# ME 57200 Aerodynamic Design

Lecture #6: Basic Equations in Aerodynamics

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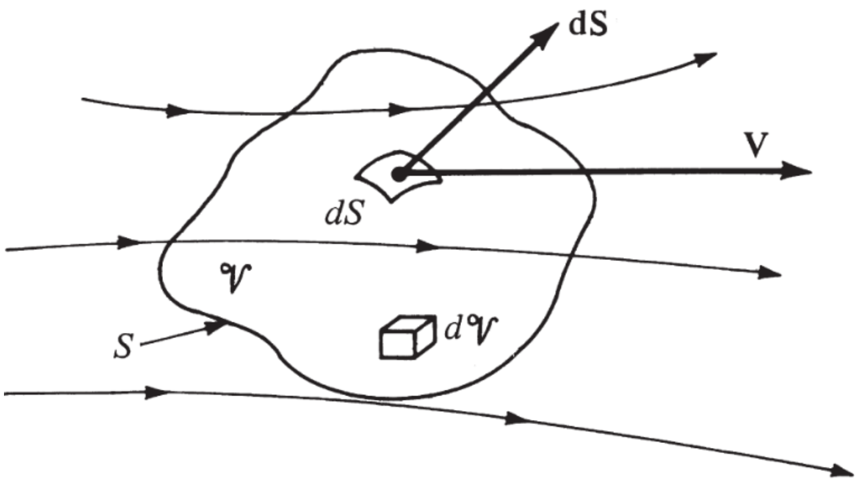
# Continuity Equation




$$\boxed{\oiint_S \rho \mathbf{V} \cdot \mathbf{dS} = -\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \, d\mathcal{V}}$$

$$\boxed{\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \, d\mathcal{V} + \oiint_S \rho \mathbf{V} \cdot \mathbf{dS} = 0}$$

$$\iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\mathcal{V} + \oiint_S \rho \mathbf{V} \cdot \mathbf{dS} = 0$$

$$\oiint_S (\rho \mathbf{V}) \cdot \mathbf{dS} = \iiint_{\mathcal{V}} \nabla \cdot (\rho \mathbf{V}) \, d\mathcal{V}$$

Divergence Theorem

$$\iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\mathcal{V} + \iiint_{\mathcal{V}} \nabla \cdot (\rho \mathbf{V}) \, d\mathcal{V} = 0$$

$$\iiint_{\mathcal{V}} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) \right] d\mathcal{V} = 0$$

Continuity equation in the form of a partial differential equation

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0}$$

# Continuity Equation

In an ***unsteady flow***, the flow field variables are a function of both spatial location and time

$$\rho = \rho(x, y, z, t)$$

In an ***steady flow***, the flow field variables are a function of spatial location only

$$\rho = \rho(x, y, z)$$

$$\iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\mathcal{V} + \oiint_{S} \rho \mathbf{V} \cdot \mathbf{dS} = 0 \quad \Rightarrow \quad \boxed{\oiint_{S} \rho \mathbf{V} \cdot \mathbf{dS} = 0}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0} \quad \Rightarrow \quad \boxed{\nabla \cdot (\rho \mathbf{V}) = 0}$$

# Momentum Equation

**Physical principle** Force = time rate of change of momentum

Force:

$$\mathbf{F} = \iiint_{\mathcal{V}} \rho \mathbf{f} \, d\mathcal{V} - \oiint_S p \, \mathbf{dS} + \mathbf{F}_{\text{viscous}}$$

Net rate of momentum out of control volume:

$$\oiint_S (\rho \mathbf{V} \cdot \mathbf{dS}) \mathbf{V}$$

Time rate of change of momentum inside control volume due to unsteady flow:

$$\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \mathbf{V} \, d\mathcal{V}$$

$$\boxed{\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \mathbf{V} \, d\mathcal{V} + \oiint_S (\rho \mathbf{V} \cdot \mathbf{dS}) \mathbf{V} = - \oiint_S p \, \mathbf{dS} + \iiint_{\mathcal{V}} \rho \mathbf{f} \, d\mathcal{V} + \mathbf{F}_{\text{viscous}}}$$

**Momentum equation in integral form**

# Momentum Equation

$$\boxed{\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho \mathbf{V}\, d\mathcal{V} + \oiint_S (\rho \mathbf{V}\cdot \mathbf{dS})\mathbf{V} = -\oiint_S p\ \mathbf{dS} + \iiint_{\mathcal{V}} \rho \mathbf{f}\, d\mathcal{V} + \mathbf{F}_{\text{viscous}}}$$

$$-\oiint_S p\ \mathbf{dS} = -\iiint_{\mathcal{V}} \nabla p\ d\mathcal{V}$$

Gradient Theorem

$$\boxed{\iiint_{\mathcal{V}} \frac{\partial(\rho \mathbf{V})}{\partial t} d\mathcal{V} + \oiint_S (\rho \mathbf{V}\cdot \mathbf{dS})\mathbf{V} = -\iiint_{\mathcal{V}} \nabla p\ d\mathcal{V} + \iiint_{\mathcal{V}} \rho \mathbf{f}\, d\mathcal{V} + \mathbf{F}_{\text{viscous}}}$$

# Momentum Equation

Let's just consider "$u$" component

$$\iiint_{V} \frac{\partial(\rho u)}{\partial t} dV + \oiint_{S} (\rho \vec{V} \cdot d\vec{S}) \cdot u = -\iiint_{V} \frac{\partial p}{\partial x} dV + \iiint_{V} \rho f_x dV + F_{x,\text{viscous}}$$

$$\oiint_{S} (\rho \vec{V} \cdot d\vec{S}) u = \oiint_{S} (\rho u \vec{V}) \cdot d\vec{S} = \iiint_{V} \nabla \cdot (\rho u \vec{V}) \cdot dV$$

$$\Rightarrow \iiint_{V} \left[ \frac{\partial(\rho u)}{\partial t} + \nabla (\rho u \vec{V}) + \frac{\partial p}{\partial x} - \rho f_x - F'_{x,\text{viscous}} \right] dV = 0$$

# Momentum Equation

$$\Rightarrow \begin{cases} \cancel{\dfrac{\partial(\rho u)}{\partial t}}^{\,0} + \nabla(\rho u \vec{V}) = -\dfrac{\partial p}{\partial x} + \cancel{\rho f_x}^{\,0} + \cancel{F'_{x,\text{viscous}}}^{\,0} \\ \cancel{\dfrac{\partial(\rho v)}{\partial t}}^{\,0} + \nabla(\rho v \vec{V}) = -\dfrac{\partial p}{\partial y} + \cancel{\rho f_y}^{\,0} + \cancel{F'_{y,\text{viscous}}}^{\,0} \\ \cancel{\dfrac{\partial(\rho w)}{\partial t}}^{\,0} + \nabla(\rho w \vec{V}) = -\dfrac{\partial p}{\partial z} + \cancel{\rho f_z}^{\,0} + \cancel{F'_{z,\text{viscous}}}^{\,0} \end{cases}$$

— Navier–Stokes Equations

# Energy Equation

$$\begin{cases} \nabla\cdot(\rho u\vec{V}) = -\dfrac{\partial p}{\partial x} \\ \nabla\cdot(\rho v\vec{V}) = -\dfrac{\partial p}{\partial y} \\ \nabla\cdot(\rho w\vec{V}) = -\dfrac{\partial p}{\partial z} \end{cases}$$ — Euler Equations

Energy Equation: $\underline{\delta q} + \underline{\delta w} = \underline{de}$

$$\begin{cases} \delta q: & \iiint_V \dot{q}\rho\, d\mathcal{V} + \dot{Q}_{viscous} \\ \underline{\delta w}: & -\oiint_S (p\, d\vec{S})\cdot\vec{V} + \iiint_V \rho(\vec{f}\cdot\vec{V})\, d\mathcal{V} + \dot{W}_{viscous} \\ de: & \oiint_S (\rho\vec{V}\cdot d\vec{S})\left(e + \frac{V^2}{2}\right) \end{cases}$$

# Energy Equation

$$\frac{\partial}{\partial t} \oiiint_{\mathcal{V}} \rho \left(e + \frac{V^2}{2}\right) d\mathcal{V}$$

$$\oiiint_{\mathcal{V}} \dot{q} \rho \, d\mathcal{V} + \dot{Q}_{viscous} - \oiint_{S} p \vec{V} \cdot d\vec{S} + \oiiint_{\mathcal{V}} \rho (\vec{f} \cdot \vec{V}) d\mathcal{V} + \dot{W}_{viscous}$$

$$= \frac{\partial}{\partial t} \oiiint_{\mathcal{V}} \rho \left(e + \frac{V^2}{2}\right) d\mathcal{V} + \oiint_{S} \rho \left(e + \frac{V^2}{2}\right) \vec{V} d\vec{S}$$

$$\oiint_{S} p \vec{V} d\vec{S} = \oiiint_{\mathcal{V}} \nabla (p \vec{V}) d\mathcal{V};$$

$$\oiint_{S} \rho \left(e + \frac{V^2}{2}\right) \vec{V} \cdot d\vec{S} = \oiiint_{\mathcal{V}} \nabla \cdot \left( \rho \left(e + \frac{V^2}{2}\right) \cdot \vec{V} \right) d\mathcal{V}$$

# Energy Equation

$$\Rightarrow \iiint_{\mathcal{V}} \left[ \underbrace{\frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right] + \nabla\cdot\left[\rho\left(e+\frac{V^2}{2}\right)\vec{V}\right] - \rho\dot{q} + \nabla(p\vec{V}) - \rho(\vec{f}\cdot\vec{V})}\right.$$

$$\left. \underbrace{- \dot{Q}'_{viscous} - W'_{viscous}} \right] = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right] + \underline{\nabla\cdot\left[\rho\left(e+\frac{V^2}{2}\right)\vec{V}\right]} = \rho\dot{q} - \nabla(p\vec{V}) + \rho(\vec{f}\cdot\vec{V})$$

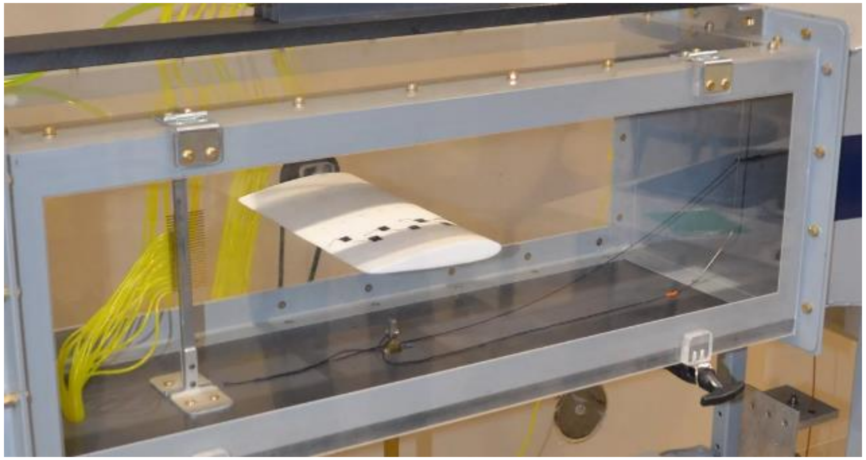
$$+ \dot{Q}'_{viscous} + W'_{viscous}$$

steady-state, inviscid, adiabatic, no body force

$$\Rightarrow \nabla\cdot\left[\rho\left(e+\frac{V^2}{2}\right)\vec{V}\right] = -\nabla(p\vec{V})$$

# In-Class Example

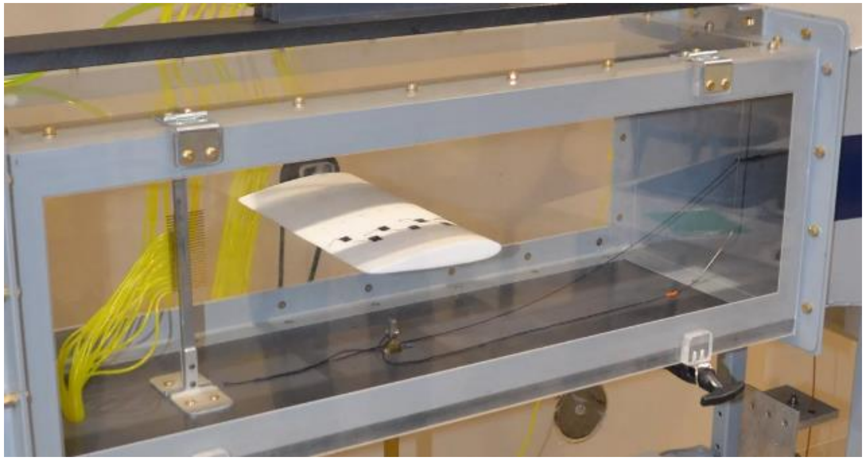
Airplane model mounted with a force balance at the bottom to measure the aerodynamic forces, i.e., lift and drag.



A wing model spanned the entire test section to establish two-dimensional flow over the wing.

*If no force balance can be mounted, how to measure the aerodynamic forces?*

# In-Class Example



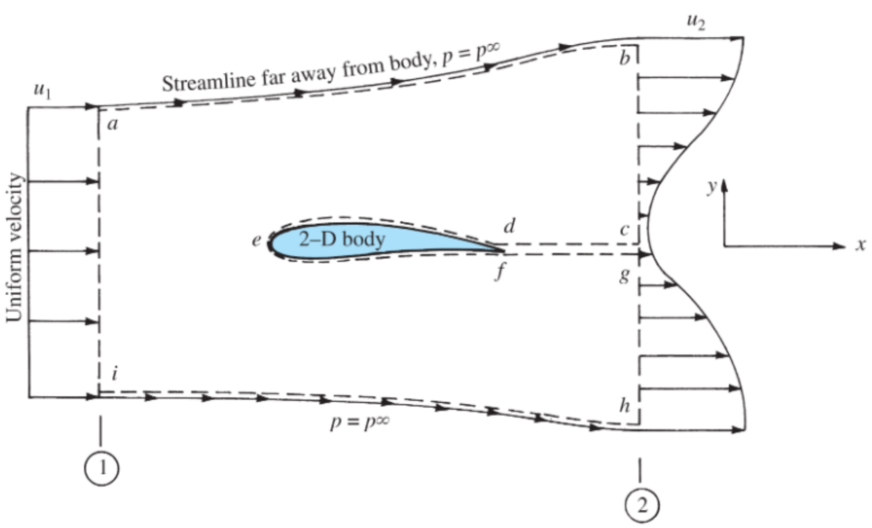
- The lift can be obtained from the pressure distributions on the ceiling and floor of the tunnel
- The drag can be obtained from measurements of the flow velocity downstream of the wing.

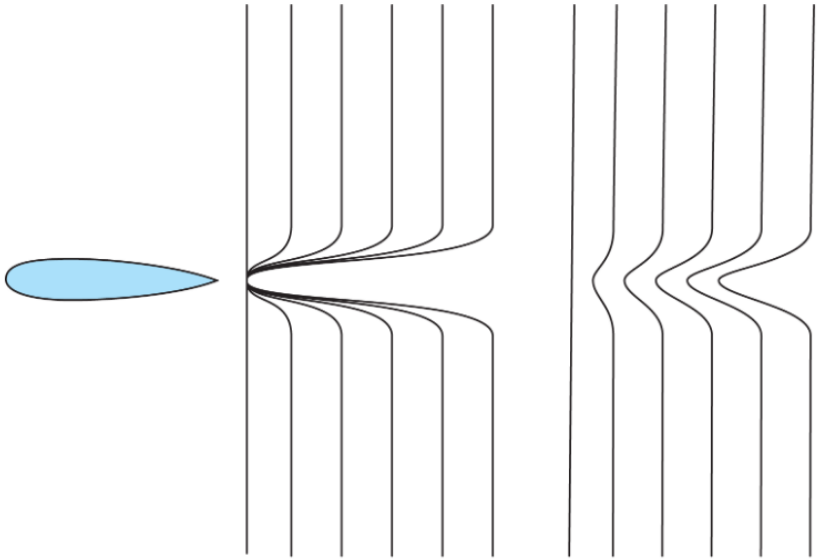
*Application of the fundamental momentum equations*

# In-Class Example

Consider a two-dimensional body in a flow. A control volume is drawn around this body, as given by the dashed lines. The control volume is bounded by:

1. The upper and lower streamlines far above and below the body (*ab* and *hi*, respectively).
2. Lines perpendicular to the flow velocity far ahead of and behind the body (*ai* and *bh*, respectively).
3. A cut that surrounds and wraps the surface of the body (*cdefg*).

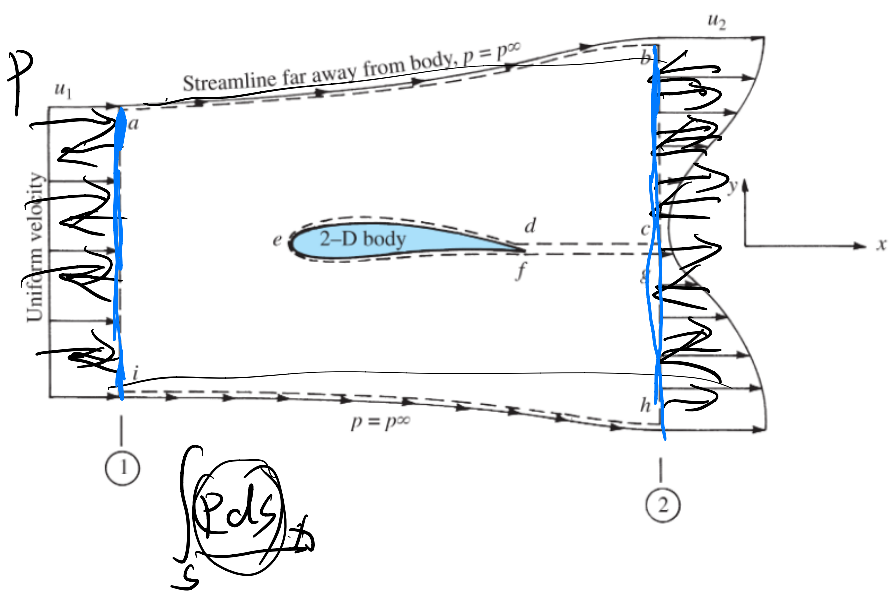


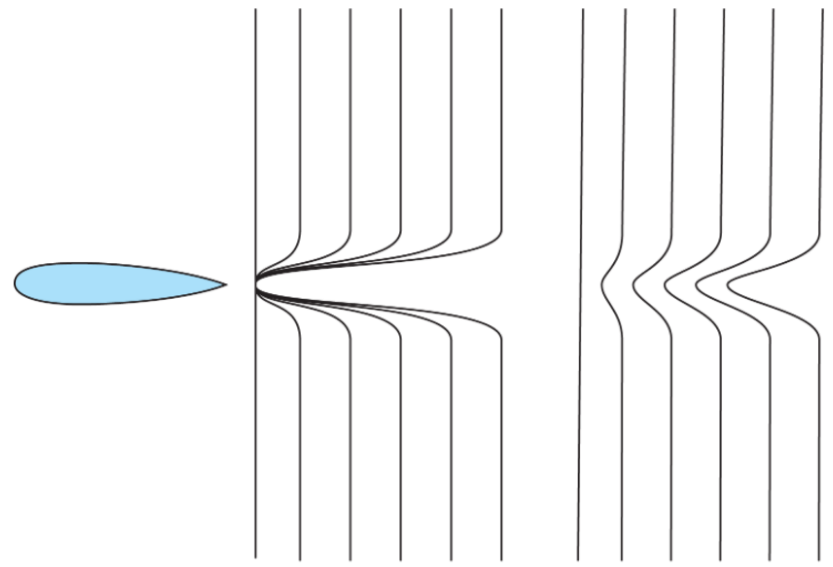



# In-Class Example

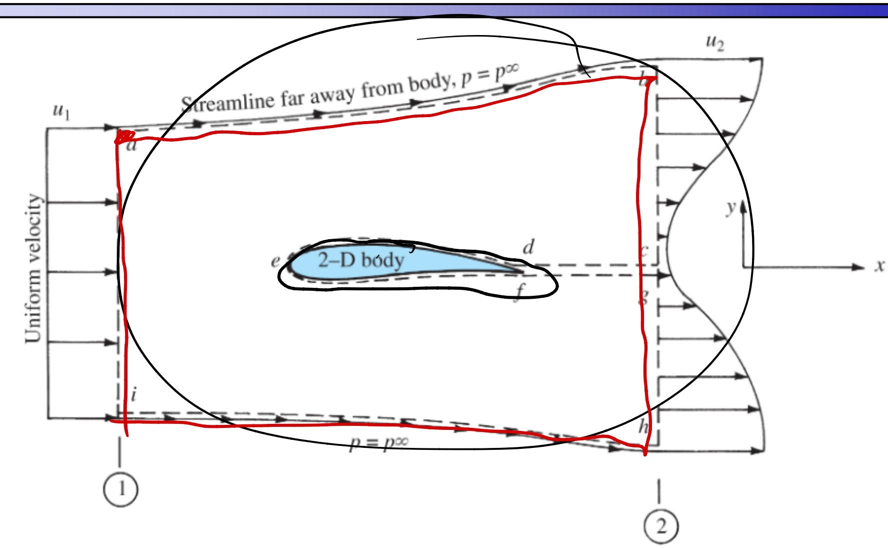
- The width of the control volume in the z direction is unity.
- Assume that the contour *abhi* is far enough from the body such that the pressure is everywhere the same on *abhi* and euqal to the freestream pressure $p = p_\infty$

**Please show how a measurement of the velocity distribution across the wake of a body can yield the drag.**



# In-Class Example




Solution:

Consider the surface forces on the control volume:

1. Pressure force: over "a b h i"

$$-\iint_{abhi} p\, d\vec{S}$$

2 Force acting on "def"


# In-Class Example

Total force : $-\iint_{abhi} p\, d\vec{S} - R'$

Momentum Equation

$$\cancel{\frac{\partial}{\partial t}\iiint_{V} \rho \vec{V}\, dV} + \oiint_{S} (\rho \vec{V} \cdot d\vec{S})\vec{V} = -\iint_{abhi} - R'$$

$$R' = -\oiint_{S} (\rho \vec{V} \cdot d\vec{S})\vec{V} - \iint_{abhi} p\, d\vec{S}$$

"x-component"

$$D' = -\oiint_{S} (\rho \vec{V} \cdot d\vec{S}) u - \iint_{abhi} (p\, dS)_x$$

# In-Class Example

Pressure "p" is constant along "abhi"

$$\iint_{abhi} (p\,ds)_x = 0$$

Along "ab" "hi" and "def" $\vec{V} \perp d\vec{s}$, $\vec{V} \cdot d\vec{s} = 0$

$$\Rightarrow \iint_S (\rho \vec{V} \cdot d\vec{s})u = -\int_i^a \rho_1 u_1^2 dy + \int_h^b \rho_2 u_2^2 dy$$

$$\Rightarrow -\int_i^a \rho_1 u_1 dy + \int_h^b \rho_2 u_2 dy = 0$$ Continuity Eq.

$$\Rightarrow \int_i^a \rho_1 u_1^2 dy = \int_h^b \rho_2 u_2 u_1 dy$$

# In-Class Example

$-D'$

$$\Rightarrow \oiint_S (\rho \vec{V} \cdot d\vec{S}) u = -\int_h^b \rho_2 u_2 u_1 dy + \int_h^b \rho_2 u_2^2 dy$$

$$= -\int_h^b \rho_2 u_2 (u_1 - u_2) dy$$

$$\Rightarrow D' = \int_h^b \rho_2 u_2 (u_1 - u_2) dy$$

If incompressibe:

$$D' = \rho \int_h^b u_2 (u_1 - u_2) dy$$

In-Class Quiz