10.14 Effects of the Earth's Oblateness on a Satellite Orbit

Equatorial radius 6378.1370 km

Polar radius 6356.7523 km

$$\text{Oblateness} = \frac{\text{Equatorial radius} - \text{Polar radius}}{\text{Equatorial radius}} \qquad (10.51)$$

$$= \frac{6378.1370 - 6356.7523}{6378.1370}$$

$$= 0.003353$$

Irregularities in a planet's gravitational field can be modelled using Potential Theory

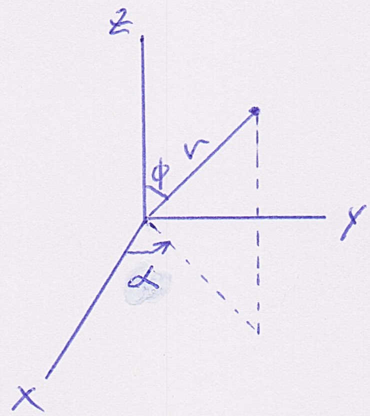
$$\bar{F}_{grav.} = \nabla U \qquad (10.52)$$

where U is the gravitational potential function [U is the negative of the potential energy] which satisfies

$$\nabla^2 U = 0 \qquad (10.53)$$

For a planet which is axisymmetric (about the z-axis) and has symmetry about the equatorial plane, the solution of (10.53) in spherical coordinates gives

$$U = \frac{\mu}{r}\left[1 - \sum_{K=2,4,6\ldots}^{\infty} \left(\frac{R}{r}\right)^{K} J_K \, P_K(\cos\phi)\right] \qquad (10.54)$$



where R = planet's equatorial radius

J_K = constant coefficients called <u>gravitational constants</u> or <u>J-numbers</u>

P_K = Legendre polynomial function of order K.

ϕ = colatitude ($=90^\circ -$ latitude)

$\frac{\mu}{r}$ = gravitational potential for a spherical planet.

The remaining terms account for latitude-dependent variations in the gravitational field. The first term in the series ($k=2$), called the oblateness term, is the largest term in the series being 1000 times as large as any of the other terms of the series

$$U_{J_2} = -\frac{\mu}{r}\left(\frac{R}{r}\right)^2 J_2 \; P_2(\cos\phi)$$

$$= -\frac{\mu R^2}{r^3} J_2 \cdot \left[\frac{1}{2}(2 - 3\sin^2\phi)\right]$$

Since $\sin^2\phi = 1 - \cos^2\phi = 1 - \frac{z^2}{r^2}$

$$U_{J_2} = -\frac{\mu R^2}{r^3} J_2 \cdot \left[\frac{1}{2}\left(\frac{3z^2}{r^2} - 1\right)\right] \qquad (10.55)$$

Planet	Oblateness	J_2
Mercury	0.000	60×10^{-6}
Venus	0.000	4.458×10^{-6}
Earth	0.003353	1.08263×10^{-3}
Mars	0.00648	1.96045×10^{-3}
Jupiter	0.06487	14.736×10^{-3}
Saturn	0.09796	16.298×10^{-3}
Uranus	0.02293	3.34343×10^{-3}
Neptune	0.01708	3.411×10^{-3}
(Earth's moon)	0.0012	202.7×10^{-6}

The perturbing force per unit mass acting on the satellite due to oblateness is

$$\overline{T}_{obl} = \nabla U_{J_2} = \frac{\partial U_{J_2}}{\partial r}\hat{i}_r + \frac{\partial U_{J_2}}{\partial z}\hat{i}_z$$

$$= -\mu J_2 R^2 \left[\left(\frac{3}{2r^4} - \frac{15z^2}{2r^6}\right)\hat{i}_r + \frac{3z}{r^5}\hat{i}_z\right]$$

The above equation may be written in terms of components and coordinates of the orbital reference frame. (See section (7.2)).

Since

$$\hat{i}_z = \sin i \sin(\omega+\theta)\hat{i}_r + \sin i \cos(\omega+\theta)\hat{i}_\theta + \cos i \, \hat{i}_h$$

and

$$z = r\cos\phi = r\sin i \sin(\omega+\theta)$$

the perturbing force per unit mass can be written as

$$\overline{T}_{obl} = T_r\hat{i}_r + T_\theta\hat{i}_\theta + T_h\hat{i}_h \qquad (10.56)$$

where

$$T_r = -\frac{\mu}{r^2}\,\frac{3}{2}\,J_2\left(\frac{R}{r}\right)^2\left[1-3\sin^2 i\,\sin^2(\omega+\theta)\right] \quad (10.56a)$$

$$T_\theta = -\frac{\mu}{r^2}\,\frac{3}{2}\,J_2\left(\frac{R}{r}\right)^2 \sin^2 i\,\sin\left[2(\omega+\theta)\right] \quad (10.56b)$$

$$T_h = -\frac{\mu}{r^2}\,\frac{3}{2}\,J_2\left(\frac{R}{r}\right)^2 \sin 2i\,\sin(\omega+\theta) \quad (10.56c)$$

If a force per unit mass $\overline{T} = T_r\hat{i}_r + T_\theta\hat{i}_\theta + T_h\hat{i}_h$ is applied to an orbiting satellite, the rate of change of the orbital elements is given by

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left[T_r\,e\sin\theta + T_\theta(1+e\cos\theta)\right] \quad (10.57)$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left\{T_r\sin\theta + \left[\frac{2\cos\theta + e\cos^2\theta + e}{1+e\cos\theta}\right]T_\theta\right\} \quad (10.58)$$

$$\frac{di}{dt} = \frac{\sqrt{1-e^2}}{na}\,\frac{\cos(\omega+\theta)}{1+e\cos\theta}\,T_h \quad (10.59)$$

$$\frac{d\Omega}{dt} = \frac{\sqrt{1-e^2}}{na}\,\frac{\sin(\omega+\theta)}{\sin i\,(1+e\cos\theta)}\,T_h \quad (10.60)$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nae}\left[\frac{\cos\theta}{1+e\cos\theta} T_r + \frac{(2+e\cos\theta)\sin\theta}{a(1-e^2)} T_\theta - \frac{e\sin(\omega+\theta)}{\tan i\,(1+e\cos\theta)} T_h\right] \qquad (10.61)$$

Notes:

1) With $T_r = T\sin\gamma = T\dfrac{e\sin\theta}{\sqrt{1+2e\cos\theta+e^2}}$

$$T_\theta = T\cos\gamma = T\frac{1+e\cos\theta}{\sqrt{1+2e\cos\theta+e^2}}$$

eqs. (10.57) and (10.58) reduce to (10.13) and (10.25), respectively.

2) $\dfrac{da}{dt}$ and $\dfrac{de}{dt}$ depend on T_r, T_θ (not T_h)

$\dfrac{di}{dt}$ and $\dfrac{d\Omega}{dt}$ depend only on T_h (not T_r & T_θ)

$\dfrac{d\omega}{dt}$ depends on T_r, T_θ, T_h

Substituting (10.56 a,b,c) into (10.57) – (10.61) gives the rate of change of the orbital elements due to the planet's oblateness.

Integrating $\dot{\Omega}$ over on complete orbit yields the average rate of change

$$\dot{\Omega}_{avg} = \frac{1}{\tau}\int_0^{\tau} \dot{\Omega}\, dt$$

where τ is the period of the orbit.

$$\boxed{\dot{\Omega}_{avg} = -\frac{3}{2}\frac{\sqrt{\mu}\, J_2 R^2}{(1-e^2)^2 a^{7/2}} \cos i} \qquad (10.62)$$

If $0° \leq i < 90°$, then $\dot{\Omega}_{avg} < 0 \Rightarrow$ for posigrade orbits, the node line drifts westward.

If $90° < i \leq 180°$, then $\dot{\Omega}_{avg} > 0 \Rightarrow$ for retrograde orbits, the node line drifts eastward.

If $i = 90°$ (polar orbit) $\Rightarrow$ the node line is stationary

Similarly, the time averaged rate of change of the argument of perigee $\dot{\omega}$ is

$$\boxed{\dot{\omega}_{avg} = -\frac{3}{2}\frac{\sqrt{\mu}\, J_2 R^2}{(1-e^2)^2 a^{7/2}}\left(\frac{5}{2}\sin^2 i - 2\right)} \quad (10.63)$$

If $0° \leq i < 63.4°$ or $116.6° < i < 180°$, then $\dot{\omega}_{avg} > 0$

$\Rightarrow$ perigee advances in the direction of motion of the satellite.

If $63.4° < i < 116.6°$, then $\dot{\omega}_{avg} < 0$

$\Rightarrow$ perigee regresses, moving opposite to the direction of motion.

If $i = 63.4°$ and $i = 116.6°$, then $\dot{\omega}_{avg} = 0$

$\Rightarrow$ These are critical inclinations where the apse line does not move.

The time-averaged rates of change for the inclination, eccentricity and semi-major axis are zero.

EXAMPLE

A spacecraft is in a 300 km × 400 km orbit inclined at 50°. Find the rates of node regression and perigee advance due to the earth's oblateness.

$$r_p = h_p + r_e = 300 + 6368 = 6668 \text{ km}$$

$$r_a = h_a + r_e = 400 + 6368 = 6768 \text{ km}$$

$$e = \frac{r_a - r_p}{r_a + r_p} = 0.007443$$

$$a = \frac{r_a + r_p}{2} = 6718 \text{ km}$$

From (10.62) the rate of node line regression is

$$\dot{\Omega}_{avg} = -\frac{3}{2} \frac{\sqrt{\mu}\, J_2 R^2}{(1-e^2)^2\, a^{7/2}} \cos i$$

$$= -\frac{3}{2} \frac{\sqrt{3.986 \times 10^5}\; 0.0010826\, (6378)^2}{(1-(0.007443)^2)^2\, (6718)^{7/2}} \cos 50^\circ$$

$$\dot{\Omega}_{avg} = -1.6784 \times 10^{-6} \cos 50^\circ$$

$$= -1.0789 \times 10^{-6} \text{ rad/sec}$$

or

$$\boxed{\dot{\Omega}_{avg} = 5.341^\circ \text{ per day to the west}}$$

From (10.63)

$$\dot{\omega}_{avg} = -\frac{3}{2} \frac{\sqrt{\mu}\, J_2 R^2}{(1-e^2)^2 a^{7/2}} \left(\frac{5}{2} \sin^2 i - 2 \right)$$

$$= -1.6784 \times 10^{-6} \left(\frac{5}{2} \sin^2 50^\circ - 2 \right)$$

$$= +8.9449 \times 10^{-7} \text{ rad/sec}$$

or

$$\boxed{\dot{\omega}_{avg} = 4.428^\circ \text{ per day in the flight direction}}$$