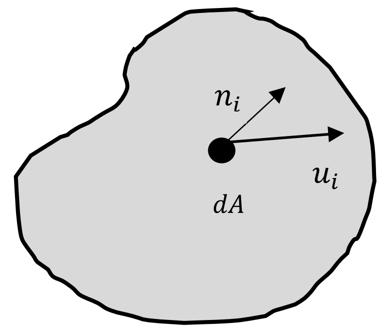
## II. Equations of Motion

The equations governing the motion of fluid flow are the mathematical representation of the conservation of mass and the conservation of momentum or Newton's 2nd law.

### II.1 <u>The Continuity Equation (Conservation of Mass)</u>

Consider a fixed control volume in an arbitrary flow field. Denote the velocity vector on the surface of the control volume as $\vec{V}$ and the unit normal pointing outward as $\hat{n}$. Then the conservation of mass statement can be written as

$$\frac{\partial}{\partial t}\int \rho \, d \vee + \int \rho n_i u_i dA = 0$$




in which $u_n$ is the normal velocity component to the surface are $(\vec{V} \cdot \hat{n})$.

Using the Divergence Theorem, the area integral can be replaced by a volume integral:

$$\int \rho n_i u_i dA = \int \frac{\partial(\rho u_i)}{\partial x_i} d \vee$$

Therefore, for a fixed control volume

$$\int \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i}\right] d \vee = 0$$

which is satisfied only when

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0 \qquad \text{(III-1)}$$

This is the *Continuity Equation*, which can be written also as the material derivative,

$$\frac{D\rho}{Dt} = 0 \tag{II-2}$$

and for incompressible fluid with $\rho = constant$,

$$\nabla \cdot \vec{V} = 0, \quad \text{or} \quad \frac{\partial u_i}{\partial x_i} = 0 \;\; (u_{i,i} = 0) \tag{II-3}$$

Using the divergent from Chapter I, Equ. (I-17) can be written in Cartesian, cylindrical and spherical coordinates as follows:

Cartesian

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{II-4a}$$

Cylindrical

$$\frac{1}{r}\frac{\partial ru}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{II-4b}$$

Spherical

$$\frac{1}{r}\frac{\partial}{\partial r}(r^2 u) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v \sin\theta) + \frac{1}{\sin\theta}\frac{\partial w}{\partial \varphi} = 0 \tag{II-4c}$$

## II.2 <u>Rate of Deformation and Vorticity</u>

The rate of displacement of a fluid particle relative to another particle depends on the difference in velocity components between the two particles $du_j$, which is given by

$$du_j = \frac{\partial u_j}{\partial x_i} dx_i$$

where $dx_i$ is the difference between the coordinates of the two fluid particles. In terms of Cartesian coordinates there are nine quantities for the term $\frac{\partial u_j}{\partial x_i}$ which can be represented by two components

$$\frac{\partial u_j}{\partial x_i} = \frac{1}{2}\left(\varepsilon_{ij} - \omega_{ij}\right)$$

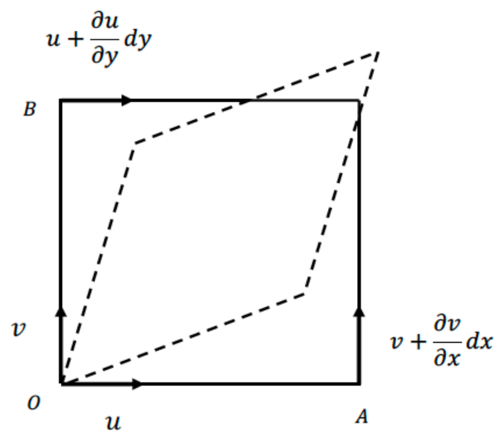
where

$$\varepsilon_{ij} = \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \tag{II-5}$$

and

$$\omega_{ij} = \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \qquad \text{(II-6)}$$

The physical meaning of $\varepsilon_{ij}$ and $\omega_{ij}$ are shown in the figures. Consider the deformation of a two-dimensional element $dx \times dy$,




The rotation of $\overline{OA}$ and $\overline{OB}$ can be represented by (CCW is positive direction)

$$\omega_{\overline{OA}} = \frac{\left(v + \frac{\partial v}{\partial x} dx\right) - v}{dx} = \frac{\partial v}{\partial x} \qquad \omega_{\overline{OB}} = -\frac{\left(u + \frac{\partial u}{\partial y} dy\right) - u}{du} = -\frac{\partial u}{\partial y}$$

Therefore,

$$\varepsilon_{12} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \qquad \omega_{12} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

Therefore, in general $\varepsilon_{ij}$ represents the deformation of the right angle at point $O$. The term $\omega_{ij}$ represents the rotation of each side of the element so that the average rotation can be defined as

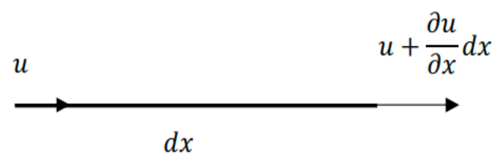
$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

Note that the angular velocity components do not include terms with repeated indices.

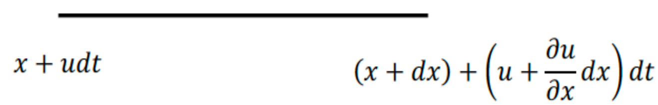
For the rate of deformation, a repeated index, say $\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}$ represents the change in length of a fluid element in the $x_1$direction.

$$\varepsilon_{xx} = \frac{Final\ Length - dx}{dx\ dt} = \frac{[(x+dx) + \left(u + \frac{\partial u}{\partial x}dx\right)dt - (x + udt)] - dx}{dxdt} = \frac{\partial u}{\partial x}$$

at time $t$




at $t + dt$ the segment changed its location and length




Therefore, $\varepsilon_{ij}$ as given by (II-5) is the *rate of deformation* of *rate of strain* tensor and $\omega_{ij}$ is the angular velocity.

Also note that $\varepsilon_{ij}$is symmetric $\varepsilon_{ij} = \varepsilon_{ji}$ and $\omega_{ij}$ is antisymmetric $\omega_{ij} = -\omega_{ji}$.

In Cartesian coordinates:

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} & 2\frac{\partial w}{\partial z} \end{pmatrix} \tag{II-7}$$

The factor 2 in the diagonal terms is not essential, but it is convenient for the purpose of interpretation as will be shown later.

The angular velocity (II-6) can be written in the form

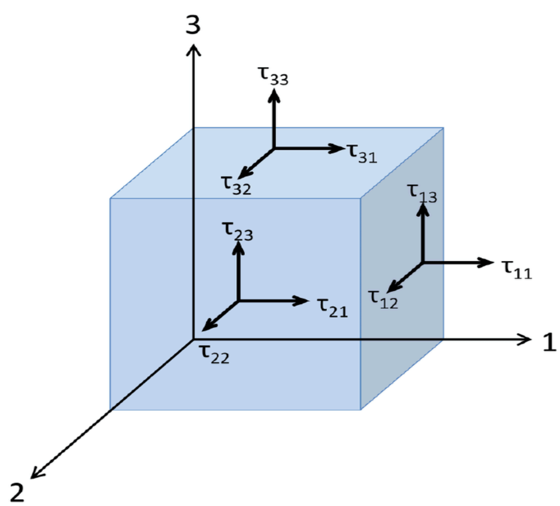
$$\omega_{ij} = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} & 0 & \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} & 0 \end{pmatrix}$$

and vorticity vector is defined by

$$\vec{\Omega} = 2\vec{\omega} = \nabla \times \vec{V} \qquad \text{(II-8)}$$

## II.3 <u>The Stress Tensor</u>

The *stress tensor* $\tau_{ij}$ acting on a fluid element in Cartesian coordinates is shown in the figure




$\tau_{ij}$ sign convention – the index $i$ indicates the direction normal to the plane on which the stress component acts, and $j$ - the direction of action. $\tau_{ij}$ is positive if both the direction of action and the normal to the surface are in the positive coordinate direction, or when both are in the negative coordinate direction. Otherwise, the stress component is negative.

When the fluid is at rest, the forces present are only hydrostatic due to body forces (gravity). Therefore, the normal stress components include the pressure term in addition to the normal stress components indicated by $\tau'_{ii}$ .

$$\tau_{ii} = -p + \tau'_{ii}$$

Since the deformation, rotation, and translation of a fluid element is driven by the stresses in the fluid, the stress tensor has to be related to the rate of strain tensor. Note that both tensor are symmetric. These relations are assumed to be linear and have to remain unchanged by the rotation of coordinate system or by redefinition of axes. In other words, the relations cannot have a preferred direction (isotropy).

Isotropy can be attained in the normal stress components if they depend on the components of the rate of strain in the same direction. This representation for each normal stress component can be written in the form:

$$\tau_{11} = -p + \lambda \frac{\partial u_i}{\partial x_i} + 2\mu \frac{\partial u_1}{\partial x_1}$$

Here the last term represents a linear dilatation and the second term represent volumetric change.

The tangential stress components are written in the form:

$$\tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

## II.4 <u>Newtonian Fluid</u>

Newtonian Fluid is defined as one in which the relations between the stress tensor and the rate of deformation is linear.

These relation in their general form has been presented by Stokes (1845)

$$\tau_{ij} = (-p + \lambda \varepsilon_{\alpha\alpha})\delta_{ij} + \mu \varepsilon_{ij}$$

For quiescent fluid (no velocity) the stress tensor becomes the isotropic pressure $-p\delta_{ij}$. When there is motion, the normal stress tensor component is proportional to the rate of volume expansion with the proportionality constant $\lambda$, while the tangential stress components are proportional to the rate of deformation.

Stokes further introduced the assumption (*Stokes Hypothesis*) $\lambda = \frac{2}{3}\mu$ resulting in the final relations

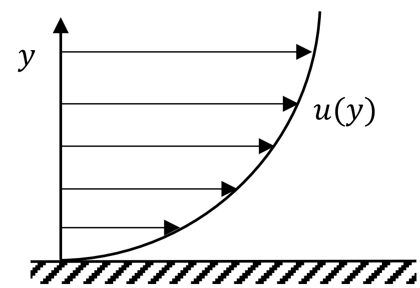
$$\tau_{ij} = \left(-p - \frac{2}{3}\mu\varepsilon_{\alpha\alpha}\right)\delta_{ij} + \mu\varepsilon_{ij} \qquad \text{(II-9)}$$

where $\mu$ is the fluid viscosity.
For compressible fluid, the *bulk viscosity* is defined as:

$$\mu' = \lambda + \frac{2}{3}\mu$$

The meaning of fluid viscosity can be illustrated in simple unidirectional motion where the tangential stress is proportional to the rate of deformation

$$\tau_{xy} = \mu\frac{du}{dy}$$




where the rate of strain represents the slop of the velocity profile

The table shows some values for the *viscosity* and the *kinematic viscosity*, $\nu = \mu/\rho$.

| | Viscosity $\mu\ [{kg}/{m-s}]$ | Kinematic Viscosity $\nu = \mu/\rho\ [m^2/s]$ |
|---|---|---|
| Air | $2\text{x}10^{-5}$ | $10^{-5}$ |
| Water | $10^{-3}$ | $2\text{x}10^{-6}$ |
| Oil | 0.3 | $3\text{x}10^{-3}$ |

## II.5 Navier-Stokes Equation

With the introduction of the stress tensor which accounts for all surface forces on a fluid element, we can now include them in Newton's 2nd Law together with a body force which acts on the entire fluid element (gravity).

If we apply Newtons' 2nd Law to a finite fluid mass moving in the $x_i$ direction, then

$$\int \rho \frac{Du_i}{Dt} d\mathsf{V} = \int \tau_{ni} dA + \int \rho g_i d\mathsf{V}$$

where $A$ is the surface area of volume $\mathsf{V}$, and $\tau_{ni}$ is the force acting on the surface area $dA$ in the $x_i$ direction, $n$ indicating the outward normal direction to the surface.

Using Green's theorem, the surface integral can be replaced with a volume integral,

$$\int \tau_{ni} dA = \int l_\alpha \tau_{\alpha i} dA = \int \frac{\partial \tau_{\alpha i}}{\partial x_\alpha} d\mathsf{V}$$

in which $l_\alpha$ are the directional cosines of the outward normal $n$.

Hence the equation becomes.

$$\int \rho \frac{Du_i}{Dt} d\mathsf{V} = \int \frac{\partial \tau_{\alpha i}}{\partial x_\alpha} d\mathsf{V} + \int \rho g_i d\mathsf{V}$$

and because the volume is arbitrary, the following must be true

$$\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial \tau_{\alpha i}}{\partial x_\alpha} \qquad \text{(II-10)}$$

Introducing the stress-strain relations (II-9) for incompressible fluids results in the *Navier Stokes Equations*

$$\rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \rho g_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_j} \right) \qquad \text{(II-11)}$$

or in vector form

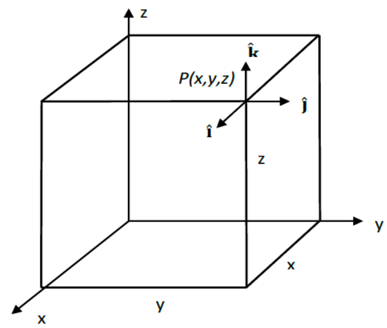
$$\rho\left(\frac{\partial\vec{V}}{\partial t}+\vec{V}\cdot\nabla\vec{V}\right)=\rho\vec{g}-\nabla p+\mu\nabla^2\vec{V} \qquad \text{(II-12)}$$

The equation and stress tensor are presented in the next two slides for three coordinate systems.

# Cartesian Coordinates

**Velocity Vector**

$\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$




**Continuity Equation**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} = 0$$

**Stress Tensor**

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\nabla\cdot\vec{V} \qquad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\nabla\cdot\vec{V} \qquad \tau_{zz} = 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\nabla\cdot\vec{V}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad \tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

**Navier Stokes Equations**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right)$$

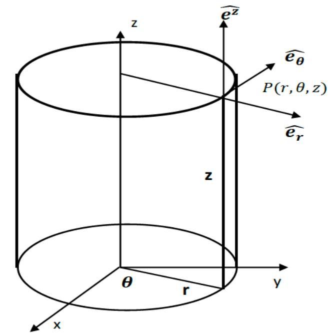
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2}\right)$$

# Cylindrical Coordinates

### Velocity Vector

$$\vec{V} = u\hat{e}_r + v\hat{e}_\theta + w\hat{e}_z$$




### Continuity Equation

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

### Cylindrical Coordinates

$$\tau_{rr} = 2\mu\frac{\partial u}{\partial r} - \frac{2}{3}\mu\nabla\cdot\vec{V} \qquad \tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right) - \frac{2}{3}\mu\nabla\cdot\vec{V} \qquad \tau_{zz} = 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\nabla\cdot\vec{V}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \theta}\right] \qquad \tau_{\theta z} = \tau_{z\theta} = \mu\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right) \qquad \tau_{zr} = \tau_{rz} = \mu\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right)$$

### Navier-Stokes Equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = g_r - \frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)\right] + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right\}$$

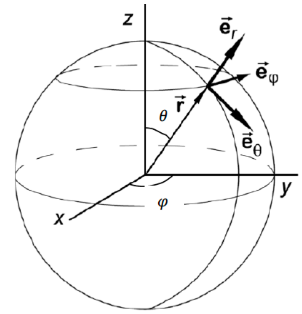
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = g_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv)\right] + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right\}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

# Spherical Coordinates

## Velocity Vector

$$\vec{V} = u\hat{e}_r + v\hat{e}_\theta + w\hat{e}_\phi$$




## Continuity Equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial\phi} = 0$$

## Stress Tensor

$$\tau_{rr} = 2\mu\frac{\partial u}{\partial r} - \frac{2}{3}\mu\nabla\cdot\vec{V} \qquad \tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r}\right) - \frac{2}{3}\mu\nabla\cdot\vec{V} \qquad \tau_{\phi\phi} = 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial w}{\partial\phi} + \frac{u}{r} + \frac{v\cot\theta}{r}\right) - \frac{2}{3}\mu\nabla\cdot\vec{V}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r} + \frac{\partial u}{\partial\theta}\right] \qquad \tau_{\theta\phi} = \tau_{\phi\theta} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{w}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v}{\partial\phi}\right] \qquad \tau_{\phi r} = \tau_{r\phi} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)\right]$$

## Navier Stokes Equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial u}{\partial\phi} - \frac{v^2 + w^2}{r} = g_r - \frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$+\nu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta) - \frac{2}{r^2\sin\theta}\frac{\partial w}{\partial\phi}\right\}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial v}{\partial\phi} + \frac{uv}{r} - \frac{w^2\cot\theta}{r} = g_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial\theta}$$
$$+\nu\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta)\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial u}{\partial\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial w}{\partial\phi}\right\}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial w}{\partial\phi} + \frac{uw}{r} + \frac{vw\cot\theta}{r} = g_\phi - \frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi}$$
$$+\nu\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(w\sin\theta)\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 w}{\partial\phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial u}{\partial\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v}{\partial\phi}\right\}$$

## II.6 Boundary Conditions

The flow conditions at the solid boundary (fluid-solid interface) or the surface between two fluids (fluid-fluid interface) are the *boundary conditions* which can be specified in terms of the tangential and normal velocity components or the stress tensor.

At the fluid-solid interface, the normal velocity component of the fluid must be zero.

$$n_i u_i|_{wall} = 0$$

Here, $n_i$ is unit normal to the surface. And if the solid wall is moving at velocity $V_i$,

$$n_i(V_i - u_i)|_{wall} = 0$$

The tangential velocity relative to the solid boundary must vanish which is the *no-slip condition* or in general,

$$u_i|_{wall} = V_i$$

which represents both the normal condition as well as the no-slip condition.

At the fluid-fluid interface (assuming the fluid are immiscible), the velocity components have to be continuous, and the stress tensor has to be balanced,

$$u_i^I = u_i^{II}$$

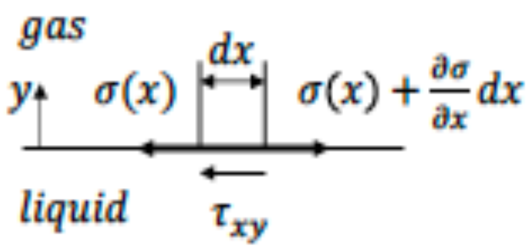
$$\tau_{ij}^I - \tau_{ij}^{II} = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\delta_{ij}$$

where the surface tension $\sigma$ is taken into account if the surface is not flat.

The Marangoni Effect (1965)

When the surface tension varies along the interface, the higher tension region induces flow along the interface from the lower tension region. Surface tension gradient can be caused by a temperature gradient or concentration gradient of surface-active molecules (surfactant).

For a liquid-gas interface, the tangential shear stress is balanced by the surface tension gradient as shown below




Then at the interface

$$\tau_{xy}dx = \left[\sigma(x) + \frac{\partial \sigma}{\partial x}dx\right] - \sigma(x)$$

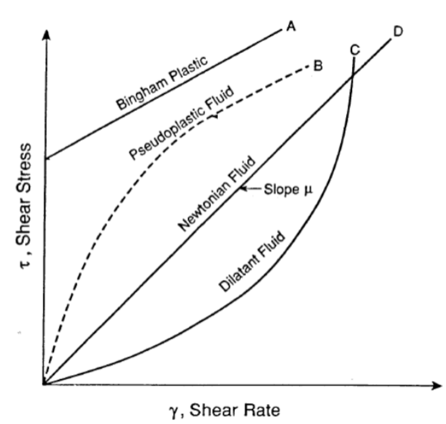
or

$$\mu\frac{\partial u}{\partial y} = \frac{\partial \sigma}{\partial x}$$

**II.7 <u>Non-Newtonian Fluids</u>**

Many industrial fluids do not obey Newton's viscosity relations. These are primarily polymers which have long-chained molecules (*macromolecules*). Mixtures or slurries and suspension also do not satisfy the Newtonian model. These are fluids like toothpaste, blood, crude oil, and others.

Unlike Newtonian fluids where the stress-strain relations are linear (D), Non-Newtonian fluids behave differently in several ways. A fluid is called *shear-thinning* when the viscosity decreases with increasing strain (B), which is typical of polymers. The viscosity of plastics on the other hand increases with increasing strain (C).



A unique case is the *Bingham Plastic* (A), which behaves like a solid material until a certain level of stress is reached, $\tau_o$, and then it behaves like a Newtonian fluid examples: whipped cream, margarine, tomato sauce).

Above this level the shear stress becomes Newtonian,

$$\tau = \tau_o + \mu \frac{\partial u}{\partial y}$$

Fluids that exhibit normal stress while shearing (usually used in extrusion) are called *viscoelastic,* and they tend to rebound if stress is applied abruptly.

The original model used to describe viscoelastic behavior was introduced by James Maxwell (183-1879). Starting with Hooke's law where the stress is proportional to the strain,

$$\tau = E\epsilon$$

Differentiating with respect to time

$$\frac{1}{E}\frac{\partial \tau}{\partial t} = \frac{\partial \epsilon}{\partial t}$$

This equation has the same dimensions for the rate of strain as ${\partial u}/{\partial y}$. Maxwell then assumed that the sum of the rate of deformation and the Newtonian stress can be added to represent viscoelastic behavior.

$$\tau + \frac{\mu}{E}\frac{\partial \tau}{\partial t} = \mu \frac{\partial u}{\partial y}$$

## II.8 The Vorticity Equation

The vorticity has been defined as $\vec{\omega} = \nabla \times \vec{V}$. It represents the rotation of the fluid at a point. Near a solid boundary where the fluid particles are not translating laterally, but they are rotating, and the rotation is propagated into the fluid away from the wall by the action of viscosity.

The equation that governs the propagation of vorticity is derived from Navier-Stokes equations

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

A more concise notation is commonly used where the derivatives are separated by a comma.

$$(u_{i,t} + u_j u_{i,j} = -\frac{1}{\rho} p_{,i} + \nu u_{i,jj})$$

Using the identity

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{1}{2} u_j u_j\right) + \varepsilon_{ijk}\omega_j u_k$$

(In vector form it is $\vec{V} \cdot \nabla \vec{V} = \frac{1}{2} \nabla \left|\vec{V}\right|^2 - \vec{V} \times \vec{\omega}$)

Therefore

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} u_j u_j\right) + \varepsilon_{ijk}\omega_j u_k = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

Taking the curl of the equation with the vorticity defined by

$$\nabla \times \vec{V} = \omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

Here $\epsilon_{pqi} \frac{\partial}{\partial x_p}$ results in the equation

$$\epsilon_{pqi}\frac{\partial^2 u_i}{\partial x_q \partial t} + \epsilon_{pqi}\frac{\partial^2}{\partial x_q \partial x_i}\left(\frac{1}{2}u_j u_j\right) + \epsilon_{pqi}\frac{\partial}{\partial x_q}\left(\varepsilon_{ijk}\omega_j u_k\right)$$
$$= -\frac{1}{\rho}\epsilon_{pqi}\frac{\partial^2 p}{\partial x_q \partial x_i} + \nu\epsilon_{pqi}\frac{\partial^3 u_i}{\partial x_j \partial x_j \partial x_q}$$

The first term represents the time derivative of the vorticity, the second term is zero because the antisymmetric alternative unit tensor is multiplied by a symmetric tensor. In vector form it represents the vector if identity $\nabla \times \nabla f = 0$.

Similarly, the pressure term is zero for the same reason. The viscous term represents the Laplacian of the vorticity.

Consider the remaining term:

$$\epsilon_{pqi}\frac{\partial}{\partial x_q}\left(\varepsilon_{ijk}\omega_j u_k\right) = \frac{\partial}{\partial x_k}\left(\omega_p u_k\right) - \frac{\partial}{\partial x_j}\left(\omega_j u_p\right) = u_k\frac{\partial \omega_p}{\partial x_k} - \omega_j\frac{\partial u_p}{\partial x_j}$$

in which the terms $\frac{\partial u_k}{\partial x_k}$ is zero due to the continuity equation, and $\frac{\partial \omega_j}{\partial x_j}$ are zero as a vector identity ($\nabla \cdot \nabla \times \vec{V} = 0$). This term in vector form is as follows

$$\nabla \times \left(\vec{V} \times \vec{\omega}\right) = \vec{\omega} \cdot \nabla\vec{V} + \vec{V}\cancel{(\nabla \cdot \vec{\omega})} - \vec{V} \cdot \nabla\omega - \vec{\omega}\cancel{\left(\nabla \cdot \vec{V}\right)}$$

Therefore, the resulting *Vorticity Equation* is

$$\frac{\partial \omega_i}{\partial t} + u_j\frac{\partial \omega_i}{\partial x_j} = \omega_j\frac{\partial u_i}{\partial x_j} + \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j} \qquad \text{(II-13)}$$

and in vector form

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \nabla\vec{V} + \nu\nabla^2\vec{\omega} \qquad \text{(II-14)}$$

The terms in the equation represent the material derivative (rate of change) of particle vorticity, the first term on the r.h.s. represents the rate of change of vortex lines and the last term is the rate of viscous diffusion of vorticity. The equation does not include pressure or body forces because they cannot produce rotation.

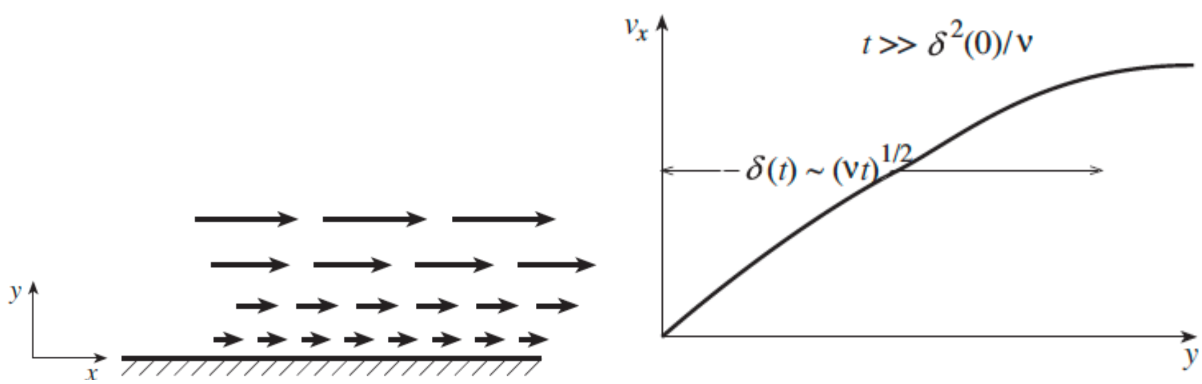
Diffusion of Vorticity

Consider an infinite plate moving horizontally from rest. The velocity in the horizontal direction depends only on the distance from the plate. In this two-dimensional flow the vorticity equation is reduced to:

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega$$

The viscous term in the equation represents diffusion of vortex lines through the fluid. The distance, $\delta,$ the vorticity spreads from the moving wall under the action of viscous stress can be estimated from the equation by

$$\frac{\omega}{t} \sim \nu \frac{\omega}{\delta^2} \qquad \delta \sim \sqrt{\nu t}$$




Therefore, when viscosity is significant, the vorticity diffuses from its source to occupy an area $\sim \nu t$ in the moving fluid after time $t$.