VI. Low Reynolds Number Hydrodynamics

Flows at low Reynolds numbers (also called Stokes Flow or Creeping Flow) has relevance to many physical events. Among them, blood flow in capillary vessels, the motion of very small dust particles or flows in small geometries called *microflows.*

The general differential equation governing low Reynolds number flows ($Re \to 0$) is

$$0 = g_i + \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad \text{(VI-1)}$$

Taking the divergence of (VI-1) results in an independent equation for the pressure distribution

$$\frac{\partial^2 p}{\partial x_i \partial x_i} = 0 \qquad \text{(VI-2)}$$

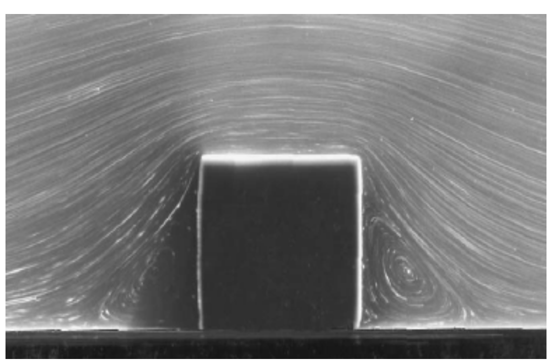
where the continuity equation eliminates the viscous terms.

In addition, the vorticity equation in low Reynolds number flow reduces into

$$\frac{\partial^2 \omega_i}{\partial x_j \partial x_j} = 0 \qquad \text{(VI-3)}$$

Equations (VI-2) and (VI-3) show that the pressure distribution and vorticity are governed entirely by viscous diffusion.

The low Reynolds number equations are linear which provides analytical solutions to many flow geometries. It also indicates that flow patterns are completely symmetrical and reversible as shown in the flow over a barrier.



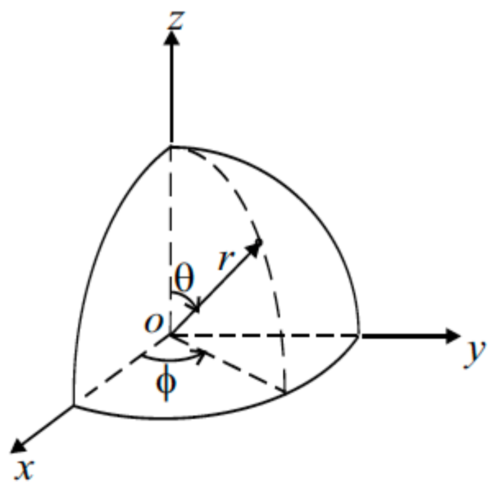
For two-dimensional flows, the vorticity can be represented by the stream function ψ as follows:

$$\omega_z = \nabla \times \vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)\hat{k}$$

So that equation (VI-3) becomes,

$$\nabla^2(\nabla^2\psi) = \nabla^4\psi = 0 \qquad \text{(VI-4)}$$

For axisymmetric flow in spherical coordinates (r, θ) the low Reynolds number equations can be derived for the stream function by first introducing the definition which satisfies the continuity equation:



$$u = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} \qquad v = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$$

Taking the curl of the governing low Reynolds number equation (V-13) we get

$$\nabla \times \nabla^2\vec{V} = \nabla \times \left[-\nabla \times \left(\nabla \times \vec{V}\right) + \nabla\left(\nabla \cdot \vec{V}\right)\right] = \nabla \times (\nabla \times \vec{\omega}) = 0$$

Using the definition of the stream function, it can be shown that the vorticity is:

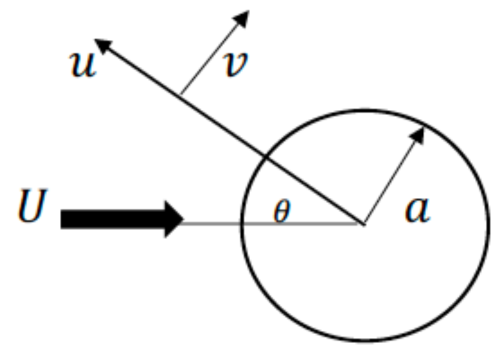
$$\vec{\omega} = \nabla \times \vec{V} = -\frac{\hat{e}_\phi}{r\ \sin\theta}\left[\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \theta}\right)\right]$$

Taking a curl again of the equation gives,

$$E^2E^2\psi = \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\psi = 0 \qquad \text{(VI-5)}$$

VI.1 <u>Flow Over a Sphere</u>

Uniform flow with velocity $U\hat{\imath}$ over a spherical particle of radius a in low Reynolds number is governed by Equ. (V-17) with the boundary conditions:



(i) at $r = a$, $\frac{\partial\psi}{\partial\theta} = \frac{\partial\psi}{\partial r} = 0$ (no-slip)

(ii) at $r \to \infty$, $u = U\cos\theta$ $v = U\sin\theta$

The stream function for the uniform flow far from the sphere is

$$\psi = -\frac{r^2}{2}U\sin^2\theta \qquad \text{as } r \to \infty$$

Equation (VI-5) can be solved by separation of variables. Assume that the stream function can be represented by

$$\psi = a^2U\sin^2\theta\, F(\rho)$$

where $\rho = \frac{r}{a}$.

Substituting into (VI-5) results in the following differential equation

$$F'''' - \frac{4}{\rho^2}F'' - \frac{8}{\rho^3}F' - \frac{8}{\rho^4}F = 0$$

This is an equidimensional ordinary differential equation which has a solution in the form ρ^n. Substituting this solution into the differential equation and solving for n generates the solution for ψ in terms of four constant coefficients which can be determined from the boundary conditions. Subsequently, the solution is

$$\psi = a^2 U \sin^2\theta \left[-\frac{1}{4\rho} + \frac{3}{4}\rho - \frac{1}{2}\rho^2\right]$$

The two velocity components can be obtained:

$$u = U\cos\theta\left(1 + \frac{1a^3}{2r^3} - \frac{3a}{2r}\right)$$

$$v = -U\sin\theta\left(1 - \frac{1a^3}{4r^3} - \frac{3a}{4r}\right)$$

The pressure can be obtained from integrating $\frac{\partial p}{\partial r}$

$$p = p_\infty - \frac{3}{2}\frac{\mu U a}{r^3}\cos\theta$$

The drag force acting on the sphere can be determined by integrating the z direction components (direction of flow) of the stress tensor over the surface area of the sphere. Namely,

$$F_z = (-p + \tau_{rr})\cos\theta - \tau_{r\theta}\sin\theta$$

Here

$$\tau_{rr} = -p_\infty + \frac{3}{2}\frac{\mu U}{a}\cos\theta$$

$$\tau_{r\theta} = -\frac{3}{2}\frac{\mu U}{a}\sin\theta$$

Integrating over the surface of the sphere:

$$F_D = \int_0^{2\pi} a\, d\phi \int_0^{\pi} d\theta\ \sin\theta\, F_z = 6\pi\mu U a$$

$$F_D = 6\pi\mu U a$$

This result is called *Stokes' Law*. It is accurate for $Re < 0.5$ and within 5% when $Re = 1$.

In general, it can be shown that the drag force on any axisymmetric body at low Reynolds number flow can be determined from the equation

$$F_D = 8\pi\mu \lim_{r\to\infty} \frac{\psi - \psi_\infty}{r\sin^2\theta}$$

VI.2 <u>Oseen Low Re Solution</u>

Oseen Approximation (1910). This is an extension of the previous solution whereby the inertial terms are included with a uniform fluid velocity as follows:

$$U\frac{\partial u_i}{\partial x_1} = g_i + \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

The solution provides an adjustment to Stokes' Law with the result for the drag force in the form,

$$F_D = 6\pi\mu U a\left(1 + \frac{3}{8}Re_a\right)$$

VI.3 <u>Force on an Oscillating Sphere</u>.

A sphere oscillating with a small amplitude and frequency ω, namely

$$U(t) = U_o \cos \omega t$$

The force on the sphere is given by the formula,

$$F_D = 6\pi\mu a \left(1 + \frac{a}{\delta}\right) U + 3\pi a^2 \rho\delta \left(1 + \frac{2a}{9\delta}\right) \frac{dU}{dt}$$

where $\delta = \sqrt{\frac{2\nu}{\omega}}$

When the frequency is large $\delta \ll a$, this result can be simplified to

$$F_D = \frac{2}{3}\pi\rho a^3 \frac{dU}{dt} + 3\pi\rho a^2 \sqrt{2\nu\omega} U$$

VI.4 <u>Force on a sphere moving at an any speed $U(t)$</u>

$$F_D = 2\pi\rho a^2 \left[\frac{1}{3}\frac{dU}{dt} + \frac{3\nu U}{a^2} + \frac{3}{a}\sqrt{\frac{\nu}{\pi}} \int_{-\infty}^{t} \frac{dU}{d\xi} \frac{d\xi}{\sqrt{t-\xi}} \right]$$

The integral covers the motion of the sphere from an earlier time, which shows that the force acting on the sphere is influence by the sphere's previous motion.

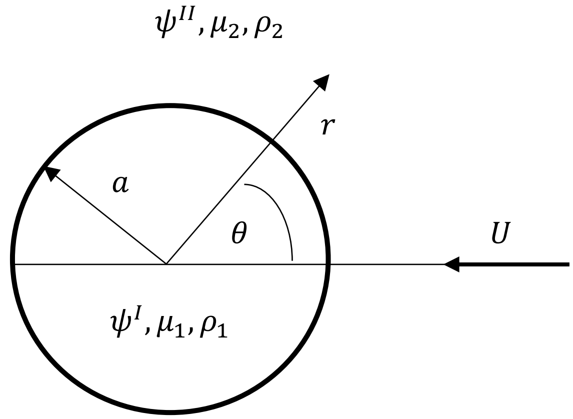
If the sphere is released from rest, the equation becomes,

$$F_D = 2\pi\rho a^2 \left[\frac{1}{3}\frac{dU}{dt} + \frac{3\nu U}{a^2} + \frac{3}{a}\sqrt{\frac{\nu}{\pi}} \int_{0}^{t} \frac{dU}{d\xi} \frac{d\xi}{\sqrt{t-\xi}} \right]$$

The solution for the sphere velocity in a fluid of lower density is presented in *Fluid Mechanics* by Yih (1979).

VI.4 Flow Past a Spherical Droplet

Consider low Reynolds number flow over a spherical droplet as shown in the figure.



The governing equations for each fluid are:

For the droplet

$$E^4\psi^I = 0$$

For exterior flow

$$E^4\psi^{II} = 0$$

The general solution for each region in the form

$$\psi = \left(C_1 r + C_2 r^2 + C_3 r^4 + \frac{C_4}{r}\right)\sin^2\theta$$

The pressure acting on the droplet from the equations:

$$-\frac{\partial p^{II}}{\partial r} = \mu_2 \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}E^2\psi^{II}$$

$$-\frac{1}{r}\frac{\partial p^{II}}{\partial\theta} = \mu_2 \frac{-1}{r\sin\theta}\frac{\partial}{\partial r}E^2\psi^{II}$$

The drag force acting on the droplet by integrating all the stresses over the surface of the droplet.

$$F_D = 2\pi a^2 \int_0^{\pi} [(-p + \tau_{rr})\cos\theta - \tau_{r\theta}\sin\theta]\sin\theta d\theta = 8\pi\mu_2 D_1$$

Boundary Conditions:

(i) $r \to \infty \quad \psi^{II} = \frac{r^2}{2}U\sin^2\theta$

(ii) $r = a \quad u^I = u^{II} = 0$ or $\frac{\partial\psi^I}{\partial\theta} = \frac{\partial\psi^{II}}{\partial\theta} = 0$

(iii) $r = a \quad \tau_{r\theta}^I = \tau_{r\theta}^{II}$ or $\mu_1\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial\theta}\right]^I = \mu_2\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial\theta}\right]^{II}$

(iv) $r = a \quad v^I = v^{II} = 0$ or $\frac{\partial\psi^I}{\partial r} = \frac{\partial\psi^{II}}{\partial r}$

(v) $r = 0$ velocity is finite.

Solutions of Governing Equations

$$\psi^I = \left(C_1 r + C_2 r^2 + C_3 r^4 + \frac{C_4}{r}\right)\sin^2\theta$$

$$\psi^{II} = \left(D_1 r + D_2 r^2 + D_3 r^4 + \frac{D_4}{r}\right)\sin^2\theta$$

Apply Boundary Conditions

(i) $D_3 = 0 \qquad D_2 = \frac{U}{2}$

(v) $C_4 = 0 \qquad C_1 = 0$

(ii) $C_2 a^2 + C_3 a^4 = D_1 a + D_2 a^2 + \frac{D_4}{a}$

(iv) $2C_2 a + 4C_3 a^3 = D_1 + 2D_2 a - \frac{D_4}{a^2}$

(iii) $\mu_1 a C_3 = \mu_2 \frac{D_4}{a^4}$

Solving for the constant coefficients:

$$D_1 = -\frac{Ua}{2}\left[1 + \frac{1}{2\left(1 + \frac{\mu_2}{\mu_1}\right)}\right] \qquad D_4 = \frac{U}{4}\frac{a^3}{1 + \frac{\mu_2}{\mu_1}}$$

$$C_2 = -\frac{U}{4}\frac{\frac{\mu_2}{\mu_1}}{1 + \frac{\mu_2}{\mu_1}} \qquad C_3 = \frac{U}{4a^2}\frac{\frac{\mu_2}{\mu_1}}{1 + \frac{\mu_2}{\mu_1}}$$

Solutions for stream functions are:

$$\psi^I = \frac{U}{4}\frac{\mu_2}{1 + \frac{\mu_2}{\mu_1}} r^2\left(\frac{r^2}{a^2} - 1\right)\sin^2\theta$$

$$\psi^{II} = \frac{Ua^2}{4}\left[\frac{\mu_2}{2(\mu_1 + \mu_2)}\frac{a}{r} - \frac{3\mu_1 + 2\mu_2}{2(\mu_1 + \mu_2)}\frac{r}{a} + \frac{r^2}{a^2}\right]\sin^2\theta$$

Determination of forces on the surface of the droplet:

<u>Pressure</u>

$$E^2\psi^{II} = \frac{Ua}{2r}\frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2}\sin^2\theta$$

$$-\frac{\partial p^{II}}{\partial r} = \mu_2\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}E^2\psi^{II} = \mu_2\frac{Ua}{2r^3}\frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2}2\cos\theta$$

$$-\frac{1}{r}\frac{\partial p^{II}}{\partial\theta} = \mu_2\frac{-1}{r\sin\theta}\frac{\partial}{\partial r}E^2\psi^{II} = \mu_2\frac{Ua}{2r^3}\frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2}\sin\theta$$

Integrating:

$$p^{II} = \mu_2\frac{Ua}{2r^2}\frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2}\cos\theta$$

Stresses at $r = a$ (outside fluid superscript II omitted)

$$p^{II} = \mu_2 \frac{U}{2a} \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \cos\theta$$

$$\tau_{r\theta} = \mu_2 \left[r \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \theta} \right]\Bigg|_{r=a} = 6\mu_2 \frac{D_4}{a^4} \sin\theta = 6\mu_2 \frac{U}{4a} \frac{\mu_1}{\mu_1 + \mu_2} \sin\theta$$

$$\tau_{rr} = 2\mu_2 \frac{\partial u}{\partial r}\bigg|_{r=a} = -2\mu_2 \frac{U}{a} \frac{\mu_1}{\mu_1 + \mu_2} \cos\theta$$

Drag Force:

$$F_D = 2\pi a^2 \int_0^\pi [(-p + \tau_{rr})\cos\theta - \tau_{r\theta}\sin\theta] \sin\theta d\theta = 8\pi\mu_2 D_1$$

$$F_D = -2\pi a \mu_2 U \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2}$$

For a solid sphere $\mu_2/\mu_1 \to 0$

$$F_D = -6\pi a \mu U$$

For a gas bubble $\mu_1/\mu_2 \to 0$

$$F_D = -4\pi a \mu U$$

Terminal Velocity

$$\frac{4}{3}\pi a^3 (\rho_1 - \rho_2) g = 2\pi a \mu_2 U \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2}$$

$$U = \frac{2}{3} \frac{a^2(\rho_1 - \rho_2) g}{\mu_2 \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2}} = \frac{3}{2} a^2 g \frac{\rho_1 - \rho_2}{\mu_2} \frac{\mu_1 + \mu_2}{3\mu_1 + 2\mu_2}$$

For a solid sphere

$$U = \frac{2}{9} a^2 g \frac{\rho_1 - \rho_2}{\mu}$$

For a gas bubble

$$U = \frac{1}{3} a^2 g \frac{\rho_1 - \rho_2}{\mu}$$