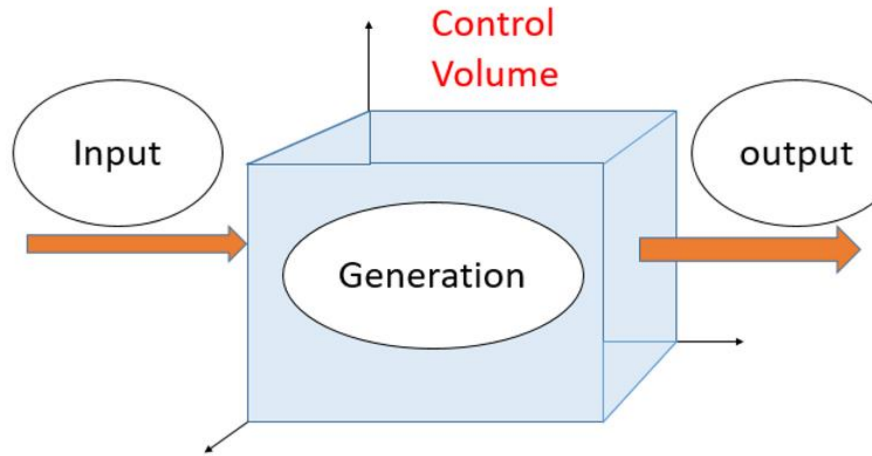
ME 57200 Aerodynamic Design

Lecture #5: Basic Equations in Aerodynamics

Dr. Yang Liu
Steinman 253
Tel: 212-650-7346
Email: yliu7@ccny.cuny.edu

Basic Equations in Aerodynamics



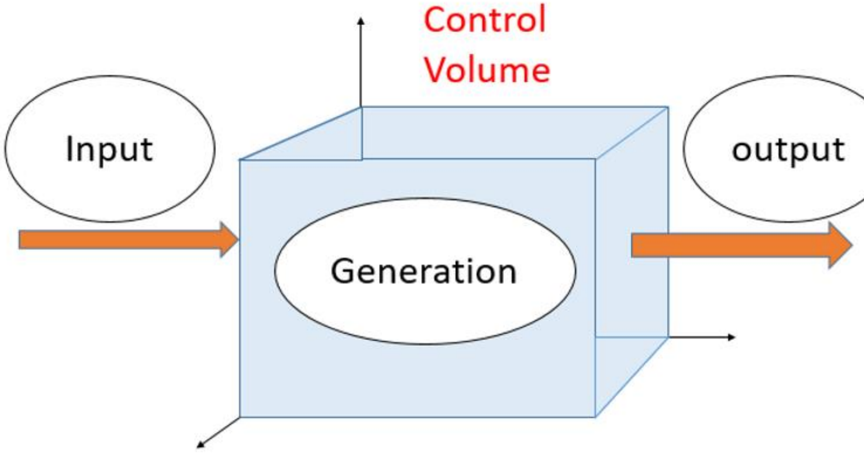
Continuity Equation

Rate of Change = Input – Output + generation

$$\frac{dm_{sys}}{dt} = \sum_{in} \dot{m}_i - \sum_{out} \dot{m}_e = 0$$

Basic Equations in Aerodynamics

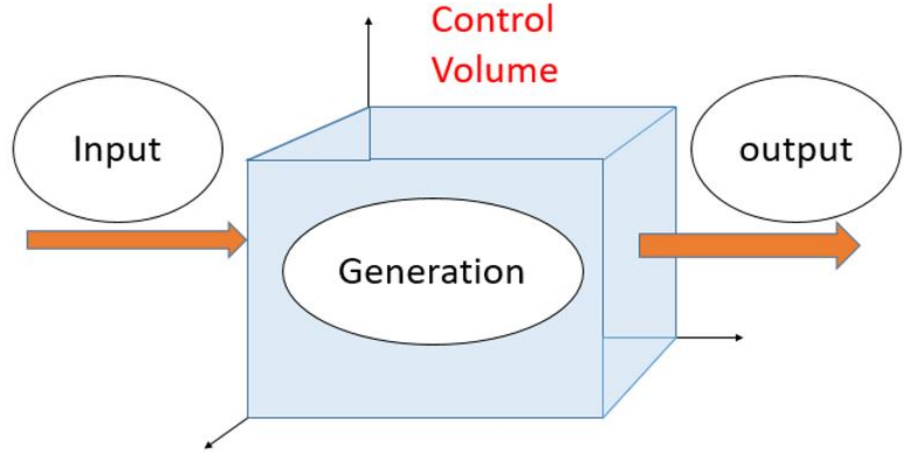
Momentum Equation



Rate of Change = Input – Output + generation

$$\frac{d\boldsymbol{p}_{sys}}{dt} = \boldsymbol{F}_{ext} + \sum_{in} \dot{m}_i \boldsymbol{V}_i - \sum_{out} \dot{m}_e \boldsymbol{V}_e$$

Basic Equations in Aerodynamics

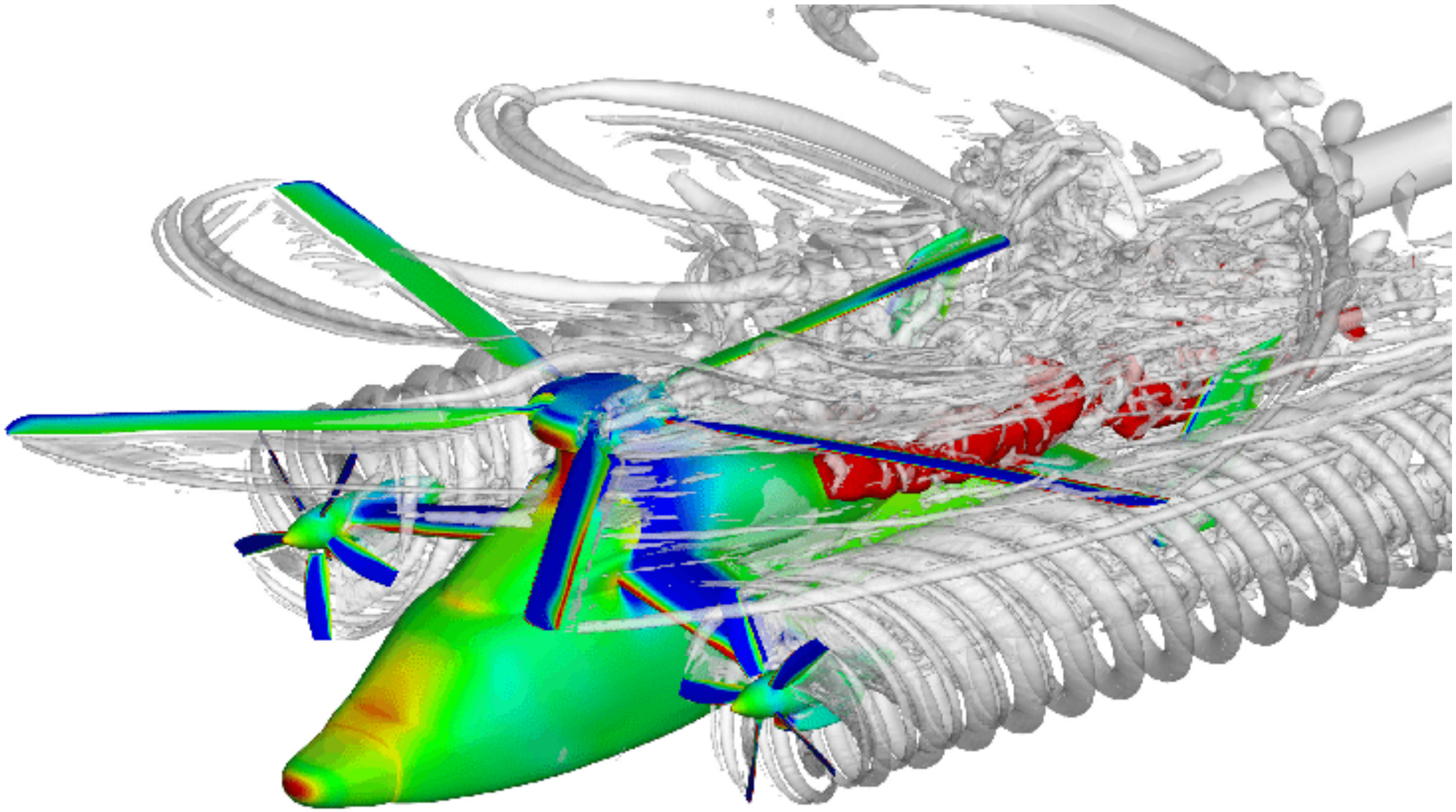


Energy Equation

Rate of Change = Input – Output + generation

$$\frac{dE_{sys}}{dt} = \dot{Q}_{net,in} + \dot{W}_{net,in} + \sum_{in} \dot{m}_i\left(u + \tfrac{1}{2}V^2 + gz\right) - \sum_{out} \dot{m}_e\left(u + \tfrac{1}{2}V^2 + gz\right)$$

Basic Equations in Aerodynamics



Review of Vector Relations

$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$, $P = P(x, y, z)$

Gradient of a scalar field:

$$\nabla \cdot P = \frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k} \quad \left(\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)$$

Divergence of a vector field:

$$\nabla \cdot \vec{V} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(V_x \vec{i} + V_y \vec{j} + V_z \vec{k}\right)$$

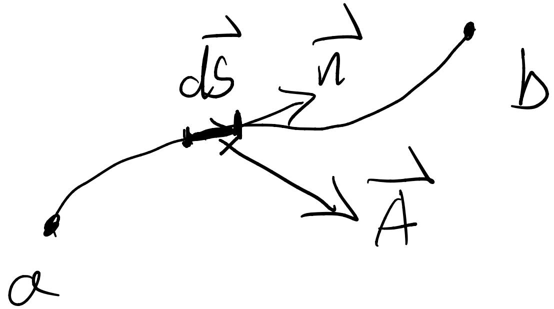
$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Curl of a vector field: $\nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$

Review of Vector Relations

Line Integrals: $\vec{A} = \vec{A}(x, y, z) = \vec{A}(r, \theta, z) = \vec{A}(r, \theta, \phi)$

Consider a curve "C" in space connecting points a and b

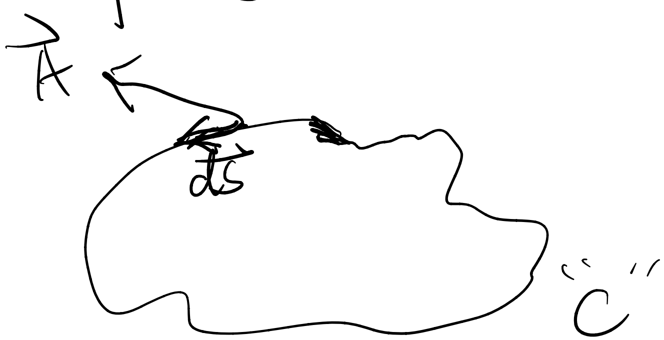


$$\vec{ds} = \vec{n} \cdot ds$$

The line integral of $\vec{A}$ along curve "C"

$$\oint_a^b \vec{A} \, \vec{ds}$$

If "C" is closed



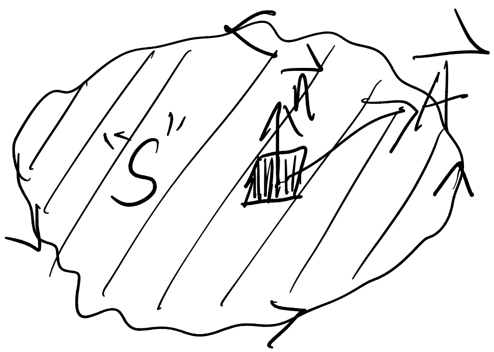
$$\oint_C \vec{A} \, \vec{ds}$$

☆: Counterclockwise direction is positive.

Review of Vector Relations

Surface integrals :

Consider an open surface "S" bounded by closed curve "C"



$$d\vec{S} = \vec{n} \cdot dS$$

$\iint_S \vec{A} \cdot d\vec{S}$ — surface integral of vector A

$\iint_S P \, d\vec{S}$ — surface integral of scalar P

If the surface is closed:

$$\oiint_S \vec{A} \, d\vec{S}$$

$$\oiint_S P \, d\vec{S}$$

Review of Vector Relations

Volume Integral

Consider a volume $\forall$ in space

$\iiint_{\forall} \vec{A} \, d\forall$ ——— volume integral of a vector $\vec{A}$

$\iiint_{\forall} \rho \, d\forall$ ——— volume integral of a Scalar ρ

Review of Vector Relations

Stokes' theorem:

$$\oint_C \vec{A}\, d\vec{s} = \iint_S (\nabla \times \vec{A})\, d\vec{S}$$

Divergence theorem:

$$\oiint_S \vec{A} \cdot d\vec{S} = \iiint_V (\nabla \cdot \vec{A})\, dV$$

Gradient theorem:

$$\oiint_S p\, d\vec{S} = \iiint_V (\nabla p)\, dV$$

Continuity Equation

Elemental mass flow rate across the elemental area:

$$d\dot{m} = \rho \vec{V} \cdot \vec{dS} = \rho V_n dS$$

Mass flowrate across the surface: $\oiint_S \rho \vec{V} \cdot \vec{dS}$

The total mass inside the volume $\mathcal{V}$: $\iiint_{\mathcal{V}} \rho \, d\mathcal{V}$

Net mass flow out of the control volume = time rate of decrease of mass inside of control volume

$$\oiint_S \rho \vec{V} \cdot \vec{dS} = -\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \, d\mathcal{V}$$

$$\Rightarrow \frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \, d\mathcal{V} + \oiint_S \rho \vec{V} \cdot \vec{dS} = 0$$

Continuity Equation

Continuity Equation

$$\left\{\begin{array}{l}\oiint_S \rho \vec{V} \cdot d\vec{S} = \iiint_{\mathcal{V}} \nabla \cdot (\rho \vec{V})\, d\mathcal{V} \\ \dfrac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho\, d\mathcal{V} = \iiint_{\mathcal{V}} \dfrac{\partial \rho}{\partial t}\, d\mathcal{V}\end{array}\right.$$

$$\Rightarrow \iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\mathcal{V} + \iiint_{\mathcal{V}} \nabla \cdot (\rho \vec{V})\, d\mathcal{V} = 0$$

$$\Rightarrow \iiint_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right] d\mathcal{V} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

Continuity Equation

Momentum Equation

Force = $\underline{\text{time rate of change of momentum}}$

Body force

Surface force

Net flow of momentum out of control volume

Time rate of change of momentum inside $\mathcal{V}$

Body force: $\iiint_{\mathcal{V}} \rho \vec{f}\, d\mathcal{V}$

Surface force:
- pressure: $-\oiint_S p\, d\vec{S}$
- viscous force: $\vec{F}_{viscous}$

$$\vec{F} = \iiint_{\mathcal{V}} \rho \vec{f}\, d\mathcal{V} - \oiint_S p\, d\vec{S} + \vec{F}_{viscous}$$

Net rate of momentum out of control volume: $\oiint_S dm \vec{V} = \oiint_S (\rho \vec{V} \cdot d\vec{S}) \vec{V}$

Momentum Equation

Time rate of change of momentum inside $\mathcal{V}$:

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho\vec{V}\,d\mathcal{V}$$

$$\Rightarrow \frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho\vec{V}\,d\mathcal{V} + \oiint_{S} (\rho\vec{V}\cdot d\vec{S})\vec{V} = -\oiint_{S} p\,d\vec{S} + \iiint_{\mathcal{V}} \rho\vec{f}\,d\mathcal{V} + \vec{F}_{viscous}$$

$$-\oiint_{S} p\,d\vec{S} = -\iiint_{\mathcal{V}} \nabla p\,d\mathcal{V}$$

$$\Rightarrow \iiint_{\mathcal{V}} \frac{\partial(\rho\vec{V})}{\partial t}d\mathcal{V} + \oiint_{S} (\rho\vec{V}\cdot d\vec{S})\vec{V} = -\iiint_{\mathcal{V}} \nabla p\,d\mathcal{V} + \iiint_{\mathcal{V}} \rho\vec{f}\,d\mathcal{V} + \vec{F}_{viscous}$$

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

In-Class Quiz