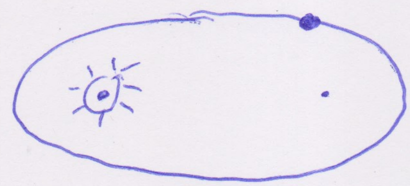
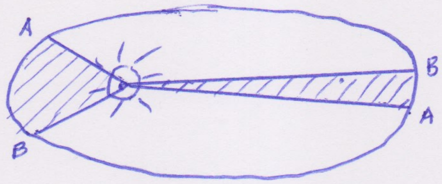
5. Two-Body Problem

Kepler's Laws - made by observation of Mars (early 17^{th} century)

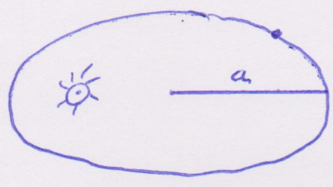
1. The orbit of each planet is an ellipse with the sun at one focus.



2. The line which joins a planet to the sun sweeps over equal areas in equal intervals of time



3. The squares of the periods of the planets are proportional to the cubes of the semi-major axes of their orbits.

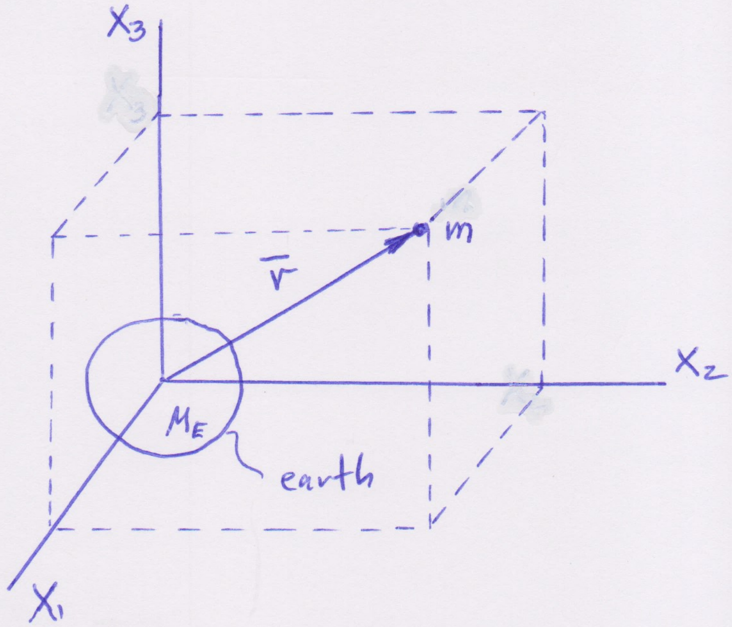


a = semi-major axis

T = period (time to complete one orbit)

$$T^2 \sim a^3$$

Satellite in Orbit About Earth (neglecting atmospheric effects)



$M_E = 5.974 \times 10^{24}$ kg

$\mu_E = G M_E$ (gravitational parameter)

$= 3.986 \times 10^5$ km^3/sec^2

Equatorial radius 6378 km
Polar radius 6357 km
Average radius 6368 km

In absence of all other bodies ($R_j = 0$, $j = 3, 4, 5 \ldots$) (4.9) reduces to

$$\frac{d^2 \bar{r}}{dt^2} = -\frac{\mu}{r^3} \bar{r} \qquad (5.1)$$

Will show that the angular momentum of the orbiting satellite remains constant

Form cross product

$$\bar{r} \times \frac{d^2 \bar{r}}{dt^2} = -\frac{\mu}{r^3} (\bar{r} \times \bar{r}) = 0 \qquad (5.2)$$

Using

$$\frac{d}{dt}\left(\bar{r} \times \frac{d\bar{r}}{dt}\right) = \cancel{\frac{d\bar{r}}{dt} \times \frac{d\bar{r}}{dt}}^{\,0} + \bar{r} \times \frac{d^2\bar{r}}{dt^2}$$

Therefore (5.2) becomes

$$\frac{d}{dt}\left(\bar{r} \times \frac{d\bar{r}}{dt}\right) = 0 \qquad (5.3)$$

Integrate

$$\bar{r} \times \frac{d\bar{r}}{dt} = \bar{h} = \text{const. (angular momentum)} \qquad (5.4)$$

Note: $\bar{h}$ is perpendicular to both $\bar{r}$ & $\frac{d\bar{r}}{dt}$

Will now show that satellite trajectory is contained in a plane which passes through the origin.

Take dot product of (5.4) with $\bar{r}$

$$\underbrace{\left(\bar{r} \times \frac{d\bar{r}}{dt}\right) \cdot \bar{r}}_{0 \text{ since } \bar{r} \perp \underbrace{\left(\bar{r} \times \frac{d\bar{r}}{dt}\right)}_{\perp\ \bar{r}\ \&\ \frac{d\bar{r}}{dt}}} = \bar{h} \cdot \bar{r}$$

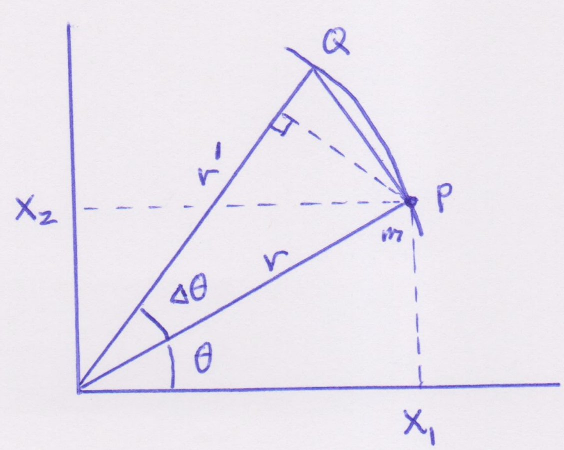
Therefore $\bar{h} \cdot \bar{r} = 0$

or $$h_1 X_1 + h_2 X_2 + h_3 X_3 = 0 \qquad (5.5)$$

(equation of plane passing through the origin)

<u>Proof of Kepler's 2nd Law</u>

Suppose particle m is moving in the $X_1 X_2$ plane as shown.



Differential area swept out by radius vector as particle moves from P to Q is

$$\Delta A = \frac{1}{2} r' (r \sin \Delta\theta)$$

Time rate of change of this area is

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r' r \frac{\Delta \theta}{\Delta t} \cdot \frac{\sin \Delta\theta}{\Delta\theta}$$

Take limit as $\Delta t \to 0$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} \qquad (5.6)$$

From figure

$$\tan\theta = \frac{x_2}{x_1}$$

$$\theta = \tan^{-1} \frac{x_2}{x_1}$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{x_2}{x_1}\right)^2} \frac{x_1 \dot{x}_2 - x_2 \dot{x}_1}{x_1^2}$$

$$\frac{d\theta}{dt} = \frac{x_1 \dot{x}_2 - x_2 \dot{x}_1}{x_1^2 + x_2^2}$$

since $\bar{r} \times \frac{d\bar{r}}{dt} = \bar{h}$ (5.4)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & 0 \\ \dot{x}_1 & \dot{x}_2 & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + (x_1\dot{x}_2 - x_2\dot{x}_1)\hat{k}$$

$$= h_3 \hat{k} = h\hat{k} \quad (\text{since } h_1 = h_2 = 0)$$

$$\Rightarrow \quad x_1\dot{x}_2 - x_2\dot{x}_1 = h$$

since $x_1^2 + x_2^2 = r^2$, get

$$\frac{d\theta}{dt} = \frac{x_1 \dot{x}_2 - x_2 \dot{x}_1}{x_1^2 + x_2^2} = \frac{h}{r^2} \qquad (5.7)$$

substitute (5.7) into (5.6)

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{h}{r^2}$$

or

$$\boxed{\frac{dA}{dt} = \frac{1}{2} h = \text{constant}} \quad (\text{Kepler's } 2^{nd} \text{ law}) \qquad (5.8)$$

Proof that the trajectory of the particle is a conic section (Kepler's 1st Law)

From (5.1)

$$\frac{d^2\bar{r}}{dt^2} = -\frac{\mu}{r^3}\bar{r}$$

$$\frac{d^2\bar{r}}{dt^2} \times \bar{h} = -\frac{\mu}{r^3}\bar{r}\times\bar{h} = -\mu\,\bar{r}\times\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)$$

Using the vector identity

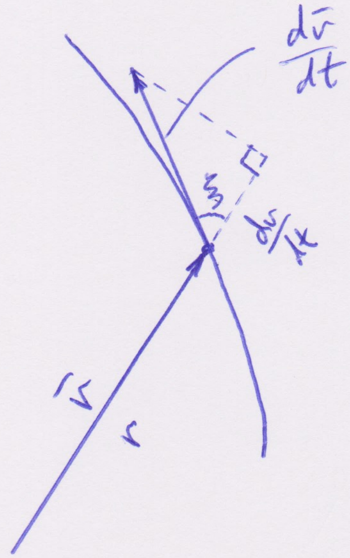
$$\bar{A}\times(\bar{B}\times\bar{C}) = (\bar{A}\cdot\bar{C})\bar{B} - (\bar{A}\cdot\bar{B})\bar{C}$$

$$\bar{r}\times\left(\bar{r}\times\frac{d\bar{r}}{dt}\right) = \left(\bar{r}\cdot\frac{d\bar{r}}{dt}\right)\bar{r} - \underbrace{(\bar{r}\cdot\bar{r})}_{r^2}\frac{d\bar{r}}{dt}$$

$$\frac{d^2\bar{r}}{dt^2}\times\bar{h} = \frac{\mu}{r^3}\left[r^2\frac{d\bar{r}}{dt} - \left(\bar{r}\cdot\frac{d\bar{r}}{dt}\right)\bar{r}\right]$$

Note: $\frac{dr}{dt}$ is component of $\frac{d\bar{r}}{dt}$ in $\bar{r}$ direction.

$$\frac{dr}{dt} \neq \left| \frac{d\bar{r}}{dt} \right|$$



$$\frac{dr}{dt} = \left| \frac{d\bar{r}}{dt} \right| \cos \xi$$

$$\bar{r} \cdot \frac{d\bar{r}}{dt} = r \left| \frac{d\bar{r}}{dt} \right| \cos \xi = r \frac{dr}{dt}$$

Returning to previous equation

$$\frac{d^2\bar{r}}{dt^2} \times \bar{h} = \frac{\mu}{r^3} \left[r^2 \frac{d\bar{r}}{dt} - r \frac{dr}{dt} \bar{r} \right]$$

$$= \mu \frac{r \frac{d\bar{r}}{dt} - \frac{dr}{dt} \bar{r}}{r^2} = \mu \frac{d}{dt} \left(\frac{\bar{r}}{r} \right)$$

Integrate

$$\frac{d\bar{v}}{dt} \times \bar{h} = \mu \left[\frac{\bar{r}}{r} + \bar{e} \right] \qquad (5.9)$$

$\bar{e}$ = constant of integration (eccentricity vector)
(lies in plane of trajectory)

Take dot product of (5.9) with $\bar{r}$ and use the identity $\bar{A} \cdot (\bar{B} \times \bar{C}) = (\bar{A} \times \bar{B}) \cdot \bar{C}$

$$\bar{r} \cdot \left(\frac{d\bar{v}}{dt} \times \bar{h} \right) = \left(\bar{r} \times \frac{d\bar{v}}{dt} \right) \cdot \bar{h} = \bar{h} \cdot \bar{h} = h^2$$

$$= \bar{r} \cdot \mu \left[\frac{\bar{r}}{r} + \bar{e} \right] = \mu \left[\bar{r} \cdot \frac{\bar{r}}{r} + \bar{r} \cdot \bar{e} \right]$$

$$= \mu \left[r + re \cos \theta \right]$$

where θ is the angle between $\bar{r}$ and $\bar{e}$ and is called the <u>true anomaly</u>. In the literature, the true anomaly is often denoted by f.

Solve for r

$$r = \frac{h^2/\mu}{1 + e\cos\theta}$$

Define the parameter p

$$p = \frac{h^2}{\mu} \qquad (5.10)$$

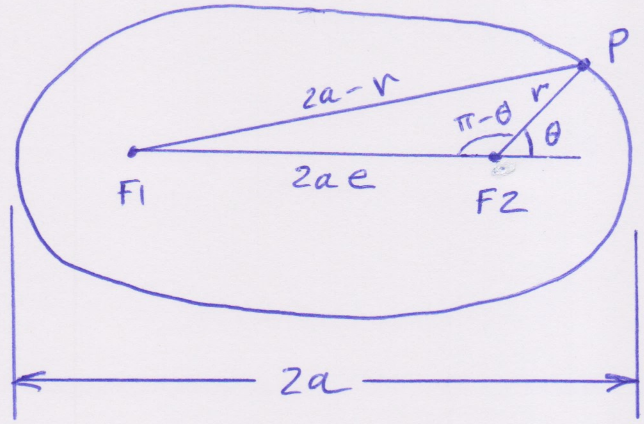
$$\boxed{r = \frac{p}{1 + e\cos\theta}} \qquad (5.11)$$

General equation for conic section in polar coordinates
(Proof of Kepler's 1st Law)

e is the <u>eccentricity</u> of the conic section

Geometric Considerations

A. Ellipse $e<1$



$$\overline{PF_1} + \overline{PF_2} = 2a$$

$$(2a-r)^2 = r^2 + (2ae)^2 - 2(2ae)(r)\cos(\pi-\theta)$$

$$4a^2 - 4ar + r^2 = r^2 + 4a^2e^2 - 4aer\underbrace{\cos(\pi-\theta)}_{-\cos\theta}$$

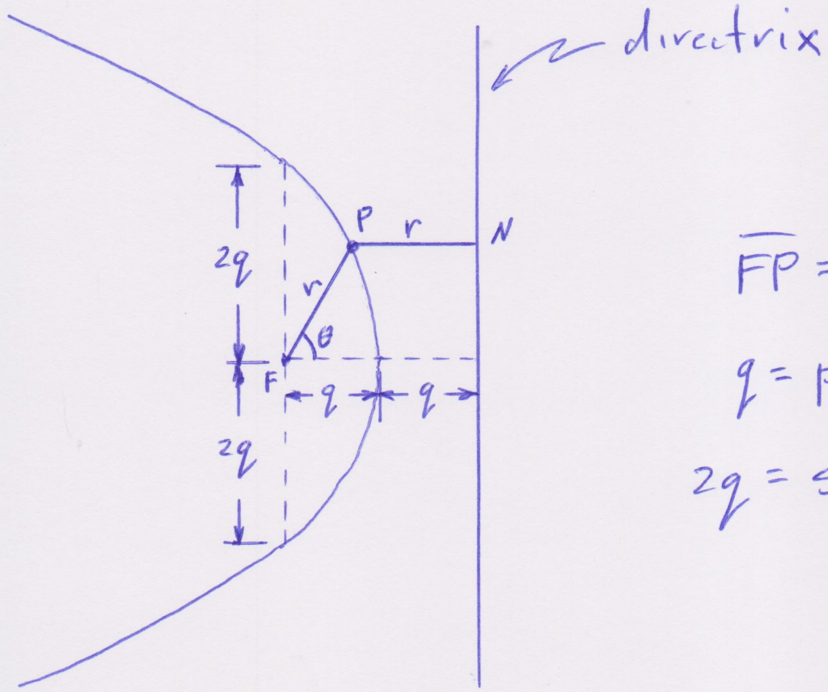
$$a - r = ae^2 + er\cos\theta$$

$$\boxed{r = \frac{a(1-e^2)}{1+e\cos\theta}} \qquad (5.12)$$

Circle is special case of ellipse with $e=0$

$r = a$

B. Parabola $e = 1$



$\overline{FP} = \overline{PN}$

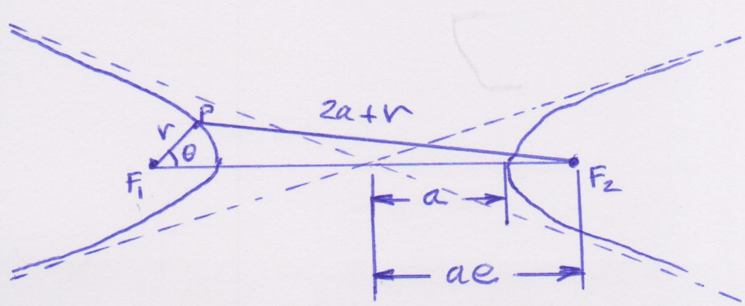
q = pericenter distance

$2q$ = semi-latus rectum

$$r + r\cos\theta = 2q$$

$$\boxed{r = \frac{2q}{1+\cos\theta}} \qquad (5.13)$$

C. Hyperbola $e > 1$



$\overline{PF_2} - \overline{PF_1} = 2a$

$$(2a+r)^2 = r^2 + (2ae)^2 - 2(r)(2ae)\cos\theta$$

$$4a^2 + 4ar + r^2 = r^2 + 4a^2e^2 - 4are\cos\theta$$

$$a + r = ae^2 - re\cos\theta$$

$$\boxed{r = \frac{a(e^2-1)}{1+e\cos\theta}} \qquad (5.14)$$

Summary

Conic Section	e	p
circle	0	a
ellipse	<1	$a(1-e^2)$
parabola	1	$2q$
hyperbola	>1	$a(e^2-1)$

Proof of Kepler's 3rd Law $\quad T^2 \sim a^3$

From (5.10) $\quad p = h^2/\mu$

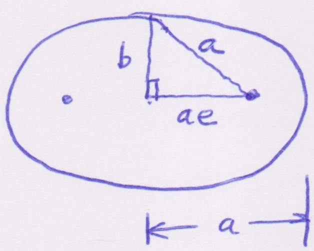
$$h = \sqrt{\mu p} = \sqrt{\mu a(1-e^2)} \quad \text{(for ellipse)} \qquad (5.15)$$

From (5.8)

$$\frac{dA}{dt} = \frac{1}{2}h$$

Area of ellipse is

$$A = \pi a b$$



$$b^2 + a^2 e^2 = a^2$$

$$b^2 = a^2(1-e^2)$$

$$A = \pi a^2 \sqrt{1-e^2} \qquad (5.16)$$

Integrate (5.8) $\frac{dA}{dt} = \frac{1}{2}h$

$$\int_0^A dA = \frac{1}{2}h \int_0^T dt$$

T = period of revolution (orbital period)

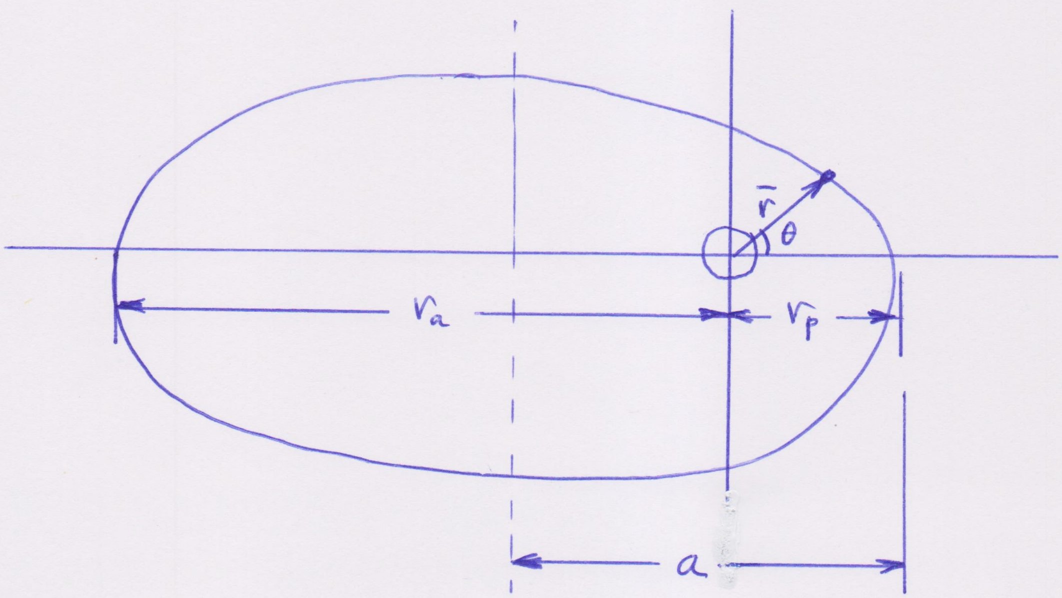
$$A = \frac{1}{2}hT \qquad (5.17)$$

Equate (5.16) to (5.17) & use (5.15)

$$\pi a^2 \cancel{\sqrt{1-e^2}} = \frac{1}{2} \overbrace{\sqrt{\mu a \cancel{(1-e^2)}}}^{h}\, T$$

$$\boxed{T^2 = \frac{4\pi^2}{\mu} a^3}$$ Kepler's 3rd Law (5.18)

Satellite in Elliptic Orbit



$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

$$r_p = r(\theta=0) = \frac{a(1-e^2)}{1+e} = a(1-e) \qquad (5.19)$$

$$r_a = r(\theta=\pi) = \frac{a(1-e^2)}{1-e} = a(1+e) \qquad (5.20)$$

Divide (5.19) by (5.20)

$$\frac{r_p}{r_a} = \frac{1-e}{1+e} \qquad (5.21)$$

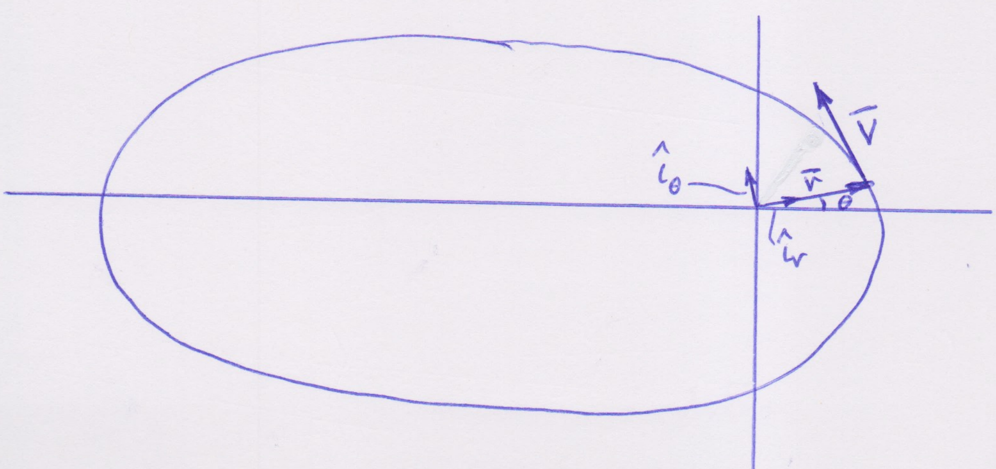
Solve for e

$$e = \frac{r_a - r_p}{r_a + r_p} \qquad (5.22)$$

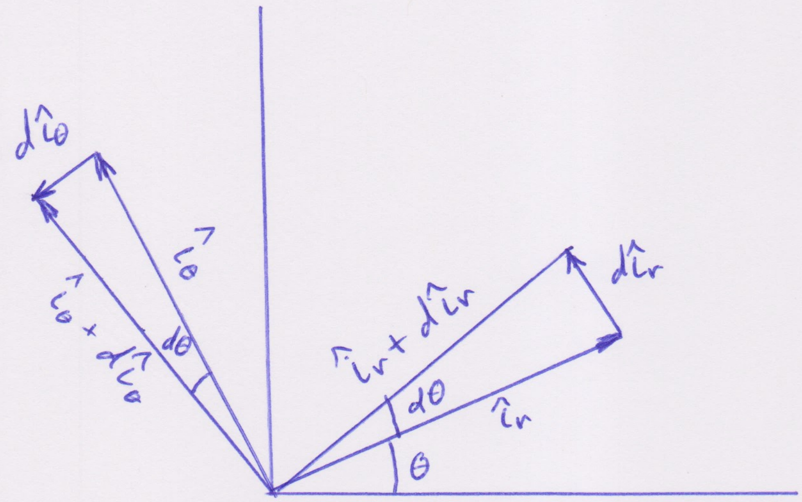
Using (5.12) and (5.19) can write

$$r = \frac{r_p(1+e)}{1+e\cos\theta} \qquad (5.23)$$

Radial and Angular Velocity Components



Note: Unit vectors $\hat{i}_r$ and $\hat{i}_\theta$ depend on θ



$$d\hat{i}_r = 1 \cdot d\theta \; \hat{i}_\theta \quad \Rightarrow \quad \frac{d\hat{i}_r}{d\theta} = \hat{i}_\theta$$

$$d\hat{i}_\theta = 1 \cdot d\theta \, (-\hat{i}_r) \quad \Rightarrow \quad \frac{d\hat{i}_\theta}{d\theta} = -\hat{i}_r$$

$$\bar{r} = r \, \hat{i}_r$$

$$\bar{V} = \frac{d\bar{r}}{dt} = \frac{dr}{dt}\hat{i}_r + r \frac{d\hat{i}_r}{d\theta}\frac{d\theta}{dt} = \frac{dr}{dt}\hat{i}_r + r\frac{d\theta}{dt}\hat{i}_\theta$$

$$V_r = \frac{dr}{dt} \qquad (5.24)$$

$$V_\theta = r\frac{d\theta}{dt} \qquad (5.25)$$

Since $r = \dfrac{a(1-e^2)}{1+e\cos\theta}$

$$\frac{dr}{dt} = -\frac{a(1-e^2)}{(1+e\cos\theta)^2}(-e\sin\theta)\frac{d\theta}{dt}$$

$$= \frac{a^2(1-e^2)^2\frac{d\theta}{dt}}{(1+e\cos\theta)^2} \cdot \frac{e\sin\theta}{a(1-e^2)}$$

$$= r^2\frac{d\theta}{dt} \cdot \frac{e\sin\theta}{a(1-e^2)}$$

Had previously found that

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}h = \text{const.} \Rightarrow h = r^2\frac{d\theta}{dt}$$

Therefore

$$\boxed{V_r = \frac{dr}{dt} = \frac{h e\sin\theta}{a(1-e^2)}} \qquad (5.26)$$

$$V_\theta = r\frac{d\theta}{dt} = \frac{h}{r}$$

$$\boxed{V_\theta = r\frac{d\theta}{dt} = \frac{h(1+e\cos\theta)}{a(1-e^2)}} \qquad (5.27)$$

At $\theta = 0$ (perigee)

$$V_r = 0 \qquad V_\theta = \frac{h(1+e)}{a(1-e^2)} = \frac{h}{a(1-e)} = \frac{h}{r_p} \Rightarrow V_p = \frac{h}{r_p}$$

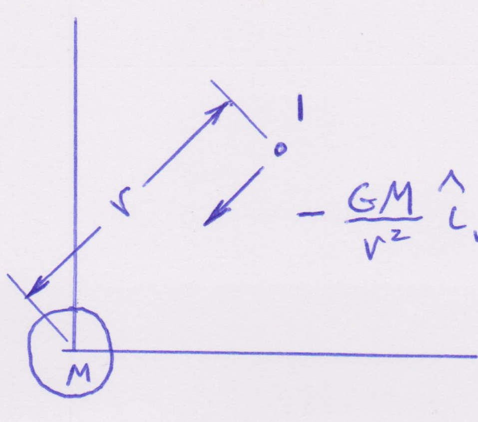
At $\theta = \pi$ (apogee)

$$V_r = 0 \qquad V_\theta = \frac{h(1-e)}{a(1-e^2)} = \frac{h}{a(1+e)} = \frac{h}{r_a} \Rightarrow V_a = \frac{h}{r_a}$$

Since h = constant

$$\boxed{r_p V_p = r_a V_a} \qquad (5.28)$$

Potential Energy



$-\frac{GM}{r^2}\hat{i}_r$ = force acting on unit mass

To move the unit mass from $r = R$ to $r = \infty$, apply force

$$+\frac{GM}{r^2}\hat{i}_r$$

Work done by the applied force is

$$W = \int_R^\infty \frac{GM}{r^2}\,dr = -\frac{GM}{r}\bigg|_R^\infty = 0 - \left(-\frac{GM}{R}\right) = \frac{GM}{R}$$

Potential energy of system is increased by $\frac{GM}{R}$

Set potential energy $V = 0$ at ∞

$$\boxed{V(R) = -\frac{GM}{R}} \qquad (5.29) \qquad \text{(per unit mass of the satellite)}$$

The Energy Integral

From equation of relative motion (5.1)

$$\frac{d^2\bar{r}}{dt^2} = -\frac{\mu}{r^3}\bar{r}$$

Dot with $\frac{d\bar{r}}{dt}$

$$\frac{d\bar{r}}{dt} \cdot \frac{d^2\bar{r}}{dt^2} = -\frac{\mu}{r^3}\bar{r} \cdot \frac{d\bar{r}}{dt}$$

Note that

$$\frac{d\bar{r}}{dt} \cdot \frac{d^2\bar{r}}{dt^2} = \frac{1}{2}\frac{d}{dt}\left(\frac{d\bar{r}}{dt} \cdot \frac{d\bar{r}}{dt}\right) = \frac{d}{dt}\left(\frac{V^2}{2}\right)$$

$$\bar{r} \cdot \frac{d\bar{r}}{dt} = r\frac{dr}{dt}$$

Therefore

$$\frac{d}{dt}\left(\frac{V^2}{2}\right) = -\frac{\mu}{r^2}\frac{dr}{dt}$$

Integrate

$$\frac{V^2}{2} = \frac{\mu}{r} + \mathcal{E} \qquad \mathcal{E} = \text{constant of integration}$$

or

$$\underbrace{\frac{V^2}{2}}_{\substack{\text{specific}\\ \text{KE}}} \underbrace{- \frac{\mu}{r}}_{\substack{\text{specific}\\ \text{PE}}} = \mathcal{E} \qquad (5.30)$$

$$T + V = \mathcal{E} = \text{const.}$$

Eq. (5.39) shows that the sum of the kinetic and potential energy of the orbiting satellite remains constant.

To evaluate $\mathcal{E}$

$$\mathcal{E} = \frac{V^2}{2} - \frac{\mu}{r} \qquad (\text{constant for all } \theta)$$

Therefore at periapse $(\theta = 0)$

$$\mathcal{E} = \frac{V_p^2}{2} - \frac{\mu}{r_p}$$

At <u>periapse</u>, regardless of the type of conic trajectory, the velocity and radius vectors are orthogonal so that using $\bar{h} = \bar{r} \times \frac{d\bar{r}}{dt}$

$$h = r_p V_p$$

Using $p = h^2/\mu \Rightarrow h = \sqrt{\mu p}$

$$r_p V_p = \sqrt{\mu p} \quad \Rightarrow \quad V_p^2 = \frac{\mu p}{r_p^2}$$

Therefore

$$\mathcal{E} = \frac{\mu p}{2 r_p^2} - \frac{\mu}{r_p} = \frac{\mu}{r_p}\left(\frac{p}{2 r_p} - 1\right)$$

Since $\quad r_p = \dfrac{p}{1+e\cos 0^\circ} = \dfrac{p}{1+e}$

$$\mathcal{E} = \frac{\mu(1+e)}{p}\left(\frac{p(1+e)}{2p} - 1\right) = \frac{\mu(1+e)(e-1)}{2p} = \frac{-\mu(1-e^2)}{2p}$$

$$\boxed{\mathcal{E} = -\frac{\mu(1-e^2)}{2p}} \qquad (5.31)$$

Valid for any conic trajectory

Conic Trajectory	e	p	$\mathcal{E}$
elliptic	<1	$a(1-e^2)$	$-\frac{\mu}{2a}$
parabolic	1	$2q$	0
hyperbolic	>1	$a(e^2-1)$	$+\frac{\mu}{2a}$

Note that the total energy for all conic trajectories is independent of eccentricity and depends only on the value of "a".

In particular, for a circular orbit of radius r_c

$$\frac{V_c^2}{2} - \frac{\mu}{r_c} = -\frac{\mu}{2r_c}$$

$$\frac{V_c^2}{2} = \frac{\mu}{2r_c}$$

$$\boxed{V_c = \sqrt{\frac{\mu}{r_c}}} \qquad (5.32)$$

For a 185 mile high circular orbit about earth:

$$r_c = 6368 \text{ km} + (185 \text{ miles})\left(\frac{1.61 \text{ km}}{1 \text{ mile}}\right) = 6670 \text{ km}$$

$$V_c = \sqrt{\frac{\mu}{r_c}} = \sqrt{\frac{3.986 \times 10^5 \ \frac{\text{km}^3}{\text{sec}^2}}{6670 \text{ km}}}$$

$$V_c = 7.730 \text{ km/sec} = 17{,}300 \text{ mph}$$

$$T = \frac{\text{circumference}}{\text{speed}} = \frac{2\pi \cdot 6670 \text{ km}}{7.730 \ \frac{\text{km}}{\text{sec}}} \cdot \frac{1 \text{ min}}{60 \text{ sec}}$$

$$T = 90 \text{ min}$$

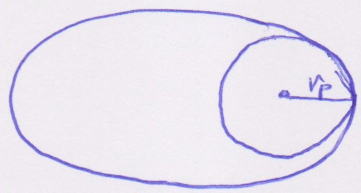
Since

$$V_p r_p = V_a r_a = h$$

$$V_p^2 = \frac{h^2}{r_p^2} = \frac{\mu p}{r_p^2} = \frac{\mu a(1-e^2)}{r_p\, a(1-e)} = \frac{\mu(1+e)}{r_p}$$

For a circular orbit at perigee height

$$V_{cp}^2 = \frac{\mu}{r_p}$$



Therefore

$$\frac{V_p^2}{V_{cp}^2} = 1+e$$

$$\boxed{e = \frac{V_p^2}{V_{cp}^2} - 1} \qquad (5.33)$$

Escape if $e \geq 1 \Rightarrow$

$$\text{escape if } \boxed{\frac{V_p}{V_{cp}} \geq \sqrt{2}} \qquad (5.34)$$

At 185 miles above earth's surface

$$V_{escape} = \sqrt{2}\, V_{cp} = \sqrt{2}\left(7.730 \frac{km}{sec}\right) = 10.90 \frac{km}{sec}$$

$$= 24{,}400 \text{ mph}$$

Also, since

$$V_a^2 = \frac{h^2}{r_a^2} = \frac{\mu a(1-e^2)}{r_a \; a(1+e)} = \frac{\mu(1-e)}{r_a}$$

$$V_{ca}^2 = \frac{\mu}{r_a}$$

$$\frac{V_a^2}{V_{ca}^2} = 1-e$$

$$\boxed{e = 1 - \frac{V_a^2}{V_{ca}^2}} \qquad (5.35)$$