ME 57200 Aerodynamic Design

Lecture #11: Elemental Flows

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Midterm Exam

- Tuesday, 3/12, from 09:30 to 10:45 am at Shepard S-308.

- The exam is open-book and open-notes.

- 5 True/False Questions: 10 pt
- 4 Math-based Problems: 40 pt
- Total: 50 pt

Laplace's Equation

- Any irrotational, incompressible flow has a velocity potential and stream function (for two-dimensional flow) that both satisfy Laplace's equation.

- Conversely, any solution of Laplace's equation represents the velocity potential or stream function (two-dimensional) for an irrotational, incompressible flow.

- Note: the sum of any particular solutions of a linear differential equation is also a solution of the equation.

$$\boxed{\nabla^2 \phi = 0}$$

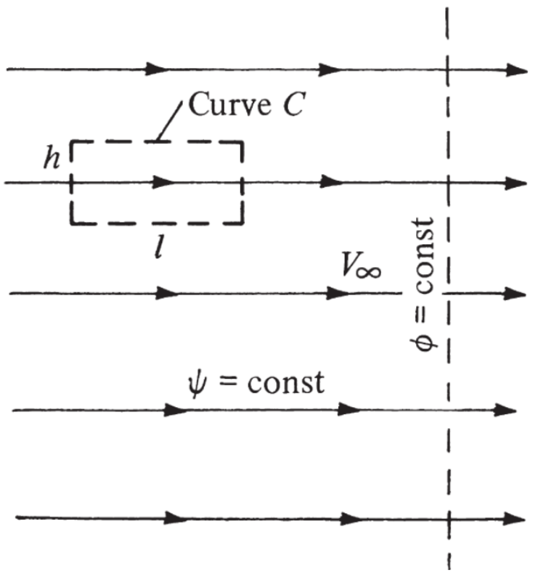
$\varphi_1, \varphi_2, \varphi_3, \varphi_4, \ldots, \varphi_n$ represent *n* separate solutions

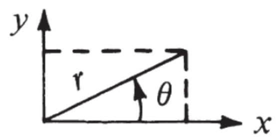
$\boxed{\varphi = \varphi_1 + \varphi_2 \ldots + \varphi_n \text{ is also a solution}}$

A complicated flow pattern for an irrotational, incompressible flow can be synthesized by adding together a number of elementary flows that are also irrotational and incompressible.

Elementary Flows

- **Uniform Flow**





$$\frac{\partial \phi}{\partial x} = u = V_\infty \quad \Rightarrow \quad \phi = V_\infty x + f(y)$$

$$\frac{\partial \phi}{\partial y} = v = 0 \quad \Rightarrow \quad \phi = \text{const} + g(x)$$

$$\phi = V_\infty x + \text{const}$$

$$\frac{\partial \psi}{\partial y} = u = V_\infty$$

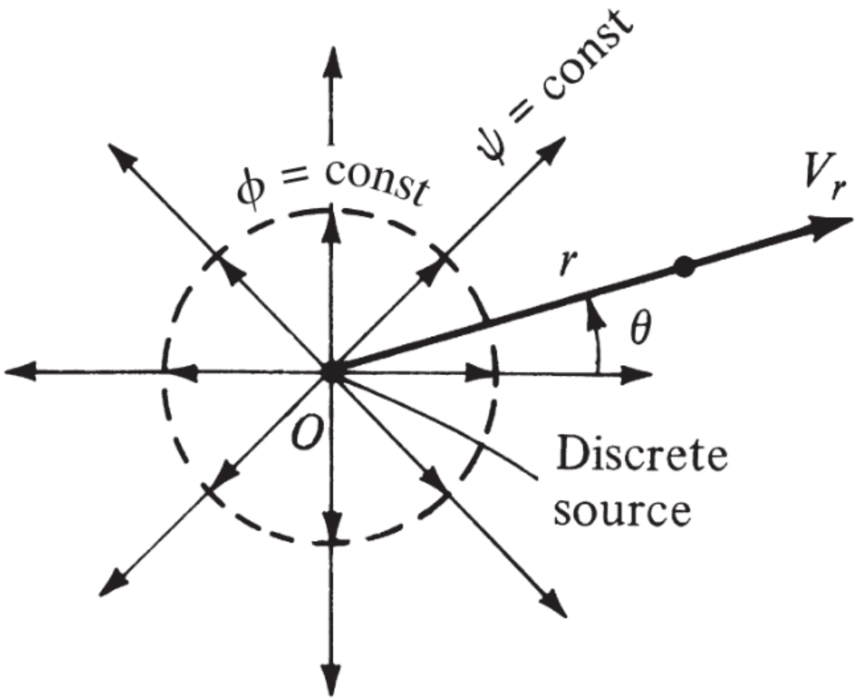
$$\frac{\partial \psi}{\partial x} = -v = 0$$

$$\Rightarrow \quad \boxed{\psi = V_\infty y}$$

$$\Gamma = -\oint_C \mathbf{V} \cdot \mathbf{ds} = -\mathbf{V}_\infty \cdot \oint_C \mathbf{ds} = \mathbf{V}_\infty \cdot 0 = 0$$

Elementary Flows

- **Source Flow**



Source flow

$$\mathbf{V}_r = \frac{c}{r}$$

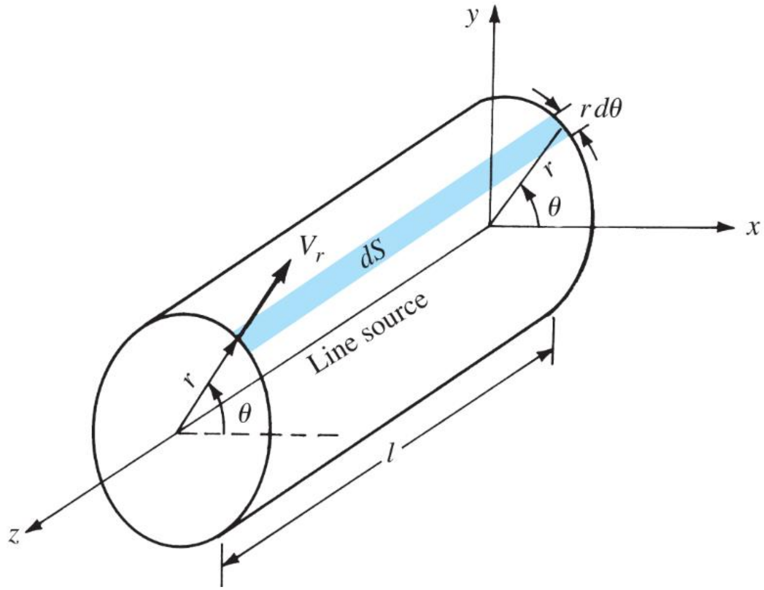
$$\mathbf{V}_\theta = 0$$

What is the value of "c"

Elementary Flows

- **Source Flow**

What is the value of "c"



$$\dot{m} = \int_0^{2\pi} \rho V_r (r\, d\theta) l = \rho r l V_r \int_0^{2\pi} d\theta = 2\pi\, r l \rho V_r$$

$$\dot{v} = \frac{\dot{m}}{\rho} = 2\pi\, r l V_r$$

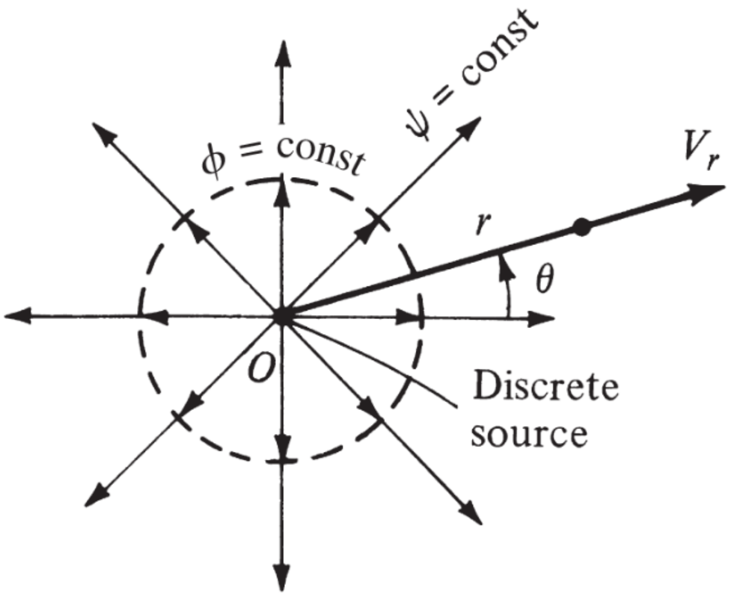
$$\Lambda = \frac{\dot{v}}{l} = 2\pi\, r V_r$$ Source Strength

$$\mathbf{V}_r = \frac{c}{r} = \frac{\Lambda}{2\pi\, r}$$

$$\boxed{V_r = \frac{\Lambda}{2\pi\, r}}$$

Elementary Flows

- **Source Flow**



Source flow

$$\Gamma = -\iint_S (\nabla \times \mathbf{V}) \cdot \mathbf{dS} = 0$$

$$\frac{\partial \phi}{\partial r} = V_r = \frac{\Lambda}{2\pi r} \Rightarrow \phi = \frac{\Lambda}{2\pi} \ln r + f(\theta)$$

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = V_\theta = 0 \Rightarrow \phi = \text{const} + f(r)$$

$$\boxed{\phi = \frac{\Lambda}{2\pi} \ln r}$$

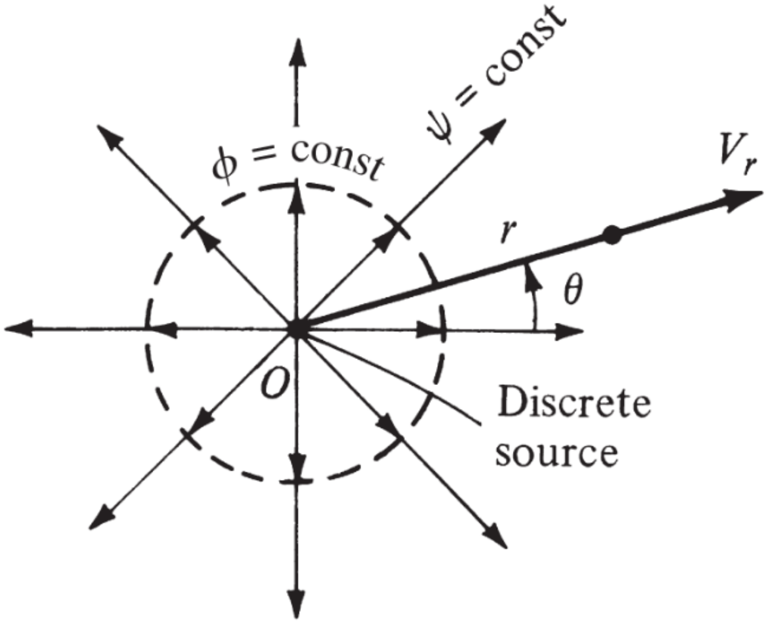
$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_r = \frac{\Lambda}{2\pi r} \Rightarrow \psi = \frac{\Lambda}{2\pi}\theta + f(r)$$

$$-\frac{\partial \psi}{\partial r} = V_\theta = 0 \Rightarrow \psi = \text{const} + f(\theta)$$

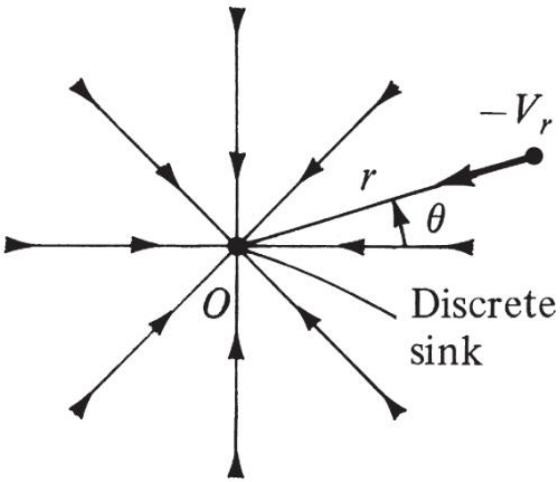
$$\boxed{\psi = \frac{\Lambda}{2\pi}\theta}$$

Elementary Flows

- **Source Flow vs. Sink Flow**



Source flow



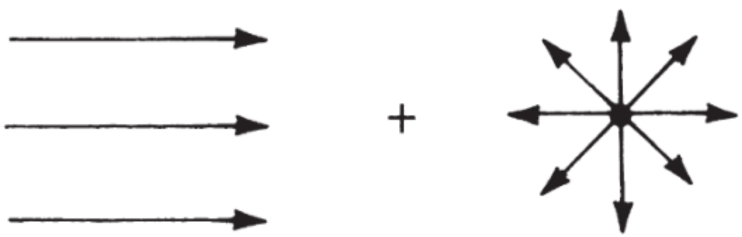
Sink flow

- In a source flow, the streamlines are directed away from the origin.
- In a sink flow, the streamlines are directed toward the origin.
- A sink flow is simply a negative source flow

Elementary Flows

- **Combination of a uniform flow with a source flow**

Consider a polar coordinate system with a source of strength Λ located at the origin. Superimpose on this flow a uniform stream with velocity V_∞ moving from left to right.



Uniform stream + Source

$$\psi = V_\infty r \sin\theta \qquad \psi = \frac{\Lambda}{2\pi}\theta$$

$$\boxed{\nabla^2 \phi = 0}$$

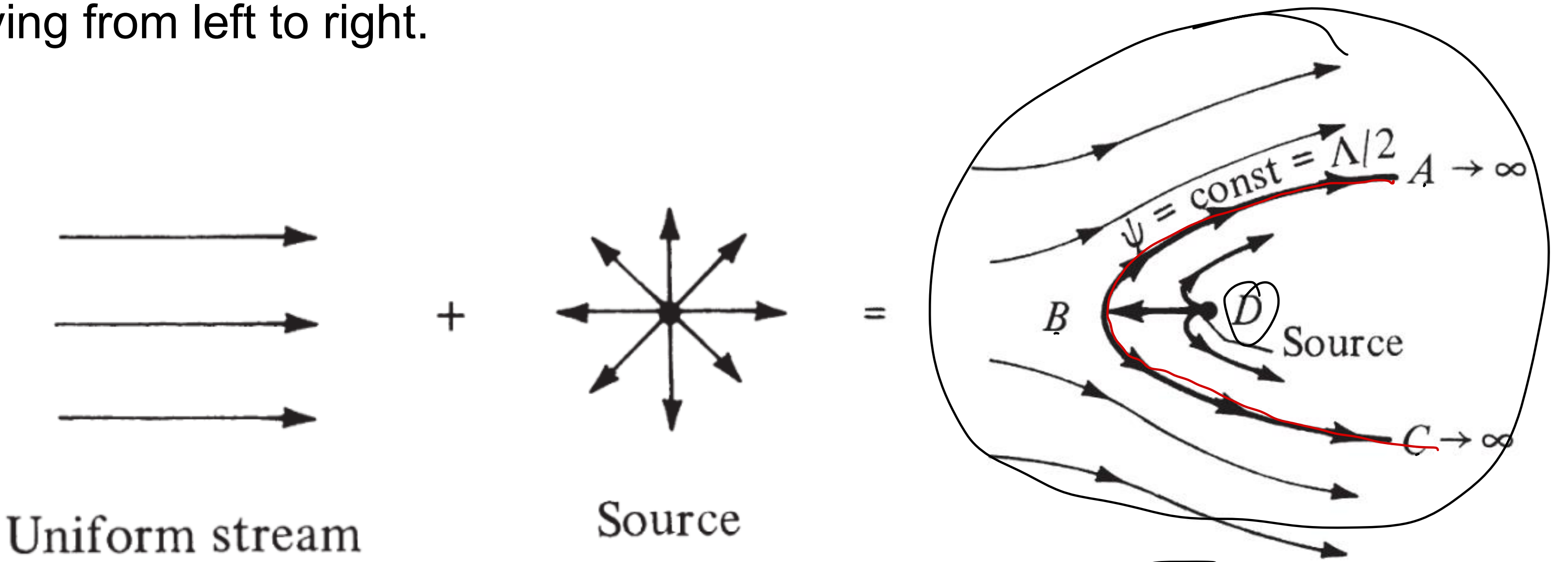
$\varphi_1, \varphi_2 \ldots, \varphi_n$ represent *n* separate solutions

$\varphi = \varphi_1 + \varphi_2 \ldots + \varphi_n$ is also a solution

Elementary Flows

- **Combination of a uniform flow with a source flow**

Consider a polar coordinate system with a source of strength Λ located at the origin. Superimpose on this flow a uniform stream with velocity V_∞ moving from left to right.



$$\psi = V_\infty r \sin\theta$$

$$\psi = \frac{\Lambda}{2\pi}\theta$$

$$\boxed{\psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}\theta}$$

Elementary Flows

- **Combination of a uniform flow with a source flow**

$$\psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}\theta$$

$$\Rightarrow \boxed{\begin{cases} \underline{V_r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_\infty \cos\theta + \frac{\Lambda}{2\pi r} \\ \underline{V_\theta} = -\frac{\partial \psi}{\partial r} = -V_\infty \sin\theta \end{cases}}$$

stagnation point in this combined flow can be obtained by setting the velocity components equal to zero.

$$\begin{cases} V_\infty \cos\theta + \frac{\Lambda}{2\pi r} = 0 \\ V_\infty \sin\theta = 0 \end{cases} \Rightarrow \begin{cases} r = \frac{\Lambda}{2\pi V_\infty} \\ \theta = \pi \end{cases}$$

Elementary Flows

$$\Rightarrow \psi = V_\infty \frac{\Lambda}{2\pi V_\infty} \sin\pi + \frac{\Lambda}{2\pi}\cdot\pi = \text{constant} = \frac{\Lambda}{2}$$

The streamline that goes through the stagnation point is described by $\psi = \frac{\Lambda}{2}$

$\cancel{\text{A}}$: There is no flow crossing the streamlines.

$\cancel{\text{A}}$: We can construct the flow over a solid semi-infinite body described by "ABC", by adding a uniform stream with velocity V_∞ to a source of strength Λ at point "D"

Elementary Flows

- **Combination of a uniform flow with a source and sink**

Uniform flow: $\psi_1 = V_\infty r \sin\theta$

Source flow: $\psi_2 = \frac{\Lambda}{2\pi}\theta_1$

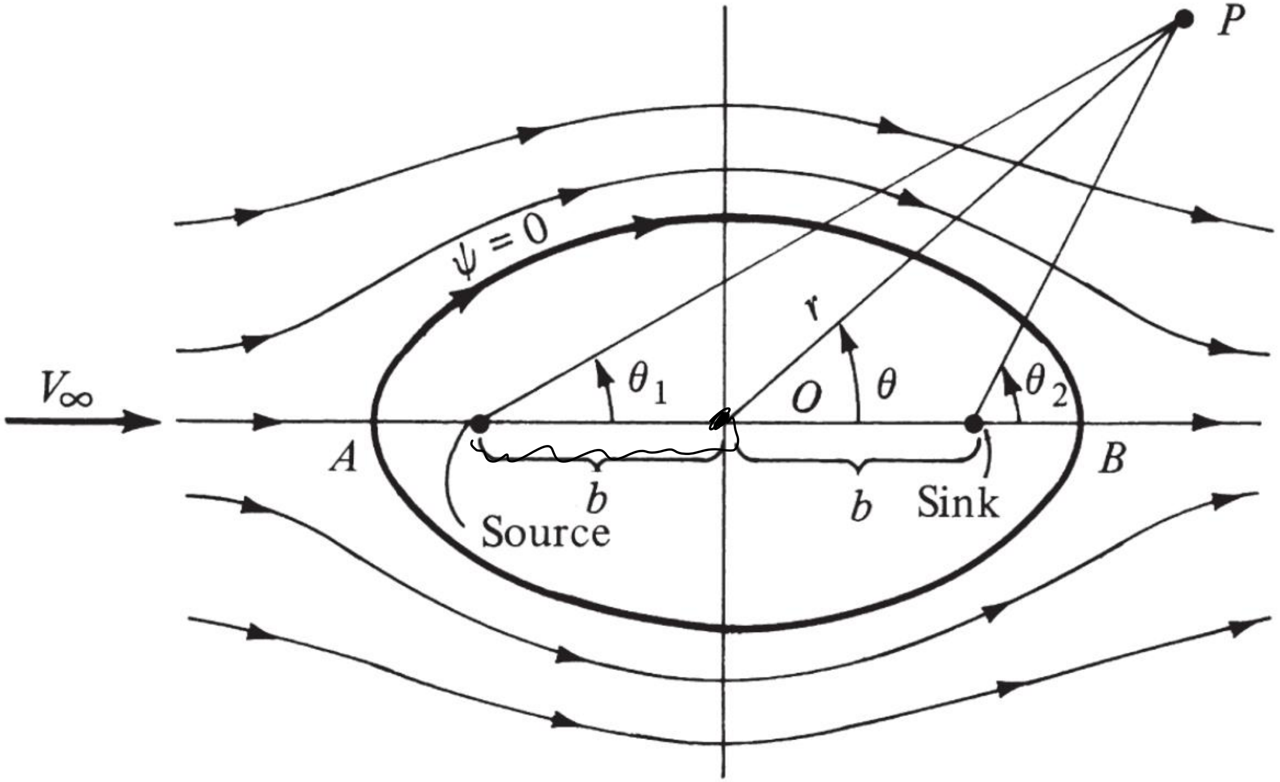
Sink flow: $\psi_3 = -\frac{\Lambda}{2\pi}\theta_2$

Combined flow: $\psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}\theta_1 - \frac{\Lambda}{2\pi}\theta_2$

$\Rightarrow \psi = V_\infty r \sin\theta + \frac{\Lambda}{2\pi}(\theta_1 - \theta_2)$

Elementary Flows

- **Combination of a uniform flow with a source and sink**



Elementary Flows

stagnation ($\vec{V}=0$)

A, B

$$\psi = V_\infty r\sin\theta + \frac{\Lambda}{2\pi}(\theta_1-\theta_2) \Rightarrow \begin{cases} V_r = 0 \\ V_\theta = 0 \end{cases}$$

$$\Rightarrow OA = OB = \sqrt{b^2+\frac{\Lambda b}{\pi V_\infty}}$$

$$\Rightarrow \psi = V_\infty r\sin\theta + \frac{\Lambda}{2\pi}(\theta_1-\theta_2) = 0$$

Stagnation streamline: $\psi = V_\infty r\sin\theta + \frac{\Lambda}{2\pi}(\theta_1-\theta_2) = 0$

"Rankine Oval"

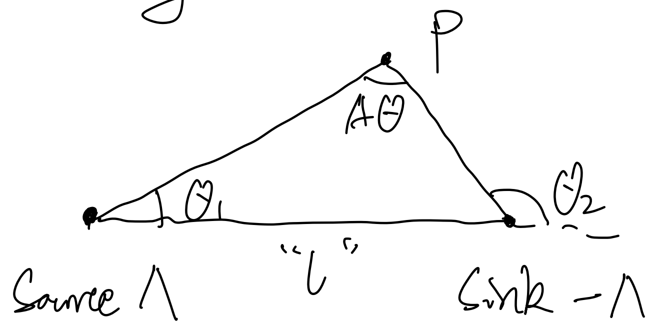
Elementary Flows

※: The region inside the oval can be replaced by a solid body with the shape given by $\psi = 0$, and the region outside the oval can be interpreted as the inviscid, irrotational, incompressible flow over the solid body.

Elementary Flows

Doublet Flow : A Source–sink flow

Consider a source of strength, Λ, and a sink of equal strength $-\Lambda$, separated by a distance l,



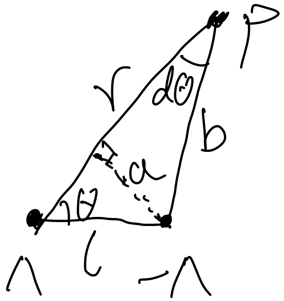
At any point "P" in the flow, the stream function is

$$\psi = \frac{\Lambda}{2\pi}(\theta_1 - \theta_2) = \underbrace{-\frac{\Lambda}{2\pi}\Delta\theta}$$

Define a "doublet" as: a flow pattern as "l" $\to 0$ while $l \cdot \Lambda$ remains constant. The strength of this doublet: $K \equiv l \cdot \Lambda$

$$\psi = \lim_{l \to 0}\left(-\frac{\Lambda}{2\pi} d\theta\right) \qquad K = l \cdot \Lambda = \text{constant.}$$

Elementary Flows



$$a = l \cdot \sin\theta, \qquad b \approx r - l\cos\theta$$

$$d\theta \approx \frac{a}{b} = \frac{l\sin\theta}{r - l\cos\theta}$$

$$\psi = \lim_{l \to 0}\left(-\frac{\Lambda}{2\pi}\frac{l\sin\theta}{r - l\cos\theta}\right) = \lim_{l \to 0}\left(-\frac{K}{2\pi}\frac{\sin\theta}{r - l\cos\theta}\right)$$

$$\Rightarrow \psi = -\frac{K}{2\pi}\frac{\sin\theta}{r}$$

streamlines of a doublet flow: $\psi = \frac{-K}{2\pi}\frac{\sin\theta}{r} = \text{constant}$

$$r = C \cdot \sin\theta$$

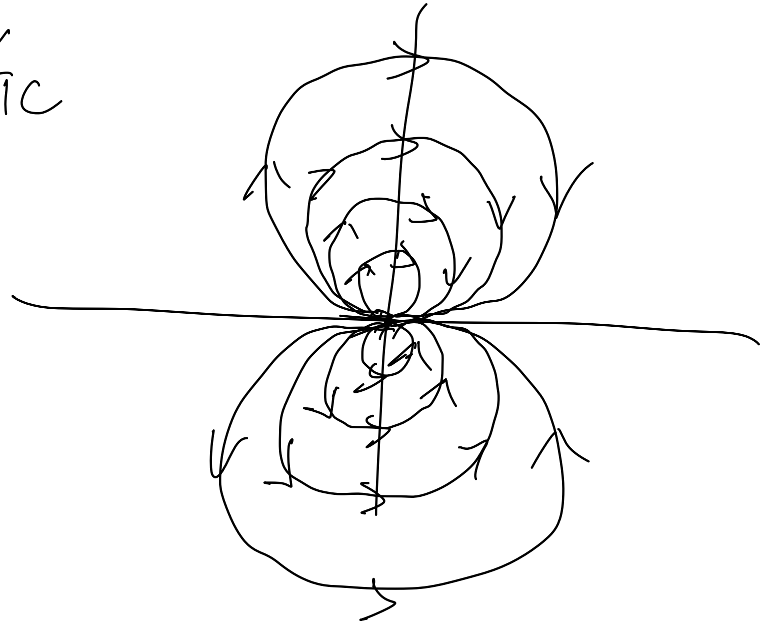
$$\Rightarrow r = -\frac{K}{2\pi C}\sin\theta$$

Elementary Flows

In polar coordinates, a circle with diameter "d" on the vertical axis and with the center located $d/2$ directly above the origin

$$r = d \cdot \sin\theta$$

$\Rightarrow$ The streamlines for a doublet flow are a family of circles with diameters $K/2\pi C$



In-Class Quiz