12. Interplanetary Trajectories

12.1. Introduction

The <u>patched conic method</u> is an approximate technique for treating the motion of a spacecraft in the presence of several celestial bodies as a sequence of two-body problems, with one body being the spacecraft. When the spacecraft is in close proximity to one celestial body, such as the earth, one can neglect the gravitational forces on the spacecraft due to the sun, moon and other planets and analyze the two-body earth-spacecraft problem. The region inside which this is valid is called the <u>sphere of influence</u> of the earth. Each celestial body has such a sphere of influence. If the spacecraft is not inside the sphere of influence of a planet or moon in the solar system, it is considered to be in orbit about the sun.

12.2. Sphere of Influence

Erroneous Definition

The spacecraft is within the earth's SOI if the gravitational force on the spacecraft due to the earth is larger than the gravitational force due to the sun.

$$\frac{G m_e m_v}{r_{ev}^2} > \frac{G m_s m_v}{r_{sv}^2} \qquad (12.1)$$

where the subscripts

$e \Rightarrow$ earth

$s \Rightarrow$ sun

$v \Rightarrow$ vehicle

Solving (12.1) for r_{ev}

$$r_{ev} < \left(\frac{m_e}{m_s}\right)^{1/2} r_{sv} \qquad (12.2)$$

If the spacecraft is between the earth and the sun

$$r_{ev} + r_{sv} = 1 \text{ au}$$

Eq. (12.2) becomes

$$r_{ev} < \left(\frac{m_e}{m_s}\right)^{1/2} (1 - r_{ev}) \qquad (\text{au})$$

$$r_{ev}\left[1 + \left(\frac{m_e}{m_s}\right)^{1/2}\right] < \left(\frac{m_e}{m_s}\right)^{1/2}$$

$$r_{ev} < \frac{\left(\frac{m_e}{m_s}\right)^{1/2}}{1 + \left(\frac{m_e}{m_s}\right)^{1/2}} \qquad (\text{au}) \qquad (12.3)$$

Since

$$\frac{m_e}{m_s} = \frac{5.974 \times 10^{24} \text{ kg}}{1.989 \times 10^{30} \text{ kg}} = 3.00 \times 10^{-6}$$

$$r_{ev} < 0.00173 \text{ au} \left(\frac{1.496 \times 10^{8} \text{ km}}{1 \text{ au}}\right)$$

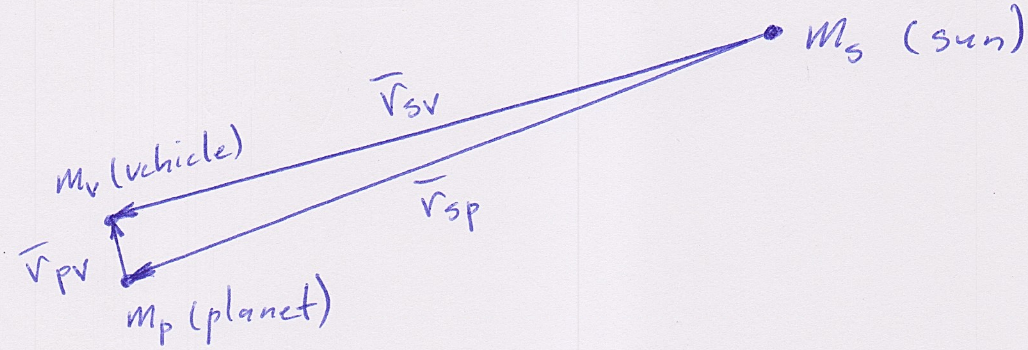
$$r_{ev} < 2.59 \times 10^{5} \text{ km}$$

The mean distance of the moon from the earth is

$$3.84 \times 10^5 \text{ km}$$

which would place the moon outside the earth's SOI as defined above. This definition of the earth's SOI is erroneous since the moon orbits the earth.

Correct Definition



Using eq. (4.5), the motion of the vehicle relative to the planet (in the presence of the sun) is described by

$$\ddot{\bar{r}}_{pv} + \frac{G(m_p + m_v)}{r_{pv}^3}\bar{r}_{pv} = -G m_s \left[\frac{\bar{r}_{sv}}{r_{sv}^3} - \frac{\bar{r}_{sp}}{r_{sp}^3} \right] \quad (12.4)$$

or

$$\ddot{\bar{r}}_{pv} - \bar{A}_p = \bar{P}_s \quad (12.5)$$

where

$\overline{A}_p$ is the gravitational acceleration due to the planet
$\overline{P}_s$ is the perturbing acceleration due to the sun.

Note that when the vehicle is close to the planet, $r_{sv} \approx r_{sp}$. Therefore $P_s << A_p$ even though $m_s >> m_p + m_v$

Using (4.5) again, the motion of the vehicle relative to the sun (in the presence of the planet) is described by

$$\ddot{\overline{r}}_{sv} + \frac{G(m_s+m_v)}{r_{sv}^3}\overline{r}_{sv} = -Gm_p\left[\frac{\overline{r}_{pv}}{r_{pv}^3} + \frac{\overline{r}_{sp}}{r_{sp}^3}\right] \quad (12.6)$$

or

$$\ddot{\overline{r}}_{sv} - \overline{A}_s = \overline{P}_p \quad (12.7)$$

where

$\overline{A}_s$ is the gravitational acceleration due to the sun
$\overline{P}_p$ is the perturbing acceleration due to the planet.

Note that when the vehicle is far from the planet, $P_p << A_s$ because $m_p << m_s$.

The sphere of influence is defined as the surface along which

$$P_p A_p = P_s A_s \qquad (12.8)$$

The spacecraft is inside the SOI of the planet if

$$P_p A_p > P_s A_s$$

The magnitude of the accelerations in (12.8) when the vehicle is in the vicinity of the planet can be estimated as follows:

$$P_p = G m_p \left| \frac{\bar{r}_{pv}}{r_{pv}^3} + \frac{\bar{r}_{sp}}{r_{sp}^3} \right| \approx G m_p \left| \frac{\bar{r}_{pv}}{r_{pv}^3} \right| = \frac{G m_p}{r_{pv}^2}$$

(since $r_{pv} << r_{sp}$)

$$P_s = G m_s \left| \frac{\bar{r}_{sv}}{r_{sv}^3} - \frac{\bar{r}_{sp}}{r_{sp}^3} \right| = G \left| \frac{\bar{r}_{pv} + \bar{r}_{sp}}{r_{sv}^3} - \frac{\bar{r}_{sp}}{r_{sp}^3} \right|$$

$$\approx G m_s \left| \frac{\bar{r}_{pv} + \bar{r}_{sp}}{r_{sp}^3} - \frac{\bar{r}_{sp}}{r_{sp}^3} \right| = G m_s \left| \frac{\bar{r}_{pv}}{r_{sp}^3} \right| = G m_s \frac{r_{pv}}{r_{sp}^3}$$

(since $r_{sv} \approx r_{sp}$)

$$A_p = \frac{G(m_p + m_v)}{r_{pv}^2} \approx \frac{G m_p}{r_{pv}^2}$$

(since $m_v \ll m_p$)

$$A_s = \frac{G(m_s + m_v)}{r_{sv}^2} \approx \frac{G m_s}{r_{sv}^2} \approx \frac{G m_s}{r_{sp}^2}$$

(since $m_v \ll m_s$) (since $r_{sv} \approx r_{sp}$)

Substituting into (12.8)

$$\left[\frac{G m_p}{r_{pv}^2}\right]\left[\frac{G m_p}{r_{pv}^2}\right] = \left[G m_s \frac{r_{pv}}{r_{sp}^3}\right]\left[\frac{G m_s}{r_{sp}^2}\right]$$

which gives

$$r_{pv} = \left(\frac{m_p}{m_s}\right)^{2/5} r_{sp}$$

or

$$\boxed{r_{SOI} = \left(\frac{m_p}{m_s}\right)^{2/5} r_{sp}} \qquad (12.9)$$

For earth

$$r_{SOI} = \left(\frac{5.974 \times 10^{24} \text{ kg}}{1.989 \times 10^{30} \text{ kg}} \right)^{2/5} (1.496 \times 10^{8} \text{ km}) = 924{,}700 \text{ km}$$

$$= \frac{924{,}700}{6378} = 145 \text{ earth radii}$$

Distance to the moon is 3.844×10^{5} km $= 60$ earth radii

The moon is within the earth's SOI.

12.3. The Patched Conic Method

EXAMPLE

Consider an earth-Venus Hohmann transfer using the patched-conic technique

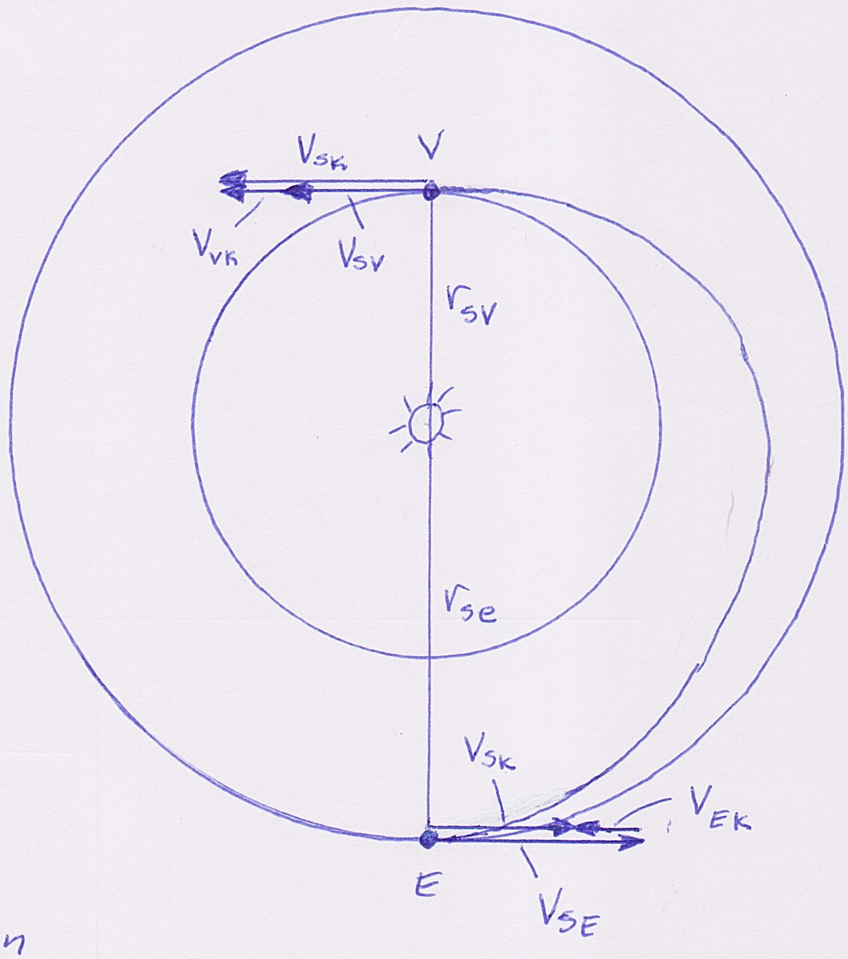
a) Calculate the actual ΔV magnitude to tangentially depart a posigrade 200 km altitude circular parking orbit about the earth and terminate at a point 500 km from the surface of Venus.

b) Determine the required angle between the departure point in the earth parking orbit and the sun-earth line.

c) Determine the required aiming radius at Venus in units of Venus radii.

Assume all orbits are coplanar

a)



S = sun

E = earth

V = Venus

K = spacecraft

For earth-venus Hohmann transfer (neglecting mass of the planets)

$$e = \frac{r_{SE} - r_{SV}}{r_{SE} + r_{SV}} = \frac{1 - 0.7233}{1 + 0.7233} = 0.1606$$

At earth

$$\Delta V_E = |V_{EK}| = V_{SE} - V_{SK} = V_{SE}\left[1-\sqrt{1-e}\right]$$

$$= \sqrt{\frac{\mu_{sun}}{r_{SE}}}\left[1-\sqrt{1-e}\right]$$

$$= \sqrt{\frac{1.327\times 10^{11}}{1.496\times 10^{8}}}\left[1-\sqrt{1-0.1606}\right] = 2.496\ \frac{km}{sec}$$

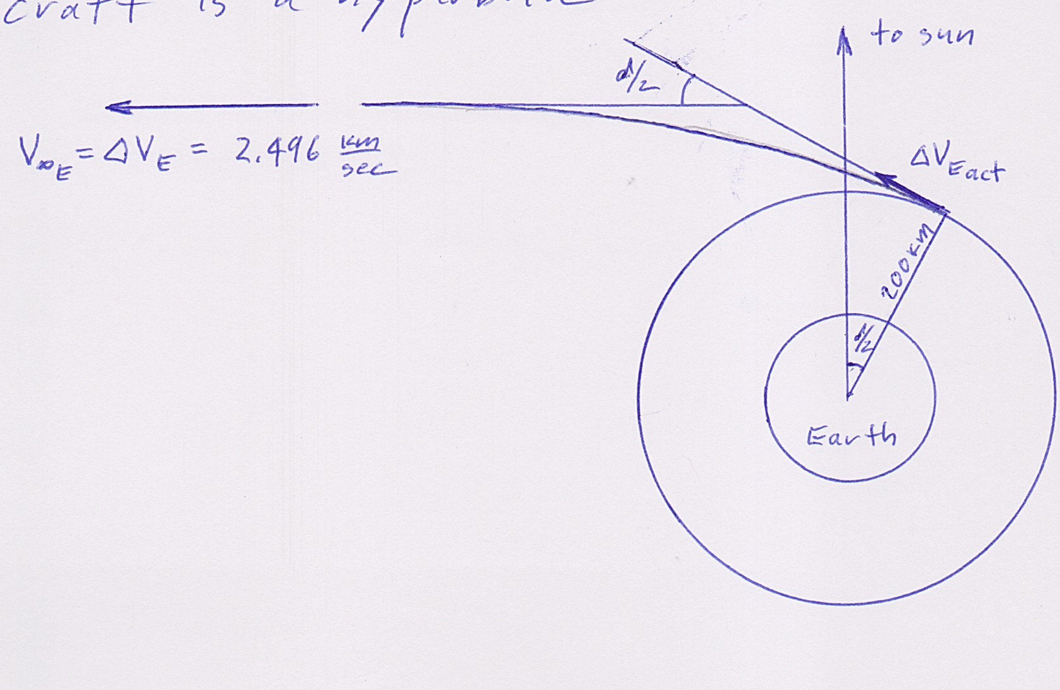
At Venus

$$\Delta V_V = |V_{VK}| = V_{SK} - V_{SV} = V_{SV}\left[\sqrt{1+e}-1\right]$$

$$= \sqrt{\frac{\mu_{sun}}{r_{SV}}}\left[\sqrt{1+e}-1\right]$$

$$= \sqrt{\frac{1.327\times 10^{11}}{(0.7233)(1.496\times 10^{8})}}\left[\sqrt{1+0.1606}-1\right] = 2.707\ \frac{km}{sec}$$

Relative to the earth, the escape trajectory of the craft is a hyperbola



For a hyperbola

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a}$$

$$\frac{V_{PE}^2}{2} - \frac{\mu_E}{r_{PE}} = \frac{V_{\infty E}^2}{2}$$

P ⇒ perigee
E ⇒ with respect to earth

$$V_{PE} = \left[\frac{2\mu_E}{r_{PE}} + V_{\infty E}^2\right]^{1/2} = \left[\frac{2(3.986\times10^5)}{6378+200} + (2.496)^2\right]^{1/2}$$

$$V_{PE} = 11.288 \text{ km/sec}$$

Circular parking orbit speed at 200 km altitude

$$V_{CE} = \sqrt{\frac{\mu_E}{r_{PE}}} = \sqrt{\frac{3.986\times10^5}{6378+200}} = 7.784 \text{ km/sec}$$

$$\Delta V_{E act} = V_{PE} - V_{CE} = 11.288 - 7.784 = \underline{\underline{3.504 \frac{\text{km}}{\text{sec}}}}$$

Significantly different from massless planet result of 2.496 km/sec.

b) <u>At earth</u>

Using formulas derived in HW 4 prob. 2

Velocity for circular orbit at earth's surface

$$V_s = \sqrt{\frac{\mu_E}{r_s}} = \sqrt{\frac{3.986 \times 10^5}{6378}} = 7.905 \text{ km/sec}$$

$$\psi = \left(\frac{V_\infty}{V_s}\right)^2 \left(\frac{r_p}{r_s}\right) = \left(\frac{2.496}{7.905}\right)^2 \left(\frac{6378+200}{6378}\right) = 0.1028$$

Eccentricity of hyperbolic trajectory inside earth's SOI

$$e = 1 + \psi = 1 + 0.1028 = 1.1028$$

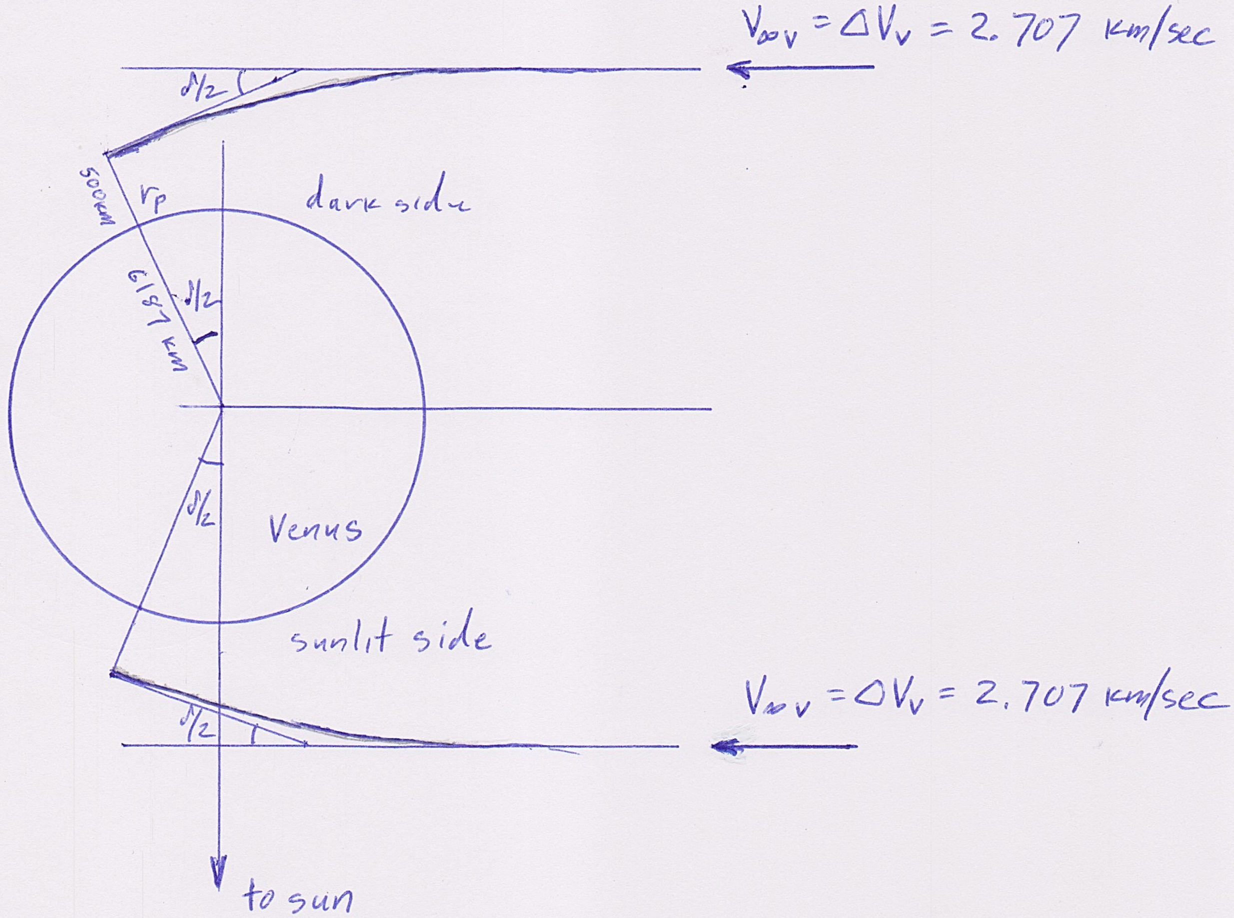
Angle between departure point r_p and earth-sun line

$$\sin\frac{\delta}{2} = \frac{1}{1+\psi} = \frac{1}{1+0.1028} = 0.9068$$

$$\boxed{\frac{\delta}{2} = 65.1^\circ}$$

c) At Venus

Spacecraft can approach Venus either from the sunlit side or the dark side



Velocity for circular orbit on surface of Venus

$$V_s = \sqrt{\frac{\mu_v}{r_s}} = \sqrt{\frac{(0.8149)(3.986 \times 10^5)}{6187}} = 7.246 \ \frac{\text{km}}{\text{sec}}$$

$$\psi = \left(\frac{V_\infty}{V_s}\right)^2 \left(\frac{r_p}{r_s}\right) = \left(\frac{2.707}{7.246}\right)^2 \left(\frac{6187 + 500}{6187}\right) = 0.1508$$

Eccentricity of hyperbolic trajectory inside Venus' SOI

$$e = 1 + \psi = 1 + 0.1508 = 1.1508$$

$$\sin\frac{\delta}{2} = \frac{1}{1+\psi} = \frac{1}{1+0.1508} = 0.8689$$

$$\frac{\delta}{2} = 60.3^{\circ}$$

Aiming radius at Venus

$$\frac{\Delta}{r_s} = \frac{r_p}{r_s}\sqrt{1+\frac{2}{\psi}} = \frac{500+6187}{6187}\sqrt{1+\frac{2}{0.1508}}$$

$$\boxed{\frac{\Delta}{r_s} = 4.082}$$

$$\Delta = 4.082(6187) = 25{,}250 \text{ km}$$