ME 51500/I5800 L | Exam 1 | P. Ganatos
75 min. | Closed Book, Open Notes* | 11/02/2023

*** Only original handwritten notes and homeworks are allowed. Photocopied notes and homework solution sheets are not permitted. Except for a hand calculator, no cell phone or electronic equipment of any kind is allowed.**

Show all work and give units in final answers.

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[50] 1. The altitude of a satellite in an elliptical orbit around the earth is 2000 km at apogee and 500 km at perigee.

a) Calculate the semi-major axis, the eccentricity, and the parameter p of the orbit. [15 points]
b) Calculate the angular momentum and total energy (per unit mass) of the satellite. [10 points]
c) Calculate the period of the orbit. [5 points]
d) Calculate the time after perigee passage that the satellite reaches an altitude of 1200 km. [20 points]

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[50] 2. Consider the earth and Mars to be in coplanar circular orbits of radii 1 au and 1.524 au, respectively. For an earth-Mars trip through a transfer angle $\Delta\theta = 90°$ along the minimum energy trajectory

a) Calculate the chord distance between the departure and arrival points and the semi-perimeter of the space triangle. [10 points]
b) Calculate the values of α and β of the trajectory. [10 points]
c) Calculate the semi-major axis, the parameter p, and eccentricity of the trajectory. [15 points]
d) Calculate the time of flight. [5 points]
e) Calculate the true anomaly at the departure and arrival points. Make sure the angles are in the correct quadrant. [10 points]

Physical constants

The Earth

Mean Radius $r_e = 6368$ km $\mu_{earth} = 3.986 \times 10^5$ km^3/sec^2

The Sun

$\mu_{sun} = 4\pi^2$ au^3/yr^2

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1) $h_a = 2000$ km $\quad h_p = 500$ km

a) $r_a = h_a + r_e = 2000 + 6368 = 8368$ km

$r_p = h_p + r_e = 500 + 6368 = 6868$ km

$a = \dfrac{r_a + r_p}{2} = \dfrac{8368 + 6868}{2} = \underline{\underline{7618 \text{ km}}}$ [5]

$e = \dfrac{r_a - r_p}{r_a + r_p} = \dfrac{8368 - 6868}{8368 + 6868} = \underline{\underline{0.098451}}$ [5]

$p = a(1-e^2) = 7618(1-(0.098451)^2) = \underline{\underline{7544.16 \text{ km}}}$ [5]

b) $h = \sqrt{\mu p} = \sqrt{(3.986\times10^5)(7544.16)} = \underline{\underline{54{,}837.1 \ \frac{\text{km}^2}{\text{sec}}}}$ [5]

$\mathcal{E} = -\dfrac{\mu}{2a} = -\dfrac{3.986\times10^5}{2(7618)} = \underline{\underline{-26.1617 \ \frac{\text{km}^2}{\text{sec}^2}}} = \underline{\underline{-26{,}161.7 \ \frac{\text{KJ}}{\text{kg}}}}$ [5]

c) $T = 2\pi\sqrt{\dfrac{a^3}{\mu}} = 2\pi\sqrt{\dfrac{(7618)^3}{3.986\times10^5}} = \underline{\underline{6617.18 \text{ sec}}} = \underline{\underline{110.286 \text{ min}}} = \underline{\underline{1.8381 \text{ hr}}}$ [5]

d) At perigee $\theta_1 = E_1 = M_1 = 0$

At $h_2 = 1200$ km $\quad r_2 = h_2 + r_e = 1200 + 6368 = 7568$ km

$r_2 = \dfrac{p}{1+e\cos\theta} \Rightarrow \cos\theta = \dfrac{\frac{p}{r_2} - 1}{e} = \dfrac{\frac{7544.16}{7568} - 1}{0.098451} = -0.031997$

$\theta_2 = 1.6020$ rad $= 91.8336^\circ$ [5]

$\frac{1}{2}\theta_2 = 45.9168^\circ$

$$\tan\tfrac{1}{2}E_2 = \sqrt{\frac{1-e}{1+e}}\ \tan\tfrac{1}{2}\theta_2 = \sqrt{\frac{1-0.098451}{1+0.098451}}\ \tan 45.9168^\circ = 0.935417$$

$$\tfrac{1}{2}E_2 = 0.752042 \text{ rad} = 43.0888^\circ \quad (\text{same quadrant as } \tfrac{1}{2}\theta_2)$$

$$E_2 = 1.50408 \text{ rad} = 86.1776^\circ \quad [5]$$

$$M_2 = E_2 - e\sin E_2 = 1.50408 - 0.098451 \sin 1.50408 = 1.40584 \text{ rad}$$

(5)

$$n = \frac{2\pi}{T} = \frac{2\pi}{6617.18} = 9.49526\times10^{-4}\ \frac{\text{rad}}{\text{sec}}$$

[5]

$$t_2 - t_1 = \frac{M_2 - M_1}{n} = \frac{1.40584 - 0}{9.49526\times10^{-4}} = \underline{\underline{1480.57 \text{ sec}}} = \underline{\underline{24.6762 \text{ min}}}$$

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2) $r_1 = 1\,\text{au} \qquad r_2 = 1.524\,\text{au} \qquad \Delta\theta = 90^\circ$

a) $$c = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\Delta\theta} = \sqrt{1^2 + 1.524^2 - 0} = 1.82279\,\text{au} \quad [5]$$

$$s = \frac{r_1 + r_2 + c}{2} = \frac{1 + 1.524 + 1.82279}{2} = 2.1734\,\text{au} \quad [5]$$

b) For minimum energy trajectory

$$\alpha_m = \underline{\pi \text{ rad}} \quad [5]$$

$$\sin\left(\frac{\beta_m}{2}\right) = \left(\frac{s-c}{s}\right)^{1/2} = \left(\frac{2.1734 - 1.82279}{2.1734}\right)^{1/2} = 0.401645$$

$$\frac{\beta_m}{2} = 0.413312 \text{ rad} \Rightarrow \beta_m = \underline{\underline{0.826625 \text{ rad}}} \quad [5]$$

c) $$a_m = \frac{s}{2} = \frac{2.1734}{2} = \underline{\underline{1.0867\,\text{au}}} \quad [5]$$

$$p = \frac{4a(s-r_1)(s-r_2)}{c^2}\sin^2\left(\frac{\alpha+\beta}{2}\right)$$

$$= \frac{4(1.0867)(2.1734-1)(2.1734-1.524)}{(1.82279)^2}\sin^2\left(\frac{\pi + 0.826625}{2}\right)$$

$p = \underline{\underline{0.836087 \text{ au}}}$ [5]

$$e = \sqrt{1 - \frac{p}{a}} = \sqrt{1 - \frac{0.836087}{1.0867}} = \underline{\underline{0.480227}}$$ [5]

d) $$t_m = \left(\frac{s^3}{8\mu}\right)^{1/2} (\pi - \beta_m + \sin \beta_m)$$

$$= \left(\frac{2.1734^3}{8(4\pi^2)}\right)^{1/2} (\pi - 0.826625 + \sin(0.826625))$$

$$= \underline{\underline{0.550012 \text{ year}}}$$ [5]

e) $$\cos \theta_1 = \frac{\frac{p}{r_1} - 1}{e} = \frac{\frac{0.836087}{1} - 1}{0.480227} = -0.34134$$

$$\theta_1 = 109.959^\circ \text{ or } 250.042^\circ$$

$$\cos \theta_2 = \frac{\frac{p}{r_2} - 1}{e} = \frac{\frac{0.836087}{1.524} - 1}{0.480227} = -0.939944$$

$$\theta_2 = 160.042^\circ \text{ or } 199.958^\circ$$

Since $\Delta\theta = \theta_2 - \theta_1 = 90^\circ$

$\theta_1 = \underline{\underline{109.96^\circ}}$ [5] $\theta_2 = \underline{\underline{199.96^\circ}}$ [5]