12.4. Sensitivity Analysis

What effects do small errors in position and velocity of the vehicle at the departure point have on the trajectory?

Specifically, for the earth-Venus transfer trajectory considered in the previous example, what change in the arrival position at Venus r_{SV} will be produced by small changes in the departure position and velocity from earth, r_{PE} and V_{PE}?

Upon arrival at Venus, the distance of the spacecraft from the sun is

$$r_{SV} = \frac{h^2/\mu_{sun}}{1+e} \qquad (12.10)$$

where

$$h = r_{SE} V_{SK}^{D} \qquad (12.11)$$

departure velocity of spacecraft relative to the sun

and

$$e = \frac{r_{SE} - r_{SV}}{r_{SE} + r_{SV}} \qquad (12.12)$$

Substitute (12.11) and (12.12) into (12.10) and solve for r_{SV}

$$r_{SV} = \frac{r_{SE}^2 \, V_{SK}^{D\,2}}{2\mu_{sun} - r_{SE} V_{SK}^{D\,2}} \qquad (12.13)$$

The change dr_{SV} in r_{SV} due to a small variation dV_{SK}^D of V_{SK}^D is

$$dr_{SV} = \frac{dr_{SV}}{dV_{SK}^D} \, dV_{SK}^D$$

$$= \frac{4 r_{SE}^2 \mu_{sun} V_{SK}^D}{\left[2\mu_{sun} - r_{SE} V_{SK}^{D\,2}\right]^2} \, dV_{SK}^D \qquad (12.14)$$

Divide (12.14) by (12.13)

$$\frac{dr_{SV}}{r_{SV}} = \frac{2}{1 - \frac{r_{SE} V_{SK}^{D\,2}}{2\mu_{sun}}} \, \frac{dV_{SK}^D}{V_{SK}^D} \qquad (12.15)$$

The departure speed of the spacecraft relative to the sun is

$$V_{SK}^{D} = V_{SE} - |V_{EK}| = V_{SE} - \Delta V_E = V_{SE} - V_{\infty E}$$

Using the energy equation for a hyperbola

$$\frac{V_{PE}^2}{2} - \frac{\mu_E}{r_{PE}} = \frac{\mu}{2a} = \frac{V_{\infty E}^2}{2}$$

from which

$$V_{\infty E} = \sqrt{V_{PE}^2 - \frac{2\mu_E}{r_{PE}}}$$

Therefore

$$V_{SK}^{D} = V_{SE} - \sqrt{V_{PE}^2 - \frac{2\mu_E}{r_{PE}}} \qquad (12.16)$$

The change in V_{SK}^{D} due to variations dr_{PE} and dV_{PE} of the departure position r_{PE} and V_{PE} is given by

$$dV_{SK}^{D} = \frac{\partial V_{SK}^{D}}{\partial r_{PE}} dr_{PE} + \frac{\partial V_{SK}^{D}}{\partial V_{PE}} dV_{PE} \qquad (12.17)$$

The partial derivatives are obtained from (12.16)

$$\frac{\partial V_{SK}^{D}}{\partial r_{PE}} = -\frac{\mu_E}{V_{\infty E}\, r_{PE}^2} \qquad \frac{\partial V_{SK}^{D}}{\partial V_{PE}} = -\frac{V_{PE}}{V_{\infty E}}$$

Therefore

$$dV_{SK}^{D} = -\frac{\mu_E}{V_{\infty E}\, r_{PE}^2} dr_{PE} - \frac{V_{PE}}{V_{\infty E}} dV_{PE} \qquad (12.18)$$

Divide by V_{SK}^{D} and again use energy equation for a hyperbola

$$\frac{dV_{SK}^{D}}{V_{SK}^{D}} = -\frac{\mu_E}{V_{SK}^{D}\, V} \frac{dr_{PE}}{r_{PE}} - \frac{V_{\infty E} + \frac{2\mu_E}{r_{PE} V_{\infty E}}}{V_{SK}^{D}} \frac{dV_{PE}}{V_{PE}}$$

$$(12.19)$$

Substituting (12.19) into (12.15) gives

$$\frac{\delta r_{SV}}{r_{SV}} = -\frac{2}{1-\frac{r_{SE}V_{SK}^{D\,2}}{2\mu_{sun}}}\left[\frac{\mu_E}{V_{SK}^D V_{\infty E} r_{PE}}\frac{\delta r_{PE}}{r_{PE}} + \frac{\left(V_{\infty E}+\frac{2\mu_E}{r_{PE}V_{\infty E}}\right)}{V_{SK}^D}\frac{\delta V_{PE}}{V_{PE}}\right] \tag{12.20}$$

Eq (12.20) gives the variation of r_{SV} due to variations in r_{PE} and V_{PE}.

For the earth-Venus mission considered in the previous example

$$\mu_{sun} = 1.327\times 10^{11}\ \text{km}^3/\text{sec}^2$$

$$\mu_E = 3.986\times 10^{5}\ \text{km}^3/\text{sec}^2$$

$$r_{SE} = 1\ \text{au} = 1.496\times 10^{8}\ \text{km}$$

$$r_{SV} = 0.7233\ \text{au} = 1.082\times 10^{8}\ \text{km}$$

$$r_{PE} = 6378 + 200 = 6578\ \text{km}$$

$$V_{SE} = \sqrt{\frac{\mu_{sun}}{r_{SE}}} = \sqrt{\frac{1.327\times 10^{11}}{1.496\times 10^{8}}} = 29.783\ \text{km/sec}$$

$$V_{SK}^{D} = \sqrt{\frac{2\mu_{sun}\, r_{SV}}{(r_{SE}+r_{SV})\,r_{SE}}} = \sqrt{\frac{2(1.327\times10^{11})(1.082\times10^{8}}{(1.496\times10^{8}+1.082\times10^{8})(1.496\times10^{8})}}$$

$$= 27.287 \text{ km/sec}$$

$$V_{\infty E} = V_{SE} - V_{SK}^{D} = 29.783 - 27.287 = 2.496 \text{ km/sec}$$

$$V_{PE} = \sqrt{V_{\infty E}^{2} + \frac{2\mu_E}{r_{PE}}} = \sqrt{(2.496)^2 + \frac{2(3.986\times10^{5})}{6578}} = 11.288 \frac{\text{km}}{\text{sec}}$$

Substituting the numerical values into (12.20)

$$\frac{\delta r_{SV}}{r_{SV}} = -3.066\,\frac{\delta r_{PE}}{r_{PE}} - 6.448\,\frac{\delta V_{PE}}{V_{PE}}$$

A 0.01% variation (1.1 m/sec) in the departure speed V_{PE} changes the target radius r_{SV} by 0.064 percent or 69,250 km.

An error of 0.01% (0.66 km) in departure radius r_{PE} changes r_{SV} by over 33,000 km.

Such errors which are likely to occur during launch must be corrected by midcourse maneuvers.