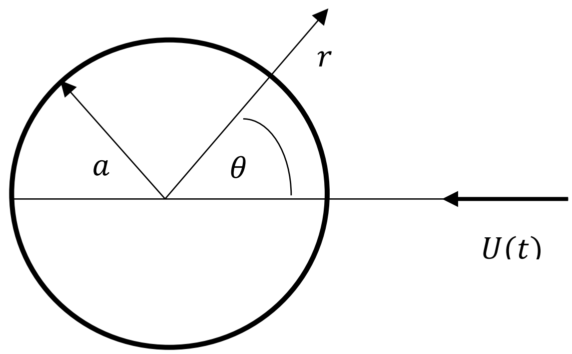
ME 55600/I0200
HW #9: Potential Flow

1. Inviscid irrotational fluid flows over a fixed sphere of radius a. The fluid velocity far from the sphere is uniform but changes with time as $U(t)$.



(a) Determine the solution for the velocity potential.
(b) Use the result from (a) and Bernoulli's equation to derive the pressure distribution on the surface of the sphere.
(c) Determine the drag force on the sphere.

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(a) Governing equation $\nabla^2\phi = 0$

$$\frac{\partial}{\partial r}\left(r^2\frac{d\phi}{dr}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0$$

Boundary Conditions:

(i) $\left.\frac{\partial\phi}{\partial r}\right|_{r\to\infty} = -U\cancel{r}\cos\theta$

(ii) $\left.\frac{\partial\phi}{\partial r}\right|_{r\to a} = 0$

Using separation of variables

$$\phi = R(r)\cos\theta$$

$$\frac{\partial}{\partial r}\left(r^2\frac{dR}{dr}\right) - 2R = 0 \qquad r^2R'' + 2rR' - 2R = 0$$

Solution (equidimensional equation)

$$R \sim r^m \qquad m(m-1) + 2m - 2 = 0, \qquad m_{1,2} = 1, -2$$

$$\phi = \left(c_1 r + \frac{c_2}{r^2}\right)\cos\theta$$

From boundary conditions: $c_1 = -U, \quad c_2 = -\frac{1}{2}Ua^3$

$$\phi = -U(t)\left(r + \frac{a^3}{2r^2}\right)\cos\theta$$

$$u = \frac{\partial\phi}{\partial r} == -U(t)\left(1 - \frac{a^3}{r^3}\right)\cos\theta$$

$$v = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = U(t)\left(1 + \frac{a^3}{2r^3}\right)\sin\theta$$

(b) Pressure distribution.

From Bernoulli's equation

$$\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}(u^2 + v^2) = F(t)$$

$$\frac{p}{\rho} = F(t) - \frac{1}{2}U^2\left[\left(1 - \frac{a^3}{r^3}\right)^2 \cos^2\theta + \left(1 + \frac{a^3}{2r^3}\right)^2 \sin^2\theta\right] + \frac{dU}{dt}\left(r + \frac{a^3}{2r^2}\right)\cos\theta$$

At $r = a$,

$$\frac{p}{\rho} = F(t) - \frac{9}{8}U^2\sin^2\theta + \frac{3}{2}a\frac{dU}{dt}\cos\theta$$

(c) The drag force on the sphere is:

$$F_D = \int p\cos\theta dA = 2\pi a^2 \int_0^\pi p(a)\cos\theta\, d\theta$$

Let $x = \cos\theta$

$$F_D = -2\pi a^2 \rho \int_{1}^{-1} \left[F(t) - \frac{9}{8} U^2 (1 - x^2) + \frac{3}{2} a \frac{dU}{dt} x \right] x dx$$

Only integrals of even functions of x contribute to the force. Hence

$$F_D = 2\pi a^2 \rho \int_{1}^{-1} \left[\frac{3}{2} a \frac{dU}{dt} x \right] x dx = 2\pi a^3 \rho \frac{dU}{dt}$$

2. A spherical gas bubble is immersed in a quiescent incompressible inviscid fl;uid of density ρ. The bubble is made to expand isothermally such that the gas inside obeys the ideal gas equation of state $pV = RT$, where V is the bubble volume, P is the pressure, R is the gas constant, and T the absolute temperature.

Derive the ordinary differential equation that governs the radius of the expanding bubble $a(t)$, if the pressure far from the bubble is constant p_∞ and the surface tension effects are negligible. Note that the radial surface velocity of the bubble is given by $\frac{da}{dt}$.

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Isothermal expansion of a bubble of radius $a(t)$.

Governing equation (Euler's)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r}$$

Continuity Equation

$$\frac{\partial}{\partial r}(r^2 u) = 0$$

From continuity

$$r^2 u = F(t)$$

Substitute into Euler's

$$\frac{F'}{r^2} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

Integrate from $r = a$ to $r \to \infty$ where $u \to 0$

$$-\frac{F'}{r}\bigg|_{r=a}^{\infty} + \frac{1}{2}u^2|_{r=a}^{\infty} = -\frac{1}{\rho}p|_{r=a}^{\infty}$$

Then,

$$\frac{F'}{a} - \frac{1}{2}(a')^2 = \frac{p_0 - p_\infty}{\rho}$$

Where p_0 is the pressure inside the bubble and the prime denotes a derivative.

Also, at $r = a$

$$a' = u|_{r=a} \quad \text{then} \quad a^2 a' = F(t)$$

Using ideal gas law

$$p_0 \frac{4}{3}\pi a^3 = RT$$

The equation for the bubble radius becomes:

$$a^3\left[\frac{d^2a}{dt^2} + \frac{3}{2}\left(\frac{da}{dt}\right)^2 + \frac{p_\infty}{\rho}\right] = \frac{3RT}{4\pi\rho}$$