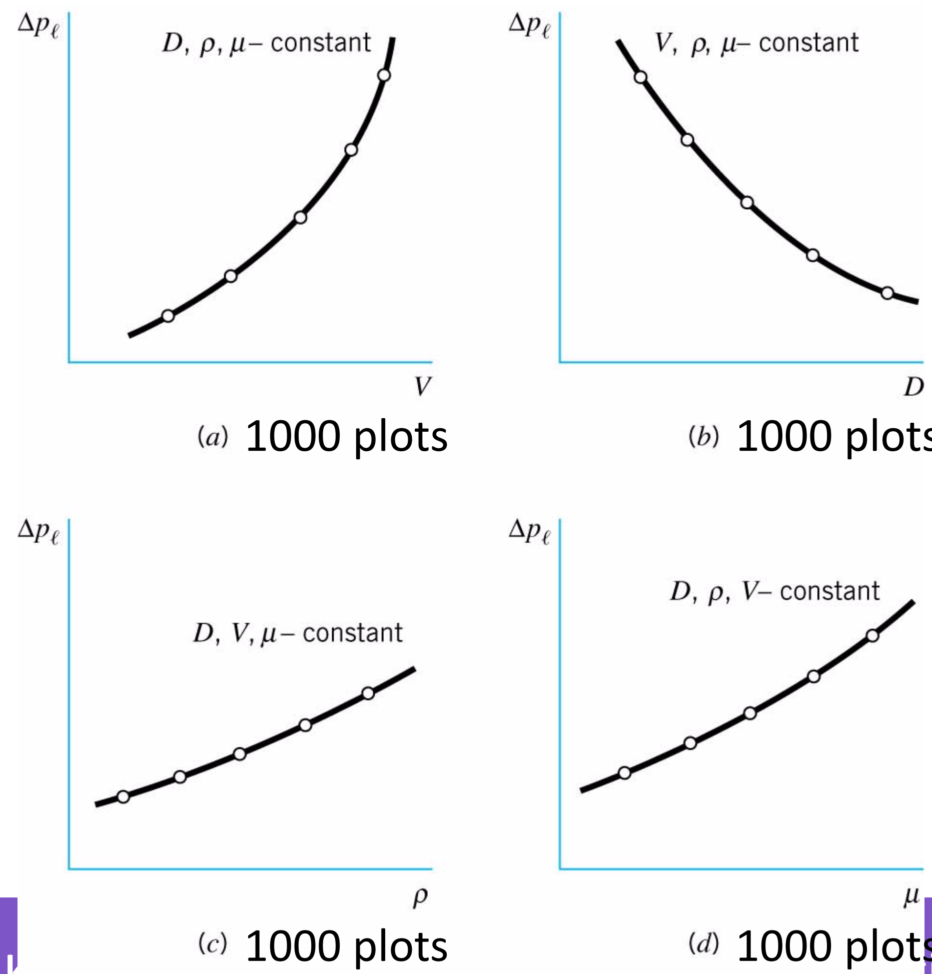
$$\Delta p_l = f(D, \rho, \mu, V)$$

Dimensional Analysis

Example: Pressure drop in pipe flow

$$\Delta p_l = f(D, \rho, \mu, V)$$

- *Diameter:* $D_1, D_2, \ldots, D_{10}$
- *Density:* $\rho_1, \rho_2, \ldots, \rho_{10}$
- *Viscosity:* $\mu_1, \mu_2, \ldots, \mu_{10}$
- *Velocity:* $V_1, V_2, \ldots, V_{10}$

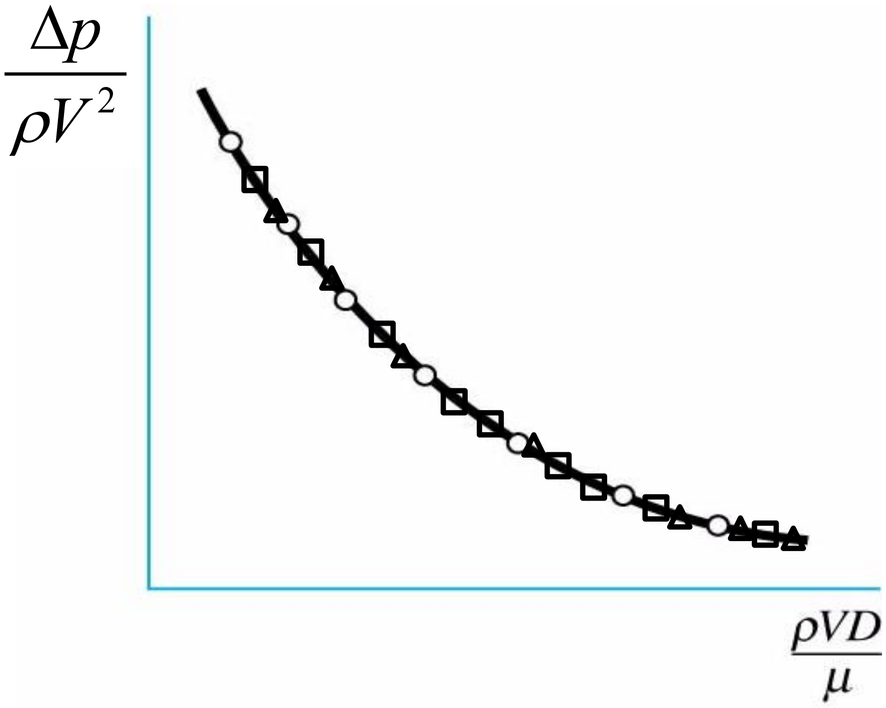


Dimensional Analysis

Example: Pressure drop in pipe flow

$$\Delta p_l = f(D, \rho, \mu, V)$$

$$\frac{\Delta p}{\rho V^2} = \Phi\left(\frac{\rho V D}{\mu}\right)$$



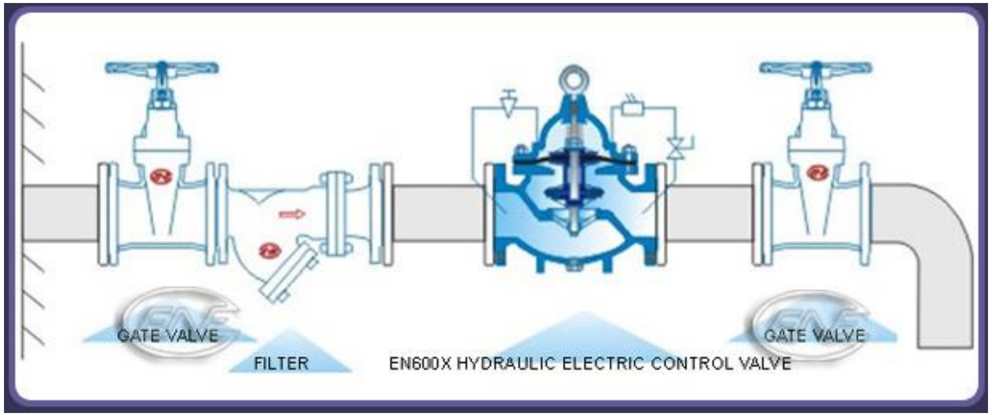
Buckingham π - Theorem

- *Step 1: List all the variables that are involved in the problem (k).*
- *Step 2: Express each of the variables in terms of basic dimensions.*
  - *Basic dimension: M, L,T*
  - *Force - $F=MLT^{-2}$, density - $\rho=ML^{-3}$*
- *Step 3: Select repeating variables.*
  - *No dependent variable.*
  - *Should contain all r dimensions (M, L and T).*
  - *No dimensionless variable*
  - *Pick simple parameters over complex parameters whenever possible.*
- *Step 4: The number of π-parameters is k-r.*
- *Step 5: Write the π-terms by combining the repeating variables with each of the remaining variables.*
- *Step 6: Solve the equations form step 5.*
- *Step 7: Write the functional relationship.*

$$\Pi_1 = \Phi(\Pi_2, \Pi_3, \ldots \Pi_{k-r})$$

Buckingham π - Theorem

- ***Step 1: List all the variables that are involved in the problem (k).***



$$\Delta p_l = f(D, \rho, \mu, V)$$

$K = 5$

Buckingham π - Theorem

- ***Step 1:*** ***List all the variables that are involved in the problem (k).***

$$\Delta p_l = f(D, \rho, \mu, V)$$

$$k = 5$$

- ***Step 2:*** ***Express each of the variables in terms of basic dimensions.***
  - ***Basic dimension: M, L,T***
  - ***Force - F=MLT^{-2}, density - ρ =ML^{-3}***

$D : [L]$

$\rho : [ML^{-3}]$

$V : [LT^{-1}]$

$\mu \, [ML^{-1}T^{-1}]$

$\Delta p \, [ML^{-1}T^{-2}]$

$$\boxed{r = 3}$$

Buckingham π - Theorem

- ***Step 1:*** *List all the variables that are involved in the problem (k).*

$$\Delta p_l = f(D, \rho, \mu, V) \qquad k = 5$$

- ***Step 2:*** *Express each of the variables in terms of basic dimensions.*

$$r = 3$$

- ***Step 3:*** *Select repeating variables.*
  - *No dependent variable.*
  - *Should contain all r dimensions (M, L and T).*
  - ~~*No dimensionless variable*~~
  - *Pick simple parameters over complex parameters whenever possible.*

$\cancel{\Delta P = [ML^{-1}T^{-2}]}$, $V = [LT^{-1}]$, $D = [L]$, $\rho = [ML^{-3}]$, $\mu = [ML^{-1}T^{-1}]$

Buckingham π - Theorem

- ***Step 1:** List all the variables that are involved in the problem (k).*
- ***Step 2:** Express each of the variables in terms of basic dimensions.*
  - *Basic dimension: M, L,T*
  - *Force - $F=MLT^{-2}$, density - $\rho = ML^{-3}$*
- ***Step 3:** Select repeating variables.*
  - *No dependent variable.*
  - *Should contain all r dimensions (M, L and T).*
  - *No dimensionless variable*
  - *Pick simple parameters over complex parameters whenever possible.*
- ***Step 4:** The number of π-parameters is k-r.*

$5 - 3 = 2$ $\quad \Pi_1 \quad \Pi_2$

Buckingham π - Theorem

- ***Step 1:** List all the variables that are involved in the problem (k).*
- ***Step 2:** Express each of the variables in terms of basic dimensions.*
  - *Basic dimension: M, L,T*
  - *Force - $F=MLT^{-2}$, density - $\rho = ML^{-3}$*
- ***Step 3:** Select repeating variables.*
  - *No dependent variable.*
  - *Should contain all r dimensions (M, L and T).*
  - *No dimensionless variable*
  - *Pick simple parameters over complex parameters whenever possible.*
- ***Step 4:** The number of π-parameters is k-r.*
- ***Step 5:** Write the π-terms by combining the repeating variables with each of the remaining variables.*

$\rho \quad V \quad D$

$\Pi_1 = \rho^{a_1} V^{b_1} D^{c_1} \cdot \Delta p \quad , \quad \Pi_2 = \rho^{a_2} V^{b_2} D^{c_2} \cdot \mu$

Buckingham π - Theorem

- ……
- ***Step 4:* *The number of π-parameters is k-r.***
- ***Step 5:* *Write the π-terms by combining the repeating variables with each of the remaining variables.***
- ***Step 6:* *Solve the equations from step 5.***

$$\pi_1 = \rho^{a_1} V^{b_1} D^{c_1} \Delta P, \qquad \pi_2 = \rho^{a_2} V^{b_2} D^{c_2} \mu$$

$$\pi_1 = [ML^{-3}]^{a_1} [LT^{-1}]^{b_1} [L]^{c_1} [ML^{-1}T^{-2}] = M^0 L^0 T^0$$

$$\begin{cases} a_1 + 1 = 0 \\ -3a_1 + b_1 + c_1 - 1 = 0 \\ -b_1 - 2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = -1 \\ b_1 = -2 \\ c_1 = 0 \end{cases} \Rightarrow \pi_1 = \rho^{-1} V^{-2} \Delta p$$

$$\pi_1 = \frac{\Delta p}{\rho V^2}$$

Buckingham π - Theorem

- ……
- ***Step 4: The number of π-parameters is k-r.***
- ***Step 5: Write the π-terms by combining the repeating variables with each of the remaining variables.***
- ***Step 6: Solve the equations from step 5.***

$$\Pi_2 = [ML^{-3}]^{a_2}[LT^{-1}]^{b_2}[L]^{c_2}[ML^{-1}T^{-1}] = M^0L^0T^0$$

$$\begin{cases} a_2+1=0 \\ -3a_2+b_2+c_2-1=0 \\ -b_2-1=0 \end{cases} \Rightarrow \begin{cases} a_2=-1 \\ b_2=-1 \\ c_2=-1 \end{cases}$$

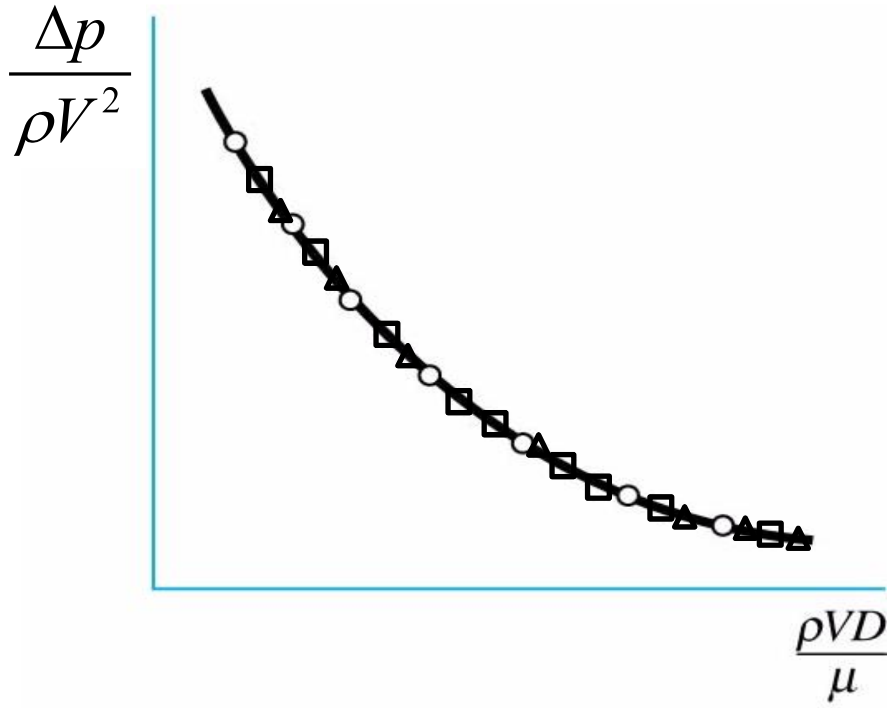
$$\Rightarrow \Pi_2 = \rho^{-1}V^{-1}D^{-1}\mu = \frac{\mu}{\rho V D} = \frac{1}{Re}$$

Dimensional Analysis

Example: Pressure drop in pipe flow

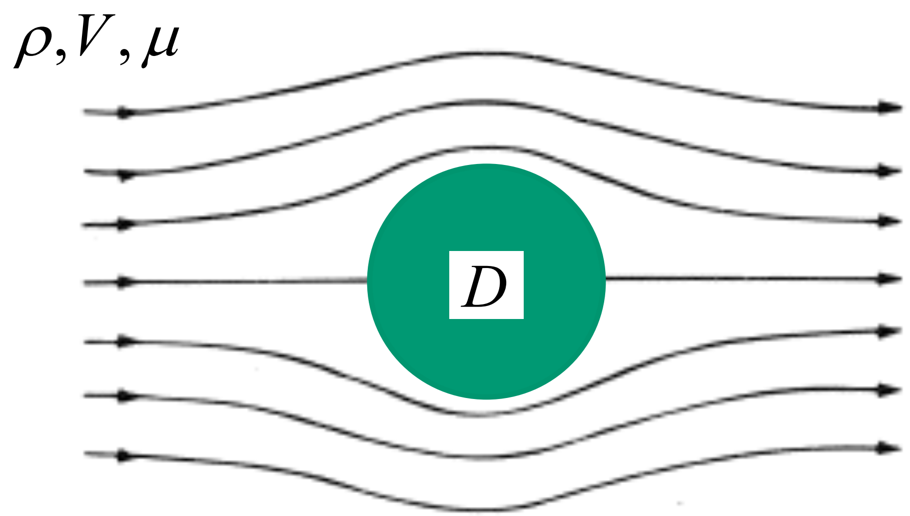
$$\Delta p_l = f(D, \rho, \mu, V)$$

$$\frac{\Delta p}{\rho V^2} = \Phi\left(\frac{\rho V D}{\mu}\right)$$



Dimensional Analysis

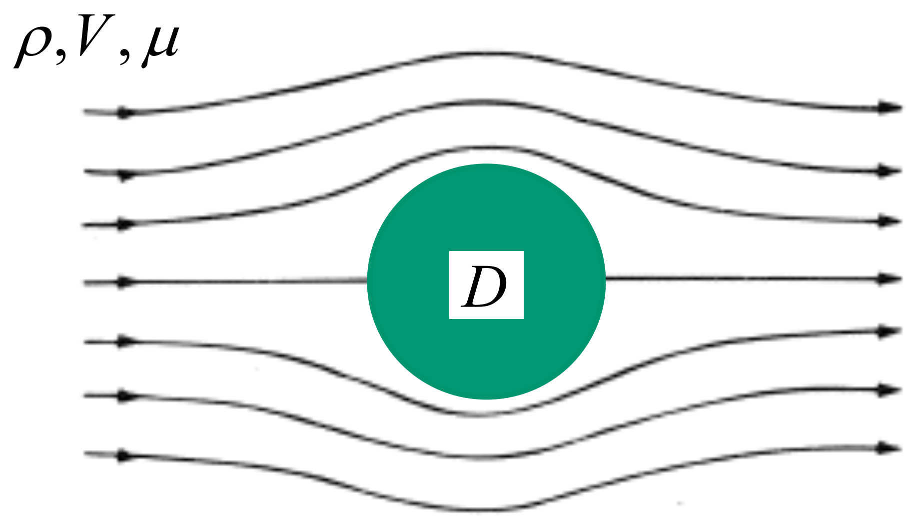
Practice: Drag of a cylinder in airflow



$$F_D = f\left(D, \rho, V, \mu\right)$$

Buckingham π - Theorem

- ***Step 1: List all the variables that are involved in the problem (k).***



$$F_D = f\left(D, \rho, V, \mu\right)$$

$$k = 5$$

Buckingham π - Theorem

- ***Step 1: List all the variables that are involved in the problem (k).***

$$\Delta p_l = f(D, \rho, \mu, V)$$

$$k = 5$$

- ***Step 2: Express each of the variables in terms of basic dimensions.***
  - ***Basic dimension: M, L,T***
  - ***Force - $F=MLT^{-2}$, density - $\rho=ML^{-3}$***

$$F_d = \left[MLT^{-2}\right] \qquad D = \left[L\right]$$

$$\rho = \left[ML^{-3}\right] \qquad \mu = \left[ML^{-1}T^{-1}\right]$$

$$V = \left[LT^{-1}\right]$$

$$r = 3$$

Buckingham π - Theorem

- ***Step 1:** List all the variables that are involved in the problem (k).*

$$F_D = f\left(D, \rho, V, \mu\right) \qquad k = 5$$

- ***Step 2:** Express each of the variables in terms of basic dimensions.*

$$r = 3$$

- ***Step 3:** Select repeating variables.*
  - *No dependent variable.*
  - *Should contain all r dimensions (M, L and T).*
  - *No dimensionless variable*

$$\boxed{F_d = \left[MLT^{-2}\right]} \qquad \boxed{D = \left[L\right]}$$

$$\boxed{\rho = \left[ML^{-3}\right]} \qquad \boxed{V = \left[LT^{-1}\right]}$$

$$\mu = \left[ML^{-1}T^{-1}\right]$$

Buckingham π - Theorem

- ***Step 1:** List all the variables that are involved in the problem (**k**).*
- ***Step 2:** Express each of the variables in terms of basic dimensions.*
  - *Basic dimension: M, L,T*
  - *Force - $F=MLT^{-2}$, density - $\rho = ML^{-3}$*
- ***Step 3:** Select repeating variables.*
  - *No dependent variable.*
  - *Should contain all **r** dimensions (M, L and T).*
  - *No dimensionless variable*
- ***Step 4:** The number of π-parameters is **k-r**.*
- ***Step 5:** Write the π-terms by combining the repeating variables with each of the remaining variables.*

$$k - r = 2$$

$$\Pi_1 = \rho^a V^b D^c F_d$$

$$\Pi_2 = \rho^a V^b D^c \mu$$

Buckingham π - Theorem

- ……
- ***Step 4: The number of π-parameters is k-r.***
- ***Step 5: Write the π-terms by combining the repeating variables with each of the remaining variables.***
- ***Step 6: Solve the equations form step 5.***

$$\rho = \left[ML^{-3}\right]$$

$$V = \left[LT^{-1}\right]$$

$$D = [L]$$

$$F_d = \left[MLT^{-2}\right]$$

$$\mu = \left[ML^{-1}T^{-1}\right]$$

$$\Pi_1 = \rho^a V^b D^c F_d \qquad \Pi_2 = \rho^a V^b D^c \mu$$

$$\Pi_1 = \left[ML^{-3}\right]^a \left[LT^{-1}\right]^b [L]^c \left[MLT^{-2}\right] = M^0 L^0 T^0$$

$$\begin{Bmatrix} a+1=0 \\ -3a+b+c+1=0 \\ -b-2=0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} a=-1 \\ b=-2 \\ c=-2 \end{Bmatrix}$$

$$\Pi_1 = \rho^{-1}V^{-2}D^{-2}F_d = \frac{F_d}{\rho V^2 D^2}$$

- ***Drag coefficient C_D***

Buckingham π - Theorem

- ……
- ***Step 4:*** ***The number of π-parameters is k-r.***
- ***Step 5:*** ***Write the π-terms by combining the repeating variables with each of the remaining variables.***
- ***Step 6:*** ***Solve the equations form step 5.***

$$\rho = \left[ML^{-3}\right]$$

$$V = \left[LT^{-1}\right]$$

$$D = \left[L\right]$$

$$\Delta p = \left[ML^{-1}T^{-2}\right]$$

$$\mu = \left[ML^{-1}T^{-1}\right]$$

$$\Pi_1 = \rho^a V^b D^c F_d$$

$$\Pi_2 = \rho^a V^b D^c \mu$$

$$\Pi_2 = \left[ML^{-3}\right]^a \left[LT^{-1}\right]^b \left[L\right]^c \left[ML^{-1}T^{-1}\right] = M^0 L^0 T^0$$

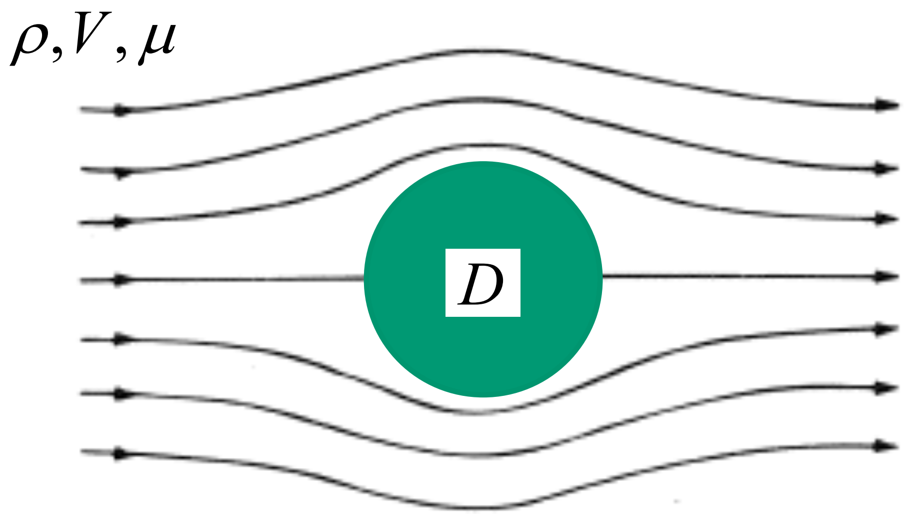
$$\begin{Bmatrix} a+1=0 \\ -3a+b+c-1=0 \\ -b-1=0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} a=-1 \\ b=-1 \\ c=-1 \end{Bmatrix}$$

$$\Pi_2 = \rho^{-1}V^{-1}D^{-1}\mu = \frac{\mu}{\rho VD}$$

$$\Pi_2 = \frac{\mu}{\rho VD} = \frac{1}{\mathrm{Re}}$$

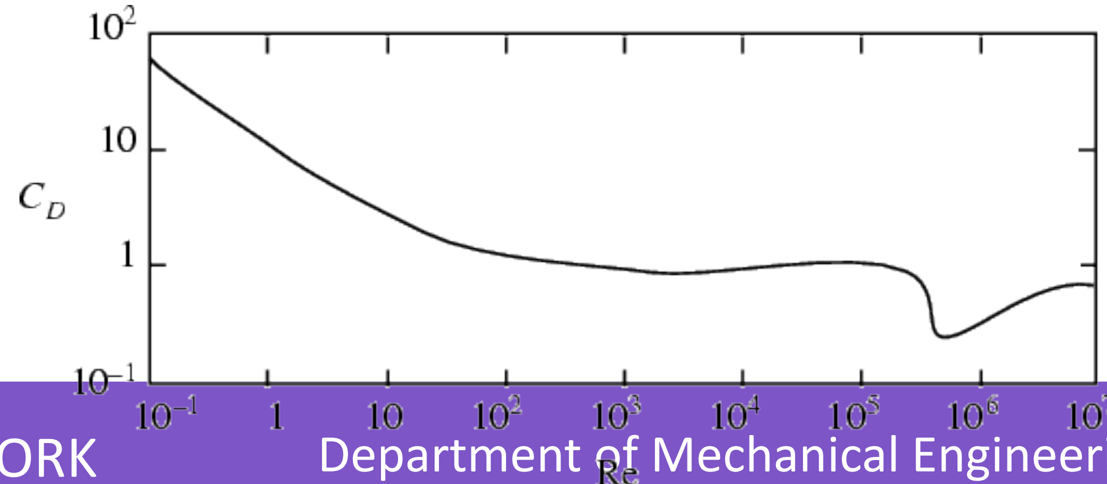
Dimensional Analysis

Practice: Drag of a cylinder in airflow



$$F_D = f\left(D, \rho, V, \mu\right)$$

$$C_D = \Phi\left(\mathrm{Re}\right)$$



Similitude

$$\text{Mach Number, M} = \frac{V}{c} \propto \frac{\text{inertial force}}{\text{compressiblity force}}$$

$$\text{Reynolds number, Re} = \frac{\rho V L}{\mu} \propto \frac{\text{inertial force}}{\text{viscous force}}$$

$$\text{Euler number, Eu} = \frac{\Delta p}{\frac{1}{2}\rho V^2} \propto \frac{\text{pressure force}}{\text{inertial force}}$$

$$\text{Drag Coefficient:} C_D = \frac{D}{\frac{1}{2}\rho V^2} = \frac{\text{Drag}}{\text{inertial force}}$$

$$\text{Lift Coefficient:} C_L = \frac{L}{\frac{1}{2}\rho V^2} = \frac{\text{Lift}}{\text{inertial force}}$$

$$\text{Prandtl Number:Pr} = \frac{V}{\gamma} = \frac{\text{momentum diffusion}}{\text{heat diffusion}}$$

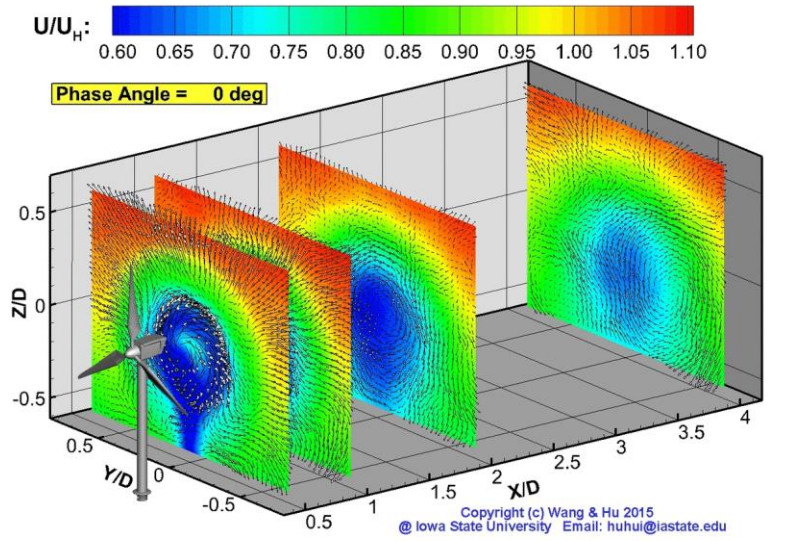
$$\text{Schmidt Number:} Sc = \frac{U}{\gamma_c} = \frac{\text{momentum}}{\text{mass}}$$

$$\text{Froude Number, Fr} = \frac{V}{\sqrt{\lg}} \propto \frac{\text{inertial force}}{\text{gravity force}}$$

$$\text{Strohal Number, Str} = \frac{l\varpi}{V} \propto \frac{\text{centrifugal force}}{\text{inertial force}}$$

$$\text{Weber Number, We} = \frac{V^2 l \rho}{\sigma} \propto \frac{\text{inertial force}}{\text{surface tension force}}$$

...



In-Class Quiz