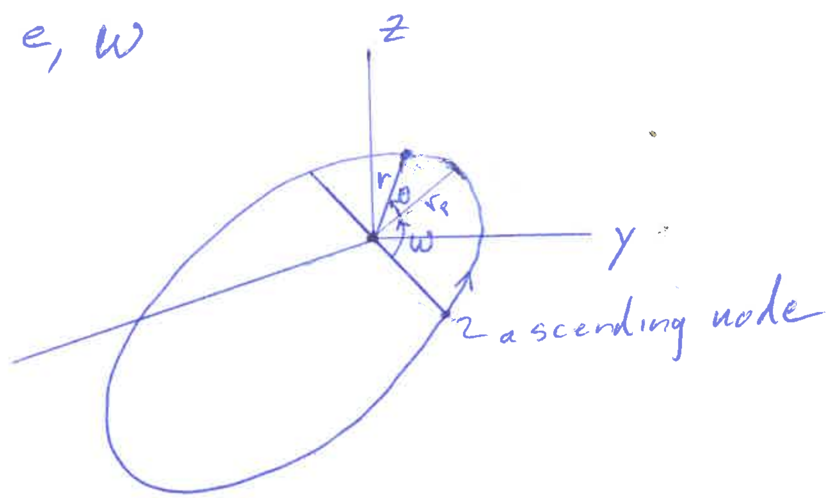
7.5. Orbit Determination

a) Orbit from 3 Coplanar Positions

Given (r_1, Θ_1), (r_2, Θ_2) and (r_3, Θ_3) where $\Theta = \omega + \theta$ (θ = true anomaly, ω = angular position of pericenter passage)

Find p, e, ω



$$r_i = \frac{p}{1 + e\cos\theta_i} = \frac{p}{1 + e\cos(\Theta_i - \omega)} \qquad i = 1, 2, 3 \qquad (7.39)$$

Three non-linear equations for 3 unknowns (p, e, ω)

To linearize, define

$$P = e\cos\omega \qquad Q = e\sin\omega \qquad (7.40)$$

$$\frac{p}{r_i} - P\cos\Theta_i - Q\sin\Theta_i = 1 \qquad i = 1, 2, 3 \qquad (7.41)$$

Solve for p, P and Q using Cramer's rule

$$p = \frac{\begin{vmatrix} 1 & -\cos\Theta_1 & -\sin\Theta_1 \\ 1 & -\cos\Theta_2 & -\sin\Theta_2 \\ 1 & -\cos\Theta_3 & -\sin\Theta_3 \end{vmatrix}}{\begin{vmatrix} \frac{1}{r_1} & -\cos\Theta_1 & -\sin\Theta_1 \\ \frac{1}{r_2} & -\cos\Theta_2 & -\sin\Theta_2 \\ \frac{1}{r_3} & -\cos\Theta_3 & -\sin\Theta_3 \end{vmatrix}}$$

$$p = \frac{r_1 r_2 r_3 \left[\sin(\Theta_3 - \Theta_2) + \sin(\Theta_1 - \Theta_3) + \sin(\Theta_2 - \Theta_1)\right]}{r_2 r_3 \sin(\Theta_3 - \Theta_2) + r_1 r_3 \sin(\Theta_1 - \Theta_3) + r_1 r_2 \sin(\Theta_2 - \Theta_1)}$$

similarly

$$P = \frac{r_1(r_2 - r_3)\sin\Theta_1 + r_2(r_3 - r_1)\sin\Theta_2 + r_3(r_1 - r_2)\sin\Theta_3}{r_2 r_3 \sin(\Theta_3 - \Theta_2) + r_1 r_3 \sin(\Theta_1 - \Theta_3) + r_1 r_2 \sin(\Theta_2 - \Theta_1)}$$

$$Q = \frac{r_1(r_3 - r_2)\cos\Theta_1 + r_2(r_1 - r_3)\cos\Theta_2 + r_3(r_2 - r_1)\cos\Theta_3}{r_2 r_3 \sin(\Theta_3 - \Theta_2) + r_1 r_3 \sin(\Theta_1 - \Theta_3) + r_1 r_2 \sin(\Theta_2 - \Theta_1)}$$

and from (7.40)

$$e = \sqrt{P^2 + Q^2} \qquad \tan\omega = \frac{Q}{P} \qquad (7.42)$$

b) Orbit from 3 Position Vectors

Given 3 successive coplanar position vectors $\bar{r}_1, \bar{r}_2, \bar{r}_3$. Find $p, \bar{e}, \bar{V}_2$

Since $\bar{r}_1, \bar{r}_2, \bar{r}_3$ are coplanar, can write

$$\bar{r}_2 = \alpha \bar{r}_1 + \beta \bar{r}_3 \qquad (7.43)$$

To get α, cross (7.43) with $\bar{r}_3$

$$\bar{r}_2 \times \bar{r}_3 = \alpha \bar{r}_1 \times \bar{r}_3 + \beta \overbrace{\cancel{\bar{r}_3 \times \bar{r}_3}}^{0} = \alpha \bar{n}$$

where

$$\bar{n} = \bar{r}_1 \times \bar{r}_3 \qquad (7.44)$$

Dot with $\bar{n}$

$$(\bar{r}_2 \times \bar{r}_3) \cdot \bar{n} = \alpha \bar{n} \cdot \bar{n} = \alpha n^2$$

$$\alpha = \frac{(\bar{r}_2 \times \bar{r}_3) \cdot \bar{n}}{n^2} \qquad (7.45)$$

To get β, cross $\bar{r}_1$ with (7.43)

$$\bar{r}_1 \times \bar{r}_2 = \alpha \overbrace{\cancel{\bar{r}_1 \times \bar{r}_1}}^{0} + \beta \bar{r}_1 \times \bar{r}_3 = \beta \bar{n}$$

Dot with $\bar{n}$

$$(\bar{r}_1 \times \bar{r}_2) \cdot \bar{n} = \beta n^2$$

$$\beta = \frac{(\bar{r}_1 \times \bar{r}_2) \cdot \bar{n}}{n^2} \qquad (7.46)$$

To get p, dot (7.43) with $\bar{e}$

$$\bar{r}_2 \cdot \bar{e} = \alpha \bar{r}_1 \cdot \bar{e} + \beta \bar{r}_3 \cdot \bar{e} \qquad (a)$$

Using $r_i = \dfrac{p}{1 + e\cos\theta_i}$

$$p = r_i + r_i e \cos\theta_i = r_i + \bar{r}_i \cdot \bar{e}$$

$$\bar{r}_i \cdot \bar{e} = p - r_i \qquad i = 1, 2, 3 \qquad (b)$$

Therefore using (b), (a) becomes

$$p - r_2 = \alpha(p - r_1) + \beta(p - r_3)$$

Solve for p

$$p = \frac{\alpha r_1 - r_2 + \beta r_3}{\alpha - 1 + \beta} \qquad (7.47)$$

To get $\bar{e}$, using the identity

$$(\bar{A} \times \bar{B}) \times \bar{C} = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{B} \cdot \bar{C})\bar{A}$$

$$\bar{n} \times \bar{e} = (\bar{r}_1 \times \bar{r}_3) \times \bar{e} = (\bar{r}_1 \cdot \bar{e})\bar{r}_3 - (\bar{r}_3 \cdot \bar{e})\bar{r}_1$$

Using (b)

$$\bar{n} \times \bar{e} = (p - r_1)\bar{r}_3 - (p - r_3)\bar{r}_1$$

Cross with $\bar{n}$ and use the identity

$$(\bar{A} \times \bar{B}) \times \bar{C} = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{B} \cdot \bar{C})\bar{A}$$

$$(\bar{n} \times \bar{e}) \times \bar{n} = n^2 \bar{e} - \underbrace{(\bar{e} \cdot \bar{n})}_{0} \bar{n}$$

since $\bar{e} \perp \bar{n}$ — $\bar{e}$ along pericenter in plane of orbit; $\bar{n}$ normal to plane of orbit.

Therefore

$$\bar{e} = \frac{1}{n^2}\left[(p - r_1)\bar{r}_3 \times \bar{n} - (p - r_3)\bar{r}_1 \times \bar{n}\right] \quad (7.48)$$

$p = \frac{h^2}{\mu} \Rightarrow h = \sqrt{\mu p} \qquad \bar{h} = h\frac{\bar{n}}{n}$

From (7.25)

$$\bar{V}_2 = \frac{\mu}{h^2}\bar{h} \times \left[\frac{\bar{r}_2}{r_2} + \bar{e}\right]$$

c) Approximate Orbit from 3 Position Fixes

The previous methods are exact but numerical error can result if angles between the given position vectors are small.

Given $\bar{r}_1, \bar{r}_2, \bar{r}_3$ at times t_1, t_2, t_3 respectively. Find $\bar{V}_2$ (from which orbital elements can be determined).

Expand $\bar{r}$ in power series to $O(t^5)$
(valid for small t)

$$\bar{r} = a_0 + t\bar{a}_1 + t^2\bar{a}_2 + t^3\bar{a}_3 + t^4\bar{a}_4 + t^5\bar{a}_5 \qquad (7.49a)$$
$$\bar{V} = \bar{a}_1 + 2t\bar{a}_2 + 3t^2\bar{a}_3 + 4t^3\bar{a}_4 + 5t^4\bar{a}_5 \qquad (7.49b)$$

Using $\dfrac{d^2\bar{r}}{dt^2} + \dfrac{\mu}{r^3}\bar{r} = 0$

write

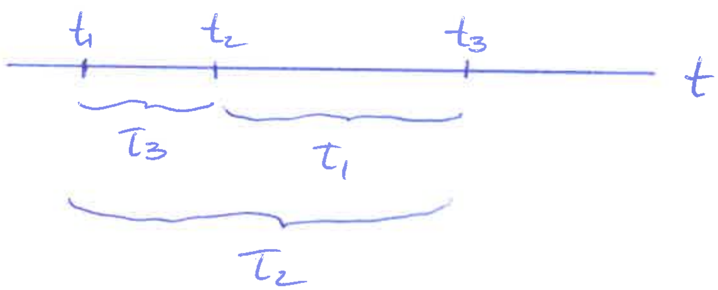
$$\frac{d^2\bar{r}}{dt^2} = -\varepsilon\bar{r} \quad \text{where } \varepsilon = \frac{\mu}{r^3}$$

Sub. (7.49 a) into above equation

$$-\varepsilon\bar{r} = 2\bar{a}_2 + 6t\,\bar{a}_3 + 12t^2\bar{a}_4 + 20t^3\bar{a}_5 \qquad (7.49c)$$

Define the time intervals

$$\tau_1 = t_3 - t_2, \quad \tau_2 = t_3 - t_1, \quad \tau_3 = t_2 - t_1$$



Set the time $t_2 = 0$

Apply (7.49) at $t = -\tau_3, 0, \tau_1$

$$\bar{r}_1 = \bar{a}_0 - \tau_3 \bar{a}_1 + \tau_3^2 \bar{a}_2 - \tau_3^3 \bar{a}_3 + \tau_3^4 \bar{a}_4 - \tau_3^5 \bar{a}_5 \quad (7.50a)$$

$$\bar{r}_2 = \bar{a}_0 \quad (7.50b)$$

$$\bar{r}_3 = \bar{a}_0 + \tau_1 \bar{a}_1 + \tau_1^2 \bar{a}_2 + \tau_1^3 \bar{a}_3 + \tau_1^4 \bar{a}_4 + \tau_1^5 \bar{a}_5 \quad (7.50c)$$

$$\bar{V}_2 = \bar{a}_1 \quad (7.50d)$$

$$-\varepsilon_1 \bar{r}_1 = 2\bar{a}_2 - 6\tau_3 \bar{a}_3 + 12\tau_3^2 \bar{a}_4 - 20\tau_3 \bar{a}_5 \quad (7.50e)$$

$$-\varepsilon_2 \bar{r}_2 = 2\bar{a}_2 \quad (7.50f)$$

$$-\varepsilon_3 \bar{r}_3 = 2\bar{a}_2 + 6\tau_1 \bar{a}_3 + 12\tau_1^2 \bar{a}_4 + 20\tau_1^3 \bar{a}_5 \quad (7.50g)$$

7 equations, 7 unknowns $\bar{a}_0, \ldots, \bar{a}_5, \bar{V}_2$

solve for $\bar{V}_2$

$$A_0 \bar{V}_2 = A_1 \bar{r}_1 + A_2 \bar{r}_2 + A_3 \bar{r}_3 \quad (7.51)$$

where

$$A_0 = 12\tau_1 \tau_2 \tau_3 (\tau_3 - \tau_1)(2\tau_1^2 + 5\tau_1\tau_3 + 2\tau_3^2)$$

$$A_1 = \tau_1^4 \left[12(2\tau_2 + 3\tau_3) + \varepsilon_1 \tau_3^2 (\tau_1 + 3\tau_2) \right]$$

$$A_2 = \varepsilon_2 \tau_1^2 \tau_2 \tau_3^2 (8\tau_2^2 + 3\tau_1\tau_3) - 12\tau_2[2\tau_2^4 - 5\tau_1\tau_3(\tau_2^2 + \tau_1\tau_3)]$$

$$A_3 = \tau_3^4 [12(3\tau_1 + 2\tau_2) + \varepsilon_3 \tau_1^2 (3\tau_2 + \tau_3)]$$

When the time intervals are equal (i.e, $\tau_1 = \tau_3 = \tau$) the solution is indeterminate.

Alternate approach which does not contain this difficulty

Truncate power series after $O(t^4)$

Have one more equation than unknowns.

Eliminate $\bar{a}_2$ by multiplying (7.50a) by τ_1^2 and (7.50c) by τ_3^2 and subtract to get

$$\tau_1^2 \bar{r}_1 - \tau_3^2 \bar{r}_3 = (\tau_1 - \tau_3)\tau_2 \bar{a}_0 - \tau_1\tau_2\tau_3 \bar{a}_1 - \tau_1^2 \tau_2 \tau_3^2 \bar{a}_3 + \tau_1^2 \tau_2 \tau_3^2 (\tau_3 - \tau_1) \bar{a}_4 \quad (7.52)$$

Let (7.52) replace (7.50a, c)

Have 6 eqs for 6 unknowns $\bar{a}_0, \ldots, \bar{a}_4, \bar{V}_2$

Solve for V_2

$$\bar{V}_2 = -\tau_1\left(\frac{1}{\tau_2\tau_3} + \frac{\varepsilon_1}{12}\right)\bar{r}_1 - (\tau_3 - \tau_1)\left(\frac{1}{\tau_1\tau_3} + \frac{\varepsilon_2}{12}\right)\bar{r}_2$$
$$+ \tau_3\left(\frac{1}{\tau_1\tau_2} + \frac{\varepsilon_3}{12}\right)\bar{r}_3 \qquad (7.53)$$

valid to 4^{th} order in time intervals

<u>Derivation of 3 useful equations</u>

$$\frac{d^2r}{dt^2} = \frac{\mu}{r^3}(p - r) \qquad (7.54) \qquad \text{(general)}$$

$$\left(\frac{dr}{dt}\right)^2 = \mu\left(\frac{2}{r} - \frac{p}{r^2} - \frac{1}{a}\right) \qquad (7.55) \qquad \text{(elliptic)}$$

$$\frac{d^2}{dt^2}(r^2) = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right) \qquad (7.56) \qquad \text{(elliptic)}$$

$$r = \frac{p}{1+e\cos\theta}$$

$$\frac{dr}{dt} = \frac{pe\sin\theta}{(1+e\cos\theta)^2}\frac{d\theta}{dt} = \frac{\cancel{r^2}}{p}e\sin\theta \cdot \frac{h}{\cancel{r^2}} = \frac{he}{p}\sin\theta$$

$$\frac{d^2r}{dt^2} = \frac{he}{p}\cos\theta\frac{d\theta}{dt} = \frac{h}{p}\frac{p-r}{r}\frac{h}{r^2} = \frac{h^2}{pr^3}(p-r) = \frac{\mu}{r^3}(p-r)$$

$$\boxed{\frac{d^2r}{dt^2} = \frac{\mu}{r^3}(p-r)}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{h^2e^2}{p^2}\sin^2\theta = \frac{\mu e^2}{p}(1-\cos^2\theta)$$

$$r + re\cos\theta = p \Rightarrow \cos\theta = \frac{p-r}{re}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{\mu e^2}{p}\left[1 - \frac{(p-r)^2}{r^2e^2}\right] = \frac{\mu}{p}\left[e^2 - \frac{p^2-2pr+r^2}{r^2}\right]$$

$$= \frac{\mu}{p}\left[e^2 - \frac{p^2}{r^2} + \frac{2p}{r} - 1\right]$$

$$\boxed{\left(\frac{dr}{dt}\right)^2 = \mu\left(\frac{2}{r} - \frac{p}{r^2} - \frac{1-e^2}{p}\right)}$$ (general)

$$\boxed{\left(\frac{dr}{dt}\right)^2 = \mu\left(\frac{2}{r} - \frac{p}{r^2} - \frac{1}{a}\right)}$$ (elliptic)

$$\frac{d}{dt}(r^2) = 2r\frac{dr}{dt}$$

$$\frac{d^2}{dt^2}(r^2) = 2r\frac{d^2r}{dt^2} + 2\left(\frac{dr}{dt}\right)^2$$

$$= 2r\frac{\mu}{r^3}(p-r) + 2\mu\left(\frac{2}{r} - \frac{p}{r^2} - \frac{1-e^2}{p}\right)$$

$$= 2\mu\left(\cancel{\frac{p}{r^2}} - \frac{1}{r} + \frac{2}{r} - \cancel{\frac{p}{r^2}} - \frac{1-e^2}{p}\right)$$

$$\boxed{\frac{d^2}{dt^2}(r^2) = 2\mu\left(\frac{1}{r} - \frac{1-e^2}{p}\right)}$$ (general)

$$\boxed{\frac{d^2}{dt^2}(r^2) = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right)}$$ (elliptic)

d) Approximate Orbit from 3 Range Measurements

Given r_1, r_2, r_3 at times t_1, t_2, t_3 respectively

Find p, a

To $O(t^4)$ write:

$$r = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \qquad (7.57)$$

Using (7.54)

$$\frac{d^2 r}{dt^2} = \varepsilon (p - r) = 2a_2 + 6a_3 t + 12 a_4 t^2 \qquad (7.58)$$

where $\varepsilon = \frac{\mu}{r^3}$

Evaluate (7.57) and (7.58) at $t = -\tau_3, 0, \tau_1$

Gives 6 equations for 6 unknowns: $a_0, \ldots, a_4, p$

Solve for p

$$p = \frac{r_1 \tau_1 (1 + \varepsilon_1 A_1) - r_2 \tau_2 (1 - \varepsilon_2 A_2) + r_3 \tau_3 (1 + \varepsilon_3 A_3)}{\tau_1 \varepsilon_1 A_1 + \tau_2 \varepsilon_2 A_2 + \tau_3 \varepsilon_3 A_3} \qquad (7.59)$$

to $O(t^4)$

where

$$\left.\begin{aligned} 12A_1 &= \tau_2\tau_3 - \tau_1^2 \\ 12A_2 &= \tau_1\tau_3 + \tau_2^2 \\ 12A_3 &= \tau_1\tau_2 - \tau_3^2 \end{aligned}\right\} \quad (7.60)$$

To get a, use

$$r^2 = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4$$

$$\frac{d^2}{dt^2}(r^2) = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right) = 2b_2 + 6b_3 t + 12 b_4 t^2$$

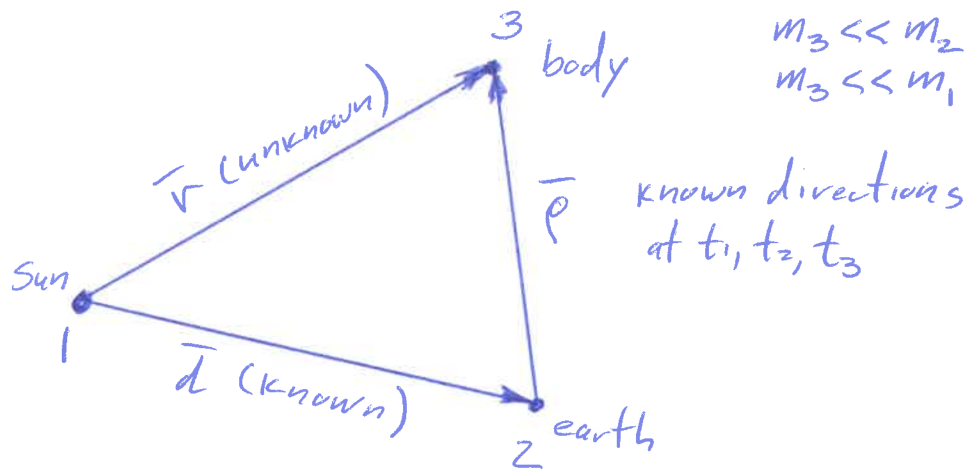
Evaluate at $t = -\tau_3, 0, \tau_1$. Get 6 equations for 6 unknowns: $b_0, \ldots, b_4, a$

Solve for μ/a

$$\frac{\mu}{a} = -\frac{r_1^2}{\tau_2\tau_3}(1 - 2\varepsilon_1 A_1) + \frac{r_2^2}{\tau_1\tau_3}(1 + 2\varepsilon_2 A_2) - \frac{r_3^2}{\tau_1\tau_2}(1 - 2\varepsilon_3 A_3)$$

(7.61)

e) Approximate Orbit from 3 Angular Observations



Given $\hat{i}_{\rho_1}, \hat{i}_{\rho_2}, \hat{i}_{\rho_3}$ at times t_1, t_2, t_3 respectively,

Find $\bar{r}$ and $\bar{V}$ at time t_2 (i.e., find $\bar{r}_2, \bar{V}_2$)

$$\bar{r} = \bar{\rho} + \bar{d} \qquad (7.62)$$

$$\frac{d^2\bar{r}}{dt^2} + \frac{\mu_1}{r^3}\bar{r} = 0 \qquad \mu_1 = Gm_1 \quad \left(\text{neglect effect of } m_2 \text{ on } m_3\right) \qquad (7.63)$$

$$\frac{d^2\bar{d}}{dt^2} + \frac{\mu_2}{d^3}\bar{d} = 0 \qquad \mu_2 = G(m_1 + m_2) \qquad (7.64)$$

Write $\bar{\rho} = \rho\,\hat{i}_\rho$

$$\frac{d\bar{\rho}}{dt} = \rho \frac{d\hat{i}_\rho}{dt} + \frac{d\rho}{dt}\hat{i}_\rho$$

$$\frac{d^2\bar{\rho}}{dt^2} = \rho \frac{d^2\hat{i}_\rho}{dt^2} + \frac{d\rho}{dt}\frac{d\hat{i}_\rho}{dt} + \frac{d\rho}{dt}\frac{d\hat{i}_\rho}{dt} + \frac{d^2\rho}{dt^2}\hat{i}_\rho$$

$$\frac{d^2\rho}{dt^2} = \frac{d^2\rho}{dt^2}\hat{i}_\rho + 2\frac{d\rho}{dt}\frac{d\hat{i}_\rho}{dt} + \rho\frac{d^2\hat{i}_\rho}{dt^2} \qquad (7.65)$$

From (7.62)

$$\frac{d^2\bar{r}}{dt^2} = \frac{d^2\bar{\rho}}{dt^2} + \frac{d^2\bar{d}}{dt^2} \qquad (7.66)$$

Substitute (7.63, 64, 65) into (7.66)

$$-\frac{\mu_1}{r^3}\bar{r} = \frac{d^2\rho}{dt^2}\hat{i}_\rho + 2\frac{d\rho}{dt}\frac{d\hat{i}_\rho}{dt} + \rho\frac{d^2\hat{i}_\rho}{dt^2} - \frac{\mu_2}{d^3}\bar{d}$$

Substitute (7.62) $\bar{r} = \rho + \bar{d}$

$$-\frac{\mu_1}{r^3}(\bar{\rho} + \bar{d}) = \frac{d^2\rho}{dt^2}\hat{i}_\rho + 2\frac{d\rho}{dt}\frac{d\hat{i}_\rho}{dt} + \rho\frac{d^2\hat{i}_\rho}{dt^2} - \frac{\mu_2}{d^3}\bar{d}$$

Since $\bar{\rho} = \rho \hat{i}_\rho$

$$-\frac{\mu_1}{r^3}\left(\rho \hat{i}_\rho + \bar{d}\right) = \frac{d^2\rho}{dt^2}\hat{i}_\rho + 2\frac{d\rho}{dt}\frac{d\hat{i}_\rho}{dt} + \rho\frac{d^2\hat{i}_\rho}{dt^2} - \frac{\mu_2}{d^3}\bar{d}$$

Rearrange

$$\left(\frac{d^2\rho}{dt^2} + \frac{\mu_1}{r^3}\rho\right)\hat{i}_\rho + 2\frac{d\rho}{dt}\frac{d\hat{i}_\rho}{dt} + \rho\frac{d^2\hat{i}_\rho}{dt^2} = \left(\frac{\mu_2}{d^3} - \frac{\mu_1}{r^3}\right)\bar{d}$$

Take dot product of each term with $\hat{i}_\rho \times \frac{d\hat{i}_\rho}{dt}$

$$0 + 0 + \rho\left(\left(\hat{i}_\rho \times \frac{d\hat{i}_\rho}{dt}\right) \cdot \frac{d^2\hat{i}_\rho}{dt^2}\right) = \left(\frac{\mu_2}{d^3} - \frac{\mu_1}{r^3}\right)\left(\left(\hat{i}_\rho \times \frac{d\hat{i}_\rho}{dt}\right) \cdot \bar{d}\right) \quad (7.67)$$

Take dot product of each term with $\hat{i}_\rho \times \frac{d^2\hat{i}_\rho}{dt^2}$

$$0 + 2\frac{d\rho}{dt}\left(\left(\hat{i}_\rho \times \frac{d^2\hat{i}_\rho}{dt^2}\right) \cdot \frac{d\hat{i}_\rho}{dt}\right) + 0 = \left(\frac{\mu_2}{d^3} - \frac{\mu_1}{r^3}\right)\left(\left(\hat{i}_\rho \times \frac{d^2\hat{i}_\rho}{dt^2}\right) \cdot \bar{d}\right)$$

Using the identity $(\bar{A} \times \bar{B}) \cdot \bar{C} = -(\bar{A} \times \bar{C}) \cdot \bar{B}$

$$2\frac{d\rho}{dt}\left(\left(\hat{i}_\rho \times \frac{d\hat{i}_\rho}{dt}\right) \cdot \frac{d^2\hat{i}_\rho}{dt^2}\right) = \left(\frac{\mu_2}{d^3} - \frac{\mu_1}{r^3}\right)\left(\left(\hat{i}_\rho \times \bar{d}\right) \cdot \frac{d^2\hat{i}_\rho}{dt^2}\right) \quad (7.68)$$

From (7.62)

$$r^2 = \rho^2 + d^2 + 2\rho\,\hat{i}_\rho \cdot \bar{d} \qquad (7.69)$$

If we can evaluate $\left.\frac{d\hat{i}_\rho}{dt}\right|_{t_2}$ and $\left.\frac{d^2\hat{i}_\rho}{dt^2}\right|_{t_2}$

evaluate (7.67) and (7.69) at t_2 and solve for r_2 and ρ_2.

Then evaluate (7.68) at t_2 and solve for $\left.\frac{d\rho}{dt}\right|_{t_2}$

Position and velocity at t_2 is given by

$$\bar{r}_2 = \rho_2\,\hat{i}_{\rho_2} + \bar{d}_2 \qquad (7.70)$$

$$\bar{V}_2 = \frac{d\rho_2}{dt}\hat{i}_{\rho_2} + \rho_2\frac{d\hat{i}_{\rho_2}}{dt} + \frac{d\bar{d}_2}{dt} \qquad (7.71)$$

From which orbital elements may be obtained.