ME 57200 Aerodynamic Design

Lecture #10: Elemental Flows

Dr. Yang Liu

Steinman 253

Tel: 212-650-7346

Email: yliu7@ccny.cuny.edu

Midterm Exam

- Tuesday, 3/12, from 09:30 to 10:45 am at Shepard S-308.

- The exam is open-book and open-notes.

- 5 True/False Questions: 10 pt
- 4 Math-based Problems: 40 pt
- Total: 50 pt

Laplace's Equation

The principle of mass conservation for an incompressible flow

$$\nabla \cdot \mathbf{V} = 0$$

For an irrotational flow $\mathbf{V} = \nabla \phi$

For an incompressible and irrotational flow $\nabla \cdot (\nabla \phi) = 0$

$$\boxed{\nabla^2 \phi = 0}$$

- Laplace's equation: a second-order linear partial differential equation
- One of the most famous and extensively studied equations in mathematical physics.
- Solutions of Laplace's equation are called *harmonic functions*

Laplace's Equation

Cartesian coordinates: $\phi = \phi(x, y, z)$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Cylindrical coordinates: $\phi = \phi(r, \theta, z)$

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Spherical coordinates: $\phi = \phi(r, \theta, \Phi)$

$$\nabla^2 \phi = \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin\theta \frac{\partial \phi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \phi}{\partial \theta} \right) + \frac{\partial}{\partial \Phi} \left(\frac{1}{\sin\theta} \frac{\partial \phi}{\partial \Phi} \right) \right] = 0$$

Laplace's Equation

For a two-dimensional incompressible flow

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)$$

$$= \frac{\partial^2 \psi}{\partial x\,\partial y} - \frac{\partial^2 \psi}{\partial y\,\partial x} = 0$$

Continuity Equation

- Stream function automatically satisfies the continuity equation.

Laplace's Equation

For a two-dimensional incompressible irrotational flow

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$$

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0}$$

- The stream function also satisfies Laplace's equation

Laplace’s Equation

- Any irrotational, incompressible flow has a velocity potential and stream function (for two-dimensional flow) that both satisfy Laplace’s equation.

- Conversely, any solution of Laplace’s equation represents the velocity potential or stream function (two-dimensional) for an irrotational, incompressible flow.

- Note: the sum of any particular solutions of a linear differential equation is also a solution of the equation.

$$\boxed{\nabla^2 \phi = 0}$$

$\varphi_1, \varphi_2, \varphi_3, \varphi_4, \ldots, \varphi_n$ represent *n* separate solutions

$\boxed{\varphi = \varphi_1 + \varphi_2 \ldots + \varphi_n \text{ is also a solution}}$

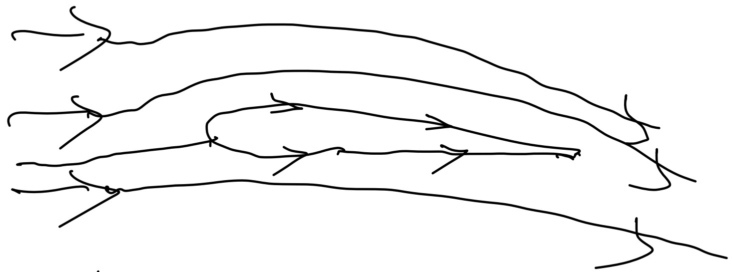
A complicated flow pattern for an irrotational, incompressible flow can be synthesized by adding together a number of elementary flows that are also irrotational and incompressible.

Laplace's Equation

How do we obtain different flows (solutions) for different bodies?

Boundary Conditions:

Consider the flow over an airfoil



The flow is bounded by

(1) the freestream flow that occurs an infinite distance away from the body.

(2) The surface of the body itself.

1. Infinity Boundary Condition

Far away from the body in all directions, the flow approaches the uniform freestream condition

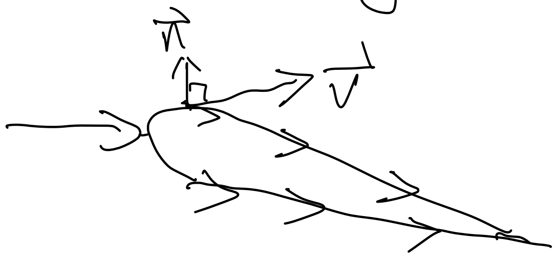
Let V_∞ be aligned with x direction, at infinity

$$\begin{cases} u = \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y} = V_\infty \\ v = \dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x} = 0 \end{cases}$$

Laplace's Equation

2. Wall Boundary Condition

Because the flow cannot penetrate the surface, velocity at the surface is always tangent to the body surface.



$$\vec{V} \cdot \vec{n} = 0$$

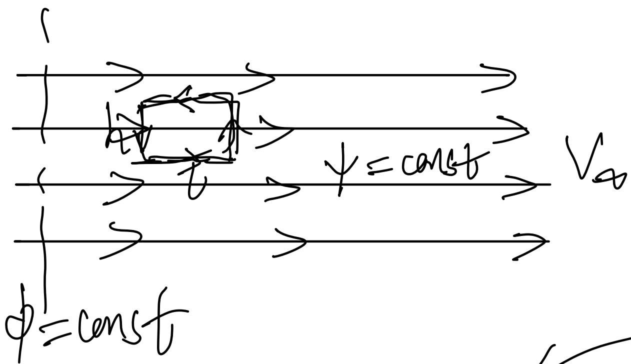
$$\Rightarrow (\nabla \phi) \cdot \vec{n} = 0 \quad \text{or} \quad \frac{\partial \phi}{\partial n} = 0$$

The body contour is a streamline: $\psi = \text{constant}$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{v}{u}\right)_{\text{surface}}$$

Elementary Flows

Unifor flow



$$\begin{cases} \nabla \cdot \vec{V} = 0 & \text{incompressible} \\ \nabla \times \vec{V} = 0 & \text{irrotational.} \end{cases}$$

$$u = V_\infty, \quad v = 0$$

$$\frac{\partial \phi}{\partial x} = V_\infty, \quad \frac{\partial \phi}{\partial y} = 0$$

$$\Rightarrow \begin{cases} \phi = V_\infty x + f(y) \\ \phi = Const + g(x) \end{cases}$$

$$\Rightarrow \begin{cases} g(x) = V_\infty x \\ f(y) = Const \end{cases} \Rightarrow \phi = V_\infty x + Const =$$

For stream function, ψ: $\dfrac{\partial \psi}{\partial y} = u = V_\infty$, $\dfrac{\partial \psi}{\partial x} = -v = 0$

Elementary Flows

$$\Rightarrow \psi = V_\infty y + \text{const}$$

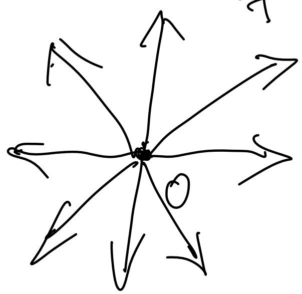
Circulation: Consider a rectangular $l \times h$ a "curve"

$$\Gamma = -\oint_C \vec{V} \cdot d\vec{s} = -V_\infty l - 0 \cdot h + V_\infty l + 0 \cdot h = 0$$

For arbitrary "c": $\Gamma = -\oint_C \vec{V} \cdot d\vec{s} = -\vec{V}_\infty \oint_C d\vec{s} = 0$

Elementary Flows

Source Flow : A two-dimensional incompressible flow where all the streamlines are straight lines from a central point "o". The velocity along each of the streamlines vary inversely with distance from point "o".



Polar coordinates : $\underline{V_r,\ V_\theta}$ $(V_\theta = 0)$

(1) Source flow is physically possible incompressible flow

$$\nabla \cdot \vec{V} = 0 \quad \text{Continuity Equation.}$$

(2) Source flow is irrotational at every point.

By definition: Source flow velocity is inversely proportional to radial distance.

$$\begin{cases} V_r = \dfrac{c}{r} & (c \text{ is a constant}) \\ V_\theta = 0 \end{cases}$$

Elementary Flows

Now extend to 3-dimensional

Consider the mass flow rate across the elementary surface of the cylinder

$$ds = (r \cdot d\theta) \cdot l$$

$$d\dot{m} = \rho \cdot V_r \cdot ds = \rho V_r (r d\theta) \cdot l$$

Total mass flow rate: $\dot{m} = \int_0^{2\pi} \rho V_r (r d\theta) l = \rho r l V_r \int_0^{2\pi} d\theta = \underline{2\pi r l \rho V_r}$

Total volumetric flow rate: $\dot{V} = \frac{\dot{m}}{\rho} = 2\pi r l V_r$

Volume flow rate per unit length: $\underline{\Lambda = \frac{\dot{V}}{l} = 2\pi r V_r}$ (source strength)

$$\Rightarrow \begin{cases} V_r = \frac{\Lambda}{2\pi r} \\ V_r = \frac{c}{r} \end{cases} \Rightarrow c = \frac{\Lambda}{2\pi}$$

Elementary Flows

The velocity potential for a source flow:

$$\begin{cases} \dfrac{\partial \phi}{\partial r} = V_r = \dfrac{\Lambda}{2\pi r} \\ \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} = V_\theta = 0 \end{cases} \Rightarrow \begin{cases} \phi = \dfrac{\Lambda}{2\pi}\ln r + f(\theta) \\ \phi = const + f(r) \end{cases}$$

$$\Rightarrow \begin{cases} f(r) = \dfrac{\Lambda}{2\pi}\ln r \\ f(\theta) = const \end{cases} \Rightarrow \boxed{\phi = \dfrac{\Lambda}{2\pi}\ln r + const}$$

Stream function

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} = V_r = \dfrac{\Lambda}{2\pi r} \\ -\dfrac{\partial \psi}{\partial r} = V_\theta = 0 \end{cases} \Rightarrow \begin{cases} \psi = \dfrac{\Lambda}{2\pi}\theta + f(r) \\ \psi = const + f(\theta) \end{cases}$$

$$\Rightarrow \boxed{\psi = \dfrac{\Lambda}{2\pi}\theta + Const}$$

In-Class Quiz