ME 51500/I5800 HW 1 SOLUTIONS P. GANATOS

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1) $$F = G\frac{m_1 m_2}{r_{12}^2} = 6.674\times10^{-11}\ \frac{m^3}{kg\text{-}sec^2}\ \frac{(80\,kg)(50\,kg)}{(0.5\,m)^2}$$

$$= 1.068\times10^{-6}\ N = \underline{\underline{1.068\ \mu N}}$$

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2) $$F = G\frac{m_1 m_2}{r_{12}^2}$$

Let $m_1 = m_p$ be the mass of the planet or moon

Let m_2 be the mass of the person whose weight on earth is W.

On earth

$$W = G\frac{m_e m_2}{r_e^2} \Rightarrow m_2 = \frac{W r_e^2}{G m_e} \quad (1)$$

Let W_p be the person's weight on the planet

$$W_p = G\frac{m_p m_2}{r_p^2} \quad (2)$$

Sub. (1) into (2)

$$W_p = \frac{G m_p}{r_p^2}\cdot\frac{W r_e^2}{G m_e}$$

$$\boxed{W_p = \frac{m_p}{m_e}\left(\frac{r_e}{r_p}\right)^2 W}$$

Using data from Curtis Table A-1

a) <u>On the moon</u>

$$W_p = \frac{73.48\times10^{21}}{5.974\times10^{24}}\left(\frac{6378}{1737}\right)^2 W = \underline{\underline{0.1658\ W}}$$

b) On Mars

$$W_p = \frac{641.9 \times 10^{21}}{5.974 \times 10^{24}} \left(\frac{6378}{3396}\right)^2 W = \underline{\underline{0.3790\ W}}$$

c) On Jupiter

$$W_p = \frac{1.899 \times 10^{27}}{5.974 \times 10^{24}} \left(\frac{6378}{71{,}490}\right)^2 W = \underline{\underline{2.530\ W}}$$

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3) Definitions:

$$\nabla_i = \frac{\partial}{\partial x_i}\hat{i} + \frac{\partial}{\partial y_i}\hat{j} + \frac{\partial}{\partial z_i}\hat{k} \qquad (1)$$

$$\bar{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k} \qquad (2)$$

$$r_{ij}^2 = |\bar{r}_j - \bar{r}_i| \cdot |\bar{r}_j - \bar{r}_i| = (x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2 \qquad (3)$$

Expand (3.4) as follows:

$$U = \frac{G}{2}\left[m_1 \sum_{j=1}^{N}{}' \frac{m_j}{r_{1j}} + m_2 \sum_{j=1}^{N}{}' \frac{m_j}{r_{2j}} + \cdots + m_N \sum_{j=1}^{N}{}' \frac{m_j}{r_{Nj}}\right] \qquad (4)$$

Further expand (4) as follows: [Note that ($'$) means omit the term $j=i$ from the summation]:

$$U = \frac{G}{2}\left[\underline{\frac{m_1 m_2}{r_{12}}} + \underline{\frac{m_1 m_3}{r_{13}}} + \cdots + \underline{\frac{m_1 m_N}{r_{1N}}}\right.$$

$$+ \underline{\frac{m_2 m_1}{r_{21}}} + \frac{m_2 m_3}{r_{23}} + \cdots + \frac{m_2 m_N}{r_{2N}}$$

$$\left. + \cdots + \underline{\frac{m_N m_1}{r_{N1}}} + \frac{m_N m_2}{r_{N2}} + \cdots + \frac{m_N m_{N-1}}{r_{N,N-1}}\right] \quad (5)$$

Compute $\frac{\partial U}{\partial x_1}$ [Note: terms in (5) which depend on x_1 are underlined]

$$\frac{\partial U}{\partial x_1} = \frac{G}{2}\left[-\frac{m_1 m_2}{r_{12}^2}\left(-\frac{x_2 - x_1}{r_{12}}\right) - \frac{m_1 m_3}{r_{13}^2}\left(-\frac{x_3 - x_1}{r_{13}}\right)\right.$$

$$- \cdots - \frac{m_1 m_N}{r_{1N}^2}\left(-\frac{x_N - x_1}{r_{1N}}\right)$$

$$\left. - \frac{m_2 m_1}{r_{21}^2}\left(\frac{x_1 - x_2}{r_{21}}\right) - \cdots - \frac{m_N m_3}{r_{N1}}\left(\frac{x_1 - x_N}{r_{N1}}\right)\right]$$

$$= G\left[\frac{m_1 m_2}{r_{12}^3}(x_2 - x_1) + \frac{m_1 m_3}{r_{13}^3}(x_3 - x_1) + \cdots + \frac{m_1 m_N}{r_{1N}^3}(x_N - x_1)\right.$$

$$= G \sum_{j=2}^{N} \frac{m_1 m_j}{r_{1j}^3}(x_j - x_1) \qquad [\text{special case } i = 1]$$

or in general

$$\frac{\partial U}{\partial x_i} = G \sum_{j=1}^{N}{}' \frac{m_i m_j}{r_{ij}^3} (x_j - x_i) \qquad (6a)$$

Similarly

$$\frac{\partial U}{\partial y_i} = G \sum_{j=1}^{N}{}' \frac{m_i m_j}{r_{ij}^3} (y_j - y_i) \qquad (6b)$$

$$\frac{\partial U}{\partial z_i} = G \sum_{j=1}^{N}{}' \frac{m_i m_j}{r_{ij}^3} (z_j - z_i) \qquad (6c)$$

Using (1) & (2) eq. (6) becomes

$$\boxed{\nabla_i U = G \sum_{j=1}^{N}{}' \frac{m_i m_j}{r_{ij}^3} (\bar{r}_j - \bar{r}_i)}$$

4) In general, the work done by a force $\bar{F}$ acting along a path C is

$$W = \int_C \bar{F} \cdot d\bar{r}$$

Consider a system of N point masses.

- The work done in bringing in particle 1 in the absence of all other particles is zero because there is no force.
- The work done in bringing in particle 2 in the presence of particle 1 is

$$W_2 = \int_{\infty}^{r_{12}} \frac{G m_1 m_2}{r^2} \cos 180^\circ \, dr = \frac{G m_1 m_2}{r} \Bigg|_{\infty}^{r_{12}} = \frac{G m_1 m_2}{r_{12}}$$

$$= \frac{G}{2} \left[\frac{m_1 m_2}{r_{12}} + \frac{m_2 m_1}{r_{21}} \right]$$

- The work done in bringing in particle 3 in the presence of particles 1 & 2 is

$$W_3 = \int_{\infty}^{r_{13}} \frac{G m_1 m_3}{r^2} \cos 180^\circ \, dr + \int_{\infty}^{r_{23}} \frac{G m_2 m_3}{r^2} \cos 180^\circ \, dr$$

$$= \frac{G m_1 m_3}{r_{13}} + \frac{G m_2 m_3}{r_{23}}$$

$$= \frac{G}{2} \left[\frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} + \frac{m_3 m_1}{r_{31}} + \frac{m_3 m_2}{r_{32}} \right]$$

- The work done in bringing in particle N in the presence of particles $1, 2, 3, \ldots \& N-1$ is

$$W_N = \frac{G m_1 m_N}{r_{1N}} + \frac{G m_2 m_N}{r_{2N}} + \frac{G m_3 m_N}{r_{3N}} + \cdots + \frac{m_{N-1} m_N}{r_{N-1,N}}$$

$$= \frac{G}{2}\left[\frac{m_1 m_N}{r_{1N}} + \frac{m_2 m_N}{r_{2N}} + \frac{m_3 m_N}{r_{3N}} + \cdots + \frac{m_{N-1} m_N}{r_{N-1,N}} \right.$$

$$\left. + \frac{m_N m_1}{r_{N1}} + \frac{m_N m_2}{r_{N2}} + \frac{m_N m_3}{r_{N3}} + \cdots + \frac{m_N m_{N-1}}{r_{N,N-1}} \right]$$

The total work in assembling all N particles is

$$W = W_1 + W_2 + W_3 + \cdots + W_N$$

$$= \frac{G}{2}\left[0 + \frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \cdots + \frac{m_1 m_N}{r_{1N}} \right.$$

$$+ \frac{m_2 m_1}{r_{21}} + 0 + \frac{m_2 m_3}{r_{23}} + \cdots + \frac{m_2 m_N}{r_{2N}}$$

$$+ \frac{m_3 m_1}{r_{31}} + \frac{m_3 m_2}{r_{32}} + 0 + \cdots + \frac{m_3 m_N}{r_{3N}}$$

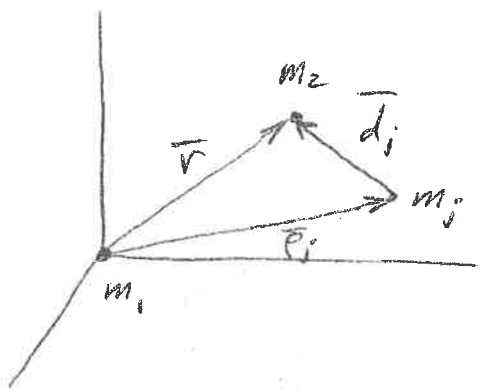
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$+ \frac{m_{N-1} m_1}{r_{N-1,1}} + \frac{m_{N-1} m_2}{r_N} + \frac{m_{N-1,} m_3}{r.} + \cdots + 0 + \frac{m_{N-1} m_N}{r_{N-1,N}}$$

$$\left. + \frac{m_N m_1}{r_{N1}} + \frac{m_N m_2}{r_{N2}} + \frac{m_N m_3}{r_{N3}} + \cdots + \frac{m_N m_{N-1}}{r_{N,N-1}} + 0 \right]$$

$$W = \frac{G}{2} \sum_{i=1}^{N} \sum_{j=1}^{N\,\prime} \frac{m_i m_j}{r_{ij}} = U$$

5) Without loss of generality, place origin at center of particle m_1



$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\bar{\rho}_j = x_j\hat{i} + y_j\hat{j} + z_j\hat{k}$$

$$\bar{d}_j = \bar{r} - \bar{\rho}_j$$

$$= (x - x_j)\hat{i} + (y - y_j)\hat{j} + (z - z_j)i$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\frac{1}{d_j} - \frac{1}{\rho_j^3}\bar{r}\cdot\bar{\rho}_j = \left[(x-x_j)^2 + (y-y_j)^2 + (z-z_j)^2\right]^{-1/2} - \frac{x x_j + y y_j + z z_j}{(x_j^2 + y_j^2 + z_j^2)^{3/2}}$$

$$-\nabla\left(\frac{1}{d_j} - \frac{1}{\rho_j^3}\bar{r}\cdot\bar{\rho}_j\right) = -\left\{-\frac{1}{\cancel{2}}\left[(x-x_j)^2 + (y-y_j)^2 + (z-z_j)^2\right]^{-3/2}\cancel{2}(x-x_j) - \frac{x_j}{(x_j^2 + y_j^2 + z_j^2)^{3/2}}\right\}\hat{i}$$

$$-\left\{-\frac{1}{\cancel{2}}\left[(x-x_j)^2 + (y-y_j)^2 + (z-z_j)^2\right]^{-3/2}\cancel{2}(y-y_j) - \frac{y_j}{(x_j^2 + y_j^2 + z_j^2)^{3/2}}\right\}\hat{j}$$

(cont'd)

$$- \left\{ -\frac{1}{\cancel{2}} \left[(x-x_j)^2 + (y-y_j)^2 + (z-z_j)^2 \right]^{-3/2} \cancel{2}(z-z_j) - \frac{z_j}{(x_j^2 + y_j^2 + z_j^2)^{3/2}} \right\} \hat{k}$$

$$-\nabla\left(\frac{1}{d_j} - \frac{1}{\rho_j^3}\bar{r}\cdot\bar{\rho}_j\right) = \frac{(x-x_j)\hat{i} + (y-y_j)\hat{j} + (z-z_j)\hat{k}}{\left[(x-x_j)^2 + (y-y_j)^2 + (z-z_j)^2\right]^{3/2}} + \frac{x_j\hat{i} + y_j\hat{j} + z_j\hat{k}}{(x_j^2 + y_j^2 + z_j^2)^{3/2}}$$

$$= \frac{\bar{d}_j}{d_j^3} + \frac{\bar{\rho}_j}{\rho_j^3}$$