ME 51500/I 5800 HW 2 SOLUTIONS P. Ganatos

1.) Curtis 2.9 Show that $v = (\mu/h)\sqrt{1 + 2e\cos\theta + e^2}$ for any orbit.

For any orbit

$$r = \frac{h^2/\mu}{1 + e\cos\theta}$$

$$\frac{dr}{dt} = -\frac{h^2/\mu}{(1+e\cos\theta)^2}(-e\sin\theta)\frac{d\theta}{dt}$$

From (5.7) $\frac{d\theta}{dt} = \frac{h}{r^2}$

$$\frac{dr}{dt} = \frac{\cancel{r}e\sin\theta}{1+e\cos\theta}\cdot\frac{h}{r\cancel{^2}} = \frac{he\sin\theta}{(1+e\cos\theta)\,\frac{h^2/\mu}{1+e\cos\theta}} = \frac{\mu}{h}e\sin\theta$$

Using (5.24) & (5.25)

$$V_r = \frac{dr}{dt} = \frac{\mu}{h}e\sin\theta$$

$$V_\theta = r\frac{d\theta}{dt} = \frac{h}{r} = \frac{h}{\frac{h^2/\mu}{1+e\cos\theta}} = \frac{\mu}{h}(1+e\cos\theta)$$

$$V = \sqrt{V_r^2 + V_\theta^2} = \sqrt{\frac{\mu^2}{h^2}e^2\sin^2\theta + \frac{\mu^2}{h^2}(1 + 2e\cos\theta + e^2\cos^2\theta)}$$

$$\boxed{V = \frac{\mu}{h}\sqrt{1 + 2e\cos\theta + e^2}}$$

2) Curtis

2.10 Relative to a nonrotating, earth-centered Cartesian coordinate system, the position and velocity vectors of a spacecraft are $\mathbf{r} = 7000\hat{\mathbf{i}} - 2000\hat{\mathbf{j}} - 4000\hat{\mathbf{k}}$ (km) and $\mathbf{v} = 3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ (km/s). Calculate the orbit's (a) eccentricity vector and (b) the true anomaly.

{Ans.: (a) $\mathbf{e} = 0.2888\hat{\mathbf{i}} + 0.08523\hat{\mathbf{j}} - 0.3840\hat{\mathbf{k}}$; (b) $\theta = 33.32°$}

a) $$\bar{h} = \bar{r} \times \frac{d\bar{r}}{dt} = \bar{r} \times \bar{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7000 & -2000 & -4000 \\ 3 & -6 & 5 \end{vmatrix} = -34{,}000\,\hat{i} - 47{,}000\,\hat{j} - 36{,}000\,\hat{k} \quad (\text{km}^2/\text{sec})$$

$$r = |\bar{r}| = \sqrt{(7000)^2 + (-2000)^2 + (-4000)^2} = 8306.6 \quad \text{km}$$

$$\bar{V} \times \bar{h} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & 5 \\ -34{,}000 & -47{,}000 & -36{,}000 \end{vmatrix} = 451{,}000\,\hat{i} - 62{,}000\,\hat{j} - 345{,}000\,\hat{k} \quad (\text{km}^4/\text{sec}^2)$$

From (5.9) in the notes

$$\frac{d\bar{r}}{dt} \times \bar{h} = \mu\left[\frac{\bar{r}}{r} + \bar{e}\right]$$

With $\bar{V} = \frac{d\bar{r}}{dt}$, solve (5.9) for $\bar{e}$

$$\bar{e} = \frac{\bar{V} \times \bar{h}}{\mu} - \frac{\bar{r}}{r}$$

$$= \frac{451{,}000\,\hat{i} - 62{,}000\,\hat{j} - 345{,}000\,\hat{k}}{3.986 \times 10^5} - \frac{7000\,\hat{i} - 2000\,\hat{j} - 4000\,\hat{k}}{8306.6}$$

$$\boxed{\bar{e} = 0.2888\,\hat{i} + 0.08523\,\hat{j} - 0.3840\,\hat{k}}$$

b) $e = |\bar{e}| = \sqrt{(0.2888)^2 + (0.08523)^2 + (-0.3840)^2} = 0.4880$

From $\bar{r} \cdot \bar{e} = re\cos\theta$

$$\cos\theta = \frac{\bar{r}\cdot\bar{e}}{re} = \frac{(7000\hat{i} - 2000\hat{j} - 4000\hat{k})\cdot(0.2888\hat{i} + 0.08523\hat{j} - 0.3840\hat{k})}{(8306.6)(0.4880)}$$

$$= 0.8356$$

$$\boxed{\theta = 33.32^\circ}$$

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3) Curtis

2.15 The specific angular momentum of a satellite in circular earth orbit is 60.000 km²/s. Calculate the period.

{Ans.: 2.372 h}

Using $r = \dfrac{h^2/\mu}{1 + e\cos\theta}$

For a circular orbit, $e = 0$

$\therefore \; r_c = \dfrac{h^2}{\mu}$

Using $T = 2\pi\sqrt{\dfrac{a^3}{\mu}}$

For a circular orbit $a = r_c$

$$T_c = 2\pi\sqrt{\frac{r_c^3}{\mu}} = 2\pi\sqrt{\frac{(h^2/\mu)^3}{\mu}} = 2\pi\frac{h^3}{\mu^2}$$

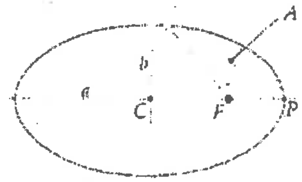
$$= 2\pi\frac{\left(60{,}000\ \frac{km^2}{sec}\right)^3}{\left(3.986\times10^5\ \frac{km^3}{sec^2}\right)^2}\cdot\frac{1\ hr}{3600\ sec}$$

$$\boxed{T_c = 2.373\ hr}$$

4) Curtis

2.17 Calculate the area A swept out during the time $t = T/4$ since periapsis, where T is the period of the elliptical orbit. See the figure below

{Ans.: $0.7854ab$}



Since the rate at which the area is swept out is constant, the area swept out at $t = T/4$ where T is the period of the orbit is 1/4 the area of the ellipse

$$\boxed{A = \frac{1}{4}\cdot \pi ab = 0.7854\, ab}$$