ME 57200 Aerodynamic Design

Lecture #9: Inviscid Incompressible Flow

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Midterm Exam

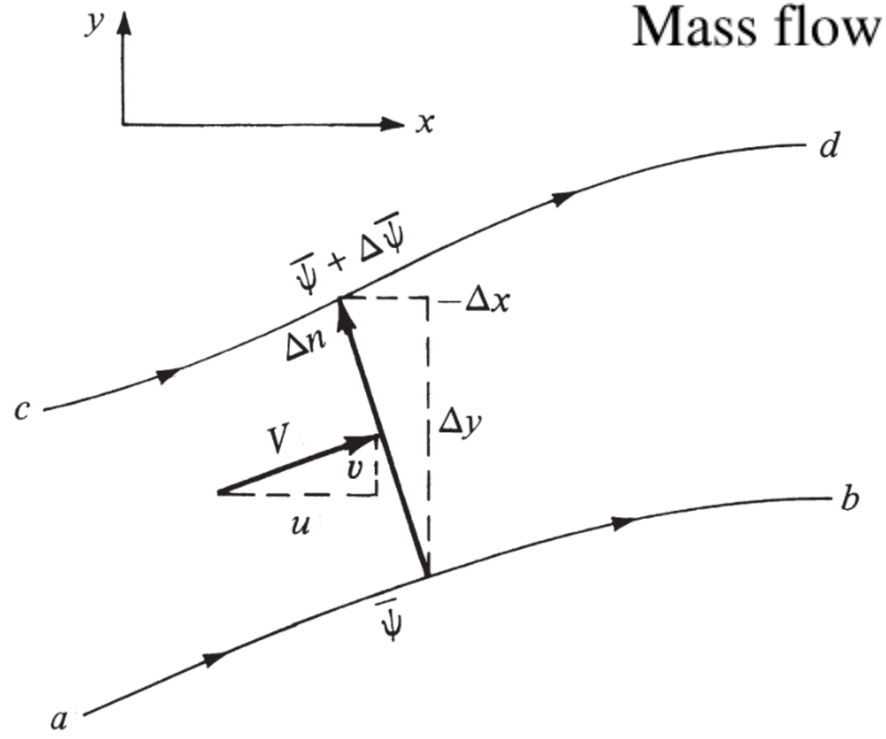
- Tuesday, 3/12, from 09:30 to 10:45 am at Shepard S-308.

- The exam is open-book and open-notes.

- 5 True/False Questions: 10 pt
- 4 Math-based Problems: 40 pt
- Total: 50 pt

Stream Function

Stream Function: $\bar{\Psi} = constant$ designates a streamline, and $\Delta\bar{\Psi}$ equals to the mass flow rate between streamlines.



$$\text{Mass flow} = \Delta\bar{\psi} = \rho V \Delta n = \rho u\, \Delta y + \rho v(-\Delta x)$$

$$d\bar{\psi} = \rho u\, dy - \rho v\, dx$$

$$d\bar{\psi} = \frac{\partial\bar{\psi}}{\partial x}dx + \frac{\partial\bar{\psi}}{\partial y}dy$$

$$\boxed{\rho u = \frac{\partial\bar{\psi}}{\partial y}} \qquad \boxed{\rho v = -\frac{\partial\bar{\psi}}{\partial x}}$$

Stream Function

Stream Function: $\bar{\Psi} = constant$ designates a streamline, and $\Delta\bar{\Psi}$ equals to the mass flow rate between streamlines.

$$\boxed{\rho u = \frac{\partial \overline{\psi}}{\partial y}}$$

$$\boxed{\rho v = -\frac{\partial \overline{\psi}}{\partial x}}$$

$$\boxed{\rho V_r = \frac{1}{r}\frac{\partial \overline{\psi}}{\partial \theta}}$$

$$\boxed{\rho V_\theta = -\frac{\partial \overline{\psi}}{\partial r}}$$

For incompressible flow: $\psi \equiv \overline{\psi}/\rho$

$$\boxed{u = \frac{\partial \psi}{\partial y}}$$

$$\boxed{v = -\frac{\partial \psi}{\partial x}}$$

$$\boxed{V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}}$$

$$\boxed{V_\theta = -\frac{\partial \psi}{\partial r}}$$

Velocity Potential

Velocity Potential: For an irrotational flow, there exists a scalar function φ such that the velocity is given by the gradient of φ. *We denote φ as the **velocity potential**.*

$$u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

Cartesion

$$\boxed{u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \quad w = \frac{\partial \phi}{\partial z}}$$

Cylindrical

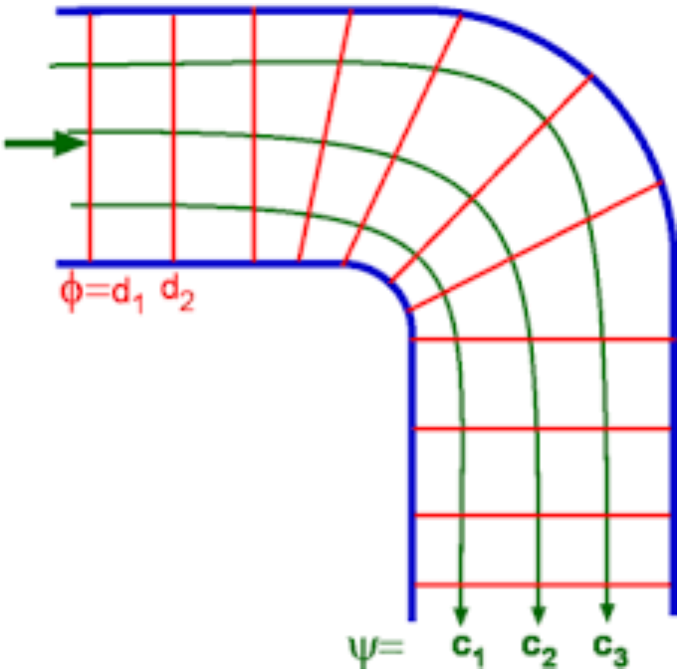
$$\boxed{V_r = \frac{\partial \phi}{\partial r} \quad V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad V_z = \frac{\partial \phi}{\partial z}}$$

Spherical

$$\boxed{V_r = \frac{\partial \phi}{\partial r} \quad V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad V_\Phi = \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \Phi}}$$

Stream Function & Velocity Potential

- A line of constant $\overline{\Psi}$: *Streamline*
- A line of constant φ: *Equipotential Line*



The differential of *ψ* along a streamline is zero.

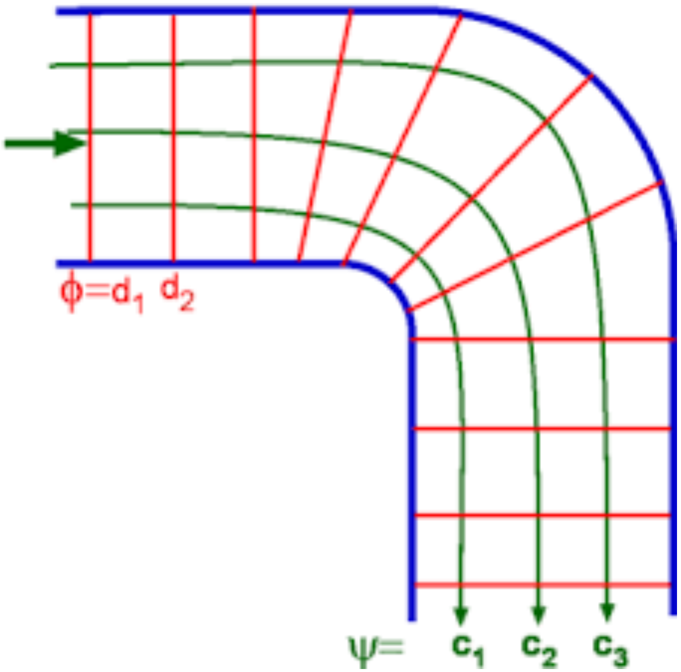
$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = 0$$

$$d\psi = -v\,dx + u\,dy = 0$$

$$\left(\frac{dy}{dx}\right)_{\psi=\text{const}} = \frac{v}{u}$$

Stream Function & Velocity Potential

- A line of constant $\bar{\Psi}$: *Streamline*
- A line of constant φ: *Equipotential Line*



The differential of φ along an equipotential line is zero.

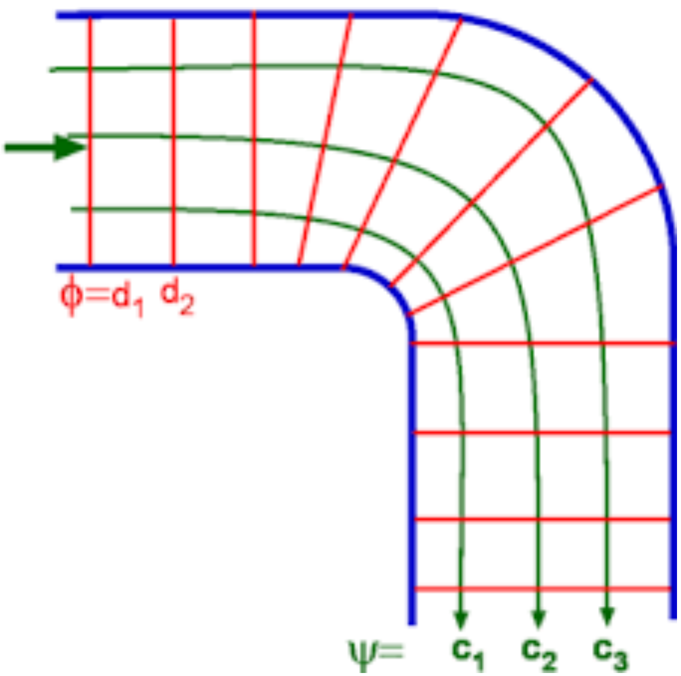
$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0$$

$$d\phi = u\,dx + v\,dy = 0$$

$$\left(\frac{dy}{dx}\right)_{\phi=\text{const}} = -\frac{u}{v}$$

Stream Function & Velocity Potential

- A line of constant $\overline{\Psi}$: *Streamline*
- A line of constant φ: *Equipotential Line*



$$\left(\frac{dy}{dx}\right)_{\psi=\text{const}} = -\frac{1}{(dy/dx)_{\phi=\text{const}}}$$

The slope of a $\Psi = constant$ line is the negative reciprocal of the slop of a $\varphi = constant$ line.

Streamlines and equipotential lines are mutually perpendicular

Stream Function & Velocity Potential

Similarity between stream function and velocity potential

- *They are both related to velocity by taking the derivative*

Differences between stream function and velocity potential

- *The flow field velocities are obtained by differentiating φ in the same direction as the velocities, whereas ψ is differentiated normal to the velocity direction.*
- *The velocity potential is defined for irrotational flow only. The stream function can be used in either rotational or irrotational flows*
- *The velocity potential can be applied to 3D flows, the stream function is defined for 2D flow only.*

Inviscid, Incompressible Flow

Bernoulli's Equation:

$$p + \frac{1}{2}\rho V^2 = \text{constant}$$

x component of the momentum equation, for inviscid flow with no body force:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x}$$

$$\Rightarrow \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}$$

For steady flow: $\frac{\partial u}{\partial t} = 0$

$$\Rightarrow u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Inviscid, Incompressible Flow

$$\Rightarrow u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx + w\frac{\partial u}{\partial z}dx = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx$$

Consider the flow along a streamline in 3-D space

$$\begin{cases} v\,dx - u\,dy = 0 \\ u\,dz - w\,dx = 0 \end{cases}$$

$$\Rightarrow \boxed{u\cdot\frac{\partial u}{\partial x}dx + u\cdot\frac{\partial u}{\partial y}dy + u\frac{\partial u}{\partial z}dz} = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

Inviscid, Incompressible Flow

$$\Rightarrow \quad u\,du = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx$$

$$\Rightarrow \begin{cases} \frac{1}{2}d(u^2) = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \\ \frac{1}{2}d(v^2) = -\frac{1}{\rho}\frac{\partial p}{\partial y}dy \\ \frac{1}{2}d(w^2) = -\frac{1}{\rho}\frac{\partial p}{\partial z}dz \end{cases}$$

$$\Rightarrow \frac{1}{2}d(u^2+v^2+w^2) = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right)$$

Inviscid, Incompressible Flow

$$\Rightarrow \frac{1}{2} d(V^2) = -\frac{1}{\rho}(dp)$$

$$\Rightarrow \underline{dp = -\rho V dV}$$ "Euler's Equation".

If the flow is incompressible: ρ = constant

$$\int_{p_1}^{p_2} dp = -\rho \int_{V_1}^{V_2} V dV$$

$$p_2 - p_1 = -\rho\left(\frac{V_2^2}{2} - \frac{V_1^2}{2}\right)$$

Inviscid, Incompressible Flow

$$\Rightarrow P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_2^2 = \text{Constant}$$

"Bernoulli's Equation"

For both rotational and irrotational flows.

Laplace's Equation

Continuity Equation : $\nabla \cdot \vec{V} = 0$

"incompressible"

For an irrotational flow : $\vec{V} = \nabla \phi$

$$\Rightarrow \quad \nabla \cdot (\nabla \phi) = 0 \quad \Rightarrow \quad \nabla^2 \phi = 0$$

Laplace's Equation

$$\phi = \phi(x, y, z)$$

$$\Rightarrow \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Laplace's Equation

For 2-D incompressible flow : $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)$$

$$= \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

Laplace's Equation

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad (\nabla \times \vec{V} = 0)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

In-Class Quiz