12.2 Optimal Staging

Want to determine the optimum distribution of masses among an N-stage rocket such that for specified m_{PL}, ΔV and Isp_i, ε_i of each stage, the total mass of the vehicle is minimized.

Mathematical Background — Lagrange Multiplier Method

Consider a function of 2 variables $f(x,y)$ which which we want to minimize. The change in f due to changes in x and y is given by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad (12.10)$$

At an extremum (maximum or minimum) $df = 0$. Since dx and dy are independent, this requires

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \qquad (12.11)$$

Suppose we wish to minimize f subject to the constraint

$$g(x,y) = 0 \qquad (12.12)$$

In addition to (12.11), this requires

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0 \qquad (12.13)$$

Thus dx and dy are not independent but must satisfy (12.13) which along with (12.10) [with df=0] gives

$$\frac{\partial f/\partial x}{\partial g/\partial x} = \frac{\partial f/\partial y}{\partial g/\partial y} \qquad (12.14)$$

Denote this ratio by $-\eta$. Then

$$\frac{\partial f}{\partial x} + \eta \frac{\partial g}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} + \eta \frac{\partial g}{\partial y} = 0 \qquad (12.15)$$

Note that these equations are the same as would result if we minimized the function

$$h(x,y,\eta) = f(x,y) + \eta\, g(x,y) \qquad (12.16)$$

yielding the 3 equations

$$\frac{\partial h}{\partial x} = 0 \qquad \frac{\partial h}{\partial y} = 0 \qquad \frac{\partial h}{\partial \eta} = g = 0 \qquad (12.17)$$

The variable η is called the <u>Lagrange multiplier</u>

For more constraints, additional Lagrange multipliers can be introduced.

Consider a two-stage rocket. Using (12.3), (12.4) & (12.6)

$$\Delta V_1 = c_1 \ln z_1 = c_1 \ln \frac{m_1 + m_2 + m_{PL}}{\varepsilon_1 m_1 + m_2 + m_{PL}} \qquad (12.18a)$$

$$\Delta V_2 = c_2 \ln z_2 = c_2 \ln \frac{m_2 + m_{PL}}{\varepsilon_2 m_2 + m_{PL}} \qquad (12.18b)$$

where $m_i = m_{s_i} + m_{P_i}$ is the sum of the structural and propellant mass of stage i

The optimization problem may be stated as

Minimize

$$f = M = m_1 + m_2 \qquad (12.19)$$

subject to the constraint

$$g = \Delta V_1 + \Delta V_2 - \Delta V_{tot} = 0 \qquad (12.20)$$

The Lagrange multiplier method [eqs. (12.17)] applied to this problem yields 3 equations

$$\frac{\partial f}{\partial m_1} + \eta \frac{\partial g}{\partial m_1} = 0$$

$$\frac{\partial f}{\partial m_2} + \eta \frac{\partial g}{\partial m_2} = 0 \qquad (12.21\ a,b,c)$$

$$g = 0$$

for the 3 unknowns m_1, m_2 and η. However, these equations are quite complicated and difficult to solve.

To simplify the solution, note that

$$\frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL}} = \frac{(1-\varepsilon_1)(m_1 + m_2 + m_{PL})}{(1-\varepsilon_1)(m_2 + m_{PL}) + \varepsilon_1 m_1 - \varepsilon_1 m_1)}$$

$$= \frac{(1-\varepsilon_1)(m_1 + m_2 + m_{PL})}{\varepsilon_1 m_1 + m_2 + m_{PL} - \varepsilon_1 (m_1 + m_2 + m_{PL})}$$

$$= \frac{(1-\varepsilon_1)\dfrac{m_1 + m_2 + m_{PL}}{\varepsilon_1 m_1 + m_2 + m_{PL}}}{1 - \varepsilon_1 \dfrac{m_1 + m_2 + m_{PL}}{\varepsilon_1 m_1 + m_2 + m_{PL}}}$$

$$= \frac{(1-\varepsilon_1) Z_1}{1 - \varepsilon_1 Z_1} \qquad (12.22)$$

and similarly

$$\frac{m_2 + m_{PL}}{m_{PL}} = \frac{(1-\varepsilon_2) Z_2}{1 - \varepsilon_2 Z_2} \qquad (12.23)$$

Using (12.22) & (12.23) can write

$$\frac{m_o}{m_{PL}} = \frac{m_1 + m_2 + m_{PL}}{m_{PL}} = \frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL}} \cdot \frac{m_2 + m_{PL}}{m_{PL}}$$

$$\frac{m_o}{m_{PL}} = \frac{(1-\varepsilon_1) Z_1}{1-\varepsilon_1 Z_1} \cdot \frac{(1-\varepsilon_2) Z_2}{1-\varepsilon_2 Z_2} \qquad (12.24)$$

Taking the log of (12.24)

$$\ln\left(\frac{m_o}{m_{PL}}\right) = \left[\ln(1-\varepsilon_1) + \ln Z_1 - \ln(1-\varepsilon_1 Z_1)\right] + \left[\ln(1-\varepsilon_2) + \ln Z_2 - \ln(1-\varepsilon_2 Z_2)\right] \qquad (12.25)$$

Note that for m_{PL} fixed, $\ln \frac{m_o}{m_{PL}}$ is a monotonically increasing function of m_o. Therefore $\ln(m_o/m_{PL})$ has a minimum when m_o does.

Therefore the optimization problem for a 2-stage rocket can be stated as follows:

Minimize

$$f = \left[\ln(1-\varepsilon_1) + \ln Z_1 - \ln(1-\varepsilon_1 Z_1)\right] + \left[\ln(1-\varepsilon_2) + \ln Z_2 - \ln(1-\varepsilon_2 Z_2)\right] \qquad (12.26)$$

subject to

$$g = \Delta V_{tot} - c_1 \ln Z_1 - c_2 \ln Z_2 = 0 \qquad (12.27)$$

or, using the Lagrange multiplier η, eqs. (12.26) & (12.27) may be combined to obtain

$$h = \left[\ln(1-\varepsilon_1) + \ln z_1 - \ln(1-\varepsilon_1 z_1)\right] + \left[\ln(1-\varepsilon_2) + \ln z_2 - \ln(1-\varepsilon_2 z_2)\right] + \eta\left[\Delta V_{tot} - c_1 \ln z_1 - c_2 \ln z_2\right] \qquad (12.28)$$

For the optimum solution, eqs. (12.17) require

$$\frac{\partial h}{\partial z_1} = \frac{1}{z_1} + \frac{\varepsilon_1}{1-\varepsilon_1 z_1} - \eta\frac{c_1}{z_1} = 0 \qquad (12.29a)$$

$$\frac{\partial h}{\partial z_2} = \frac{1}{z_2} + \frac{\varepsilon_2}{1-\varepsilon_2 z_2} - \eta\frac{c_2}{z_2} = 0 \qquad (12.29b)$$

$$\frac{\partial h}{\partial \eta} = \Delta V_{tot} - c_1 \ln z_1 - c_2 \ln z_2 = 0 \qquad (12.29c)$$

Eqs. (12.29a,b) give

$$z_1 = \frac{c_1\eta - 1}{c_1\varepsilon_1\eta} \qquad z_2 = \frac{c_2\eta - 1}{c_2\varepsilon_2\eta} \qquad (12.30a,b)$$

Eqs. (12.30a,b) are substituted into (12.29c) to give

$$\Delta V_{tot} - c_1 \ln\frac{c_1\eta - 1}{c_1\varepsilon_1\eta} - c_2 \ln\frac{c_2\eta - 1}{c_2\varepsilon_2\eta} = 0 \qquad (12.31)$$

For a given required ΔV_{tot}, eq. (12.31) may be solved numerically for η using any root finding technique. The root is then substituted into (12.30 a,b) to give the mass ratios of the stages. The optimal mass for each stage can then be determined if the payload mass is given.

$$Z_2 = \frac{m_2 + m_{PL}}{\varepsilon_2 m_2 + m_{PL}} \quad \Rightarrow \quad m_2 = \frac{Z_2 - 1}{1 - \varepsilon_2 Z_2} m_{PL} \qquad (12.32a)$$

$$Z_1 = \frac{m_1 + m_2 + m_{PL}}{\varepsilon_1 m_1 + m_2 + m_{PL}} \quad \Rightarrow \quad m_1 = \frac{Z_1 - 1}{1 - \varepsilon_1 Z_1} (m_{PL} + m_2) \qquad (12.32b)$$

The procedure may be generalized for an N-stage vehicle

Minimize

$$f = \sum_{i=1}^{N} \left[\ln(1-\varepsilon_i) + \ln Z_i - \ln(1 - \varepsilon_i Z_i) \right] \qquad (12.33)$$

subject to

$$g = \Delta V_{tot} - \sum_{i=1}^{N} c_i \ln Z_i = 0 \qquad (12.34)$$

For an extremum of f, require

$$\frac{\partial f}{\partial z_i} + \eta \frac{\partial g}{\partial z_i} = \left[\frac{1}{z_i} + \frac{\varepsilon_i}{1-\varepsilon_i z_i}\right] + \eta\left[-\frac{c_i}{z_i}\right] = 0 \qquad i = 1, 2, \dots N \qquad (12.35)$$

yielding

$$z_i = \frac{\eta c_i - 1}{\eta c_i \varepsilon_i} \qquad (12.36)$$

where η is obtained by solving

$$g = \Delta V_{tot} - \sum_{i=1}^{N} c_i \ln\left[\frac{\eta c_i - 1}{\eta c_i \varepsilon_i}\right] = 0 \qquad (12.37)$$

numerically.