1) a) What is the apogee altitude of a ballistic missile fired on an optimum trajectory across a 6000 km range?

b) What is the apogee altitude of the missile when fired with the same velocity to a range of 3000 km?

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a) $S_{max} = 6000 \text{ km}$

$$\phi_{max} = \frac{S_{max}}{2r_o} = \frac{6000}{2(6368)} = 0.4711 \text{ radians} = 26.99^\circ$$

$$h_{max} = \frac{r_o}{2}\left[\cos\phi_{max} + \sin\phi_{max} - 1\right] = \frac{6368}{2}\left[\cos 26.99^\circ + \sin 26.99^\circ - 1\right]$$

$$\boxed{h_{max} = 1098 \text{ km}}$$

b) $$v_{min} = \sqrt{\frac{2\sin\phi_{max}}{1+\sin\phi_{max}}} = \sqrt{\frac{2\sin 26.99^\circ}{1+\sin 26.99^\circ}} = 0.7902$$

$v_o = 0.7902$

$S = 3000 \text{ km}$

$$\phi_o = \frac{S}{2r_o} = \frac{3000}{2(6368)} = 0.2356 \text{ radians} = 13.50^\circ$$

Need γ_o in terms of ϕ_o and v_o. From class notes:

$$\tan\phi_o = \frac{(\tan\gamma_o)(v_o^2\cos^2\gamma_o)}{1-v_o^2\cos^2\gamma_o} = \frac{v_o^2\tan\gamma_o}{\sec^2\gamma_o - v_o^2} = \frac{v_o^2\tan\gamma_o}{\tan^2\gamma_o + 1 - v_o^2}$$

$$\tan\phi_o\tan^2\gamma_o - v_o^2\tan\gamma_o + (1-v_o^2)\tan\phi_o = 0$$

Solve for $\tan\gamma_o$ using quadratic formula

$$\tan\gamma_o = \frac{\nu_o^2 \pm \sqrt{\nu_o^4 - 4(1-\nu_o^2)\tan^2\phi_o}}{2\tan\phi_o}$$

$$= \frac{(0.7902)^2 \pm \sqrt{(0.7902)^4 - 4(1-(0.7902)^2)\tan^2 13.50^\circ}}{2\tan 13.50^\circ}$$

Find two possible solutions

$\gamma_{o1} = 1.1829 \text{ rad} = 67.77^\circ$ $\quad\quad$ $\gamma_{o2} = 0.1523 \text{ rad} = 8.72^\circ$

For $\gamma_{o1} = 67.77^\circ$

$$h_{a1} = \frac{r_o}{2-\nu_o^2}\left[\sqrt{1-\nu_o^2(2-\nu_o^2)\cos^2\gamma_{o1}} - (1-\nu_o^2)\right]$$

$$= \frac{6368}{2-(0.7902)^2}\left[\sqrt{1-(0.7902)^2(2-(0.7902)^2)\cos^2 67.77^\circ} - (1-(0.7902)^2)\right]$$

$$\boxed{h_{a1} = 2597 \text{ km}}$$

For $\gamma_{o2} = 8.72^\circ$

$$h_{a2} = \frac{r_o}{2-\nu_o^2}\left[\sqrt{1-\nu_o^2(2-\nu_o^2)\cos^2\gamma_{o2}} - (1-\nu_o^2)\right]$$

$$= \frac{6368}{2-(0.7902)^2}\left[\sqrt{1-(0.7902)^2(2-(0.7902)^2)\cos^2 8.72^\circ} - (1-(0.7902)^2)\right]$$

$$\boxed{h_{a2} = 118 \text{ km}}$$

2) Determine the range of initial velocities and elevation angles which will carry a projectile over a 30,000 km range on the earth.

$$\phi_o = \frac{s}{2r_o} = \frac{30,000}{2(6368)} = 2.356 \text{ rad} = 135^\circ \quad (90^\circ < \phi_o < 180^\circ)$$

From (6.29) in the class notes

$$\gamma_o < \frac{\pi - \phi_o}{2} = \frac{\pi - 2.356}{2} = 0.3928 \text{ rad} = 22.5^\circ$$

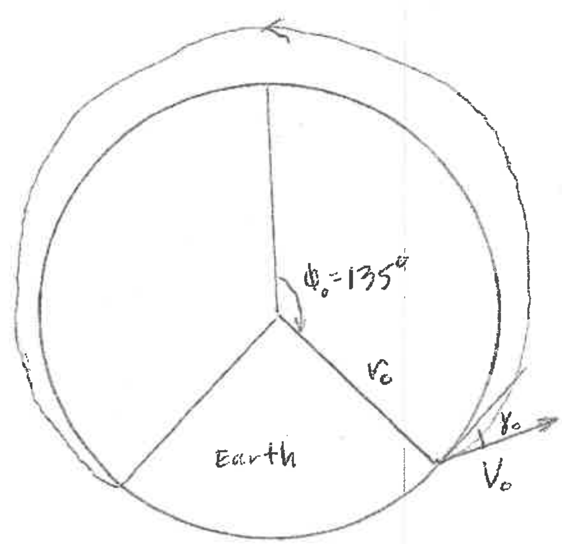
$$V_{co} = \sqrt{\frac{\mu}{r_o}} = \sqrt{\frac{3.986 \times 10^5}{6368}} = 7.9117 \text{ km/sec}$$

Using (6.28) in the class notes

$$\nu_o^2 = \frac{\tan\phi_o}{\sin\gamma_o\cos\gamma_o + \cos^2\gamma_o\tan\phi_o} = \frac{V_o^2}{V_{co}^2}$$

a table of possible solutions is shown below

γ_o	ν_o^2	V_o (km/sec)
0°	1	7.9117
5°	1.1041	8.3133
10°	1.2515	8.8507
15°	1.4634	9.5708
20°	1.7792	10.5530
22.5°	2	11.1888



3) Curtis

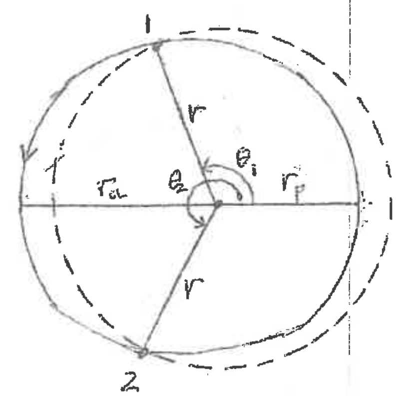
3.8 A satellite is in earth orbit for which the perigee altitude is 200 km and the apogee altitude is 600 km. Find the time interval during which the satellite remains above an altitude of 400 km.

{Ans.: 47.15 min}

$r_p = 6378 + 200 = 6578 \text{ km}$

$r_a = 6378 + 600 = 6978 \text{ km}$

$r = 6378 + 400 = 6778 \text{ km}$



$$a = \frac{r_a + r_p}{2} = \frac{6978 + 6578}{2} = 6778 \text{ km}$$

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{6978 - 6578}{6978 + 6578} = 0.029507$$

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \Rightarrow \cos\theta = \frac{a(1-e^2) - r}{re}$$

$$\cos\theta = \frac{6778\left(1 - (0.029507)^2\right) - 6778}{(6778)(0.029507)} = -0.029507$$

$\theta_1 = 1.60031 \text{ rad} = 91.6909°$ $\qquad \theta_2 = 4.68288 \text{ rad} = 268.309°$

$$\tan\frac{E_1}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_1}{2} = \sqrt{\frac{1-0.029507}{1+0.029507}}\tan\frac{91.6909°}{2} = 1 \Rightarrow E_1 = 1.5708 \text{ rad}$$

$$\tan\frac{E_2}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_2}{2} = \sqrt{\frac{1-0.029507}{1+0.029507}}\tan\frac{268.309°}{2} = -1 \Rightarrow E_2 = 4.71239 \text{ rad}$$

$$M_1 = E_1 - e\sin E_1 = 1.5708 - 0.029507\sin 1.5708 = 1.54129 \text{ rad}$$

$$M_2 = E_2 - e\sin E_2 = 4.71239 - 0.029507\sin 4.71239 = 4.74188 \text{ rad}$$

$$n = \sqrt{\frac{\mu}{a^3}} = \sqrt{\frac{3.986 \times 10^5}{(6778)^3}} = 0.0011314 \ \text{sec}^{-1}$$

$$t_2 - t_1 = \frac{M_2 - M_1}{n} = \frac{4.74188 - 1.54129}{0.0011314}$$

$$\boxed{t_2 - t_1 = 2828.88 \ \text{sec} = 47.148 \ \text{min}}$$

4) Curtis

3.11 A satellite in earth orbit has perigee and apogee radii of $r_p = 7500$ km and $r_a = 16{,}000$ km, respectively. Find its true anomaly 40 min after passing the true anomaly of 80°.

{Ans.: 174.7°}

$$a = \frac{r_a + r_p}{2} = \frac{16{,}000 + 7500}{2} = 11{,}750 \text{ km}$$

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{16{,}000 - 7500}{16{,}000 + 7500} = 0.3617$$

$$\tan\frac{E_1}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta_1}{2} = \sqrt{\frac{1-0.3617}{1+0.3617}}\tan\frac{80^\circ}{2} = 0.574494 \Rightarrow E_1 = 1.04291 \text{ rad}$$

$$M_1 = E_1 - e\sin E_1 = 1.04291 - 0.3617\sin 1.04291 = 0.730447 \text{ rad}$$

$$n = \sqrt{\frac{\mu}{a^3}} = \sqrt{\frac{3.986\times10^5}{(11{,}750)^3}} = 0.000495692 \text{ sec}^{-1}$$

$$t_1 = \frac{M_1}{n} = \frac{0.730447}{0.000495692} = 1473.59 \text{ sec}$$ (time to travel from perigee ($\theta = 0$) to $\theta_1 = 80^\circ$)

$$t_2 = 1473.59 + 40(60) = 3873.59 \text{ sec}$$

$$M_2 = n t_2 = (0.000495692)(3873.59) = 1.92011 \text{ rad}$$

To solve $E_2 - e\sin E_2 = M_2$ for E_2, use the Newton-Raphson method

$$f(E_2) = E_2 - e\sin E_2 - M_2 = 0$$

$$f'(E_2) = 1 - e\cos E_2$$

$$E_{z_{n+1}} = E_{z_n} - \frac{f(E_{z_n})}{f'(E_{z_n})} = E_{z_n} - \frac{E_{z_n} - e\sin E_{z_n} - M_z}{1 - e\cos E_{z_n}}$$

Starting with an initial guess $E_{z_0} = M_z = 1.92011$ rad, the iterations are shown below

i	E_{z_i}
0	1.92011
1	2.22253
2	2.21035
3	2.21033
4	2.21033

$E_z = 2.21033$ rad

$$\tan\frac{\theta_z}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E_z}{2} = \sqrt{\frac{1+0.3617}{1-0.3617}}\tan\frac{2.21033}{2} = 2.90675$$

$$\boxed{\theta_z = 2.47891 \text{ rad} = 142.03^\circ}$$