ME 57200 Aerodynamic Design

Lecture #20: Compressible Flow

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Compressible Flow

Review of Thermodynamics

- Second Law of Thermodynamics

$$\boxed{T ds = de + p dv}$$

$$\boxed{T ds = dh - v dp}$$

$de = c_v \, dT$ and $dh = c_p \, dT$.

$$\boxed{s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}}$$

$$\boxed{s_2 - s_1 = c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1}}$$

Compressible Flow

Review of Thermodynamics

- Second Law of Thermodynamics

Isentropic Relations $\delta q = 0.$ $ds_{\text{irrev}} = 0.$

$$0 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

$$\boxed{\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}}$$

Practice Example

- Consider the gas in the reservoir of the supersonic wind tunnel in the previous example. The pressure and temperature of the air in the reservoir are 20 atm and 300 K, respectively. The air in the reservoir expands through the wind tunnel duct. At a certain location in the duct, the pressure is 1 atm. Calculate the air temperature at this location if: (a) the expansion is isentropic and (b) the expansion is non-isentropic with an entropy increase through the duct to this location of 320 J/(kg K).

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(a) the expansion is isentropic

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}$$

$$T_2 = T_1\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = 300\left(\frac{1}{20}\right)^{\frac{0.4}{1.4}} = 300(0.05)^{0.2857}$$

$$= 300(0.4249) = \boxed{127.5\text{K}}$$

Practice Example

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(b) the expansion is non-isentropic with an entropy increase through the duct to this location of 320 J/(kg K)

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

$$c_p = \frac{\gamma R}{\gamma - 1} = \frac{(1.4)(287)}{0.4} = 1004.5\frac{\text{J}}{\text{kg}\cdot\text{K}}$$

$$320 = 1004.5\ln\left(\frac{T_2}{300}\right) - (287)\ln\left(\frac{1}{20}\right)$$

$$= 1004.5\ln\left(\frac{T_2}{300}\right) - (-859.78)$$

$$\ln\left(\frac{T_2}{300}\right) = \frac{320 - 859.78}{1004.5} = -0.5374$$

$$\frac{T_2}{300} = e^{-0.5374} = 0.5843$$

$$T_2 = (0.5843)(300) = \boxed{175.3\text{K}}$$

Practice Example

- Consider a Boeing 747 flying at a standard altitude of 36,000 ft. The pressure at a point on the wing is 400 lb/ft^2. Assuming isentropic flow over the wing, calculate the temperature at tis point.

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Solution

From **Appendix E**, at a standard altitude of 36,000 ft, p_∞ = 476 lb/ft^2 and T_∞ = 391 °R.

$$\frac{p}{p_\infty} = \left(\frac{T}{T_\infty}\right)^{\gamma/(\gamma-1)}$$

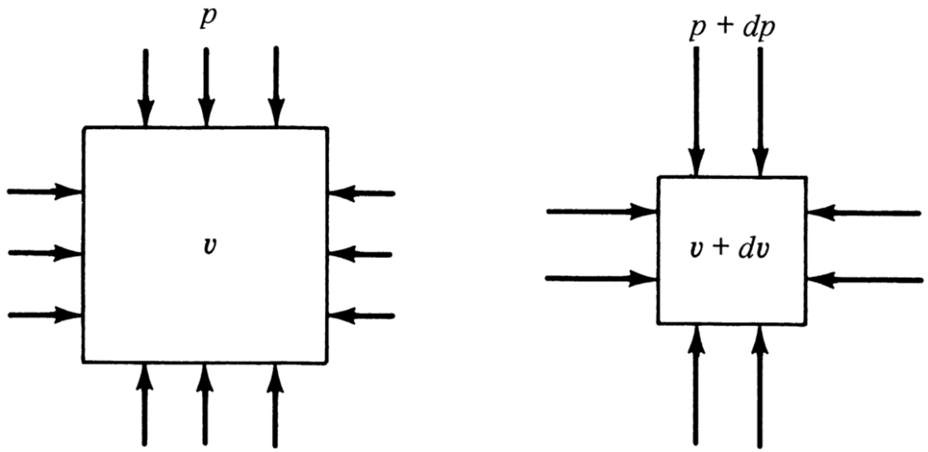
$$T = T_\infty\left(\frac{p}{p_\infty}\right)^{(\gamma-1)/\gamma} = 391\left(\frac{400}{476}\right)^{0.4/1.4} = \boxed{372°\mathrm{R}}$$

Compressible Flow

Compressibility

- Consider a small element of fluid of volume v. The pressure exerted on the sides of the element is p. Assume the pressure is now increased by an infinitesimal amount dp. The volume of the element will change by a corresponding amount dv. Here, the volume will decrease, therefore dv is a negative quantity.

The compressibility τ of the fluid is: $$\tau = -\frac{1}{v}\frac{dv}{dp}$$



Compressible Flow

Compressibility

- If the temperature of the fluid element is held constant, due to heat transfer, τ is identified as the isothermal compressibility, τ_T

$$\tau_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T$$

Constant Temperature

- If there is no heat transfer and no friction, τ is identified as the isentropic compressibility, τ_s

$$\tau_s = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_s$$

Constant Entropy

- *The compressibility of gases is several orders of magnitude larger than that of liquids*

Compressible Flow

Compressibility

- The compressibility τ of the fluid is:

$$\tau = -\frac{1}{v}\frac{dv}{dp}$$

$$v = 1/\rho \quad \Longrightarrow \quad \tau = \frac{1}{\rho}\frac{d\rho}{dp} \quad \Longrightarrow \quad d\rho = \rho\tau\, dp$$

- For liquid, compressibility τ is very small, for a given pressure change dp from one point to another in the flow, $d\rho$ will be negligibly small, and we can assume that ρ is constant.
- For gas, compressibility τ is large, for a given pressure change dp from one point to another in the flow, $d\rho$ can be large, and ρ is NOT constant.
  - However for low-speed flow, dp is small, the value of $d\rho$ is dominated by dp, we can still assume that ρ is constant

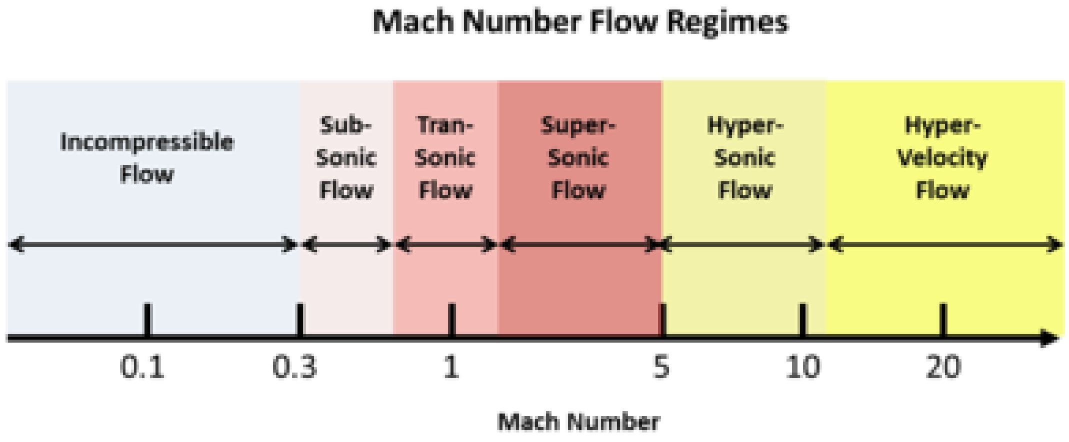
Compressible Flow

Mach number

- The ratio of local flow velocity V to the local speed of sound a:

$$M \equiv \frac{V}{a}$$

- When $M > 0.3$, the gas flow should be considered compressible.



Compressible Flow

Governing equations for inviscid, compressible flow

- The primary dependent variables for the study of compressible flow:

  P, V, ρ, e, and T

- We need five governing equations

- Continuity $\displaystyle \frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \, d\mathcal{V} + \oiint_{S} \rho \mathbf{V} \cdot \mathbf{dS} = 0 \Longrightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$

- Momentum $\displaystyle \frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \mathbf{V} \, d\mathcal{V} + \oiint_{S} (\rho \mathbf{V} \cdot \mathbf{dS}) \mathbf{V} = - \oiint_{S} p \, \mathbf{dS} + \iiint_{\mathcal{V}} \rho \mathbf{f} \, d\mathcal{V}$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x \qquad \rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y \qquad \rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z$$

Compressible Flow

Governing equations for inviscid, compressible flow

- Energy

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}}\rho\left(e+\frac{V^2}{2}\right)d\mathcal{V}+\oiint_{S}\rho\left(e+\frac{V^2}{2}\right)\mathbf{V}\cdot\mathbf{dS}$$

$$=\iiint_{\mathcal{V}}\dot{q}\rho\,d\mathcal{V}-\oiint_{S}p\mathbf{V}\cdot\mathbf{dS}+\iiint_{\mathcal{V}}\rho(\mathbf{f}\cdot\mathbf{V})d\mathcal{V}$$

$$\Longrightarrow\qquad \rho\frac{D(e+V^2/2)}{Dt}=\rho\dot{q}-\nabla\cdot p\mathbf{V}+\rho(\mathbf{f}\cdot\mathbf{V})$$

Compressible Flow

Governing equations for inviscid, compressible flow

- Continuity $$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$
- Momentum $$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x \quad \rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y \quad \rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z$$
- Energy $$\rho \frac{D(e + V^2/2)}{Dt} = \rho \dot{q} - \nabla \cdot p\mathbf{V} + \rho(\mathbf{f} \cdot \mathbf{V})$$

Additional two equation are needed to complete the system

- Equation of state $$p = \rho R T$$
- Internal energy $$e = c_v T$$

Compressible Flow

Total (Stagnation) Conditions

$$\underset{\substack{\text{static}\\ \text{pressure}}}{p_1} + \underset{\substack{\text{dynamic}\\ \text{pressure}}}{\tfrac{1}{2}\rho V_1^2} = \underset{\substack{\text{total}\\ \text{pressure}}}{p_0}$$

For incompressible flow

- Consider a fluid element passing through a given point in a flow where the local pressure, temperature, density, Mach number, and velocity are p, T, ρ, M, and V. Here, p, T, and ρ are **static quantities**.
  - They are the pressure, temperature and density you feel when you ride along with the gas at the local flow velocity.
- When the fluid element is adiabatically slowed down to zero velocity, the local temperature at the zero velocity is *total temperature*, T_0
- The corresponding value of enthalpy is *total enthalpy*, $h_0 = c_p T_0$

Compressible Flow

Total (Stagnation) Conditions

- Assume that the flow is adiabatic and that body forces are negligible. The energy equation becomes:

$$\rho \frac{D(e + V^2/2)}{Dt} = -\nabla \cdot p\mathbf{V}$$

- We also have: $$\rho \frac{D(p/\rho)}{Dt} = \frac{Dp}{Dt} + p\nabla \cdot \mathbf{V} = \frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + p\nabla \cdot \mathbf{V}$$

$$\rho \frac{D}{Dt}\left(e + \frac{p}{\rho} + \frac{V^2}{2}\right) = -p\nabla \cdot \mathbf{V} - \mathbf{V} \cdot \nabla p + \frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + p\nabla \cdot \mathbf{V}$$

$$e + \frac{p}{\rho} = e + pv \equiv h \quad \Longrightarrow \quad \rho \frac{D(h + V^2/2)}{Dt} = \frac{\partial p}{\partial t}$$

Compressible Flow

Total (Stagnation) Conditions

- If the flow is steady,

$$\rho \frac{D(h + V^2/2)}{Dt} = \frac{\partial p}{\partial t} \Longrightarrow \rho \frac{D(h + V^2/2)}{Dt} = 0$$

$$\boxed{h + \frac{V^2}{2} = \text{const}} \Longrightarrow \boxed{h + \frac{V^2}{2} = h_0}$$

- At any point in a flow, the total enthalpy is given by the sum of the static enthalpy plus the kinetic energy.
- For a steady, adiabatic, inviscid flow, the total enthalpy is constant along a streamline.
- *Total temperature* $h_0 = c_p T_0$ $\boxed{T_0 = \text{const}}$
- The total temperature is constant throughout the steady, adiabatic, inviscid flow of a calorically perfect gas.

Compressible Flow

Total (Stagnation) Conditions

- When the fluid element is brought to rest isentropically (both adiabatically and reversibly), the resulting pressure and density are defined as the total pressure p_0 and total density ρ_0
- If the general flow field is isentropic throughout, both p_0 and ρ_0 are constant throughout the flow.

Compressible Flow

Example Practice 1:

At a point in an airflow, the pressure, temperature, and velocity are 1 atm, 320 K, and 1000 m/s. Calculate the total temperature and total pressure at this point.

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- Solution

$$h + \frac{V^2}{2} = h_0$$

$$h = c_p T \qquad c_p = \frac{\gamma R}{\gamma - 1}$$

$$\Longrightarrow \qquad c_p T + \frac{V^2}{2} = c_p T_0$$

$$T_0 = T + \frac{V^2}{2c_p} = T + \left(\frac{\gamma-1}{2\gamma R}\right) V^2$$

$$T_0 = 320 + \left[\frac{0.4}{2(1.4)(287)}\right](1000)^2 = 320 + 497.8$$

$$T_0 = \boxed{817.8\text{K}}$$

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

Hence, $$p_0 = p\left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = (1\text{atm})\left(\frac{817.8}{320}\right)^{\frac{1.4}{0.4}}$$

$$P_0 = \boxed{26.7 \text{ atm}}$$

Example Practice 2:

An airplane is flying at a standard altitude of 10,000 ft. A pitot tube mounted at the nose measures a pressure of 2220 lb/ft^2. The airplane is flying at a high subsonic speed, and the flow should be considered compressible. Calculate the velocity of the airplane.

Compressible Flow

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- Solution

$$\frac{p_0}{p_\infty} = \left(\frac{T_0}{T_\infty}\right)^{\gamma/(\gamma-1)} \quad \Longrightarrow \quad T_0 = T_\infty \left(\frac{p_0}{p_\infty}\right)^{(\gamma-1)/\gamma}$$

the pressure and temperature at a standard altitude of 10,000 ft are 1455.6 lb/ft^2 and 483.04 °R.

$$T_0 = (483.04)\left(\frac{2220}{1455.6}\right)^{0.4/1.4} = 544.9\ \text{°R}$$

Compressible Flow

Example Practice 2:

$$c_p T_\infty + \frac{V_\infty^2}{2} = c_p T_0$$

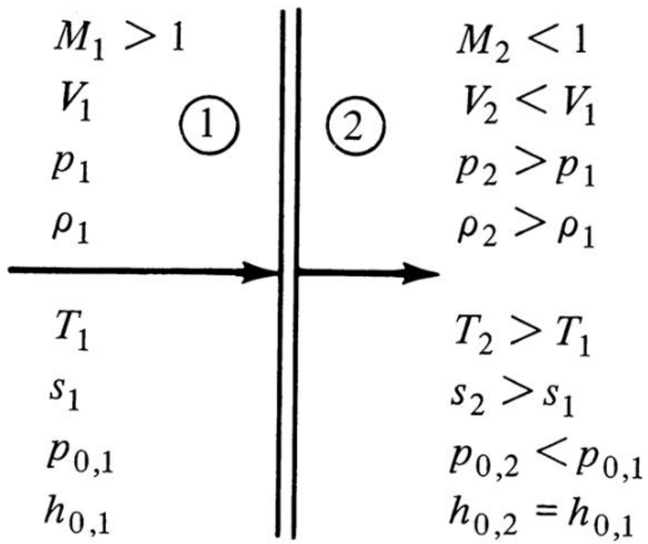
$$c_p = \frac{\gamma R}{\gamma - 1} = \frac{(1.4)(1716)}{0.4} = 6006 \frac{\text{ft} \cdot \text{lb}}{\text{slug} \cdot {}^\circ\text{R}}$$

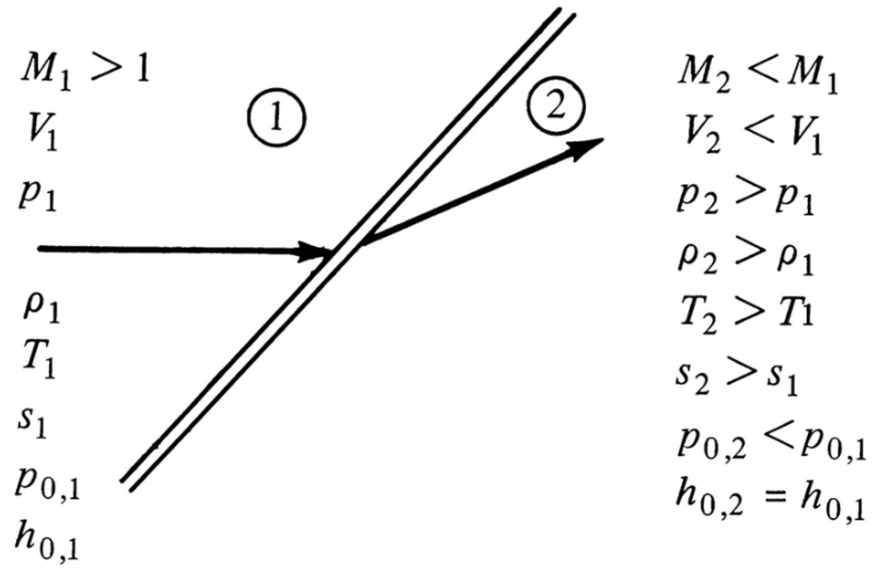
$$\begin{aligned} V_\infty &= [2\, c_p\, (T_0 - T_\infty)]^{1/2} \\ &= [2\,(6006)(544.9 - 483.04]^{1/2} \\ &= \boxed{862 \text{ ft/s}} \end{aligned}$$

Compressible Flow

Shock Waves

- A shock wave is an extremely thin region, typically on the order of 10^{-5} cm, across which the flow properties can change drastically.
  - Normal shock
  - Oblique shock

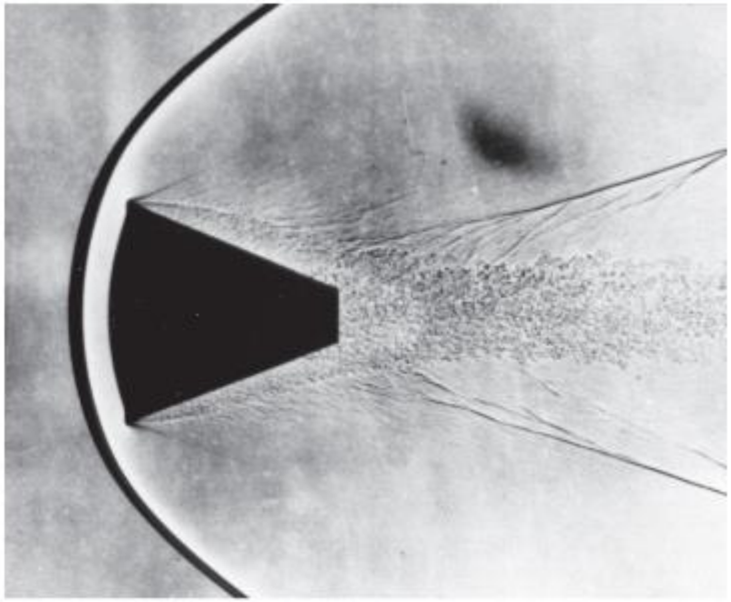


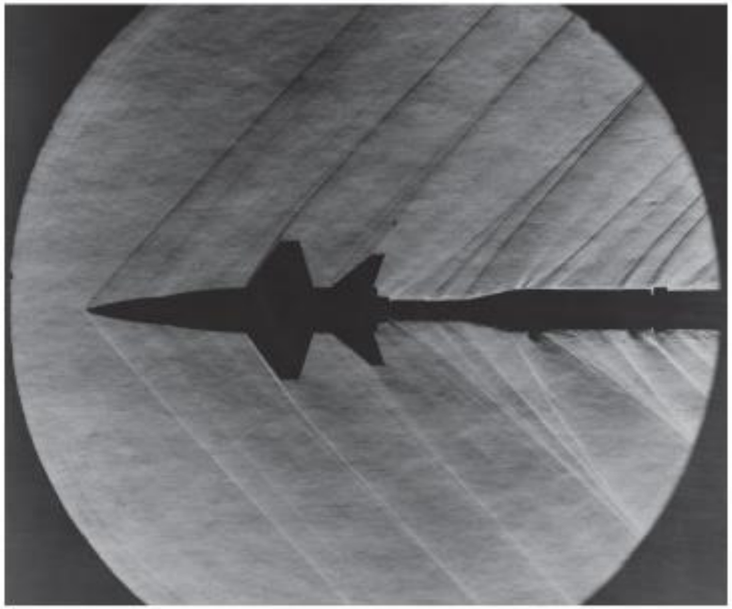


- Physically, the flow across a shock wave is adiabatic – the total enthalpy is constant across the wave

Compressible Flow

<u>Shock Waves</u>





- Air is transparent, we cannot usually see shock waves with the naked eye.
- But, due to the density changes across the shock wave, light rays propagating through the flow can be refracted across the shock
  - Schlieren and Shadowgraphs (ME 59911 Experimental Method in Fluid Mechanics)