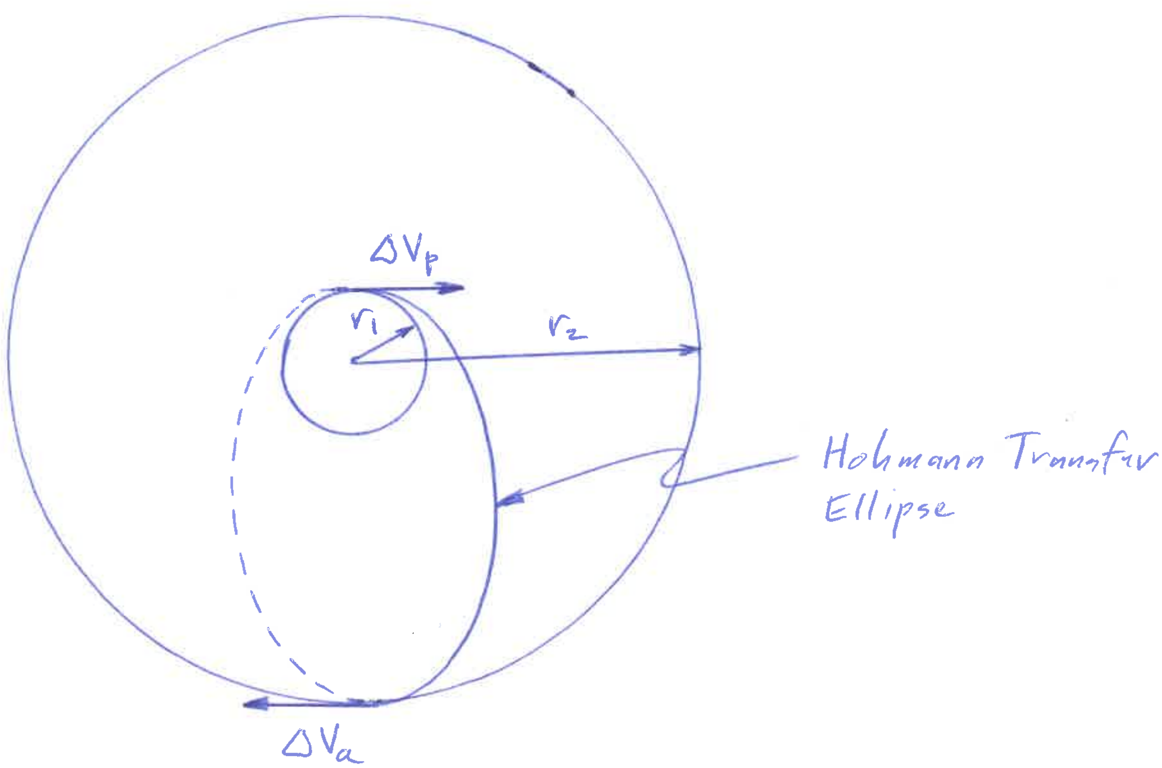
9. Orbital Maneuvers, Interception and Rendezvous

Will assume that the rocket burns used to change orbital velocity are of relatively short duration so that the changes in path are instantaneous.

9.1. Transfer Between Two Concentric, Coplanar Circular Orbits

Hohmann Transfer

The most efficient way to transfer between two concentric, coplanar circular orbits (utilizing 2 burns).



Let r_1 = radius of smaller circular orbit
= perigee radius of transfer ellipse

r_2 = radius of larger circular orbit
= apogee radius of transfer ellipse

a = semi-major axis of transfer ellipse
e = eccentricity of transfer ellipse

$$r_1 = a(1-e)$$
$$r_2 = a(1+e)$$

$$\frac{r_1}{r_2} = \frac{1-e}{1+e}$$

$$\boxed{e = \frac{r_2 - r_1}{r_2 + r_1}} \qquad (9.1)$$

e is now known.

At perigee of transfer ellipse

$$\frac{V_p^2}{V_{cp}^2} - 1 = e \qquad (5.33)$$

$$V_p^2 = V_{cp}^2 (1+e)$$

$$V_p = V_{cp} \sqrt{1+e}$$

$$\Delta V_p = V_p - V_{cp}$$

$$= V_{cp}\sqrt{1+e} - V_{cp}$$

$$= V_{cp}\left[\sqrt{1+e} - 1\right]$$

$$\boxed{\Delta V_p = \sqrt{\frac{\mu}{r_1}}\left[\sqrt{1+e} - 1\right]} \qquad (9.2)$$

At apogee of transfer ellipse

$$1 - \frac{V_a^2}{V_{ca}^2} = e \qquad (5.35)$$

$$V_a = V_{ca}\sqrt{1-e}$$

$$\Delta V_a = V_{ca} - V_a$$

$$= V_{ca} - V_{ca}\sqrt{1-e}$$

$$= V_{ca}\left[1 - \sqrt{1-e}\right]$$

$$\boxed{\Delta V_a = \sqrt{\frac{\mu}{r_2}}\left[1 - \sqrt{1-e}\right]} \qquad (9.3)$$

Total velocity impulse

$$\boxed{\Delta V_{total} = \Delta V_p + \Delta V_a} \qquad (9.4)$$

Energy of satellite in inner circular orbit

$$\mathcal{E}_1 = -\frac{\mu}{2r_1}$$

Energy of satellite in outer circular orbit

$$\mathcal{E}_2 = -\frac{\mu}{2r_2}$$

Energy increment

$$\Delta\mathcal{E} = \mathcal{E}_2 - \mathcal{E}_1$$

$$\boxed{\Delta\mathcal{E} = \frac{\mu}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \qquad (9.5)$$

Kinetic energy imparted to the vehicle during the two burns

$$\Delta KE = \frac{V_p^2 - V_{cp}^2}{2} + \frac{V_{ca}^2 - V_a^2}{2}$$

$$= \frac{V_{cp}^2\left[(1+e) - 1\right] + V_{ca}^2\left[(1-(1-e)\right]}{2}$$

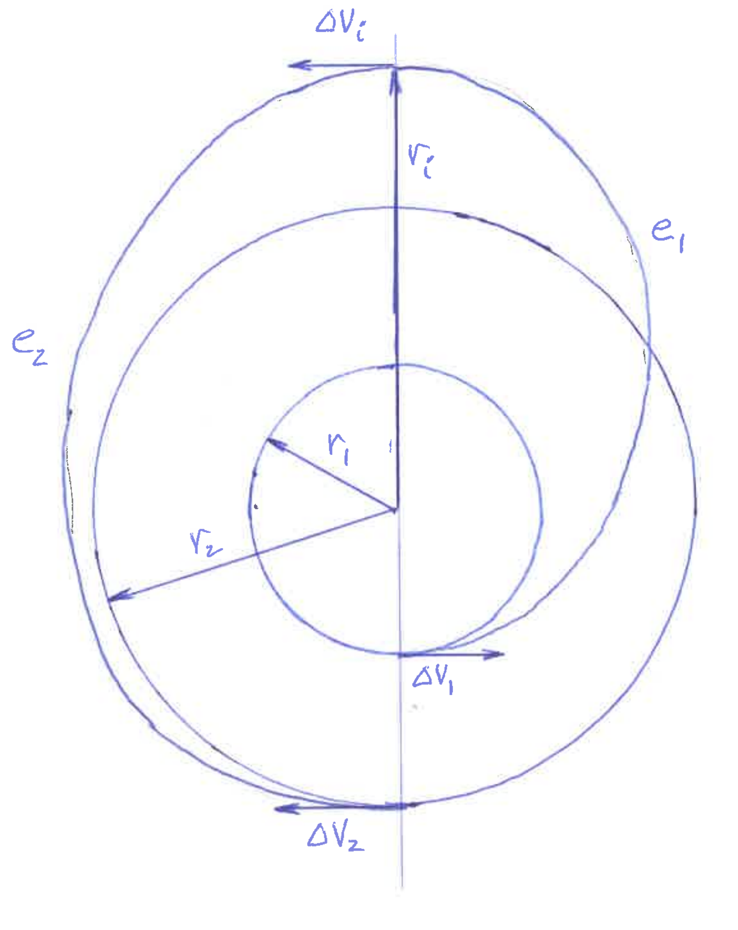
$$\Delta KE = \frac{V_{cp}^2 e + V_{ca}^2 e}{2} = \frac{\frac{\mu}{r_1} e + \frac{\mu}{r_2} e}{2} = \frac{\mu}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) e$$

$$= \frac{\mu}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\left(\frac{r_2 - r_1}{r_2 + r_1}\right) = \frac{\mu}{2}\cancel{\left(\frac{1}{r_1} + \frac{1}{r_2}\right)}\left(\frac{\frac{1}{r_1} - \frac{1}{r_2}}{\cancel{\frac{1}{r_1} + \frac{1}{r_2}}}\right)$$

$$= \frac{\mu}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \Delta \mathcal{E}$$

Bi-elliptic Transfer

If r_2/r_1 is sufficiently large, the bi-elliptic (3-burn) transfer is more efficient than a Hohmann transfer. Note that it requires more than double the transfer time for a Hohmann transfer.



Using (9.2)

$$\Delta V_1 = \sqrt{\frac{\mu}{r_1}}\left[\sqrt{1+e_1} - 1\right] \qquad (9.6)$$

where

$$e_1 = \frac{r_i - r_1}{r_i + r_1} \qquad (9.7)$$

Using (5.35)

$$\Delta V_i = \sqrt{\frac{\mu}{r_i}}\left[\sqrt{1-e_2} - \sqrt{1-e_1}\right] \qquad (9.8)$$

where

$$e_2 = \frac{r_i - r_2}{r_i + r_2} \qquad (9.9)$$

Using (9.2)

$$\Delta V_2 = \sqrt{\frac{\mu}{r_2}}\left[\sqrt{1+e_2} - 1\right] \qquad (9.10)$$

$$\Delta V_{total} = \Delta V_1 + \Delta V_i + \Delta V_2 \qquad (9.11)$$

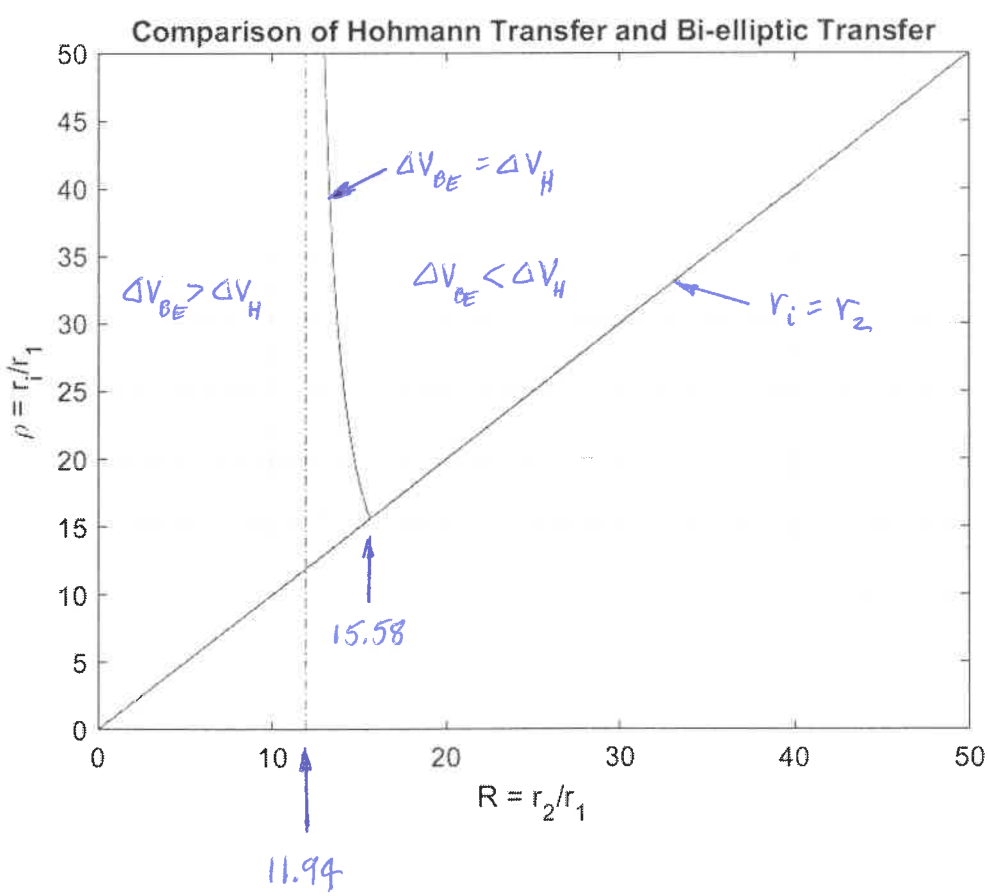
Define

$$R = \frac{r_2}{r_1} \qquad \rho = \frac{r_i}{r_1} \qquad V_{c1} = \sqrt{\frac{\mu}{r_1}}$$

Eq. (9.4) for the Hohmann transfer and (9.11) for the bielliptic transfer may be written as

$$\frac{\Delta V_H}{V_{c1}} = \frac{1}{\sqrt{R}} - 1 - \sqrt{\frac{2}{R(1+R)}}\,(1-R) \qquad (9.12)$$

$$\frac{\Delta V_{BE}}{V_{c1}} = \sqrt{\frac{2(R+\rho)}{\rho R}} - \frac{1}{\sqrt{R}} - 1 - \sqrt{\frac{2}{\rho(1+\rho)}}\,(1-\rho) \qquad (9.13)$$

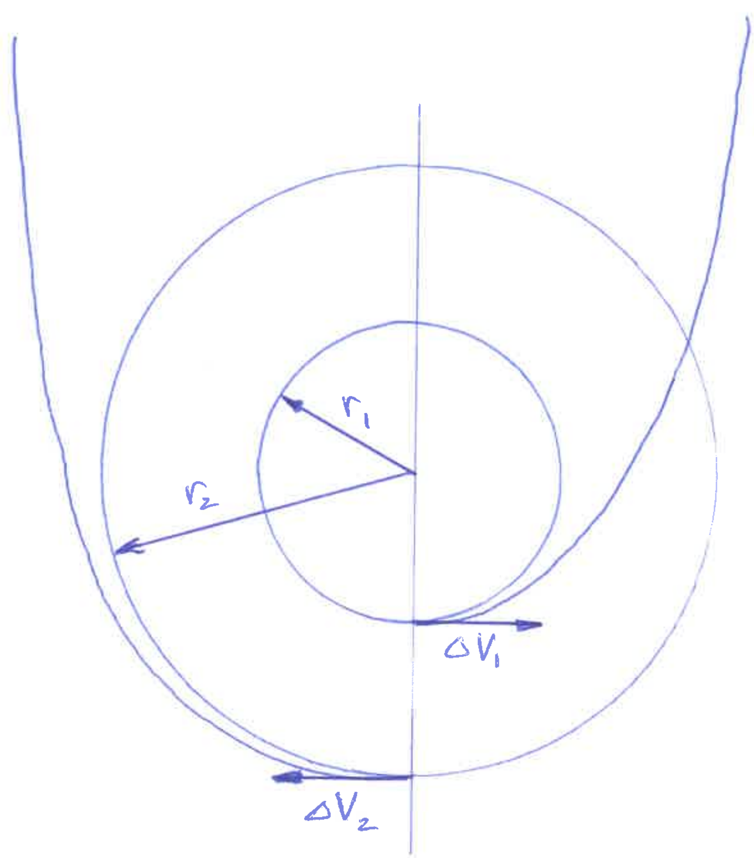


The bi-elliptic transfer is more efficient than the Hohmann transfer when $\frac{r_2}{r_1} > 15.58$ for any $r_i > r_2$. (Applicable for missions to Uranus, Neptune and Pluto).

The Hohmann transfer is always more efficient when $\frac{r_2}{r_1} < 11.94$.

Bi-parabolic Transfer

Limiting case of the bi-elliptic transfer as $r_i \rightarrow \infty$



$$\Delta V_{total} = \Delta V_1 + \cancel{\Delta V_i}^{\,0} + \Delta V_2$$

ΔV_1 and ΔV_2 are obtained from (9.6) & (9.10) with $e_1 = e_2 = 1$.

$$\Delta V_{total} = \sqrt{\frac{\mu}{r_1}}\left[\sqrt{2}-1\right] + \sqrt{\frac{\mu}{r_2}}\left[\sqrt{2}-1\right]$$

or

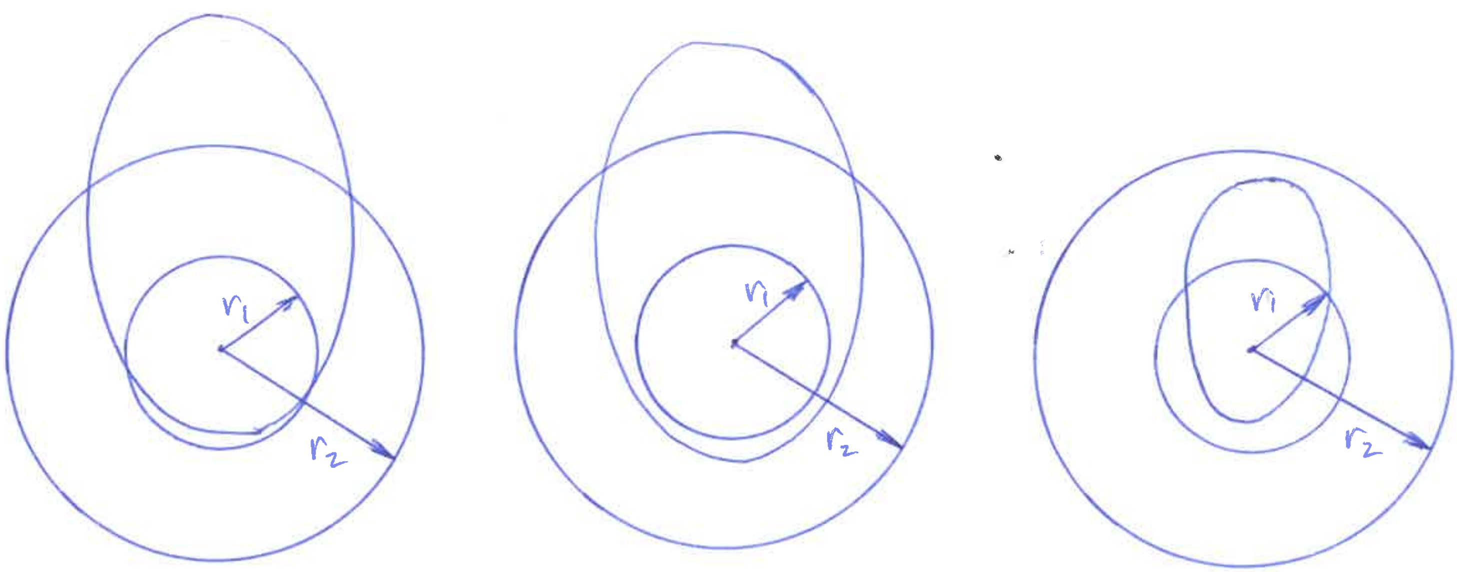
$$\frac{\Delta V_{BP}}{V_{c1}} = (\sqrt{2}-1)\left(1+\frac{1}{\sqrt{R}}\right) \qquad (9.14)$$

More efficient than a Hohmann transfer when $\frac{r_2}{r_1} > 11.94$.

Impractical because it requires an infinite transfer time.

General Coplanar Transfer Between Concentric, Coplanar, Circular Orbits

Any trajectory which intersects both circular orbits can be used as a transfer trajectory but none are as efficient as the Hohmann transfer.



Possible because $r_p < r_1$ and $r_a > r_2$

Impossible because $r_p > r_1$

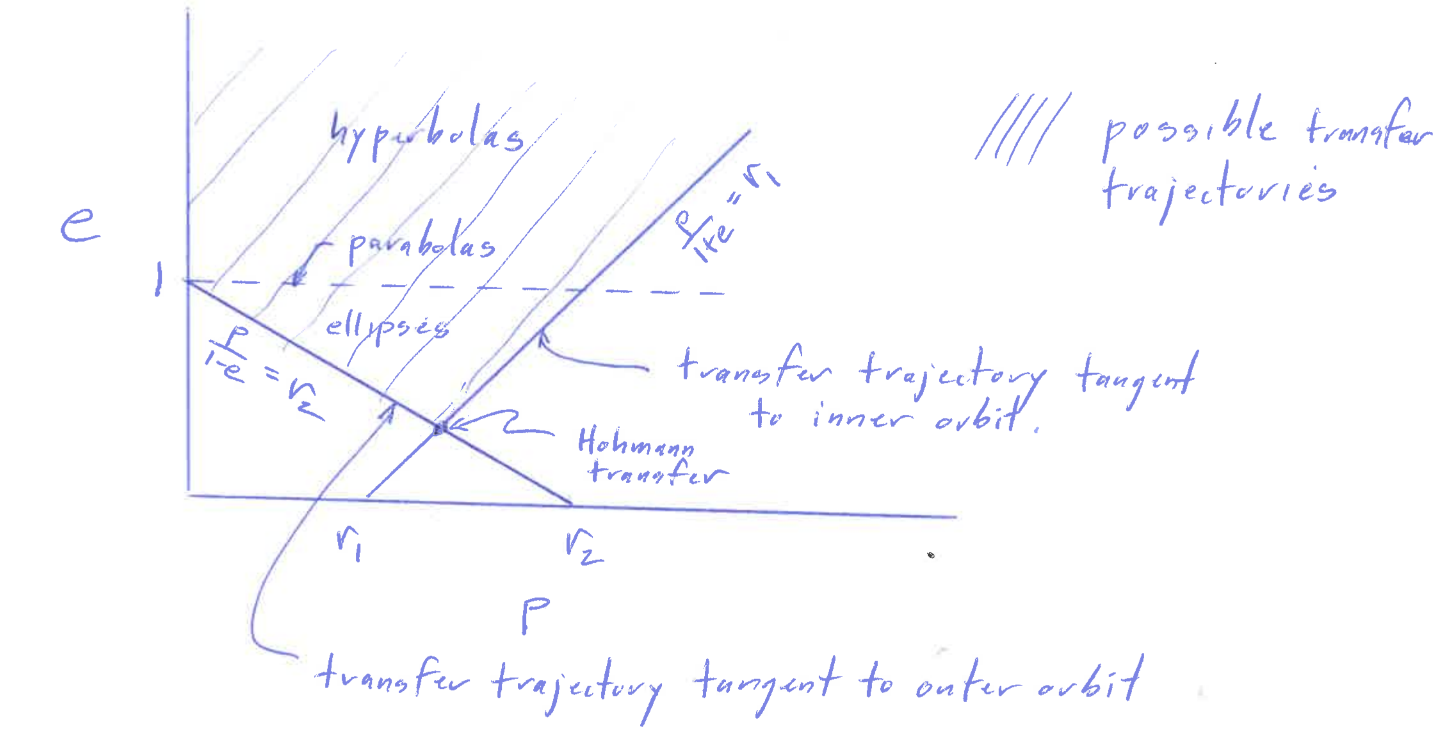
Impossible because $r_a < r_2$

Mathematically, trajectory can be used as a transfer trajectory if

$$r_p = \frac{p}{1+e} \leq r_1$$

$$r_a = \frac{p}{1-e} \geq r_2$$

Graphically



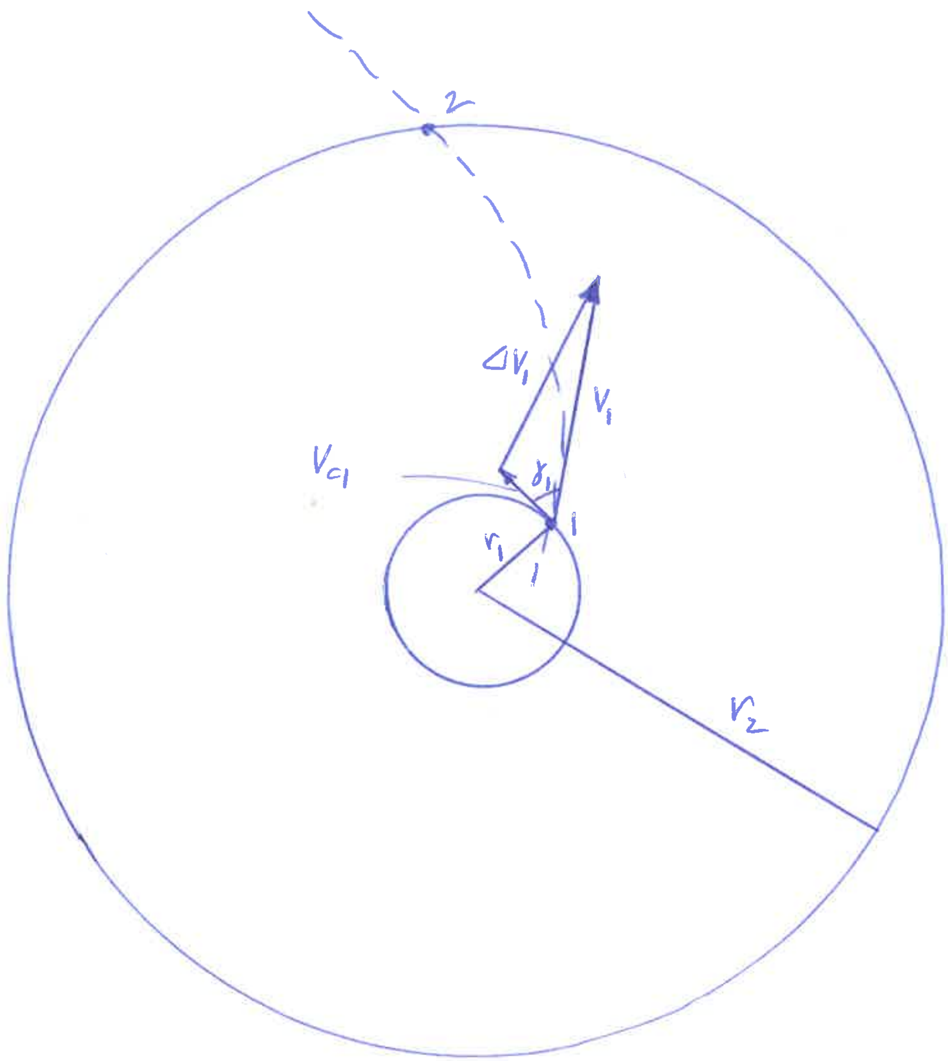
Procedure

1) Pick values of p and e of the transfer trajectory which satisfy the conditions above.

2) Calculate

$$\mathcal{E}_t = -\frac{\mu(1-e^2)}{2p} \qquad h_t = \sqrt{\mu p}$$

3) obtain ΔV_1



Since $h_t = r_1 V_1 \cos\gamma_1 \Rightarrow \cos\gamma_1 = \dfrac{h_t}{r_1 V_1}$

From $\dfrac{V_1^2}{2} - \dfrac{\mu}{r_1} = \mathcal{E}_t \Rightarrow V_1 = \sqrt{2\left(\dfrac{\mu}{r_1} + \mathcal{E}_t\right)}$

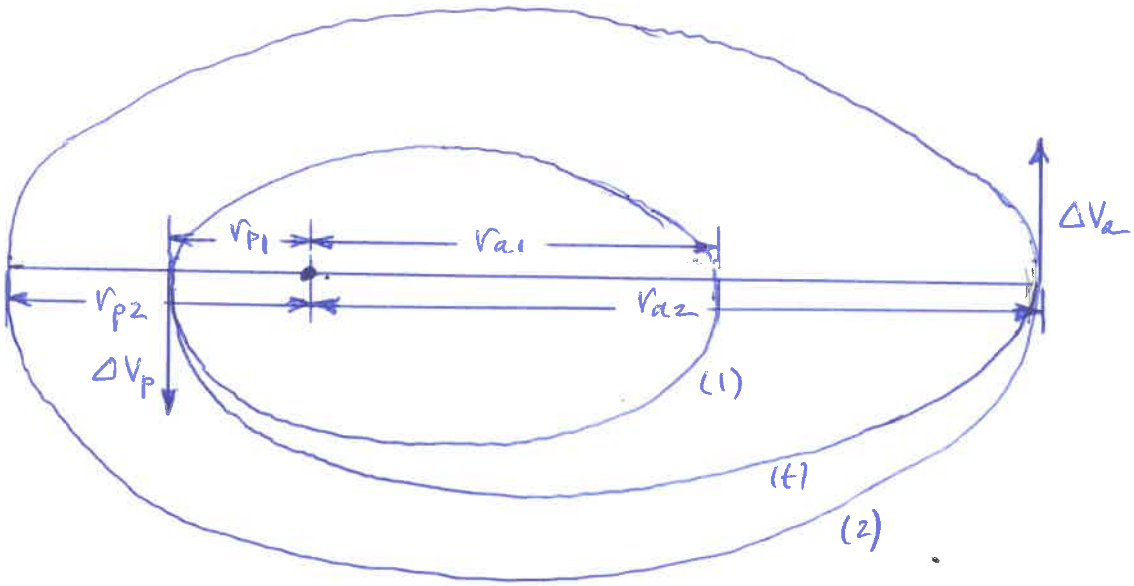
$$V_{c1} = \sqrt{\frac{\mu}{r_1}}$$

$$\Delta V_1 = \sqrt{V_1^2 + V_{c1}^2 - 2 V_1 V_{c1} \cos\gamma_1}$$

Hohmann transfer is special case with $\gamma_1 = 0$

4) Similarly obtain ΔV_2.

9.2. Transfer Between Coplanar, Confocal, Elliptic orbits



Velocity at perigee of inner elliptic orbit,

$$V_{p1} = \sqrt{\frac{\mu}{r_{p1}}(1+e_1)} \qquad (9.15)$$

Velocity at perigee of transfer ellipse

$$V_{pt} = \sqrt{\frac{\mu}{r_{p1}}(1+e_t)} \qquad (9.16)$$

where

$$e_t = \frac{r_{a2} - r_{p1}}{r_{a2} + r_{p1}} \qquad (9.17)$$

Velocity at apogee of transfer ellipse

$$V_{a_t} = \sqrt{\frac{\mu}{r_{a2}}(1-e_t)} \qquad (9.18)$$

Velocity at apogee of outer elliptic orbit

$$V_{a2} = \sqrt{\frac{\mu}{r_{a2}}(1-e_2)} \qquad (9.19)$$

Velocity increment at perigee of inner elliptic orbit

$$\Delta V_p = |V_{p_t} - V_{p1}| \qquad (9.20)$$

Velocity increment at apogee of transfer ellipse

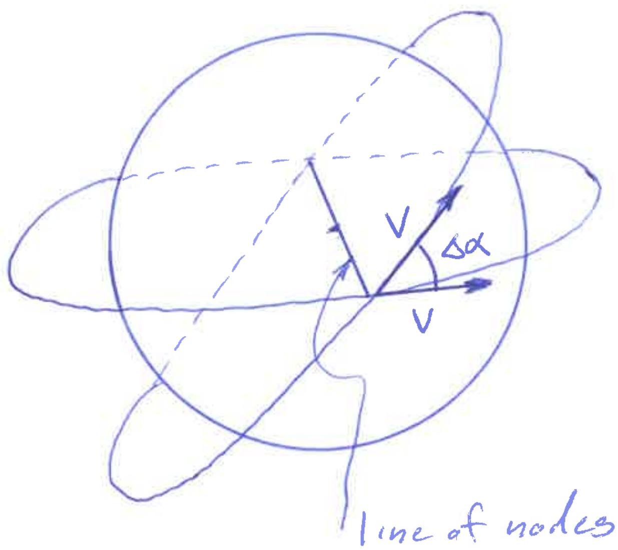
$$\Delta V_a = |V_{a2} - V_{a_t}| \qquad (9.21)$$

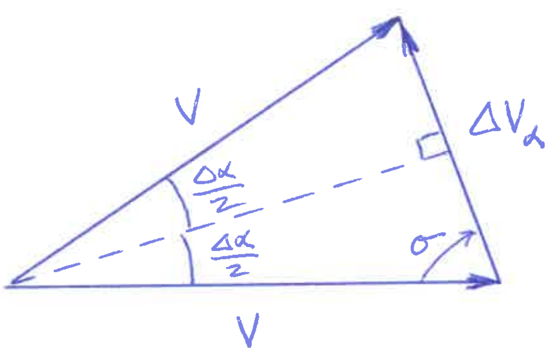
Total velocity increment

$$\Delta V_{total} = \Delta V_p + \Delta V_a \qquad (9.22)$$

For minimum ΔV_{total} (whenever possible) perform burns at perigee of inner ellipse and apogee of outer ellipse

9.3. Transfer Between Non-Coplanar Circular Orbits of the Same Radii





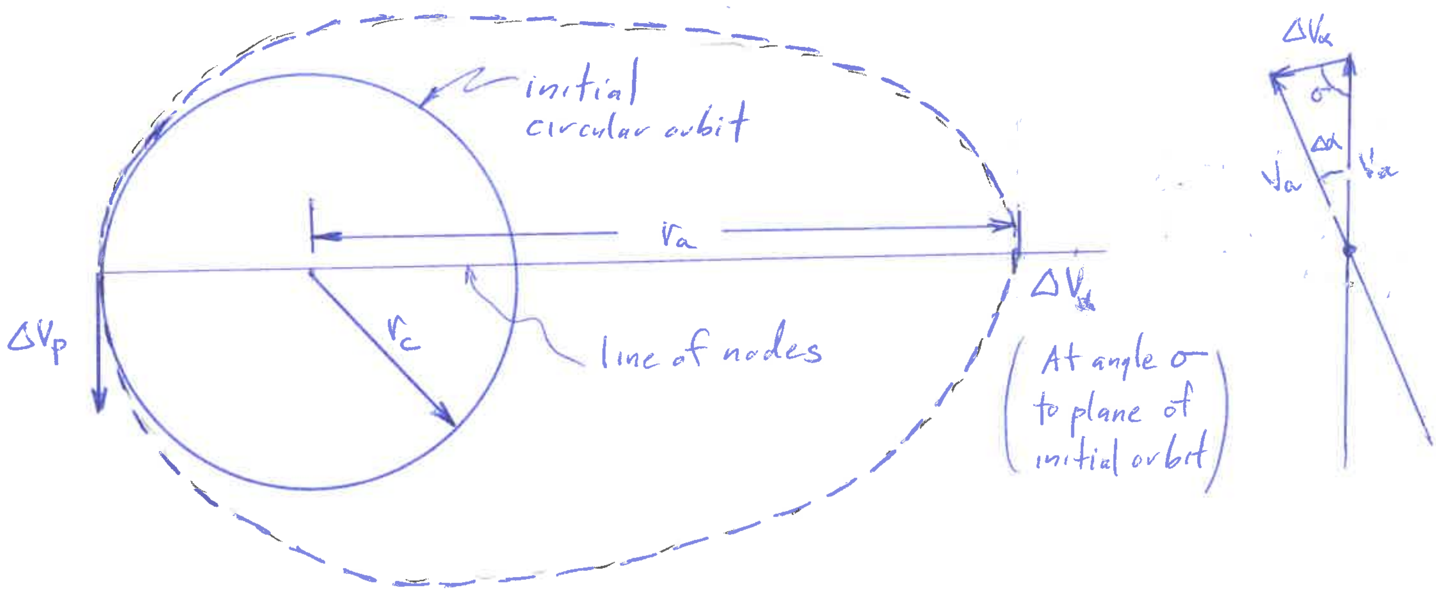
Velocity impulse required for plane change

$$\Delta V_\alpha = 2V \sin \tfrac{1}{2}\Delta\alpha \qquad (9.23)$$

Orientation of velocity impulse

$$\sigma = \frac{\pi}{2} - \frac{\Delta\alpha}{2} \qquad (9.24)$$

Since $V = \sqrt{\frac{\mu}{r}}$ for a circular orbit of radius r, increasing r decreases V thus decreasing ΔV_α (energy) required for the plane change maneuver. However, the energy required to place the vehicle in a higher orbit and then bring it back down increases with r. An optimum r must exist in which the sum of the two energy requirements is a minimum.



Velocity increment to change from circular orbit to transfer orbit.

$$\Delta V_p = \sqrt{\frac{\mu}{r_c}}\left[\sqrt{1+e}-1\right] \qquad (9.25)$$

where e = eccentricity of transfer orbit

$$e = \frac{r_a - r_c}{r_a + r_c} \qquad (9.26)$$

Sub. (9.26) into (9.25)

$$\Delta V_p = \sqrt{\frac{\mu}{r_c}}\left[\left(\frac{2 r_a}{r_a + r_c}\right)^{1/2} - 1\right] \qquad (9.27)$$

At apogee of transfer orbit, required velocity increment for plane change maneuver is

$$\Delta V_\alpha = 2 V_a \sin \tfrac{1}{2}\alpha \qquad (9.28)$$

Need to determine V_a in terms of r_a and known quantities.

$$V_a = \frac{r_c}{r_a} V_p = \frac{r_c}{r_a}\left(V_c + \Delta V_p\right) = \frac{r_c}{r_a}\left[\cancel{\sqrt{\frac{\mu}{r_c}}} + \sqrt{\frac{\mu}{r_c}}\left(\left(\frac{2r_a}{r_a+r_c}\right)^{1/2} - \cancel{1}\right)\right]$$

$$V_a = \sqrt{\frac{2\mu r_c}{r_a(r_c+r_a)}} \qquad (9.29)$$

Sub. (9.29) into (9.28)

$$\Delta V_\alpha = 2\sqrt{\frac{2\mu r_c}{r_a(r_c+r_a)}}\ \sin\tfrac{1}{2}\Delta\alpha \qquad (9.30)$$

At perigee of transfer orbit (which is now inclined) velocity must be decreased by ΔV_p to place the vehicle back into a circular orbit of radius r_c.

The total velocity impulse needed for the complete maneuver is

$$\Delta V_{total} = 2\left|\Delta V_p\right| + \left|\Delta V_\alpha\right| \qquad (9.31)$$

Sub. (9.27) & (9.30) into (9.31)

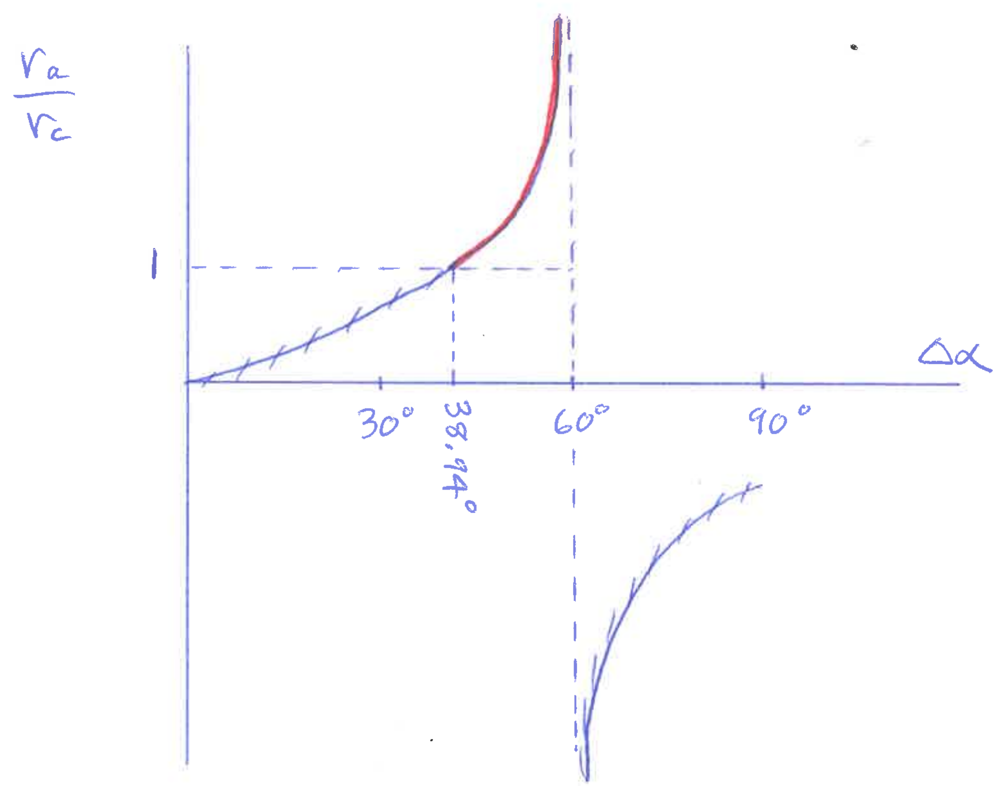
$$\Delta V_{total} = 2\sqrt{\frac{\mu}{r_c}}\left[\left(\frac{2r_a}{r_a+r_c}\right)^{1/2} - 1 + \left(\frac{2r_c^2}{r_a(r_a+r_c)}\right)^{1/2}\sin\tfrac{1}{2}\Delta\alpha\right]$$

To find optimum r_a where ΔV_{total} is minimum

set $\dfrac{d\,\Delta V_{total}}{d r_a} = 0$

Get

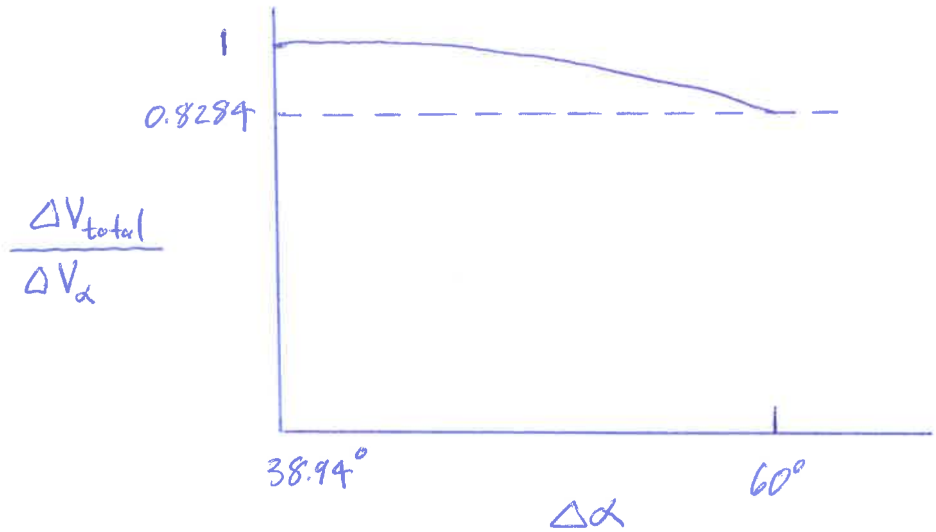
$$\frac{r_a}{r_c} = \frac{\sin\frac{1}{2}\Delta\alpha}{1 - 2\sin\frac{1}{2}\Delta\alpha} \qquad (9.32)$$



If $0 \le \Delta\alpha < 38.94^\circ$ this maneuver does not save energy

If $38.94^\circ \le \Delta\alpha < 60^\circ$ optimum r_a is given by (9.32)

If $\Delta\alpha \ge 60^\circ$ optimum path is a parabola because plane change at infinity requires zero impulse (impractical)

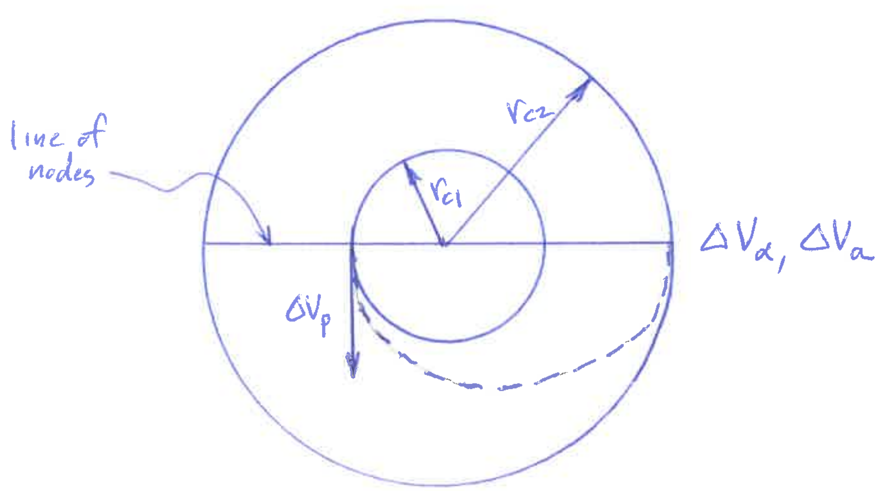


ΔV_{total} calculated from (9.31)

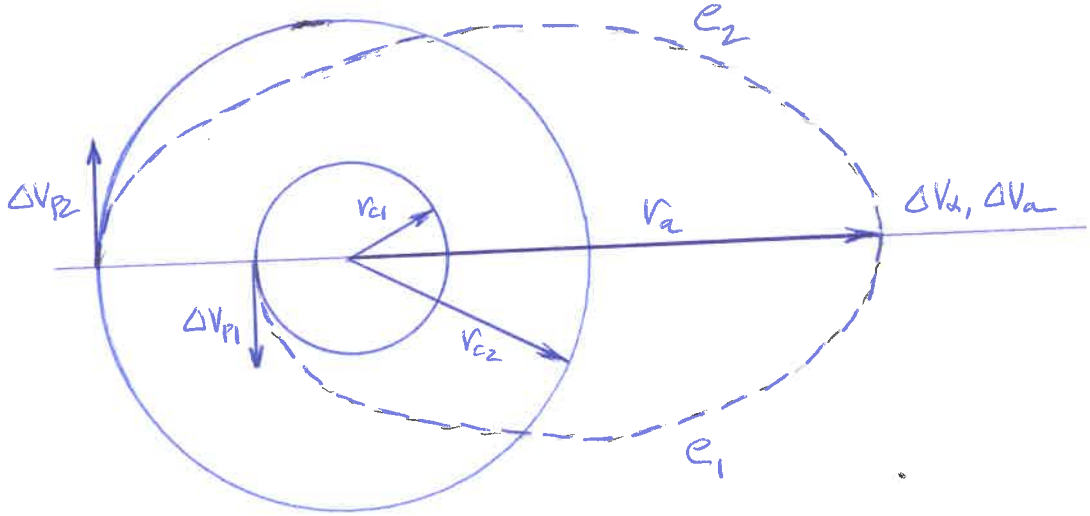
ΔV_α calculated from (9.23)

9.4. Transfer Between Non-coplanar Circular Orbits of Different Radii

Simplest Method



Method Requiring Less Energy



Same as section 9.3 except that at apogee of transfer ellipse, need ΔV_a for vehicle to enter a new ellipse whose perigee altitude is altitude of desired final circular orbit.

$$\Delta V_{P1} = \sqrt{\frac{\mu}{r_{c1}}}\left[\left(\frac{2r_a}{r_a + r_{c1}}\right)^{1/2} - 1\right] \quad \text{(see 9.27)} \qquad (9.33)$$

$$\Delta V_\alpha = 2\sqrt{\frac{2\mu r_{c1}}{r_a(r_{c1}+r_a)}}\,\sin\tfrac{1}{2}\Theta\alpha \quad \text{(see 9.30)} \qquad (9.34)$$

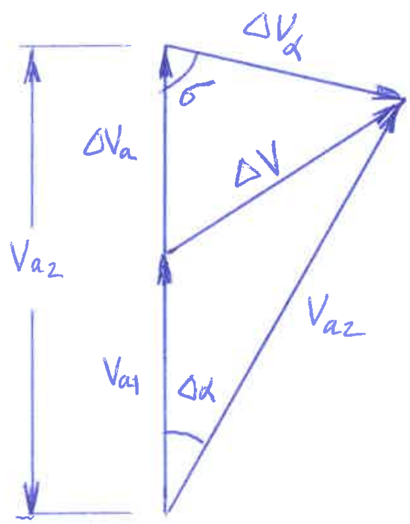
$$\Delta V_a = \sqrt{\frac{\mu}{r_a}}\left[(1-e_2)^{1/2} - \left(\frac{2r_{c1}}{r_{c1}+r_a}\right)^{1/2}\right] \quad \text{(see 9.21)} \qquad (9.35)$$

where $$e_2 = \frac{r_a - r_{c2}}{r_a + r_{c2}} \qquad (9.36)$$

$$\Delta V_{P2} = V_{c2} - V_{P2} = \sqrt{\frac{\mu}{V_{c2}}}\left[1 - \left(\frac{2V_a}{V_{c2}+V_a}\right)^{1/2}\right] \qquad (9.37)$$

$$\Delta V_{total} = |\Delta V_{P1}| + |\Delta V| + |\Delta V_{P2}| \qquad (9.38)$$

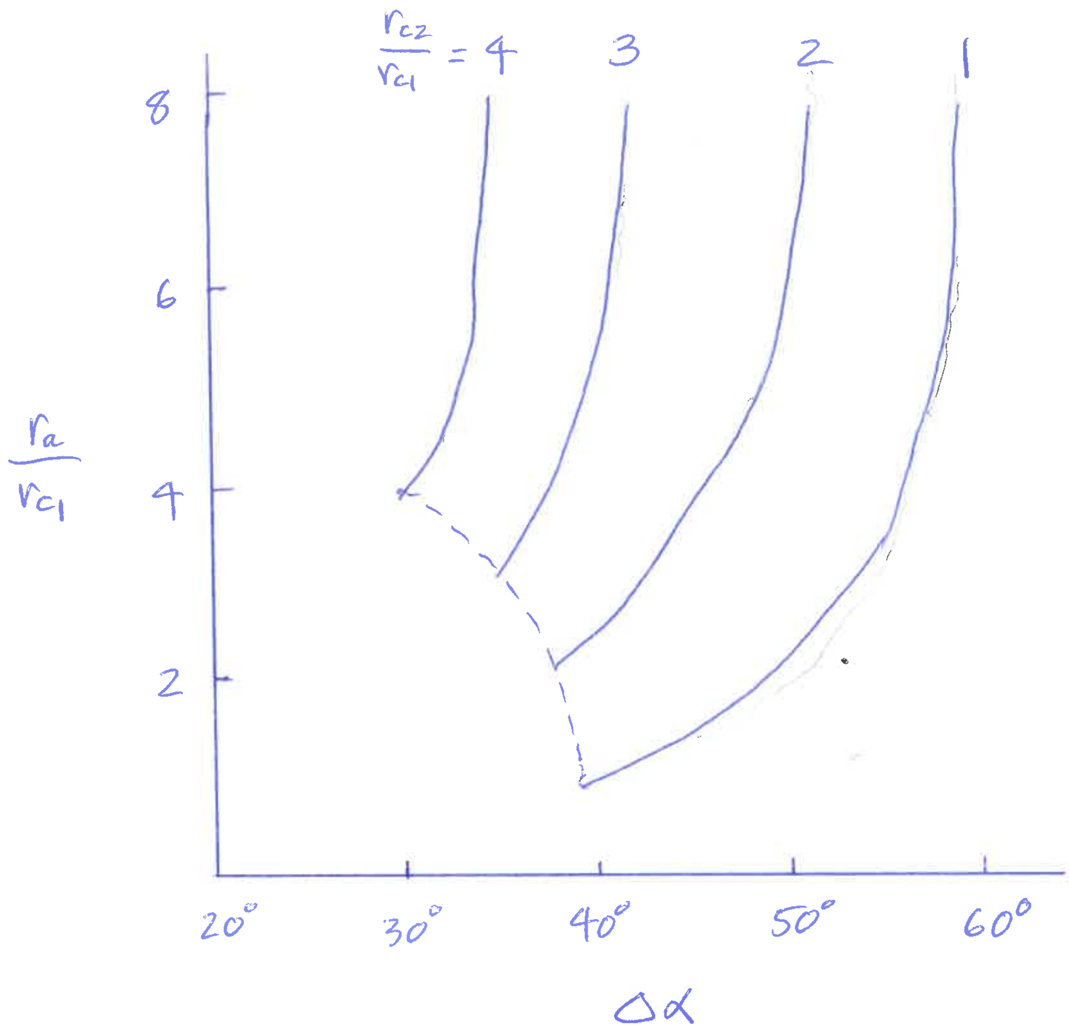
where $|\Delta V|$ is the vectorial combination of ΔV_a and ΔV_α



$$(\Delta V)^2 = (\Delta V_a)^2 + (\Delta V_\alpha)^2 - 2\left(|\Delta V_a||\Delta V_\alpha|\right)\cos\sigma$$

$$\sigma = \frac{\pi}{2} - \frac{\Delta\alpha}{2}$$

Set $\dfrac{d\,\Delta V_{total}}{d V_a} = 0$



As $\frac{r_{c2}}{r_{c1}}$ increases, the lower limit of $\Delta\alpha$ for which the manenver can be used decreases. However, the range of $\Delta\alpha$ over which the manenver can be used also decreases.

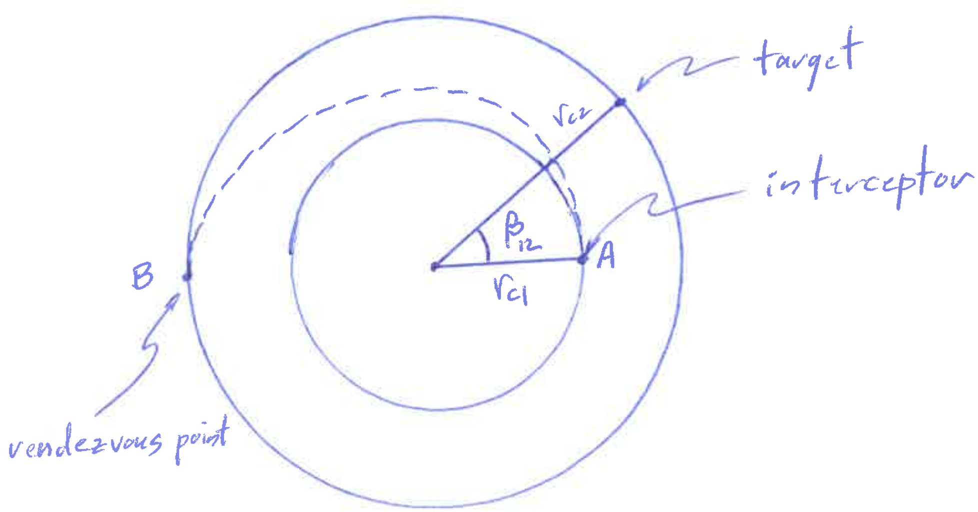
9.5. Rendezvous or Interception Between Circular, Non-coplanar Orbits.

In rendezvous, final orbits and arrival times must be matched.

In interception, final orbits need not be matched.

In the methods which follow, assume that the orbit of the target is circular and that the interceptor is initially in a smaller circular orbit inclined at $\Delta\alpha$ with respect to the target.

Method A (Employing Hohmann transfer)



1) Plane change at line of nodes of the 2 orbits

$$\Delta V_\alpha = 2 V_{c1} \sin \tfrac{1}{2} \Delta \alpha \qquad (9.39)$$

where $V_{c1} = \sqrt{\dfrac{\mu}{r_{c1}}}$

2) Begin Hohmann transfer at A with interceptor trailing target by angle β_{12}.

$$\Delta V_p = \sqrt{\frac{\mu}{r_{c1}}} \left[\left(\frac{2 r_{c2}}{r_{c1} + r_{c2}} \right)^{1/2} - 1 \right] \qquad (9.40)$$

3) Circularize orbit at B.

$$\Delta V_a = \sqrt{\frac{\mu}{r_{c2}}} \left[1 - \left(\frac{2 r_{c1}}{r_{c1} + r_{c2}} \right)^{1/2} \right] \qquad (9.41)$$

Total velocity increment for rendezvous is

$$\Delta V_{total} = \Delta V_\alpha + \Delta V_p + \Delta V_a \qquad (9.42)$$

The time required for interceptor to complete semi-ellipse from A to B is

$$\tau = \pi \sqrt{\frac{a^3}{\mu}} = \frac{\pi}{\sqrt{\mu}} \left(\frac{r_{c1} + r_{c2}}{2} \right)^{3/2} \qquad (9.43)$$

The corresponding time for the target to move through $\pi - \beta_{12}$ radians is

$$\tau = \frac{\pi - \beta_{12}}{\sqrt{\mu}} r_{c2}^{3/2} \qquad (9.44)$$

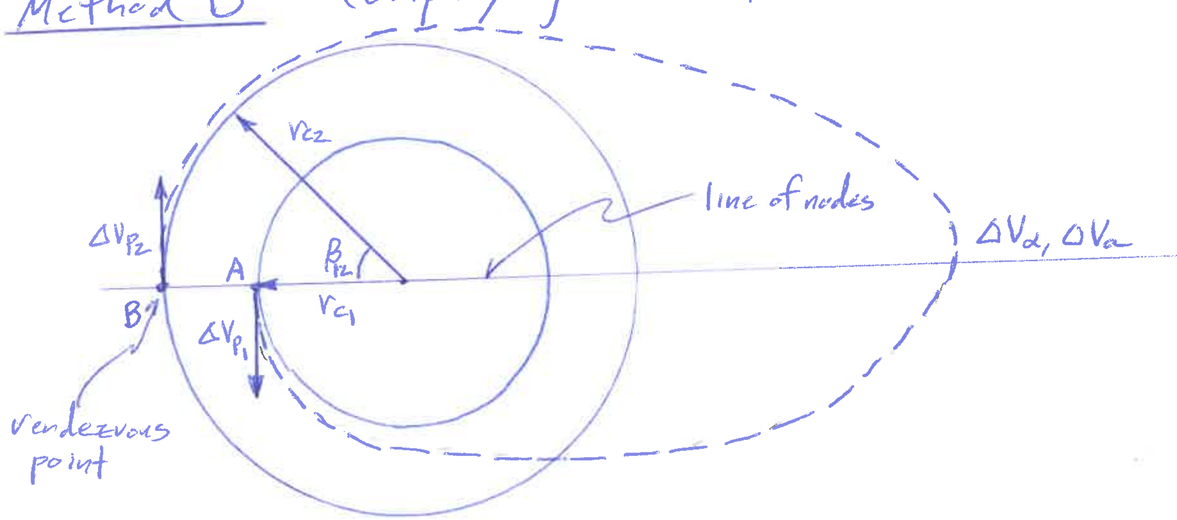
For vehicle to arrive at B simultaneously, equate (9.43) to (9.44). The value of β_{12} necessary at the start of the transfer ellipse is

$$\beta_{12} = \pi\left[1 - \left\{\frac{1}{2}\left(1 + \frac{r_{c1}}{r_{c2}}\right)\right\}^{3/2}\right] \qquad (9.45)$$

For interception, (9.45) still applies but (9.42) becomes

$$\Delta V_{total} = \Delta V_\alpha + \Delta V_p \qquad (9.46)$$

<u>Method B</u> (Employing bi-elliptic transfer)



Perform ΔV_{P1} when interceptor crosses line of nodes <u>and</u> target trails by angle β_R (to be determined)

For rendezvous

$$\Delta V_{total} = \Delta V_{P_1} + \underbrace{\Delta V_\alpha + \Delta V_a}_{\text{combine vectorially}} + \Delta V_{P2} \qquad (9.47)$$

Time for interceptor to complete first semi-ellipse is

$$\tau_1 = \frac{\pi}{\sqrt{\mu}}\left(\frac{r_{c1} + r_a}{2}\right)^{3/2} \qquad (9.48)$$

and to complete second semi-ellipse is

$$\tau_2 = \frac{\pi}{\sqrt{\mu}}\left(\frac{r_a + r_{c2}}{2}\right)^{3/2} \qquad (9.49)$$

The total time for the interceptor to move from A to B is

$$\tau = \frac{\pi}{\sqrt{\mu}}\left[\left(\frac{r_{c1} + r_a}{2}\right)^{3/2} + \left(\frac{r_a + r_{c2}}{2}\right)^{3/2}\right] \qquad (9.50)$$

Assuming that in the time it takes for the rendezvous maneuvers to be completed the target completes 1 orbit + β_{12}

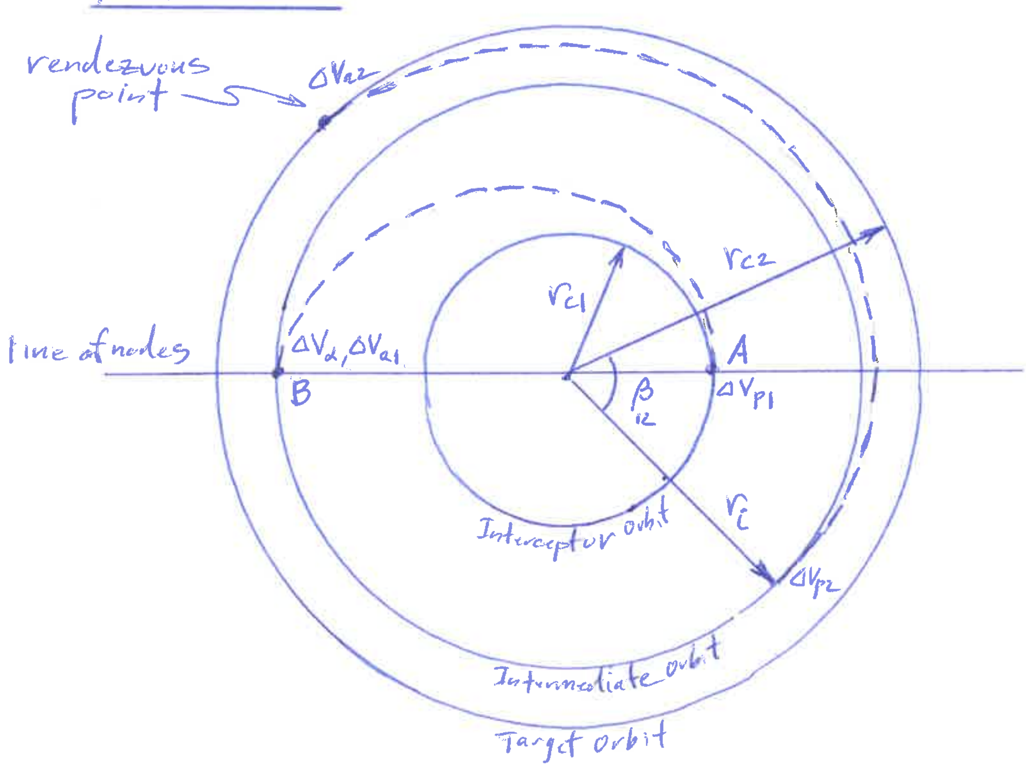
$$\tau = \frac{2\pi + \beta_{12}}{\sqrt{\mu}} r_{c2}^{3/2} \qquad (9.51)$$

For rendezvous, equate (9.50) to (9.51)

$$\beta_{12} = \pi\left[\left(\frac{r_{c1}+r_a}{2r_{c2}}\right)^{3/2} + \left(\frac{r_a + r_{c2}}{2r_{c2}}\right)^{3/2} - 2\right] \qquad (9.52)$$

The method is advantageous for $\Delta\alpha$ given in section (9.4).

<u>Method C</u>



1) Begin Hohmann transfer at A

2) Circularize orbit and perform plane change at B.

3) Second Hohmann transfer begins when interceptor trails target by angle β_{12}.

More accurate than previous methods since errors may be assessed and corrected while interceptor is in intermediate orbit.

Velocity increment for plane change ΔV_α is smaller than in Method A.

Relative velocity between target and interceptor is small at final approach.

For rendezvous

$$\Delta V_{total} = |\Delta V_{P1}| + \underbrace{|\Delta V_\alpha| + |\Delta V_{a1}|}_{\text{combine vectorially}} + |\Delta V_{P2}| + |\Delta V_{a2}| \quad (9.53)$$

For interception

$$\Delta V_{total} = |\Delta V_{P1}| + \underbrace{|\Delta V_\alpha| + |\Delta V_{a1}|}_{\text{combine vectorially}} + |\Delta V_{P2}| \quad (9.54)$$

where

$$\Delta V_{P1} = \sqrt{\frac{\mu}{r_{c1}}}\left[\left(\frac{2r_i}{r_{c1}+r_i}\right)^{1/2} - 1\right] \qquad (9.55)$$

$$\Delta V_\alpha = 2V_a \sin\tfrac{1}{2}\Delta\alpha = 2\sqrt{\frac{2\mu r_{c1}}{r_i(r_{c1}+r_i)}}\sin\tfrac{1}{2}\Delta\alpha \qquad (9.56)$$

$$\Delta V_{a1} = \sqrt{\frac{\mu}{r_i}}\left[1 - \left(\frac{2r_{c1}}{r_{c1}+r_i}\right)^{1/2}\right] \qquad (9.57)$$

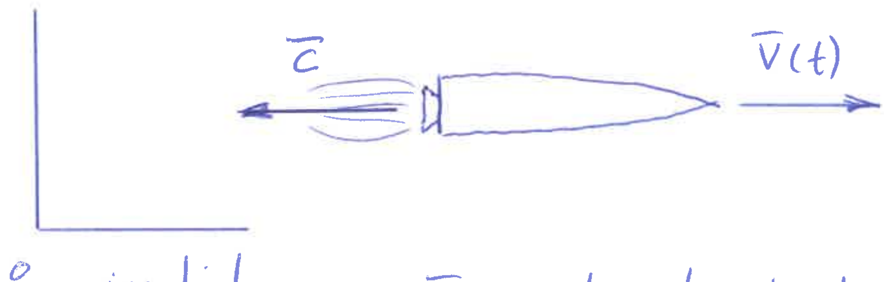
$$\Delta V_{P2} = \sqrt{\frac{\mu}{r_i}}\left[\left(\frac{2r_{c2}}{r_i+r_{c2}}\right)^{1/2} - 1\right] \qquad (9.58)$$

$$\Delta V_{a2} = \sqrt{\frac{\mu}{r_{c2}}}\left[1 - \left(\frac{2r_i}{r_i+r_{c2}}\right)^{1/2}\right] \qquad (9.59)$$

For rendezvous or interception

$$\beta_{12} = \pi\left[1 - \left\{\frac{1}{2}\left(1+\frac{r_i}{r_{c2}}\right)\right\}^{3/2}\right] \qquad (9.60)$$

9.6 Relation Between Velocity Impulse ΔV and Propellant Usage.



$\bar{c}$ = exhaust velocity (relative to rocket)

For conservation of mass (CV moving with rocket)

$$\frac{dm}{dt} = -\dot{m}_e \qquad (9.61)$$

where

$\frac{dm}{dt}$ = rate of change of mass of rocket

$\dot{m}_e$ = mass flow rate of exhaust gases (relative to rocket)

Since mass of rocket is not constant, must use more general form of Newton's Second Law in an inertial reference frame.

$$\bar{F}_{ext} = \frac{d\bar{P}}{dt} \qquad (9.62)$$

where

$\bar{P} = m\bar{V}$ (linear momentum)

$\bar{F}_{ext}$ = sum of external forces acting on rocket other than thrust (e.g. atmospheric drag)

During a short time interval Δt

$$\Delta\bar{P} = \bar{F}_{ext}\,\Delta t \qquad (9.63)$$

Before rocket firing

$$\bar{P} \text{ at time } t = m\bar{V}$$

After rocket firing

$$\bar{P} \text{ at time } t+\Delta t = \underbrace{(m - \dot{m}_e\Delta t)(\bar{V}+\Delta\bar{V})}_{\text{rocket}} + \underbrace{(\dot{m}_e\Delta t)(\bar{V}+\bar{c})}_{\text{exhaust gases}}$$

Sub. into (9.63)

$$(m-\dot{m}_e\Delta t)(\bar{V}+\Delta\bar{V})+(\dot{m}_e\Delta t)(\bar{V}+\bar{c})-m\bar{V}=\bar{F}_{ext}\Delta t$$

$$\cancel{m\bar{V}}+m\Delta\bar{V}-\cancel{\dot{m}_e\Delta t\,\bar{V}}-\dot{m}_e\Delta t\Delta\bar{V}+\cancel{\dot{m}_e\Delta t\,\bar{V}}+\dot{m}_e\Delta t\bar{c}-\cancel{m\bar{V}}$$
$$=\bar{F}_{ext}\Delta t$$

Divide by Δt

$$m\frac{\Delta\bar{V}}{\Delta t}=\bar{F}_{ext}-\dot{m}_e\bar{c}+\dot{m}_e\Delta\bar{V}$$

Take limit as $\Delta t\to 0$

$$m\frac{d\bar{V}}{dt}=\bar{F}_{ext}-\dot{m}_e\bar{c} \qquad (9.64)$$

where $-\dot{m}_e\bar{c}$ is the thrust of the rocket (directed opposite to exhaust velocity $\bar{c}$).

For $\bar{F}_{ext}=0$

$$m\frac{d\bar{V}}{dt}=-\dot{m}_e\bar{c}=\frac{dm}{dt}\bar{c}$$

↑ rate of change of mass of rocket (using 9.61)

$$d\bar{V} = \bar{c}\,\frac{dm}{m}$$

Integrating for a constant thrust rocket

$$\bar{V}\Big|_{\bar{V}_0}^{\bar{V}} = \bar{c}\,\ln m\Big|_{m_0}^{m}$$

$$\Delta\bar{V} = \bar{V} - \bar{V}_0 = -\bar{c}\,\ln\left(\frac{m_0}{m}\right) \qquad (9.65)$$

Take magnitude

$$\Delta V = c\,\ln\left(\frac{m_0}{m}\right) \qquad (9.66)$$

$$\frac{m}{m_0} = e^{-\frac{\Delta V}{c}}$$

Define $\Delta m = m_0 - m$ (the amount of propellant consumed)

$$\frac{\Delta m}{m_0} = 1 - \frac{m}{m_0} = 1 - e^{-\frac{\Delta V}{c}} \qquad (9.67)$$

Consider 2 successive maneuvers (burns).

From (9.66)

$$\Delta V_1 = c \ln \frac{m_0}{m_1}$$

$$\Delta V_2 = c \ln \frac{m_1}{m_2}$$

$$\Delta V_{total} = \Delta V_1 + \Delta V_2$$

$$= c \ln \frac{m_0}{m_1} + c \ln \frac{m_1}{m_2}$$

$$= c \ln \frac{m_0}{m_2}$$

or in general

$$\Delta V_{total} = c \ln \frac{m_0}{m_f} \qquad (9.68)$$

<u>Specific Impulse of a Rocket Engine</u>

$$I_{sp} = \frac{\text{total impulse}}{\text{weight of propellant consumed}}$$

$$= \frac{\overbrace{(\dot{m}_e c)}^{\text{thrust}} \Delta t}{\underbrace{(\dot{m}_e \Delta t)}_{\text{mass of propellant consumed}} \cdot \underbrace{g}_{\text{at earth's surface}}} = \frac{c}{g} \quad (\text{sec}) \qquad (9.69)$$

For high thrust (chemical rockets)

$I_{sp} = 200 - 500$ sec

e.g. SLS

2 Solid Rocket Motors $I_{sp} = 265$ sec

4 RS-25 Liquid fuel Engines $I_{sp} = 452$ sec

Low thrust engines (high exhaust velocity, low mass flow rate)

e.g. Ion propulsion $I_{sp} = 10{,}000$ sec

(Ideal for deep space missions)

EXAMPLE (Chase Maneuver)

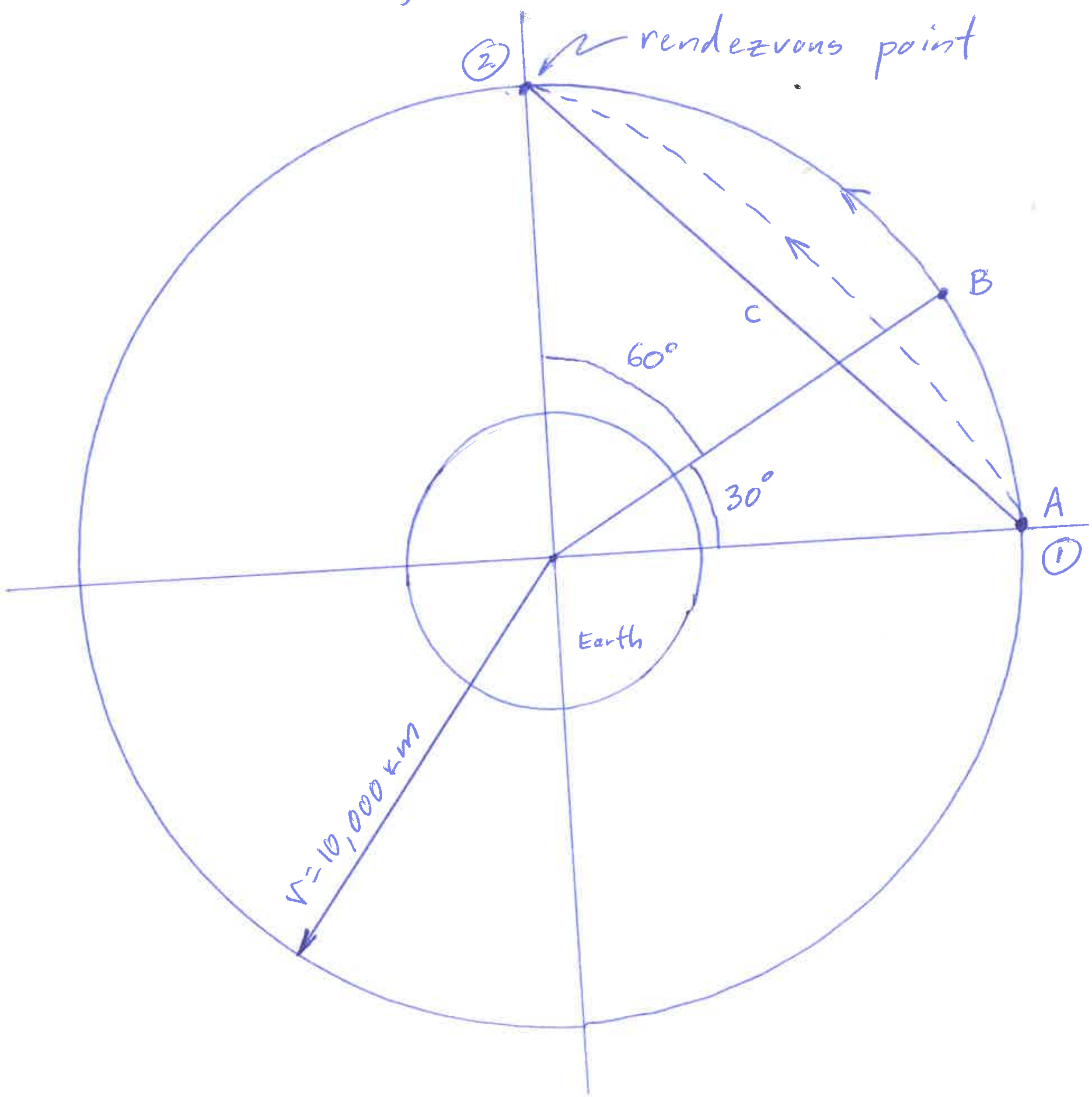
Two spacecraft, A and B, are in the same circular orbit of radius 10,000 km with spacecraft B 30° ahead of spacecraft A.

a) Verify that a transfer ellipse exists which would allow spacecraft A to rendezvous with spacecraft B when spacecraft B has traveled 60°.

b) Determine a and e of the transfer ellipse

c) Calculate the total ΔV needed for rendezvous.

d) If spacecraft A is equipped with a constant thrust engine having $I_{sp} = 200$ sec, calculate the percentage reduction of the mass of the vehicle as a result of propellant expended in performing the maneuvers.



a) The time required to reach the rendezvous point

$$t_2 - t_1 = \frac{60°}{360°} \cdot 2\pi \sqrt{\frac{r^3}{\mu}} = \frac{1}{6} \cdot 2\pi \sqrt{\frac{(10,000)^3}{3.986 \times 10^5}} = 1658.67 \text{ sec}$$

Since $\Delta\theta = 90°$

$$c = \sqrt{(10,000)^2 + (10,000)^2} = 14,142.1 \text{ km}$$

$$S = \frac{1}{2}(r + r + c) = \frac{1}{2}(10,000 + 10,000 + 14,142.1) = 17,071.1 \text{ km}$$

The flight time along a parabolic trajectory is

$$t_p = \frac{\sqrt{2}}{3\sqrt{\mu}}\left[S^{3/2} - \text{sgn}(\sin\Delta\theta)(s-c)^{3/2}\right] \qquad \Delta\theta = 90°$$

$$= \frac{\sqrt{2}}{3\sqrt{3.986 \times 10^5}}\left[(17,071.1)^{3/2} - (17,071.1 - 14,142.1)^{3/2}\right]$$

$$t_p = 1547.04 \text{ sec}$$

$t_2 - t_1 > t_p \Rightarrow$ elliptic transfer ellipse is possible

b) $$\sin\left(\frac{\alpha}{2}\right) = \left(\frac{s}{2a}\right)^{1/2} \qquad (1)$$

$$\sin\left(\frac{\beta}{2}\right) = \left(\frac{s-c}{2a}\right)^{1/2} \qquad (2)$$

$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[\alpha - \beta - (\sin\alpha - \sin\beta)\right] \qquad (3)$$

Numerical solution of (1), (2) & (3) gives

$$\boxed{a = 42{,}466.13 \text{ km}}$$

$$\alpha = 0.929784$$

$$\beta = 0.373574$$

$$p = \frac{4a(s-r)^2}{c^2} \sin^2\left(\frac{\alpha+\beta}{2}\right)$$

$$= \frac{4(42{,}466.13)(17{,}071.1 - 10{,}000)^2}{(14{,}142.1)^2} \sin^2\left(\frac{0.929784 + 0.373574}{2}\right)$$

$$= 15{,}621.97 \text{ km}$$

$$p = a(1-e^2) \Rightarrow e = \sqrt{1 - \frac{p}{a}} = \sqrt{1 - \frac{15{,}621.97}{42{,}466.13}} = \underline{\underline{0.795067}}$$

c) $\hat{u}_1 = \frac{\bar{r}_1}{r_1} = \hat{i}$ $\qquad \hat{u}_2 = \frac{\bar{r}_2}{r_2} = \hat{j}$

$$\hat{u}_c = \frac{(\bar{r}_2 - \bar{r}_1)}{c} = \frac{(\hat{j} - \hat{i})\,10{,}000}{14{,}142.1} = \frac{\hat{j} - \hat{i}}{\sqrt{2}}$$

$$A = \left(\frac{\mu}{4a}\right)^{1/2} \cot\left(\frac{\alpha}{2}\right) = \left(\frac{3.986\times10^5}{4(42{,}466.13)}\right)^{1/2} \cot\left(\frac{0.929784}{2}\right) = 3.0542$$

$$B = \left(\frac{\mu}{4a}\right)^{1/2} \cot\left(\frac{\beta}{2}\right) = \left(\frac{3.986\times10^5}{4(42{,}466.13)}\right)^{1/2} \cot\left(\frac{0.373574}{2}\right) = 8.10547$$

$$\bar{V}_1 = (B+A)\,\hat{u}_c + (B-A)\,\hat{u}_1$$

$$= (8.10547 + 3.0542)\,\frac{\hat{j}-\hat{i}}{\sqrt{2}} + (8.10547 - 3.0542)\,\hat{i}$$

$$\bar{V}_1 = 2.52564\,\hat{i} + 7.89108\,\hat{j} \quad (km/s)$$

$$\bar{V}_{c1} = \sqrt{\frac{\mu}{r^3}}\,\hat{j} = \sqrt{\frac{3.986 \times 10^5}{10{,}000}}\,\hat{j} = 6.31348\,\hat{j} \quad (km/s)$$

$$\Delta\bar{V}_1 = \bar{V}_1 - \bar{V}_{c1} = (2.52564\,\hat{i} + 7.89108\,\hat{j}) - (6.31348\,\hat{j})$$

$$\Delta\bar{V}_1 = 2.52564\,\hat{i} + 1.5776\,\hat{j} \quad (km/sec)$$

$$\Delta V_1 = \sqrt{(2.52564)^2 + (1.5776)^2} = 2.97786 \text{ km/sec}$$

$$\bar{V}_2 = (B+A)\,\hat{u}_c - (B-A)\,\hat{u}_2$$

$$= (8.10547 + 3.0542)\,\frac{\hat{j}-\hat{i}}{\sqrt{2}} - (8.10547 - 3.0542)\,\hat{j}$$

$$= -7.89108\,\hat{i} + 2.83981\,\hat{j} \quad (km/sec)$$

$$\bar{V}_{c2} = -\sqrt{\frac{\mu}{r^3}}\,\hat{i} = -6.31348\,\hat{i} \quad (km/sec)$$

$$\Delta\bar{V}_2 = \bar{V}_{c2} - \bar{V}_2 = -6.31348\,\hat{i} - (-7.89108\,\hat{i} + 2.83981\,\hat{j})$$

$$= 1.5776\,\hat{i} - 2.83981\,\hat{j} \quad (km/sec)$$

$$\Delta V_2 = \sqrt{(1.5776)^2 + (2.83981)^2} = 3.24859 \text{ km/sec}$$

$$\Delta V_{TOT} = \Delta V_1 + \Delta V_2 = 2.97786 + 3.24859 = \underline{\underline{6.22645 \frac{km}{sec}}}$$

d) $c = I_{sp}\, g = (200 \text{ sec})\left(9.8066 \frac{m}{sec^2}\right) = 1961.32 \frac{m}{sec} = 1.96132 \frac{km}{sec}$

$$\frac{\Delta m}{m_0} = 1 - e^{-\frac{\Delta V_{TOT}}{c}} = 1 - e^{-\frac{6.22645}{1.96132}} = 0.95819 = \underline{\underline{95.819\%}}$$