1) A chaser vehicle in a circular orbit of radius 8590 km lags 2 degrees behind a target vehicle which is in a coplanar circular orbit of radius 8600 km.

   a) Compute the relative position and velocity of the chaser in the CW frame.
   b) Determine the chaser's relative position and velocity components 30 minutes later.
   c) Compute the total velocity increment required for a 30 minute, 2-impulse rendezvous maneuver starting from the initial state given in part (a).

2) An astronaut taking a spacewalk outside the International Space Station which is in a circular orbit about the earth at an altitude of 420 km discards a spent battery by throwing it radially outward away from the ISS and the earth with a velocity of 10 meters/sec.

   a) Draw the trajectory of the battery relative to the ISS over one complete orbit of the ISS.
   b) Was it a good idea to discard the battery this way?

1) Exact Solution

a) At $t = 0$

A = target vehicle $\quad r_A = 8600$ km

B = chaser vehicle $\quad r_B = 8590$ km

$\theta_{B_0} - \theta_{A_0} = -2°$

From class notes (p. 176)

$$\bar{r}_{rel} = [r_B \cos(\theta_B - \theta_A) - r_A]\hat{i} + r_B \sin(\theta_B - \theta_A)\hat{j} \quad (1)$$

$$\bar{r}_{rel_0} = [8590 \cos(-2°) - 8600]\hat{i} + 8590 \sin(-2°)\hat{j}$$

$$\boxed{\bar{r}_{rel_0} = -15.2328\,\hat{i} - 299.787\,\hat{j}} \quad (\text{km}) \quad (2)$$

$$\Omega = \frac{v_A}{r_A} = \frac{\sqrt{\frac{\mu}{r_A}}}{r_A} = \sqrt{\frac{\mu}{r_A^3}} = n_A$$

$$\bar{\Omega} = n_A \hat{k} \quad \text{where} \quad n_A = \sqrt{\frac{\mu}{r_A^3}} \quad (3)$$

Using (1b) & (2b) from p. 173 (class notes) with (6a, b) p. 174 (class notes)

$$\bar{V}_A = \sqrt{\frac{\mu}{r_A}}\,\hat{j} \quad (4)$$

$$\bar{V}_B = \sqrt{\frac{\mu}{r_B}}\left[-\sin(\theta_B - \theta_A)\hat{i} + \cos(\theta_B - \theta_A)\hat{j}\right] \quad (5)$$

Using (9.80) from class notes

$$\bar{V}_{rel} = \bar{V}_B - \bar{V}_A - \Omega \times \bar{r}_{rel} \qquad (6)$$

Sub (1), (3), (4) & (5) into (6)

$$\bar{V}_{rel} = r_B (n_B - n_A)\left[-\sin(\theta_B - \theta_A)\hat{i} + \cos(\theta_B - \theta_A)\hat{j}\right] \qquad (7)$$

where $n_B = \sqrt{\dfrac{\mu}{r_B^3}}$

$$n_A = \sqrt{\frac{\mu}{r_A^3}} = \sqrt{\frac{3.986 \times 10^5}{(8600)^3}} = 0.000791628 \text{ rad/sec} \qquad (8)$$

$$n_B = \sqrt{\frac{\mu}{r_B^3}} = \sqrt{\frac{3.986 \times 10^5}{(8590)^3}} = 0.00079301 \text{ rad/sec} \qquad (9)$$

$$\bar{V}_{rel_0} = (8590)(0.00079301 - 0.000791628)\left[-\sin(-2°)\hat{i} + \cos(-2°)\hat{j}\right]$$

$$\boxed{\bar{V}_{rel_0} = 0.00041453\,\hat{i} + 0.0118706\,\hat{j}} \quad (\text{km/sec}) \qquad (10)$$

b) <u>At $t = 30\,min = 1800\,sec$</u>

For circular orbits

$$\Delta\theta_A = n_A \Delta t = \theta_A - \theta_{A0} = n_A (t-0) \quad (11)$$

$$\Delta\theta_B = n_B \Delta t = \theta_B - \theta_{B0} = n_B (t-0) \quad (12)$$

subtract (11) from (12)

$$\theta_B - \theta_A = (\theta_{B0} - \theta_{A0}) + (n_B - n_A)\,t \quad (13)$$

$$= -2^\circ + (0.00079301 - 0.000791628)(1800)\left(\frac{180^\circ}{\pi}\right)$$

$$= -1.85747^\circ$$

Using (1)

$$\bar{r}_{rel} = \left[8590\cos(-1.85747^\circ) - 8600\right]\hat{i} + 8590\sin(-1.85747^\circ)\hat{j}$$

$$\boxed{\bar{r}_{rel} = -14.5132\,\hat{i} - 278.4186\,\hat{j}} \quad (km) \quad (14)$$

Using (7)

$$\bar{V}_{rel} = (8590)(0.00079301 - 0.000791628)\left[-\sin(-1.85747^\circ)\hat{i} + \cos(-1.85747^\circ)\hat{j}\right]$$

$$\boxed{\bar{V}_{rel} = 0.00038498\,\hat{i} + 0.0118716\,\hat{j}} \quad (km/sec) \quad (15)$$

1) Linear Theory

a) At $t=0$

A = target vehicle $\qquad r_A = 8600$ km

B = chaser vehicle $\qquad r_B = 8590$ km

$$\theta_{B0} - \theta_{A0} = -2^\circ$$

To the degree of accuracy of the linear theory

$$\delta x_0 = r_B - r_A = 8590 - 8600 = \underline{\underline{-10 \text{ km}}} \quad (11)$$

$$\delta y_0 = r_A(\theta_{B0} - \theta_{A0}) = 8600(-2^\circ)\left(\frac{\pi}{180^\circ}\right) = \underline{\underline{-300.20 \text{ km}}} \quad (12)$$

Since both orbits are circular, the radial component of the velocities are zero.

$$\boxed{\delta u_0 = 0} \quad (13)$$

Using (9.105a) $\quad \delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y} = 0$

since the radial component of the velocity remains zero, $\delta\ddot{x} = 0$. Thus

$$\delta\dot{y} = -\frac{3}{2}n\,\delta x \quad (14)$$

$$\delta v_0 = -\frac{3}{2}n\,\delta x_0 \quad (15)$$

$$n = n_A = \sqrt{\frac{\mu}{r_A^3}} = \sqrt{\frac{3.986 \times 10^5}{(8600)^3}} = 0.000791628 \ \frac{\text{rad}}{\text{sec}}$$

$$\delta v_0 = -\frac{3}{2}(0.000791628)(-10)$$

$$\boxed{\delta v_0 = 0.011874 \ \text{km/sec}} \qquad (16)$$

b) At $t = 30$ min $= 1800$ sec

$$\cos nt = \cos((0.000791628)(1800)) = 0.145349$$

$$\sin nt = \sin((0.000791628)(1800)) = 0.98938$$

Using (9.115a)

$$\delta x = (4 - 3\cos nt)\delta x_0 + \frac{1}{n}\sin nt \, \delta u_0 + \frac{2}{n}(1 - \cos nt)\delta v_0$$

$$\delta x = (4 - 3(0.145349))(-10) + \frac{0.98938}{0.000791628}(0)$$

$$+ \frac{2}{0.000791628}(1 - 0.145349)(0.011874)$$

$$\boxed{\delta x = -10 \ \text{km}}$$

Using (9.115b)

$$\delta y = 6(\sin nt - nt)\delta x_0 + \delta y_0 + \frac{2}{n}(\cos nt - 1)\delta u_0 + \frac{1}{n}(4\sin nt - 3nt)\delta v_0$$

$$\delta y = 6\left(0.98938 - (0.000791628)(1800)\right)(-10) + (-300.29)$$

$$+ \frac{2}{0.000791628}(0.145349 - 1)(0)$$

$$+ \frac{1}{0.000791628}\left(4(0.98938) - 3(0.000791628)(1800)\right)(0.011874)$$

$$\boxed{\delta y = -278.82 \text{ km}}$$

Using (9.116 a)

$$\delta u = 3n \sin nt\, \delta x_0 + \cos nt\, \delta u_0 + 2 \sin nt\, \delta v_0$$

$$= 3(0.000791628)(0.98938)(-10) + (0.145349)(0)$$

$$+ 2(0.98938)(0.011874)$$

$$\boxed{\delta u = 0}$$

Using (9.116b)

$$\delta v = 6n(\cos nt - 1)\delta x_0 - 2 \sin nt\, \delta u_0 + (4 \cos nt - 3)\, \delta v_0$$

$$= 6(0.000791628)(0.145349) - 2(0.98938)(0)$$

$$+ (4(0.145349) - 3)(0.011874)$$

$$\boxed{\delta v = 0.011874 \text{ km/sec}}$$

The table below compares the exact and linear theory solutions for the relative position and relative velocity of the chaser at t=0 and t=30 min.

	t=0 Exact	t=0 Linear Theory	t=30 min Exact	t=30 min Linear Theory
x	-15.2328 km	-10 km	-14.5132 km	-10
y	-299.787 km	-300.20 km	-278.4186 km	-278.82 km
$\dot{x}$	0.00041453 $\frac{km}{sec}$	0	0.00038498 $\frac{km}{sec}$	0
$\dot{y}$	0.0118706 $\frac{km}{sec}$	0.011874 $\frac{km}{sec}$	0.0118716 $\frac{km}{sec}$	0.011874 $\frac{km}{sec}$

c) $$\delta\bar{r}_0 = \begin{bmatrix} \delta x_0 \\ \delta y_0 \end{bmatrix} = \begin{bmatrix} -10 \\ -300.20 \end{bmatrix} \quad (km)$$

$$\bar{\Phi}_{rr} = \begin{bmatrix} 4-3\cos nt & 0 \\ 6(\sin nt - nt) & 1 \end{bmatrix} = \begin{bmatrix} 4-3(0.145349) & 0 \\ 6(0.98938 - (0.00079628)(1800)) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.5713 & 0 \\ -2.6261 & 1 \end{bmatrix}$$

$$\bar{\bar{\Phi}}_{rv} = \begin{bmatrix} \frac{1}{n}\sin nt & \frac{2}{n}(1-\cos nt) \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0.98938}{0.000791628} & \\ \frac{2}{0.000791628}(0.145349-1) & \frac{1}{0.000791628}[4(0.98938)-3(0.000791628)(1800)] \end{bmatrix}$$

$$= \begin{bmatrix} 1248.0782 & 2161.6694 \\ -2161.6694 & -407.6872 \end{bmatrix}$$

$$\bar{\bar{\Phi}}_{rv}^{-1} = \begin{bmatrix} 0.0000979078 & -0.000519134 \\ 0.000519134 & 0.000299731 \end{bmatrix}$$

$$\delta \bar{V}_0^{\,req} = -\bar{\bar{\Phi}}_{rv}^{-1}\,\Phi_{rr}\,\delta\bar{r}_0$$

$$= \begin{bmatrix} 0.0000979078 & -0.000519134 \\ 0.000519134 & 0.000299731 \end{bmatrix} \begin{bmatrix} 3.5713 & 0 \\ -2.6261 & 1 \end{bmatrix} \begin{bmatrix} -10 \\ -300.20 \end{bmatrix}$$

$$= \begin{bmatrix} -0.145708 \\ 0.100648 \end{bmatrix} \quad (\text{km/sec})$$

$$\delta \bar{V}_0 = \begin{bmatrix} 0 \\ 0.011874 \end{bmatrix} \text{(km/sec)}$$

$$\Delta \bar{V}_0 = \delta \bar{V}_0^{req} - \delta \bar{V}_0 = \begin{bmatrix} -0.145708 \\ 0.100648 \end{bmatrix} - \begin{bmatrix} 0 \\ 0.011874 \end{bmatrix}$$

$$= \begin{bmatrix} -0.145708 \\ 0.088774 \end{bmatrix} \text{(km/sec)}$$

$$|\Delta \bar{V}_0| = \sqrt{(-0.145708)^2 + (0.088774)^2}$$

$$|\Delta \bar{V}_0| = 0.170621 \text{ km/sec}$$

$$\bar{\bar{\Phi}}_{vr} = \begin{bmatrix} 3n \sin nt & 0 \\ 6n(\cos nt - 1) & 0 \end{bmatrix} = \begin{bmatrix} 3(0.000791628)(0.98938) & 0 \\ 6(0.000791628)(0.145349-1) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.00235462 & 0 \\ -0.00407820 & 0 \end{bmatrix}$$

$$\bar{\bar{\Phi}}_{vv} = \begin{bmatrix} \cos nt & 2\sin nt \\ -2\sin nt & 4\cos nt - 3 \end{bmatrix} = \begin{bmatrix} 0.145349 & 2(0.98938) \\ -2(0.98938) & 4(0.145349) - 3 \end{bmatrix}$$

$$\bar{\bar{\Phi}}_{vv} = \begin{bmatrix} 0.142887 & 1.979478 \\ -1.979478 & -2.428453 \end{bmatrix}$$

$$\delta \bar{V} = \bar{\bar{\Phi}}_{vr}\, \delta \bar{r}_0 + \bar{\bar{\Phi}}_{vv}\, \delta \bar{V}_0^{req}$$

$$= \begin{bmatrix} 0.00235462 & 0 \\ -0.00407820 & 0 \end{bmatrix} \begin{bmatrix} -10 \\ -300.20 \end{bmatrix} + \begin{bmatrix} 0.142887 & 1.979478 \\ -1.979478 & -2.428453 \end{bmatrix} \begin{bmatrix} -0.145708 \\ 0.100648 \end{bmatrix}$$

$$= \begin{bmatrix} 0.154865 \\ 0.084788 \end{bmatrix} \text{ (km/sec)}$$

$$\Delta \bar{V}_f = \bar{0} - \delta \bar{V}$$

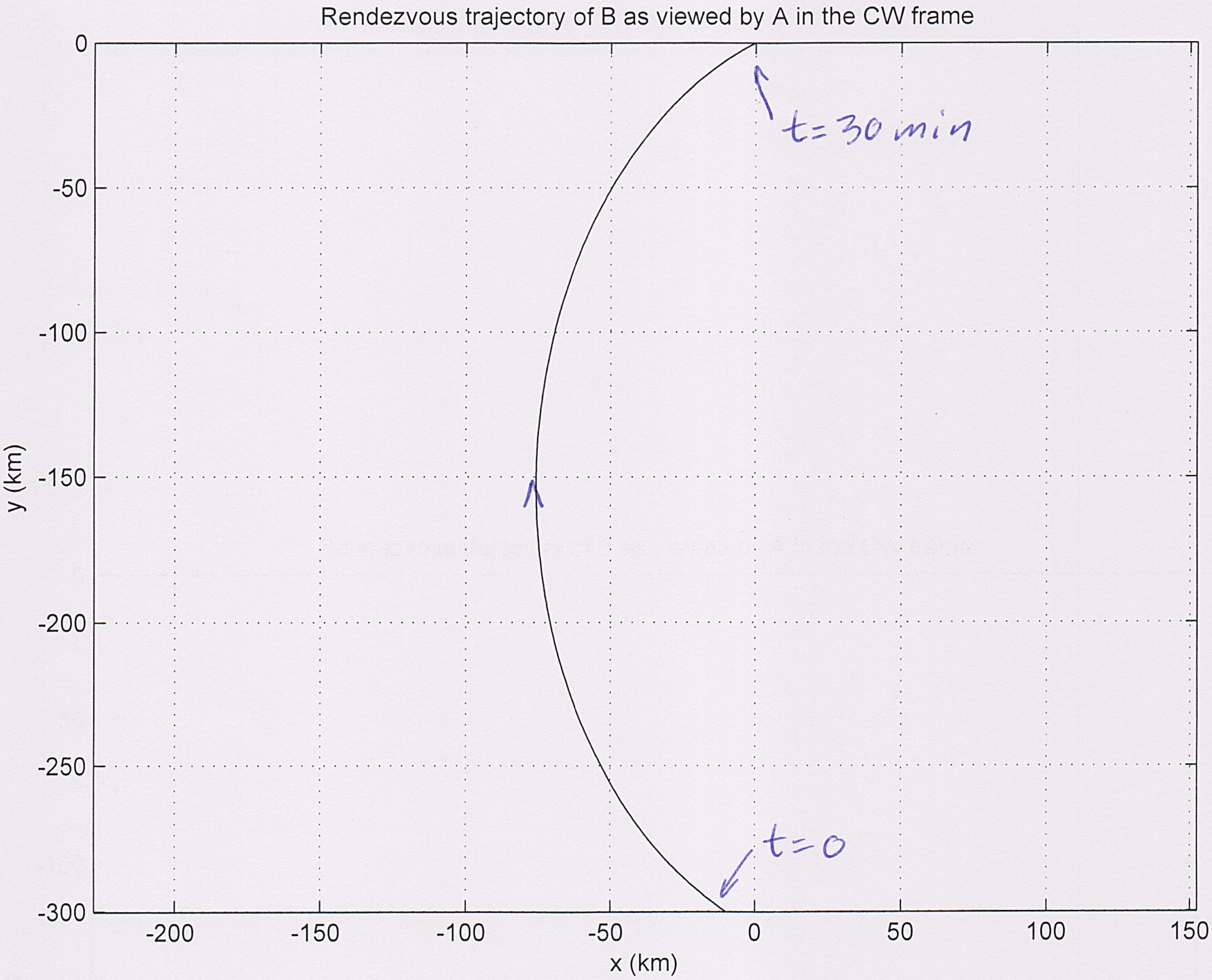
$$= \begin{bmatrix} -0.154865 \\ -0.084788 \end{bmatrix}$$

$$|\Delta \bar{V}_f| = \sqrt{(-0.154865)^2 + (-0.084788)^2}$$

$$|\Delta \bar{V}_f| = 0.176556 \text{ km/sec}$$

$$\Delta V_{total} = |\Delta \bar{V}_0| + |\Delta \bar{V}_f| = 0.170621 + 0.176556$$

$$\boxed{\Delta V_{total} = 0.347177 \text{ km/sec}}$$



2) a) Initial Conditions

$\delta x_0 = 0$

$\delta y_0 = 0$

$\delta u_0 = 10 \text{ m/sec} = 0.01 \text{ km/sec}$

$\delta v_0 = 0$

Radius of ISS (and astronaut's) orbit

$$R = 6368 + 420 = 6788 \text{ km}$$

$$n = \sqrt{\frac{\mu}{R^3}} = \sqrt{\frac{3.986 \times 10^5}{(6788)^3}} = 0.0011289 \text{ rad/sec}$$

Period of the orbit

$$T = \frac{2\pi}{n} = \frac{2\pi}{0.0011289} = 5565.76 \text{ sec} = 92.76 \text{ min}$$

The trajectory of the battery as viewed by the astronaut is obtained from (9.115 a,b) in the notes

$$\delta x = \frac{1}{n} \sin nt \, \delta u_0$$

$$\delta y = \frac{2}{n} (\cos nt - 1) \, \delta u_0$$

At $t = T$ $\delta x = 0$

$\delta y = 0$

The velocity of the battery as viewed by the astronaut is obtained from (9.116 a,b)

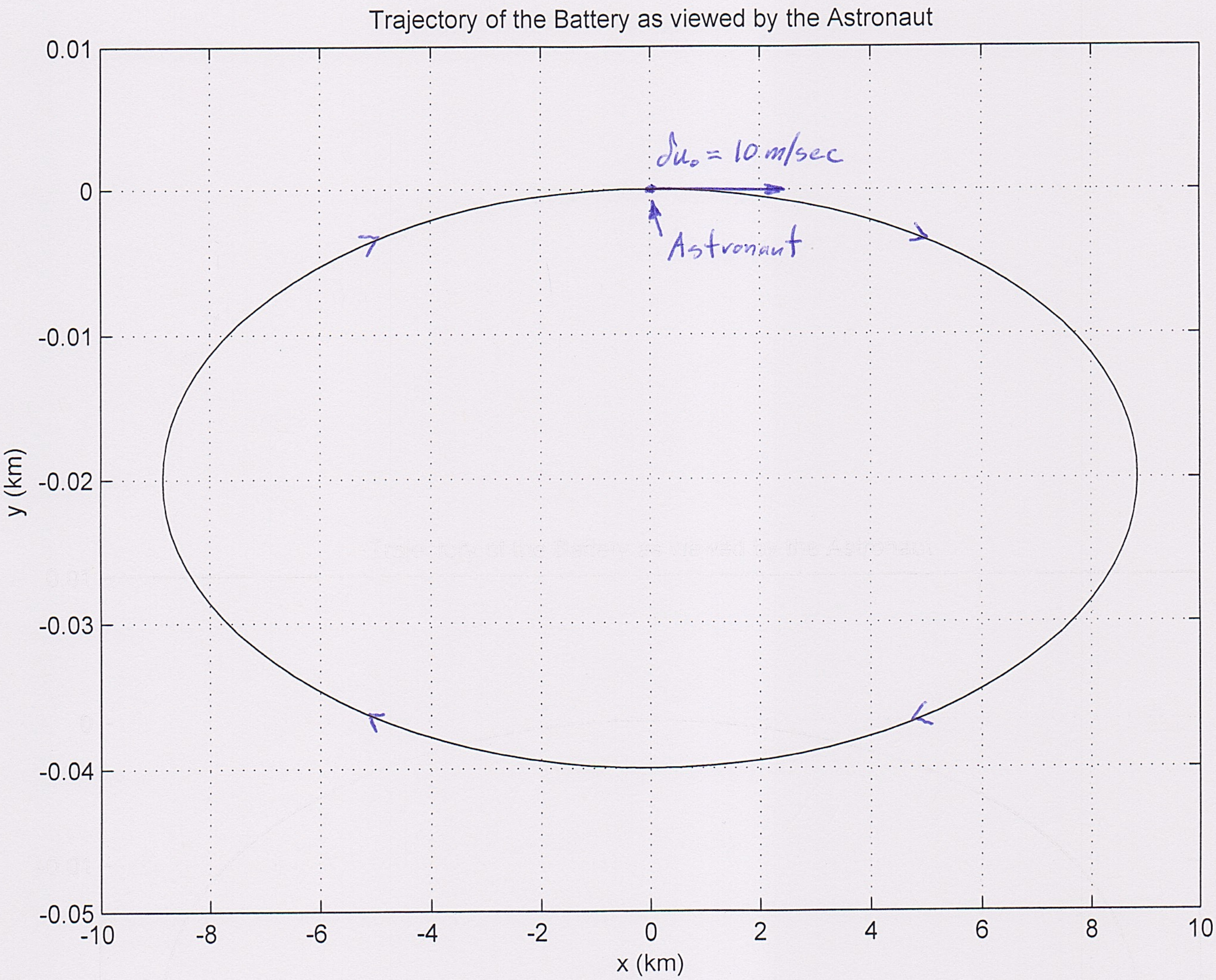
$\delta u = \cos nt \, \delta u_0$

$\delta v = -2 \sin nt \, \delta u_0$

At $t=T$ $\delta u = \delta u_0 = 10 \text{ m/sec}$

$\delta v = 0$

The trajectory of the battery as viewed by the astronaut is shown below.



b) At $t=T=92.76$ min, the battery will strike the astronaut in the back with the same speed as he/she threw it.