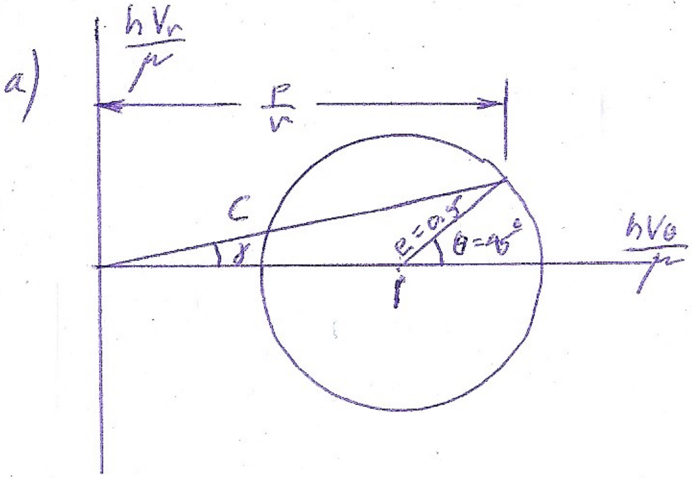
1) Using the velocity hodograph for an elliptic orbit of eccentricity $e = 0.5$, graphically determine the following quantities and verify their values using the appropriate formulas.

   a) The value of the flight path angle γ at $\theta = 45°$.
   b) The minimum value of the flight path angle on the orbit.
   c) The value of the radius r for part (a) in terms of the parameter p.
   d) The value of r for part (b).
   e) Will specification of the value of semimajor axis a determine numerical values to parts (c) and (d)?

a)



Using Law of Cosines

$$c^2 = (1)^2 + (0.5)^2 - 2(1)(0.5)\cos 135°$$

$$c = 1.399$$

Using Law of Sines

$$\frac{0.5}{\sin\gamma} = \frac{1.399}{\sin 135°}$$

$$\sin\gamma = 0.2527$$

$$\boxed{\gamma = 14.64°}$$

To verify

$$\tan\gamma = \frac{V_r}{V_\theta}$$

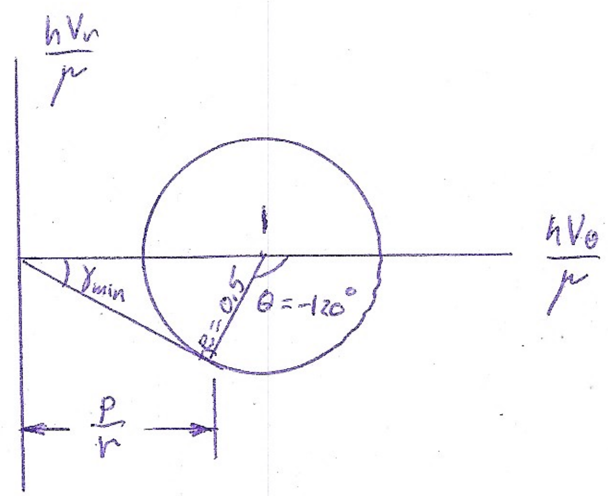
Using (5.26) & (5.27) from the notes

$$\tan\gamma = \frac{\dfrac{he\sin\theta}{a(1-e^2)}}{\dfrac{h(1+\cos\theta)}{a(1-e^2)}} = \frac{e\sin\theta}{1+e\cos\theta}$$

$$\tan\gamma = \frac{(0.5)\sin 45^\circ}{1+(0.5)\cos 45^\circ} = 0.2612$$

$$\boxed{\gamma = 14.64^\circ}$$

b)



γ_{min} occurs when $\theta = -120^\circ$

$$\sin\gamma_{min} = -\frac{0.5}{1} = -0.5$$

$$\boxed{\gamma_{min} = -30^\circ}$$

To verify, find γ_{min} by differentiating

$$\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta}$$

$$\sec^2\gamma \frac{d\gamma}{d\theta} = \frac{(1+e\cos\theta)e\cos\theta + e^2\sin^2\theta}{(1+e\cos\theta)^2} = 0$$

$$e\cos\theta + e^2\cos^2\theta + e^2\sin^2\theta = 0$$

$$e\cos\theta + e^2 = 0$$

$$\cos\theta = -e = -0.5 \Rightarrow \theta = -120°$$

$$\tan\gamma_{min} = \frac{e\sin\theta}{1+e\cos\theta} = \frac{(0.5)\sin(-120°)}{1+0.5\cos(-120°)}$$

$$= \frac{(0.5)(-0.8660)}{1+0.5(-0.5)} = -0.57735$$

$$\boxed{\gamma_{min} = -30°}$$

c) From the hodograph in part (a)

$$\frac{p}{r} = 1 + 0.5\cos 45° = 1.3536$$

$$\boxed{r = 0.7388\ p}$$

To verify, evaluate

$$r = \frac{p}{1+e\cos\theta} = \frac{p}{1+0.5\cos 45^\circ}$$

$$\boxed{r = 0.7388\,p}$$

d) From the hodograph in part (b)

$$\frac{p}{r} = 1-0.5\cos 60^\circ = 0.75$$

$$\boxed{r = 1.3333\,p}$$

To verify, evaluate

$$r = \frac{p}{1+e\cos\theta} = \frac{p}{1+0.5\cos(-120^\circ)}$$

$$\boxed{r = 1.3333\,p}$$

e) The results of parts (c) and (d) are expressed in terms of p. For an elliptic orbit, $p = a(1-e^2)$. Since $e = 0.5$, specifying "a" will determine a numerical value for p and thus numerical values to parts (c) and (d).

2) Curtis

4.3 Find the orbital elements of a geocentric satellite whose inertial position and velocity vectors in a geocentric equatorial frame are

$$\mathbf{r} = 2500\hat{\mathbf{I}} + 16{,}000\hat{\mathbf{J}} + 4000\hat{\mathbf{K}}\ (\text{km})$$
$$\mathbf{v} = -3\hat{\mathbf{I}} - \hat{\mathbf{J}} + 5\hat{\mathbf{K}}\ (\text{km/s})$$

{Ans.: $e = 0.4658$, $h = 98{,}623\ \text{km}^2/\text{s}$, $i = 62.52°$, $\Omega = 73.74°$, $\omega = 22.08°$, $\theta = 353.6°$}

To do this problem, see algorithm on pp. 191-193 in the text. Also find the semi-major axis a and time of perigee passage τ (in terms of t). Often, as in the class notes, a and τ are used as orbital elements in place of h and θ.

$$\bar{r} = 2500\hat{i} + 16{,}000\hat{j} + 4000\hat{k} \quad (\text{km})$$
$$\bar{V} = -3\hat{i} - \hat{j} + 5\hat{k} \quad (\text{km/sec})$$

1. $r = |\bar{r}| = \sqrt{(2500)^2 + (16{,}000)^2 + (4000)^2} = 16{,}681 \text{ km}$

2. $V = |\bar{V}| = \sqrt{(-3)^2 + (-1)^2 + (5)^2} = 5.9161 \text{ km/sec}$

3. $\bar{r} \cdot \bar{V} = (2500)(-3) + (16{,}000)(-1) + (4000)(5) = -3500 \text{ km}^2/\text{sec}$

$$\bar{r} \cdot \frac{d\bar{r}}{dt} = r\frac{dr}{dt} \Rightarrow \bar{r} \cdot \bar{V} = r V_r \Rightarrow V_r = \frac{\bar{r} \cdot \bar{V}}{r} = \frac{-3500}{16{,}681} = -0.20982 \frac{\text{km}}{\text{sec}}$$

$V_r < 0 \Rightarrow$ satellite is flying toward perigee

4. $\bar{h} = \bar{r} \times \bar{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2500 & 16{,}000 & 4{,}000 \\ -3 & -1 & 5 \end{vmatrix}$

$$= \hat{i}\left[(16{,}000)(5) - (4000)(-1)\right] - \hat{j}\left[(2500)(5) - (4000)(-3)\right] + \hat{k}\left[(2500)(-1) - (16{,}000)(-3)\right]$$

$$= 84{,}000\hat{i} - 24{,}500\hat{j} + 45{,}500\hat{k} \quad (\text{km}^2/\text{sec})$$

5. $h = |\bar{h}| = \sqrt{(84{,}000)^2 + (-24{,}500)^2 + (45{,}500)^2} = \underline{\underline{98{,}623 \text{ km}^2/\text{sec}}}$

6. $i = \cos^{-1}\left(\frac{h_z}{h}\right) = \cos^{-1}\left(\frac{45,500}{48,623}\right) = \underline{\underline{62.526^\circ}}$

$0 < i < 90^\circ \Rightarrow$ orbit is posigrade

7. $\bar{n} = \hat{k} \times \bar{h} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ 84,000 & -24,500 & 45,500 \end{vmatrix} =$

$= \hat{i}\left[(0)(45,500) - (1)(-24,500)\right] - \hat{j}\left[(0)(45,500) - (1)(84,000)\right] + \hat{k}\left[0\right]$

$= 24,500\,\hat{i} + 84,000\,\hat{j}$ (km²/sec)

8. $n = |\bar{n}| = \sqrt{(24,500)^2 + (84,000)^2} = 87,500$ km²/sec

9. $\Omega = \cos^{-1}\left(\frac{n_x}{n}\right) = \cos^{-1}\left(\frac{24,500}{87,500}\right) = \underline{\underline{73.740^\circ}}$

If $n_y > 0$ $0 \le \Omega < 180^\circ$. If $n_y < 0$ $180^\circ \le \Omega < 360^\circ$

Since $n_y = 84,000 > 0 \Rightarrow 0^\circ < \Omega < 180^\circ$

10. Using (5.9) in the notes

$$\frac{d\bar{v}}{dt} \times \bar{h} = \mu\left[\frac{\bar{r}}{r} + \bar{e}\right] \quad (5.9)$$

Replace $\frac{d\bar{v}}{dt}$ by $\bar{V}$ and solve for $\bar{e}$

$$\bar{e} = \frac{1}{\mu}\left[\bar{V} \times \bar{h} - \mu\frac{\bar{r}}{r}\right] = \frac{1}{\mu}\left[\bar{V} \times (\bar{r} \times \bar{V}) - \mu\frac{\bar{r}}{r}\right]$$

Using the identity $\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C}$

$$\bar{e} = \frac{1}{\mu}\left[(\bar{V} \cdot \bar{V})\bar{r} - (\bar{V} \cdot \bar{r})\bar{V} - \mu\frac{\bar{r}}{r}\right]$$

$$\bar{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\bar{r} - (\bar{r}\cdot\bar{v})\bar{v}\right]$$

$$= \frac{1}{3.986\times10^5}\left[\left((5.9161)^2 - \frac{3.986\times10^5}{16,681}\right)(2500\hat{i} + 16,000\hat{j} + 4000\hat{k}) - (-3500)(-3\hat{i} - \hat{j} + 5\hat{k})\right]$$

$$= 0.0433065\hat{i} + 0.436971\hat{j} + 0.155342\hat{k}$$

11. $e = |\bar{e}| = \sqrt{(0.0433065)^2 + (0.436971)^2 + (0.155342)^2} = \underline{\underline{0.46578}}$

12. $\omega = \cos^{-1}\left(\frac{\bar{n}\cdot\bar{e}}{ne}\right) = \cos^{-1}\left[\frac{(24,500)(0.0433065) + (84,000)(0.436971) + 0}{(87,500)(0.46578)}\right]$

$= 22.081°$ If $e_z \geq 0$ $0 \leq \omega \leq 180°$. If $e_z < 0$ $180° < \omega < 360°$

Since $e_z = 0.155342 > 0$ $\omega = \underline{\underline{22.081°}}$

13. $\bar{r}\cdot\bar{e} = re\cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{\bar{r}\cdot\bar{e}}{re}\right)$

$$\theta = \cos^{-1}\left[\frac{(2500)(0.0433065) + (16,000)(0.436971) + (4000)(0.155342)}{(16,681)(0.46578)}\right] = 6.4056°$$

If $\bar{r}\cdot\bar{V} \geq 0$ $0 \leq \theta \leq 180°$ If $\bar{r}\cdot\bar{V} < 0$ $180 < \theta < 360°$

Since $\bar{r}\cdot\bar{V} = -3500 < 0$ $\theta = 360° - 6.4056° = \underline{\underline{353.59°}}$

Find a

$$\frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \Rightarrow a = \frac{r}{2 - \frac{rV^2}{\mu}} = \frac{16,681}{2 - \frac{16,681(5.9161)^2}{3.986\times10^5}} = \underline{\underline{31,163 \text{ km}}}$$

Find τ (in terms of t)

Given that at time t the true anomaly is θ,

$$\tan\frac{E}{2} = \left(\frac{1-e}{1+e}\right)^{1/2} \tan\frac{\theta}{2} \quad \text{gives } E$$

$$M = E - e\sin E \quad \text{gives } M$$

Then from $M = n(t-\tau)$

$$\boxed{\tau = t - \frac{M}{n}} \quad \text{where } n = \sqrt{\frac{\mu}{a^3}}$$

3) Curtis

4.14 At time t_0 (relative to perigee passage), the position **r** and velocity **v** of a satellite in the geocentric equatorial frame are

$$\mathbf{r} = -5000\hat{\mathbf{I}} - 8000\hat{\mathbf{J}} - 2100\hat{\mathbf{K}}\ (\text{km})$$
$$\mathbf{v} = -4\hat{\mathbf{I}} + 3.5\hat{\mathbf{J}} - 3\hat{\mathbf{K}}\ (\text{km/s})$$

Find **r** and **v** at time t_0 + 50 min.
{Ans.: $\mathbf{r} = -1717\hat{\mathbf{I}} + 7604\hat{\mathbf{J}} - 2101\hat{\mathbf{K}}$ (km); $\mathbf{v} = 6.075\hat{\mathbf{I}} + 1.925\hat{\mathbf{J}} + 3.591\hat{\mathbf{K}}$ (km/s)}

$$\bar{r}_o = -5000\hat{i} - 8000\hat{j} - 2100\hat{k} \quad (\text{km})$$

$$\bar{V}_o = -4\hat{i} + 3.5\hat{j} - 3\hat{k} \quad (\text{km/sec})$$

$$r_o = |\bar{r}_o| = \sqrt{(-5000)^2 + (-8000)^2 + (-2100)^2} = 9664.9 \text{ km}$$

$$V_o = |\bar{V}_o| = \sqrt{(-4)^2 + (3.5)^2 + (-3)^2} = 6.1033 \text{ km/sec}$$

$$\bar{r}_o \cdot \bar{V}_o = (-5000)(-4) + (-8000)(3.5) + (-2100)(-3) = -1700 \text{ km}^2/\text{sec}$$

$$\sigma_o = \frac{\bar{r}_o \cdot \bar{V}_o}{\sqrt{\mu}} = \frac{-1700}{\sqrt{3.986 \times 10^5}} = -2.69265 \text{ km}^{1/2}$$

$$\bar{h} = \bar{r}_o \times \bar{V}_o = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5000 & -8000 & -2100 \\ -4 & 3.5 & -3 \end{vmatrix}$$

$$= \hat{i}\left[(-8000)(-3) - (-2100)(3.5)\right] - \hat{j}\left[(-5000)(-3) - (-2100)(-4)\right]$$
$$+ \hat{k}\left[(-5000)(3.5) - (-8000)(-4)\right]$$

$$= 31{,}350\hat{i} - 6600\hat{j} - 49{,}500\hat{k} \quad (\text{km}^2/\text{sec})$$

$$h = |\bar{h}| = \sqrt{(31{,}350)^2 + (-6600)^2 + (-49{,}500)^2} = 58{,}963 \text{ km}^2/\text{sec}$$

$$p = \frac{h^2}{\mu} = \frac{(58,963)^2}{3.986 \times 10^5} = 8722.1 \text{ km}$$

$$\mathcal{E} = \frac{V_0^2}{2} - \frac{\mu}{r_0} = \frac{(6.1033)^2}{2} - \frac{3.986 \times 10^5}{9664.9} = -22.6171 \text{ km}^2/\text{sec}^2$$

$\mathcal{E} < 0 \Rightarrow$ orbit is elliptic

$$\mathcal{E} = -\frac{\mu}{2a} \Rightarrow a = -\frac{\mu}{2\mathcal{E}} = -\frac{3.986 \times 10^5}{2(-22.6171)} = 8811.9 \text{ km}$$

$$p = a(1-e^2) \Rightarrow e = \sqrt{1 - \frac{p}{a}} = \sqrt{1 - \frac{8722.1}{8811.9}} = 0.10096$$

$$\bar{r}_0 \cdot \frac{d\bar{r}_0}{dt} = r_0 \frac{dr_0}{dt} \Rightarrow \bar{r}_0 \cdot \bar{V} = r_0 V_{r_0} \Rightarrow V_{r_0} = \frac{\bar{r}_0 \cdot \bar{V}_0}{r_0} = \frac{-1700}{9664.9} = -0.17589 \text{ km/sec}$$

since $V_{r_0} < 0$, satellite is flying toward perigee ($\pi \le \theta_0 < 2\pi$)

$$r_0 = \frac{p}{1 + e\cos\theta_0} \Rightarrow \theta_0 = \cos^{-1}\left(\frac{p - r_0}{r_0 e}\right) \text{ (principal value)}$$

$$\theta_0 = \begin{cases} \cos^{-1}\left(\frac{p - r_0}{r_0 e}\right) & \text{if } V_r \ge 0 \\ 2\pi - \cos^{-1}\left(\frac{p - r_0}{r_0 e}\right) & \text{if } V_r < 0 \end{cases}$$

$$\theta_0 = 2\pi - \cos^{-1}\left(\frac{p - r_0}{r_0 e}\right) = 2\pi - \cos^{-1}\left(\frac{8722.1 - 9664.9}{(9664.9)(0.10096)}\right) = 3.40228 \text{ rad} = 194.94^\circ$$

$$E_0 = 2\tan^{-1}\left[\left(\frac{1-e}{1+e}\right)^{1/2} \tan\left(\frac{\theta_0}{2}\right)\right] = 2\tan^{-1}\left[\left(\frac{1 - 0.10096}{1 + 0.10096}\right)^{1/2} \tan\left(\frac{3.40228}{2}\right)\right] = -2.8535 \text{ rad}$$

$$M_0 = E_0 - e\sin E_0 = -2.8535 - (0.10096)\sin(-2.8535) = -2.8248 \text{ rad}$$

$$n = \sqrt{\frac{\mu}{a^3}} = \sqrt{\frac{3.986 \times 10^5}{(8811.9)^3}} = 7.63246 \times 10^{-4} \text{ sec}^{-1}$$

$$t_0 = \frac{M_0}{n} = \frac{-2.8248}{7.63246 \times 10^{-4}} = -3701.1 \text{ sec}$$ (negative sign means time until perigee passage)

At $t_0 + 50$ min

$$t = t_0 + 50\,\text{min} = -3701.1\,\text{sec} + 50\,\text{min}\left(\frac{60\,\text{sec}}{1\,\text{min}}\right) = -701.1\,\text{sec}$$

$$M = nt = (7.63246 \times 10^{-4})(-701.1) = -0.53509\,\text{rad}$$

$$f(E) = E - e\sin E - M = E - (0.10096)\sin E + 0.53509 = 0$$

$$f'(E) = 1 - (0.10096)\cos E$$

$$E_{n+1} = E_n - \frac{f(E_n)}{f'(E_n)} = E_n - \frac{E_n - (0.10096)\sin E_n + 0.53509}{1 - (0.10096)\cos E_n}$$

n	E_n
0	-0.53509
1	-0.59147
2	-0.59138
3	-0.59138

$E = -0.59138$ rad

$$\theta = 2\tan^{-1}\left[\left(\frac{1+e}{1-e}\right)^{1/2}\tan\left(\frac{E}{2}\right)\right] = 2\tan^{-1}\left[\left(\frac{1+0.10096}{1-0.10096}\right)^{1/2}\tan\left(\frac{-0.59138}{2}\right)\right] =$$

$$= -0.650256\,\text{rad} = -37.257^\circ$$

$$r = \frac{p}{1+e\cos\theta} = \frac{8722.1}{1+(0.10096)\cos(-0.650256)} = 8073.38\,\text{km}$$

Lagrange coefficients

$$F = 1 - \frac{r}{p}\left(1-\cos(\theta-\theta_0)\right) = 1 - \frac{8073.38}{8722.1}\left(1-\cos(-0.650256-3.40228)\right)$$

$$= -0.493043$$

$$G = \frac{r r_0}{\sqrt{\mu p}}\sin(\theta-\theta_0) = \frac{(8073.38)(9664.9)}{\sqrt{(3.986\times10^5)(8722.1)}}\sin(-0.650256-3.40228)$$

$$= 1045.53$$

$$F_t = \frac{\sqrt{\mu}}{r_0 p}\left[\sigma_0\left(1-\cos(\theta-\theta_0)\right) - \sqrt{p}\,\sin(\theta-\theta_0)\right]$$

$$= \frac{\sqrt{3.986\times10^5}}{(9664.9)(8722.1)}\Big[(-2.69265)\left(1-\cos(-0.650256-3.40228)\right)$$

$$-\sqrt{8722.1}\,\sin(-0.650256-3.40228)\Big]$$

$$= -5.85149\times10^{-4}$$

$$G_t = 1-\frac{r_0}{p}\left(1-\cos(\theta-\theta_0)\right) = 1-\frac{9664.9}{8722.1}\left(1-\cos(-0.650256-3.40228)\right)$$

$$= -0.787368$$

$$\bar{r} = F\bar{r}_0 + G\bar{V}_0 = (-0.493043)(-5000\hat{i}-8000\hat{j}-2100\hat{k})$$

$$+(1045.53)(-4\hat{i}+3.5\hat{j}-3\hat{k})$$

$$\boxed{\bar{r} = -1717\hat{i}+7604\hat{j}-2101\hat{k} \quad (\text{km})}$$

$$\bar{V} = F_t\bar{r}_0 + G_t\bar{V}_0 = (-5.85149\times10^{-4})(-5000\hat{i}-8000\hat{j}-2100\hat{k})$$

$$+(-0.787368)(-4\hat{i}+3.5\hat{j}-3\hat{k})$$

$$\boxed{\bar{V} = 6.075\hat{i}+1.925\hat{j}+3.591\hat{k} \quad (\text{km/sec})}$$