Homework #3

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ME 515: Orbital Mechanics
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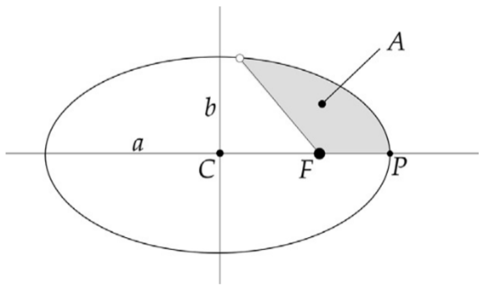
Curtis Chapter 2, #s 18, 19, 20, & 37

2.18 Determine the true anomaly θ of the point(s) on an elliptical orbit at which the speed equals the speed of a circular orbit with the same radius (i.e., $v_{ellipse} = v_{circle}$).



Ans.

$$v_{circle} = \sqrt{\frac{\mu}{r}}$$

$$\frac{v_{ellipse}^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$\implies v_{ellipse} = \sqrt{\frac{\mu(2a-r)}{ar}}$$

Equate v_{circle} and $v_{ellipse}$

$$\sqrt{\frac{\cancel{\mu}}{\cancel{r}}} = \sqrt{\frac{\cancel{\mu}(2a-r)}{a\cancel{r}}}$$

$$r = a$$

Use definition of r

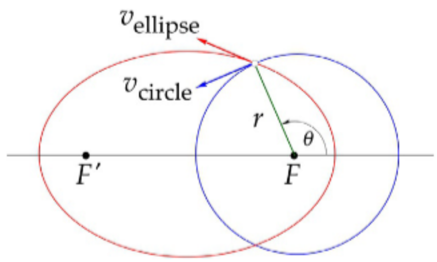
$$r = a\frac{1-e^2}{1+e\cos(\theta)}$$

$$\Rightarrow \cancel{(a)} = \cancel{a}\frac{1-e^2}{1+e\cos\theta}$$

$$\cancel{1} + \cancel{e}\cos(\theta) = \cancel{1} - e^{\cancel{2}}$$

$$\boxed{\theta = \cos^{-1}(-e)}$$

2.19 Calculate the flight path angle at the locations found in Problem 2.19.



Ans.

From problem 2.18 $\implies$ $\underline{\theta = \cos^{-1}(-e)}$
Plug into:

$$\tan(\gamma) = \frac{e\sin(\theta)}{1 + e\cos(\theta)}$$

$$= \frac{e\sin(\cos^{-1}(-e))}{1 + e\cos(\cos^{-1}(-e))} \qquad \begin{cases} \cdot\ \underbrace{\sin(\cos^{-1}(-e)) = \sqrt{1-e^2}}_{\text{Identity: } \sin(\cos^{-1}(x)) = \sqrt{1-x^2}} \\ \cdot\ \cos(\cos^{-1}(-e)) = -e \end{cases}$$

$$= \frac{e\sqrt{1-e^2}}{1-e^2}$$

$$= \frac{e}{\sqrt{1-e^2}}$$

$$\Rightarrow \boxed{\gamma = \tan^{-1}\left(\frac{e}{\sqrt{1-e^2}}\right)}$$

2.20 An unmanned satellite orbits the earth with a perigee radius of 10,000 km and an apogee radius of 100,000 km. Calculate:

- **(a)** the eccentricity of the orbit;
- **(b)** the semimajor axis of the orbit (km)
- **(c)** the period of the orbit (h);
- **(d)** the specific energy of the orbit ($\frac{km^2}{s^2}$) ;
- **(e)** the true anomaly (degrees) at which the altitude is 10,000 km ;
- **(f)** v_r and $v_\perp$ ($\frac{km}{s}$) at the points found in part (e);
- **(g)** the speed at perigee and apogee ($\frac{km}{s}$);

Ans.

(a)

r_p = 10,000 km, r_a = 100,000 km

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{100,000 - 10,000}{100,000 + 10,000} \Rightarrow \boxed{e = \frac{9}{11} = 0.8182}$$

(b)

Find a, the semimajor axis

$$2a = r_a + r_p$$

$$a = \frac{1}{2}(100,000 + 10,000)$$

$$a = 55,000\ km$$

Use a to find semiminor axis, b

$$b = a\sqrt{1 - e^2}$$

$$b = (55,000)\sqrt{1 - \left(\frac{9}{11}\right)^2}$$

$$\boxed{b = 31,623\ km}$$

(c)

$$T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}, \quad \mu = 398600 \ \frac{km^3}{s^2}$$

$$\Rightarrow T = \frac{2\pi}{\sqrt{398600}}(55,000)^{\frac{3}{2}}$$

$$T = 128,368 \text{ seconds}$$

$$= 2,139.5 \text{ minutes}$$

$$= \boxed{35.66 \text{ hours}}$$

(d)
Specific energy is given by

$$\varepsilon = -\frac{\mu}{2a} = -\frac{(398,600)}{2(55,000)}$$

$$\boxed{\varepsilon = 3.624 \ \frac{km^2}{s^2}}$$

(e)
z = 10,000 km $\Rightarrow r = r_e + z = 6368 + 10,000 = 16368 km$, where r_e is the radius of the earth
Plug into

$$r = a\frac{1 - e^2}{1 + e\cos(\theta)}$$

$$(16,368) = (55,000)\frac{1 - (\frac{9}{11})^2}{1 + (\frac{9}{11})\cos(\theta)}$$

$$1 + \left(\frac{9}{11}\right)\cos(\theta) = \frac{55,000}{16,368}\left(1 - \left(\frac{9}{11}\right)^2\right)$$

Solve for θ

$$\cos(\theta) = 0.1108$$

$$\boxed{\theta = 82.22^o}$$

(f)
Find h from:

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2}$$

$$\Rightarrow h = \sqrt{a\mu(1-e^2)} = \sqrt{(55,000)(398,600)\left(1-\left(\frac{9}{11}\right)^2\right)} = 85130\ \frac{km^2}{s}$$

Plug into equation for v_r

$$v_r = \frac{\mu}{h} e \sin{(\theta)} = \frac{398,600}{85130}\left(\frac{9}{11}\right)\sin{(82.22^o)}$$

$$\boxed{v_r = 3.796\ \frac{km}{s}}$$

Similarly for $v_\perp$

$$v_\perp = \frac{\mu}{h} e \cos{(\theta)} = \frac{398,600}{85,130}\left(\frac{9}{11}\right)\cos{(82.22^o)}$$

$$\boxed{v_\perp = 0.5186\ \frac{km}{s}}$$

(g)
Solve for v

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \Rightarrow \frac{v^2}{2} = \frac{\mu}{r} - \frac{\mu}{2a}$$

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

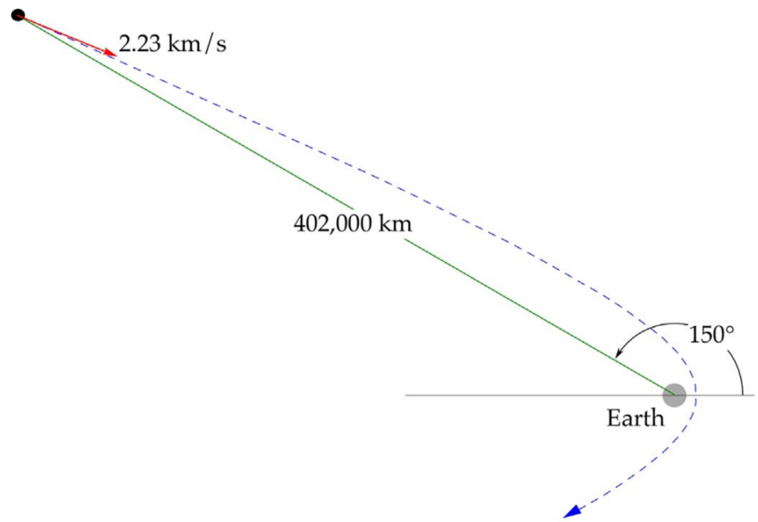
Plug in r_p to find v_p and r_a to find v_a

$$v_p = \sqrt{(398,600))\left(\frac{2}{(10,000)} - \frac{1}{(55,000)}\right)} \Rightarrow \boxed{v_p = 8.513\ \frac{km}{s}}$$

$$v_a = \sqrt{(398,600))\left(\frac{2}{(100,000)} - \frac{1}{(55,000)}\right)} \Rightarrow \boxed{v_a = 0.8513\ \frac{km}{s}}$$

2.37 A meteoroid is first observed approaching the earth when it is 402,000 km from the center of the earth with a true anomaly of 150^o. If the speed of the meteoroid at that time is 2.23 km/s, calculate:

(a) the eccentricity of the trajectory;
(b) the altitude at closest approach;
(c) the speed at the closest approach.



Ans.

(a)
Find semimajor axis a from:

$$\frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2a}$$

$$\frac{(2.23)^2}{2} - \frac{(398,600)}{(402,000)} = \frac{(398600))}{2a}$$

$$1.495 = \frac{(398600))}{2a} \Rightarrow a = 133,319\ km$$

Plug into orbit equation for hyperbolic trajectory

$$r = a\frac{e^2 - 1}{1 + e\cos(\theta)}$$

$$(402,000) = (133,319)\frac{e^2 - 1}{1 + e\cos(150^o)}$$

$$1 + e\cos(150^o) = \underbrace{\frac{133,319}{402,000}}_{3.015}(e^2 - 1)$$

$$\Rightarrow 0 = e^2 - e\frac{\cos(150^o)}{3.105} - 1 - \frac{1}{3.015}$$

$$0 = e^2 + 0.287e - 1.332$$

This is a quadratic equation, so we will have two solutions

$$e = \frac{-(0.287) \ \pm \ \sqrt{(0.287)^2 - 4(1)(-1.332)}}{2(1)}$$

$$\boxed{e = 1.086} \ , \ \cancel{e = -3.697}$$

(b)

The altitude of closest approach occurs at perigee radius

$$r_p = a(e - 1)$$

$$r_p = (133,319)((1.086) - 1)$$

$$r_p = 11,465.4 \ km$$

The altitude is the perigee radius subtracted by the radius of the earth

$$z_p = r_p - r_e$$

$$z_p = (11,465.4) - (6368)$$

$$\boxed{z_p = 5094 \ km}$$

(c)

Find v_p by using r_p (speed at closest approach)

$$\frac{v_p^2}{2} - \frac{\mu}{r_p} = \frac{\mu}{2a}$$

$$\Rightarrow v_p = \sqrt{\mu(\frac{2}{r_p} - \frac{1}{a})}$$

$$\boxed{v_p = 8.516\ \frac{km}{s}}$$
