10. Perturbations of a Satellite Orbit

10.1 The Effect of Atmospheric Drag on Satellite Orbits

The drag force encountered when an object moves through a fluid is given by the relationship:

$$D = \frac{1}{2} C_D \rho V^2 A \qquad (10.1)$$

where:

D = drag force (opposite in direction to the direction of motion)

C_D = drag coefficient (dimensionless) depends on Re, M and geometric shape of the object. ($C_D \approx 2$ if the mean free path of the atmospheric molecules is large compared to the size of the satellite)

ρ = density of the fluid

V = velocity of the object relative to the fluid

A = largest cross-sectional area perpendicular to the fluid stream. (If the satellite is tumbling, each orientation is assumed equally probable and $\frac{1}{4}$ the total surface area of the object is used as the reference area.

10.2. Factors which affect the atmosphere and consequently the satellite orbit.

- Rotation of the earth (lifetime of retrograde orbits is shorter)
- Gravitational effects of the moon and sun (tidal motions)
- Solar activity (affects atmospheric density)

10.3 Simplifying Assumptions

1. Two-body problem.
2. Atmosphere will be assumed to be stationary. (Thus velocity of vehicle relative to the atmosphere will be equal to the total velocity.

10.4. Density Variation as a Function of Altitude

Hydrostatic equation

$$\frac{dP}{dr} = -\rho g$$

Equation of state

$$P = \rho R T$$

T varies with r but for small changes in r it can be assumed constant.

Also assume g is constant

$$RT\frac{d\rho}{dr} = -\rho g$$

$$\frac{d\rho}{\rho} = -\frac{g}{RT}\,dr$$

$$\ln\rho = -\frac{g}{RT}r + \ln C$$

$$\rho = C e^{-\frac{g}{RT}r}$$

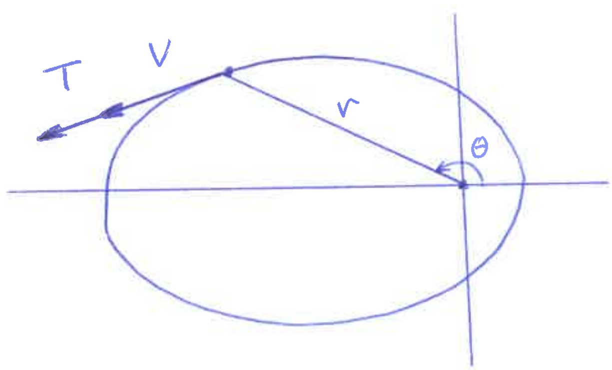
Define $H = \frac{RT}{g}$ (scale height)

$$\rho = C e^{-\frac{r}{H}}$$

At $r = r_p$ $\rho = \rho_p$ (air density at perigee)

$$\rho = \rho_p \, e^{-\frac{(r - r_p)}{H}} \qquad (10.2)$$

10.5 Ballistic Coefficient β



Consider a satellite in an elliptic orbit being acted upon by a force per unit mass T (T is positive in the direction of V)

$$T = -\frac{D}{m} = -\frac{C_D \rho V^2 A}{2m} \qquad (10.3)$$

Define the ballistic coefficient β

$$\beta = \frac{C_D A}{2m} \qquad (10.4)$$

Therefore (10.3) can be written as

$$T = -\beta \rho V^2 \qquad (10.5)$$

10.6 Semi-major axis variation

The energy change per unit mass as the satellite moves a distance ds along its path is equal to the work done by the applied force per unit mass T

$$d\mathcal{E} = T\,ds \qquad (10.6)$$

But

$$ds = V\,dt \qquad (10.7)$$

Therefore

$$d\mathcal{E} = TV\,dt \qquad (10.8)$$

The total energy per unit mass of a satellite in an elliptic orbit is

$$\mathcal{E} = -\frac{\mu}{2a} \qquad (10.9)$$

$$d\mathcal{E} = \frac{\mu}{2a^2}\, da \qquad (10.10)$$

Equate (10.8) to (10.10)

$$TV\,dt = \frac{\mu}{2a^2}\, da$$

$$\frac{da}{dt} = \frac{2a^2}{\mu} VT \qquad (10.11)$$

The total velocity is given by

$$V = \sqrt{V_r^2 + V_\theta^2}$$

Using (5.26) & (5.27) with $p = \frac{h^2}{\mu} = a(1-e^2)$ gives

$$V = \sqrt{\frac{\mu}{a(1-e^2)}}\,(1 + 2e\cos\theta + e^2)^{1/2} \qquad (10.12)$$

Sub. (10.12) into (10.11)

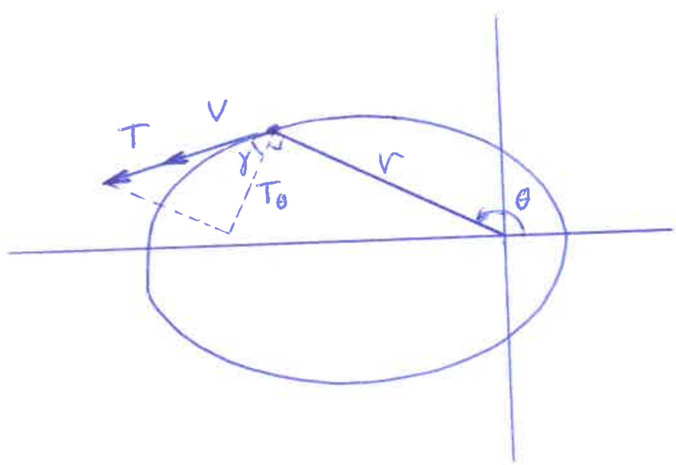
$$\frac{da}{dt} = 2\sqrt{\frac{a^3}{\mu}} \, \frac{\sqrt{1+2e\cos\theta+e^2}}{\sqrt{1-e^2}} \, T$$

or

$$\boxed{\frac{da}{dt} = \frac{2\sqrt{1+2e\cos\theta+e^2}}{n\sqrt{1-e^2}} \, T} \qquad (10.13)$$

where $n = \sqrt{\frac{\mu}{a^3}}$ (mean angular velocity) (10.14)

10.7. Angular Momentum Variation



Applied torque = Rate of change of angular momentum

$$rT_\theta = \frac{dh}{dt} \qquad (10.15)$$

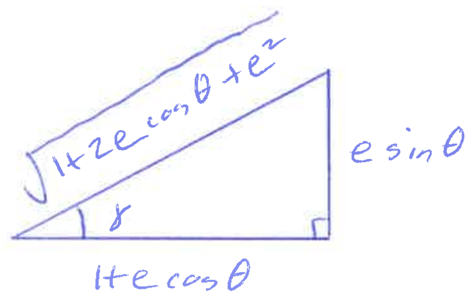
$$T_\theta = T\cos\gamma \qquad (10.16)$$

Therefore

$$rT\cos\gamma = \frac{dh}{dt} \qquad (10.17).$$

From (6.26)

$$\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta}$$



$$\cos\gamma = \frac{1+e\cos\theta}{\sqrt{1+2e\cos\theta+e^2}} \qquad (10.18)$$

Also

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \qquad (10.19)$$

Sub. (10.18), (10.19) into (10.17)

$$\boxed{\frac{dh}{dt} = \frac{a(1-e^2)}{\sqrt{1+2e\cos\theta+e^2}}\,T} \qquad (10.20)$$

10.8. Eccentricity Variation

From $p = \frac{h^2}{\mu} = a(1-e^2)$

$$e^2 = 1 - \frac{h^2}{\mu a} \qquad (10.21)$$

$$e = e(h, a)$$

$$\frac{de}{dt} = \left(\frac{\partial e}{\partial h}\right)_a \frac{dh}{dt} + \left(\frac{\partial e}{\partial a}\right)_h \frac{da}{dt} \qquad (10.22)$$

From (10.21)

$$\left(\frac{\partial e}{\partial h}\right)_a = -\frac{h}{\mu a e} = -\frac{\sqrt{1-e^2}}{\sqrt{\mu a}\; e} \qquad (10.23)$$

$$\left(\frac{\partial e}{\partial a}\right)_h = \frac{h^2}{2\mu a^2 e} = \frac{1-e^2}{2ae} \qquad (10.24)$$

Sub. (10.13), (10.20), (10.23) & (10.24) into (10.22)

$$\boxed{\frac{de}{dt} = \frac{2\sqrt{1-e^2}\,(\cos\theta + e)}{na\sqrt{1+2e\cos\theta+e^2}}\,T} \qquad (10.25)$$

10.9 Period Variation

$$\tau = 2\pi\sqrt{\frac{a^3}{\mu}}$$

$$\frac{d\tau}{dt} = 3\pi\sqrt{\frac{a}{\mu}}\,\frac{da}{dt} = \frac{3}{2}\,\frac{\tau}{a}\,\frac{da}{dt}$$

$$\boxed{\frac{d\tau}{dt} = \frac{3}{2}\,\frac{\tau}{a}\,\frac{da}{dt}} \qquad (10.26)$$

$\frac{da}{dt}$ is given by (10.13)

10.10 Apogee and Perigee Variation

Altitude to apogee & perigee

$$h_a = r_a - r_e$$
$$h_p = r_p - r_e$$

r_e = radius of earth

But

$$r_a = a(1+e)$$

$$r_p = a(1-e)$$

Therefore

$$h_a = a(1+e) - r_e$$

$$h_p = a(1-e) - r_e$$

Differentiate with respect to time

$$\boxed{\frac{dh_a}{dt} = (1+e)\frac{da}{dt} + a\frac{de}{dt}} \qquad (10.27)$$

$$\boxed{\frac{dh_p}{dt} = (1-e)\frac{da}{dt} - a\frac{de}{dt}} \qquad (10.28)$$

$\frac{da}{dt}$ and $\frac{de}{dt}$ are given by (10.13) and (10.25)

From $r^2 \frac{d\theta}{dt} = h$

using $p = \frac{h^2}{\mu} = a(1-e^2)$ and $r = \frac{a(1-e^2)}{1+e\cos\theta}$

get

$$\frac{d\theta}{dt} = \frac{h}{r^2} = \frac{\sqrt{\mu a(1-e^2)}}{\left[\frac{a(1-e^2)}{1+e\cos\theta}\right]^2} = \sqrt{\frac{\mu}{a^3}}\,\frac{(1+e\cos\theta)^2}{(1-e^2)^{3/2}}$$

or

$$\boxed{\frac{d\theta}{dt} = \frac{n\,(1+e\cos\theta)^2}{(1-e^2)^{3/2}}} \qquad (10.29)$$

Recall equations (10.13) & (10.25)

$$\boxed{\frac{da}{dt} = \frac{2\sqrt{1+2e\cos\theta+e^2}}{n\sqrt{1-e^2}}\,T} \qquad (10.13)$$

$$\boxed{\frac{de}{dt} = \frac{2\sqrt{1-e^2}\,(\cos\theta+e)}{na\sqrt{1+2e\cos\theta+e^2}}\,T} \qquad (10.25)$$

If the applied force per unit mass is known, and initial conditions are prescribed for θ, a & e, eqs. (10.13), (10.25) & (10.29) may be solved numerically to obtain the variation of θ, a & e with time.

10.11 Decay of a highly eccentric satellite orbit

For a highly eccentric satellite orbit, most of the drag occurs at perigee of the orbit (velocity and density are highest).

To find the maximum variation of the orbital parameters a & e, set $\theta \sim 0$ (where the drag force is maximum).

Eqs. (10.13) & (10.25) become

$$\frac{da}{dt} = \frac{2(1+e)}{n\sqrt{1-e^2}} T \qquad (10.30)$$

$$\frac{de}{dt} = \frac{2\sqrt{1-e^2}}{na} T \qquad (10.31)$$

Divide (10.30) by (10.31)

$$\frac{da/dt}{de/dt} = \frac{a}{1-e}$$

or

$$\frac{da}{dt} = \frac{a}{1-e}\frac{de}{dt} \qquad (10.32)$$

Sub. (10.32) into (10.27) & (10.28)

$$\frac{dh_a}{dt} = (1+e)\frac{da}{dt} + a\frac{de}{dt}$$

$$= (1+e)\left(\frac{a}{1-e}\right)\frac{de}{dt} + a\frac{de}{dt}$$

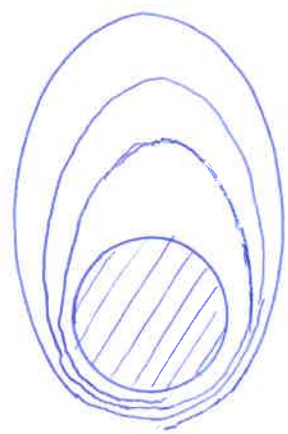
$$\boxed{\frac{dh_a}{dt} = \frac{2a}{1-e}\frac{de}{dt} = 2\frac{da}{dt}} \qquad (10.33)$$

$$\frac{dh_p}{dt} = (1-e)\frac{da}{dt} - a\frac{de}{dt}$$

$$= (1-e)\left(\frac{a}{1-e}\right)\frac{de}{dt} - a\frac{de}{dt}$$

$$\boxed{\frac{dh_p}{dt} = 0} \quad \text{(to a first approximation)} \qquad (10.34)$$

From (10.33) & (10.34) conclude that elliptic orbits tend to become circular as they decay.



10.12 The Variation of a and e over one Complete Revolution

Sub. (10.5) $[T = -\beta\rho V^2]$ into (10.13) & (10.25)

and use (10.14) $\left[n = \sqrt{\frac{\mu}{a^3}}\right]$

$$\frac{da}{dt} = -2\beta\rho V^2 \sqrt{\frac{a^3}{\mu}} \frac{\sqrt{1+2e\cos\theta+e^2}}{\sqrt{1-e^2}} \qquad (10.35)$$

$$\frac{de}{dt} = -\frac{2\beta\rho V^2}{a} \sqrt{\frac{a^3}{\mu}} \frac{\sqrt{1-e^2}\,(\cos\theta+e)}{\sqrt{1+2e\cos\theta+e^2}} \qquad (10.36)$$

From (10.12)

$$V^2 = \frac{\mu}{a} \frac{1+2e\cos\theta+e^2}{1-e^2} \qquad (10.37)$$

Since

$$\frac{da}{d\theta} = \frac{da}{dt} \cdot \frac{dt}{d\theta} = \frac{da/dt}{d\theta/dt} \qquad (10.38)$$

$$\frac{de}{d\theta} = \frac{de}{dt} \cdot \frac{dt}{d\theta} = \frac{de/dt}{d\theta/dt} \qquad (10.39)$$

Sub. (10.37) into (10.35) & (10.36). Then sub. (10.29), (10.35) & (10.36) into (10.38) & (10.39)

$$\frac{da}{d\theta} = -2\beta\rho a^2 \frac{(1+2e\cos\theta+e^2)^{3/2}}{(1+e\cos\theta)^2} \qquad (10.40)$$

$$\frac{de}{d\theta} = -2\beta\rho a(1-e^2) \frac{(1+2e\cos\theta+e^2)^{1/2}(\cos\theta+e)}{(1+e\cos\theta)^2} \qquad (10.41)$$

For a given density distribution and appropriate initial conditions for a & e, eqs. (10.40) & (10.41) may be solved numerically to obtain the variation of a & e with θ.

Some simplification of (10.40) & (10.41) can be obtained by writing them in terms of the eccentric anomaly E.

Since

$$\cos\theta = \frac{\cos E - e}{1 - e\cos E} \qquad (7.6a)$$

$$\frac{d\theta}{dE} = \frac{(1-e^2)\sin E}{(1-e\cos E)^2 \sin\theta}$$

Using (7.6b) $\left[\sin\theta = \frac{(1-e^2)^{1/2}\sin E}{1-e\cos E} \right]$

$$\frac{d\theta}{dE} = \frac{\sqrt{1-e^2}}{1-e\cos E} \qquad (10.42)$$

Therefore

$$\frac{da}{dE} = \frac{da}{d\theta}\frac{d\theta}{dE} = -2\beta\rho a^2 \frac{(1+e\cos E)^{3/2}}{(1-e\cos E)^{1/2}} \qquad (10.43)$$

$$\frac{de}{dE} = \frac{de}{d\theta}\frac{d\theta}{dE} = -2\beta\rho a(1-e^2)\frac{(1+e\cos E)^{1/2}}{(1-e\cos E)^{1/2}}\cos E \qquad (10.44)$$

For a given density distribution and appropriate initial conditions for a & e, (10.43) & (10.44) represent two coupled o.d.e's which may be solved numerically to give the variation of a & e as a function of E.

For slowly decaying orbits, a & e may be assumed approximately constant over one revolution. Equations (10.43) & (10.44) may be integrated to give

$$\frac{\Delta a}{rev} = -2\beta a^2 \int_0^{2\pi} \rho \frac{(1+e\cos E)^{3/2}}{(1-e\cos E)^{1/2}}\, dE \qquad (10.45)$$

$$\frac{\Delta e}{rev} = -2\beta a(1-e^2) \int_0^{2\pi} \rho \frac{(1+e\cos E)^{1/2}}{(1-e\cos E)^{1/2}} \cos E\, dE \qquad (10.46)$$

Note: $\rho = \rho(r) \qquad r = a(1-e\cos E)$

10.13 Decay of Circular Orbits

A circular orbit is a special case of an elliptic orbit with

$$a = r_c \qquad e = 0$$

Eqs. (10.45) & (10.46) reduce to

$$\frac{\Delta r_c}{rev} = -2\beta r_c^2 \int_0^{2\pi} \rho \, dE \qquad (10.47)$$

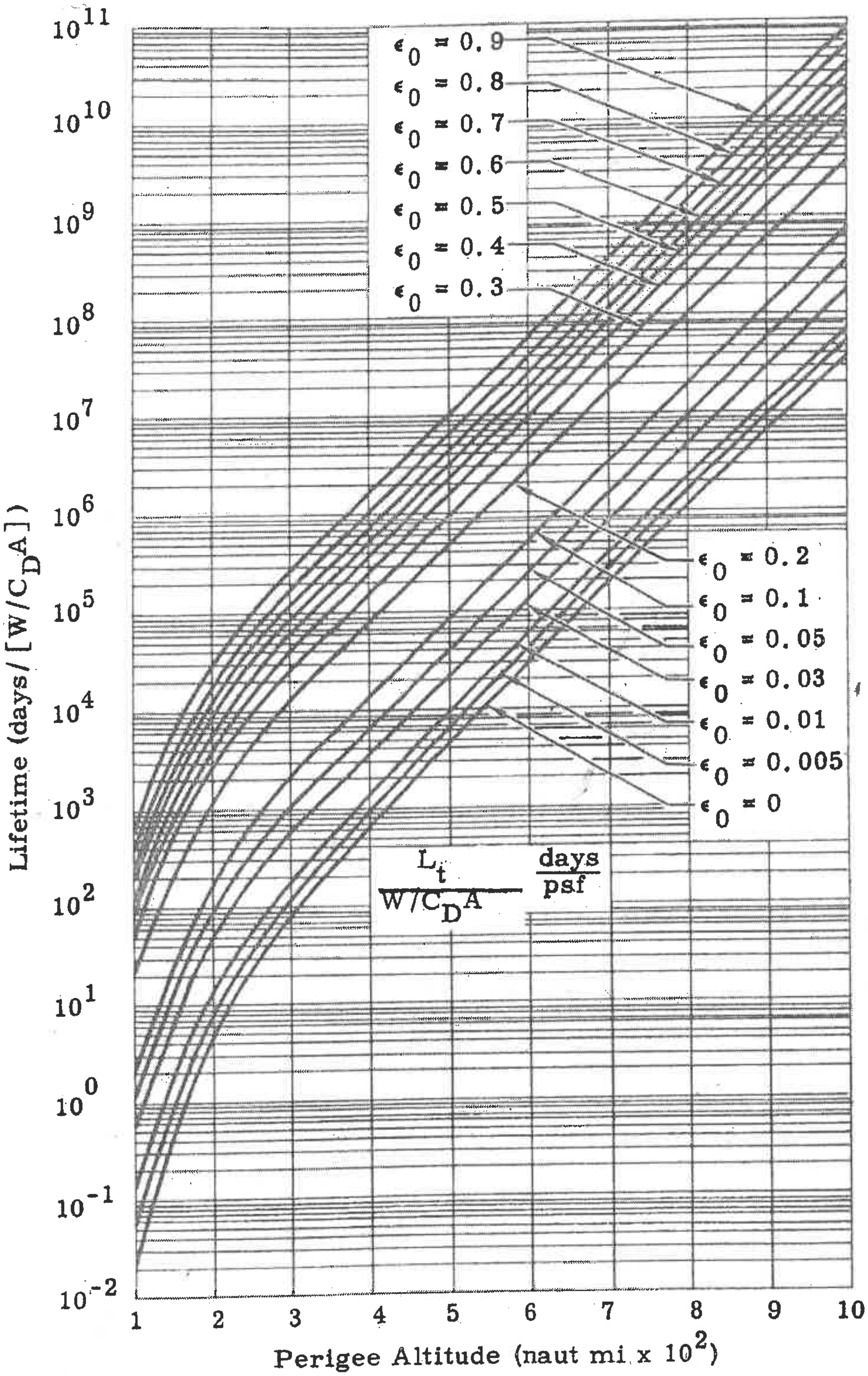
$$\frac{\Delta e}{rev} = -2\beta r_c \int_0^{2\pi} \rho \cos E \, dE \qquad (10.48)$$

Since (10.47) & (10.48) are valid for slowly decaying circular orbits, assume $\rho \sim$ constant.

$$\frac{\Delta r_c}{rev} = -4\pi \beta \rho r_c^2 \qquad (10.49)$$

$$\frac{\Delta e}{rev} = 0 \qquad (10.50)$$

Eq. (10.50) shows that circular orbits remain circular as they decay.



$\epsilon_0 = 0.9$
$\epsilon_0 = 0.8$
$\epsilon_0 = 0.7$
$\epsilon_0 = 0.6$
$\epsilon_0 = 0.5$
$\epsilon_0 = 0.4$
$\epsilon_0 = 0.3$
$\epsilon_0 = 0.2$
$\epsilon_0 = 0.1$
$\epsilon_0 = 0.05$
$\epsilon_0 = 0.03$
$\epsilon_0 = 0.01$
$\epsilon_0 = 0.005$
$\epsilon_0 = 0$
$\frac{L_t}{W/C_D A}$ $\frac{\text{days}}{\text{psf}}$
Lifetime (days/ [$W/C_D A$])
10^{11}
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
1
2
3
4
5
6
7
8
9
10
Perigee Altitude (naut mi x 10^2)

EXAMPLE

Vanguard 1D was launched on March 17, 1958 into a 356×2074 nautical mile orbit. Although communication was lost in 1964, it remains the oldest man-made satellite still in orbit. The satellite is a 6.4 inch diameter sphere weighing 3.2 lbs. Estimate its orbital lifetime.

$$r_e = 6368\text{ km}\left(\frac{3281\text{ ft}}{1\text{ km}}\right)\left(\frac{1\text{ n-mi}}{6076\text{ ft}}\right) = 3439\text{ n-mi}$$

$$e = \frac{r_a - r_p}{r_a + r_p} = \frac{h_a - h_p}{h_a + h_p + 2r_e} = \frac{2074 - 356}{2074 + 356 + 2(3439)} = 0.1846$$

$$A = \frac{\pi d^2}{4} = \frac{\pi\left(\frac{6.4}{12}\right)^2}{4} = 0.2234\text{ ft}^2$$

$$\frac{W}{C_D A} = \frac{3.2}{2(0.2234)} = 7.162\text{ psf}$$

From orbital lifetime chart, for $h_p = 356$ n-mi & $e = 0.1846$

$$\frac{L_t}{W/C_D A} = 2\times10^4\ \frac{\text{days}}{\text{psf}}$$

$$L_t = (2\times10^4)(7.162)$$

$$\boxed{L_t = 143{,}200 \text{ days} = 392 \text{ years}}$$