ME 57200 Aerodynamic Design

Lecture #7: Basic Concepts in Aerodynamics

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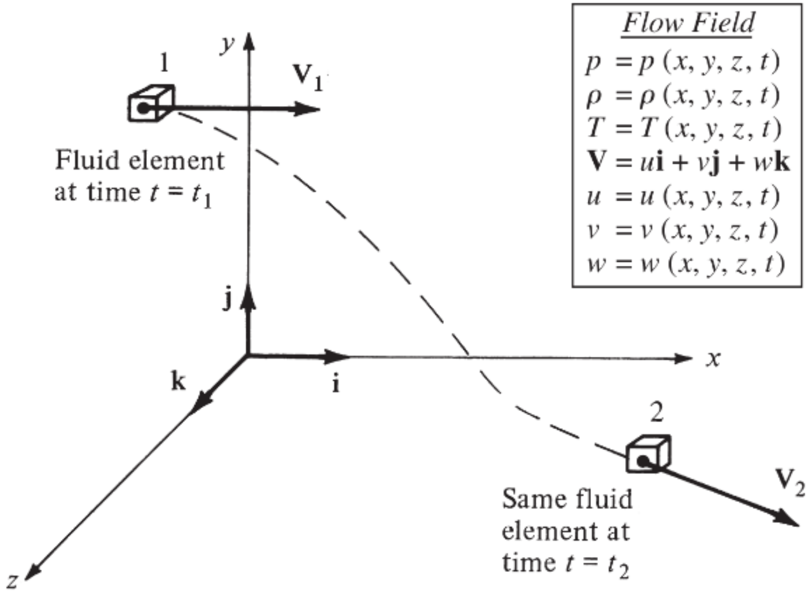
Substantial Derivative



$$\rho_1 = \rho(x_1, y_1, z_1, t_1)$$

$$\rho_2 = \rho(x_2, y_2, z_2, t_2)$$

Since $\rho = \rho(x, y, z, t)$, we can expand this function in a Taylor series about point 1 as follows:

$$\rho_2 = \rho_1 + \left(\frac{\partial \rho}{\partial x}\right)_1 (x_2 - x_1) + \left(\frac{\partial \rho}{\partial y}\right)_1 (y_2 - y_1) + \left(\frac{\partial \rho}{\partial z}\right)_1 (z_2 - z_1)$$
$$+ \left(\frac{\partial \rho}{\partial t}\right)_1 (t_2 - t_1) + \text{higher-order terms}$$

Substantial Derivative

$$\rho_2 = \rho_1 + \left(\frac{\partial \rho}{\partial x}\right)_1 (x_2 - x_1) + \left(\frac{\partial \rho}{\partial y}\right)_1 (y_2 - y_1) + \left(\frac{\partial \rho}{\partial z}\right)_1 (z_2 - z_1) + \left(\frac{\partial \rho}{\partial t}\right)_1 (t_2 - t_1) + \text{higher-order terms}$$

Dividing by $t_2 - t_1$, and ignoring the higher-order terms, we have

$$\frac{\rho_2 - \rho_1}{t_2 - t_1} = \left(\frac{\partial \rho}{\partial x}\right)_1 \frac{x_2 - x_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial y}\right)_1 \left(\frac{y_2 - y_1}{t_2 - t_1}\right) + \left(\frac{\partial \rho}{\partial z}\right)_1 \frac{z_2 - z_1}{t_2 - t_1} + \left(\frac{\partial \rho}{\partial t}\right)_1$$

Average time rate of change in density of the fluid element as it moves from point 1 to point 2

In the limit, as t_2 approaches t_1, this term becomes

$$\lim_{t_2 \to t_1} \frac{\rho_2 - \rho_1}{t_2 - t_1} = \frac{D\rho}{Dt}$$

Instantaneous time rate of change in density of the fluid element as it moves through point 1

Substantial Derivative

$$\frac{\rho_2-\rho_1}{t_2-t_1}=\left(\frac{\partial\rho}{\partial x}\right)_1\frac{x_2-x_1}{t_2-t_1}+\left(\frac{\partial\rho}{\partial y}\right)_1\left(\frac{y_2-y_1}{t_2-t_1}\right)+\left(\frac{\partial\rho}{\partial z}\right)_1\frac{z_2-z_1}{t_2-t_1}+\left(\frac{\partial\rho}{\partial t}\right)_1$$

$$\lim_{t_2\to t_1}\frac{\rho_2-\rho_1}{t_2-t_1}=\frac{D\rho}{Dt}$$

What is the difference between Dρ/Dt and ∂ρ/∂t?

- *Dρ/Dt* is the time rate of change of density of a given fluid element as it moves through space.
- *∂ρ/∂t* is the time rate of change of density at a fixed point.

$$\lim_{t_2\to t_1}\frac{x_2-x_1}{t_2-t_1}\equiv u$$

$$\lim_{t_2\to t_1}\frac{y_2-y_1}{t_2-t_1}\equiv v$$

$$\lim_{t_2\to t_1}\frac{z_2-z_1}{t_2-t_1}\equiv w$$

$$\Rightarrow \quad \boxed{\frac{D\rho}{Dt}=u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}+\frac{\partial\rho}{\partial t}}$$

Substantial Derivative

The expression for the substantial derivative in Cartesian coordinates

$$\boxed{\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}}$$

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \qquad V = u\mathbf{i} + v\mathbf{j} + w\mathbf{k},$$

$$\boxed{\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla)}$$

The substantial derivative is physically the time rate of change following a moving fluid element.

$$\frac{DT}{Dt} \equiv \underbrace{\frac{\partial T}{\partial t}}_{\substack{\text{local}\\ \text{derivative}}} + \underbrace{(\mathbf{V} \cdot \nabla)T}_{\substack{\text{convective}\\ \text{derivative}}} \equiv \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

Substantial Derivative

$$\frac{DT}{Dt} \equiv \underbrace{\frac{\partial T}{\partial t}}_{\substack{\text{local} \\ \text{derivative}}} + \underbrace{(\mathbf{V} \cdot \nabla)T}_{\substack{\text{convective} \\ \text{derivative}}} \equiv \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

- **Local derivative**: physically the time rate of change at a fixed point
- **Convective derivative**: physically the time rate of change due to the movement of the fluid element from one location to another in the flow field where the flow properties are spatially different.

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho} + \rho \nabla \cdot \mathbf{V} = 0$$

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0}$$

- **<u>the continuity equation written in terms of the substantial derivative.</u>**

Momentum Equation

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{V}) = -\frac{\partial p}{\partial x} + \rho f_x + (\mathscr{F}_x)_{\text{viscous}}$$

$$\frac{\partial(\rho u)}{\partial t} = \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t}$$

$$\nabla \cdot (\rho u \mathbf{V}) \equiv \nabla \cdot [u(\rho \mathbf{V})] = u \nabla \cdot (\rho \mathbf{V}) + (\rho \mathbf{V}) \cdot \nabla u$$

$$\Downarrow$$

$$\rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t} + u \nabla \cdot (\rho \mathbf{V}) + (\rho \mathbf{V}) \cdot \nabla u = -\frac{\partial p}{\partial x} + \rho f_x + (\mathscr{F}_x)_{\text{viscous}}$$

$$\rho \frac{\partial u}{\partial t} + u \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) \right] + (\rho \mathbf{V}) \cdot \nabla u = -\frac{\partial p}{\partial x} + \rho f_x + (\mathscr{F}_x)_{\text{viscous}}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

- <u>**the continuity equation**</u>

$$\boxed{\rho \frac{\partial u}{\partial t} + \rho \mathbf{V} \cdot \nabla u = -\frac{\partial p}{\partial x} + \rho f_x + (\mathscr{F}_x)_{\text{viscous}}}$$

Momentum Equation

$$\boxed{\rho \frac{\partial u}{\partial t} + \rho \mathbf{V} \cdot \nabla u = -\frac{\partial p}{\partial x} + \rho f_x + (\mathscr{F}_x)_{\text{viscous}}}$$

$$\rho \left(\frac{\partial u}{\partial t} + \mathbf{V} \cdot \nabla u \right) = -\frac{\partial p}{\partial x} + \rho f_x + (\mathscr{F}_x)_{\text{viscous}}$$

$$\boxed{\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x + (\mathscr{F}_x)_{\text{viscous}}}$$

$$\boxed{\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y + (\mathscr{F}_y)_{\text{viscous}}}$$

$$\boxed{\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z + (\mathscr{F}_z)_{\text{viscous}}}$$

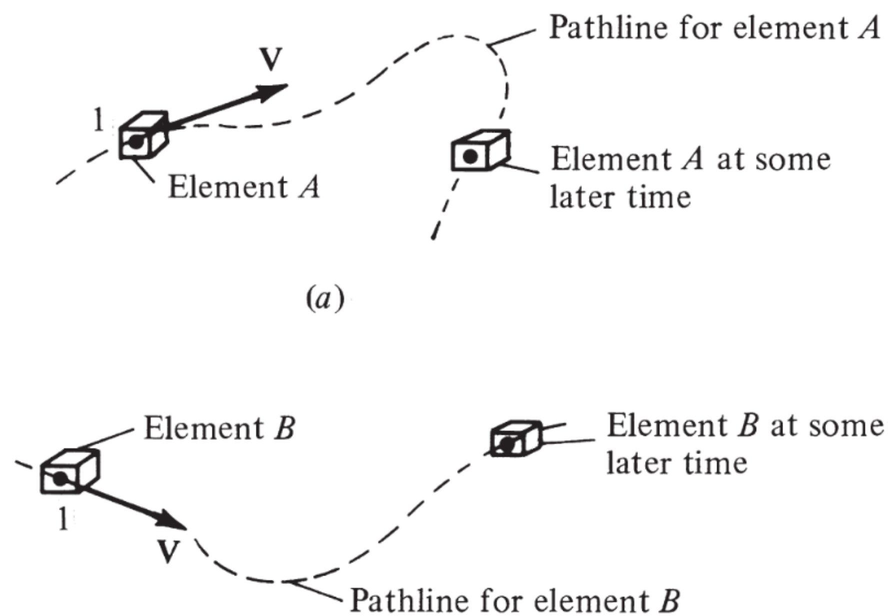
- **the momentum equation written in terms of the substantial derivative.**

Energy Equation

$$\boxed{\rho \frac{D(e + V^2/2)}{Dt} = \rho \dot{q} - \nabla \cdot (p\mathbf{V}) + \rho(\mathbf{f} \cdot \mathbf{V}) + \dot{Q}'_{\text{viscous}} + \dot{W}'_{\text{viscous}}}$$

Pathlines and Streamlines

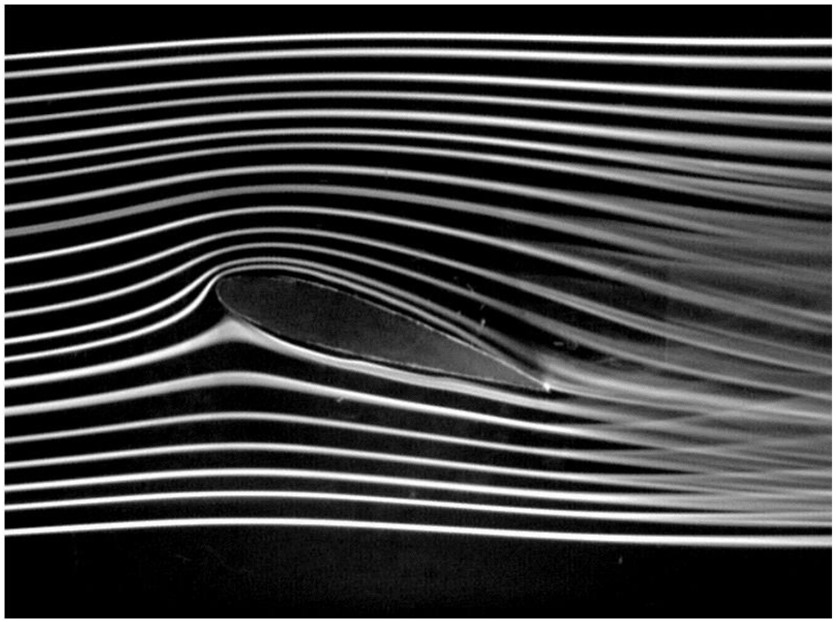
- **Pathlines:** a time-exposure photograph of a given fluid element

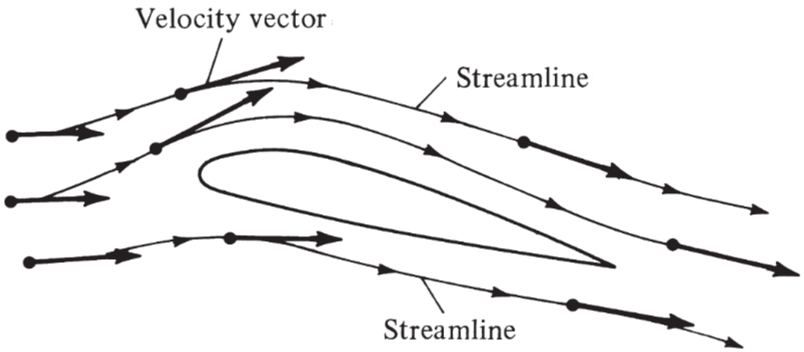


For <u>unsteady flow</u>, the pathlines for different fluid elements passing through the same point are not the same.

Pathlines and Streamlines

- **<u>Streamlines:</u>** a single frame of a motion picture of the flow



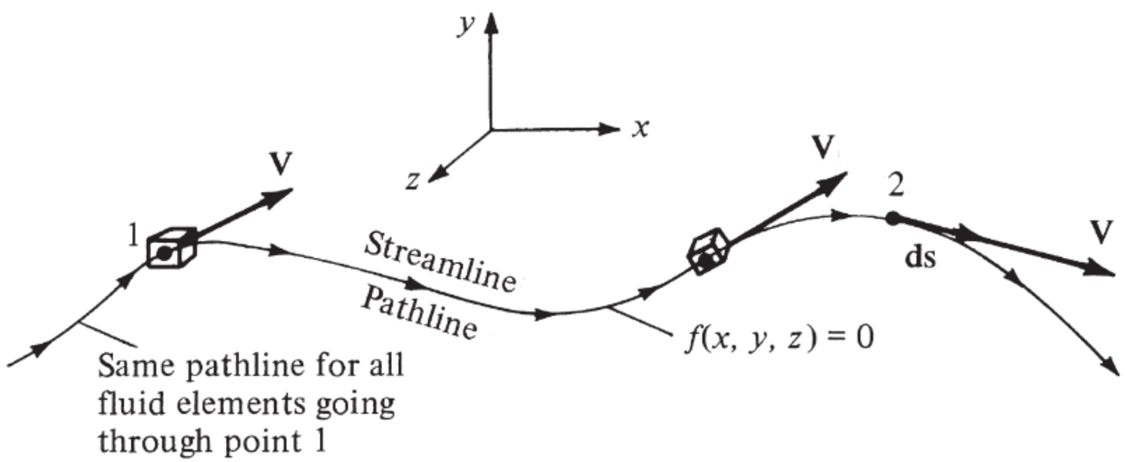


- *A streamline is a curve whose tangent at any point is in the direction of the velocity vector at that point.*
- *For <u>unsteady flow</u>, the streamline pattern is different at different times*

Pathlines and Streamlines

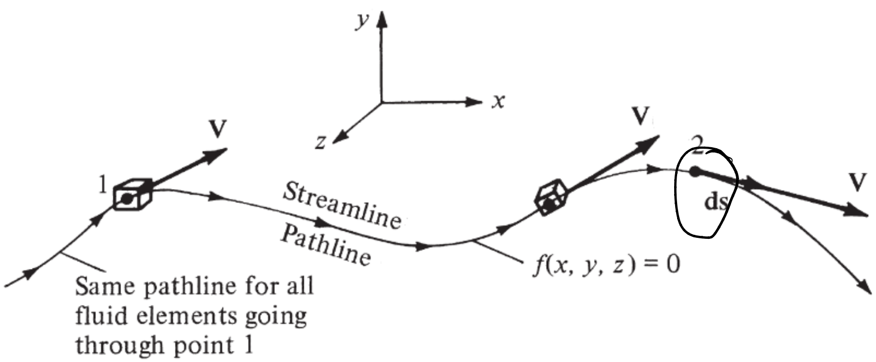
- *What is the relationship between pathlines and streamlines in steady-state flow?*

  - *The magnitude and direction of the velocity vectors at all points are fixed, invariant with time.*
  - *The pathlines and streamlines are identical.*



Pathlines and Streamlines

- *How to obtain the mathematical equation for a streamline?*



$\vec{ds}$, $\vec{V}$, parallel

$$\vec{ds}\times\vec{V} = |\vec{ds}|\,|\vec{V}|\sin\theta\,\vec{e} = 0$$

$$\vec{ds} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k}$$

$$\Rightarrow \begin{cases} w\,dy - v\,dz = 0 \\ u\,dz - w\,dx = 0 \\ v\,dx - u\,dy = 0 \end{cases}$$

For 2-D

$$v\,dx = u\,dy$$

$$\boxed{\frac{dy}{dx} = \frac{v}{u}}$$

$$\vec{ds}\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ dx & dy & dz \\ u & v & w \end{vmatrix} = \vec{i}\,(w\,dy - v\,dz) + \vec{j}\,(u\,dz - w\,dx) + \vec{k}\,(v\,dx - u\,dy) = 0$$

In-Class Example

Consider a velocity field: $u = \frac{y}{x^2+y^2}$, $v = -\frac{x}{x^2+y^2}$

Calculate the equation of the streamline passing through (0, 5)

Solution:

$$\frac{dy}{dx} = \frac{v}{u} = \frac{\left(-\frac{x}{x^2+y^2}\right)}{\left(\frac{y}{x^2+y^2}\right)} = -\frac{x}{y}$$

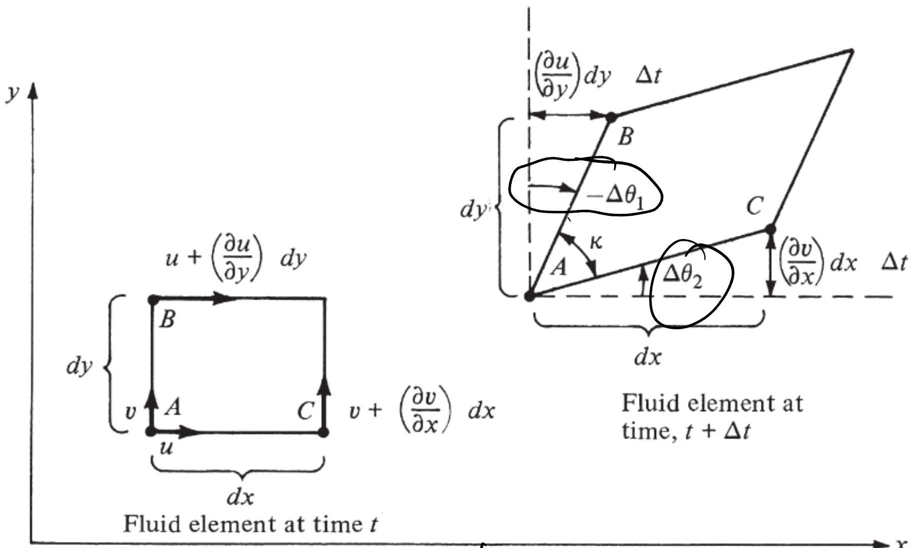
$$\Rightarrow y\,dy = -x\,dx$$

$$\Rightarrow \int y\,dy = \int -x\,dx \Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + \text{Constant}$$

$$\Rightarrow y^2 = -x^2 + C \qquad (0, 5) \Rightarrow C = 25$$

$$\Rightarrow \boxed{y^2 = -x^2 + 25}$$

Angular Velocity and Vorticity



At "t", "A" — "v"

"C" — "$v + \frac{\partial v}{\partial x} dx$"

At "$t + \Delta t$", "A" — "$v \cdot \Delta t$"

"C" — "$(v + \frac{\partial v}{\partial x} dx)\Delta t$"

Relative displacement between "A" and "C"

$$\left(v + \frac{\partial v}{\partial x} dx\right)\Delta t - v\Delta t$$

$$= \frac{\partial v}{\partial x} \cdot dx \cdot \Delta t$$

$$\tan \Delta\theta_2 = \frac{\frac{\partial v}{\partial x} \cdot dx \, \Delta t}{dx}$$

$$= \frac{\partial v}{\partial x} \Delta t$$

$$\Delta\theta_2 = \frac{\partial v}{\partial x} \cdot \Delta t$$

Angular Velocity and Vorticity

Similarly: $\Delta\theta_1 = -\frac{\partial u}{\partial y}\Delta t$

$$\Rightarrow \begin{cases} \frac{d\theta_1}{dt} = -\frac{\partial u}{\partial y} \\ \frac{d\theta_2}{dt} = \frac{\partial v}{\partial x} \end{cases}$$

Angular velocity: $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

3-D case
$$\begin{cases} \vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k} \\ \vec{\omega} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}\right] \end{cases}$$

Angular Velocity and Vorticity

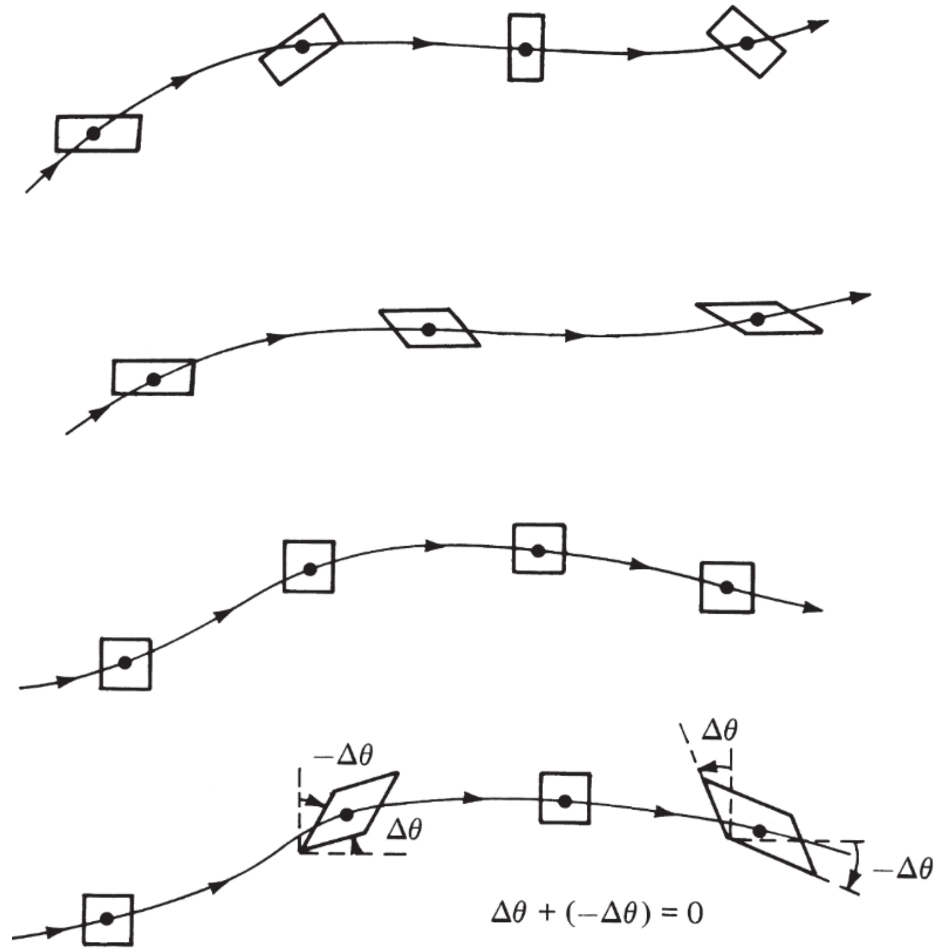
Vorticity $\quad \vec{\xi} = 2\vec{\omega}$

$$\Rightarrow \boxed{\vec{\xi} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}}$$

$$\Rightarrow \quad \vec{\xi} = \nabla \times \vec{V}$$

$$\begin{cases} \text{If } \nabla \times \vec{V} \neq 0 \text{ at every point in a flow — Rotational} \\ \text{If } \nabla \times \vec{V} = 0 \text{ at every point in a flow — Irrotational} \end{cases}$$

Angular Velocity and Vorticity



In-Class Quiz