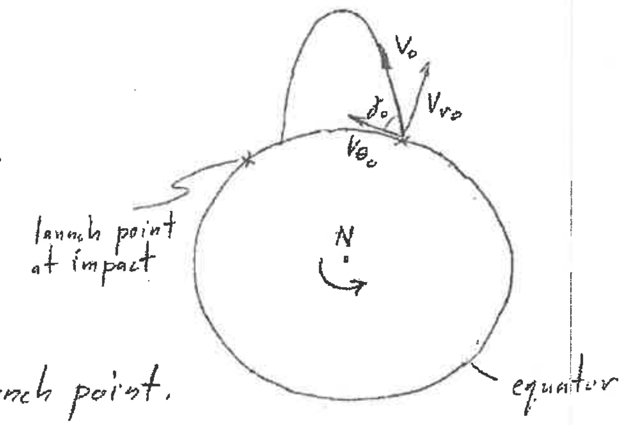
ME 51500/I5800 HW 5 SOLUTIONS P. GANATOS

1) A projectile is launched straight up from the equator with an initial velocity (radial) of 6 km/sec. Will it come down in the same place or not? Substantiate your answer with a numerical calculation. Neglect the effect of atmospheric drag but include the effect of the earth's rotation.



Since $r^2 \frac{d\theta}{dt} = h = \text{const.}$, as r is increased, $\frac{d\theta}{dt}$ (angular velocity of the projectile) must decrease. Since the earth rotates with constant angular velocity, the projectile will land west of the launch point.

Numerical calculation:

$V_{r0} = 6 \text{ km/sec}$

$$V_{\theta 0} = \frac{2\pi r_0}{1 \text{ sidereal day}} = \frac{2\pi (6378 \text{ km})}{86{,}164 \text{ sec}} = 0.4651 \text{ km/sec}$$

$$V_0^2 = V_{r0}^2 + V_{\theta 0}^2 = (6)^2 + (0.4651)^2 = 36.2163 \text{ km}^2/\text{sec}^2$$

$$V_{c0}^2 = \frac{\mu}{r_0} = \frac{3.986016 \times 10^5 \text{ km}^3/\text{sec}^2}{6378 \text{ km}} = 62.4963 \text{ km}^2/\text{sec}^2$$

$$\nu_0^2 = \frac{V_0^2}{V_{c0}^2} = \frac{36.2163}{62.4963} = 0.5795$$

$$\tan \gamma_0 = \frac{V_{r0}}{V_{\theta 0}} = \frac{6}{0.4651} = 12.901 \Rightarrow \gamma_0 = 85.58^\circ$$

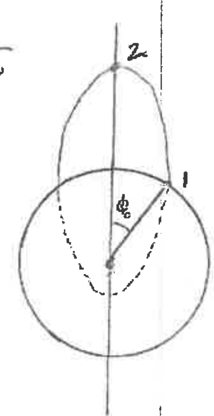
Compute ground range to impact as viewed from a stationary reference frame

$$\tan \phi_0 = \frac{\sin 2\gamma_0}{\frac{2}{\nu_0^2} - 1 - \cos 2\gamma_0} = \frac{0.1541}{\frac{2}{0.5795} - 1 + 0.9881} = 0.04481$$

$$\phi_0 = 0.04478 \text{ radians} = 2.565^\circ$$

$$S = r_0(2\phi_0) = (6378)[2(0.04478)] = \underline{\underline{571 \text{ km}}}$$

To compute how far the launch point has travelled due to the earth's rotation, need to compute the time of flight τ.



$$e = \sqrt{1 - \nu_0^2(2-\nu_0^2)\cos^2\gamma_0} = \sqrt{1 - 0.5795(2-0.5795)\cos^2 85.58°}$$

$$= 0.9976$$

$$\tan\frac{E_1}{2} = \left[\frac{1-e}{1+e}\right]^{1/2} \tan\frac{\pi-\phi_0}{2} = \left[\frac{1-0.9976}{1+0.9976}\right]^{1/2} \tan\frac{\pi-0.04481}{2}$$

$$= 1.5468 \quad\Rightarrow\quad E_1 = 1.994 \text{ radians} = 114.2°$$

$$M_1 = E_1 - e\sin E_1 = 1.994 - (0.9976)\sin 1.994 = 1.084 \text{ radians} = 62.13°$$

$$\tan\frac{E_2}{2} = \left[\frac{1-e}{1+e}\right]^{1/2} \tan\frac{\pi}{2} \quad\Rightarrow\quad E_2 = \pi \text{ radians}$$

$$M_2 = E_2 - e\sin E_2 = \pi - 0.9976\sin\pi = \pi \text{ radians}$$

$$a = \frac{r_0}{2-\nu_0^2} = \frac{6378}{2-0.5795} = 4490 \text{ km}$$

$$n = \sqrt{\frac{\mu}{a^3}} = \sqrt{\frac{3.986\times10^5}{(4490)^3}} = 0.002098 \text{ rad/sec}$$

$$M_2 - M_1 = n(t_2 - t_1) = n\frac{\tau}{2}$$

$$\tau = \frac{2(M_2-M_1)}{n} = \frac{2(\pi - 1.084)}{0.002098} = \underline{\underline{1961 \text{ sec}}}$$

Launch point travels

$$V_{\oplus_0}\tau = (0.4651)(1961) = \underline{\underline{912 \text{ km}}}$$

$$\Delta s = 912 - 571 = \underline{\underline{341 \text{ km}}}$$

$\boxed{\text{The projectile lands 341 km west of the launch point}}$

2) Show that for a parabolic trajectory, the time after pericenter passage is

$$t = \frac{1}{2\sqrt{\mu}}\left[pD + \frac{1}{3}D^3\right]$$

where $D = \sqrt{p}\tan\frac{\theta}{2}$.

From (5.8) in the class notes

$$\frac{dA}{dt} = \frac{1}{2}h = \text{const.} \qquad (1)$$

Integrate (1)

$$A = \frac{1}{2}ht = \frac{1}{2}\sqrt{\mu p}\; t \qquad (2)$$

where the condition $A=0$ at $t=0$ has been used

The area swept out by r is

$$A = \frac{1}{2}\int_0^{\theta} r^2\, d\theta \qquad (3)$$

For a parabolic trajectory

$$r = \frac{p}{1+\cos\theta} \qquad (4)$$

Sub. (4) into (3)

$$A = \frac{p^2}{2}\int_0^{\theta}\frac{d\theta}{(1+\cos\theta)^2} \qquad (5)$$

Using the identities

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2} \qquad 1+\tan^2\alpha = \sec^2\alpha$$

eq (5) may be written as

$$A = \frac{P^2}{8}\int_0^\theta \frac{d\theta}{\cos^4\frac{\theta}{2}} = \frac{P^2}{8}\int_0^\theta \sec^4\frac{\theta}{2}\,d\theta = \frac{P^2}{8}\int_0^\theta \left(1+\tan^2\frac{\theta}{2}\right)^2 d\theta \qquad (6)$$

Let

$$\boxed{D = \sqrt{P}\tan\frac{\theta}{2}} \qquad (7)$$

$$\tan\frac{\theta}{2} = \frac{D}{\sqrt{P}} \qquad (8)$$

$$dD = \frac{\sqrt{P}}{2}\sec^2\frac{\theta}{2}\,d\theta = \frac{\sqrt{\ }}{2}\left(1+\tan^2\frac{\theta}{2}\right)d\theta = \frac{\sqrt{P}}{2}\left(1+\frac{D^2}{P}\right)d\theta$$

$$d\theta = \frac{dD}{\frac{\sqrt{P}}{2}\left(1+\frac{D^2}{P}\right)} \qquad (9)$$

Sub (8) & (9) into (6)

$$A = \frac{P^2}{8}\int_0^D \left(1+\frac{D^2}{P}\right)^2 \frac{dD}{\frac{\sqrt{P}}{2}\left(1+\frac{D^2}{P}\right)} = \frac{\sqrt{P}}{4}\int_0^D (P+D^2)\,dD$$

$$A = \frac{\sqrt{p}}{4}\left(pD + \frac{D^3}{3}\right) \qquad (10)$$

Equate (2) to (10)

$$\frac{1}{2}\sqrt{\mu p}\, t = \frac{\sqrt{p}}{4}\left(pD + \frac{D^3}{3}\right)$$

$$\boxed{t = \frac{1}{2\sqrt{\mu}}\left(pD + \frac{D^3}{3}\right)} \qquad (11)$$

3) Starting with the relation

$$\bar{r}\cdot\frac{d\bar{r}}{dt} = r\frac{dr}{dt}$$

show that for a parabolic trajectory

$$\bar{r}\cdot\bar{V} = \sqrt{\mu p}\,\frac{\sin\theta}{1+\cos\theta} = \sqrt{\mu p}\,\tan\frac{\theta}{2}$$

and that the parabolic eccentric anomaly

$$D = \frac{\bar{r}\cdot\bar{V}}{\sqrt{\mu}}$$

The above equation is a convenient expression to calculate D if $\bar{r}$ and $\bar{V}$ are known.

It was shown in class that

$$\bar{r}\,\frac{d\bar{r}}{dt} = r\,\frac{dr}{dt}$$

Since

$$\bar{V} = \frac{d\bar{r}}{dt}$$

can write

$$\bar{r}\cdot\bar{V} = r\,\frac{dr}{dt} \qquad (1)$$

For a parabolic trajectory

$$r = \frac{p}{1+\cos\theta} \qquad (2)$$

$$\frac{dr}{dt} = -\frac{p}{(1+\cos\theta)^2}(-\sin\theta)\frac{d\theta}{dt} = \frac{p\sin\theta}{(1+\cos\theta)^2}\frac{d\theta}{dt} = \frac{p\sin\theta}{p^2/r^2}\frac{d\theta}{dt}$$

$$= \frac{r^2\frac{d\theta}{dt}}{p}\sin\theta = \frac{h}{p}\sin\theta = \frac{\sqrt{\mu p}}{p}\sin\theta = \sqrt{\frac{\mu}{p}}\sin\theta \qquad (3)$$

Sub. (2) & (3) into (1)

$$\bar{r} \cdot \bar{V} = r \frac{dr}{dt} = \frac{p}{1+\cos\theta} \sqrt{\frac{\mu}{p}} \sin\theta = \sqrt{\mu p} \frac{\sin\theta}{1+\cos\theta}$$

$$\boxed{\bar{r} \cdot \bar{V} = \sqrt{\mu p} \frac{\sin\theta}{1+\cos\theta}} \quad (4)$$

Using the identities

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \cos^2\alpha - (1-\cos^2\alpha) = 2\cos^2\alpha - 1$$

Set $\alpha = \frac{\theta}{2}$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \quad (5)$$

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 \quad (6)$$

Therefore

$$\frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+\left(2\cos^2\frac{\theta}{2}-1\right)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2} \quad (7)$$

Sub. (7) into (4)

$$\boxed{\bar{r} \cdot \bar{V} = \sqrt{\mu p} \tan\frac{\theta}{2}} \quad (8)$$

or

$$\sqrt{p} \tan\frac{\theta}{2} = \frac{\bar{r} \cdot \bar{V}}{\sqrt{\mu}} \quad (9)$$

The parabolic eccentric anomaly D is defined in the notes by

$$D = \sqrt{p}\,\tan\frac{\theta}{2} \qquad (10)$$

Sub (9) into (10)

$$\boxed{D = \frac{\bar{r}\cdot\bar{V}}{\sqrt{\mu}}} \qquad (11)$$

4) Curtis: **4.22** A satellite in earth orbit has the following orbital parameters: $a = 7016$ km, $e = 0.05$, $i = 45°$, $\Omega = 0°$, $\omega = 20°$, and $\theta = 10°$. Find the position vector in the geocentric equatorial frame. {Ans.: $\mathbf{r} = 5776.4\hat{\mathbf{I}} + 2358.2\hat{\mathbf{J}} + 2358.2\hat{\mathbf{K}}$ (km)}

$a = 7016$ km $\quad e = 0.05 \quad i = 45° \quad \Omega = 0° \quad \omega = 20° \quad \theta = 10°$

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{7016\left[1-(0.05)^2\right]}{1+0.05\cos 10°} = 6670.03 \text{ km}$$

Using (7.18) in the notes

$$x' = r\cos\theta = (6670.03)\cos 10° = 6568.70 \text{ km}$$
$$y' = r\sin\theta = (6670.03)\sin 10° = 1158.24 \text{ km}$$
$$z' = 0$$

Using equations for coefficients following (7.16)

$$a_{11} = \cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega = \cos 20°\cos 0° - \sin 20°\cos 45°\sin 0° = 0.939693$$
$$a_{12} = \cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega = \cos 20°\sin 0° + \sin 20°\cos 45°\cos 0° = 0.241845$$
$$a_{13} = \sin\omega\sin i = \sin 20°\sin 45° = 0.241845$$
$$a_{21} = -\sin\omega\cos\Omega - \cos\omega\cos i\sin\Omega = -\sin 20°\cos 0° - \cos 20°\cos 45°\sin 0° = -0.34202$$
$$a_{22} = \cos\omega\cos i\cos\Omega - \sin\omega\sin\Omega = \cos 20°\cos 45°\cos 0° - \sin 20°\sin 0° = 0.664463$$
$$a_{23} = \cos\omega\sin i = \cos 20°\sin 45° = 0.664463$$
$$a_{31} = \sin i\sin\Omega = \sin 45°\sin 0° = 0$$
$$a_{32} = -\sin i\cos\Omega = -\sin 45°\cos 0° = -0.707107$$
$$a_{33} = \cos i = \cos 45° = 0.707107$$

Using (7.17)

$x = a_{11}x' + a_{21}y' + a_{31}z'$

$= (0.939693)(6568.70) + (-0.34202)(1158.24) + (0)(0) = 5776.42$ km

$y = a_{12}x' + a_{22}y' + a_{32}z'$

$= (0.241845)(6568.70) + (0.664463)(1158.24) + (0)(0) = 2358.21$ km

$z = a_{13}x' + a_{23}y' + a_{33}z'$

$= (0.241845)(6568.70) + (0.664463)(1158.24) + (0.707107)(0) = 2358.21$ km

$\therefore \boxed{\bar{r} = 5776.42\,\hat{i} + 2358.21\,\hat{j} + 2358.21\,\hat{k} \quad (\text{km})}$