ME 57200 Aerodynamic Design

Lecture #4: Review of Vector Relations

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Navier-Stokes Equations

- **The Navier–Stokes equations** are partial differential equations which describe the motion of viscous fluid flows.

- **The Navier–Stokes equations** mathematically express <u>momentum balance</u> for Newtonian fluids and making use of <u>conservation of mass.</u> They are sometimes accompanied by an equation of state relating pressure, temperature and density.

- **The Navier–Stokes equations** may be used to model the weather, ocean currents, water flow in a pipe and air flow around a wing, help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution, and many other problems.

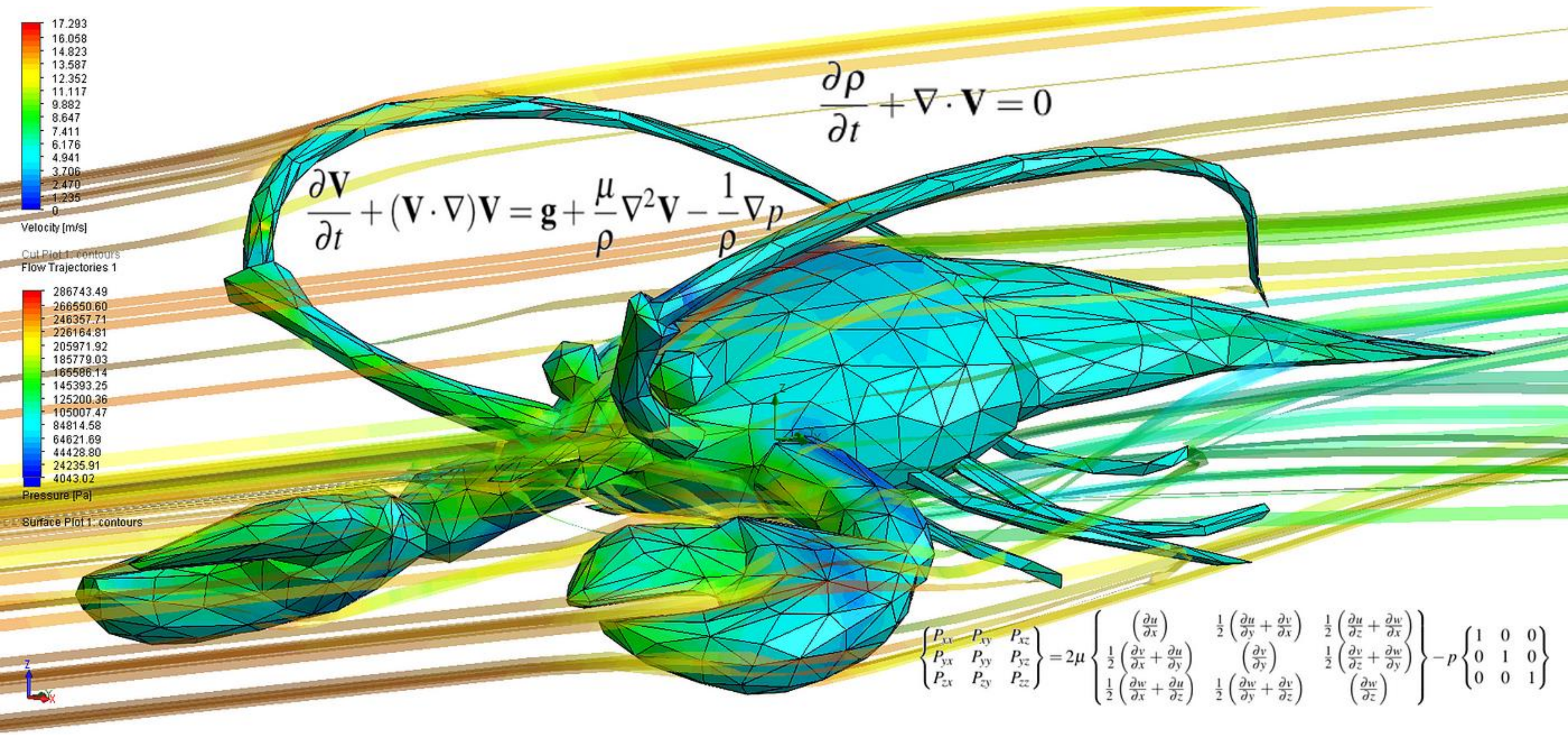
Navier-Stokes Equations

- **The Navier–Stokes equations** are of great interest in a purely mathematical sense.

- Despite their wide range of practical uses, it has not yet been proven whether **smooth solutions always exist in three dimensions**—i.e., whether they are infinitely differentiable (or even just bounded) at all points in the domain.

- This is called the **Navier–Stokes existence and smoothness problem**. The Clay Mathematics Institute has called this **one of the seven most important open problems** in mathematics and has offered a US $1 million prize for a solution or a counterexample.

Navier-Stokes Equations

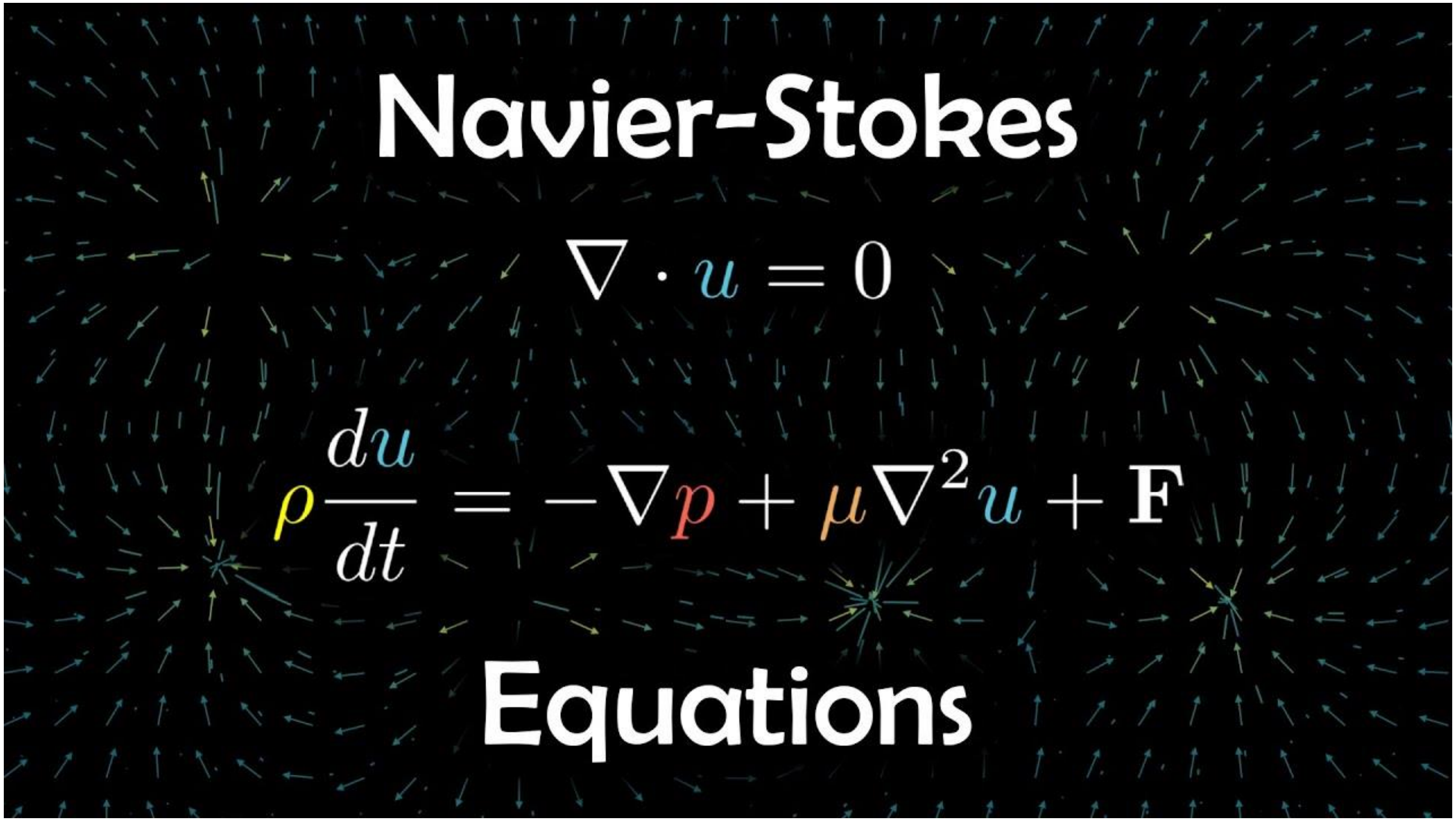


$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{V} = 0$$

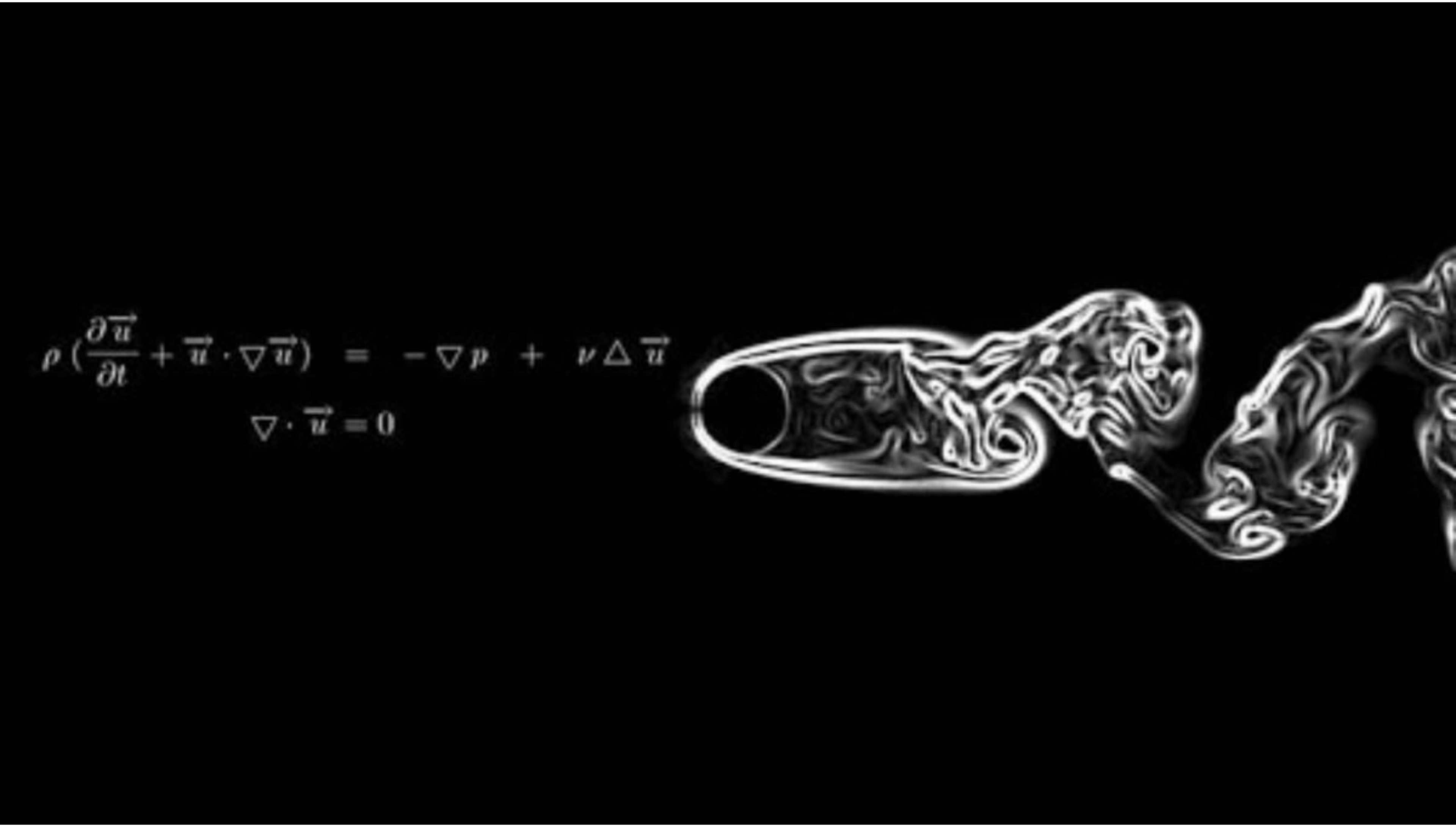
$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = \mathbf{g} + \frac{\mu}{\rho}\nabla^2 \mathbf{V} - \frac{1}{\rho}\nabla p$$

$$\begin{Bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{Bmatrix} = 2\mu \begin{Bmatrix} \left(\frac{\partial u}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \left(\frac{\partial v}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \left(\frac{\partial w}{\partial z}\right) \end{Bmatrix} - p \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

Navier-Stokes Equations



A Brief History of the Navier-Stokes Equations



Review of Vector Relations

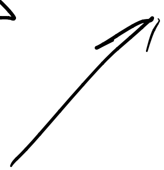
Scalar Quantity: A quantity which does not depend on direction

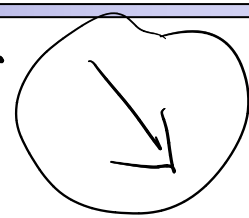
- Mass, volume, density, pressure, temperature, energy, enthalpy

Vector Quantity: A quantity which depends on direction

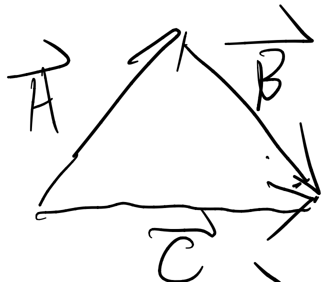
- Force, momentum, displacement, velocity, acceleration, vorticity…

Some Vector Algebra

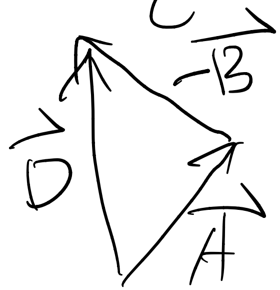
$\vec{A}$ 

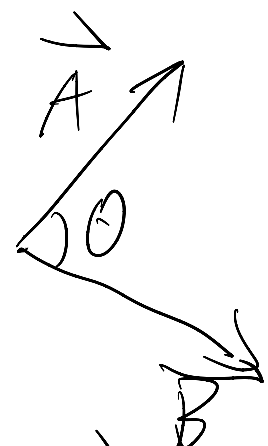
$\vec{B}$ 

$\vec{A} + \vec{B} :$



$\vec{A} - \vec{B} :$





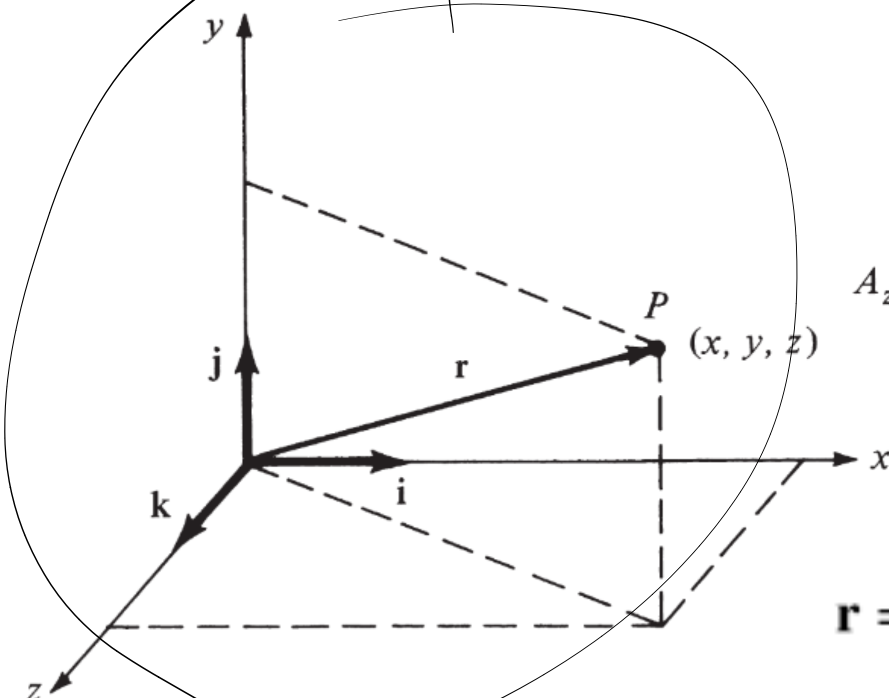
$\vec{A} \cdot \vec{B} : |\vec{A}| \cdot |\vec{B}| \cdot \cos\theta$

$\overline{\vec{A} \times \vec{B}} : |\vec{A}| \cdot |\vec{B}| \cdot \sin\theta \cdot \vec{e}$

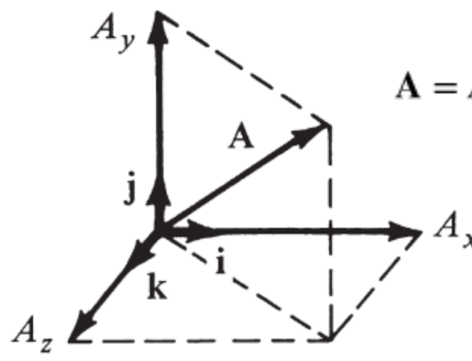
Typical Orthogonal Coordinate Systems

Cartesian coordinate system

$$\vec{A} \times \vec{B} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$



$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



(b)

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

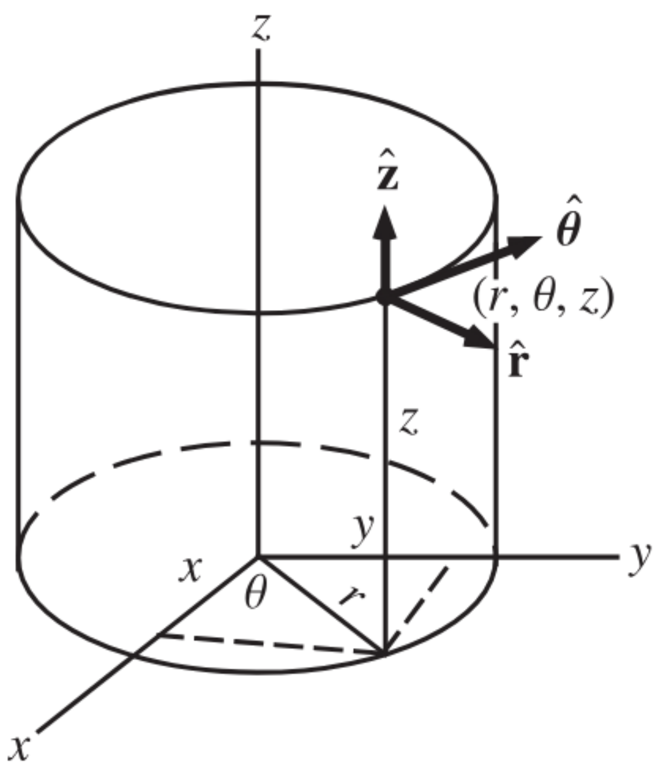
$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$$

$$\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$$

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Typical Orthogonal Coordinate Systems

Cylindrical coordinate system



$$\vec{A} = A_r\vec{e_r} + A_\theta\vec{e_\theta} + A_z\vec{e_z}$$

$$\vec{B} = B_r\vec{e_r} + B_\theta\vec{e_\theta} + B_z\vec{e_z}$$

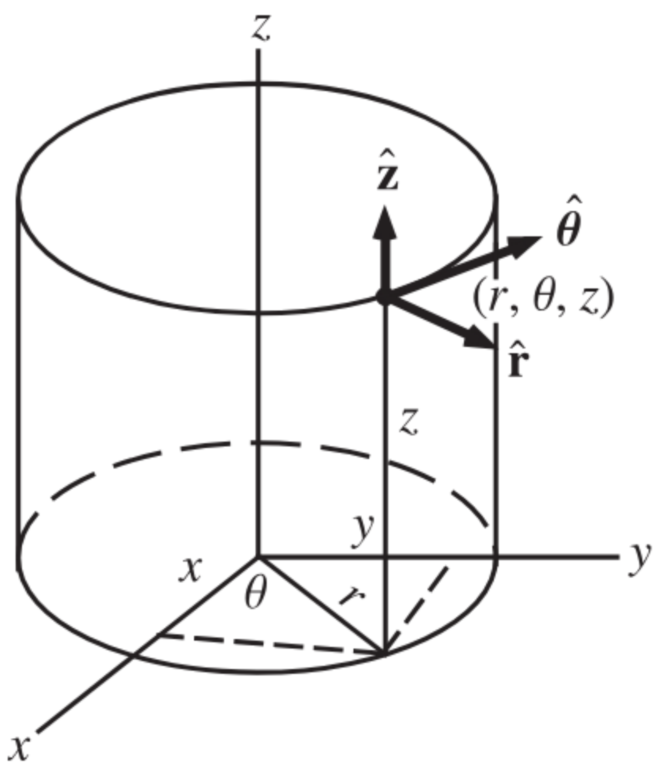
$$\mathbf{A} = \mathbf{A}_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_z\mathbf{e}_z$$

$$\vec{A}\cdot\vec{B} = A_rB_r + A_\theta B_\theta + A_zB_z$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{e_r} & \vec{e_\theta} & \vec{e_z} \\ A_r & A_\theta & A_z \\ B_r & B_\theta & B_z \end{vmatrix}$$

Typical Orthogonal Coordinate Systems

Cylindrical coordinate system

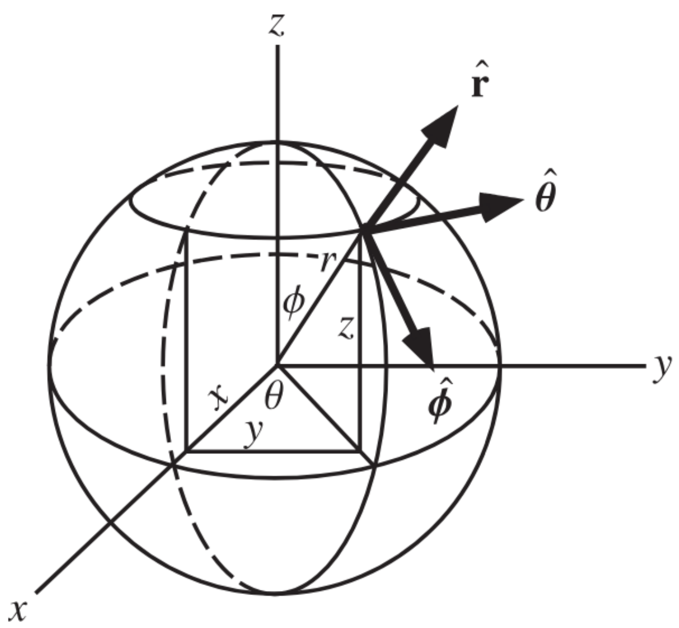


$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z
\end{aligned}
$$

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\theta &= \arctan\frac{y}{x} \\
z &= z
\end{aligned}
$$

Typical Orthogonal Coordinate Systems

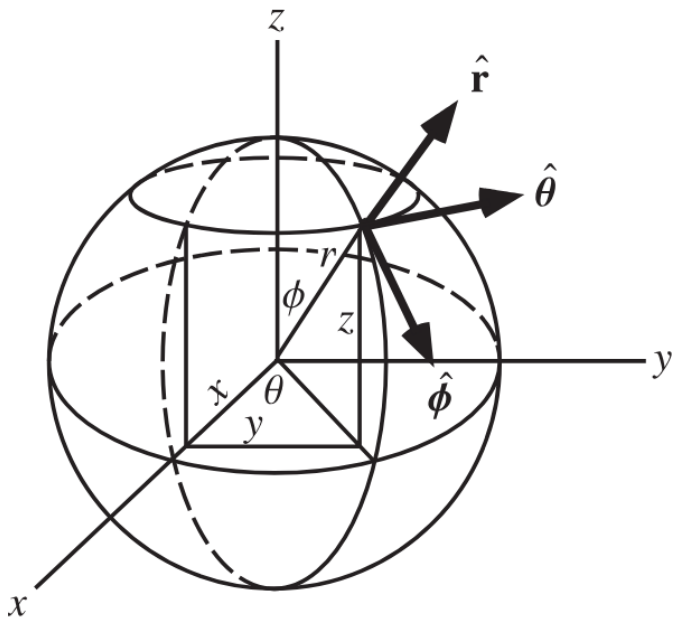
Spherical coordinate system



$$\mathbf{A} = A_r \mathbf{e}_r + A_\theta \mathbf{e}_\theta + A_\Phi \mathbf{e}_\Phi$$

Typical Orthogonal Coordinate Systems

Spherical coordinate system



$$
\begin{aligned}
x &= r\sin\theta\cos\Phi \\
y &= r\sin\theta\sin\Phi \\
z &= r\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
r &= \sqrt{x^2+y^2+z^2} \\
\theta &= \arccos\frac{z}{r} = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} \\
\Phi &= \arccos\frac{x}{\sqrt{x^2+y^2}}
\end{aligned}
$$

Scalar and Vector Fields

Scalar Field: A scalar quantity given as a function of coordinate space and time.

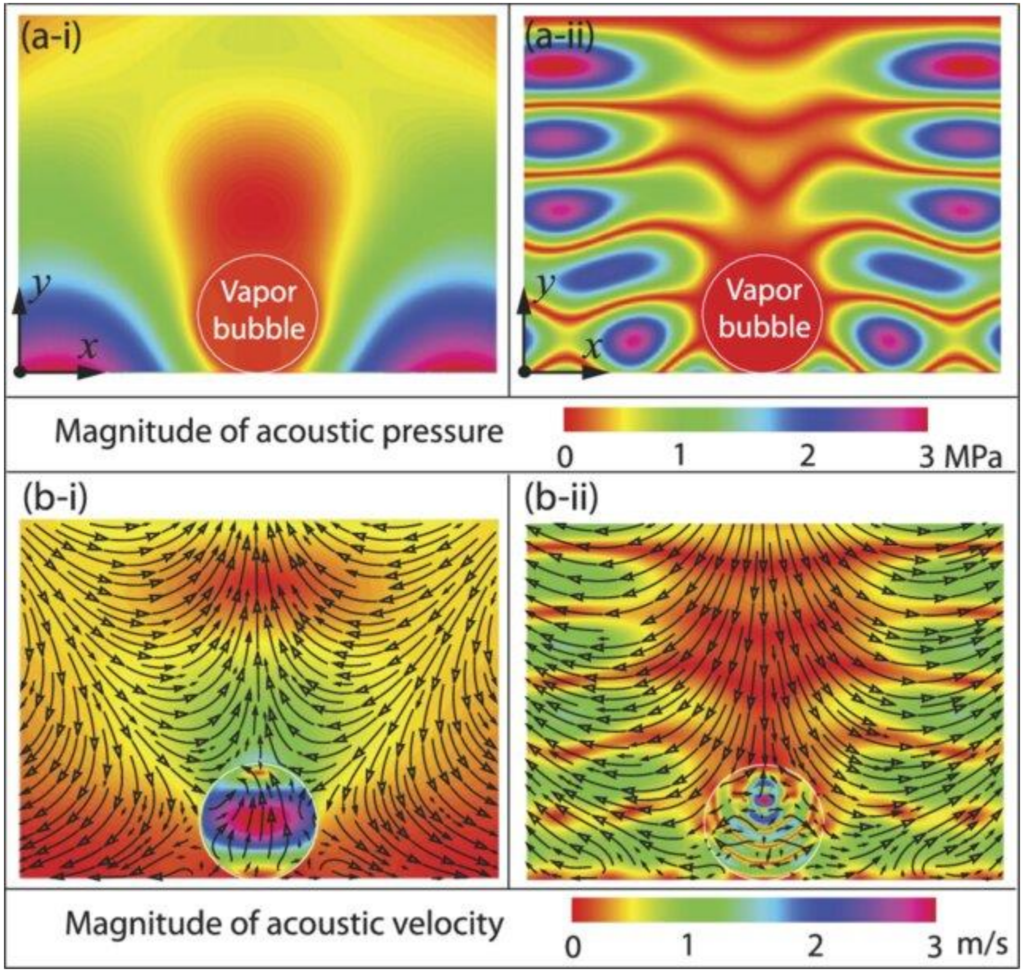
$$
\begin{aligned}
p &= p_1(x, y, z, t) = p_2(r, \theta, z, t) = p_3(r, \theta, \Phi, t) \\
\rho &= \rho_1(x, y, z, t) = \rho_2(r, \theta, z, t) = \rho_3(r, \theta, \Phi, t) \\
T &= T_1(x, y, z, t) = T_2(r, \theta, z, t) = T_3(r, \theta, \Phi, t)
\end{aligned}
$$

Vector Field: A vector quantity given as a function of coordinate space and time.

$$\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$$

$$
\begin{aligned}
V_x &= V_x(x, y, z, t) \\
V_y &= V_y(x, y, z, t) \\
V_z &= V_z(x, y, z, t)
\end{aligned}
$$

Scalar and Vector Fields



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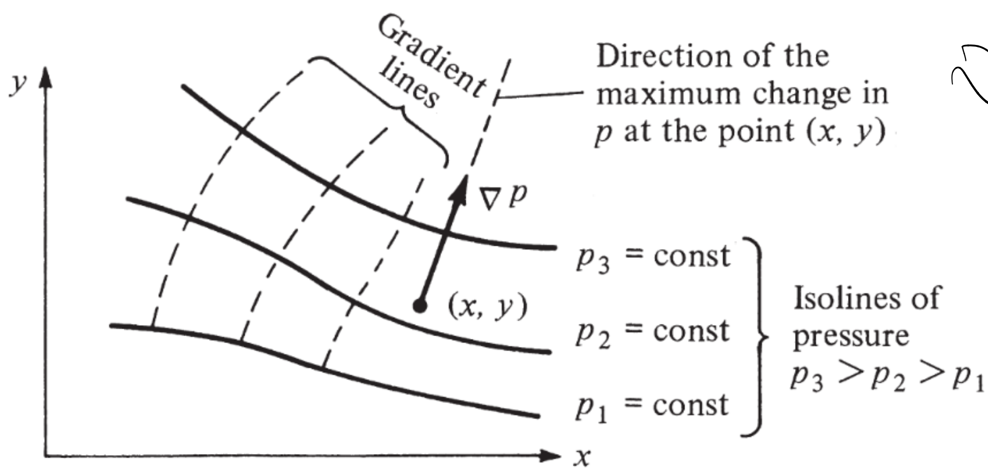
Gradient of a Scalar Field

The *gradient* of p, ∇p, at a given point in space is defined as a vector such that:

1. Its magnitude is the maximum rate of change of p per unit length of the coordinate space at the given point.
2. Its direction is that of the maximum rate of change of p at the given point.

$$\vec{\nabla} P = \frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}$$



$$P = P(r, \theta, z)$$

$$\nabla P = \frac{\partial P}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial P}{\partial \theta}\vec{e}_\theta + \frac{\partial P}{\partial z}\vec{e}_z$$

$$P(r, \theta, \phi) \Rightarrow \nabla P = \frac{\partial P}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial P}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}\vec{e}_\phi$$

Divergence of a Vector Field

The time rate of change of the volume of a moving fluid element of fixed mass, per unit volume of that element.

<u>The divergence of a vector is a scalar quantity</u>

$$\vec{V} = \vec{V}(x, y, z) = V_x \vec{i} + V_y \vec{j} + V_z \vec{k}$$

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \right)$$

Divergence of a Vector Field

$$\vec{V} = \vec{V}(r, \theta, z)$$

$$\Rightarrow \vec{\nabla} \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$$

$$\vec{V} = \vec{V}(r, \theta, \phi)$$

$$\Rightarrow \vec{\nabla} \cdot \vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(V_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}$$

In-Class Quiz