ME 57200 Aerodynamic Design

Lecture #15: Thin Airfoil Theory

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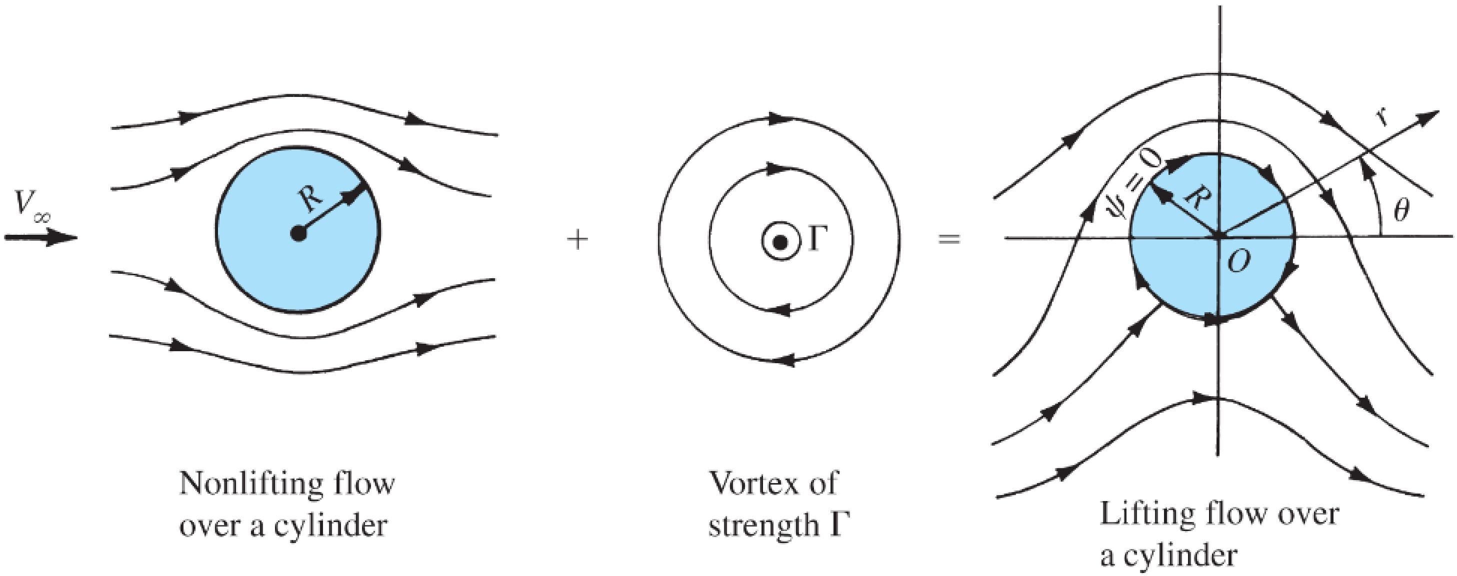
Lifting Flow over a Cylinder



$$\boxed{\psi = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\frac{r}{R}}$$

Lifting Flow over a Cylinder

$r = R$:

$$V = V_\theta = -2V_\infty \sin\theta - \frac{\Gamma}{2\pi R}$$

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2 = 1 - \left(-2\sin\theta - \frac{\Gamma}{2\pi R V_\infty}\right)^2$$

$$\boxed{C_p = 1 - \left[4\sin^2\theta + \frac{2\Gamma\sin\theta}{\pi R V_\infty} + \left(\frac{\Gamma}{2\pi R V_\infty}\right)^2\right]}$$

Lifting Flow over a Cylinder

$$c_l = c_n = \frac{1}{c}\int_0^c C_{p,l}\,dx - \frac{1}{c}\int_0^c C_{p,u}\,dx$$

$$c_l = -\frac{1}{2}\int_\pi^{2\pi} C_{p,l}\sin\theta\,d\theta + \frac{1}{2}\int_\pi^0 C_{p,u}\sin\theta\,d\theta$$

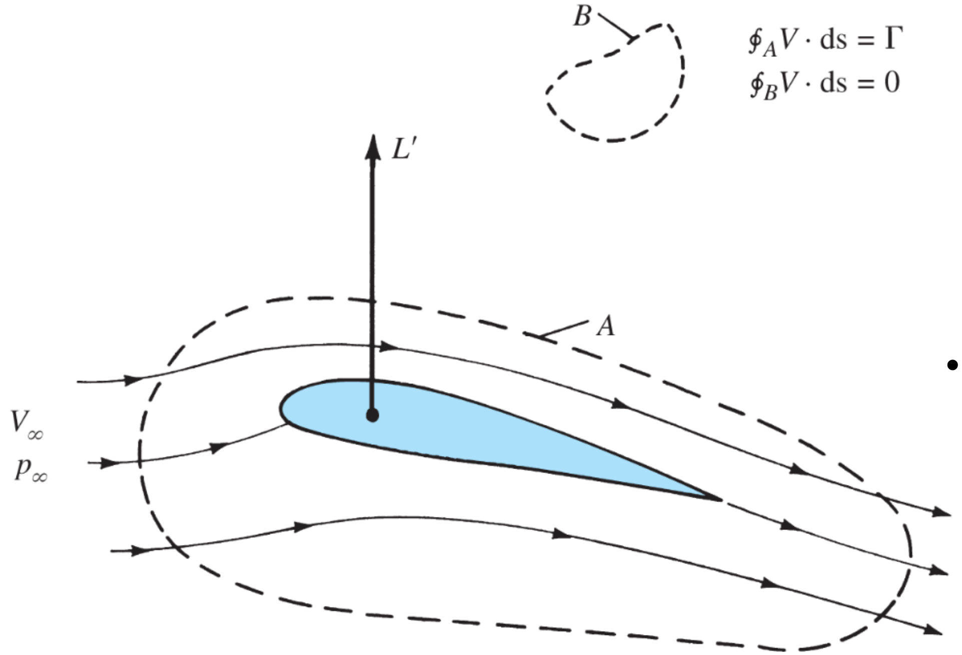
$$c_l = -\frac{1}{2}\int_0^{2\pi} C_p\sin\theta\,d\theta$$

$$c_l = \frac{\Gamma}{RV_\infty} \qquad L' = q_\infty S c_l = \frac{1}{2}\rho_\infty V_\infty^2 S c_l$$

$$\boxed{L' = \frac{1}{2}\rho_\infty V_\infty^2 2R \frac{\Gamma}{RV_\infty}} \qquad \boxed{L' = \rho_\infty V_\infty \Gamma}$$

The Kutta-Joukowski Theorem

Consider the incompressible flow over an airfoil section. Let curve A be a curve in the flow enclosing the airfoil.



- If the airfoil is producing lift, the velocity field around the airfoil will be such that the line integral of velocity around A will be finite, that is, the circulation is finite

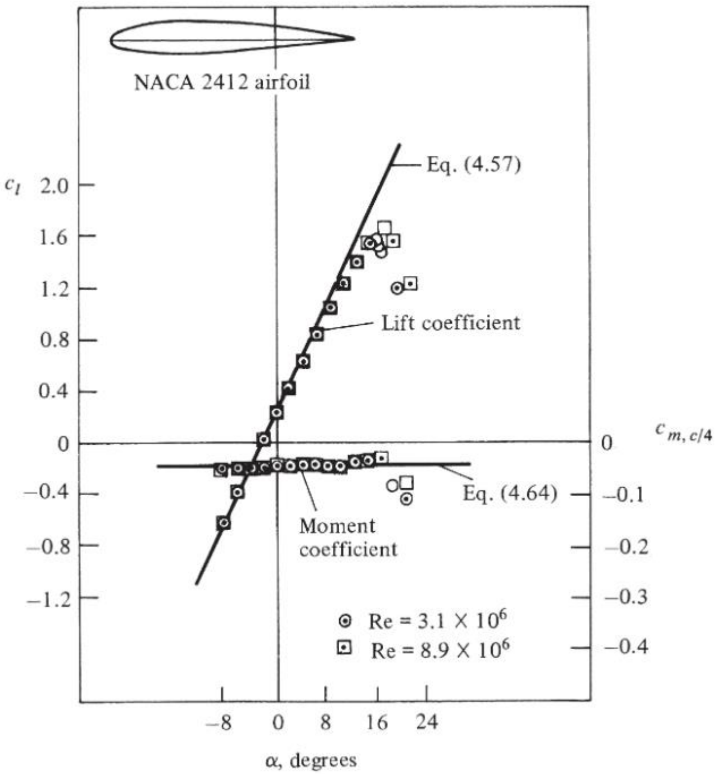
$$\Gamma \equiv \oint_A \mathbf{V} \cdot \mathbf{ds}$$

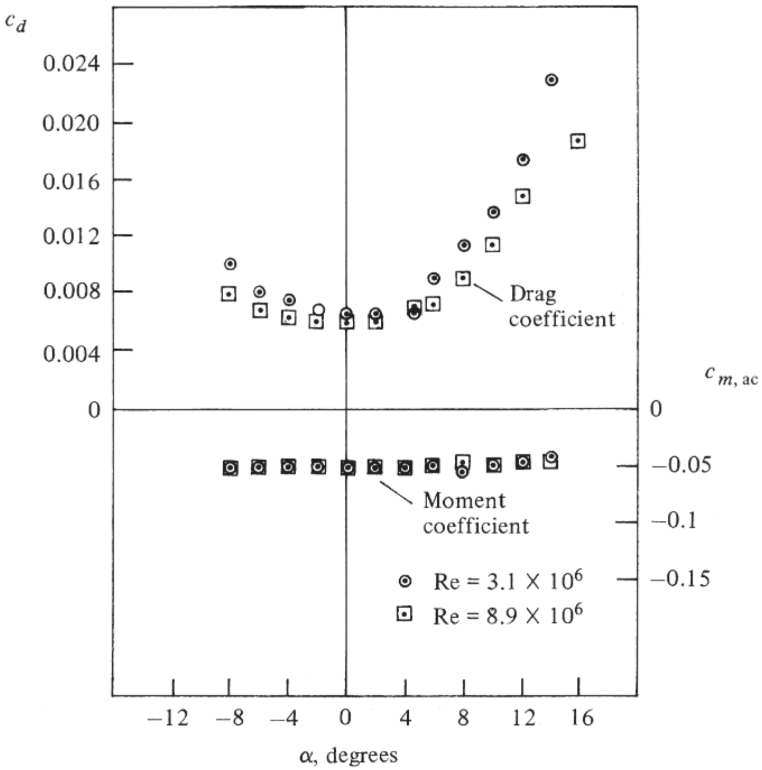
- The lift per unit span on the airfoil will be give by *Kutta-Joukowski theorem*

$$\boxed{L' = \rho_\infty V_\infty \Gamma}$$

Airfoil Characteristics

- Experimental measurements: Some typical results through wind tunnel measurements.



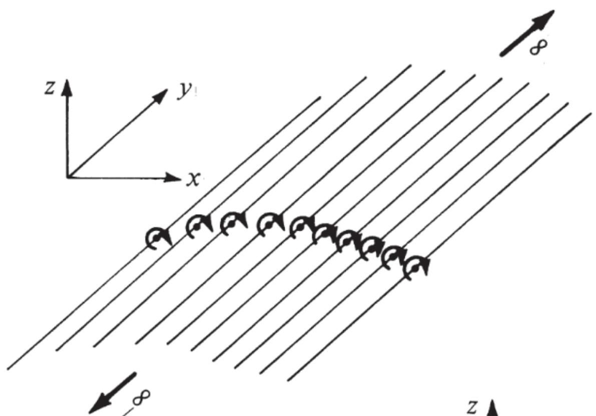


Theoretical Solutions for Flow over Airfoils

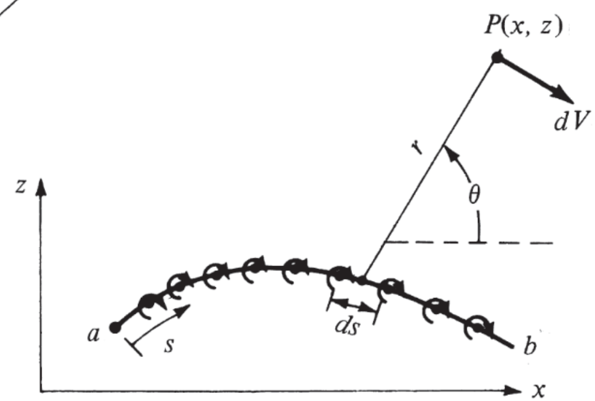
- <u>Calculation: Circulation theory of lift</u>

Consider an infinite number of straight vortex filaments side by side, where the strength of each filament is infinitesimally small.

- Let *s* be the distance measured along the vortex sheet.
- Define $\gamma = \gamma(s)$ as the strength of the vortex sheet, per unit length along *s*.



Vortex sheet in perspective



Edge view of sheet

<u>The strength of an infinitesimal portion ds of the sheet is γds</u>

<u>The induced velocity at point P</u>

$$dV = -\frac{\gamma\, ds}{2\pi r}$$

Theoretical Solutions for Flow over Airfoils

- <u>Calculation: Circulation theory of lift</u>

<u>The strength of an infinitesimal portion ds of the sheet is γds</u>

<u>The induced velocity at point P</u> $$dV = -\frac{\gamma\, ds}{2\pi r}$$

<u>The increment in velocity potential dφ</u> $$d\phi = -\frac{\gamma\, ds}{2\pi}\theta$$

The velocity potential at P due to the entire vortex sheet from a to b

$$\boxed{\phi(x, z) = -\frac{1}{2\pi}\int_a^b \theta\gamma\, ds}$$

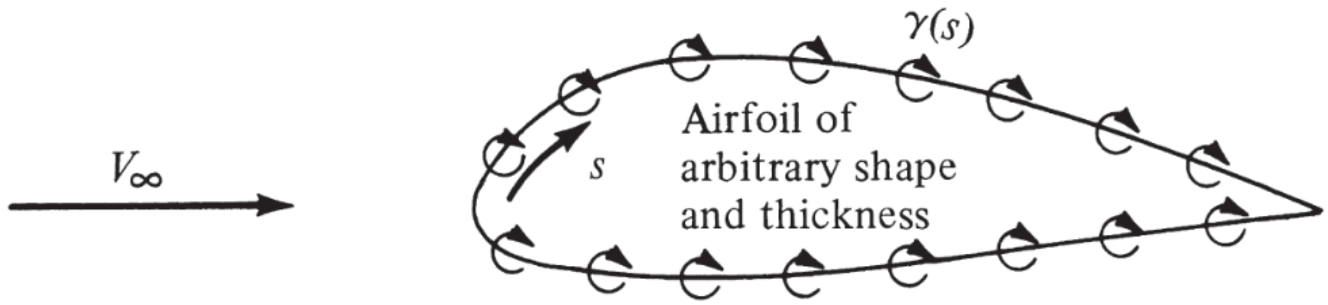
The circulation around the vortex sheet is the sum of the strengths of the elemental vortices from a to b

$$\boxed{\Gamma = \int_a^b \gamma\, ds}$$

Theoretical Solutions for Flow over Airfoils

- <u>Calculation: Circulation theory of lift</u>

Consider an airfoil of arbitrary shape and thickness in a freestream with velocity V_∞. Replace the airfoil surface with a vortex sheet of variable strength $\gamma(s)$



- The circulation around the airfoil will be given by

$$\Gamma = \int \gamma \, ds$$

- The resulting lift is given by the Kutta-Joukowski theorem:

$$L' = \rho_\infty V_\infty \Gamma$$

Theoretical Solutions for Flow over Airfoils

- <u>Calculation: Circulation theory of lift</u>

Consider an airfoil of arbitrary shape and thickness in a freestream with velocity V_∞. Replace the airfoil surface with a vortex sheet of variable strength $\gamma(s)$

<u>The need for analytical solutions for $\gamma = \gamma(s)$</u>

No general analytical solution exists…

- The circulation around the airfoil will be given by

$$\Gamma = \int \gamma \, ds$$

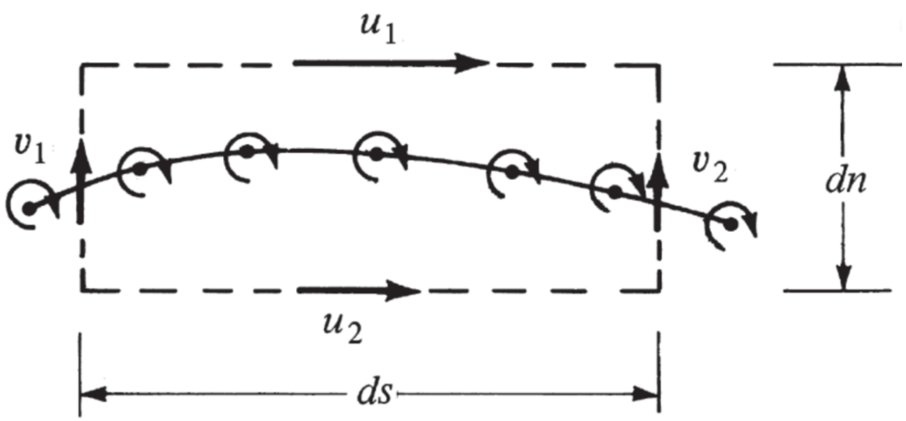
- The resulting lift is given by the Kutta-Joukowski theorem:

$$L' = \rho_\infty V_\infty \Gamma$$

Theoretical Solutions for Flow over Airfoils

- <u>Calculation: Circulation theory of lift</u>

Consider a rectangular dashed path enclosing a section of the sheet of length *ds*.



- Velocities tangential to the top and bottom are u_1 and u_2, and the velocities tangential to the left and right sides are v_1 and v_2. The circulation around the dashed path is:

$$\Gamma = -(v_2\, dn - u_1\, ds - v_1\, dn + u_2\, ds)$$

- Since the strength of the vortex sheet contained inside the dashed path is

$$\Gamma = \gamma\, ds$$

Theoretical Solutions for Flow over Airfoils

- <u>Calculation: Circulation theory of lift</u>

$$\Gamma = -(v_2\,dn - u_1\,ds - v_1\,dn + u_2\,ds) \qquad \Gamma = \gamma\,ds$$

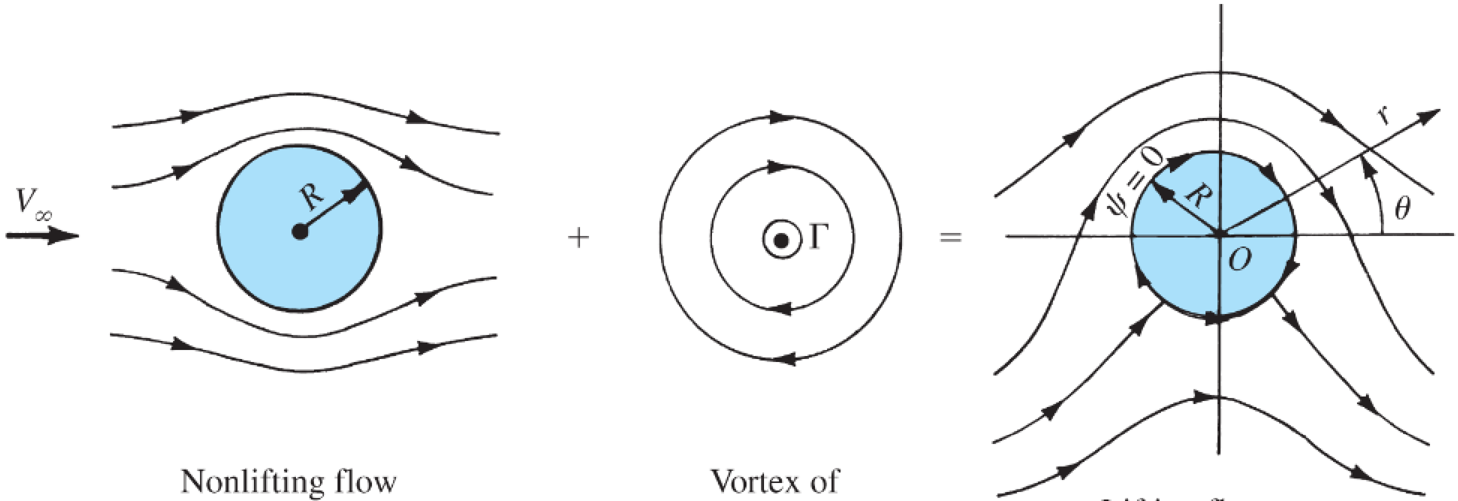
$$\gamma\,ds = (u_1 - u_2)\,ds + (v_1 - v_2)\,dn$$

- Let *dn* → *0* $\quad \gamma\,ds = (u_1 - u_2)\,ds \qquad \boxed{\gamma = u_1 - u_2}$

The local jump in tangential velocity across the vortex sheet is equal to the local sheet strength.

Theoretical Solutions for Flow over Airfoils

- <u>The Kutta Condition</u>



Nonlifting flow over a cylinder

Vortex of strength Γ

Lifting flow over a cylinder

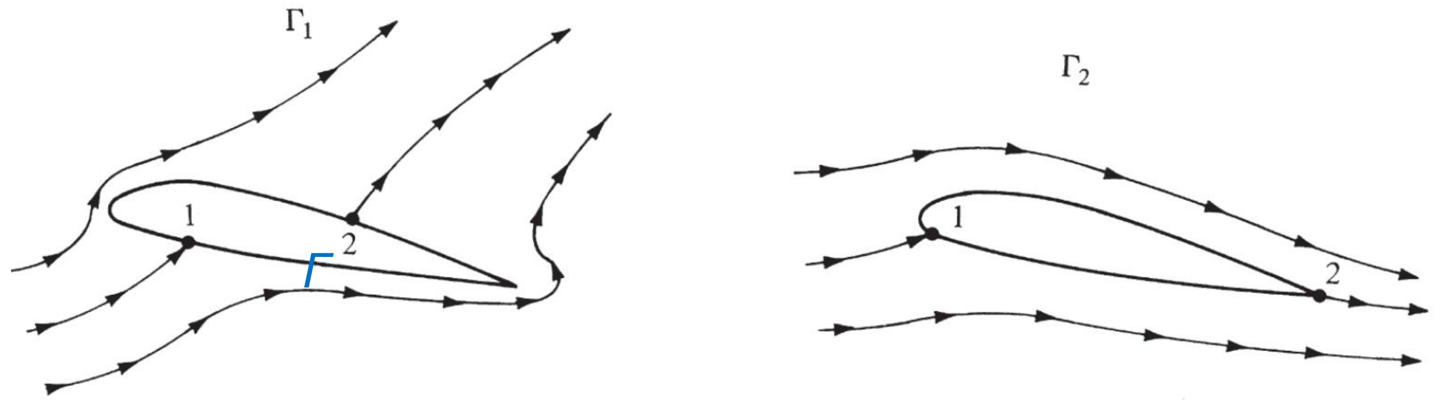
$$\boxed{\psi = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi} \ln \frac{r}{R}}$$

An infinite number of potential flow solutions are possible, corresponding to the infinite choice of Γ

Theoretical Solutions for Flow over Airfoils

- <u>The Kutta Condition</u>

For a given airfoil at a given angle of attack, there are an infinite number of valid theoretical solutions, corresponding to an infinite choice of Γ



We know from experience that a given airfoil at a given angle of attack under a given incoming flow condition produces a single value of lift.

How to pick the particular solution?

We need an additional condition that fixes *Γ for a given airfoil at a given α!!!*

Theoretical Solutions for Flow over Airfoils

- <u>Experimental evidence</u>

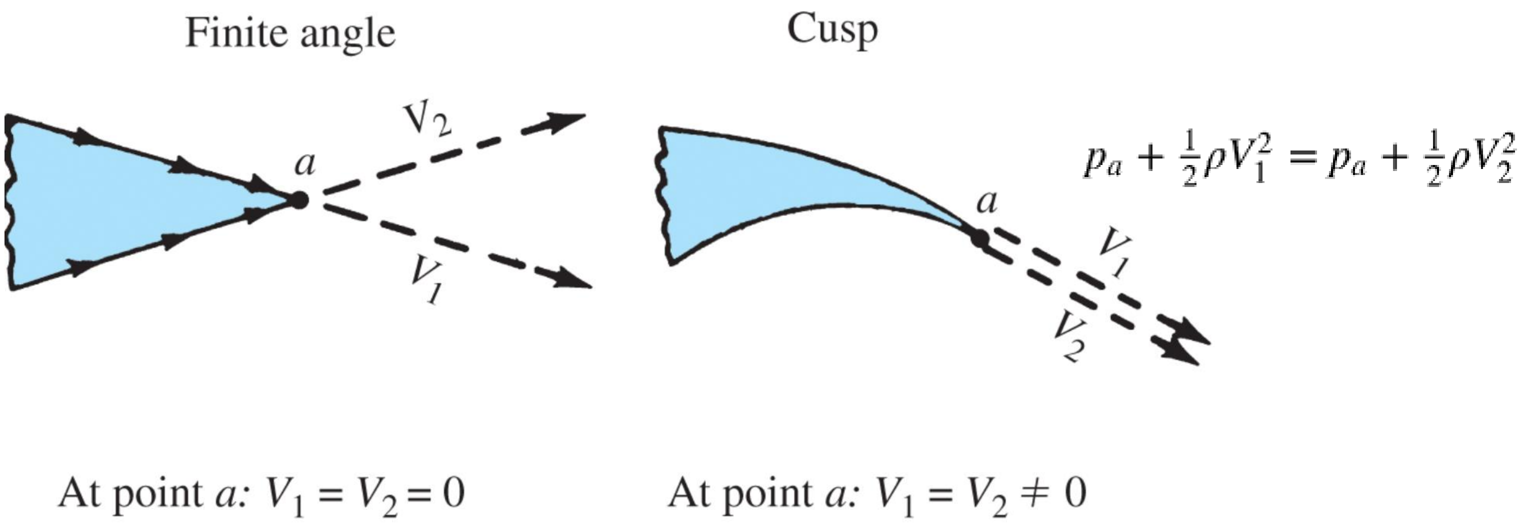
In a steady state condition, the flow is smoothly leaving the top and the bottom surfaces of the airfoil at the trailing edge

A particular value of *Γ which results in the flow leaving smoothly at the trailing edge* – *Kutta Condition*

Theoretical Solutions for Flow over Airfoils

- <u>Theoretical analysis</u>

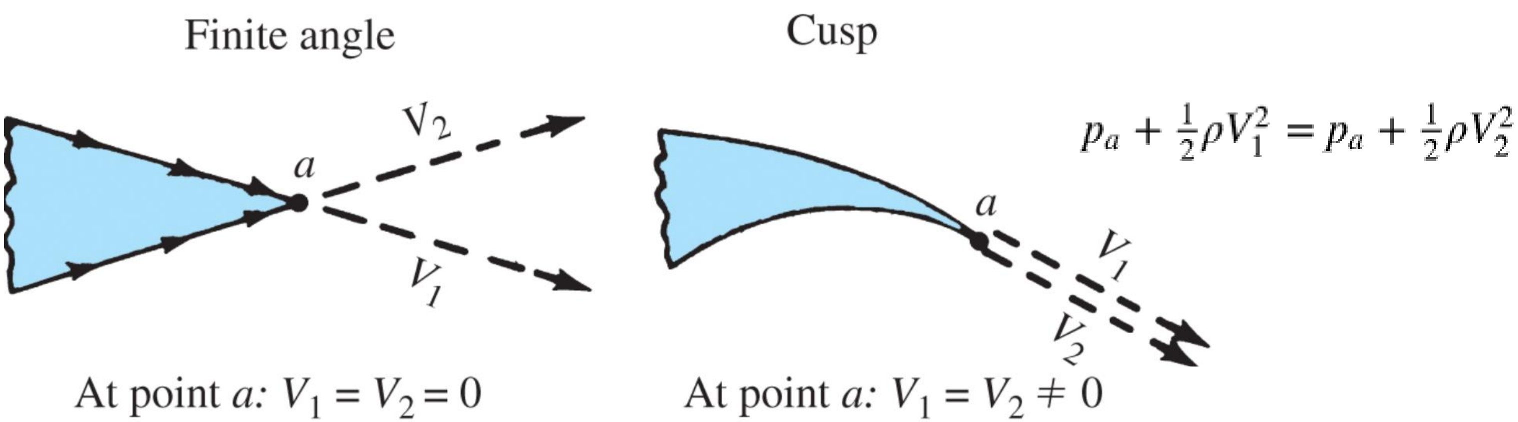
In a steady state condition, the flow is smoothly leaving the top and the bottom surfaces of the airfoil at the trailing edge



Theoretical Solutions for Flow over Airfoils

- <u>Kutta condition</u>

  - For a given airfoil at a given angle of attack, the value of Γ around the airfoil is such that the flow leaves the trailing edge smoothly.
  - If the trailing-edge is finite, then the trailing edge is a stagnation point.
  - If the trailing-edge is cusped, then the velocities leaving the top and bottom surfaces at the trailing edge are finite and equal in magnitude and direction.



$\gamma(\mathrm{TE}) = \gamma(a) = V_1 - V_2$

$\boxed{\gamma(\mathrm{TE}) = 0}$

Theoretical Solutions for Flow over Airfoils

- *Could we have lift without friction in the real world?*

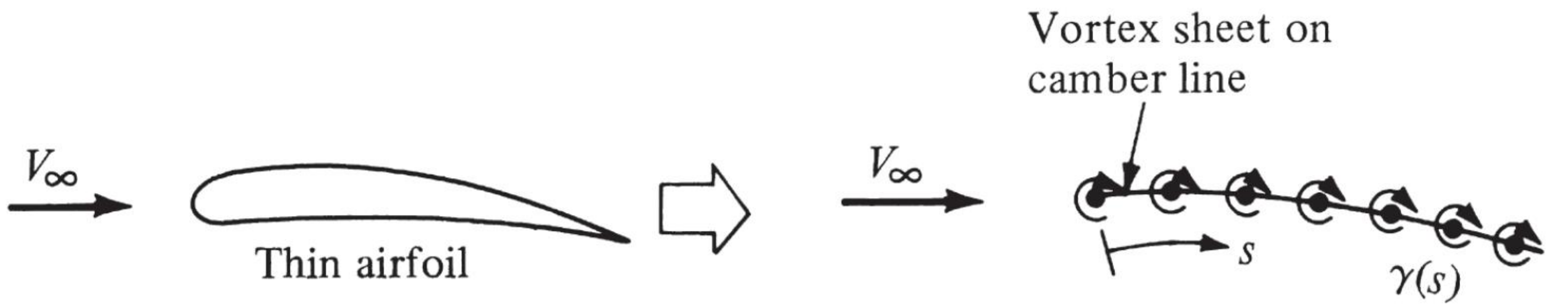
Nature enforces the Kutta condition by means of friction.

- *If there were no boundary layer (no friction), there would be no physical mechanism in the real world to achieve the Kutta condition.*

Theoretical Solutions for Flow over Airfoils

- <u>Thin Airfoil</u>

- Imagine the airfoil is made very thin – the vortex sheet on the top and bottom surface of the airfoil would almost coincide.
- We can approximate a thin airfoil by replacing it with a single vortex sheet, $\gamma(s)$ distributed over the camber line of the airfoil.



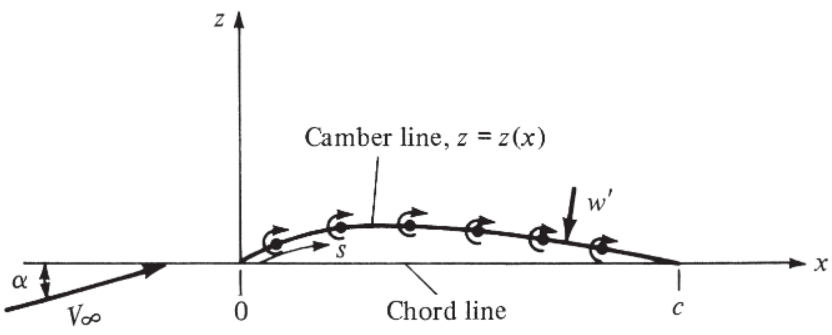
Yielding a closed-form analytical solution, such that:

- Camber line becomes a streamline of the flow
- The Kutta condition is satisfied at the trailing edge: $\gamma(LE) = 0$

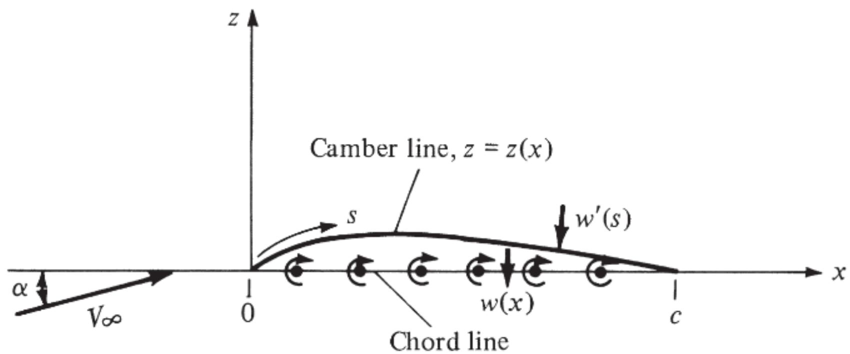
Theoretical Solutions for Flow over Airfoils

- <u>Thin Airfoil</u>

- If the airfoil is thin, the camber line is close to the chord line, the vortex sheet appears to fall approximately on the chord line.



(a) Vortex sheet on the camber line

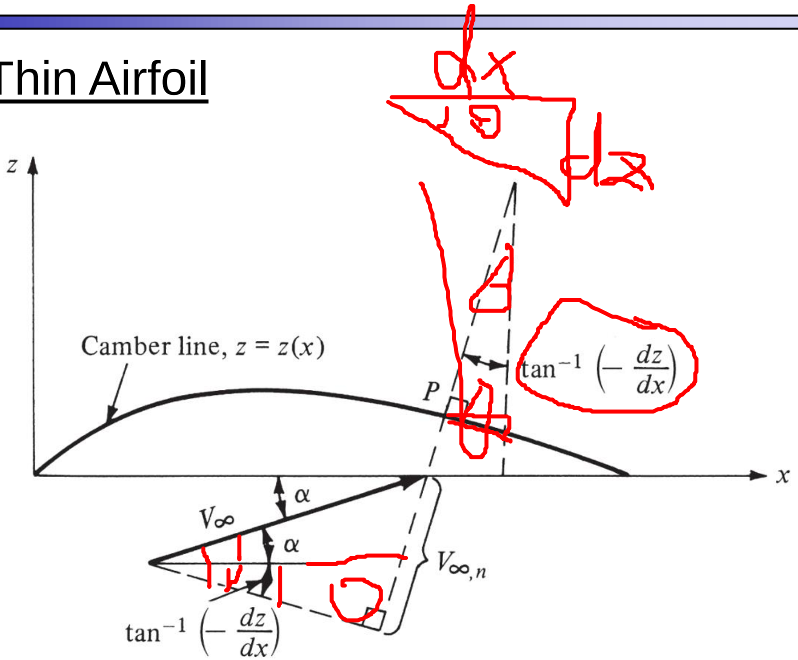


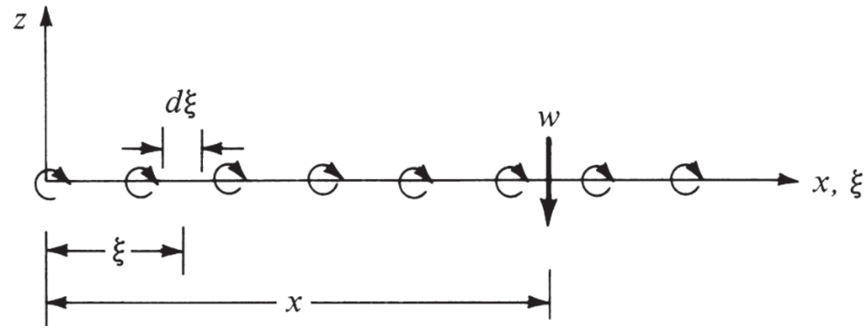
(b) Vortex sheet on the chord line

- Camber line becomes a streamline of the flow
- The Kutta condition is satisfied at the trailing edge: $\gamma(c) = 0$

Theoretical Solutions for Flow over Airfoils

- Thin Airfoil





$$V_{\infty,n} + w'(s) = 0$$

$$w'(s) \approx w(x)$$

$$V_{\infty,n} = V_\infty \sin\left[\alpha + \tan^{-1}\left(-\frac{dz}{dx}\right)\right]$$

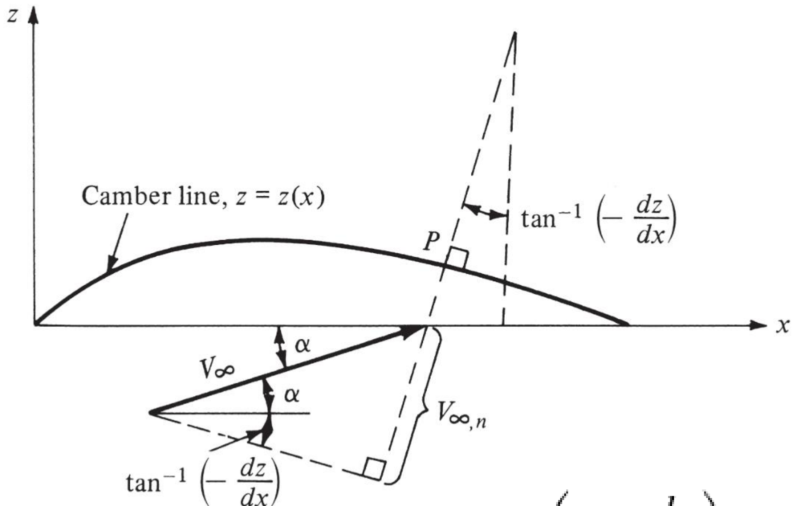
$$dw = -\frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}$$

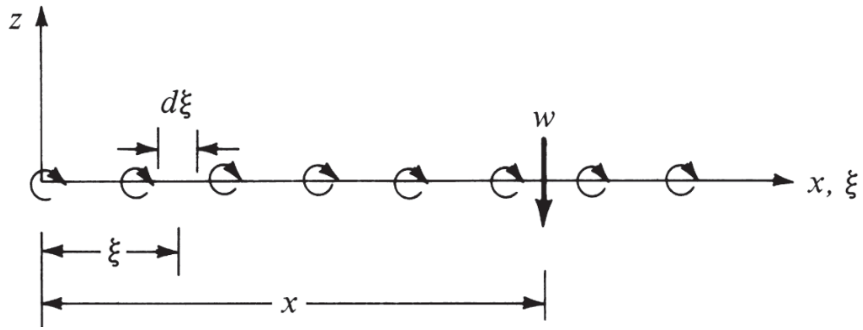
$$V_{\infty,n} = V_\infty\left(\alpha - \frac{dz}{dx}\right)$$

$$w(x) = -\int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}$$

Theoretical Solutions for Flow over Airfoils

- Thin Airfoil





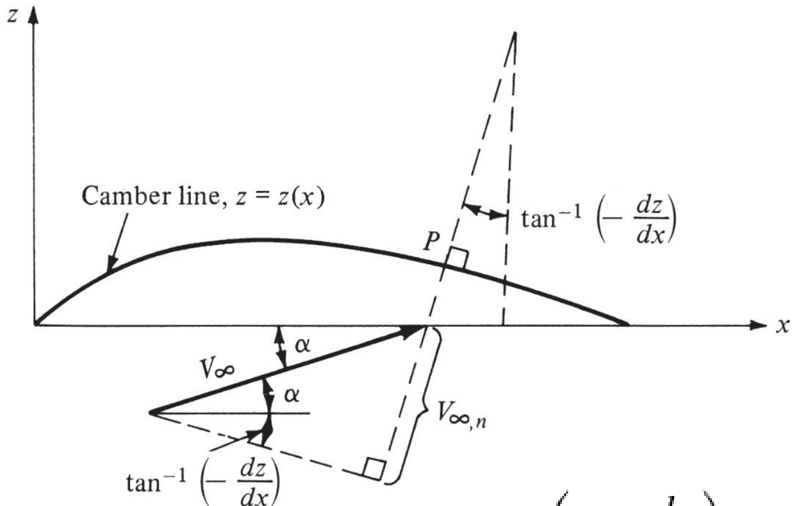
$$V_\infty\left(\alpha - \frac{dz}{dx}\right) - \int_0^c \frac{\gamma(\xi)\,d\xi}{2\pi(x-\xi)} = 0$$

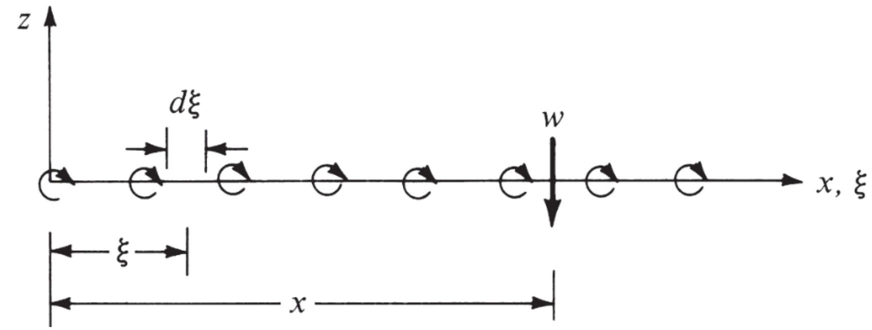
$$\boxed{\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right)}$$

- Camber line becomes a streamline of the flow

Theoretical Solutions for Flow over Airfoils

- Thin Airfoil





$$V_\infty\left(\alpha - \frac{dz}{dx}\right) - \int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)} = 0$$

$$\boxed{\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right)}$$

$dz/dx = 0,$

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty \alpha$$

Theoretical Solutions for Flow over Airfoils

- <u>Thin Airfoil</u>

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty \alpha$$

$$\xi = \frac{c}{2}(1-\cos\theta) \qquad d\xi = \frac{c}{2}\sin\theta\, d\theta$$

$$x = \frac{c}{2}(1-\cos\theta_0)$$

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\, d\theta}{\cos\theta - \cos\theta_0} = V_\infty \alpha$$

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

$$\boxed{\gamma(\theta) = 2\alpha V_\infty \frac{(1+\cos\theta)}{\sin\theta}}$$

Using L'Hospital's rule $\qquad \gamma(\pi) = 2\alpha V_\infty \frac{-\sin\pi}{\cos\pi} = 0$

- The Kutta condition is satisfied at the trailing edge: $\gamma(c) = \gamma(\pi) = 0$

Theoretical Solutions for Flow over Airfoils

- <u>Thin Airfoil</u>

$$\Gamma = \int_0^c \gamma(\xi)\, d\xi$$

$$\Gamma = \frac{c}{2} \int_0^\pi \gamma(\theta) \sin\theta\, d\theta$$

$$\Gamma = \alpha c V_\infty \int_0^\pi (1 + \cos\theta)\ d\theta = \pi \alpha c V_\infty$$

- Kutta-Joukowski theorem

$$L' = \rho_\infty V_\infty \Gamma = \pi \alpha c \rho_\infty V_\infty^2$$

$$c_l = \frac{\pi \alpha c \rho_\infty V_\infty^2}{\frac{1}{2}\rho_\infty V_\infty^2 c(1)}$$

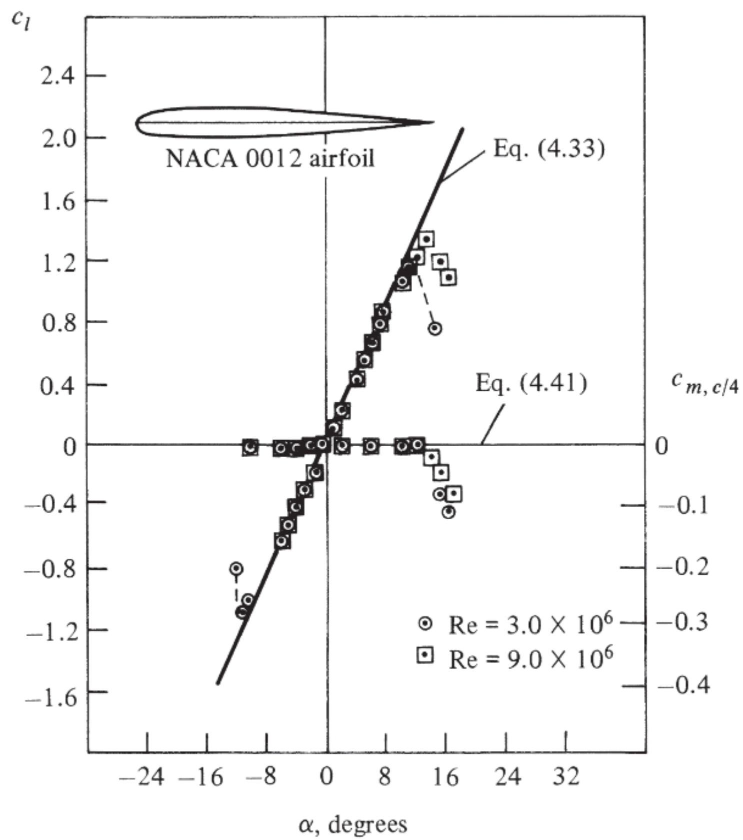
$$\boxed{c_l = 2\pi\alpha}$$

$$\boxed{\text{Liftslope} = \frac{dc_l}{d\alpha} = 2\pi}$$

Theoretical Solutions for Flow over Airfoils

- Thin Airfoil

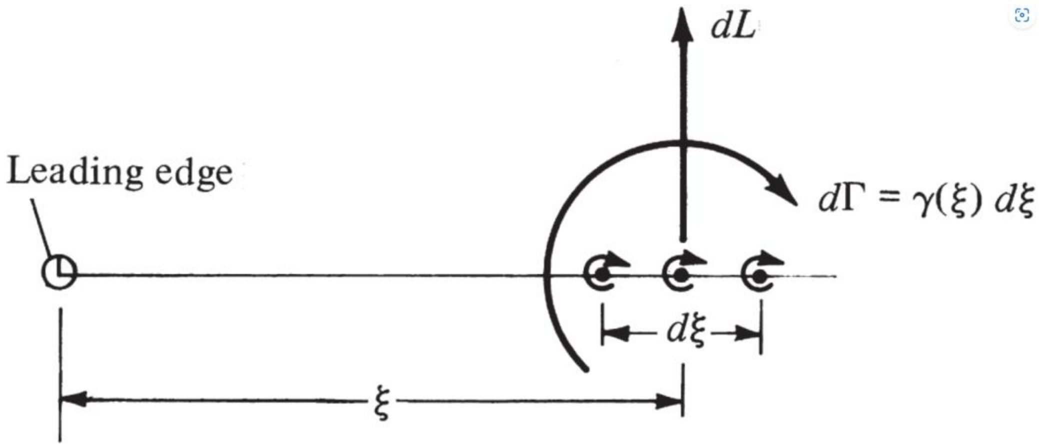
$\boxed{c_l = 2\pi\alpha}$ accurately predicts c_l over a large range of angle attack.



Theoretical Solutions for Flow over Airfoils

- <u>Thin Airfoil</u>

The total moment about the leading edge



$$M'_{LE} = -\int_0^c \xi(dL) = -\rho_\infty V_\infty \int_0^c \xi\gamma(\xi)\,d\xi$$

$$M'_{LE} = -q_\infty c^2 \frac{\pi\alpha}{2}$$

Theoretical Solutions for Flow over Airfoils

- <u>Thin Airfoil</u>

The moment coefficient about the leading edge is $$c_{m,\mathrm{le}} = \frac{M'_{\mathrm{LE}}}{q_\infty c^2} = -\frac{\pi\alpha}{2}$$

$$\boxed{c_l = 2\pi\alpha}$$

$$\boxed{c_{m,\mathrm{le}} = -\frac{c_l}{4}}$$

The moment coefficient about the quarter-chord point is $$c_{m,c/4} = c_{m,\mathrm{le}} + \frac{c_l}{4}$$

$$\boxed{c_{m,c/4} = 0}$$

The quarter-chord point is the center of pressure and the aerodynamic center for a symmetric airfoil

Theoretical Solutions for Flow over Airfoils

- Example Practice

Consider a thin airfoil at 5 deg. angle of attack. Calculate the (1) lift coefficient, (2) moment coefficient about the leading edge, and (3) moment coefficient about the quarter-chord point.

Theoretical Solutions for Flow over Airfoils

- <u>Example Practice</u>

Consider a thin airfoil at 5 deg. angle of attack. Calculate the (1) lift coefficient, (2) moment coefficient about the leading edge, and (3) moment coefficient about the quarter-chord point.

$$c_\ell = 2\pi\alpha$$

$$\alpha = \frac{5}{57.3} = 0.0873 \text{ rad}$$

$$c_\ell = 2\pi(0.0873) = \boxed{0.5485}$$

$$c_{m,\ell e} = -\frac{c_\ell}{4} = -\frac{0.5485}{4} = \boxed{-0.137}$$

$$c_{m,c/4} = \boxed{0}$$