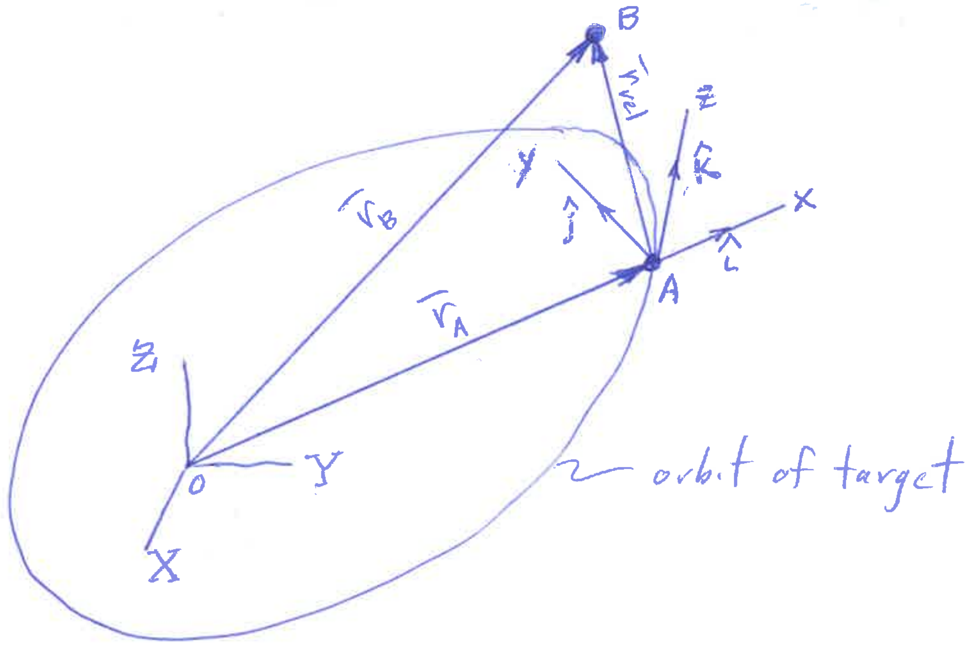
9.7. Relative Motion of Two Vehicles in Orbit

Vehicle A: target vehicle

Vehicle B: interceptor (chaser) vehicle

Wish to determine the motion of B as viewed from A.



Inertial reference frame (X, Y, Z) has its origin at the focus.

Moving reference frame (x, y, z) has its origin at the target vehicle

Define the unit vectors

$$\hat{i} = \frac{\bar{r}_A}{r_A} \quad \text{(directed along } \bar{r}_A\text{)} \qquad (9.70)$$

$$\hat{k} = \frac{\bar{h}_A}{h_A} \quad \text{(directed perpendicular to plane of target orbit)} \tag{9.71}$$

$$\hat{j} = \hat{k} \times \hat{i} \quad \text{(lies in plane of target orbit)} \tag{9.72}$$

The position, velocity and acceleration of B as viewed by A is

$$\bar{r}_{rel} = x\hat{i} + y\hat{j} + z\hat{k} \tag{9.73a}$$

$$\bar{V}_{rel} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} \tag{9.73b}$$

$$\bar{a}_{rel} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k} \tag{9.73c}$$

Note: The unit vectors $\hat{i}, \hat{j}, \hat{k}$ are not changing direction when viewed from the moving frame.

The angular velocity $\bar{\Omega}$ of the moving frame which is just the angular velocity of $\bar{r}_A$ is obtained from the fact that

$$\bar{h}_A = \bar{r}_A \times \bar{V}_A = r_A^2 \dot{\theta}_A \hat{k} = r_A^2 \bar{\Omega}_A = r_A^2 \bar{\Omega}$$

$$\text{or} \quad \bar{\Omega} = \frac{\bar{h}_A}{r_A^2} = \frac{\bar{r}_A \times \bar{V}_A}{r_A^2} \tag{9.74}$$

The angular acceleration of the moving frame is obtained by differentiating (9.74)

$$\dot{\overline{\Omega}} = \overline{h}_A \frac{d}{dt}\left(\frac{1}{r_A^2}\right) = -2\,\frac{\overline{h}_A}{r_A^3}\,\dot{r}_A \qquad (9.75)$$

Using the equations prior to (5.24)

$$\overline{r} = r\,\hat{i}_r$$

$$\overline{V} = \frac{dr}{dt}\hat{i}_r + r\frac{d\theta}{dt}\hat{i}_\theta$$

can write

$$\overline{r}\cdot\overline{V} = r\frac{dr}{dt}$$

from which

$$\dot{r}_A = \frac{\overline{r}_A \cdot \overline{V}_A}{r_A} \qquad (9.76)$$

Therefore substituting (9.76) into (9.75)

$$\dot{\overline{\Omega}} = -2\,\frac{\overline{r}_A\cdot\overline{V}_A}{r_A^4}\,\overline{h}_A \qquad (9.77)$$

Using (9.74), eq. (9.77) can be written as

$$\dot{\bar{\Omega}} = -2\,\frac{\bar{r}_A \cdot \bar{V}_A}{r_A^2}\,\bar{\Omega} \qquad (9.78)$$

The position $\bar{r}_{vel}$, velocity $\bar{V}_{vel}$ and acceleration $\bar{a}_{vel}$ of vehicle B as viewed from vehicle A can be determined from

$$\bar{r}_{vel} = \bar{r}_B - \bar{r}_A \qquad (9.79)$$

$$\bar{V}_{vel} = \bar{V}_B - \bar{V}_A - \bar{\Omega} \times \bar{r}_{vel} \qquad (9.80)$$

$$\bar{a}_{vel} = \bar{a}_B - \bar{a}_A - \underbrace{\dot{\bar{\Omega}} \times \bar{r}_{vel}}_{\text{angular acceleration}} - \underbrace{\bar{\Omega} \times (\bar{\Omega} \times \bar{r}_{vel})}_{\text{centripetal acceleration}} - \underbrace{2\bar{\Omega} \times \bar{r}_{vel}}_{\text{Coreolis' acceleration}} \qquad (9.81)$$

where

$\bar{r}_A, \bar{V}_A, \bar{a}_A$ are the position, velocity and acceleration of vehicle A as viewed from the inertial reference frame.

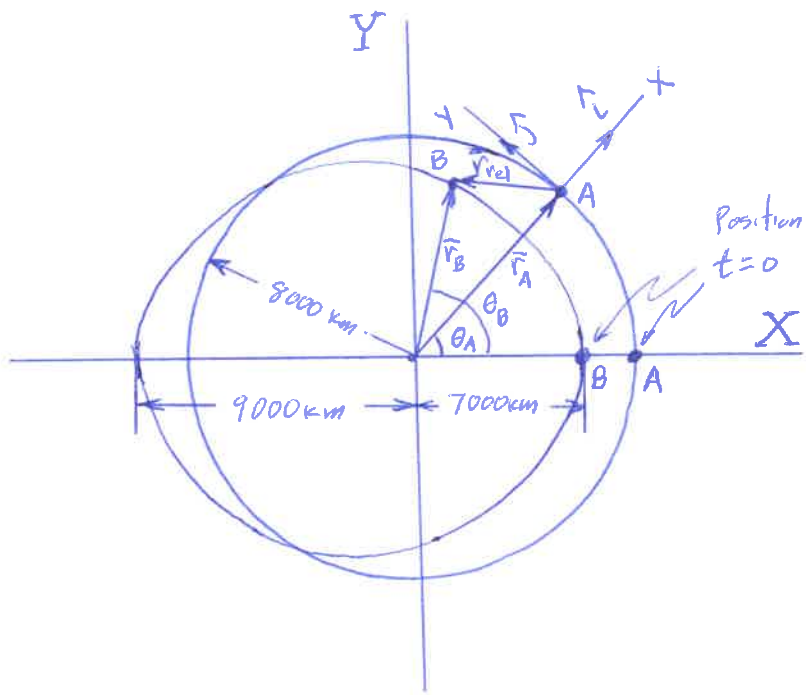
$\bar{r}_B, \bar{V}_B, \bar{a}_B$ are the position, velocity and acceleration of vehicle B as viewed from the inertial reference frame

$\bar{\Omega}$ and $\dot{\bar{\Omega}}$ are given by (9.74) & (9.78), respectively.

EXAMPLE

Vehicle A in circular orbit $r_A = 8000$ km

Vehicle B in coplanar elliptic orbit $a = 8000$ km, $e = 0.125$



Since $r_A = a$, period of both orbits

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\frac{(8000)^3}{3.986\times10^5}} = 7121 \text{ sec} = 1.9781 \text{ hr}$$

From (7.26) the state vectors are given by

$$\bar{r}_A = r_A\left[\cos\theta_A\,\hat{I} + \sin\theta_A\,\hat{J}\right] \qquad (1a)$$

$$\bar{V}_A = \sqrt{\frac{\mu}{r_A}}\left[-\sin\theta_A\,\hat{I} + \cos\theta_A\,\hat{J}\right] \qquad (1b)$$

$$\bar{r}_B = r_B\left[\cos\theta_B\,\hat{I} + \sin\theta_B\,\hat{J}\right] \qquad (2a)$$

$$\bar{V}_B = \sqrt{\frac{\mu}{a(1-e^2)}}\left[-\sin\theta_B\,\hat{I} + (e+\cos\theta_B)\,\hat{J}\right] \qquad (2b)$$

$$\bar{h}_A = \bar{r}_A \times \bar{V}_A = \begin{vmatrix} \hat{I} & \hat{J} & \hat{K} \\ r_A \cos\theta_A & r_A \sin\theta_A & 0 \\ \sqrt{\frac{\mu}{r_A}} \sin\theta_A & \sqrt{\frac{\mu}{r_A}} \cos\theta_A & 0 \end{vmatrix}$$

$$\bar{h}_A = \left(r_A \sqrt{\frac{\mu}{r_A}} \cos^2\theta_A + r_A \sqrt{\frac{\mu}{r_A}} \sin^2\theta_A \right) \hat{K} = r_A \sqrt{\frac{\mu}{r_A}} \hat{K} = \sqrt{\mu r_A}\, \hat{K} \quad (3)$$

$$h_A = \sqrt{\mu r_A} \quad (4)$$

From (9.70), (9.71) & (9.72)

$$\hat{i} = \frac{\bar{r}_A}{r_A} = \cos\theta_A \hat{I} + \sin\theta_A \hat{J} \quad (5a)$$

$$\hat{k} = \frac{\bar{h}_A}{h_A} = \hat{K} \quad (5b)$$

$$\hat{j} = \hat{k} \times \hat{i} = \begin{vmatrix} \hat{I} & \hat{J} & \hat{K} \\ 0 & 0 & 1 \\ \cos\theta_A & \sin\theta_A & 0 \end{vmatrix} = -\sin\theta_A \hat{I} + \cos\theta_A \hat{J} \quad (5c)$$

From which

$$\hat{I} = \cos\theta_A \hat{i} - \sin\theta_A \hat{j} \quad (6a)$$

$$\hat{J} = \sin\theta_A \hat{i} + \cos\theta_A \hat{j} \quad (6b)$$

$$\hat{K} = \hat{k} \quad (6c)$$

From (9.74)

$$\bar{\Omega} = \frac{\bar{h}_A}{r_A^2} = \frac{\sqrt{\mu r_A}\,\hat{K}}{r_A^2} = \sqrt{\frac{\mu}{r_A^3}}\,\hat{K} \qquad (7)$$

From state vector for A

$$\bar{r}_A \cdot \bar{V}_A = r_A\sqrt{\frac{\mu}{r_A}}\left[-\sin\theta_A\cos\theta_A + \sin\theta_A\cos\theta_A\right] = 0$$

From (9.78)

$$\dot{\bar{\Omega}} = -2\,\frac{\cancelto{0}{\bar{r}_A\cdot\bar{V}_A}}{r_A^2}\,\bar{\Omega} = 0 \qquad (8)$$

From equation of motion for 2-body problem (5.1)

$$\bar{a}_A = -\frac{\mu}{r_A^3}\,\bar{r}_A = -\frac{\mu}{r_A^3}\,r_A\left[\cos\theta_A\,\hat{I} + \sin\theta_A\,\hat{J}\right]$$

$$= -\frac{\mu}{r_A^2}\left[\cos\theta_A\,\hat{I} + \sin\theta_A\,\hat{J}\right] \qquad (9)$$

$$\bar{a}_B = -\frac{\mu}{r_B^3}\,\bar{r}_B = -\frac{\mu}{r_B^3}\,r_B\left[\cos\theta_B\,\hat{I} + \sin\theta_B\,\hat{J}\right]$$

$$= -\frac{\mu}{r_B^2}\left[\cos\theta_B\,\hat{I} + \sin\theta_B\,\hat{J}\right] \qquad (10)$$

Using (6) the position vectors $\bar{r}_A, \bar{r}_B$ can be written in terms of their components in the moving frame as

$$\bar{r}_A = r_A \hat{\imath} \qquad (11)$$

$$\bar{r}_B = r_B \cos(\theta_B - \theta_A)\hat{\imath} + r_B \sin(\theta_B - \theta_A)\hat{\jmath} \qquad (12)$$

Sub. (11) & (12) into (9.79)

$$\bar{r}_{rel} = \bar{r}_B - \bar{r}_A$$

$$\boxed{\bar{r}_{rel} = \left(r_B \cos(\theta_B - \theta_A) - r_A\right)\hat{\imath} + r_B \sin(\theta_B - \theta_A)\hat{\jmath}} \qquad (13)$$

where

$$\boxed{r_B = \frac{a(1-e^2)}{1 + e\cos\theta_B}} \qquad (14)$$

Need to relate θ_B to θ_A

For A to travel in its circular orbit from $\theta_A = 0$ to $\theta_A = \theta_A$, the required time is

$$t_A = \frac{\theta_A}{2\pi} \cdot 2\pi \sqrt{\frac{r_A{}^3}{\mu}} = \theta_A \sqrt{\frac{r_A{}^3}{\mu}} \qquad (15)$$

For B to travel in its elliptic orbit from $\theta_B = 0$ to $\theta_B = \theta_B$, the required time is

$$t_B = \frac{M_B}{n} \qquad (16a)$$

where

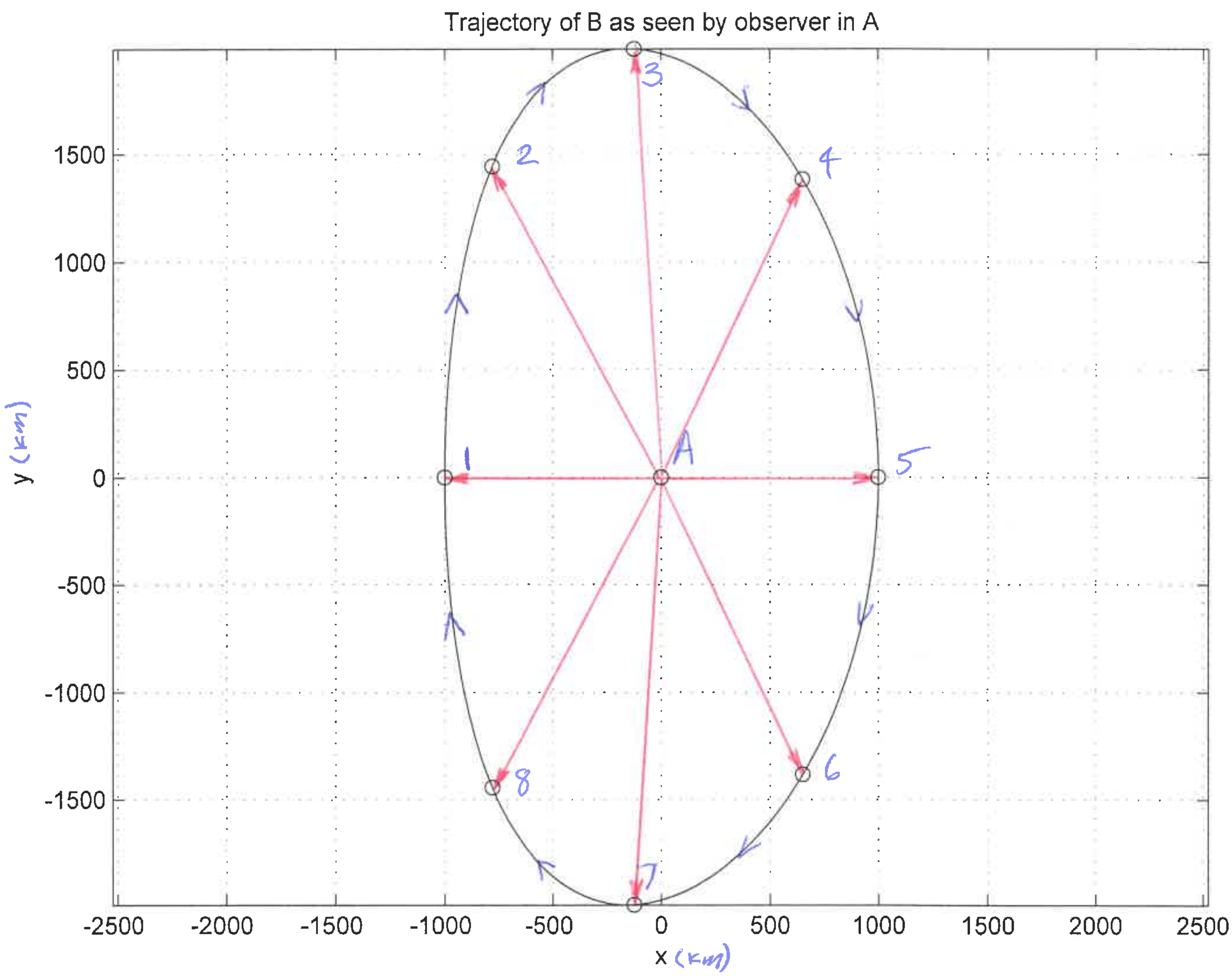
$$n = \sqrt{\frac{\mu}{a^3}} \qquad (16b)$$

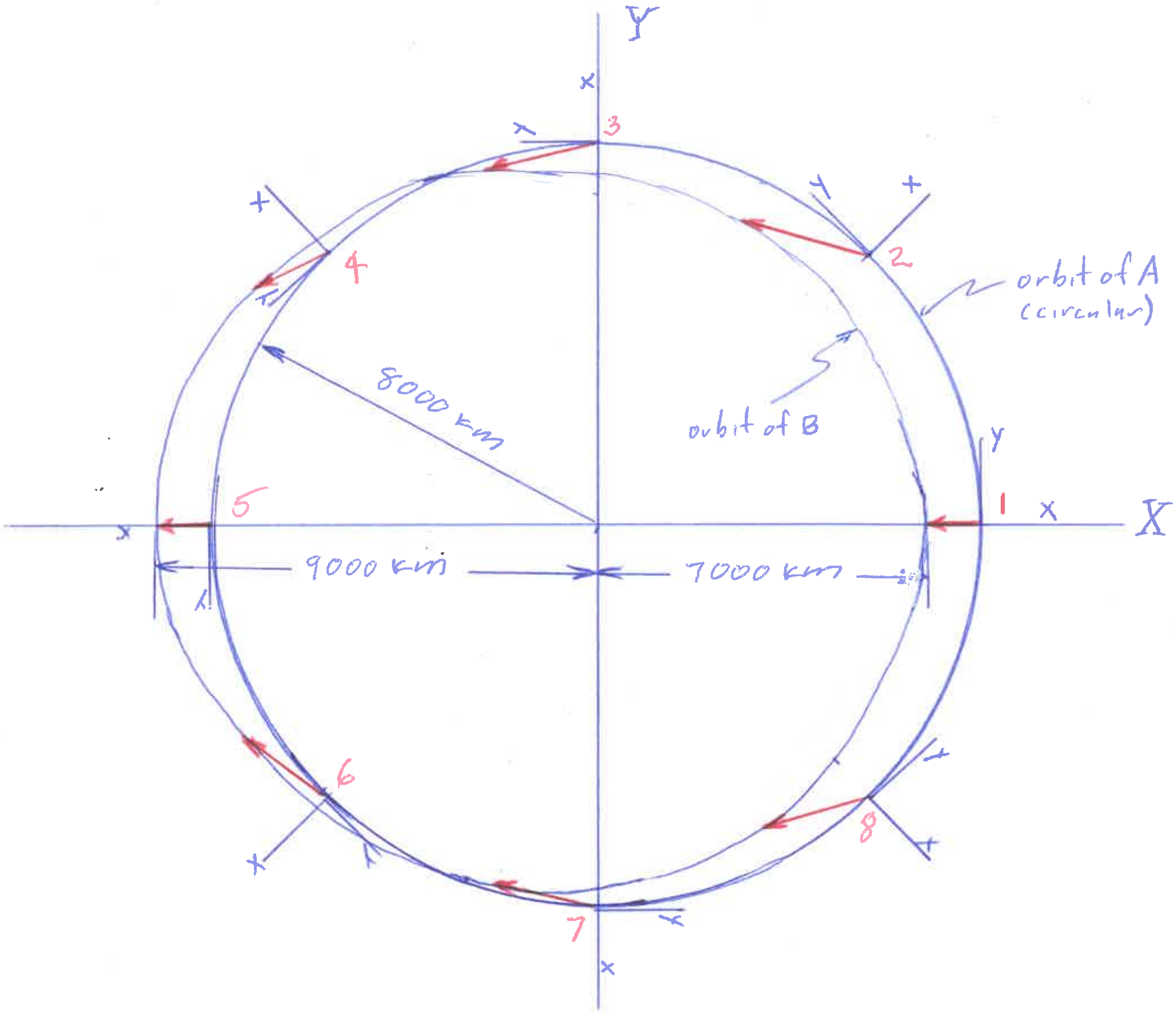
$$M_B = E_B - e \sin E_B \qquad (16c)$$

$$\tan \tfrac{1}{2} E_B = \sqrt{\frac{1-e}{1+e}} \tan \tfrac{1}{2} \theta_B \qquad (16d)$$

With $t_A = t_B$, eqs. (15) & (16) relate relate θ_B to θ_A.

Position of B	θ_A	t (hr)	θ_B	X (km)	Y (km)
1	0°	0	0°	−1000	0
2	45°	0.2473	56.3047°	−778.6	1443.6
3	90°	0.4945	104.1779°	−123.7	1989.8
4	135°	0.7418	144.0799°	652.2	1382.7
5	180°	0.9890	180°	1000	0
6	225°	1.2363	215.9201°	652.2	−1382.7
7	270°	1.4836	255.8221	−123.7	−1989.8
8	315°	1.7308	303.6953	−778.6	−1443.6
9	360°	1.9781	360°	−1000	0


Trajectory of B as seen by observer in A
y (km)
x (km)
1
2
3
4
5
6
7
8
A
1500
1000
500
0
-500
-1000
-1500
-2500
-2000
-1500
-1000
-500
0
500
1000
1500
2000
2500



Orbits of A and B in inertial frame

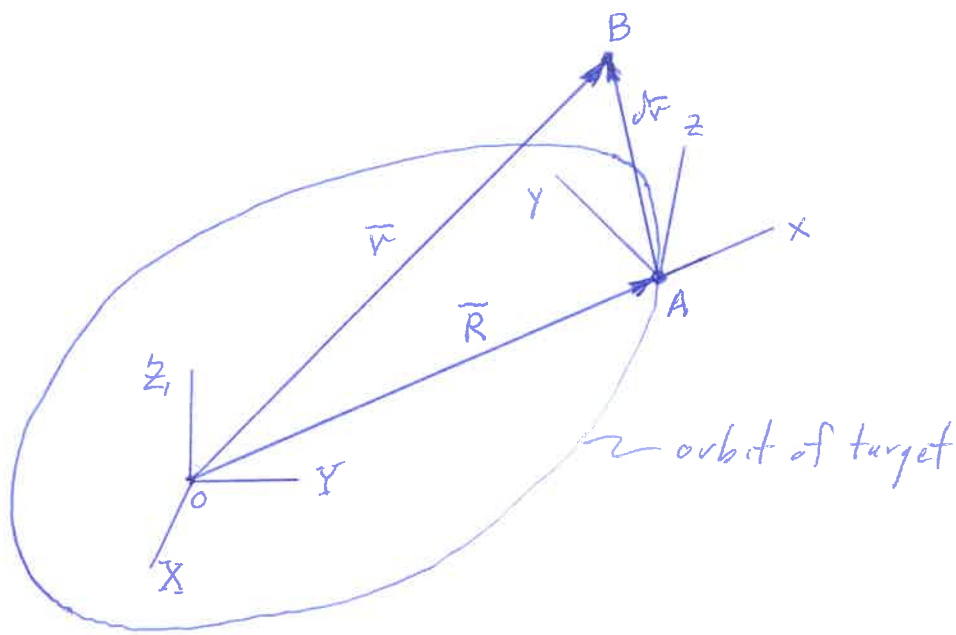
position of B relative to A

9.8 Linearization of the Equations of Relative Motion in Orbit.

Vehicle A: target vehicle

Vehicle B: interceptor (chaser) vehicle

Wish to determine the motion of B as viewed from A when the distance between A and B is small.



$\bar{R}$ = position of A in inertial reference frame

$\bar{r}$ = position of B in inertial reference fram

$\delta\bar{r}$ = position of B relative to A (assumed small)

$$\bar{r} = \bar{R} + \delta\bar{r} \qquad (9.82)$$

Assume that A and B are in close proximity to each other so that

$$\frac{\delta r}{R} \ll 1 \quad \left(\text{and} \quad \frac{\delta r}{r} \ll 1 \right) \qquad (9.83)$$

The equation of motion of B in the inertial reference frame is

$$\ddot{\bar{r}} = -\mu \frac{\bar{r}}{r^3} \qquad (9.84)$$

Sub. (9.82) into (9.84)

$$\delta\ddot{\bar{r}} = -\ddot{\bar{R}} - \mu \frac{\bar{R} + \delta\bar{r}}{r^3} \qquad (9.85)$$

where $r = |\bar{R} + \delta\bar{r}|$

Write

$$r^2 = \bar{r} \cdot \bar{r} = (\bar{R} + \delta\bar{r}) \cdot (\bar{R} + \delta\bar{r})$$

$$= \bar{R} \cdot \bar{R} + 2\bar{R} \cdot \delta\bar{r} + \delta\bar{r} \cdot \delta\bar{r}$$

Since $\bar{R} \cdot \bar{R} = R^2$ and $d\bar{r} \cdot d\bar{r} = dr^2$

$$r^2 = R^2 + 2\bar{R} \cdot d\bar{r} + dr^2$$

$$= R^2 \left[1 + \frac{2\bar{R} \cdot d\bar{r}}{R^2} + \left(\frac{dr}{R}\right)^2 \right]$$

Since $\frac{dr}{R} \ll 1$, the last term can be neglected

$$r^2 = R^2 \left(1 + \frac{2\bar{R} \cdot d\bar{r}}{R^2} \right) \qquad (9.86)$$

Since

$$\frac{1}{r^3} = (r^2)^{-3/2}$$

can write

$$\frac{1}{r^3} = \frac{1}{R^3} \left(1 + \frac{2\bar{R} \cdot d\bar{r}}{R^2} \right)^{-3/2} \qquad (9.87)$$

Using the binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$$

with $\frac{dr}{R} \ll 1$

$$\left(1+\frac{2\bar{R}\cdot d\bar{r}}{R^2}\right) \approx 1 - \frac{3}{2}\left(\frac{2\bar{R}\cdot d\bar{r}}{R^2}\right)$$

so that (9.87) can be written as

$$\frac{1}{r^3} = \frac{1}{R^3}\left(1 - \frac{3}{R^2}\,\bar{R}\cdot d\bar{r}\right)$$

or

$$\frac{1}{r^3} = \frac{1}{R^3} - \frac{3}{R^5}\,\bar{R}\cdot d\bar{r} \qquad (9.88)$$

Sub. (9.88) into (9.85)

$$\begin{aligned}
d\ddot{\bar{r}} &= -\ddot{\bar{R}} - \mu\left(\frac{1}{R^3} - \frac{3}{R^5}\,\bar{R}\cdot d\bar{r}\right)(\bar{R}+d\bar{r}) \\
&= -\ddot{\bar{R}} - \mu\left(\frac{\bar{R}+d\bar{r}}{R^3} - \frac{3}{R^5}(\bar{R}\cdot d\bar{r})(\bar{R}+d\bar{r})\right) \\
&= -\ddot{\bar{R}} - \mu\left(\frac{\bar{R}}{R^3} + \frac{d\bar{r}}{R^3} - \frac{3}{R^5}(\bar{R}\cdot d\bar{r})\bar{R} \underbrace{- \frac{3}{R^5}(\bar{R}\cdot d\bar{r})d\bar{r}}_{\text{neglect } O\left(\frac{dr}{R}\right)^2}\right)
\end{aligned}$$

so that

$$\delta\ddot{\bar{r}} = -\ddot{\bar{R}} - \mu\frac{\bar{R}}{R^3} - \frac{\mu}{R^3}\left[\delta\bar{r} - \frac{3}{R^2}(\bar{R}\cdot\delta\bar{r})\bar{R}\right] \quad (9.89)$$

The equation of motion of A in the inertial reference frame is

$$\ddot{\bar{R}} = -\mu\frac{\bar{R}}{R^3} \quad (9.90)$$

Sub. (9.90) into (9.89)

$$\boxed{\delta\ddot{\bar{r}} = -\frac{\mu}{R^3}\left[\delta\bar{r} - \frac{3}{R^2}(\bar{R}\cdot\delta\bar{r})\bar{R}\right]} \quad (9.91)$$

Eq. (9.91) is the linearized equation of motion which describes the motion of B as viewed from an observer on A.

In the moving frame, write

$$\bar{R} = R\,\hat{i} \quad (9.92)$$

$$\delta\bar{r} = \delta x\,\hat{i} + \delta y\,\hat{j} + \delta z\,\hat{k} \quad (9.93)$$

Sub. (9.92) & (9.93) into (9.91)

$$\delta\ddot{\bar{r}} = -\frac{\mu}{R^3}\left\{(\delta x\hat{i}+\delta y\hat{j}+\delta z\hat{k}) - \frac{3}{R^2}\left[(R\hat{i})\cdot(\delta x\hat{i}+\delta y\hat{j}+\delta z\hat{k})\right](R\hat{i})\right\}$$

$$= -\frac{\mu}{R^3}\left[(\delta x\hat{i}+\delta y\hat{j}+\delta z\hat{k}) - 3\delta x\,\hat{i}\right]$$

$$\delta\ddot{\bar{r}} = -\frac{\mu}{R^3}\left(-2\delta x\,\hat{i}+\delta y\,\hat{j}+\delta z\,\hat{k}\right) \qquad (9.94)$$

In eq. (9.94) $\delta\ddot{\bar{r}}$ represents the acceleration of B relative to A as measured in the inertial frame.

The velocity and acceleration of B relative to A as measured in the moving frame can be found using (9.80) & (9.81)

$$\delta\bar{V}_{rel} = \delta\dot{\bar{r}} - \bar{\Omega}\times\delta\bar{r} \qquad (9.95)$$

$$\delta\bar{a}_{rel} = \delta\ddot{\bar{r}} - \dot{\bar{\Omega}}\times\delta\bar{r} - \bar{\Omega}\times(\bar{\Omega}\times\delta\bar{r}) - 2\bar{\Omega}\times\delta\bar{V}_{rel} \qquad (9.96)$$

From (9.74) & (9.7)

$$\bar{\Omega} = \frac{h}{R^2}\hat{k} \qquad (9.97)$$

$$\dot{\bar{\Omega}} = -\frac{2(\bar{R}\cdot\bar{V})h}{R^4}\hat{k} \qquad (9.98)$$

where h is the angular momentum of the target vehicle A and $\bar{V} = \dot{\bar{R}}$

Now proceed to evaluate each term in (9.96)

$$\dot{\bar{\Omega}} \times d\bar{r} = \left[-\frac{2(\bar{R}\cdot\bar{V})h}{R^4}\hat{k}\right] \times (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \frac{2(\bar{R}\cdot\bar{V})h}{R^4}(dy\,\hat{i} - dx\,\hat{j}) \qquad (9.99)$$

$$\bar{\Omega} \times (\bar{\Omega} \times d\bar{r}) = \frac{h}{R^2}\hat{k} \times \left[\frac{h}{R^2}\hat{k} \times (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})\right]$$

$$= -\frac{h^2}{R^4}(dx\,\hat{i} + dy\,\hat{j}) \qquad (9.100)$$

Using (9.73b) $d\bar{V}_{rel} = d\dot{x}\,\hat{i} + d\dot{y}\,\hat{j} + d\dot{z}\,\hat{k}$

$$2\bar{\Omega} \times d\bar{V}_{rel} = 2\frac{h}{R^2}\hat{k} \times (d\dot{x}\hat{i} + d\dot{y}\hat{j} + d\dot{z}\hat{k})$$

$$= 2\frac{h}{R^2}(d\dot{x}\hat{j} - d\dot{y}\hat{k}) \qquad (9.101)$$

Sub. (9.94), (9.99), (9.100) & (9.101) into (9.96)

$$d\bar{a}_{rel} = \overbrace{-\frac{\mu}{R^3}(-2dx\hat{i} + dy\hat{j} + dz\hat{k})}^{d\ddot{\bar{r}}} - \overbrace{\frac{2(\bar{R}\cdot\bar{V})h}{R^4}(dy\hat{i} - dx\hat{j})}^{\dot{\bar{\Omega}} \times d\bar{r}}$$

$$- \overbrace{\left[-\frac{h^2}{R^4}(dx\hat{i} + dy\hat{j})\right]}^{\bar{\Omega} \times (\bar{\Omega} \times d\bar{r})} - \overbrace{2\frac{h}{R^2}(d\dot{x}\hat{j} - d\dot{y}\hat{i})}^{2\bar{\Omega} \times d\bar{V}_{rel}}$$

Using (9.73c) $d\bar{a}_{rel} = d\ddot{x}\hat{i} + d\ddot{y}\hat{j} + d\ddot{z}\hat{k}$ and collecting terms on right hand side

$$d\ddot{x}\hat{i} + d\ddot{y}\hat{j} + d\ddot{z}\hat{k} = \left[\left(\frac{2\mu}{R^3} + \frac{h^2}{R^4}\right)dx - \frac{2(\bar{R}\cdot\bar{V})h}{R^4}dy + 2\frac{h}{R^2}d\dot{y}\right]\hat{i}$$

$$+ \left[\left(\frac{h^2}{R^4} - \frac{\mu}{R^3}\right)dy + \frac{2(\bar{R}\cdot\bar{V})h}{R^4}dx - 2\frac{h}{R^2}d\dot{x}\right]\hat{j}$$

$$- \frac{\mu}{R^3}dz\hat{k} \qquad (9.102)$$

or written as 3 scalar equations

$$\boxed{\begin{aligned} &\delta\ddot{x} - \left(\frac{2\mu}{R^3} + \frac{h^2}{R^4}\right)\delta x + \frac{2(\bar{R}\cdot\bar{V})h}{R^4}\delta y - 2\frac{h}{R^2}\delta\dot{y} = 0 && (9.103a)\\ &\delta\ddot{y} + \left(\frac{\mu}{R^3} - \frac{h^2}{R^4}\right)\delta y - \frac{2(\bar{R}\cdot\bar{V})h}{R^4}\delta x + 2\frac{h}{R^2}\delta\dot{x} = 0 && (9.103b)\\ &\delta\ddot{z} + \frac{\mu}{R^3}\delta z = 0 && (9.103c)\end{aligned}}$$

The solution of (9.103) gives the position of B $(\delta x, \delta y, \delta z)$ as viewed by an observer on A as a function of time.

Eqs. (9.103 a, b) are coupled since δx & δy appears in both equations.

δz appears only in (9.103c) meaning that the motion in z can be obtained independently of that in x & y.

$\bar{R}$ & $\bar{V}$ vary with time so, in general, eqs. (9.103) cannot be solved analytically and must be solved numerically.