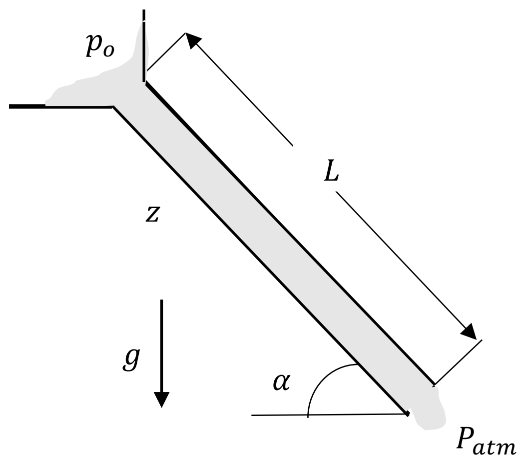
ME 55600/I0200
HW #3: Pipe Flow

1. Consider a pipe of radius a and length L inclined by an angle α, as shown in the figure. The inlet pressure to the pipe p_i and the outlet pressure is atmospheric. Determine the inlet pressure for which the flow is arrested.



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In the pipe, the governing equations is the Navier-Stokes equation in cylindrical coordinates which describes Poiseuille flow due to pressure gradient and gravitational force. Therefore,

$$\nu \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = \frac{1}{\rho}\frac{\partial p}{\partial z} - g\sin\alpha$$

where

$$\frac{\partial p}{\partial z} = \frac{p_{atm} - p_i}{L} = \frac{\Delta p_i}{L}$$

Integrating twice

$$w(r) = \left(\frac{\Delta p_o}{\rho L} - g\sin\alpha\right)\frac{r^2}{4\nu} + C_1 \ln r + C_2$$

Boundary conditions

(i) $w(a) = 0$ $\qquad C_2 = -\left(\frac{\Delta p_o}{\rho L} - g\sin\alpha\right)\frac{a^2}{4\nu}$

(ii) $w(0)$ is finite $\qquad C_1 = 0$

Velocity profile

$$w = \frac{\rho g L \sin\alpha - \Delta p_i}{4\mu L}(a^2 - r^2)$$

The volumetric flow rate is

$$Q = \int_0^a w 2\pi r\, dr = \pi \frac{\rho g L \sin\alpha - \Delta p_i}{8\mu L} a^4$$

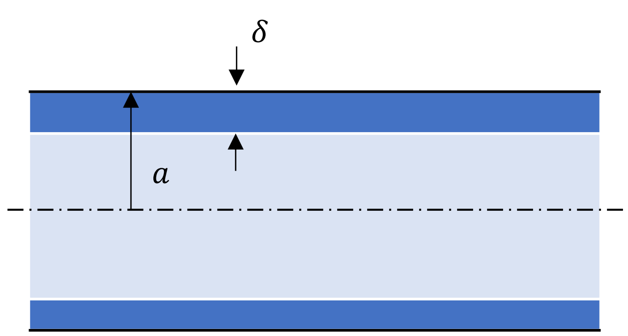
Thus, the flow can be zero when,

$$p_i = p_{atm} - \rho g L \sin\alpha$$

2. <u>Coaxial Poiseuille Flow</u>. In arterial blood flow a plasma layer of viscosity μ_p flows adjacent to the arterial wall, while the axial core has viscosity μ. Assuming the pressure gradient is given by

$$\beta = -\frac{\Delta p}{L}$$

Where Δp is the pressure difference of a section of length L, determine the velocity profile in each region and the combined flow rate.



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The governing equations for each region are:

$$\mu \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \beta \qquad\qquad 0 \le r \le a - \delta$$

$$\mu_p \frac{1}{r}\frac{d}{dr}\left(r\frac{du_p}{dr}\right) = \beta \qquad\qquad a - \delta \le r \le a$$

Boundary Conditions:

(i) $u(0)$ is finite
(ii) $u(a-\delta) = u_p(a-\delta)$
(iii) $\mu_p \frac{du_p}{dr} = \mu \frac{du}{dr}$ at $r = a-\delta$
(iv) $u_p(a) = 0$

Solution of differential equations

$$u = \frac{\beta}{4\mu} r^2 + C_1 \ln r + C_2$$
$$u_p = \frac{\beta}{4\mu_p} r^2 + D_1 \ln r + D_2$$

Apply boundary conditions.

(i) $C_1 = 0$
(ii) $\beta \frac{(a-\delta)^2}{4\mu} + C_2 = \beta \frac{(a-\delta)^2}{4\mu_p} + D_1 \ln(a-\delta) + D_2$
(iii) $\mu\beta \frac{a-\delta}{2\mu} = \mu_p \beta \frac{a-\delta}{2\mu_p} + D_1 \frac{\mu_p}{a-\delta}$ $\qquad D_1 = 0$
(iv) $D_2 = -\beta \frac{a^2}{4\mu_p}$

$$C_2 = \frac{\beta}{4}\left[(a-\delta)^2\left(\frac{1}{\mu_p} - \frac{1}{\mu}\right) - \frac{a^2}{\mu_p}\right]$$

Velocity profile are (with $k = \mu/\mu_p$):

$$u = \frac{\beta}{4\mu}[r^2 + (a-\delta)^2(k-1) - ka^2]$$

$$u_p = \frac{\beta}{4\mu_p}(r^2 - a^2)$$

The flowrate is:

$$Q = \int_0^{a-\delta} u 2\pi r dr + \int_{a-\delta}^{a} u_p 2\pi r dr$$

$$Q = 2\pi\frac{\beta}{4\mu}\left[\frac{a^4}{4} + \frac{a^2}{2}(a-\delta)^2(k-1) - k\frac{a^4}{2}\right] + 2\pi\frac{\beta}{4\mu_p}\left[\frac{a^4}{4} - \frac{a^4}{2} - \frac{(a-\delta)^4}{4} + \frac{(a-\delta)^4}{2}\right]$$

$$Q = \frac{\pi\beta a^4}{8\mu}\left[1 - 3k + 2(k-1)\left(1-\frac{\delta}{a}\right)^2 + k\left(1-\frac{\delta}{a}\right)^4\right]$$

If $\delta = 0$

$$Q = -\frac{\pi\Delta p a^4}{8\mu L}$$