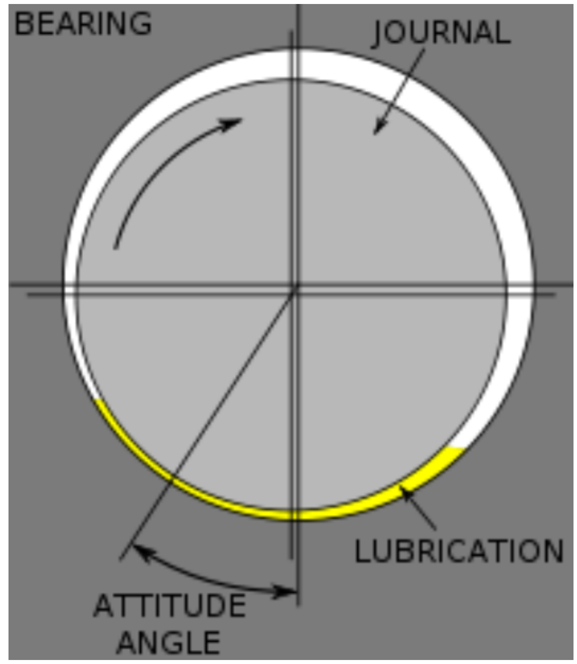
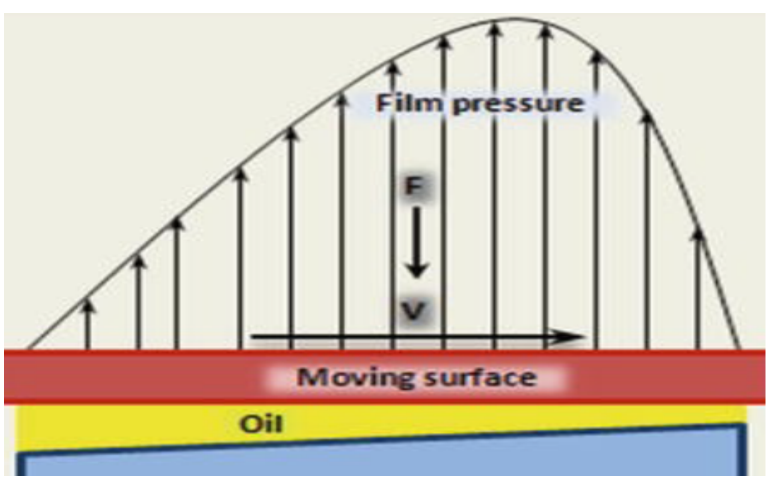
## V. Lubrication Theory

When two solid surfaces slide over one another, the presence of a lubricating fluid between the two surfaces reduces friction and the force needed to support this motion. Lubrication theory offers an approximate solution to this general problem that can be applied to thrust bearings (Kingsbury) and journal bearing.




In these types of bearings, the two moving surfaces are not parallel but inclined at a small angle.

The lubricant present between the two moving surface acts to increase the pressure which separates the surfaces and support the required load.



The typical geometry associated with such systems involve a long channel of small tapering gap thickness. Therefore, there are two important characteristic lengths in this problem of different magnitude: the length of the channel $L$, and its average gap width $h$, such that $h \ll L$. The characteristic horizontal velocity is the plate velocity $U$, but we have no information on the characteristic velocity in the vertical direction. The gravitational force is also negligible because the small thickness of the channel.

## V.1 The Lubrication Approximation

With this in mind, we can now rescale the dimensionless variable in the two-dimensional steady-state Continuity equation and Navier-Stokes equations as follows.

$$(x, y)^* = \left(\frac{x}{L}, \frac{y}{h}\right)$$
$$(u, v)^* = \left(\frac{u}{U}, \frac{v}{V}\right)$$

where $V$ is yet unknow, and the pressure can be scaled by the pressure in the bearing chamber $p_o$.

Substituting these dimensionless variables into the governing equations results in the following Continuity equation.

$$\frac{U}{L}\left(\frac{\partial u}{\partial x}\right)^* + \frac{V}{h}\left(\frac{\partial v}{\partial x}\right)^* = 0$$

The flow in the tapering channel has both velocity components. Therefore, to retain both velocities we need

$$V = \frac{h}{L}U$$

Let also denote

$$\epsilon = \frac{h}{L} \ll 1$$

Continuing with the Navier Stokes Equations:

$$\frac{U^2}{L}\left(u\frac{\partial u}{\partial x}\right)^* + \epsilon\frac{U^2}{h}\left(v\frac{\partial u}{\partial y}\right)^* = -\frac{p_o}{\rho L}\left(\frac{\partial p}{\partial x}\right)^* + \nu\frac{U}{L^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2 u}{\partial y^2}\right)^*$$

$$\epsilon\frac{U^2}{L}\left(u\frac{\partial v}{\partial x}\right)^* + \epsilon^2\frac{U^2}{h}\left(v\frac{\partial v}{\partial y}\right)^* = -\frac{p_o}{\rho h}\left(\frac{\partial p}{\partial y}\right)^* + \nu\frac{\epsilon U}{L^2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2 v}{\partial y^2}\right)^*$$

Rearranging

$$\left(u\frac{\partial u}{\partial x}\right)^* + \left(v\frac{\partial u}{\partial y}\right)^* = -\frac{p_o}{\rho U^2}\left(\frac{\partial p}{\partial x}\right)^* + \frac{\nu}{Uh}\left(\epsilon\frac{\partial^2 u}{\partial x^2} + \frac{1}{\epsilon}\frac{\partial^2 u}{\partial y^2}\right)^*$$

$$\epsilon\left(u\frac{\partial v}{\partial x}\right)^* + \epsilon\left(v\frac{\partial v}{\partial y}\right)^* = -\frac{1}{\epsilon}\frac{p_o}{\rho U^2}\left(\frac{\partial p}{\partial y}\right)^* + \epsilon^2\frac{\nu}{Uh}\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2 v}{\partial y^2}\right)^*$$

Since $\epsilon \ll 1$, we assume the following requirements to retain a viscous term and the pressure gradient.

$$Re = \frac{Uh}{\nu} \sim O(1)$$

$$E = \frac{p_o}{\rho U^2} \sim O(\epsilon^{-1})$$

Only terms that are $O(1)$ in both equations are retained. The result in dimensional form is the *Lubrication Equation*:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$0 = \frac{\partial p}{\partial y}$$

or:

$$\mu\frac{\partial^2 u}{\partial y^2} = \frac{dp}{dx} \qquad \text{(V-1)}$$

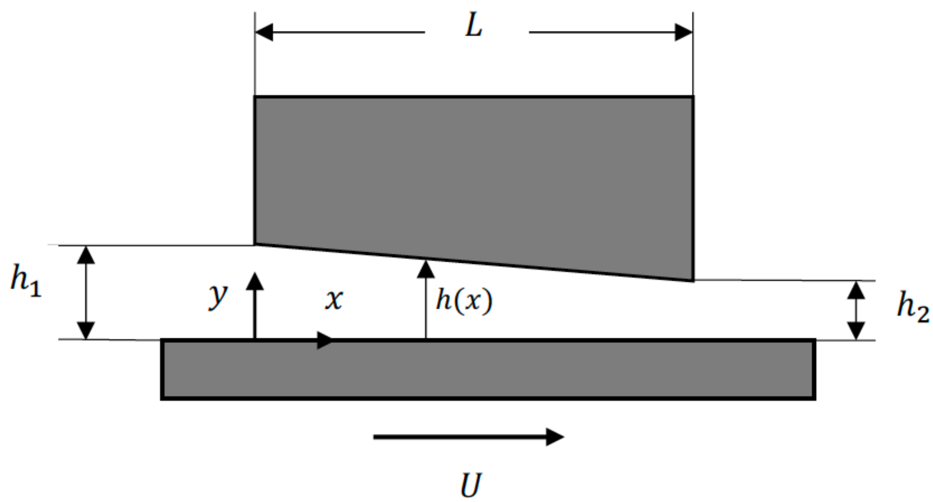
It is important to note that from the dimensionless analysis and scaling the pressure can be estimated from (V-1) or from the dimensionless parameters as

$$p \sim \mu\frac{UL}{h^2}$$

## V.2 Thrust Bearing

Consider a thrust bearing as shown in the figure below. To simplify the geometry the moving part is placed at $y = 0$, and the inclined pad is stationary above. The profile of the pad is linear given by the equation:

$$h(x) = h_1 - \frac{h_2 - h_1}{L} x$$




To obtain a solution to this problem, we integrate (V-1) twice subject to the boundary conditions:

(i) $u(0, x) = U$
(ii) $u(h, x) = 0$

Therefore,

$$u(x, y) = \frac{1}{2\mu} \frac{dp}{dx} y^2 + C_1 y + C_2$$

Applying the boundary conditions provides the solutions for the constants of integration.

$$C_2 = U$$

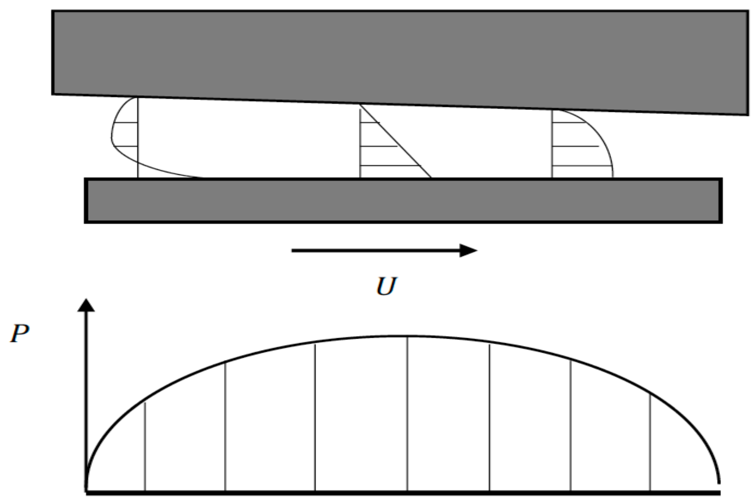
$$C_1 = -\frac{U}{h} - \frac{1}{2\mu} \frac{dp}{dx} h$$

Substituting the constants into the solution for the velocity profile gives

$$u(x,y) = \frac{1}{2\mu}\frac{dp}{dx}y(y-h) + U\left(1-\frac{y}{h}\right) \tag{V-3}$$

The velocity profile resembles the profile associated with Couette flow with positive, zero, or adverse pressure gradient.

Velocity and pressure distribution are shown schematically below. The velocity profile coincides with the Couette flow solution for a flow between two plates with positive, zero, and negative pressure gradients along the channel.




The volumetric flowrate can be determined now by integrating (V-3)

$$Q = \int_0^h u(x,y)dy = -\frac{1}{12\mu}\frac{dp}{dx}h^3 + \frac{Uh}{2} \tag{V-4}$$

Equation (V-4) can now be solved for the pressure gradient.

$$\frac{dp}{dx} = -6\mu\left(\frac{2Q}{h^3} - \frac{U}{h^2}\right) \tag{V-5}$$

Integrating (V-6) using the linear profile for $h(x)$ given by (V-2) provides the result for the pressure distribution:

$$p(x) - p(0) = \frac{6\mu L}{(h_1 - h_2)}\left[U\left(\frac{1}{h} - \frac{1}{h_1}\right) - Q\left(\frac{1}{h^2} - \frac{1}{h_1^2}\right)\right] \tag{V-6}$$

If we assume that the pressure at $x = 0$ is atmospheric, as well as the pressure at $x = L$, equations (V-6) provide the solution for the flow rate

$$Q = U\frac{h_1 h_2}{h_1+h_2} \tag{V-7}$$

and using this result (V-6) can be rewritten,

$$p(x) - p(0) = \frac{6\mu L}{h_1^2 - h_2^2}\left(\frac{h_1}{h} - 1\right)\left(1 - \frac{h_2}{h}\right) \tag{V-8}$$

Integrating (V-8) from $x = 0$ to $x = L$ gives the result for the total load supported by the pad per unit width.

$$F = \frac{6\mu U L^2}{(h_1 - h_2)^2}\left[\ln\frac{h_1}{h_2} - 2\left(\frac{h_1 - h_2}{h_1 + h_2}\right)\right] \tag{V-9}$$

The maximum force per unit width can be determined by evaluating the maximum of $F(h_1)$. The result is:

$$h_1 = 2.2\, h_2 \tag{V-10}$$

and for this value, the force becomes simply,

$$F_{max} = 0.16\mu U\left(\frac{L}{h_2}\right)^2 \tag{V-11}$$

For multiple pads, $n,$ of width $W$, the total force the thrust bearing can support is therefore,

$$F_{max} = 0.16\mu U\left(\frac{L}{h_2}\right)^2 Wn \tag{V-12}$$

It is interesting also to determine the effective *Coefficient of Friction* for a thrust bearing. To this end, we first determine the tangential force per unit width acting on the pad:

$$F_t = \int_o^L \left.\frac{\partial u}{\partial y}\right|_{y=0} dx = \frac{2\mu U L}{(h_1 - h_2)}\left[2\ln\frac{h_1}{h_2} - 3\left(\frac{h_1 - h_2}{h_1 + h_2}\right)\right]$$

The effective coefficient of friction is then the ratio between the tangential and the normal force:

$$C_f = \frac{F}{F_t} = \frac{2(h_1 + h_2)}{3L} G(k)$$

where,

$$k = \frac{h_1 - h_2}{h_1 + h_2}$$

$$G(k) = k \frac{\ln\left(\frac{1+k}{1-k}\right) - \frac{3k}{2}}{\ln\left(\frac{1+k}{1-k}\right) - 2k}$$

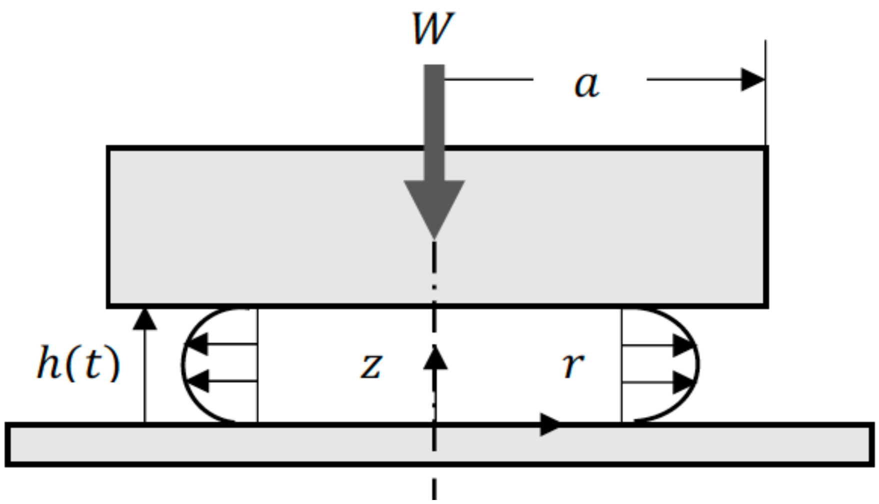
Since $0 < h_2 < h_1$, then $0 < k < 1$. the function $G(k)$ decreases monotonically with $k$. If $k \sim O(1)$ then,

$$C_f \sim O\left(\frac{h_1 + h_2}{2L}\right) \ll 1$$

The result shows that $C_f$ is independent of the fluid viscosity and is much smaller than 1. On the other hand, in the absence of a fluid layer for two solid surfaces moving relative to each other, the Coefficient of Friction is closer to unity. Thus, the presence of a fluid layer reduces significantly the drag acting on the solid surface.

## V.3 <u>Viscous (Stefan) Adhesion</u>

Consider a cylindrical disk descending towards a flat surface in a viscous fluid film.




The governing equations are ($\dot{h}$ is a time derivative):

Continuity $$-\dot{h}\pi r^2 = \int_0^{h(t)} 2\pi r u(t,z)dz$$

Lubrication equation $$\frac{dp}{dr} = \mu\frac{\partial^2 u}{\partial z^2}$$

With the boundary conditions

$$u(t,0) = 0$$
$$u(t,h) = 0$$

The solution of the lubrication equations is

$$u(t,r) = \frac{1}{2\mu}\frac{dp}{dr}z^2 + C_1 z + C_2$$

Applying the two boundary conditions determines the constants of integration:

$$C_1 = -\frac{1}{2\mu}\frac{dp}{dr}h \qquad C_2 = 0$$

The velocity profile is then,

$$u(t,z) = \frac{1}{2\mu}\frac{dp}{dr}z(z-h)$$

Using the continuity equation with the above velocity profile gives

$$-\dot{h}r = 2\int_0^h \frac{1}{2\mu}\frac{dp}{dr}z(z-h)dz = -\frac{1}{6\mu}\frac{dp}{dr}h^3$$

Solve for the pressure gradient and integrate,

$$\frac{dp}{dr} = \frac{6\mu\dot{h}r}{h^3}$$

$$p(a) - p(r) = \frac{6\mu\dot{h}}{h^3}\int_r^a rdr = \frac{3\mu\dot{h}}{h^3}(a^2 - r^2)$$

Assuming the pressure outside the disk is atmospheric the pressure distribution in the fluid film is

$$p(r) = -\frac{3\mu\dot{h}}{h^3}(a^2 - r^2)$$

The force balance on the descending disk can be written as follows:

$$W = \int_0^a p2\pi rdr = -\frac{3}{2}\mu\pi a^4\frac{\dot{h}}{h^3}$$

Solving this equation for $h(t)$:

$$\int_{h_o}^{h} -\frac{dh}{h^3} = \int_0^t \frac{2W}{3\mu a^4}dt$$

$$\frac{1}{h^2} - \frac{1}{h_o^2} = \frac{4Wt}{3\mu a^4}$$

Finally

$$h(t) = \frac{1}{\sqrt{\frac{1}{h_0^2} + \frac{4Wt}{3\mu a^4}}}$$

This result describes the rate of descent of a cylindrical load toward a flat surface in the presence of a fluid film. $h_0$is the initial height of the cylinder at $t = 0$.