11. Three-body Problem

The general three-body problem must be solved numerically.

Two special cases where an analytic solution is possible:

a) Lagrange's solution of the 3-body problem for special configurations

b) The restricted 3-body problem (Earth-Moon-Spaceship problem)
One of the masses is much smaller than the other two.

11.1. Lagrange's Solution of the Three-body Problem for Special Configurations

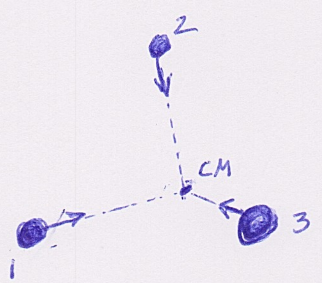
There are cases where the geometrical form of the three-body configuration does not change, although the scale can change and the figure can rotate:

Case 1: The particles are colinear

Case 2: The particles form an equilateral triangle

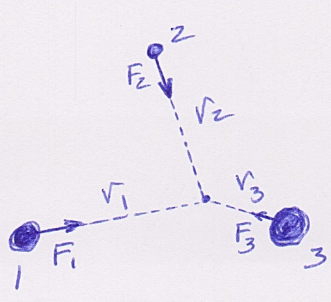
In 1772, Lagrange showed that for 3 particles of arbitrary mass to exist in such configurations, the following conditions must be true:

1) The resultant force on each mass passes through the center of mass of the system.



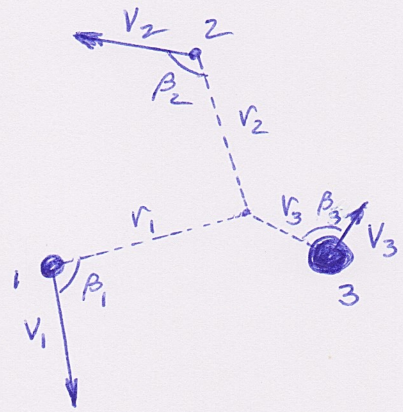
2) The resultant force per unit mass is directly proportional to the distance of each mass from the center of mass

F = force per unit mass (acceleration)



$$\frac{F_1}{r_1} = \frac{F_2}{r_2} = \frac{F_3}{r_3}$$

3) The velocities are proportional in magnitude to the respective distance of each particle to the center of mass, and make equal angles with the radius vectors to the particles from the center of mass.



$$\frac{V_1}{r_1} = \frac{V_2}{r_2} = \frac{V_3}{r_3}$$

$$\beta_1 = \beta_2 = \beta_3$$

From (3.2) the equations of motion for the 3 particles are

$$m_i \frac{d^2 \bar{r}_i}{dt^2} = G m_i \sum_{j=1}^{3}{}' \frac{m_j (\bar{r}_j - \bar{r}_i)}{r_{ij}^{\;3}} \qquad (11.1)$$

$$i = 1, 2, 3$$

where

$$r_{ij} = |\bar{r}_j - \bar{r}_i|$$

The center of mass is at

$$\bar{r}_{cm} = \frac{\sum_{i=1}^{3} m_i \bar{r}_i}{\sum_{i=1}^{3} m_i} \qquad (11.2)$$

If the origin is placed at the center of mass

$$\bar{r}_{cm} = 0$$

$$\sum_{i=1}^{3} m_i \bar{r}_i = 0$$

or $$m_1 \bar{r}_1 + m_2 \bar{r}_2 + m_3 \bar{r}_3 = 0 \qquad (11.3)$$

Add and subtract $m_2 \bar{r}_1$ & $m_3 \bar{r}_1$ which yields

$$(m_1 + m_2 + m_3)\bar{r}_1 + m_2(\bar{r}_2 - \bar{r}_1) + m_3(\bar{r}_3 - \bar{r}_1) = 0$$

or

$$M\bar{r}_1 = -m_2 \bar{r}_{12} - m_3 \bar{r}_{13} \qquad (11.4)$$

where

$$M = m_1 + m_2 + m_3$$

Take the dot product of each side of (11.4) with itself

$$M^2 r_1^2 = m_2^2 r_{12}^2 + m_3^2 r_{13}^2 + 2 m_2 m_3 \bar{r}_{12} \cdot \bar{r}_{13} \quad (11.5)$$

If the shape of the configuration is not to change, the relative distances r_{12}, r_{23} and r_{13} are given by

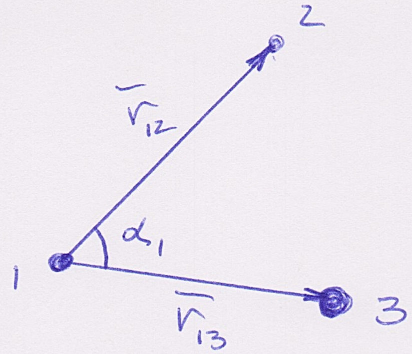
$$\frac{r_{12}}{(r_{12})_0} = \frac{r_{23}}{(r_{23})_0} = \frac{r_{13}}{(r_{13})_0} = f(t) \quad (11.6)$$

where $(r_{ij})_0 = r_{ij}$ at $t=0$ (the time when the particles were placed in the required configuration)

Also, if the angles between $\bar{r}_{12}$ and $\bar{r}_{13}$ and $\bar{r}_{12}$ and $\bar{r}_{23}$, etc. are to remain constant

$$\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}(t) \quad (11.7)$$

where $\dot{\theta}_i$ = angular velocity of line joining center of mass to m_i



Let α_1 = angle between $\bar{r}_{12}$ and $\bar{r}_{13}$ (constant)

Write $\bar{r}_{12} \cdot \bar{r}_{13} = r_{12} r_{13} \cos\alpha_1$

Using (11.6), can write (11.5) as follows:

$$M^2 r_1^2 = [f(t)]^2 \left[m_2^2 (r_{12})_0^2 + m_3^2 (r_{13})_0^2 + 2 m_2 m_3 (r_{12})_0 (r_{13})_0 \cos\alpha_1 \right] \tag{11.8}$$

But from (11.6) $f(0) = 1$

Therefore at $t=0$, (11.8) becomes

$$M^2 (r_1)_0^2 = \left[m_2^2 (r_{12})_0^2 + m_3^2 (r_{13})_0^2 + 2 m_2 m_3 (r_{12})_0 (r_{13})_0 \cos\alpha_1 \right] \tag{11.9}$$

Hence

$$r_1^2 = (r_1)_0^2 \, [f(t)]^2$$

or

$$r_1 = (r_1)_0 f(t)$$

and in general

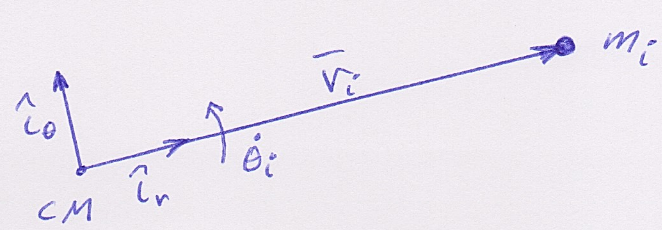
$$r_i = (r_i)_0 f(t) \qquad i=1,2,3 \qquad (11.10)$$

We will now prove that the resultant force on each particle passes through the center of mass of the system but to do that, we will first show that the angular momentum of <u>each</u> particle is conserved.

It was shown in the n-body derivation that the total angular momentum of the system is conserved (see (3.8))

$$\bar{C} = \sum_{i=1}^{3} m_i \bar{r}_i \times \frac{d\bar{r}_i}{dt} \qquad (11.11)$$

For the i^{th} mass m_i



$$\bar{r}_i = r_i \hat{i}_r$$

$$\bar{V}_i = \frac{d\bar{r}_i}{dt} = \frac{dr_i}{dt}\hat{i}_r + r_i \frac{d\hat{i}_r}{dt} = \frac{dr_i}{dt}\hat{i}_r + r_i \overbrace{\frac{d\hat{i}_r}{d\theta}}^{\hat{i}_\theta} \frac{d\theta_i}{dt}$$

$$= \frac{dr_i}{dt}\hat{i}_r + r_i \frac{d\theta_i}{dt}\hat{i}_\theta$$

$$\bar{a}_i = \frac{d\bar{V}_i}{dt} = \frac{d^2 r_i}{dt^2}\hat{i}_r + \frac{dr_i}{dt}\frac{d\theta_i}{dt}\hat{i}_\theta + \frac{dr_i}{dt}\frac{d\theta_i}{dt}\hat{i}_\theta$$

$$+ r_i \frac{d^2\theta_i}{dt^2}\hat{i}_\theta - r_i\left(\frac{d\theta_i}{dt}\right)^2 \hat{i}_r$$

$$= \left[\frac{d^2 r_i}{dt^2} - r_i\left(\frac{d\theta_i}{dt}\right)^2\right]\hat{i}_r + \left[r_i \frac{d^2\theta_i}{dt^2} + 2\frac{dr_i}{dt}\frac{d\theta_i}{dt}\right]\hat{i}_\theta$$

radial acceleration — centripetal acceleration — angular acceleration — Coriolis acceleration

To compute the angular momentum of the system

$$\bar{r}_i \times \frac{d\bar{r}_i}{dt} = \begin{vmatrix} \hat{i}_r & \hat{i}_\theta & \hat{i}_z \\ r_i & 0 & 0 \\ \frac{dr_i}{dt} & r_i \frac{d\theta_i}{dt} & 0 \end{vmatrix} = r_i^2 \frac{d\theta_i}{dt}\hat{i}_z$$

$$\bar{C} = \sum_{i=1}^{3} m_i \bar{r}_i \times \frac{d\bar{r}_i}{dt} = \sum_{i=1}^{3} m_i r_i^2 \frac{d\theta_i}{dt} \hat{i}_z$$

Take magnitude

$$C = \sum_{i=1}^{3} m_i r_i^2 \frac{d\theta_i}{dt} \qquad (11.12)$$

Sub. (11.7) and (11.10) into (11.12)

$$C = \sum_{i=1}^{3} \left[m_i (r_i)_0^2 \right] f^2 \frac{d\theta}{dt} \qquad (11.13)$$

Eq. (11.13) shows that

$$f^2 \frac{d\theta}{dt} = \text{const.}$$

The angular momentum of the i^{th} particle is

$$C_i = m_i r_i^2 \frac{d\theta_i}{dt} = m_i (r_i)_0^2 \underbrace{f^2 \frac{d\theta}{dt}}_{\text{const}} = \text{const.} \qquad (11.14)$$

The angular momentum <u>of each particle</u> is conserved.

Recall that

Applied torque = rate of change of angular momentum

$$m_i r_i F_{\theta i} = \frac{dc_i}{dt}$$

per unit mass

Using (11.14) conclude that $F_{\theta i} = 0$, i.e.

$$\boxed{\text{The resultant force acting on each mass passes through the center of mass (Lagrange's 1st condition)}}$$

If F_i is the resultant force per unit mass acting on the mass m_i (directed toward the center of mass of the system), then its equation of motion is

$$m_i F_i = m_i \left[\frac{d^2 r_i}{dt^2} - r_i \left(\frac{d\theta_i}{dt} \right)^2 \right]$$

Using (11.7) $[\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}]$ and (11.10) $r_i = (r_i)_0 f(t)$

$$m_i F_i = m_i \left[(r_i)_0 \frac{d^2 f}{dt^2} - (r_i)_0 f \left(\frac{d\theta}{dt}\right)^2 \right]$$

$$= m_i (r_i)_0 f \left[\frac{d^2 f/dt^2}{f} - \left(\frac{d\theta}{dt}\right)^2 \right]$$

$$= m_i r_i \underbrace{\left[\frac{d^2 f/dt^2}{f} - \left(\frac{d\theta}{dt}\right)^2 \right]}_{\text{same for all particles}}$$

Therefore at each instant of time

$$\boxed{\frac{F_1}{r_1} = \frac{F_2}{r_2} = \frac{F_3}{r_3}} \qquad (11.15)$$

(Lagrange's 2nd condition)

Will now prove first part of Lagrange's 3rd condition $[V_i \sim r_i]$

Since

$$\bar{V}_i = \frac{d\bar{r}_i}{dt} = \frac{dr_i}{dt} \hat{i}_{r_i} + r_i \frac{d\theta_i}{dt} \hat{i}_{\theta_i}$$

$$V_i = \sqrt{\left(\frac{dr_i}{dt}\right)^2 + r_i^2\left(\frac{d\theta_i}{dt}\right)^2}$$

Using (11.7) and (11.10)

$$V_i = \sqrt{(r_i)_0^2\left(\frac{df}{dt}\right)^2 + (r_i)_0^2 f^2\left(\frac{d\theta}{dt}\right)^2}$$

$$= (r_i)_0 \sqrt{\left(\frac{df}{dt}\right)^2 + f^2\left(\frac{d\theta}{dt}\right)^2}$$

$$= r_i \underbrace{\frac{1}{f}\sqrt{\left(\frac{df}{dt}\right)^2 + f^2\left(\frac{d\theta}{dt}\right)^2}}_{\text{same for all particles}}$$

Therefore at each instant of time

$$\boxed{\frac{V_1}{r_1} = \frac{V_2}{r_2} = \frac{V_3}{r_3}} \qquad (11.16)$$

(First part of Lagrange's 3rd condition)

Will now prove 2nd part of Lagrange's 3rd condition, that angles between velocity and radius vectors are all equal.

Since

$$m_i \overline{F}_i = m_i r_i \left[\frac{\ddot{f}}{f} - \dot{\theta}^2 \right] \hat{i}_{r_i} \qquad \text{(see eq. prior to (11.15))}$$

and

$$\overline{V}_i = \frac{dr_i}{dt} \hat{i}_{r_i} + r_i \frac{d\theta_i}{dt} \hat{i}_{\theta_i} = \underbrace{(r_i)_o}_{\frac{r_i}{f}} \dot{f} \hat{i}_{r_i} + r_i \dot{\theta} \hat{i}_{\theta_i}$$

$$= r_i \left[\frac{\dot{f}}{f} \hat{i}_{r_i} + \dot{\theta} \hat{i}_{\theta_i} \right]$$

$$\overline{F}_i \cdot \overline{V}_i = r_i^2 \left[\frac{\ddot{f}}{f} - \dot{\theta}^2 \right] \frac{\dot{f}}{f}$$

$$= \underbrace{r_i \left[\frac{\ddot{f}}{f} - \dot{\theta}^2 \right]}_{|F_i|} \cdot \underbrace{r_i \left[\left(\frac{\dot{f}}{f} \right)^2 - \dot{\theta}^2 \right]^{1/2}}_{|\overline{V}_i|} \cos \beta_i$$

where β_i is the angle between F_i and $\overline{V}_i$

$$\cos \beta_i = \underbrace{\frac{\dot{f}/f}{\left[\left(\frac{\dot{f}}{f} \right)^2 + \dot{\theta}^2 \right]^{1/2}}}_{\text{sume for all particles}}$$

Therefore at each instant of time

$$\boxed{\beta_1 = \beta_2 = \beta_3} \qquad (11.17)$$

Second part of Lagrange's 3rd condition

We will now obtain the geometric configurations of the 3 particles which satisfy Lagrange's conditions

If the resultant force on each particle passes through the center of mass of the system

$$\bar{r}_i \times \bar{F}_i = 0 \qquad (11.18)$$

or

$$\bar{r}_i \times \ddot{\bar{r}}_i = 0 \qquad (11.19)$$

From (11.1)

$$\cancel{m_i}\,\ddot{\bar{r}}_i = G\,\cancel{m_i} \sum_{j=1}^{3}{}' \frac{m_j(\bar{r}_j - \bar{r}_i)}{r_{ij}^{\,3}}$$

For $i=1$

$$\bar{r}_1 \times \ddot{\bar{r}}_1 = \bar{r}_1 \times G\left(\frac{m_2(\bar{r}_2-\bar{r}_1)}{r_{12}^3} + \frac{m_3(\bar{r}_3-\bar{r}_1)}{r_{13}^3}\right)$$

$$= \bar{r}_1 \times G\left(\frac{m_2\bar{r}_2}{r_{12}^3} + \frac{m_3\bar{r}_3}{r_{13}^3}\right) = 0$$

$$\bar{r}_1 \times \left(\frac{m_2\bar{r}_2}{r_{12}^3} + \frac{m_3\bar{r}_3}{r_{13}^3}\right) = 0$$

From (11.3)

$$m_3\bar{r}_3 = -m_1\bar{r}_1 - m_2\bar{r}_2$$

Therefore

$$\bar{r}_1 \times \left(\frac{m_2\bar{r}_2}{r_{12}^3} - \cancel{\frac{m_1\bar{r}_1}{r_{13}^3}} - \frac{m_2\bar{r}_2}{r_{13}^3}\right) = 0$$

or

$$\bar{r}_1 \times \left(\frac{m_2\bar{r}_2}{r_{12}^3} - \frac{m_2\bar{r}_2}{r_{13}^3}\right) = 0$$

or

$$\bar{r}_1 \times \bar{r}_2 \left(\frac{1}{r_{12}^3} - \frac{1}{r_{13}^3} \right) = 0 \qquad (11.20)$$

Similarly <u>for $i=2$</u>

$$\bar{r}_2 \times \bar{r}_3 \left(\frac{1}{r_{23}^3} - \frac{1}{r_{12}^3} \right) = 0 \qquad (11.21)$$

and <u>for $i=3$</u>

$$\bar{r}_3 \times \bar{r}_1 \left(\frac{1}{r_{13}^3} - \frac{1}{r_{23}^3} \right) = 0 \qquad (11.22)$$

The above conditions (11.20)–(11.22) are satisfied if either of the following conditions hold:

i) $r_{12} = r_{23} = r_{13} = r$ (equilateral triangle)

ii) $\bar{r}_1 \times \bar{r}_2 = \bar{r}_2 \times \bar{r}_3 = \bar{r}_3 \times \bar{r}_1 = 0$ (particles are in a straight line)

For the equilateral triangle configuration, (11.1) with $i=1$ becomes

$$m_1 \ddot{\bar{r}}_1 = \frac{G m_1}{r^3} \left(m_2 \bar{r}_{12} + m_3 \bar{r}_{13} \right)$$

Using (11.4) $\left(M \bar{r}_1 = -m_2 \bar{r}_{12} - m_3 \bar{r}_{13} \right)$

$$m_1 \ddot{\bar{r}}_1 + \frac{GM}{r^3} \bar{r}_1 = 0 \qquad (11.23)$$

Using (11.5) $\left(M^2 r_1^2 = m_2^2 r_{12}^2 + m_3^2 r_{13}^2 + 2 m_2 m_3 \bar{r}_{12} \cdot \bar{r}_{13} \right)$

$$M^2 r_1^2 = m_2^2 r^2 + m_3^2 r^2 + 2 m_2 m_3 r^2 \cos 60^\circ$$

$$= \left(m_2^2 + m_3^2 + m_2 m_3 \right) r^2$$

or

$$r = \frac{M r_1}{\left(m_2^2 + m_3^2 + m_2 m_3 \right)^{1/2}}$$

Sub. into (11.23)

$$\boxed{\ddot{\bar{r}}_1 + \frac{GM_1}{r_1^3}\bar{r}_1 = 0} \qquad (11.24)$$

where

$$\boxed{M_1 = \frac{(m_2^2 + m_3^2 + m_2 m_3)^{3/2}}{(m_1 + m_2 + m_3)^2}} \qquad (11.25)$$

Compare (11.24) with (5.1) which is the 2-body equation of motion. Eq (11.24) shows that m_1 moves about the center of mass in a trajectory (ellipse, parabola, hyperbola) as if a mass M_1 were placed there.

Similar equations can be obtained for m_2 & m_3.

For the colinear case (assume the particles lie along the x-axis) the force per unit mass on m_1 is

$$F_1 = \frac{Gm_2}{x_{12}^2} + \frac{Gm_3}{x_{13}^2}$$

Using (11.10)

$$X_i = (X_i)_0 \, f(t)$$

$$F_1 = \frac{G}{[f(t)]^2}\left[\frac{m_2}{X_{12}^2} + \frac{m_3}{X_{13}^2}\right]_0 = \frac{\text{const.}}{[f(t)]^2}$$

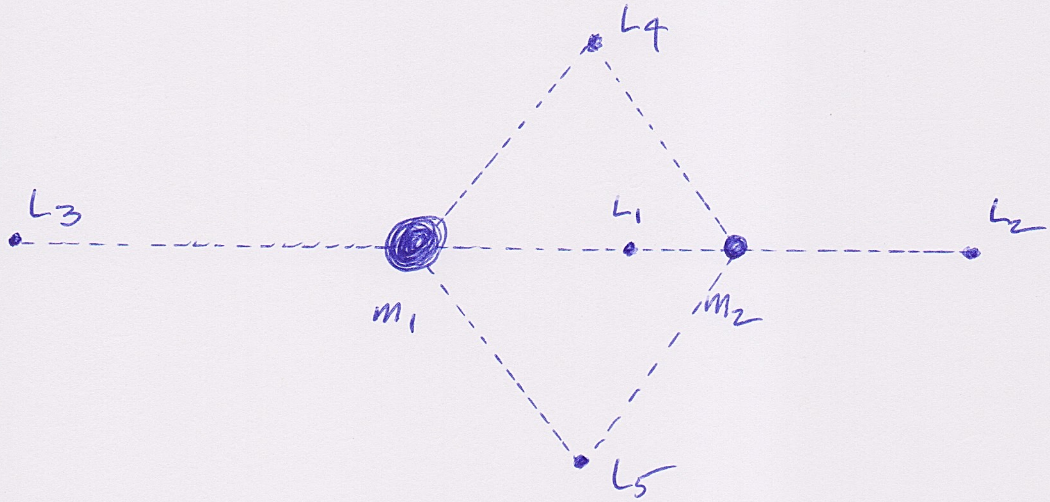
Since f is proportional to the distance from the center of mass, m_1 is acted upon by an inverse square law central force.

Therefore it's trajectory is a conic section with focus that the center of mass.

If $f(t) = 1$, there is no change in scale. The solutions are called <u>stationary</u>.

The entire configuration can rotate about the center of mass with constant angular velocity.

For a given position of m_1 & m_2, there are 5 positions of m_3 which will give a stationary configuration



L_i's are called <u>libration points</u> or <u>Lagrange points</u>.

L_1, L_2, L_3 are unstable

L_4, L_5 are stable if $\frac{m_1}{m_2} + \frac{m_2}{m_1} \geq 25$

<u>EXAMPLE</u>

$m_1 \longrightarrow$ sun

$m_2 \longrightarrow$ Jupiter

~ 20 asteroids at L_4 & L_5 called Trojans

$$\frac{m_1}{m_2} = \frac{1.989 \times 10^{30} \text{ kg}}{1.899 \times 10^{27} \text{ kg}} = 1047$$