ME 57200 Aerodynamic Design

Lecture #22: Shock Waves

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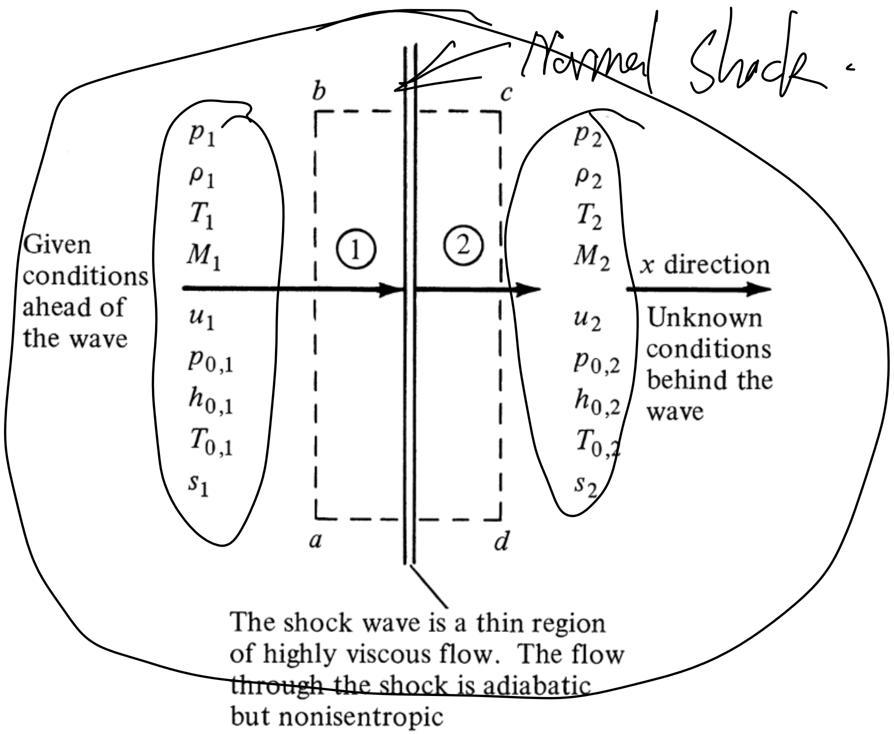
Shock Waves

Normal Shock Waves

- Given the flow properties upstream of the wave, how to calculate the flow properties downstream of the wave?



- The flow is steady
- The flow is adiabatic
- There are no viscous effects on the sides of the control volume
- There are no body forces

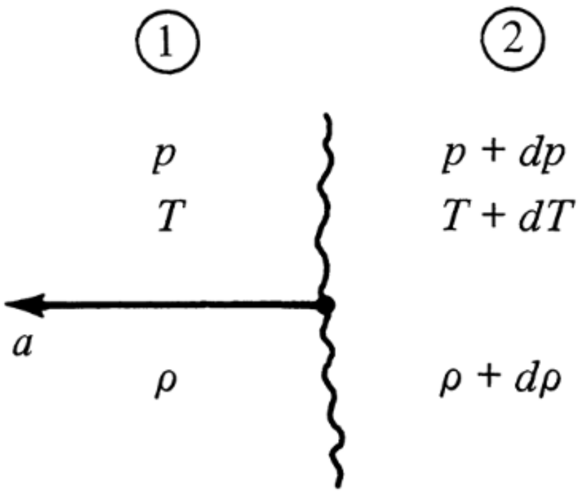
Shock Waves

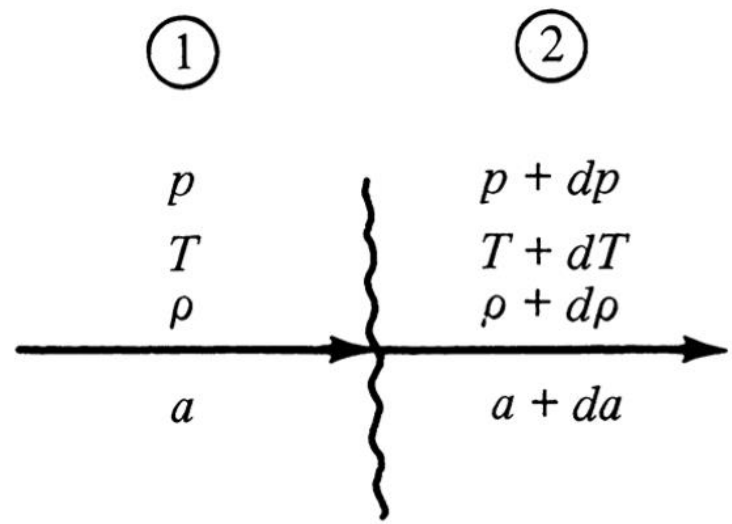
Normal Shock Waves

- Continuity equation $$\boxed{\rho_1 u_1 = \rho_2 u_2}$$
- Momentum equation $$\boxed{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2}$$
- Energy equation $$\boxed{h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}}$$
- Enthalpy $$h_2 = c_p T_2$$
- Equation of State $$p_2 = \rho_2 R T_2$$

Speed of Sound

Consider a sound wave propagating through a gas with velocity a. (from right to left)





Speed of Sound

- Assume the gas is calorically perfect, isentropic relation can be applied

$$\frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma} \Longrightarrow p = c\rho^{\gamma}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = c\gamma\rho^{\gamma-1}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \left(\frac{p}{\rho^{\gamma}}\right)\gamma\rho^{\gamma-1} = \frac{\gamma p}{\rho}$$

$$\boxed{a = \sqrt{\frac{\gamma p}{\rho}}} \xrightarrow{\frac{p}{\rho} = RT} \boxed{a = \sqrt{\gamma RT}}$$

- The speed of sound in a calorically perfect gas is a function of temperature only

Speed of Sound

Relation between the speed of sound and the compressibility of a gas

$$a = \sqrt{\frac{1}{\rho \tau_s}}$$

- The lower the compressibility, the higher the speed of sound.
    - *The speed of sound in a theoretically incompressible fluid is infinite*
    - *In turn, for an incompressible flow with finite velocity, V, the Mach number, M = V/a, is zero.*

Speed of Sound

Mach Number: consider a fluid element moving along a streamline, the ratio between the kinetic and internal energies is

$$\frac{V^2/2}{e} = \frac{V^2/2}{c_v T} = \frac{V^2/2}{RT/(\gamma-1)} = \frac{(\gamma/2)V^2}{a^2/(\gamma-1)} = \frac{\gamma(\gamma-1)}{2} M^2$$

- The square of Mach number is proportional to the ratio of kinetic energy to internal energy of a gas flow.
- The Mach number is a measure of the directed motion of the gas compared with the random thermal motion of the molecules.

Spatial Forms of Energy Equation

$$\frac{\gamma R T_1}{\gamma - 1} + \frac{u_1^2}{2} = \frac{\gamma R T_2}{\gamma - 1} + \frac{u_2^2}{2}$$

$$a = \sqrt{\gamma R T},$$

$$\boxed{\frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma - 1} + \frac{u_2^2}{2}}$$

At the stagnation point, the stagnation speed of sound is a_0

$$\boxed{\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a_0^2}{\gamma - 1}}$$

Spatial Forms of Energy Equation

$$\boxed{\frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma - 1} + \frac{u_2^2}{2}}$$

In a sonic flow, where, $u = a^*$

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a^{*2}}{\gamma - 1} + \frac{a^{*2}}{2}$$

$$\boxed{\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2}}$$

$$\boxed{\frac{\gamma + 1}{2(\gamma - 1)} a^{*2} = \frac{a_0^2}{\gamma - 1} = \text{const}}$$

Spatial Forms of Energy Equation

$$\boxed{c_p T + \frac{u^2}{2} = c_p T_0}$$

$$\frac{T_0}{T} = 1 + \frac{u^2}{2c_p T}$$

$$\frac{T_0}{T} = 1 + \frac{u^2}{2\gamma RT/(\gamma - 1)} = 1 + \frac{u^2}{2a^2/(\gamma - 1)}$$

$$\boxed{\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2}$$

Only the Mach number dictates the ratio of total temperature to static temperature.

Spatial Forms of Energy Equation

For isentropic compression of the flow to zero velocity

$$\frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\gamma/(\gamma-1)}$$

$$\boxed{\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)}}$$

$$\boxed{\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)}}$$

Hence, for a given gas (i.e., given γ), the ratios T_0/T, p_0/p, and ρ_0/ρ depend only on Mach number.

Spatial Forms of Energy Equation

For a sonic flow, M = 1

$$\boxed{\frac{T^*}{T_0} = \frac{2}{\gamma + 1}}$$

$$\boxed{\frac{p^*}{p_0} = \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)}}$$

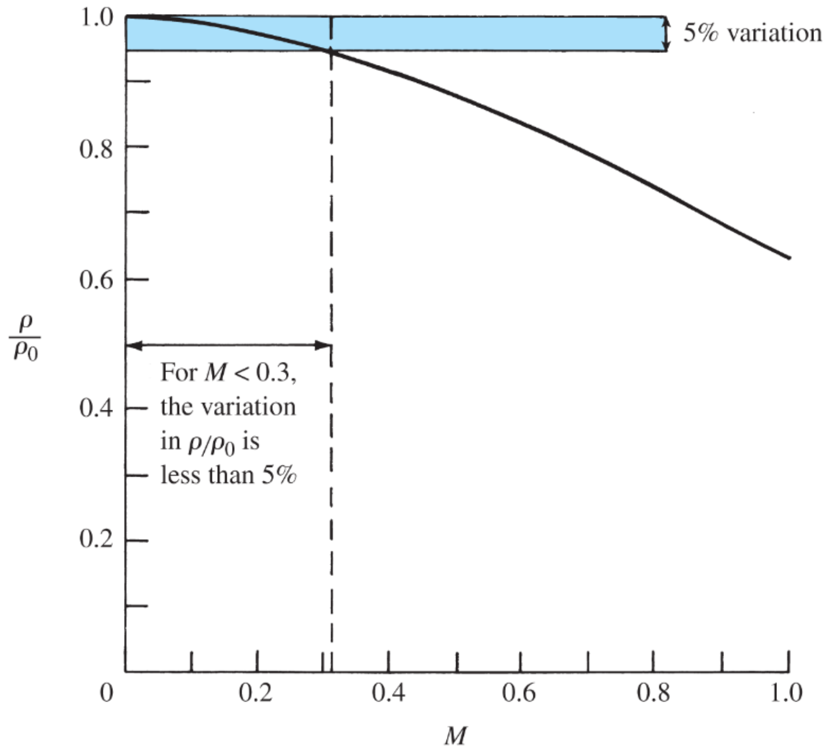
$$\boxed{\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma + 1}\right)^{1/(\gamma-1)}}$$

For $\gamma = 1.4$, these ratios are $\frac{T^*}{T_0} = 0.833$ $\frac{p^*}{p_0} = 0.528$ $\frac{\rho^*}{\rho_0} = 0.634$

Spatial Forms of Energy Equation

When is a flow compressible?

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{1/(\gamma-1)}$$



Characteristic Mach Number

$$M^* \equiv \frac{u}{a^*}$$

where a^* is the value of the speed of sound at sonic conditions, *not* the actual local value.

The value of a^* is given by $a^* = \sqrt{\gamma R T^*}$.

$$\boxed{\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2}}$$

$$\frac{(a/u)^2}{\gamma - 1} + \frac{1}{2} = \frac{\gamma + 1}{2(\gamma - 1)} \left(\frac{a^*}{u} \right)^2$$

$$\frac{(1/M)^2}{\gamma - 1} = \frac{\gamma + 1}{2(\gamma - 1)} \left(\frac{1}{M^*} \right)^2 - \frac{1}{2}$$

$$\boxed{M^2 = \frac{2}{(\gamma + 1)/M^{*2} - (\gamma - 1)}}$$

$$\boxed{M^{*2} = \frac{(\gamma + 1) M^2}{2 + (\gamma - 1) M^2}}$$

Characteristic Mach Number

$$\boxed{M^{*2} = \frac{(\gamma + 1)M^2}{2 + (\gamma - 1)M^2}}$$

$$\begin{aligned} M^* &= 1 \quad \text{if } M = 1 \\ M^* &< 1 \quad \text{if } M < 1 \\ M^* &> 1 \quad \text{if } M > 1 \\ M^* &\rightarrow \sqrt{\tfrac{\gamma+1}{\gamma-1}} \quad \text{if } M \rightarrow \infty \end{aligned}$$

Shock Waves

Normal Shock Waves

- Continuity equation $$\boxed{\rho_1 u_1 = \rho_2 u_2}$$
- Momentum equation $$\boxed{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2}$$
- Energy equation $$\boxed{h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}}$$
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Shock Waves

Momentum Equation: $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$

$$\frac{p_1}{\rho_1 u_1} + u_1 = \frac{p_2}{\rho_2 u_2} + u_2$$

$$\Rightarrow \frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = u_2 - u_1$$

$$a = \sqrt{\gamma R T} = \sqrt{\gamma \cdot \frac{p}{\rho}}$$

$$\Rightarrow \boxed{\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1}$$

Shock Waves

$$a_1^2 + \frac{\gamma-1}{2} u_1^2 = \frac{\gamma+1}{2} a^{*2}$$

$$\Rightarrow \begin{cases} \underline{a_1^2} = \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} u_1^2 \\ \underline{a_2^2} = \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} u_2^2 \end{cases}$$

$$\Rightarrow \left[\frac{\gamma+1}{2} \cdot \frac{a^{*2}}{\gamma u_1} - \frac{\gamma-1}{2\gamma} u_1\right] - \left[\frac{\gamma+1}{2} \frac{a^{*2}}{\gamma u_2} + \frac{\gamma-1}{2\gamma} u_2\right] = u_2 - u_1$$

$$\Rightarrow \frac{\gamma+1}{2\gamma u_1 u_2} \underline{(u_2 - u_1)} \cdot a^{*2} + \frac{\gamma-1}{2\gamma} \underline{(u_2 - u_1)} = \underline{u_2 - u_1}$$

Shock Waves

$$\Rightarrow \quad \frac{\gamma+1}{2\gamma u_1 u_2} a^{*2} + \frac{\gamma-1}{2\gamma} = 1$$

$$\Rightarrow \quad \boxed{a^{*2} = u_1 \cdot u_2}$$

$$1 = \frac{u_1}{a^*} \cdot \frac{u_2}{a^*}$$

$$\Rightarrow \quad 1 = M_1^* \cdot M_2^*$$

$$\Rightarrow \quad M_2^* = \frac{1}{M_1^*}$$

$$M^* = \frac{(\gamma+1) \cdot M^2}{2+(\gamma-1) M^2}$$

Shock Waves

$$\frac{(\gamma+1)\cdot M_2^2}{2+(\gamma-1)M_2^2} = \left[\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}\right]^{-1}$$

$$\Rightarrow \quad M_2^2 = \frac{1+[(\gamma-1)/2]M_1^2}{\gamma M_1^2-(\gamma-1)/2}$$

$$\Rightarrow \quad \boxed{M_2 = \sqrt{\frac{1+[(\gamma-1)/2]M_1^2}{\gamma M_1^2-(\gamma-1)/2}}}$$

$$\frac{\rho_2}{\rho_1}, \quad \frac{p_2}{p_1}, \quad T_2/T_1$$

Shock Waves

$$\rho_1 u_1 = \rho_2 u_2 \qquad \text{(Continuity Equation)}$$

$$\Rightarrow \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1 \cdot u_1}{u_2 \cdot u_1} = \frac{u_1^2}{u_1 \cdot u_2} = \frac{u_1^2}{a^{*2}} = \underline{M_1^{*2}}$$

$$\Rightarrow \boxed{\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1) M_1^2}{2+(\gamma-1) M_1^2}}$$

$$\underline{p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2} = \rho_1 u_1^2 - \rho_1 u_1 u_2 = \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

$$\frac{p_2 - p_1}{p_1} = \frac{\gamma \rho_1 u_1^2}{\gamma p_1} \left(1 - \frac{u_2}{u_1}\right)$$

Shock Waves

$$\frac{\frac{\gamma \rho_1 u_1^2}{\gamma p_1}}{} = \frac{\gamma u_1^2}{\gamma \frac{p_1}{\rho_1}} \qquad a_1^2 = \frac{\gamma p_1}{\rho_1}$$

$$\Rightarrow \gamma \frac{u_1^2}{a_1^2}$$

$$\Rightarrow \frac{p_2 - p_1}{p_1} = \gamma M_1^2 \left[1 - \frac{2 + (\gamma - 1) M_1^2}{(\gamma + 1) M_1^2} \right]$$

$$\Rightarrow \boxed{\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1} (M_1^2 - 1)}$$

Shock Waves

$$P = \rho R T$$

$$\Rightarrow \frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right) \cdot \left(\frac{\rho_1}{\rho_2}\right)$$

$$\boxed{\frac{T_2}{T_1} = \left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)\right] \cdot \left[\frac{2 + (\gamma-1)M_1^2}{(\gamma+1)M_1^2}\right]}$$

$$h = C_p \cdot T$$

$$\boxed{\frac{h_2}{h_1}} = \frac{T_2}{T_1}$$

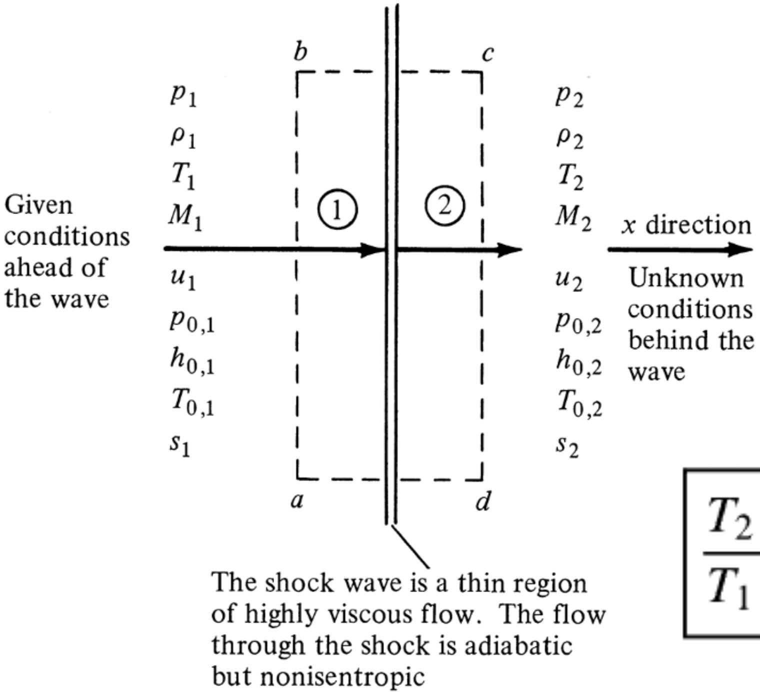
Shock Waves

- ρ_2/ρ_1, p_2/p_1, and T_2/T_1 are functions of the upstream Mach number M_1 only

M_2 is a function of M_1 only.

Shock Waves

Normal Shock Waves



$$\boxed{M_2^2 = \frac{1 + [(\gamma - 1)/2]M_1^2}{\gamma M_1^2 - (\gamma - 1)/2}}$$

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}}$$

$$\boxed{\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)}$$

$$\boxed{\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right] \frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}}$$

$$\boxed{T_{0,1} = T_{0,2}}$$

$$\boxed{s_2 - s_1 = -R \ln \frac{p_{0,2}}{p_{0,1}}}$$

Shock Waves

APPENDIX B

Normal Shock Properties

M	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{0_2}}{p_{0_1}}$	$\frac{p_{0_2}}{p_1}$	M_2
0.1000 + 01	0.1000 + 01	0.1000 + 01	0.1000 + 01	0.1000 + 01	0.1893 + 01	0.1000 + 01
0.1020 + 01	0.1047 + 01	0.1033 + 01	0.1013 + 01	0.1000 + 01	0.1938 + 01	0.9805 + 00
0.1040 + 01	0.1095 + 01	0.1067 + 01	0.1026 + 01	0.9999 + 00	0.1984 + 01	0.9620 + 00
0.1060 + 01	0.1144 + 01	0.1101 + 01	0.1039 + 01	0.9998 + 00	0.2032 + 01	0.9444 + 00
0.1080 + 01	0.1194 + 01	0.1135 + 01	0.1052 + 01	0.9994 + 01	0.2082 + 01	0.9277 + 00
0.1100 + 01	0.1245 + 01	0.1169 + 01	0.1065 + 01	0.9989 + 00	0.2133 + 01	0.9118 + 00
0.1120 + 01	0.1297 + 01	0.1203 + 01	0.1078 + 01	0.9982 + 00	0.2185 + 01	0.8966 + 00
0.1140 + 01	0.1350 + 01	0.1238 + 01	0.1090 + 01	0.9973 + 00	0.2239 + 01	0.8820 + 00

Example Practice:

Consider a normal shock wave in air where the upstream flow properties are u_1 = 680 m/s, T_1 = 288 K, and p_1 = 1 atom. Calculate the velocity, temperature, and pressure downstream of the shock.

Solution:

$$M_1 = \frac{u_1}{a_1}$$

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4 \cdot 287 \cdot 288} = 340 \, m/s$$

$$\Rightarrow M_1 = \frac{u_1}{a_1} = \frac{680}{340} = 2$$

$$P_2 = \frac{P_2}{P_1} \cdot P_1 = 4.5 \cdot (1 \text{ atm}) = 4.5 \text{ atm}$$

$$T_2 = \left(\frac{T_2}{T_1}\right) \cdot T_1 = 1.687 \, (288K) = 486 K$$

$$a_2 = \sqrt{\gamma R T_2} = 442 \, m/s$$

$$u_2 = M_2 \cdot a_2 = 0.5774 \cdot 442$$