ME 57200 Aerodynamic Design

Lecture #16: Flow over Finite Wings

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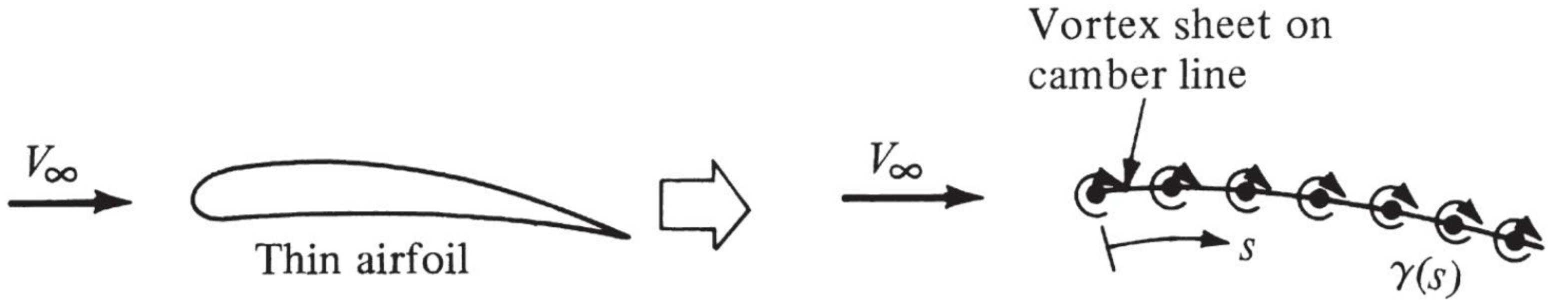
Theoretical Solutions for Flow over Airfoils

- <u>Thin Airfoil</u>

- Imagine the airfoil is made very thin – the vortex sheet on the top and bottom surface of the airfoil would almost coincide.
- We can approximate a thin airfoil by replacing it with a single vortex sheet, $\gamma(s)$ distributed over the camber line of the airfoil.

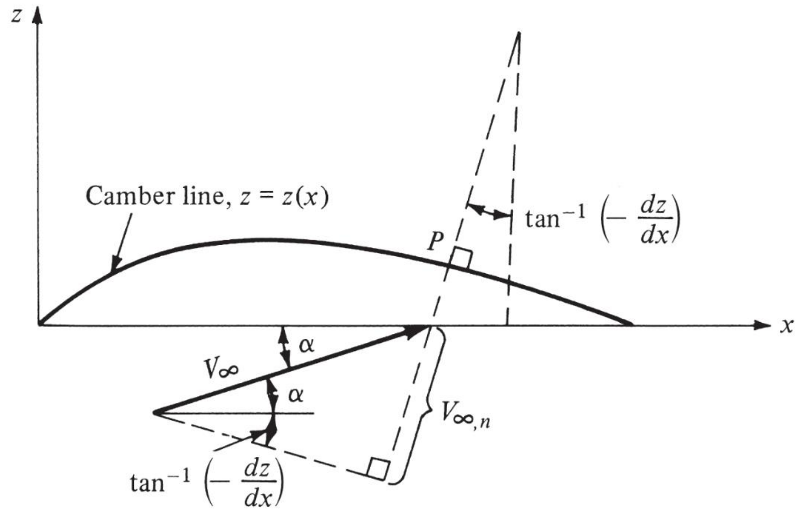


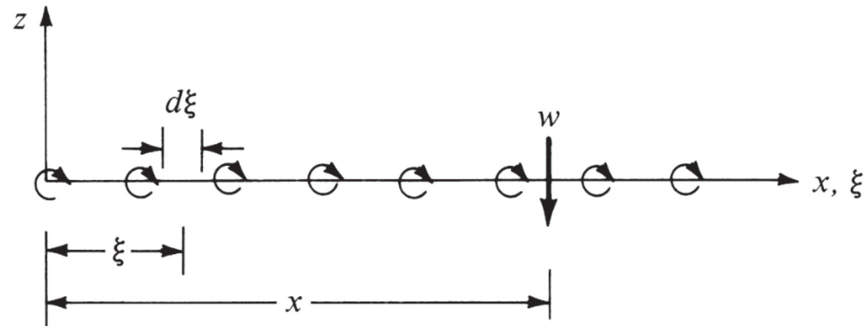
Yielding a closed-form analytical solution, such that:

- Camber line becomes a streamline of the flow
- The Kutta condition is satisfied at the trailing edge: $\gamma(LE) = 0$

Theoretical Solutions for Flow over Airfoils

- Thin Airfoil





$$V_{\infty,n} + w'(s) = 0$$

$$w'(s) \approx w(x)$$

$$V_{\infty,n} = V_\infty \sin\left[\alpha + \tan^{-1}\left(-\frac{dz}{dx}\right)\right]$$

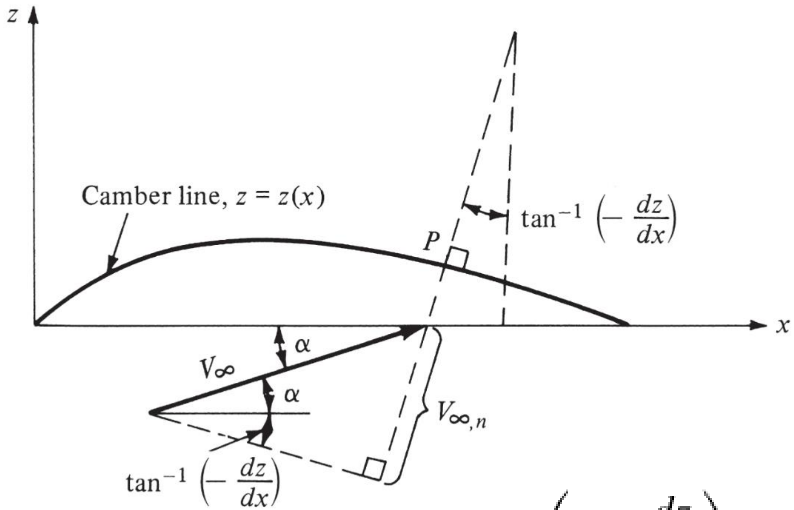
$$dw = -\frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}$$

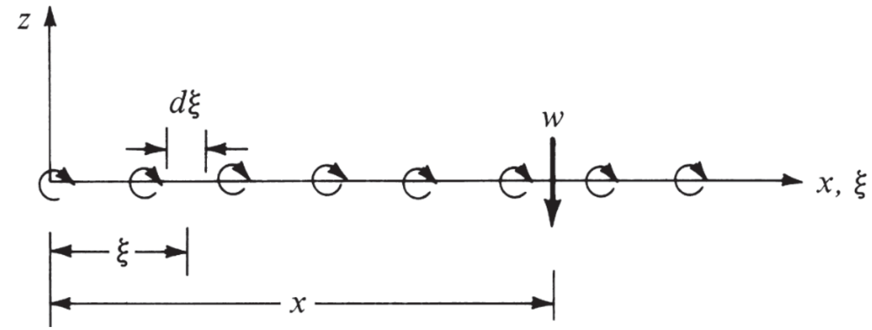
$$V_{\infty,n} = V_\infty\left(\alpha - \frac{dz}{dx}\right)$$

$$w(x) = -\int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}$$

Theoretical Solutions for Flow over Airfoils

- <u>Thin Airfoil</u>





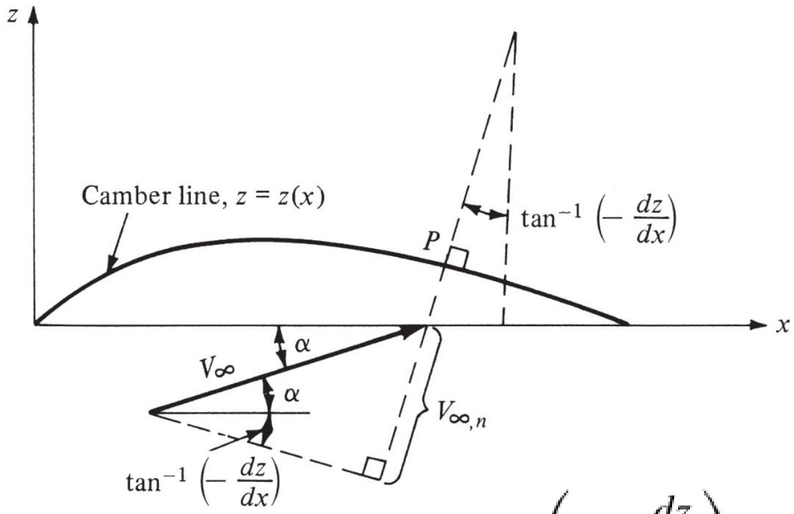
$$V_\infty\left(\alpha - \frac{dz}{dx}\right) - \int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)} = 0$$

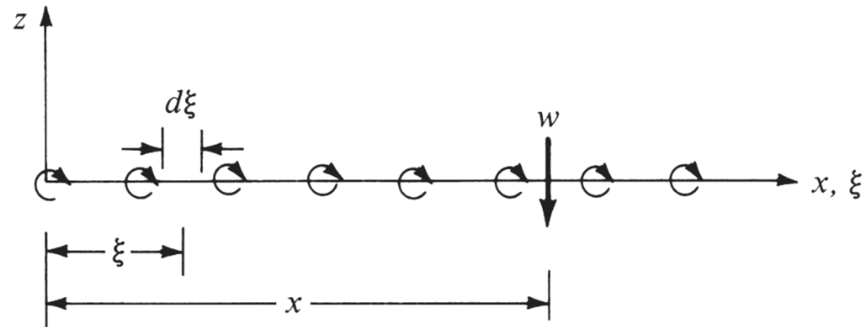
$$\boxed{\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right)}$$

- Camber line becomes a streamline of the flow

Theoretical Solutions for Flow over Airfoils

- Thin Airfoil





$$V_\infty\left(\alpha - \frac{dz}{dx}\right) - \int_0^c \frac{\gamma(\xi)\,d\xi}{2\pi(x-\xi)} = 0$$

$$\boxed{\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right)}$$

$\boxed{dz/dx = 0,}$

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty \alpha$$

Theoretical Solutions for Flow over Airfoils

- Thin Airfoil

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty \alpha$$

$$\xi = \frac{c}{2}(1-\cos\theta) \qquad d\xi = \frac{c}{2}\sin\theta\, d\theta$$

$$x = \frac{c}{2}(1-\cos\theta_0)$$

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\, d\theta}{\cos\theta - \cos\theta_0} = V_\infty \alpha$$

$$\boxed{\gamma(\theta) = 2\alpha V_\infty \frac{(1+\cos\theta)}{\sin\theta}}$$

Using L'Hospital's rule $$\gamma(\pi) = 2\alpha V_\infty \frac{-\sin\pi}{\cos\pi} = 0$$

- The Kutta condition is satisfied at the trailing edge: $\gamma(c) = \gamma(\pi) = 0$

Theoretical Solutions for Flow over Airfoils

- Thin Airfoil

$$\Gamma = \int_0^c \gamma(\xi)\, d\xi$$

$$\Gamma = \frac{c}{2} \int_0^\pi \gamma(\theta) \sin\theta\, d\theta$$

$$\Gamma = \alpha c V_\infty \int_0^\pi (1 + \cos\theta)\; d\theta = \pi \alpha c V_\infty$$

  - Kutta-Joukowski theorem

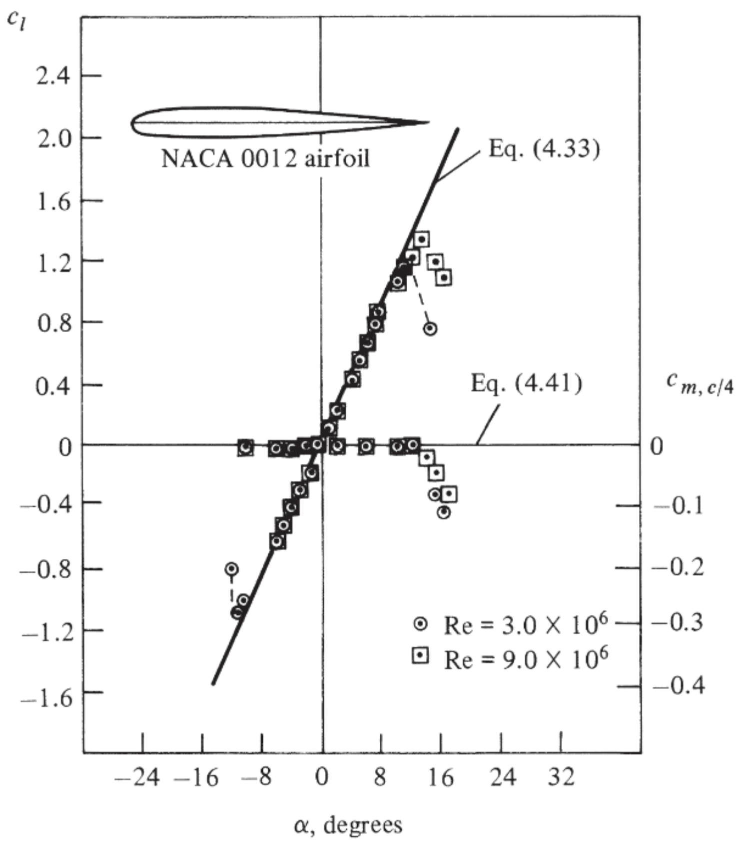
$$L' = \rho_\infty V_\infty \Gamma = \pi \alpha c \rho_\infty V_\infty^2$$

$$c_l = \frac{\pi \alpha c \rho_\infty V_\infty^2}{\frac{1}{2}\rho_\infty V_\infty^2 c(1)} \qquad \boxed{c_l = 2\pi\alpha} \qquad \boxed{\text{Liftslope} = \frac{dc_l}{d\alpha} = 2\pi}$$

Theoretical Solutions for Flow over Airfoils

- <u>Thin Airfoil</u>

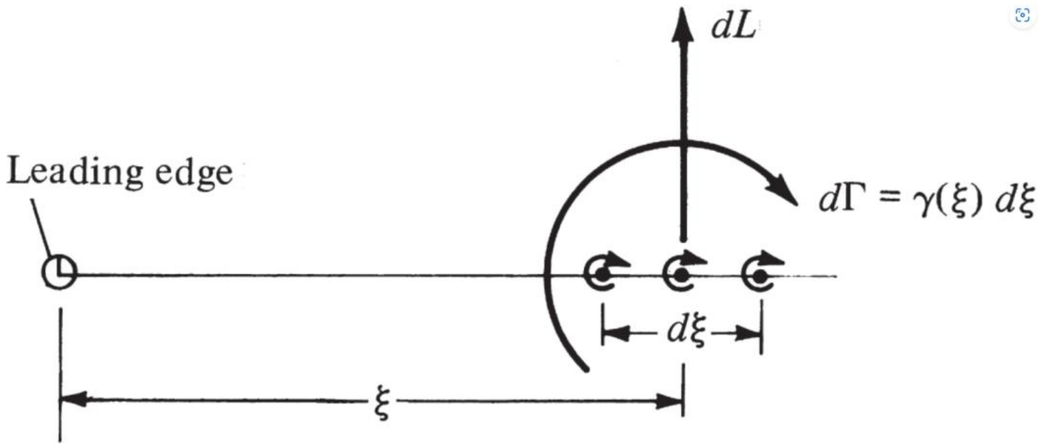
$\boxed{c_l = 2\pi\alpha}$ accurately predicts c_l over a large range of angle attack.



Theoretical Solutions for Flow over Airfoils

- Thin Airfoil

The total moment about the leading edge



$$M'_{\mathrm{LE}} = -\int_0^c \xi(dL) = -\rho_\infty V_\infty \int_0^c \xi\gamma(\xi)\, d\xi$$

$$M'_{\mathrm{LE}} = -q_\infty c^2 \frac{\pi\alpha}{2}$$

Theoretical Solutions for Flow over Airfoils

- <u>Thin Airfoil</u>

The moment coefficient about the leading edge is $c_{m,\mathrm{le}} = \frac{M'_{\mathrm{LE}}}{q_\infty c^2} = -\frac{\pi\alpha}{2}$

$$\boxed{c_l = 2\pi\alpha}$$

$$\boxed{c_{m,\mathrm{le}} = -\frac{c_l}{4}}$$

The moment coefficient about the quarter-chord point is $c_{m,c/4} = c_{m,\mathrm{le}} + \frac{c_l}{4}$

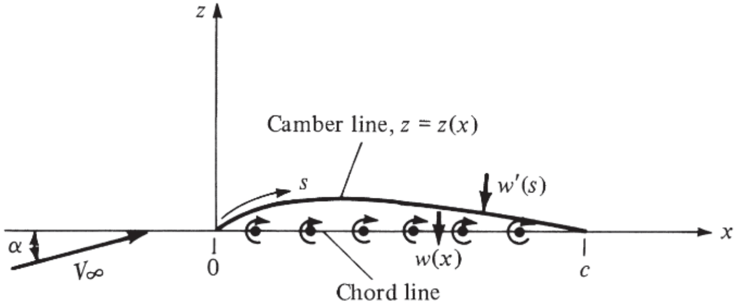
$$\boxed{c_{m,c/4} = 0}$$

The quarter-chord point is the center of pressure and the aerodynamic center for a symmetric airfoil

Theoretical Solutions for Flow over Airfoils

- The Cambered Airfoil

Thin airfoil theory for a cambered airfoil: a generalization of the method for a symmetric airfoil.



(b) Vortex sheet on the chord line

$$V_\infty\left(\alpha - \frac{dz}{dx}\right) - \int_0^c \frac{\gamma(\xi)\,d\xi}{2\pi(x-\xi)} = 0$$

$$\boxed{\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right)}$$

For a cambered airfoil, *dz/dx* is finite

Theoretical Solutions for Flow over Airfoils

- <u>The Cambered Airfoil</u>

Thin airfoil theory for a cambered airfoil: a generalization of the method for a symmetric airfoil.

$$\boxed{\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right)}$$

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\,d\theta}{\cos\theta - \cos\theta_0} = V_\infty\left(\alpha - \frac{dz}{dx}\right)$$

$$\gamma(\theta) = 2V_\infty\left(A_0\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right)$$

- A_0 depends on both dz/dx and α
- A_n depends on dz/dx

$$\boxed{c_l = 2\pi\left[\alpha + \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta_0 - 1)\,d\theta_0\right]}$$

$$\boxed{\text{Lift slope} \equiv \frac{dc_l}{d\alpha} = 2\pi}$$

From thin airfoil theory: $dc_l/d\alpha = 2\pi$ is valid for any shape airfoil

Incompressible Flow over Finite Wings

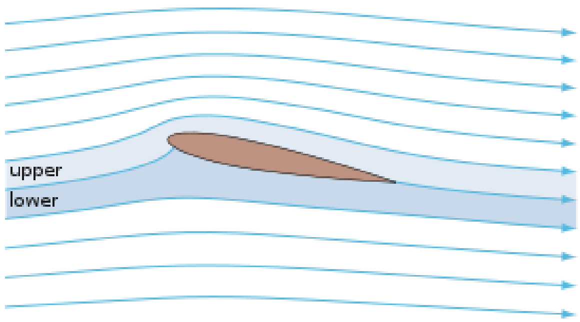
Incompressible Flow over Finite Wings

Are C_L and C_D for a finite wing the same as those for the airfoil (cross section of the wing)?

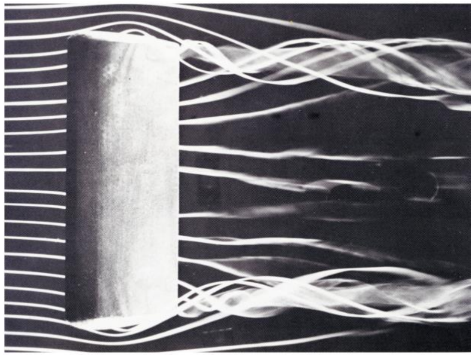
The answer is NO!

Why?

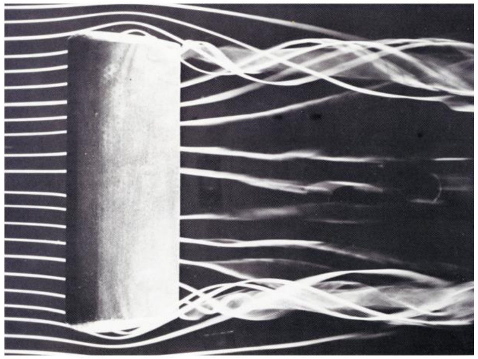
The flow over an airfoil is 2-D



The flow over a finite wing is 3-D



Incompressible Flow over Finite Wings



Why is the flow over a finite wing three-dimensional?

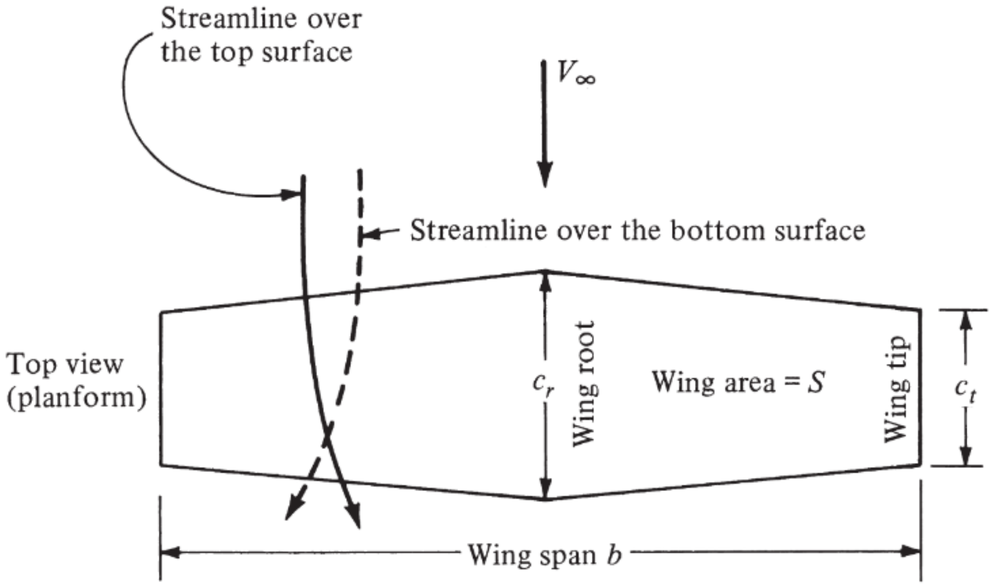
How is the lift generated by the wing?

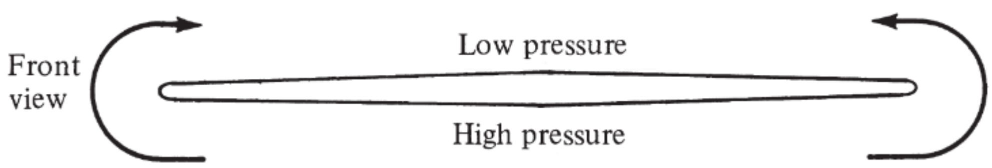
The net imbalance of the pressure distribution creates the lift

As a by-product of the pressure imbalance, the flow near the wing tips tends to curl around the tips, being forced from the high-pressure region underneath the tips to the low-pressure region on top.

- On the top surface of the wing, there is a spanwise component of flow from the tip toward the wing root, the streamlines bend toward the root.
- On the bottom surface of the wing, there is a spanwise component of flow from the root toward the wing tip, the streamlines bend toward the tip.

Incompressible Flow over Finite Wings

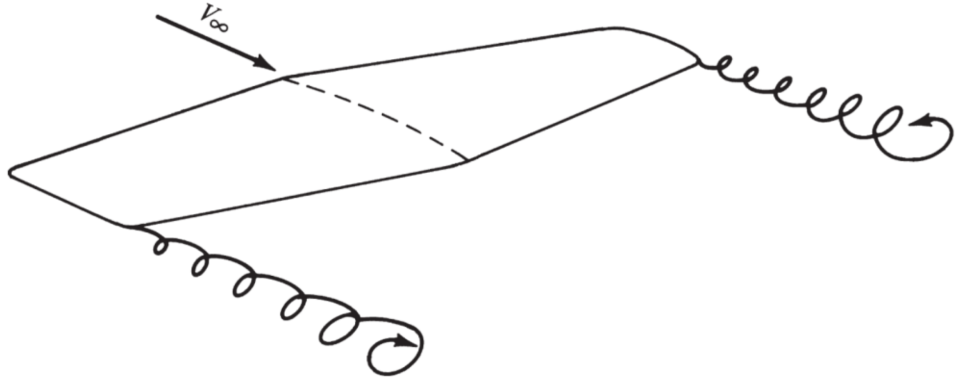




Incompressible Flow over Finite Wings

Wing-tip vortices

The flow “leak” around the wing tips leads to the formation of wing-tip vortices

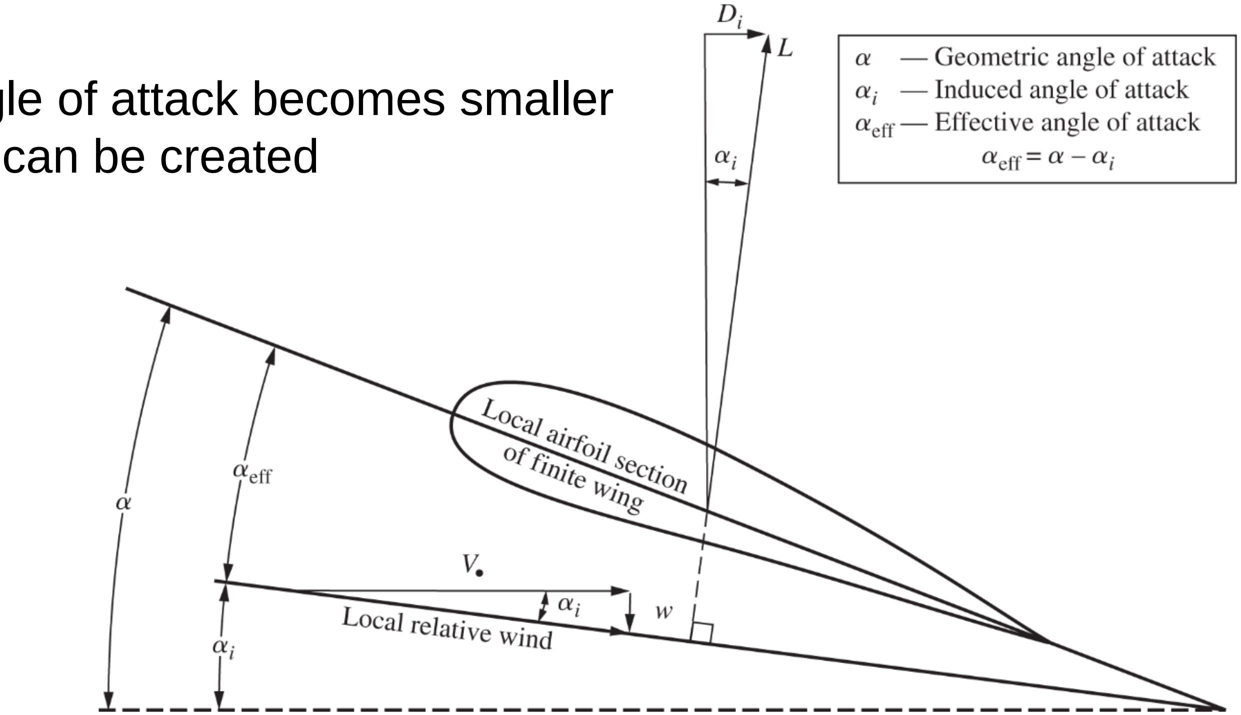


Downwash: the velocity component in the downward direction at the wing due to the wing-tip vortices

Incompressible Flow over Finite Wings

The presence of downwash, and its effect on inclining the local relative wind in the downward direction, has two import effects on the local airfoil section:

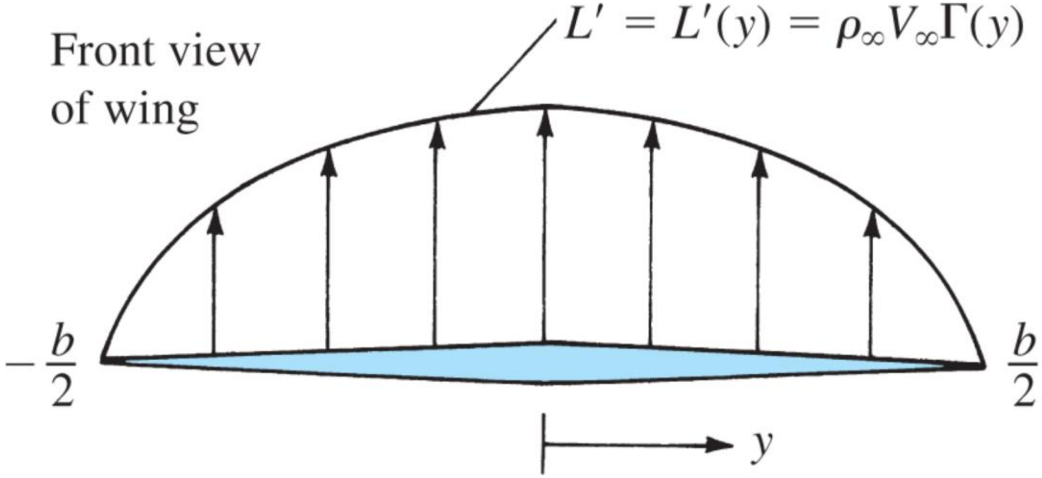
- The effective angle of attack becomes smaller
- An induced drag can be created



How is the lift distributed for a finite wing?
How to calculate the induced drag and the total lift?

Incompressible Flow over Finite Wings

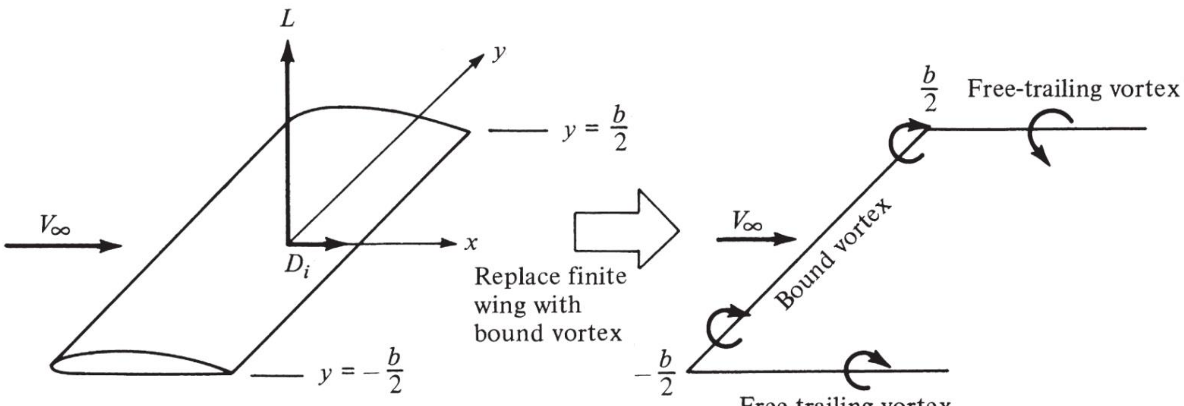
Lift distribution



- Most finite wings have a variable chord
- Many wings are geometrically twisted so that α is different at different spanwise locations
- Many wings have different airfoil sections along the span with different values of $\alpha_{L=0}$
- Zero lift at the tips?

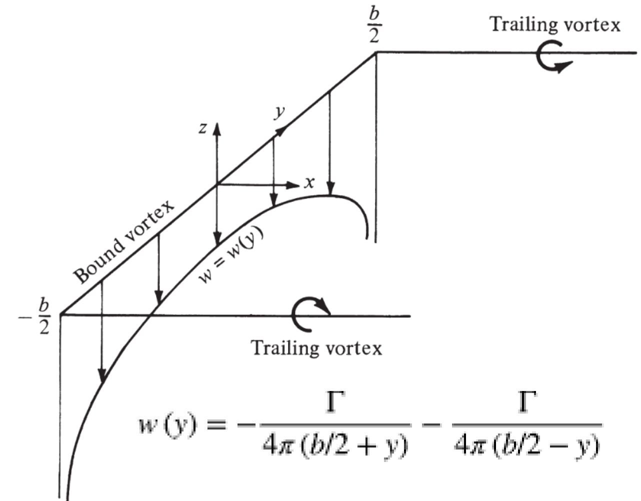
Incompressible Flow over Finite Wings

Prandtl's classical lifting-line theory



Finite wing

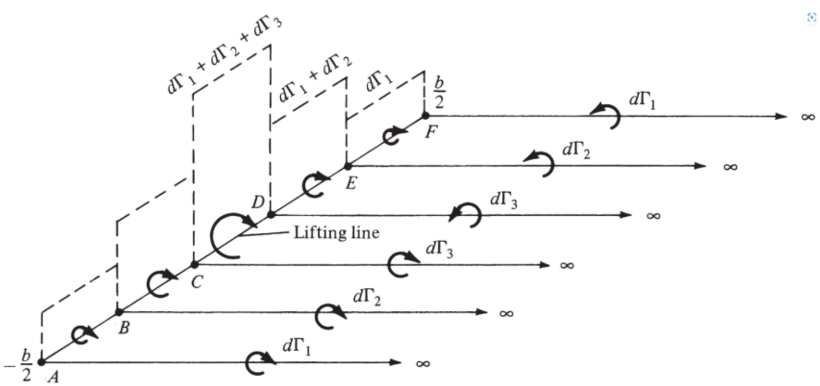
Horseshoe vortex

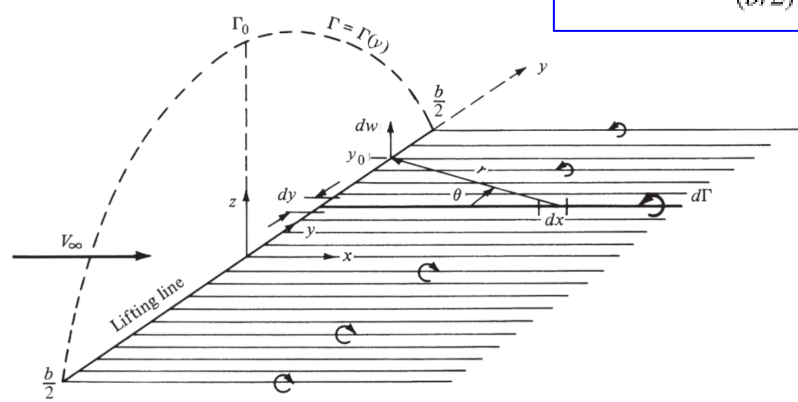


$$w(y) = -\frac{\Gamma}{4\pi(b/2 + y)} - \frac{\Gamma}{4\pi(b/2 - y)}$$

$$\boxed{w(y) = -\frac{\Gamma}{4\pi}\frac{b}{(b/2)^2 - y^2}}$$

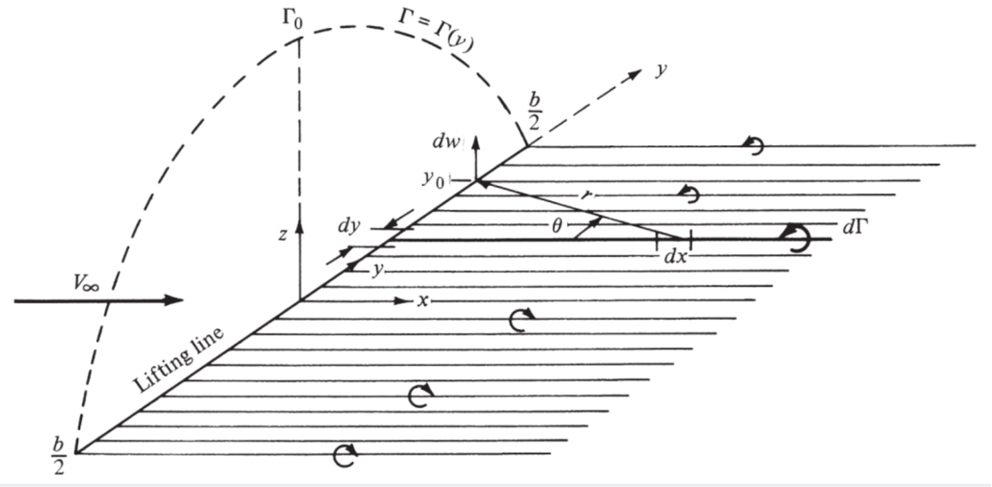
- Note that *w* approaches -∞ as y approaches –*b/2* or *b/2*





Incompressible Flow over Finite Wings

Prandtl's classical lifting-line theory



$$dw = -\frac{(d\Gamma/dy)\,dy}{4\pi(y_0 - y)}$$

$$\boxed{w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)\,dy}{y_0 - y}}$$

$$\boxed{\alpha_i(y_0) = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)\,dy}{y_0 - y}}$$

Geometric AoA

$$\boxed{\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)\,dy}{y_0 - y}}$$

effective angle induced angle

The solution $\Gamma = \Gamma(y_0)$ can be obtained!

Incompressible Flow over Finite Wings

Prandtl's classical lifting-line theory

With the solution $\Gamma = \Gamma(y_0)$

- The lift distribution can be obtained from the Kutta-Joukowski theorem

$$L'(y_0) = \rho_\infty V_\infty \Gamma(y_0)$$

- The total lift can be obtained by integrating over the span

$$L = \rho_\infty V_\infty \int_{-b/2}^{b/2} \Gamma(y)\, dy$$

- The induced drag can be obtained

$$D_i' = L_i' \sin \alpha_i$$

$$D_i = \rho_\infty V_\infty \int_{-b/2}^{b/2} \Gamma(y)\, \alpha_i(y)\, dy$$

Incompressible Flow over Finite Wings

Elliptical Lift Distribution

Consider a circulation distribution given by $\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}$

$$\Gamma(b/2) = \Gamma(-b/2) = 0.$$

$$L'(y) = \rho_\infty V_\infty \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}$$

$$w(y_0) = \frac{\Gamma_0}{\pi b^2} \int_{-b/2}^{b/2} \frac{y}{\left(1 - 4y^2/b^2\right)^{1/2} (y_0 - y)} dy$$

$$y = \frac{b}{2}\cos\theta \qquad w(\theta_0) = -\frac{\Gamma_0}{2\pi b} \int_{\pi}^{0} \frac{\cos\theta}{\cos\theta_0 - \cos\theta} d\theta$$

$$\boxed{w(\theta_0) = -\frac{\Gamma_0}{2b}}$$

Downwash is constant over the span for an elliptical lift distribution

Incompressible Flow over Finite Wings

Elliptical Lift Distribution

$$\alpha_i = -\frac{w}{V_\infty} = \frac{\Gamma_0}{2bV_\infty}$$

Induced angle of attack is also constant over the span for an elliptical lift distribution

$$L = \rho_\infty V_\infty \Gamma_0 \frac{b}{2} \int_0^\pi \sin^2\theta d\theta = \rho_\infty V_\infty \Gamma_0 \frac{b}{4}\pi$$

$$\Gamma_0 = \frac{4L}{\rho_\infty V_\infty b\pi}$$

$$L = \tfrac{1}{2}\rho_\infty V_\infty^2 S C_L. \qquad \Gamma_0 = \frac{2V_\infty S C_L}{b\pi}$$

$$\alpha_i = \frac{2V_\infty S C_L}{b\pi}\frac{1}{2bV_\infty} \qquad \alpha_i = \frac{SC_L}{\pi b^2}$$

Aspect Ratio: $\mathrm{AR} \equiv \frac{b^2}{S}$

$$\boxed{\alpha_i = \frac{C_L}{\pi \mathrm{AR}}}$$

$$\boxed{C_{D,i} = \frac{C_L^2}{\pi \mathrm{AR}}}$$

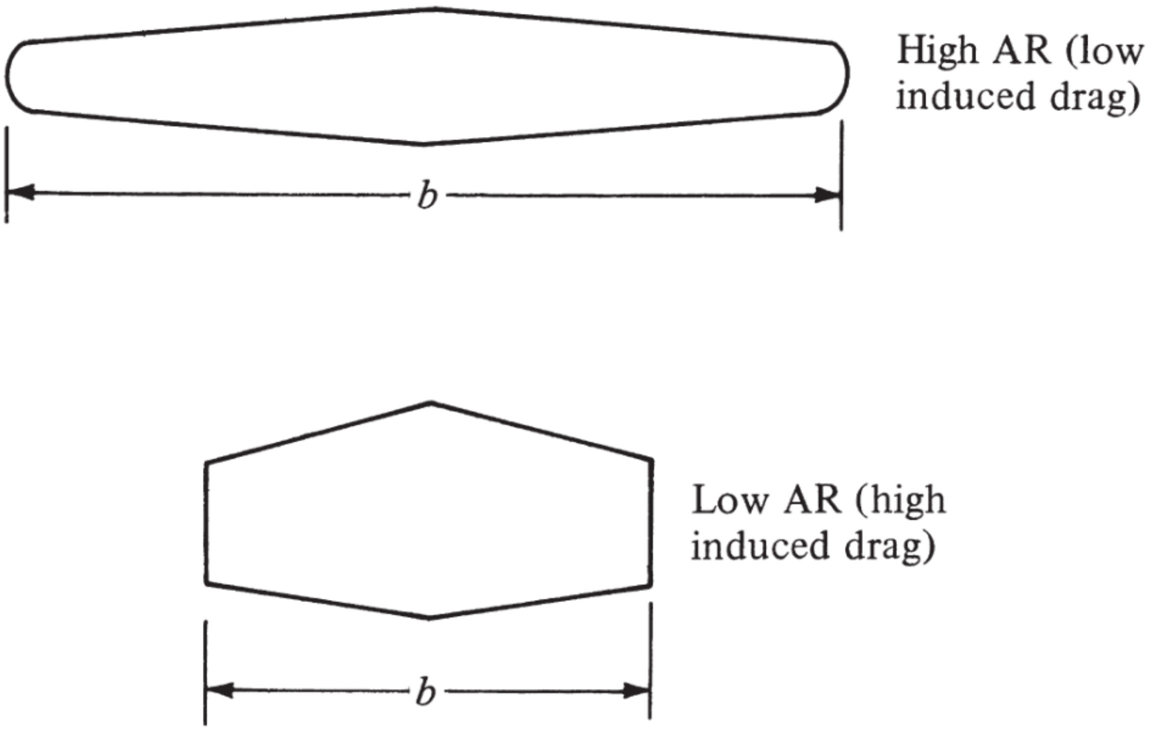
- The induced drag coefficient is directly proportional to the square of the lift coefficient
- The induced drag coefficient is inversely proportional to aspect ratio

Incompressible Flow over Finite Wings

Elliptical Lift Distribution

$$\boxed{C_{D,i} = \frac{C_L^2}{\pi \text{AR}}}$$

$$\boxed{\text{AR} = b^2/S}$$



Term Project Update