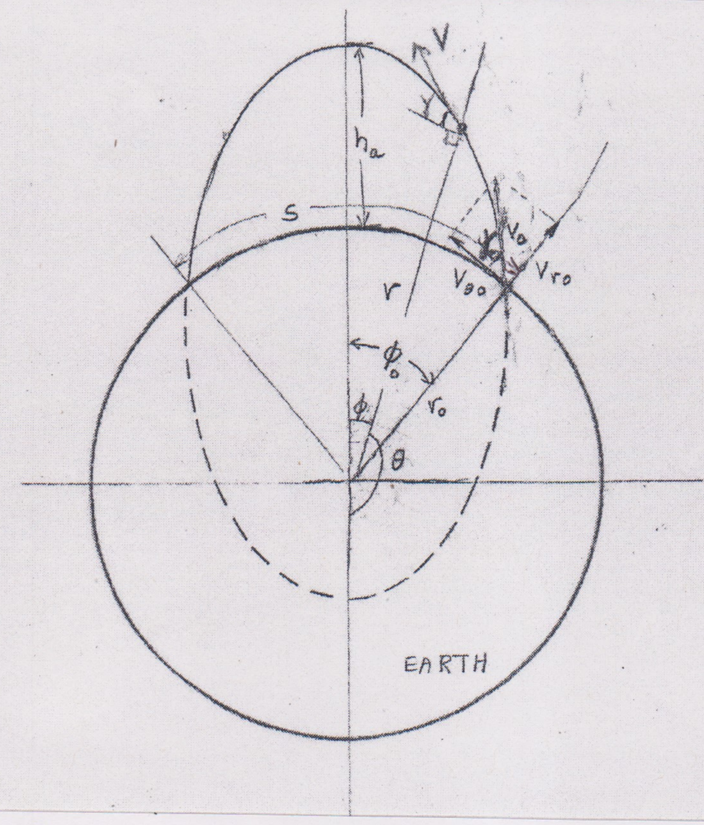
6. Suborbital Flight (Ballistic Trajectory)

In the analysis which follows, neglect
a) atmospheric drag
b) rotation of the earth



S = surface range

ϕ_o = semi-range angle, $\phi_o = \frac{S}{2 r_o}$

V_o = initial velocity (burnout velocity)

γ_o = elevation angle

γ = flight path angle

θ = true anomaly

$\phi = \pi - \theta$

h_a = apogee altitude

A. For a given initial velocity V_0, what value of γ_0 will give maximum range S (maximum Φ_0) if $\Phi_0 < \frac{\pi}{2}$, $\gamma_0 < \frac{\pi}{2}$

Need relation between Φ_0 and γ_0.

$$r = \frac{P}{1 + e\cos\theta}$$

$$e\cos\theta = \frac{P}{r} - 1 \qquad (6.1)$$

Differentiate (6.1) with respect to time

$$-e\dot{\theta}\sin\theta = -\frac{P}{r^2}\dot{r}$$

$$e\dot{\theta}\sin\theta = \frac{P}{r^2} V_r \qquad (6.2)$$

Divide (6.2) by (6.1)

$$\dot{\theta}\tan\theta = \frac{\frac{P}{r^2} V_r}{\frac{P}{r} - 1}$$

$$\tan\theta = V_r \frac{P}{r \underbrace{r\dot{\theta}}_{V_\theta}} \frac{1}{\frac{P}{r} - 1} = \frac{V_r}{V_\theta} \frac{P/r}{\frac{P}{r} - 1} \qquad (6.3)$$

Note that $h = r^2 \dot{\theta}$

$$\frac{P}{r} = \frac{h^2}{\mu r} = \frac{r^2 (r\dot{\theta})^2}{\mu r} = \frac{r^2 V_\theta^2}{\mu r} = \frac{r^2 V^2 \cos^2 \gamma}{\mu r} = \frac{V^2 \cos^2 \gamma}{\mu / r} \quad (6.4)$$

Define a dimensionless velocity

$$v \equiv \frac{V}{V_c} \quad (6.5)$$

where

$V_c^2 = \frac{\mu}{r}$ (required velocity for circular orbit of radius r)

Therefore (6.4) becomes

$$\frac{P}{r} = v^2 \cos^2 \gamma \quad (6.6)$$

Sub. (6.6) into (6.3)

$$\tan \theta = \frac{V_r}{V_\theta} \frac{v^2 \cos^2 \gamma}{v^2 \cos^2 \gamma - 1} \quad (6.7)$$

But

$$\frac{V_r}{V_\theta} = \tan \gamma$$

$$\tan \theta = \tan(\pi - \phi) = -\tan \phi$$

Sub. into (6.7)

$$\tan \phi = \frac{(\tan \gamma)(v^2 \cos^2 \gamma)}{1 - v^2 \cos^2 \gamma}$$

$$= \frac{v^2 \sin \gamma \cos \gamma}{1 - v^2 \cos^2 \gamma}$$

$$= \frac{v^2 \frac{1}{2} \sin 2\gamma}{1 - v^2 \left(\frac{1 + \cos 2\gamma}{2} \right)}$$

$$\tan \phi = \frac{\sin 2\gamma}{\left(\frac{2}{v^2} - 1 \right) - \cos 2\gamma}$$

At burnout

$$\boxed{\tan \phi_0 = \frac{\sin 2\gamma_0}{\left(\frac{2}{v_0^2} - 1\right) - \cos 2\gamma_0}} \qquad (6.8)$$

where $$v_0 = \frac{V_0}{V_{c0}} = \frac{V_0}{\sqrt{\mu / r_0}} \qquad (6.9)$$

Eq. (6.8) gives the semi-range angle as a function of the initial velocity and elevation angle.

The trajectory parameters can be found in terms of the burnout conditions

From (6.6)

$$\boxed{p = r_0 v_0^2 \cos^2 \gamma_0} \qquad (6.10)$$

From energy considerations

$$\frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

At burnout

$$\frac{V_0^2}{2} - \frac{\mu}{r_0} = -\frac{\mu}{2a}$$

$$\boxed{a = \frac{r_0}{2 - v_0^2}} \qquad (6.11)$$

Since for elliptic trajectory

$$p = a(1-e^2)$$

$$e = \sqrt{1 - \frac{p}{a}}$$

Sub. (6.10) & (6.11)

$$e = \sqrt{1 - \frac{r_0 v_0^2 \cos^2\gamma_0}{r_0/(2-v_0^2)}}$$

$$\boxed{e = \sqrt{1 - v_0^2(2-v_0^2)\cos^2\gamma_0}}$$

Note: If $v_0 \geq \sqrt{2} \Rightarrow$ escape regardless of γ_0

The apogee altitude is

$$h_a = r_a - r_0 = a(1+e) - r_0$$

Sub. (6.11) & (6.12)

$$h_a = \frac{r_0}{2-v_0^2}\left[1+\sqrt{1-v_0^2(2-v_0^2)\cos^2\gamma_0}\right] - r_0$$

$$\boxed{h_a = \frac{r_0}{2-v_0^2}\left[\sqrt{1-v_0^2(2-v_0^2)\cos^2\gamma_0} - (1-v_0^2)\right]} \qquad (6.13)$$

To find elevation angle γ_0 for maximum range (maximum ϕ_0)

Find $\frac{d\phi_0}{d\gamma_0}$ from (6.8) and set to zero

$$\sec^2\phi_0 \frac{d\phi_0}{d\gamma_0} = \frac{\left[\left(\frac{2}{v_0^2}-1\right)-\cos 2\gamma_0\right]2\cos 2\gamma_0 - \sin 2\gamma_0\left[2\sin 2\gamma_0\right]}{\left[\left(\frac{2}{v_0^2}-1\right)-\cos 2\gamma_0\right]^2} = 0$$

$$\left[\left(\frac{2}{v_0^2}-1\right)-\cos 2\gamma_0\right]\cos 2\gamma_0 - \sin^2 2\gamma_0 = 0$$

Using (6.8)

$$\frac{\sin 2\gamma_o}{\tan\phi_o}\cos 2\gamma_o - \sin^2 2\gamma_o = 0$$

$$\frac{\sin 2\gamma_o}{\cos 2\gamma_o} = \frac{1}{\tan\phi_o}$$

$$\boxed{\tan 2\gamma_{opt} = \cot\phi_o} \qquad (6.14)$$

$$\frac{\sin 2\gamma_{opt}}{\cos 2\gamma_{opt}} = \frac{\cos\phi_o}{\sin\phi_o}$$

$$\cos 2\gamma_{opt}\cos\phi_o - \sin 2\gamma_{opt}\sin\phi_o = 0$$

$$\cos(2\gamma_{opt} + \phi_o) = 0$$

$$2\gamma_{opt} + \phi_o = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

Since $\phi_o < \frac{\pi}{2}$ & $\gamma_{opt} < \frac{\pi}{2}$, only 1st solution has physical meaning

$$2\gamma_{opt} + \phi_o = \frac{\pi}{2}$$

$$\boxed{\gamma_{opt} = \frac{\pi}{4} - \frac{\phi_o}{2}} \qquad (6.15)$$

Note: As $\phi_0 \to 0$ (small range)

$$\gamma_{opt} \to \frac{\pi}{4} = 45^\circ$$

To find maximum range, from (6.8)

$$\tan\phi_{max} = \frac{\sin 2\gamma_{opt}}{\left(\frac{2}{v_0^2}-1\right) - \cos 2\gamma_{opt}} \qquad \gamma_{opt} = \frac{\pi}{4} - \frac{\phi_0}{2}$$

$$= \frac{\sin 2\left(\frac{\pi}{4} - \frac{\phi_{max}}{2}\right)}{\left(\frac{2}{v_0^2}-1\right) - \cos 2\left(\frac{\pi}{4} - \frac{\phi_{max}}{2}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{2} - \phi_{max}\right)}{\left(\frac{2}{v_0^2}-1\right) - \cos\left(\frac{\pi}{2} - \phi_{max}\right)}$$

$$\sin\left(\frac{\pi}{2} - \phi_{max}\right) = \cos\phi_{max}$$

$$\cos\left(\frac{\pi}{2} - \phi_{max}\right) = \sin\phi_{max}$$

$$\tan\phi_{max} = \frac{\cos\phi_{max}}{\left(\frac{2}{v_0^2}-1\right) - \sin\phi_{max}}$$

$$\frac{\sin\phi_{max}}{\cos\phi_{max}} = \frac{\cos\phi_{max}}{\left(\frac{2}{v_o^2}-1\right)-\sin\phi_{max}}$$

$$\left(\frac{2}{v_o^2}-1\right)\sin\phi_{max} - \sin^2\phi_{max} = \cos^2\phi_{max}$$

$$\left(\frac{2}{v_o^2}-1\right)\sin\phi_{max} = 1$$

$$\sin\phi_{max} = \frac{1}{\frac{2}{v_o^2}-1}$$

$$\boxed{\sin\phi_{max} = \frac{v_o^2}{2-v_o^2}} \qquad (6.17)$$

The corresponding apogee altitude for <u>maximum range</u> is (from 6.13)

$$h_{max} = \frac{r_o}{2-v_o^2}\left[\sqrt{1-v_o^2(2-v_o^2)\cos^2\gamma_{opt}} - (1-v_o^2)\right]$$

$$\cos^2\gamma_{opt} = \frac{1+\cos 2\gamma_{opt}}{2} = \frac{1+\cos 2\left(\frac{\pi}{4}-\frac{\phi_{max}}{2}\right)}{2} = \frac{1+\sin\phi_{max}}{2}$$

$$= \frac{1+\frac{v_0^2}{2-v_0^2}}{2} = \frac{1}{2-v_0^2}$$

$$\boxed{h_{max} = \frac{v_0}{2-v_0^2}\left[\sqrt{1-v_0^2}-(1-v_0^2)\right]} \qquad (6.18)$$

or in terms of the maximum range [sub (6.17) into (6.18)]

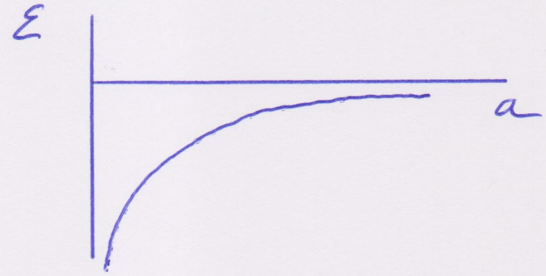
$$\boxed{h_{max} = \frac{v_0}{2}\left[\cos\phi_{max}+\sin\phi_{max}-1\right]} \qquad (6.19)$$

B. Minimum Energy Trajectory (MET)

For a given range S, which elliptic trajectory requires the lowest energy?

Total energy of a projectile in elliptic trajectory

$$\mathcal{E} = -\frac{\mu}{2a}$$



Ellipse with smallest "a" has lowest energy.

$$r = \frac{p}{1+e\cos\theta} \qquad (6.20)$$

For $\theta = \pi - \phi_0$ $r = r_0$

$$r_0 = \frac{p}{1-e\cos\phi_0}$$

$$p = r_0(1-e\cos\phi_0) \qquad (6.21)$$

Sub (6.21) into (6.20)

$$r = \frac{r_0(1 - e\cos\phi_0)}{1 + e\cos\theta} \qquad (6.22)$$

At apogee $(\theta = \pi)$

$$r_a = a(1+e) = \frac{r_0(1 - e\cos\phi_0)}{1-e}$$

Therefore

$$\boxed{a = \frac{r_0(1 - e\cos\phi_0)}{1-e^2}} \qquad (6.23)$$

Eq. (6.23) gives a as a function of e.

Wish to find the value of e which gives minimum "a" and therefore minimum energy $\mathcal{E}$.

$$\frac{da}{de} = \frac{(1-e^2)(-r_0\cos\phi_0) - r_0(1 - e\cos\phi_0)(-2e)}{(1-e^2)^2} = 0$$

$$-(1-e^2)\cos\phi_0 + 2e(1 - e\cos\phi_0) = 0$$

$$e = \frac{2 \pm \sqrt{4 - 4\cos^2\phi_0}}{2\cos\phi_0}$$

$$e = \frac{1 \pm \sin\phi_0}{\cos\phi_0}$$

Since $e < 1$ for ellipse and $\sin\phi_0 > 0$, only negative sign gives meaningful result

$$\boxed{e_{MET} = \frac{1 - \sin\phi_0}{\cos\phi_0}} \qquad (6.24)$$

Wish to find launch elevation angle γ_0 for MET.

$$\tan\gamma = \frac{V_r}{V_\theta} \qquad (6.25)$$

From (5.26) & (5.27)

$$V_r = \frac{he\sin\theta}{a(1-e^2)} \qquad V_\theta = \frac{h(1+e\cos\theta)}{a(1-e^2)}$$

Therefore

$$\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta} \qquad (6.26)$$

At launch (burnout)

$$\tan \gamma_o = \frac{e \sin \theta_o}{1 + e \cos \theta_o} = \frac{e \sin(\pi - \phi_o)}{1 + e \cos(\pi - \phi_o)} = \frac{e \sin \phi_o}{1 - e \cos \phi_o}$$

For minimum energy trajectory

$$\tan \gamma_{o_{MET}} = \frac{e_{MET} \sin \phi_o}{1 - e_{MET} \cos \phi_o}$$

$$= \frac{e_{MET} \sin \phi_o}{1 - \frac{1 - \sin \phi_o}{\cos \phi_o} \cos \phi_o}$$

$$\boxed{\tan 2\gamma_{o_{MET}} = e_{MET}} \qquad (6.27)$$

Want to show that (6.27) is equivalent to (6.14)

Using the trigonometric identity

$$\tan 2\gamma = \frac{2 \tan \gamma}{1 - \tan^2 \gamma}$$

$$\tan 2\gamma_{0\,MET} = \frac{2\, e_{MET}}{1 - e_{MET}^2} = \frac{2\left(\frac{1-\sin\phi_0}{\cos\phi_0}\right)}{1 - \left(\frac{1-\sin\phi_0}{\cos\phi_0}\right)^2}$$

$$= \frac{2\left(\frac{1-\sin\phi_0}{\cos\phi_0}\right)}{1 - \left(\frac{1 - 2\sin\phi_0 + \sin^2\phi_0}{\cos^2\phi_0}\right)}$$

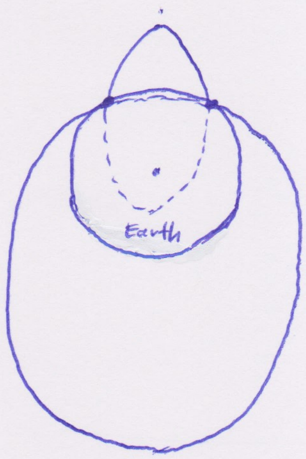
$$= \frac{2(1-\sin\phi_0)\cos\phi_0}{2(1-\sin\phi_0)\sin\phi_0}$$

$$\boxed{\tan 2\gamma_{MET} = \cot\phi_0}$$

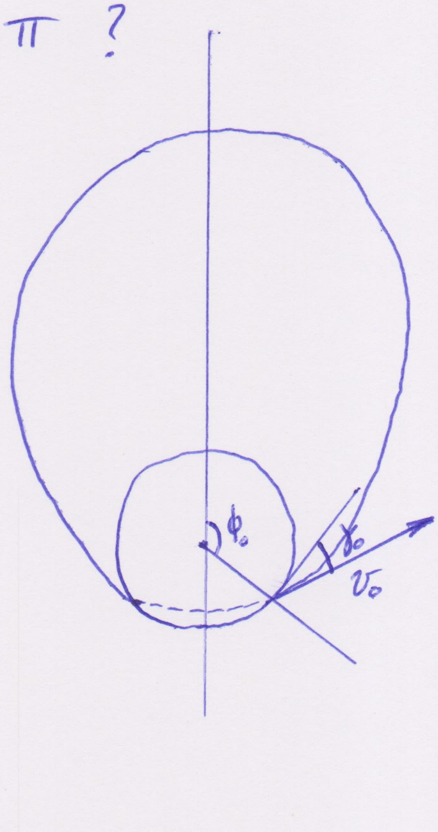
Note:

a) From (6.24) as $\phi_0 \to 0$ (small ranges) $e \to 1$
(parabolic trajectory)

b) From (6.27) as $e \to 1$ $\gamma_0 \to 45^\circ$



c. Given v_0, what value of γ_0 will make ϕ_0 maximum if $\frac{\pi}{2} < \phi_0 < \pi$?



$0 < \gamma_0 < \frac{\pi}{2}$

Eq. (6.8) still holds

$$\tan \phi_0 = \frac{\sin 2\gamma_0}{\left(\frac{2}{v_0^2} - 1\right) - \cos 2\gamma_0} \qquad (6.8)$$

Set $\frac{d\phi_0}{d\gamma_0} = 0$, find

$$2\gamma_{opt} + \phi_0 = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

Since $\frac{\pi}{2} < \phi_0 < \pi$ and $0 < \gamma_{opt} < \frac{\pi}{2}$

$$\frac{\pi}{2} < 2\gamma_{opt} + \phi_0 < 2\pi$$

Therefore the only solution which can have physical meaning is

$$2\gamma_{opt} + \phi_0 = \frac{3\pi}{2}$$

$$\gamma_{opt} = \frac{3\pi}{4} - \frac{\phi_0}{2}$$

the maximum range is obtained from

$$\tan\phi_{max} = \frac{\sin 2\gamma_{opt}}{\left(\frac{2}{v_0^2} - 1\right) - \cos 2\gamma_{opt}}$$

Get

$$\sin\phi_{max} = -\frac{v_o^2}{2 - v_o^2}$$

Since $\frac{\pi}{2} < \phi_{max} < \pi \Rightarrow \sin\phi_{max} > 0 \Rightarrow v_o^2 > 2 \Rightarrow$ escape (contradiction)

Therefore $2\gamma_o + \phi_o = \frac{3\pi}{2}$ is <u>not</u> a solution.

To obtain range of possible solutions, return to (6.8)

$$\tan\phi_o = \frac{\sin 2\gamma_o}{\left(\frac{2}{v_o^2} - 1\right) - \cos 2\gamma_o} = \frac{v_o^2 \sin\gamma_o \cos\gamma_o}{1 - v_o^2 \cos^2\gamma_o}$$

$$\tan\phi_o - v_o^2\cos^2\gamma_o \tan\phi_o = v_o^2 \sin\gamma_o \cos\gamma_o$$

$$v_o^2 = \frac{\tan\phi_o}{\sin\gamma_o\cos\gamma_o + \cos^2\gamma_o \tan\phi_o} \qquad (6.28)$$

Eq. (6.28) gives initial velocity required for given range ϕ_o and launch angle γ_o.

Require $0 < v_0^2 < 2$

For $v_0^2 > 0$

Since $\frac{\pi}{2} < \phi_0 < \pi$, $\tan\phi_0 < 0$

For $v_0^2 > 0$, (6.28) requires

$$\underbrace{\sin\gamma_0 \cos\gamma_0}_{\text{always} > 0} + \underbrace{\cos^2\gamma_0 \tan\phi_0}_{\text{always} < 0} < 0$$

Need

$$\sin\gamma_0 \cos\gamma_0 < \cos^2\gamma_0 \, |\tan\phi_0|$$

$$\tan\gamma_0 < |\tan\phi_0|$$

$$\tan\gamma_0 < \tan(\pi - \phi_0)$$

$$\boxed{\gamma_0 < \pi - \phi_0} \qquad \text{for } v_0^2 > 0$$

For $v_0^2 < 2$

$$v_0^2 = \frac{\overbrace{\tan\phi_0}^{<0}}{\underbrace{\sin\gamma_0\cos\gamma_0 + \cos^2\gamma_0\tan\phi_0}_{\therefore <0}} < 2$$

$$\tan\phi_0 > 2\sin\gamma_0\cos\gamma_0 + 2\cos^2\gamma_0\tan\phi_0$$

$$\tan\phi_0 > \sin 2\gamma_0 + (1+\cos 2\gamma_0)\tan\phi_0$$

$$\sin 2\gamma_0 + \tan\phi_0\cos 2\gamma_0 < 0$$

Divide by $\sin 2\gamma_0$ (>0)

$$1 + \frac{\tan\phi_0}{\tan 2\gamma_0} < 0$$

$$\frac{\tan\phi_0}{\tan 2\gamma_0} < -1$$

Take reciprocal

$$\frac{\tan 2\gamma_0}{\underbrace{\tan\phi_0}_{<0}} > -1$$

$$\tan 2\gamma_0 < -\tan\phi_0$$

$$\tan 2\gamma_0 < \tan(\pi - \phi_0)$$

$$2\gamma_0 < \pi - \phi_0$$

$$\boxed{\gamma_0 < \frac{\pi - \phi_0}{2}} \qquad \text{for } 0 < v_0^2 < 2 \qquad (6.29)$$

EXAMPLE

Suppose $\phi_0 = 170°$

$$\gamma_0 < \frac{\pi - \phi_0}{2} = 5°$$

$$\mathcal{E} = \frac{v_0^2}{2} - \frac{\mu}{r_0}$$

γ_0	v_0^2 (eq. 6.28)	
0°	1.00	← Minimum energy
4°	1.66	Admissible values
5°	2.00	
6°	2.48