HW Assignment 1

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(1) Curtis (1.6): An 80 kg man and a 50 kg woman stand 0.5 m from each other. What is the force of gravitational attraction between the couple?

$$F = \frac{Gm_1m_2}{r_{12}^2} \qquad \begin{cases} \mathrm{G} = 6.67259 \times 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg\text{ - }sec}^2} \\ m_1 = 80\ \mathrm{kg} \\ m_2 = 50\ \mathrm{kg} \\ r_{12} = 0.5\ \mathrm{m} \end{cases}$$

$$= \frac{6.67259 \times 10^{-11})(80)(50)}{(0.5)}$$

$$= \boxed{1.068\ \mu N}$$

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(2) Curtis (1.8): If a person's weight is W on the surface of the earth, calculate what it would be, in terms of W, at the surface of a) The moon; b) Mars; c) Jupiter.

On surface of Earth:

$$F = W = \frac{Gm_1m_E}{r_E^2} = m_1\left(\frac{Gm_E}{r_E^2}\right)$$

Knowing that $G = 6.67259 \times 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg\text{ - }sec}^2}$, $m_E = 5.9722 \times 10^{24}\ \mathrm{kg}$, and $r_E = 6.3781 \times 10^6\ \mathrm{m}:$

$$W = m_1\left(\frac{(6.67259\times10^{-11})(5.9722\times10^{24})}{(6.3781\times10^{6})^2}\right) = m_1(\underbrace{9.796\ \frac{\mathrm{m}}{\mathrm{s}^2}}_{g})$$

$$\Rightarrow \underline{m_1 = \frac{W}{9.796\ \frac{\mathrm{m}}{\mathrm{s}^2}}}$$

a) On the moon;

$$g_{moon} = 1.62\ \frac{\mathrm{m}}{\mathrm{s}^2}$$

$$\Rightarrow W_{moon} = m_1 g_{moon} = \left(\frac{W}{9.796\ \frac{\mathrm{m}}{\mathrm{s}^2}}\right)\left(1.62\ \frac{\mathrm{m}}{\mathrm{s}^2}\right)$$

$$= \boxed{0.165W}$$

b) On Mars;

$$g_{Mars} = 3.71\ \frac{\mathrm{m}}{\mathrm{s}^2}$$

$$\Rightarrow W_{Mars} = m_1 g_{Mars} = \left(\frac{W}{9.796\ \frac{\mathrm{m}}{\mathrm{s}^2}}\right)\left(3.71\ \frac{\mathrm{m}}{\mathrm{s}^2}\right)$$

$$= \boxed{0.379W}$$

c) On Jupiter;

$$g_{Jupiter} = 24.79\ \frac{\mathrm{m}}{\mathrm{s}^2}$$

$$\Rightarrow W_{Jupiter} = m_1 g_{Jupiter} = \left(\frac{W}{9.796\ \frac{\mathrm{m}}{\mathrm{s}^2}}\right)\left(24.79\ \frac{\mathrm{m}}{\mathrm{s}^2}\right)$$

$$= \boxed{2.53W}$$

(3) Prove that equations (3.3) and (3.4) given in the class notes are equivalent to (3.2).

$$m_i \frac{d\bar{r}_i}{dt^2} = G m_i {\sum_{j=1}^{N}}' \frac{m_j}{r_{ij}} (\bar{r}_j - \bar{r}_i) \qquad (3.2)$$

$$m_i \frac{d\bar{r}_i}{dt^2} = \nabla_i U \qquad (3.3)$$

$$U = \frac{G}{2} \sum_{i=1}^{N} {\sum_{j=1}^{N}}' \frac{m_i m_j}{r_{ij}} \qquad (3.4)$$

From (3.4), we can get rid of the i summation because we only want the force of attraction on mass m_i , and not on every possible pair. To prove this - at some point in expanding the double summation, we will get similar terms that add up. e.g.

$$\frac{m_1 m_2}{r_{12}} = \frac{m_2 m_1}{r_{21}} = 2\left[\frac{m_1 m_2}{r_{12}}\right]$$

Since this will be true for each pair, and by keeping i = 1, we get

$$\begin{aligned} U &= \frac{G}{2} {\sum_{j=1}^{N}}' 2\left[\frac{m_i m_j}{r_{ij}}\right] \\ &= G m_i {\sum_{j=1}^{N}}' \frac{m_j}{r_{ij}} \end{aligned}$$

Now we can plug the force function U into (3.3)

$$m_i \frac{d\bar{r}_i}{dt^2} = \nabla_i \left(G m_i {\sum_{j=1}^{N}}' \frac{m_j}{r_{ij}} \right)$$

To evaluate the gradient, we must take the partial of U with each direction

$$\nabla_i U = \frac{\partial U}{\partial x_i}\hat{i} + \frac{\partial U}{\partial y_i}\hat{i} + \frac{\partial U}{\partial z_i}\hat{i}$$

Let us evaluate partial x first. Since $G, m_i,$ and m_j are independent of direction, we can group them outside of the gradient

$$\begin{aligned}\frac{\partial U}{\partial x_i}\hat{i} &= \frac{\partial}{\partial x_i}\left(Gm_i\sum_{j=1}^{N}{}' \frac{m_j}{r_{ij}}\right)\hat{i}\\ &= Gm_i\sum_{j=1}^{N}{}' m_j\frac{\partial}{\partial x_i}\left(\frac{1}{r_{ij}}\right)\hat{i}\end{aligned}$$

We know that

$$\begin{aligned}r_{ij} &= \mid \bar{r}_j - \bar{r}_i \mid\\ &= ((\bar{r}_j - \bar{r}_i)^2)^{1/2}\\ &= ((x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2)^{1/2}\end{aligned}$$

So that partial x is equal to

$$\begin{aligned}&= \frac{\partial}{\partial x_i}\left(((x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2)^{-1/2}\right)\hat{i}\\ &= \left[-\frac{1}{2}((x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2)^{-3/2}\right]\left[2(x_j - x_i)\right]\left[-1\right]\hat{i}\\ &= \frac{(x_j - x_i)}{((x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2)^{3/2}}\\ &= \frac{(x_j - x_i)}{r_{ij}^3}\hat{i}\end{aligned}$$

Similarly, for partial y and partial z

$$\begin{aligned}\frac{\partial}{\partial y_i}\left(\frac{1}{r_{ij}}\right)\hat{j} &= \frac{(y_j - y_i)}{r_{ij}^3}\hat{j}\\ \frac{\partial}{\partial z_i}\left(\frac{1}{r_{ij}}\right)\hat{k} &= \frac{(z_j - z_i)}{r_{ij}^3}\hat{k}\end{aligned}$$

We can take out the r_{ij}^3 term since it is the same in all 3 components of the gradient. We are left with

$$Gm_i \sum_{j=1}^{N}{}' \frac{m_j}{r_{ij}^3} \underbrace{\left((x_j - x_i)\hat{i} \ + (y_j - y_i)\hat{j} \ + (z_j - z_i)\hat{k}\right)}_{(\bar{r}_j - \bar{r}_i)}$$

$$= \boxed{Gm_i \sum_{j=1}^{N}{}' \frac{m_j}{r_{ij}^3}(\bar{r}_j - \bar{r}_i)}$$

which is identical to the right hand side of equation (3.2)

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(4) Prove that the force function U given by equation (3.4) in the class notes is equal to the total work done by the gravitational forces in assembling a system of N point masses from a state of infinite dispersion to a given configuration.

$$U = \frac{G}{2} \sum_{i=1}^{N} \sum_{j=1}^{N}{}' \frac{m_i m_j}{r_{ij}} \qquad (3.4)$$

Total work is given by

$$-\int F * \ dr$$

Force due to gravitational attraction on mass m_i

$$F_i = \frac{Gm_i m_j}{r_{ij}^2}$$

Since we want to bring point masses from infinity to a given config, we do

$$
\begin{aligned}
-\int_{\infty}^{r_{ij}} F\,dr &= -\int_{\infty}^{r_{ij}} \frac{Gm_i m_j}{r^2}\,dr \\
&= -Gm_i m_j \int_{\infty}^{r_{ij}} r^{-2}\,dr \\
&= -Gm_i m_j \left[-\frac{1}{r}\right]_{\infty}^{r_{ij}} \\
&= -Gm_i m_j \left[\left(-\frac{1}{r_{ij}}\right) + \underbrace{\left(-\frac{1}{\infty}\right)}_{\to 0}\right] \\
&= \frac{Gm_i m_j}{r_{ij}}
\end{aligned}
$$

This is only for 2 masses. If we want N masses, we include summations. Similar to problem (3), we will have a summation iterating the i^{th} mass and another iterating the j^{th} mass (where $j \neq i$). As previously stated, this type of notation will lead to accounting each pair of masses twice, hence dividing the entire equation by 2

$$
\Rightarrow \boxed{\frac{G}{2} \sum_{i=1}^{N} {\sum_{j=1}^{N}}' \frac{m_i m_j}{r_{ij}}}
$$

which is identical to equation (3.4)

(5) Prove equation (4.7) given in the class notes.

$$\frac{\bar{d}_j}{d_j^3}+\frac{\bar{\rho}_j}{\rho_j^3}=-\nabla\left(\frac{1}{d_j}-\frac{1}{\rho_j^3}\bar{r}\cdot\bar{\rho}_j\right) \qquad (4.7)$$

We know that

$$\bar{r}=\bar{r}_2-\bar{r}_1$$

If we set m_1 to 0 in all directions (origin), then $\bar{r}=0$

$$\begin{aligned}\bar{r}&=\bar{r}_2-(0)=\bar{r}_2\\ \bar{\rho}_j&=\bar{r}_j-(0)=\bar{r}_j\\ \bar{d}_j&=\bar{r}_2-\bar{r}_j\end{aligned}$$

We can separate this gradient into

$$-\nabla\left(\frac{1}{d_j}\right)-\nabla\left(-\frac{1}{\rho_j^3}\bar{r}\cdot\bar{\rho}_j\right)$$

The left part will be

$$\begin{aligned}&=-\nabla\left(((x_2-x_j)^2+(y_2-y_j)^2+(z_2-z_j)^2)^{-1/2}\right)\\ &=\frac{1}{d_j^3}((x_2-x_j)\hat{i}+(y_2-y_j)\hat{j}+(z_2-z_j)\hat{k})\\ &\boxed{\frac{\bar{d}_j}{d_j^3}}\end{aligned}$$

For the right side, we have

$$\bar{r}\cdot\bar{\rho}_j=(x\cdot x_j\;\hat{i})+(y\cdot y_j\;\hat{j})+(z\cdot z_j\;\hat{k})$$

So that

$$= -\nabla \left(-\frac{(x \cdot x_j\ \hat{i}) + (y \cdot y_j\ \hat{j}) + (z \cdot z_j\ \hat{k})}{\left((x_j^2 + y_j^2 + z_j^2)^{1/2}\right)} \right)$$

$$= \frac{1}{\rho_j^3}(x_j\hat{i} + y_j\hat{j} + z_j\hat{k})$$

$$= \boxed{\frac{\bar{\rho}_j}{\rho_j^3}}$$

The steps in taking the derivative were skipped due to similarity of the derivate taken in question (3). Combining the left and right part we get

$$\boxed{\frac{\bar{d}_j}{d_j^3} + \frac{\bar{\rho}_j}{\rho_j^3}}$$