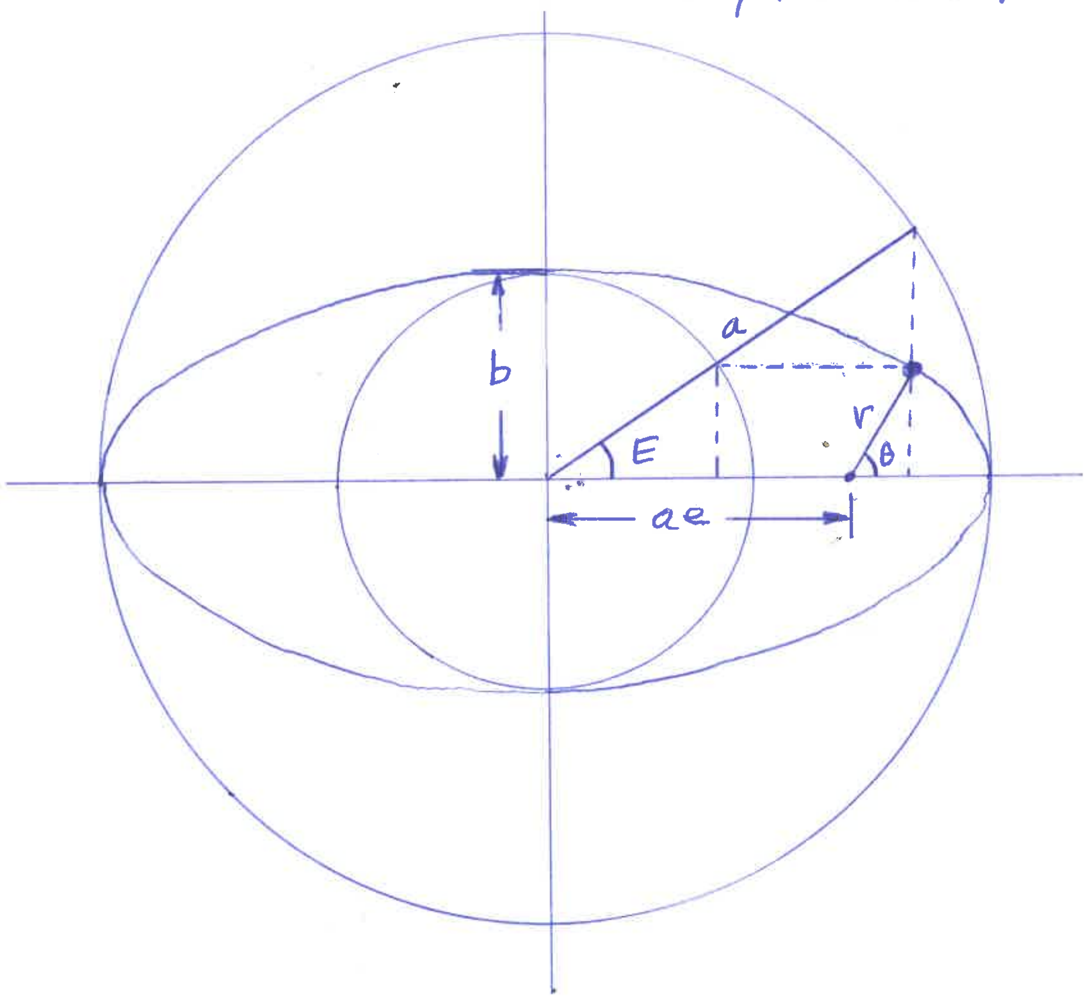
7. Location of a Satellite in Space

7.1 Eccentric Anomaly and Mean Anomaly

Consider a satellite in an elliptic orbit



θ is called the true anomaly

E is called the eccentric anomaly

From the figure

$$a \cos E = ae + r\cos\theta \qquad (7.1)$$

$$b \sin E = r \sin\theta \qquad (7.2)$$

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \qquad (7.3)$$

$$b = a(1-e^2)^{1/2} \qquad (7.4)$$

Substitute (7.3) & (7.4) into (7.1) & (7.2)

$$\cos E = \frac{e+\cos\theta}{1+e\cos\theta} \qquad (7.5a)$$

$$\sin E = \frac{(1-e^2)^{1/2}\sin\theta}{1+e\cos\theta} \qquad (7.5b)$$

From (7.5a, b)

$$\cos\theta = \frac{\cos E - e}{1-e\cos E} \qquad (7.6a)$$

$$\sin\theta = \frac{(1-e^2)^{1/2}\sin E}{1-e\cos E} \qquad (7.6b)$$

substitute (7.6a) into (7.3)

$$r = \frac{a(1-e^2)}{1+e\,\frac{\cos E - e}{1-e\cos E}} = \frac{a(1-e^2)(1-e\cos E)}{1-\cancel{e\cos E}+\cancel{e\cos E}-e^2}$$

$$r = a(1 - e\cos E) \qquad (7.7)$$

Using the identities

$$\sin^2 \tfrac{1}{2}\theta = \frac{1-\cos\theta}{2}$$

$$\cos^2 \tfrac{1}{2}\theta = \frac{1+\cos\theta}{2}$$

with (7.6a) and (7.7) gives

$$\sin^2 \tfrac{1}{2}\theta = \frac{a(1+e)}{r} \sin^2 \tfrac{1}{2}E \qquad (7.8a)$$

$$\cos^2 \tfrac{1}{2}\theta = \frac{a(1-e)}{r} \cos^2 \tfrac{1}{2}E \qquad (7.8b)$$

Divide (7.8a) by (7.8b)

$$\boxed{\tan \tfrac{1}{2}\theta = \sqrt{\frac{1+e}{1-e}}\ \tan\tfrac{1}{2}E} \qquad (7.9)$$

Eq (7.9) is useful because $\frac{1}{2}\theta$ and $\frac{1}{2}E$ are always in the same quadrant.

The area swept out by r

$$A = \frac{1}{2}\int_0^\theta r^2\, d\theta = \frac{1}{2}\int_0^\theta \left[\frac{a(1-e^2)}{1+e\cos\theta}\right]^2 d\theta \qquad (7.10)$$

The integral is difficult to evaluate. Write in terms of E.

From (7.6 a)

$$-\sin\theta\, d\theta = \frac{(1-e\cos E)(-\sin E)-(\cos E - e)(e\sin E)}{(1-e\cos E)^2}\, dE$$

Using (7.6 b)

$$-\frac{(1-e^2)^{1/2}\sin E}{1-e\cos E}\, d\theta = -\frac{(1-\cancel{e\cos E}+\cancel{e\cos E}-e^2)\sin E}{(1-e\cos E)^2}\, dE$$

$$d\theta = \frac{(1-e^2)^{1/2}}{1-e\cos E}\, dE = \frac{a(1-e^2)^{1/2}}{a(1-e\cos E)}\, dE = \frac{b}{r}\, dE$$

$$\Rightarrow \quad r\, d\theta = b\, dE$$

Thus the area swept out by r

$$A = \frac{1}{2}\int_0^\theta r^2\, d\theta = \frac{1}{2}\int_0^E r\, b\, dE = \frac{b}{2}\int_0^E r\, dE$$

Using (7.7)

$$A = \frac{ab}{2} \int_0^E (1 - e \cos E)\, dE$$

$$\boxed{A = \frac{ab}{2} (E - e \sin E)} \qquad (7.11)$$

Define

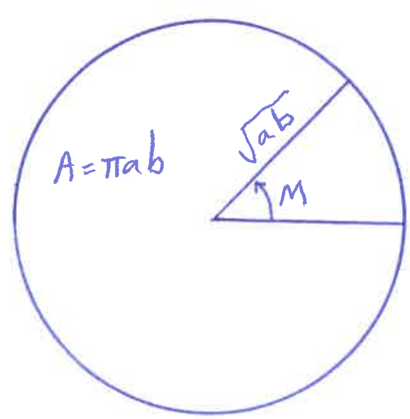
$$\boxed{M = E - e \sin E} \qquad (7.12)$$

M is called the <u>mean anomaly</u> and (7.12) is known as <u>Kepler's equation.</u>

Eq. (7.11) may be written as

$$A = \pi a b \frac{M}{2\pi}$$

Thus M is the central angle swept out on a circle (not an orbit) having the same total area is the elliptic orbit.



The period of one elliptic orbit (and one sweep around the circle) is

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

Since $\frac{dA}{dt}$ = constant (Kepler's 2nd law), the mean angular velocity

$$n \equiv \frac{dM}{dt} = \text{const.} = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}} \qquad (7.13)$$

Assuming $t=0$ at pericenter passage where $M=0$, (7.13) may be integrated to give

$$\boxed{M = nt} \quad \text{or} \quad \boxed{t = \frac{M}{n}} \qquad (7.14)$$

<u>Two types of Problems</u>

<u>Case 1</u> Given θ, find t

(7.9) gives E
(7.12) gives M
(7.14) gives t

<u>Case 2</u> Given t, find θ

(7.14) gives M

(7.12) must be solved numerically for E

(7.9) gives θ

Eq (7.14) may be applied at 2 points along an orbit.

$$M_1 = n t_1 \qquad t_1 = \frac{M_1}{n}$$

$$M_2 = n t_2 \qquad t_2 = \frac{M_2}{n}$$

Subtracting

$$\boxed{(M_2 - M_1) = n(t_2 - t_1)} \qquad \boxed{t_2 - t_1 = \frac{M_2 - M_1}{n}}$$

Substituting (7.12) & (7.13) into (7.14) gives the time after pericenter passage as

$$t = \sqrt{\frac{a^3}{\mu}}\,(E - e\sin E)$$

Following a similar procedure, the time after pericenter passage for parabolic and hyperbolic trajectories is given by

Parabolic

$$t = \frac{1}{2\sqrt{\mu}}\left[pD + \frac{1}{3}D^3\right]$$

where $p = \frac{h^2}{\mu} = 2q$

$D = \sqrt{p}\,\tan\frac{\theta}{2}$ D is called the parabolic eccentric anomaly

Hyperbolic

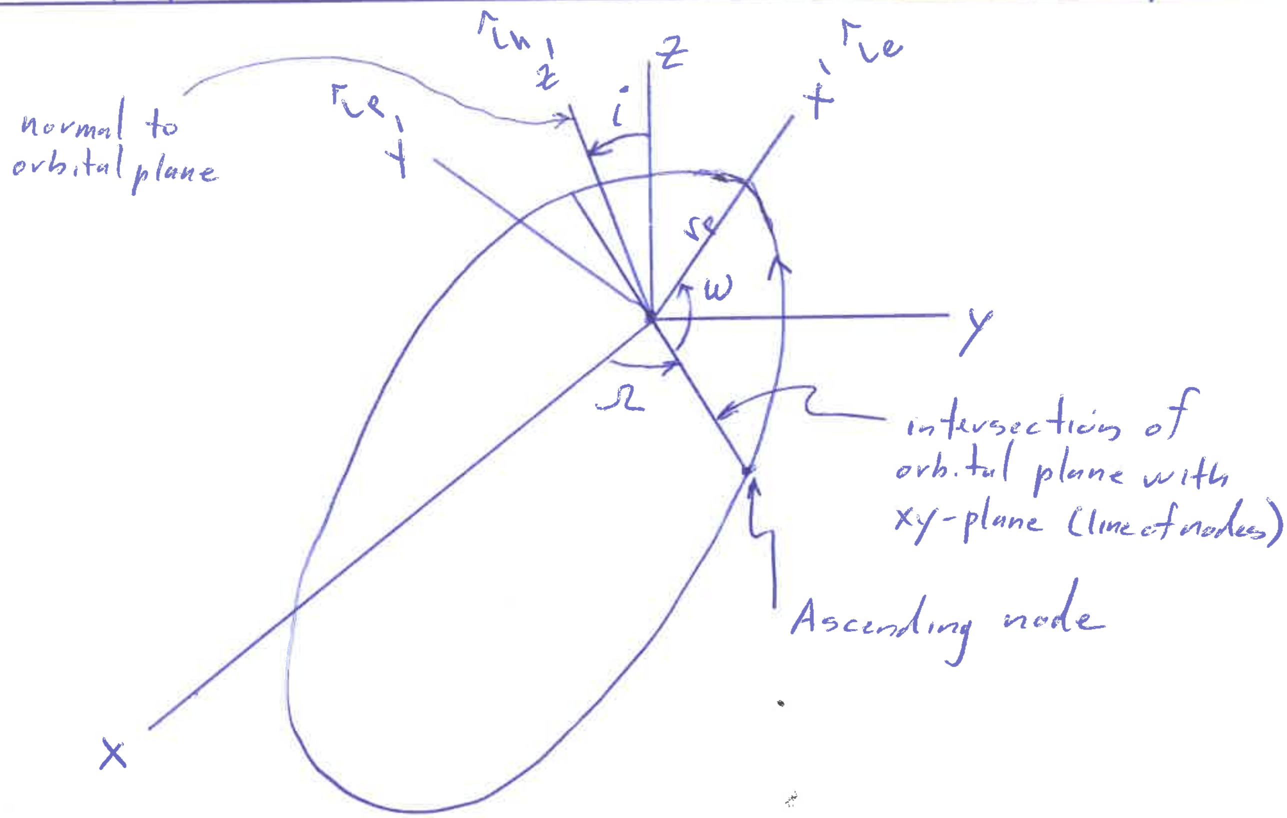
$$t = \sqrt{\frac{a^3}{\mu}}\,(e\sinh F - F)$$

where $\cosh F = \frac{e+\cos\theta}{1+e\cos\theta}$ $\tan\frac{\theta}{2}$ F is called the hyperbolic eccentric anomaly

or $F = \ln\left[y + \sqrt{y^2-1}\right]$ For $0 < \theta < \pi \Rightarrow F > 0$

where $y = \cosh F$ For $\pi < \theta < 2\pi \Rightarrow F < 0$

7.2. Orientation of Orbits in Three-Dimensional Space



Let x, y, z be reference coordinate system

x', y', z' be orbital axes coordinate system

Orbital Elements

Ω = longitude of the ascending node
ω = angular position of pericenter passage
i = inclination of orbit.

} Euler angles determine orientation of orbital plane

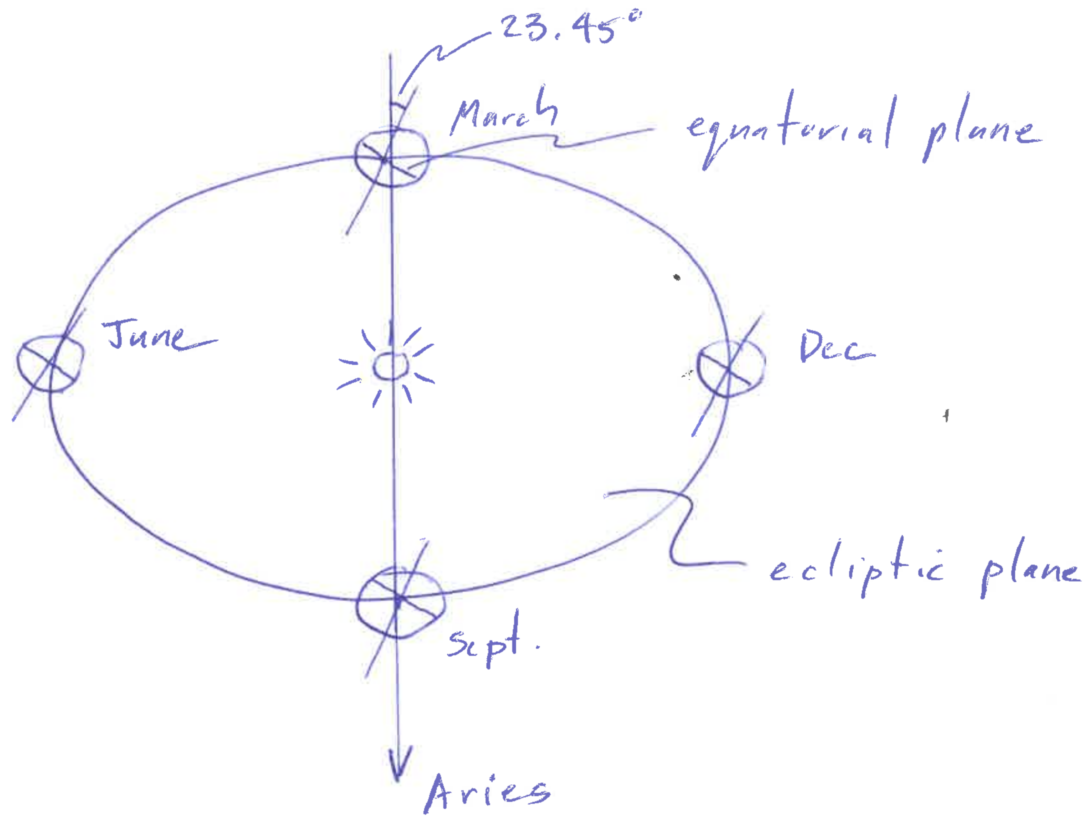
a = semi-major axis
e = eccentricity

} determine size and shape of orbit.

τ = time of pericenter passage

For heliocentric orbits (Ecliptic Coordinate System)
xy-plane is the ecliptic plane (plane of earth's orbit)

For geocentric orbits (Equatorial Coordinate System)
xy-plane is the equatorial plane.



In both systems the x-axis is in the direction of the vernal equinox and points toward the constellation Aries.

The z-axis is normal to the xy-plane in the northern direction

When $\Omega = i = \omega = 0$, $x'y'z'$ coincides with x, y, z

The coordinate transformation is

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{7.15}$$

or

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \overline{\overline{A}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{7.16}$$

where

$$a_{11} = \cos\omega \cos\Omega - \sin\omega \cos i \sin\Omega$$

$$a_{12} = \cos\omega \sin\Omega + \sin\omega \cos i \cos\Omega$$

$$a_{13} = \sin\omega \sin i$$

$$a_{21} = -\sin\omega \cos\Omega - \cos\omega \cos i \sin\Omega$$

$$a_{22} = \cos\omega \cos i \cos\Omega - \sin\omega \sin\Omega$$

$$a_{23} = \cos\omega \sin i$$

$$a_{31} = \sin i \sin\Omega$$

$$a_{32} = -\sin i \cos\Omega$$

$$a_{33} = \cos i$$

The inverse transformation is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \qquad (7.17)$$

$$= \bar{\bar{A}}^{-1} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

Note: $\bar{\bar{A}}$ is orthogonal, i.e. $\bar{\bar{A}}^{-1} = \bar{\bar{A}}^{T}$

The position of the body in the orbital axes (x', y', z') is

$$\left.\begin{aligned} x' &= r\cos\theta = a(\cos E - e) \\ y' &= r\sin\theta = a\sqrt{1-e^2}\,\sin E \\ z' &= 0 \end{aligned}\right\} \qquad (7.18)$$

The position in the reference axes (x, y, z) is

$$\left.\begin{aligned} x &= a\left[a_{11}(\cos E - e) + a_{21}\sqrt{1-e^2}\,\sin E\right] \\ y &= a\left[a_{12}(\cos E - e) + a_{22}\sqrt{1-e^2}\,\sin E\right] \\ z &= a\left[a_{13}(\cos E - e) + a_{23}\sqrt{1-e^2}\,\sin E\right] \end{aligned}\right\} \qquad (7.19)$$

Procedure for determining the position of a satellite at time t given the 6 orbital elements $a, e, \tau, \Omega, i, \omega$

1) Compute the mean angular velocity from

$$n = \sqrt{\frac{\mu}{a^3}} \qquad (7.20)$$

2) Determine the mean anomaly for the time in question from

$$M = n(t - \tau) \qquad (7.21)$$

3) Obtain the eccentric anomaly from Kepler's equation

$$M - E + e \sin E = 0 \qquad (7.22)$$

4) Use (7.19) to obtain Cartesian components (x, y, z).

The velocity components are found by differentiating (7.17) & (7.18)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \bar{\bar{A}}^{-1} \begin{bmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{bmatrix} = \bar{\bar{A}}^{-1} \begin{bmatrix} -a \sin E \, \dot{E} \\ a\sqrt{1-e^2} \cos E \, \dot{E} \\ 0 \end{bmatrix}$$

Differentiate (7.21)

$$\dot{M} = n$$

Differentiate (7.22)

$$\dot{M} - \dot{E} + e \cos E \, \dot{E} = 0$$

$$\therefore \quad \dot{E} = \frac{n}{1 - e\cos E}$$

Therefore

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \bar{\bar{A}}^{-1} \begin{bmatrix} -\dfrac{na \sin E}{1-e\cos E} \\ \dfrac{na\sqrt{1-e^2}\cos E}{1-e\cos E} \\ 0 \end{bmatrix} \qquad (7.23)$$

or in terms of θ

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \bar{\bar{A}}^{-1} \begin{bmatrix} \dfrac{-na \sin\theta}{(1+e)\sqrt{1-e^2}} \\ \dfrac{na}{\sqrt{1-e^2}}(\cos\theta + e) \\ 0 \end{bmatrix} \qquad (7.24)$$

7.3. The Hodographic Plane

Using (5.9)

$$\frac{d\bar{r}}{dt} \times \bar{h} = \mu\left[\frac{\bar{r}}{r} + \bar{e}\right] \qquad (5.9)$$

$$\bar{V} \times \bar{h} = \mu\left[\frac{\bar{r}}{r} + \bar{e}\right]$$

Using the identity

$$\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A}\cdot\bar{C})\bar{B} - (\bar{A}\cdot\bar{B})\bar{C}$$

$$\bar{h} \times (\bar{V} \times \bar{h}) = h^2\bar{V} - \underbrace{(\bar{h}\cdot\bar{V})}_{0}\bar{h} = h^2\bar{V}$$

$$\therefore\ h^2\bar{V} = \mu\,\bar{h} \times \left[\frac{\bar{r}}{r} + \bar{e}\right]$$

$$\bar{V} = \frac{\mu}{h^2}\,\bar{h} \times \left[\frac{\bar{r}}{r} + \bar{e}\right] \qquad (7.25)$$

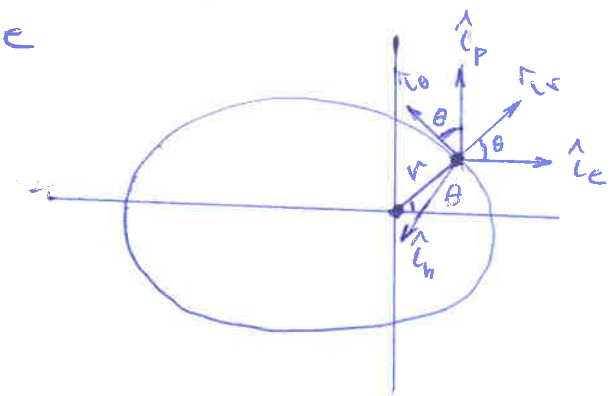
Graphical interpretation of (7.25):

Rewrite (7.25) as

$$\frac{h\bar{V}}{\mu} = \hat{i}_h \times (\hat{i}_r + e\,\hat{i}_e)$$

$$= \hat{i}_h \times \hat{i}_r + e\,\hat{i}_h \times \hat{i}_e$$

$$= \hat{i}_\theta + e\,\hat{i}_p$$



Since $\hat{i}_p = \sin\theta\,\hat{i}_r + \cos\theta\,\hat{i}_\theta$

$$\frac{h\bar{V}}{\mu} = \hat{i}_\theta + e(\sin\theta\,\hat{i}_r + \cos\theta\,\hat{i}_\theta)$$

$$\frac{h\bar{V}}{\mu} = e\sin\theta\,\hat{i}_r + (1 + e\cos\theta)\,\hat{i}_\theta$$

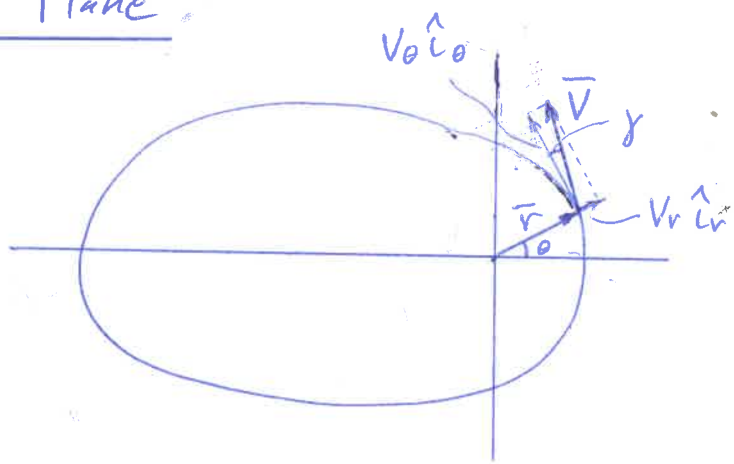
or $\quad \dfrac{hV_r}{\mu} = e\sin\theta \qquad \dfrac{hV_\theta}{\mu} = 1 + e\cos\theta$

$$\left(\frac{hV_\theta}{\mu} - 1\right)^2 + \left(\frac{hV_r}{\mu}\right)^2 = e^2$$

Circle of radius e with center at

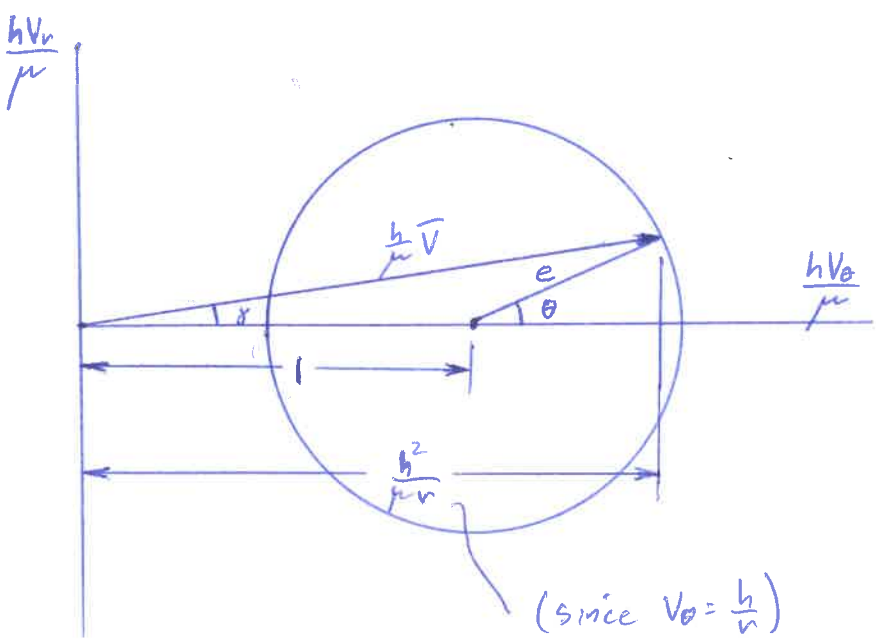
$$\frac{hV_\theta}{\mu} = 1 \qquad \frac{hV_r}{\mu} = 0$$

Physical Plane



$$\tan\gamma = \frac{V_r}{V_\theta}$$

Hodographic Plane



(since $V_\theta = \frac{h}{r}$)
(see eq. following 5.26)

special cases

circle $e = 0$

parabola $e = 1$
(circle tangent to $\frac{hV_r}{\mu}$ axis)

hyperbola $e > 1$
(circle crosses $\frac{hV_r}{\mu}$ axis)

(hodograph consists of part of circle in right half plane)

7.4 The Lagrangian Coefficients

Two Body Problem

Recall (5.1)

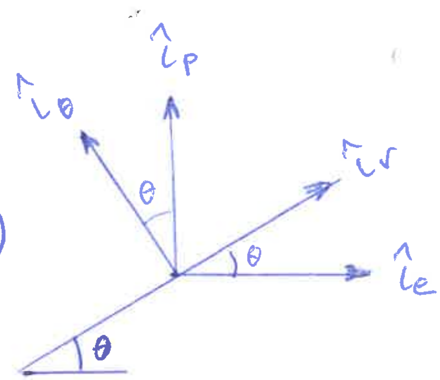
$$\frac{d^2\bar{r}}{dt^2} + \frac{\mu}{r^3}\bar{r} = 0 \qquad (5.1)$$

Given $\bar{r} = \bar{r}_o$ and $\bar{V} = \frac{d\bar{r}}{dt} = \bar{V}_o$ at $t = t_o$

Want to find $\bar{r}(t)$ and $\bar{V}(t)$ in terms of $\bar{r}_o$ and $\bar{V}_o$.

Write

$$\bar{r} = r\cos\theta\,\hat{i}_e + r\sin\theta\,\hat{i}_p \qquad (7.26a)$$



Using (7.25)

$$\bar{V} = \frac{\mu}{h^2}\bar{h} \times \left[\frac{\bar{r}}{r} + \bar{e}\right] \qquad (7.25)$$

$$= \frac{\mu}{h^2} h\hat{i}_h \times \left[\hat{i}_r + e\,\hat{i}_e\right]$$

$$= \frac{\mu}{h}\hat{i}_\theta + \frac{\mu e}{h}\hat{i}_p$$

$$= \frac{\mu}{h}\left(\cos\theta\,\hat{i}_p - \sin\theta\,\hat{i}_e\right) + \frac{\mu e}{h}\hat{i}_p$$

$$\bar{V} = -\frac{\mu}{h}\sin\theta\, \hat{i}_e + \frac{\mu}{h}(e+\cos\theta)\,\hat{i}_p \qquad (7.26\,b)$$

At $t=t_0$, (7.26 a, b) become

$$\left.\begin{aligned} \bar{r}_0 &= r_0\cos\theta_0\,\hat{i}_e + r_0\sin\theta_0\,\hat{i}_p \\ \bar{V}_0 &= -\frac{\mu}{h}\sin\theta_0\,\hat{i}_e + \frac{\mu}{h}(e+\cos\theta_0)\,\hat{i}_p \end{aligned}\right\} \quad (7.27)$$

To invert (7.27) (i.e., express $\hat{i}_e, \hat{i}_p$ in terms of $\bar{r}_0, \bar{V}_0$) use Cramer's rule

$$\begin{vmatrix} r_0\cos\theta_0 & r_0\sin\theta_0 \\ -\frac{\mu}{h}\sin\theta_0 & \frac{\mu}{h}(e+\cos\theta_0) \end{vmatrix} = \frac{\mu r_0}{h} e\cos\theta_0 + \frac{\mu r_0}{h}\cos^2\theta_0 + \frac{\mu r_0}{h}\sin^2\theta_0$$

$$= \frac{\mu r_0}{h}(1+e\cos\theta_0)$$

$$= \frac{\mu \dfrac{h^2/\mu}{1+e\cos\theta_0}(1+e\cos\theta_0)}{h}$$

$$= h$$

$$\hat{i}_e = \frac{1}{h}\begin{vmatrix} \bar{r}_0 & r_0 \sin\theta_0 \\ \bar{V}_0 & \frac{\mu}{h}(e+\cos\theta_0) \end{vmatrix} = \frac{\mu}{h^2}(e+\cos\theta_0)\bar{r}_0 - \frac{r_0}{h}\sin\theta_0 \bar{V}_0$$

$$\hat{i}_p = \frac{1}{h}\begin{vmatrix} r_0\cos\theta_0 & \bar{r}_0 \\ -\frac{\mu}{h}\sin\theta_0 & \bar{V}_0 \end{vmatrix} = \frac{\mu}{h^2}\sin\theta_0 \bar{r}_0 + \frac{r_0}{h}\cos\theta_0 \bar{V}_0 \qquad (7.28)$$

Sub. (7.28) into (7.26a)

$$\bar{r} = \frac{\mu r}{h^2}(e\cos\theta + \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0)\bar{r}_0$$

$$+ \frac{r r_0}{h}(\sin\theta\cos\theta_0 - \cos\theta\sin\theta_0)\bar{V}_0$$

$$\bar{r} = \frac{r}{p}\Big(e\cos\theta + \cos(\theta-\theta_0)\Big)\bar{r}_0 + \frac{r r_0}{\sqrt{\mu p}}\sin(\theta-\theta_0)\bar{V}_0$$

Using $r = \frac{p}{1+e\cos\theta} \Rightarrow e\cos\theta = \frac{p}{r} - 1$

$$\bar{r} = \left[1 - \frac{r}{p}\Big(1-\cos(\theta-\theta_0)\Big)\right]\bar{r}_0 + \frac{r r_0}{\sqrt{\mu p}}\sin(\theta-\theta_0)\bar{V}_0$$

(7.29a)

Sub. (7.28) into (7.26b)

$$\bar{V} = \frac{\mu^2}{h^3}\left(e\sin\theta_0 + \cos\theta\sin\theta_0 - e\sin\theta - \sin\theta\cos\theta_0\right)\bar{r}_0$$

$$+ \frac{\mu r_0}{h^2}\left(e\cos\theta_0 + \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\right)\bar{V}_0$$

$$= \sqrt{\frac{\mu}{p^3}}\left(e\sin\theta_0 - e\sin\theta - \sin(\theta-\theta_0)\right)\bar{r}_0$$

$$+ \frac{r_0}{p}\left(e\cos\theta_0 + \cos(\theta-\theta_0)\right)\bar{V}_0$$

Using (7.27) calculate

$$\bar{r}_0\cdot\bar{V}_0 = \cancel{-\frac{\mu r_0}{h}\cos\theta_0\sin\theta_0} + \frac{\mu r_0}{h}\left(e+\cancel{\cos\theta_0}\right)\sin\theta_0$$

$$= \frac{\mu r_0}{h}e\sin\theta_0$$

$$e\sin\theta_0 = \frac{h}{\mu r_0}\,\bar{r}_0\cdot\bar{V}_0$$

Define $\sigma_0 \equiv \dfrac{\bar{r}_0\cdot\bar{V}_0}{\sqrt{\mu}}$ (7.30)

$$e \sin\theta_0 = \frac{\sigma_0 \sqrt{p}}{r_0} \qquad (7.31)$$

$$e \cos\theta_0 = \frac{p}{r_0} - 1 \qquad (7.32)$$

Write

$$\sin\theta = \sin\left((\theta-\theta_0)+\theta_0\right)$$

$$= \sin(\theta-\theta_0)\cos\theta_0 + \cos(\theta-\theta_0)\sin\theta_0$$

$$\therefore \bar{V} = \sqrt{\frac{\mu}{p^3}}\left(\frac{\sigma_0\sqrt{p}}{r_0} - \left(\frac{p}{r_0} - \cancel{1}\right)\sin(\theta-\theta_0) - \frac{\sigma_0\sqrt{p}}{r_0}\cos(\theta-\theta_0)\right.$$

$$\left.\cancel{-\sin(\theta-\theta_0)}\right)\bar{r}_0 + \frac{r_0}{p}\left(\frac{p}{r_0} - 1 + \cos(\theta-\theta_0)\right)\bar{V}_0$$

$$\bar{V} = \frac{\sqrt{\mu}}{r_0 p}\left[\sigma_0\left(1-\cos(\theta-\theta_0)\right) - \sqrt{p}\sin(\theta-\theta_0)\right]\bar{r}_0$$

$$+\left[1 - \frac{r_0}{p}\left(1-\cos(\theta-\theta_0)\right)\right]\bar{V}_0 \qquad (7.29b)$$

Eqs. (7.29 a, b) are of the form:

$$\left.\begin{aligned} \bar{r} &= F\,\bar{r}_o + G\,\bar{V}_o \\ \bar{V} &= F_t\,\bar{r}_o + G_t\,\bar{V}_o \end{aligned}\right\} \quad (7.33)$$

where

$$F = 1 - \frac{r}{p}\left(1-\cos(\theta-\theta_o)\right) \qquad G = \frac{r r_o}{\sqrt{\mu p}} \sin(\theta-\theta_o)$$

$$F_t = \frac{\sqrt{\mu}}{r_o p}\left[\sigma_o\left(1-\cos(\theta-\theta_o)\right) - \sqrt{p}\sin(\theta-\theta_o)\right] \qquad G_t = 1 - \frac{r_o}{p}\left(1-\cos(\theta-\theta_o)\right)$$

(7.34)

are called <u>Lagrangian coefficients</u>

Eq. (7.33) can be written in matrix form

$$\begin{bmatrix} \bar{r} \\ \bar{V} \end{bmatrix} = \begin{bmatrix} F & G \\ F_t & G_t \end{bmatrix} \begin{bmatrix} \bar{r}_o \\ \bar{V}_o \end{bmatrix}$$

where $\begin{bmatrix} \bar{r} \\ \bar{V} \end{bmatrix}$ is called the <u>state vector</u>

Define the <u>transition matrix</u>

$$\bar{\bar{\Phi}} \equiv \begin{bmatrix} F & G \\ F_t & G_t \end{bmatrix} \qquad (7.35)$$

<u>Properties of $\bar{\bar{\Phi}}$</u>

1) $\det \bar{\bar{\Phi}} = F G_t - G F_t = 1$

<u>Proof:</u> From conservation of angular momentum

$$\bar{r} \times \bar{V} = (F \bar{r}_o + G \bar{V}_o) \times (F_t \bar{r}_o + G_t \bar{V}_o)$$

$$= F G_t (\bar{r}_o \times \bar{V}_o) + G F_t (\bar{V}_o \times \bar{r}_o)$$

$$= (F G_t - G F_t)(\bar{r}_o \times \bar{V}_o) \Rightarrow F G_t - G F_t = 1$$

2) Inverse of $\bar{\bar{\Phi}}$ is

$$\bar{\bar{\Phi}}^{-1} = \begin{bmatrix} G_t & -G \\ -F_t & F \end{bmatrix}$$

3) For any 3 points on an orbit $\bar{r}_0, \bar{r}_1, \bar{r}_2$

$$\bar{\bar{\Phi}}_{2,0} = \bar{\bar{\Phi}}_{2,1}\,\bar{\bar{\Phi}}_{1,0} \qquad (7.38)$$

Proof:

$$\begin{bmatrix} \bar{r}_2 \\ \bar{V}_2 \end{bmatrix} = \bar{\bar{\Phi}}_{2,0} \begin{bmatrix} \bar{r}_0 \\ \bar{V}_0 \end{bmatrix}$$

$$\begin{bmatrix} \bar{r}_2 \\ \bar{V}_2 \end{bmatrix} = \bar{\bar{\Phi}}_{2,1} \begin{bmatrix} \bar{r}_1 \\ \bar{V}_1 \end{bmatrix} = \bar{\bar{\Phi}}_{2,1}\,\bar{\bar{\Phi}}_{1,0} \begin{bmatrix} \bar{r}_0 \\ \bar{V}_0 \end{bmatrix}$$

$$\therefore\ \bar{\bar{\Phi}}_{2,0} = \bar{\bar{\Phi}}_{2,1}\,\bar{\bar{\Phi}}_{1,0}$$