Derivation of Four Relations Needed in Using the Patched Conic Method

Prussing & Conway, prob. 1.14 In terms of V_s, which is circular orbit speed at the surface of a planet of radius r_s, and r_p, which is periapse radius of a hyperbolic orbit about the planet, show that

a) $\mu = r_s V_s^2$

b) $e = 1 + \psi$ where $\psi \equiv \left(\frac{V_\infty}{V_s}\right)^2 \left(\frac{r_p}{r_s}\right)$

c) $\sin\left(\frac{\delta}{2}\right) = \frac{1}{1+\psi}$

d) $\frac{\Delta}{r_s} = \frac{r_p}{r_s}\sqrt{1 + \frac{2}{\psi}}$

a) For a circular orbit

$$V_c = \sqrt{\frac{\mu}{r_c}}$$

or

$$\mu = r_c V_c^2$$

On the surface of a planet $r_c = r_s$, $V_c = V_s$

$$\therefore \boxed{\mu = r_s V_s^2} \qquad (1)$$

b) The energy integral for a hyperbolic trajectory is

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a}$$

As $r \to \infty$, $V = V_\infty$ (V_∞ is called the <u>hyperbolic excess speed</u>)

Therefore

$$\frac{V_\infty^2}{2} = \frac{\mu}{2a}$$

or

$$a = \frac{\mu}{V_\infty^2} \qquad (2)$$

Sub. (1) into (2)

$$a = r_s \left(\frac{V_s}{V_\infty}\right)^2 \qquad (3)$$

Also for a hyperbola

$$r = \frac{a(e^2-1)}{1+e\cos\theta} \qquad (4)$$

When $\theta = 0$, $r = r_p$

$$r_p = \frac{a(e^2-1)}{1+e} = a(e-1) \qquad (5)$$

or

$$e = 1 + \frac{r_p}{a} \qquad (6)$$

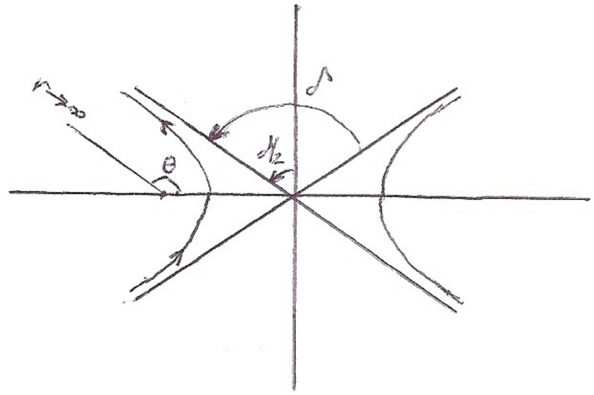
Sub. (3) into (6)

$$e = 1 + \frac{r_p}{r_s}\left(\frac{V_\infty}{V_s}\right)^2$$

or

$$\boxed{e = 1 + \psi} \quad (7) \qquad \text{where} \quad \boxed{\psi \equiv \left(\frac{V_\infty}{V_s}\right)^2\left(\frac{r_p}{r_s}\right)} \quad (8)$$

c)



δ is called the <u>turn angle</u>

From the above figure, as $r \to \infty$, $\theta \to \frac{\pi}{2} + \frac{\delta}{2}$

From (4) as $r \to \infty$

$$1 + e\cos\theta = 0$$

$$1 + e\cos\left(\frac{\pi}{2} + \frac{\delta}{2}\right) = 0$$

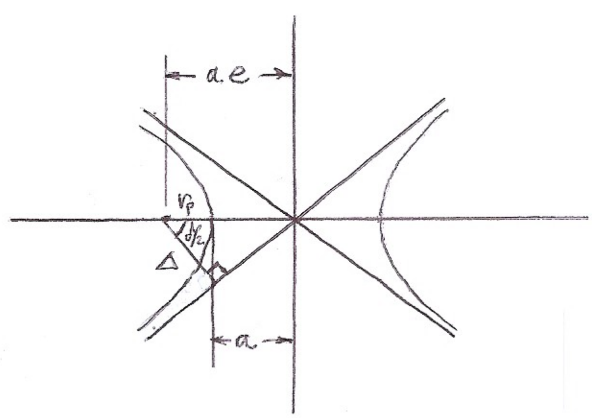
$$1 - e\sin\left(\frac{\delta}{2}\right) = 0$$

$$\sin\left(\frac{\delta}{2}\right) = \frac{1}{e} \qquad (9)$$

Sub. (7) into (9)

$$\boxed{\sin\left(\frac{\delta}{2}\right) = \frac{1}{1+\varphi}} \qquad (10)$$

d)



Δ is called the <u>aiming radius</u>

From the above figure

$$\cos\left(\frac{\delta}{2}\right) = \frac{\Delta}{ae}$$

$$\Delta = ae\cos\left(\frac{\delta}{2}\right) = ae\sqrt{1-\sin^2\left(\frac{\delta}{2}\right)} \qquad (11)$$

Solve (5) for a

$$a = \frac{r_p}{e-1} \qquad (12)$$

Sub. (10) & (12) into (11)

$$\Delta = \frac{r_p}{e-1}\, e\sqrt{1-\frac{1}{(1+\psi)^2}} = r_p \frac{e}{e-1}\sqrt{\frac{(1+\psi)^2-1}{(1+\psi)^2}}$$

$$= r_p \frac{e}{e-1}\,\frac{1}{1+\psi}\sqrt{\psi(\psi+2)} \qquad (13)$$

Sub (7) into (13)

$$\Delta = r_p \frac{\cancel{1+\psi}}{\psi}\,\frac{1}{\cancel{1+\psi}}\sqrt{\psi(\psi+2)}$$

$$\Delta = r_p\sqrt{\frac{\psi+2}{\psi}} = r_p\sqrt{1+\frac{2}{\psi}}$$

or

$$\boxed{\frac{\Delta}{r_s} = \frac{r_p}{r_s}\sqrt{1+\frac{2}{\psi}}}$$