1) a) For a given space triangle, determine expressions for the terminal velocity vectors $\bar{V}_{1m}$ and $\bar{V}_{2m}$ on the minimum energy orbit between P_1 and P_2 in terms of the unit vectors $\hat{u}_c$, $\hat{u}_1$ and $\hat{u}_2$.

b) Interpret the directions of these velocity vectors geometrically in terms of the unit vector directions

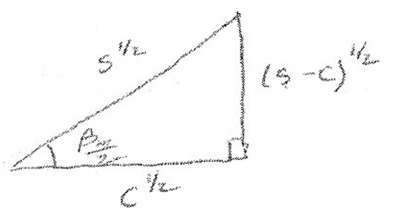
a) For the minimum energy trajectory

$$a = a_m = \frac{s}{2} \qquad \text{where} \qquad s = \frac{r_1 + r_2 + c}{2}$$

Eqs. (8.16) and (8.17) in the notes reduce to

$$\sin\left(\frac{\alpha_m}{2}\right) = \left(\frac{s}{2a_m}\right)^{1/2} = \left(\frac{s}{2\left(\frac{s}{2}\right)}\right)^{1/2} = 1 \Rightarrow \alpha_m = \pi$$

$$\sin\left(\frac{\beta_m}{2}\right) = \left(\frac{s-c}{2a_m}\right)^{1/2} = \left(\frac{s-c}{2\left(\frac{s}{2}\right)}\right)^{1/2} = \left(\frac{s-c}{s}\right)^{1/2}$$



$$\cot\left(\frac{\beta_m}{2}\right) = \left(\frac{c}{s-c}\right)^{1/2}$$

Eqs. (8.23a, b) in the notes reduce to

$$A_m = \left(\frac{\mu}{4a_m}\right)^{1/2}\cot\left(\frac{\alpha_m}{2}\right) = \left(\frac{\mu}{4\left(\frac{s}{2}\right)}\right)^{1/2}\cos\left(\frac{\pi}{2}\right) = 0$$

$$B_m = \left(\frac{\mu}{4a_m}\right)^{1/2}\cot\left(\frac{\beta_m}{2}\right) = \left(\frac{\mu}{4\left(\frac{s}{2}\right)}\right)^{1/2}\left(\frac{c}{s-c}\right)^{1/2} = \left(\frac{\mu c}{2s(s-c)}\right)^{1/2}$$

Eqs. (8.21) and (8.22) in the notes reduce to

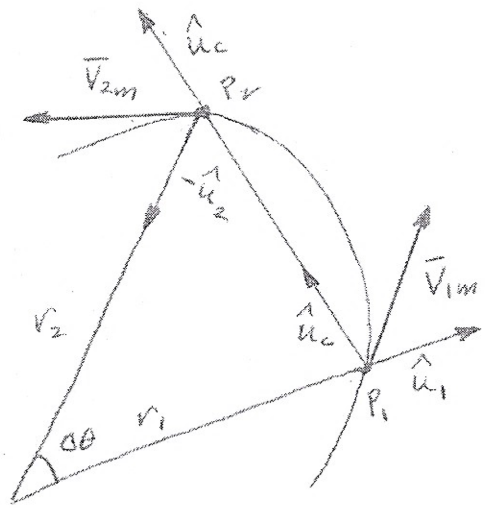
$$\bar{V}_{1m} = (B_m + A_m)\,\hat{u}_c + (B_m - A_m)\,\hat{u}_1 = B_m\left(\hat{u}_c + \hat{u}_1\right)$$

$$\bar{V}_{2m} = (B_m + A_m)\,\hat{u}_c - (B_m - A_m)\,\hat{u}_2 = B_m\left(\hat{u}_c - \hat{u}_2\right)$$

$$\boxed{\bar{V}_{1m} = \left(\frac{\mu c}{2s(s-c)}\right)^{1/2} (\hat{u}_c + \hat{u}_1)}$$

$$\boxed{\bar{V}_{2m} = \left(\frac{\mu c}{2s(s-c)}\right)^{1/2} (\hat{u}_c - \hat{u}_2)}$$

b)



Since the magnitude of the components of $\bar{V}_{1m}$ along $\hat{u}_1$ and $\hat{u}_c$ are equal, $\bar{V}_{1m}$ bisects the angle between $\hat{u}_1$ and $\hat{u}_c$

Since the magnitude of the components of $\bar{V}_{2m}$ along $\hat{u}_c$ and $-\hat{u}_2$ are equal, $\bar{V}_{2m}$ bisects the angle between $\hat{u}_c$ and $-\hat{u}_2$.

2) Consider the earth and Jupiter to be in coplanar circular orbits of radii 1 au and 5.2 au, respectively.

a) Considering the transfer angle $\Delta\theta$ as a variable, determine the range of values of a_m for all possible earth-Jupiter transfer ellipses.

b) For $\Delta\theta = 150°$ and $a = 5$ au, calculate the values of a_m (in au), t_m, t_F, $t_F^{\#}$, and t_p (in years).

c) Calculate $\bar{V}_1$ and $\bar{V}_1^{\#}$ (in EMOS) for the two transfer ellipses of (b).

d) Calculate the magnitudes of $\bar{V}_1$ and $\bar{V}_1^{\#}$.

e) Calculate p and $\tilde{p}$ (in au) along with e and $\tilde{e}$.

f) For the two ellipses, perform the graphical construction for α and β described in the text.

a) $r_1 = 1$ au $\quad r_2 = 5.2$ au

$$c = [r_1^2 + r_2^2 - 2r_1 r_2 \cos\Delta\theta]^{1/2} \qquad 0 \le \Delta\theta < 2\pi \qquad (1)$$

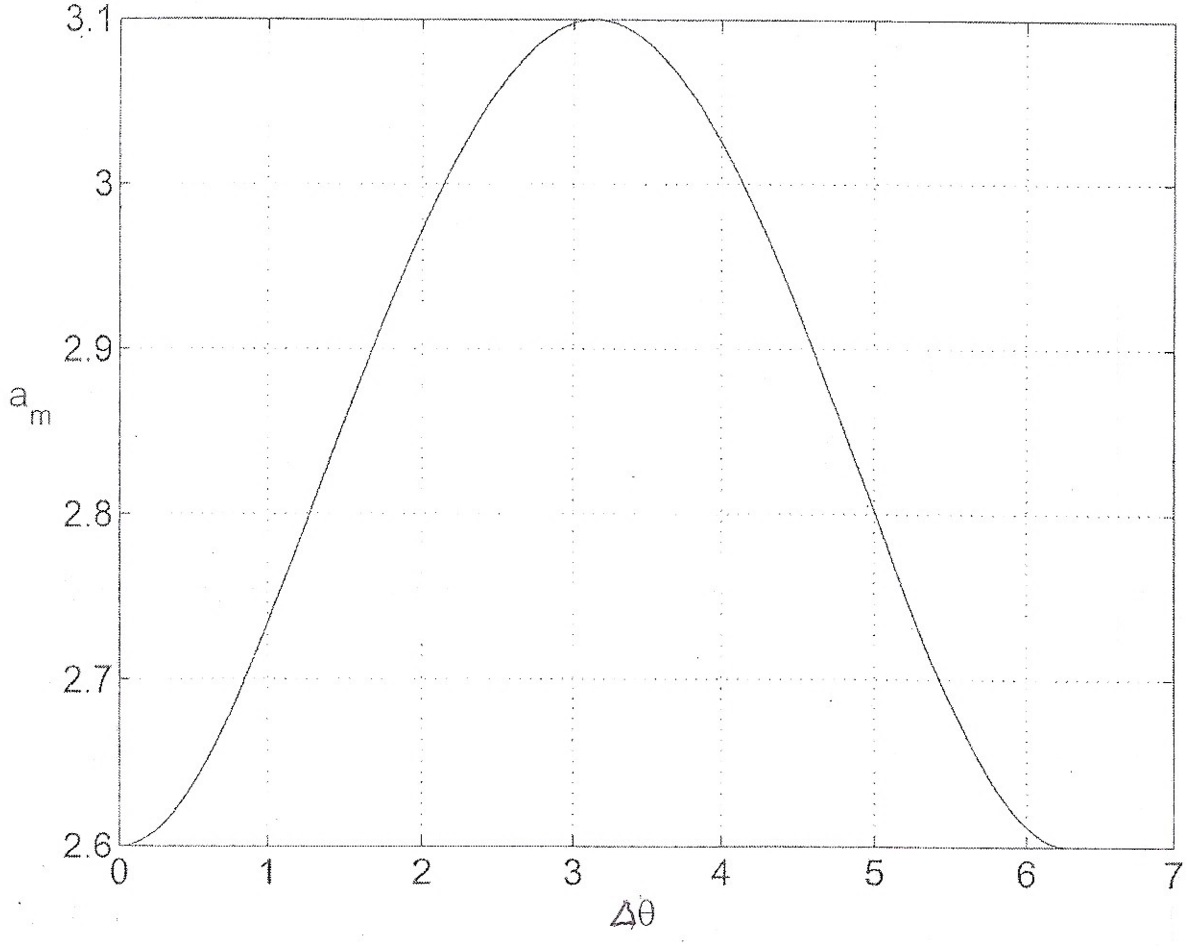
$$s = \frac{r_1 + r_2 + c}{2} \qquad (2)$$

$$a_m = \frac{s}{2} \qquad (3)$$

Sub. (1) into (2) and (2) into (3) gives

$$a_m = \frac{r_1 + r_2 + [r_1^2 + r_2^2 - 2r_1 r_2 \cos\Delta\theta]^{1/2}}{4} \qquad (4)$$

A plot of eq (4) for $r_1 = 1$ au, $r_2 = 5.2$ au is shown on the following page


3.1
3
2.9
2.8
2.7
2.6
a_m
0
1
2
3
4
5
6
7
$\Delta\theta$

$(a_m)_{min}$ occurs for $\theta = 0$ where

$$c = r_2 - r_1$$

$$s = r_2$$

$$(a_m)_{min} = \frac{r_2}{2} = \frac{5.2}{2} = 2.6 \text{ au}$$

$(a_m)_{max}$ occurs for $\Delta\theta = \pi$ where

$$c = r_1 + r_2$$

$$s = r_1 + r_2$$

$$(a_m)_{max} = \frac{r_1 + r_2}{2} = \frac{1 + 5.2}{2} = 3.1 \text{ au}$$

Therefore as the plot shows

$$\boxed{2.6 \text{ au} \leq a_m \leq 3.1 \text{ au}}$$

b) For $\Delta\theta = 150°$, eqs (1), (2) & (3) give

$$c = \left[1^2 + 5.2^2 - 2(1)(5.2)\cos 150°\right]^{1/2} = 6.0866 \text{ au}$$

$$s = \frac{1 + 5.2 + 6.0866}{2} = 6.1433 \text{ au}$$

$$a_m = \frac{6.1433}{2} = \underline{\underline{3.0716 \text{ au}}}$$

Using (8.24) in the notes:

$$\sin\left(\frac{\beta_{mo}}{2}\right) = \left(\frac{s-c}{s}\right)^{1/2} = \left(\frac{6.1433 - 6.0866}{6.1433}\right)^{1/2} = 0.096071$$

$$\frac{\beta_{mo}}{2} = 0.0962 \implies \beta_{mo} = 0.1924 \text{ radians} = 11.02°$$

Since $0 \le \Delta\theta \le \pi$ $\beta_m = \beta_{m_0} = 0.1924$ radians

Using (8.25) in the notes:

$$t_m = \left(\frac{s^3}{8\mu}\right)^{1/2}(\pi - \beta_m + \sin\beta_m)$$

$$= \left(\frac{(6.1433)^3}{8\,(4\pi^2)}\right)^{1/2}(\pi - 0.1924 + \sin(0.1924)) = \underline{\underline{2.6907 \text{ years}}}$$

Using (8.16) in the notes:

$$\sin\left(\frac{\alpha_0}{2}\right) = \left(\frac{s}{2a}\right)^{1/2} = \left(\frac{6.1433}{2(5)}\right)^{1/2} = 0.783792$$

$$\frac{\alpha_0}{2} = 0.900749 \Rightarrow \alpha_0 = 1.8015 \text{ radians} = 103.2^\circ$$

Using (8.17) in the notes:

$$\sin\left(\frac{\beta_0}{2}\right) = \left(\frac{s-c}{2a}\right)^{1/2} = \left(\frac{6.1433 - 6.0866}{2(5)}\right)^{1/2} = 0.075299$$

$$\frac{\beta_0}{2} = 0.07537 \Rightarrow \beta_0 = 0.1507 \text{ radians} = 8.63^\circ$$

Using (8.18) in the notes:

$$\alpha = \alpha_0 = 1.8015$$

Using (8.18) in the notes

$$\beta = \beta_0 = 0.1507$$

Using (8.15) in the notes

$$t_F = \left(\frac{a^3}{\mu}\right)^{1/2}(\alpha - \beta - (\sin\alpha - \sin\beta))$$

$$= \left(\frac{5^3}{4\pi^2}\right)^{1/2}(1.8015 - 0.1507 - (\sin 1.8015 - \sin 0.1507))$$

$$\boxed{t_F = 1.4723 \text{ years}}$$

Using (8.18) in the notes:

$$\alpha = 2\pi - \alpha_0 = 2\pi - 1.8015 = 4.48169 \text{ radians} = 256.78°$$

Using (8.15) in the notes:

$$t_F^{\#} = \left(\frac{a^3}{\mu}\right)^{1/2}\left(\alpha - \beta - (\sin\alpha - \sin\beta)\right)$$

$$= \left(\frac{5^3}{4\pi^2}\right)^{1/2}\left(4.48169 - 0.1507 - (\sin 4.48169 - \sin 0.1507)\right)$$

$$\boxed{t_F^{\#} = 9.7060 \text{ years}}$$

Using (8.26) in the notes:

$$t_p = \frac{1}{3}\left(\frac{2}{\mu}\right)^{1/2}\left(s^{3/2} - \text{sign}(\sin\theta)(s-c)^{3/2}\right)$$

$$= \frac{1}{3}\left(\frac{2}{4\pi^2}\right)^{1/2}\left(6.1433^{3/2} - \underbrace{\text{sign}(\sin 150°)}_{+1}(6.1433 - 6.0866)^{3/2}\right)$$

$$\boxed{t_p = 1.1414 \text{ years}}$$

c) For $t_2 - t_1 = t_F < t_m$

$\alpha = 1.8015 \quad \beta = 0.1507$

Using (8.23) in the notes:

$$A = \left(\frac{\mu}{4a}\right)^{1/2}\cot\left(\frac{\alpha}{2}\right) = \left(\frac{4\pi^2}{4(5)}\right)^{1/2}\cot\left(\frac{1.8015}{2}\right) = 1.1132 \text{ au/year}$$

$$B = \left(\frac{\mu}{4a}\right)^{1/2}\cot\left(\frac{\beta}{2}\right) = \left(\frac{4\pi^2}{4(5)}\right)^{1/2}\cot\left(\frac{0.1507}{2}\right) = 18.6051 \text{ au/year}$$

Using (8.21) in the notes:

$$\bar{V}_1 = (B+A)\hat{u}_c + (B-A)\hat{u}_1$$

$$= (18.6051 + 1.1132)\hat{u}_c + (18.6051 - 1.1132)\hat{u}_1$$

$$= 19.7183\,\hat{u}_c + 17.4919\,\hat{u}_1 \quad \text{(au/year)}$$

1 EMOS $= 2\pi$ au/year

$$\boxed{\bar{V}_1 = 3.13826\,\hat{u}_c + 2.78392\,\hat{u}_1 \quad \text{(EMOS)}}$$

For $t_2 - t_1 = t_F^{\#} > t_m$

$\alpha = 4.48169 \qquad \beta = 0.1507$

Using (8.23) in the notes:

$$A = \left(\frac{\mu}{4a}\right)^{1/2}\cot\left(\frac{\alpha}{2}\right) = \left(\frac{4\pi^2}{4(5)}\right)^{1/2}\cot\left(\frac{4.48169}{2}\right) = -1.1132 \text{ au/year}$$

$$B = \left(\frac{\mu}{4a}\right)^{1/2}\cot\left(\frac{\beta}{2}\right) = \left(\frac{4\pi^2}{4(5)}\right)^{1/2}\cot\left(\frac{0.1507}{2}\right) = 18.6051 \text{ au/year}$$

Using (8.21) in the notes:

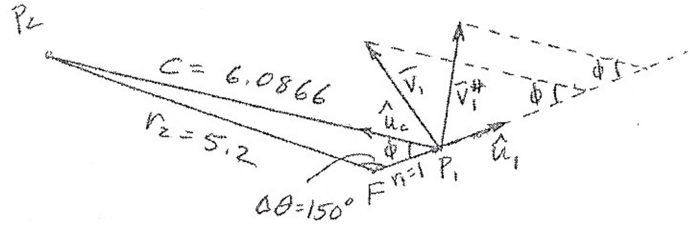
$$\bar{V}_1^{\#} = (B+A)\hat{u}_c + (B-A)\hat{u}_1$$

$$= (18.6051 + (-1.1132))\hat{u}_c + (18.6051 - (-1.1132))\hat{u}_1$$

$$= 17.4919\,\hat{u}_c + 19.7183\,\hat{u}_1 \quad \text{(au/year)}$$

$$\boxed{\bar{V}_1^{\#} = 2.78392\,\hat{u}_c + 3.13826\,\hat{u}_1 \quad \text{(EMOS)}}$$

d)



$$\frac{5.2}{\sin\phi} = \frac{6.0866}{\sin 150°} \Rightarrow \phi = 0.441358 \text{ radians} = 25.288°$$

$$V_1 = \left[3.13826^2 + 2.78392^2 - 2(3.13826)(2.78392)\cos 0.441358\right]^{1/2}$$

$$\boxed{V_1 = 1.34163 \text{ EMOS}}$$

$$V_1^{\#} = \left[2.78392^2 + 3.13826^2 - 2(2.78392)(3.13826)\cos 0.441358\right]^{1/2}$$

$$\boxed{V_1^{\#} = 1.34163 \text{ EMOS}}$$

e) Using (8.19) in the notes:

$$p = \frac{4a(s-r_1)(s-r_2)}{c^2}\sin^2\left(\frac{\alpha+\beta}{2}\right)$$

For $\alpha = 1.8015$ $\beta = 0.1507$

$$p = \frac{4(5)(6.1433-1)(6.1433-5.2)}{6.0866^2}\sin^2\left(\frac{1.8015+0.1507}{2}\right)$$

$$\boxed{p = 1.7971 \text{ au}}$$

From $p = a(1-e^2)$

$$e = \sqrt{1 - \frac{p}{a}} = \sqrt{1 - \frac{1.7971}{5}} = \underline{\underline{0.8004}}$$

For $\alpha = 4.48169$ $\beta = 0.1507$

$$\tilde{p} = \frac{4(5)(6.1433-1)(6.1433-5.2)}{6.0866^2} \sin^2\left(\frac{4.48169 + 0.1507}{2}\right)$$

$$\boxed{\tilde{p} = 1.4142 \text{ au}}$$

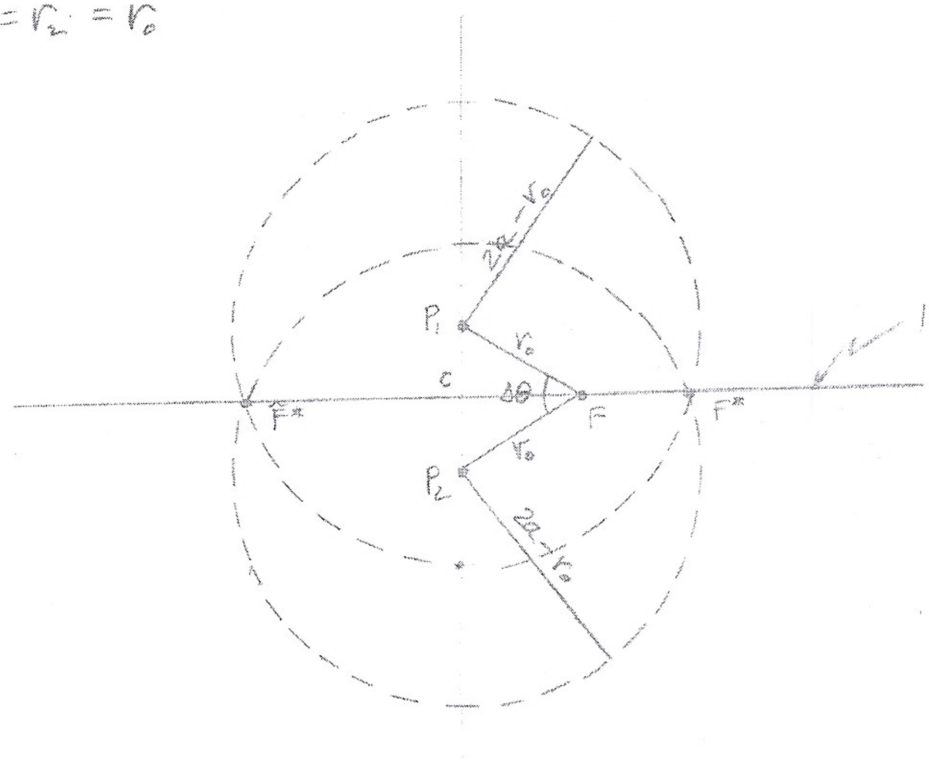
$$\tilde{e} = \sqrt{1 - \frac{\tilde{p}}{a}} = \sqrt{1 - \frac{1.4142}{5}} = \underline{\underline{0.8469}}$$

3) For the case $r_1 = r_2 = r_0$ and arbitrary transfer angle $\Delta\theta$

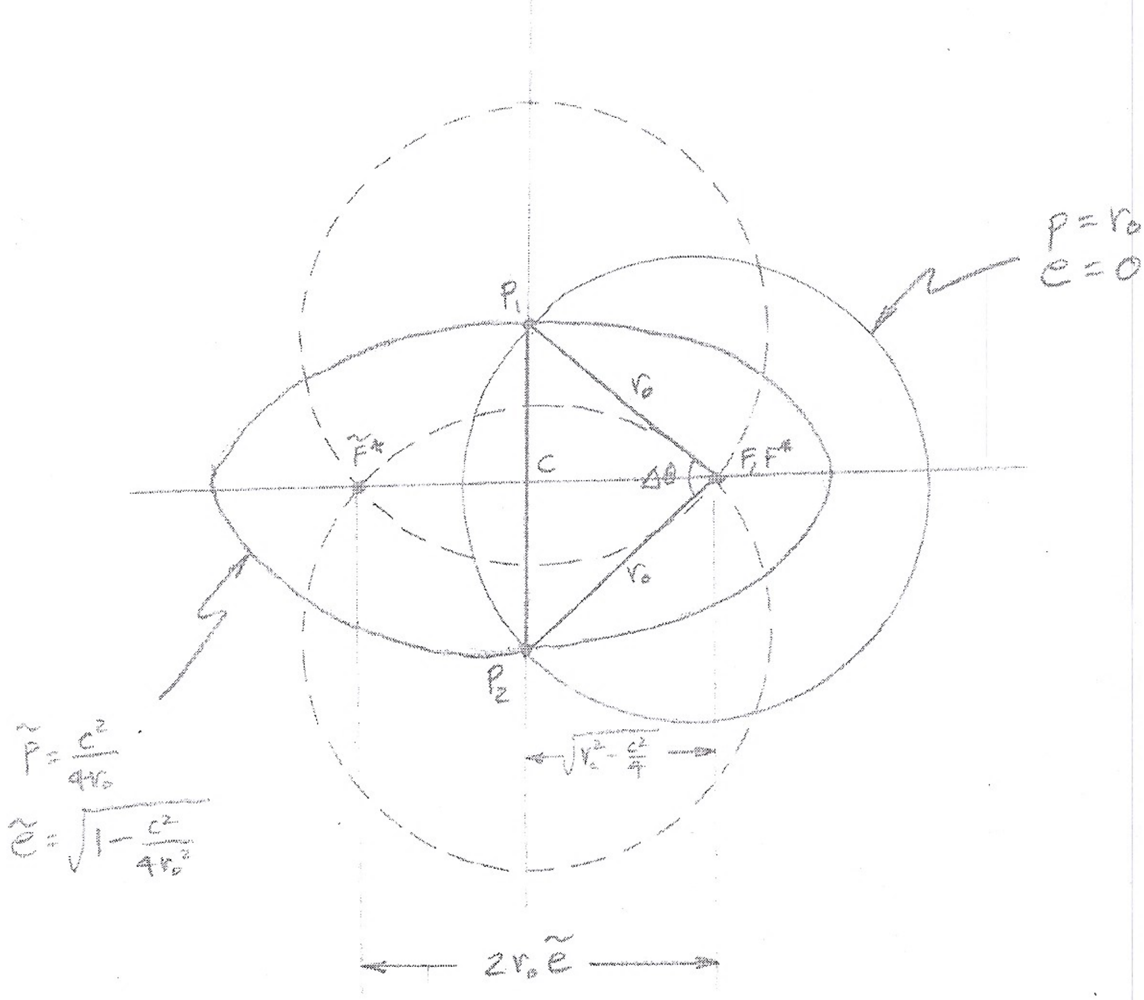
a) Construct the locus of the focus.

b) For a value of a equal to r_0 determine the values of e and $\tilde{e}$ and the corresponding values of p and $\tilde{p}$.

a) $r_1 = r_2 = r_0$



b) $r_1 = r_2 = r_0 = a$



from the figure:

$$2r_0\tilde{e} = 2\sqrt{r_0^2 - \frac{c^2}{4}} \quad \Rightarrow \quad \tilde{e} = \sqrt{1 - \frac{c^2}{4r_0^2}}$$

$$\tilde{p} = a(1 - \tilde{e}^2) = r_0\left[1 - \left(1 - \frac{c^2}{4r_0^2}\right)\right] = \frac{c^2}{4r_0}$$