9.9. The Clohessy-Wiltshire Equations

If the orbit of the target vehicle A is circular, the moving reference from is called a Clohessy-Wiltshire (CW) frame.

For a circular target orbit

$$\bar{R} \cdot \bar{V} = 0 \quad \text{and} \quad h = \sqrt{\mu R}$$

Eqs. (9.103) simplify to

$$\delta\ddot{x} - 3\frac{\mu}{R^3}\delta x - 2\sqrt{\frac{\mu}{R^3}}\,\delta\dot{y} = 0 \qquad (9.104a)$$

$$\delta\ddot{y} + 2\sqrt{\frac{\mu}{R^3}}\,\delta\dot{x} = 0 \qquad (9.104b)$$

$$\delta\ddot{z} + \frac{\mu}{R^3}\delta z = 0 \qquad (9.104c)$$

Furthermore, for a circular orbit, the mean angular velocity

$$n = \sqrt{\frac{\mu}{R^3}} = \text{const.} \qquad (\text{see } 7.13)$$

Therefore eqs. (9.104) may be written as

$$\boxed{\begin{aligned} \delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y} &= 0 \\ \delta\ddot{y} + 2n\delta\dot{x} &= 0 \\ \delta\ddot{z} + n^2\delta z &= 0 \end{aligned}}$$

(9.105a)
(9.105b)
(9.105c)

These equations are called the <u>Clohessy-Wiltshire (CW) equations.</u> They can be solved analytically.

Integrate (9.105b)

$$\delta\dot{y} + 2n\delta x = C_1$$

where C_1 is a constant of integration. Write

$$\delta\dot{y} = C_1 - 2n\delta x \qquad (9.106)$$

Sub. (9.106) into (9.105a)

$$\delta\ddot{x} + n^2\delta x = 2nC_1 \qquad (9.107)$$

The solution of (9.107) is

$$\delta x = \frac{2}{n} C_1 + C_2 \sin nt + C_3 \cos nt \qquad (9.108)$$

Differentiating (9.108) gives the x component of the relative velocity

$$\delta \dot{x} = C_2 n \cos nt - C_3 n \sin nt \qquad (9.109)$$

Sub. (9.108) into (9.106) gives the y component of the relative velocity

$$\delta \dot{y} = -3C_1 - 2C_2 n \sin nt - 2C_3 n \cos nt \qquad (9.110)$$

Integrating (9.110)

$$\delta y = -3C_1 t + 2C_2 \cos nt - 2C_3 \sin nt + C_4 \qquad (9.111)$$

The solution of (9.105c) is

$$\delta z = C_5 \sin nt + C_6 \cos nt \qquad (9.112)$$

Differentiating (9.112) gives the z component of the relative velocity

$$\delta\dot{z} = C_5 n \cos nt - C_6 n \sin nt \qquad (9.113)$$

The constants $C_1 - C_6$ are found by applying the initial conditions

At $t=0$ $\quad \delta x = \delta x_0, \ \delta y = \delta y_0, \ \delta z = \delta z_0$

$\qquad \delta\dot{x} = \delta u_0, \ \delta\dot{y} = \delta v_0, \ \delta\dot{z} = \delta w_0$

where $\delta u_0, \delta v_0, \delta w_0$ are the initial velocity components of B as seen from A in the CW frame.

Evaluating (9.108)–(9.13) at $t=0$ gives

$$\frac{2}{n} C_1 + C_3 = \delta x_0$$

$$C_2 n = \delta u_0$$

$$-3C_1 - 2C_3 n = \delta v_0$$

$$2C_2 + C_4 = \delta y_0$$

$$C_6 = \delta z_0$$

$$C_5 n = \delta w_0$$

Solving simultaneously

$$c_1 = 2n\,\delta x_0 + \delta v_0 \qquad (9.114a)$$

$$c_2 = \frac{1}{n}\,\delta u_0 \qquad (9.114b)$$

$$c_3 = -3\,\delta x_0 - \frac{2}{n}\,\delta v_0 \qquad (9.114c)$$

$$c_4 = -\frac{2}{n}\,\delta u_0 + \delta y_0 \qquad (9.114d)$$

$$c_5 = \frac{1}{n}\,\delta w_0 \qquad (9.114e)$$

$$c_6 = \delta z_0 \qquad (9.114f)$$

Substituting (9.114) into (9.108), (9.111) & (9.112) gives the trajectory of B in the CW frame.

$$\boxed{\begin{aligned} \delta x &= (4 - 3\cos nt)\,\delta x_0 + \frac{1}{n}\sin nt\,\delta u_0 + \frac{2}{n}(1-\cos nt)\,\delta v_0 \\ \delta y &= 6(\sin nt - nt)\,\delta x_0 + \delta y_0 + \frac{2}{n}(\cos nt - 1)\,\delta u_0 + \frac{1}{n}(4\sin nt - 3nt)\,\delta v_0 \\ \delta z &= \cos nt\,\delta z_0 + \frac{1}{n}\sin nt\,\delta w_0 \qquad (9.115a,b,c) \end{aligned}}$$

Note that all 3 components of $\delta\bar{r}$ oscillate with frequency n. Also note that δy has a secular term which grows linearly with t. Thus B will move further and further away from A and eventually the solution will break down because it will violate the condition $\frac{\delta r}{R} \ll 1$ on which it is based.

Differentiating (9.115) with respect to time gives the velocity components of B in the CW frame

$$\boxed{\begin{aligned} \delta u &= 3n \sin nt\, \delta x_0 + \cos nt\, \delta u_0 + 2 \sin nt\, \delta v_0 \\ \delta v &= 6n(\cos nt - 1)\,\delta x_0 - 2 \sin nt\, \delta u_0 + (4\cos nt - 3)\,\delta v_0 \\ \delta w &= -n \sin nt\, \delta z_0 + \cos nt\, \delta w_0 \qquad (9.116\ a,b,c) \end{aligned}}$$

EXAMPLE

Calculate the trajectory of the chaser (B) in the previous example based on the CW linearized solution.

The initial conditions for the problem are found as follows.

Evaluating (1) & (2) in that example at $t=0$ where $\theta_A = \theta_B = 0$

$$\bar{r}_A = r_A \hat{I} \qquad \bar{r}_B = r_B \hat{I}$$

$$\bar{V}_A = \sqrt{\frac{\mu}{r_A}}\, \hat{J} \qquad \bar{V}_B = \sqrt{\frac{\mu}{a(1-e^2)}}\,(e+1)\, \hat{J} = \sqrt{\frac{\mu(1+e)}{a(1-e)}}\, \hat{J}$$

Using (9.79) at $t=0$

$$\bar{r}_{rel} = \bar{r}_B - \bar{r}_A = (r_B - r_A)\,\hat{I}$$

Using (7) from example

$$\bar{\Omega} = \sqrt{\frac{\mu}{r_A^3}}\,\hat{K}$$

Using (9.74) at $t=0$

$$\bar{V}_{rel} = \bar{V}_B - \bar{V}_A - \bar{\Omega}\times\bar{r}_{rel}$$

$$= \sqrt{\frac{\mu(1+e)}{a(1-e)}}\,\hat{J} - \sqrt{\frac{\mu}{r_A}}\,\hat{J} - \left(\sqrt{\frac{\mu}{r_A^3}}\,\hat{K}\right)\times(r_B - r_A)\,\hat{I}$$

$$= \sqrt{\frac{\mu(1+e)}{a(1-e)}}\,\hat{J} - \cancel{\sqrt{\frac{\mu}{r_A}}\,\hat{J}} - \sqrt{\frac{\mu}{r_A^3}}\,r_B\,\hat{J} + \cancel{\sqrt{\frac{\mu}{r_A}}\,\hat{J}}$$

$$= \left(\sqrt{\frac{\mu(1+e)}{a(1-e)}} - \sqrt{\frac{\mu}{r_A^3}}\,r_B\right)\hat{J}$$

Using (6) from example at $t=0$

$$\hat{I} = \hat{i}$$
$$\hat{J} = \hat{j}$$
$$\hat{K} = \hat{k}$$

Therefore the initial conditions are

$$\delta x_0 = r_B - r_A = 7000 - 8000 = -1000 \text{ km}$$

$$\delta y_0 = 0$$

$$\delta z_0 = 0$$

$$\delta u_0 = 0$$

$$\delta v_0 = \sqrt{\frac{\mu(1+e)}{a(1-e)}} - \sqrt{\frac{\mu}{r_A^3}}\, r_B$$

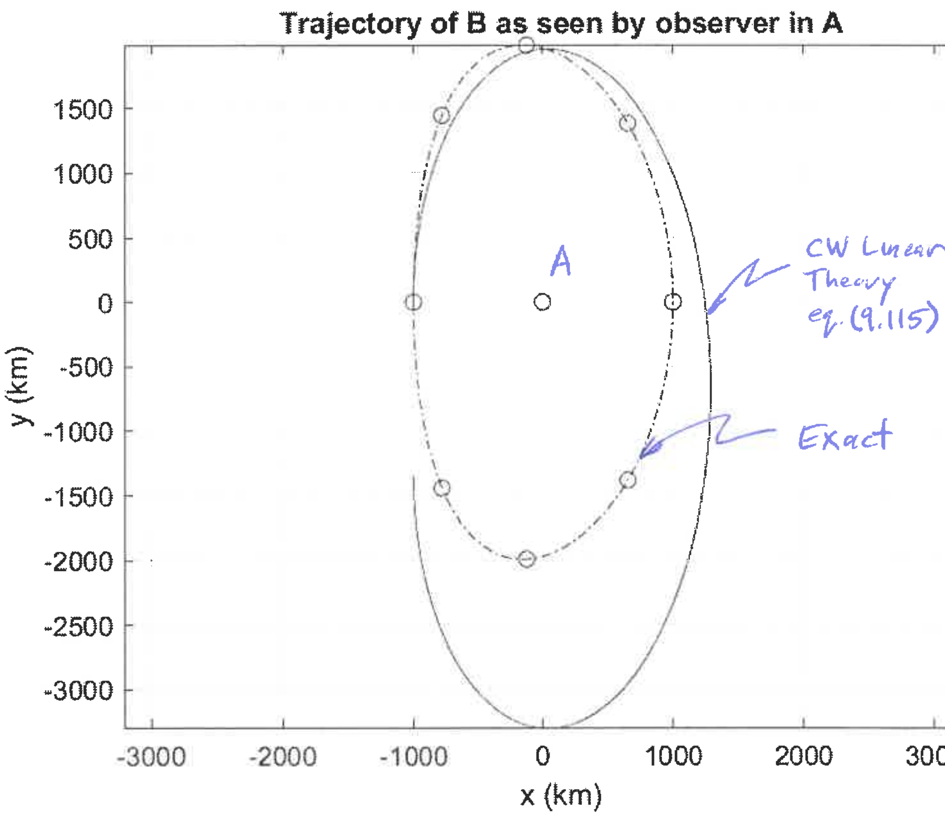
$$= \sqrt{\frac{(3.986\times10^5)(1+0.125)}{1-0.125}} - \sqrt{\frac{3.986\times10^5}{(8000)^3}}\,(7000)$$

$$= 7.54604 - 6.17635$$

$$= 1.36969 \text{ km/sec}$$

$$\delta w_0 = 0$$

A plot of the trajectory is shown on the following page


Trajectory of B as seen by observer in A
1500
1000
500
0
-500
-1000
-1500
-2000
-2500
-3000
y (km)
-3000
-2000
-1000
0
1000
2000
3000
x (km)
A
CW Linear Theory eq. (9.115)
Exact

To facilitate solutions for orbital rendezvous, the Clohessy-Wiltshire equations (9.115) & (9.116) will be written in matrix-vector format.

Define the relative position and velocity vectors

$$\delta\bar{r} = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} \qquad \delta\bar{V} = \begin{bmatrix} \delta u \\ \delta v \\ \delta w \end{bmatrix}$$

and their values at $t=0$

$$\delta\bar{r}_0 = \begin{bmatrix} \delta x_0 \\ \delta y_0 \\ \delta z_0 \end{bmatrix} \qquad \delta\bar{V}_0 = \begin{bmatrix} \delta u_0 \\ \delta v_0 \\ \delta w_0 \end{bmatrix}$$

Eqs. (9.115) & (9.116) may be written as

$$\delta\bar{r} = \bar{\bar{\Phi}}_{rr}\,\delta\bar{r}_0 + \bar{\bar{\Phi}}_{rv}\,\delta\bar{V}_0 \qquad (9.117a)$$

$$\delta\bar{V} = \bar{\bar{\Phi}}_{vr}\,\delta\bar{r}_0 + \bar{\bar{\Phi}}_{vv}\,\delta\bar{V}_0 \qquad (9.117b)$$

where the transition matrices (CW matrices) $\bar{\bar{\Phi}}$ which depend only on t are given by

$$\bar{\bar{\Phi}}_{rr} = \left[\begin{array}{cc|c} 4-3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ \hline 0 & 0 & \cos nt \end{array}\right] \tag{9.118a}$$

$$\bar{\bar{\Phi}}_{rv} = \left[\begin{array}{cc|c} \frac{1}{n}\sin nt & \frac{2}{n}(1-\cos nt) & 0 \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) & 0 \\ \hline 0 & 0 & \frac{1}{n}\sin nt \end{array}\right] \tag{9.118b}$$

$$\bar{\bar{\Phi}}_{vr} = \left[\begin{array}{cc|c} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ \hline 0 & 0 & -n\sin nt \end{array}\right] \tag{9.118c}$$

$$\bar{\bar{\Phi}}_{vv} = \left[\begin{array}{cc|c} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ \hline 0 & 0 & \cos nt \end{array}\right] \tag{9.118d}$$

The subscripts on Φ denote which of the vectors $\delta\bar{r}$ & $\delta\bar{V}$ that coefficient relates to which of the initial conditions $\delta\bar{r}_0$ & $\delta\bar{V}_0$

In problems where there is no motion in z (coplanar orbit problems) $\delta z_0 = \delta w_0 = 0$ and only the elements in the 2×2 submatrices on the upper left hand corner are needed.

Also note that

$$\frac{d}{dt}\bar{\bar{\Phi}}_{vr} = \bar{\bar{\Phi}}_{\dot{v}r} \quad \text{and} \quad \frac{d}{dt}\bar{\bar{\Phi}}_{vv} = \bar{\bar{\Phi}}_{\dot{v}v} \qquad (9.119)$$

9.10 Orbital Rendezvous Using the CW Equations

Suppose a chaser vehicle is at an arbitrary initial state $\delta\bar{r}_0, \delta\bar{V}_0$ in the CW frame and we wish to rendezvous with the target vehicle at the origin at time t.

The required initial relative velocity of the chaser $\delta\bar{V}_0^{req}$ is obtained by solving (9.117a) for $\delta\bar{V}_0$ with $\delta\bar{r}$ (at time t) $= 0$.

$$0 = \bar{\bar{\Phi}}_{rr}\, \delta\bar{r}_0 + \bar{\bar{\Phi}}_{rv}\, \delta\bar{V}_0^{req}$$

$$\bar{\bar{\Phi}}_{rv}\, \delta\bar{V}_0^{req} = -\bar{\bar{\Phi}}_{rr}\, \delta\bar{r}_0$$

$$\delta\bar{V}_0^{req} = -\bar{\bar{\Phi}}_{rv}^{-1}\, \bar{\bar{\Phi}}_{rr}\, \delta\bar{r}_0 \qquad (9.120)$$

where $\bar{\bar{\Phi}}_{rv}^{-1}$ is the inverse of the matrix given by (9.118b)

Since the initial relative velocity of the chaser at $t=0$ was $\delta\bar{V}_0$, the required velocity impulse at $t=0$ is

$$\Delta\bar{V}_0 = \delta V_0^{req} - \delta\bar{V}_0 \qquad (9.121)$$

Note that $\Delta\bar{V}_0$ is the same whether viewed in the inertial or CW frame. This can be seen from (9.95)

$$\underbrace{\delta\bar{V}_{rel}}_{\substack{\text{velocity of}\\ \text{B w.r.t. A}\\ \text{measured in}\\ \text{CW frame}}} = \underbrace{\delta\dot{\bar{r}}}_{\substack{\text{velocity of}\\ \text{B w.r.t. A}\\ \text{measured in}\\ \text{inertial frame}}} - \bar{\Omega} \times \delta\bar{r} \qquad (9.95)$$

Apply (9.95) before (−) and after (+) the burn

$$\delta V_{rel}^{-} = \delta \dot{\bar{r}}^{-} - \Omega^{-} \times \delta \bar{r}^{-} \qquad (9.122a)$$

$$\delta V_{rel}^{+} = \delta \dot{\bar{r}}^{+} - \Omega^{+} \times \delta \bar{r}^{+} \qquad (9.122b)$$

The burn by the chaser has no effect on the motion of the target so $\Omega^{+} = \Omega^{-}$. Also, since the burn is impulsive (instantaneous) there is no change in position, so $\delta\bar{r}^{+} = \delta\bar{r}^{-}$. Substracting (9.122a) from (9.122b)

$$\delta V_{rel}^{+} - \delta V_{rel}^{-} = \delta \dot{\bar{r}}^{+} - \delta \dot{\bar{r}}^{-}$$

or $\quad \Delta \bar{V}_{rel} = \Delta \bar{V}$

as measured in CW frame

as measured in inertial frame

With this burn at $t=0$, the chaser coasts and reaches the target located at the origin of the CW frame at time t.

The chaser's terminal relative velocity when it arrives at the origin can be found from (9.117b)

$$\delta \bar{V} = \bar{\bar{\Phi}}_{Vr}\, \delta \bar{r}_0 + \bar{\bar{\Phi}}_{VV}\, \delta \bar{V}_0^{req} \qquad (9.123)$$

For rendezvous with the target, a second velocity impulse is needed to cancel this residual relative velocity

$$\Delta \bar{V}_f = \bar{0} - \delta \bar{V}$$

or

$$\Delta \bar{V}_f = -\delta \bar{V} \qquad (9.124)$$

The magnitude of the total velocity increment for the two-implulse maneuver is

$$\Delta V_{total} = |\Delta \bar{V}_0| + |\Delta \bar{V}_f| \qquad (9.125)$$

Note that to compute the initial relative velocity for rendezvous using (9.120) requires the inverse of $\bar{\bar{\Phi}}_{rv}$. If the transfer time t is an exact multiple of the period of the target's orbit, i.e. $t = kT \quad k = 1, 2, 3 \ldots$, the sinusoid terms $\sin nt = 0$ and $\cos nt = 1$ and $\bar{\bar{\Phi}}_{rv}$ becomes

$$\bar{\bar{\Phi}}_{rv} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3kT & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and it's inverse does not exist. The easiest remedy is to simply avoid these points.

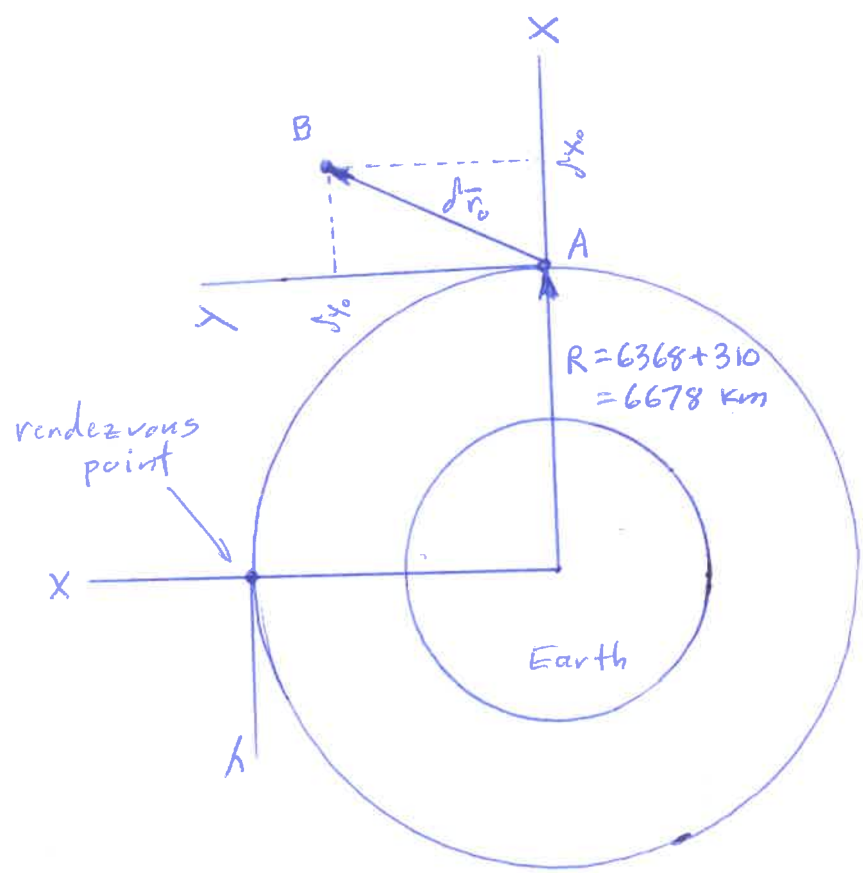
eg. for $t = 3T$, use $t = 2.99T$ or $3.01T$

EXAMPLE

Chaser vehicle B is 20 km above and 40 km ahead of target vehicle A which is in a 310 km high circular orbit. Determine

a) the chaser's relative velocity at this instant to rendezvous with the target in one-quarter of the period of the target's orbit.

b) the chaser's residual velocity at the rendezvous point.



a) The initial relative velocity required for rendezvous is given by (9.120)

$$\delta \bar{v}_0^{req} = -\bar{\bar{\Phi}}_{rv}^{-1}\, \bar{\bar{\Phi}}_{rr}\, \delta \bar{r}_0$$

where the initial relative position vector is

$$\delta \bar{r}_0 = \begin{bmatrix} \delta x_0 \\ \delta y_0 \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \end{bmatrix} \quad (\text{km})$$

For coplanar orbits, the matrices $\bar{\bar{\Phi}}_{rr}$ and $\bar{\bar{\Phi}}_{rv}$ are obtained from (9.118 a, b)

$$\bar{\bar{\Phi}}_{rr} = \begin{bmatrix} 4-3\cos nt & 0 \\ 6(\sin nt - nt) & 1 \end{bmatrix}$$

$$\bar{\bar{\Phi}}_{rv} = \begin{bmatrix} \frac{1}{n}\sin nt & \frac{2}{n}(1-\cos nt) \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) \end{bmatrix}$$

The angular velocity of the target orbit is

$$n = \sqrt{\frac{\mu}{R^3}} = \sqrt{\frac{3.986\times 10^5}{(6678)^3}} = 1.1569 \times 10^{-3} \text{ rad/sec}$$

and its orbital period is

$$T = \frac{2\pi}{n} = \frac{2\pi}{1.1569 \times 10^{-3}} = 5431 \text{ sec} = 90.52 \text{ min}$$

The desired rendezvous time is

$$t = \frac{1}{4} T = \frac{1}{4}(5431 \text{ sec}) = 1358 \text{ sec} = 22.63 \text{ min}$$

Evaluating the transition matrices for these values of n and t, get

$$\bar{\bar{\Phi}}_{rr} = \begin{bmatrix} 4 & 0 \\ -3.4248 & 1 \end{bmatrix} \qquad \bar{\bar{\Phi}}_{rv} = \begin{bmatrix} 864.3726 & 1728.7451 \\ -1728.7451 & -615.7695 \end{bmatrix}$$

The inverse of $\bar{\bar{\Phi}}_{rv}$ is

$$\bar{\bar{\Phi}}_{rv}^{-1} = \begin{bmatrix} -2.5069 \times 10^{-4} & -7.0380 \times 10^{-4} \\ 7.0380 \times 10^{-4} & 3.5190 \times 10^{-4} \end{bmatrix}$$

Thus

$$\delta \bar{V}_0^{req} = -\bar{\bar{\Phi}}_{rv}^{-1} \bar{\bar{\Phi}}_{rr} \delta \bar{r}_0$$

$$= -\begin{bmatrix} -2.5069\times10^{-4} & -7.0380\times10^{-4} \\ 7.0380\times10^{-4} & 3.5190\times10^{-4} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ -3.4248 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -0.0463 \end{bmatrix} \quad (\text{km/sec})$$

b) The chaser's residual velocity when it arrives at the target is obtained from (9.123)

$$\delta \bar{V} = \bar{\bar{\Phi}}_{vr} \delta \bar{r}_0 + \bar{\bar{\Phi}}_{vv} \delta \bar{V}_0^{req}$$

The matrices $\bar{\bar{\Phi}}_{vr}$ and $\bar{\bar{\Phi}}_{vv}$ are obtained from (9.118 c, d)

$$\bar{\bar{\Phi}}_{vr} = \begin{bmatrix} 3n \sin nt & 0 \\ 6n(\cos nt - 1) & 0 \end{bmatrix} \qquad \bar{\bar{\Phi}}_{vv} = \begin{bmatrix} \cos nt & 2\sin nt \\ -2\sin nt & 4\cos nt - 3 \end{bmatrix}$$

For $n = 1.1569 \times 10^{-3}$ rad/sec & $t = T/4 = 5431$ sec

$$\bar{\bar{\Phi}}_{vr} = \begin{bmatrix} 3.4707\times10^{-3} & 0 \\ -6.9415\times10^{-3} & 0 \end{bmatrix} \qquad \bar{\bar{\Phi}}_{vv} = \begin{bmatrix} 0 & 2 \\ -2 & -3 \end{bmatrix}$$

$$\delta\bar{V} = \bar{\bar{\Phi}}_{vr}\,\delta\bar{r}_0 + \bar{\bar{\Phi}}_{vv}\,\delta\bar{V}_0^{req}$$

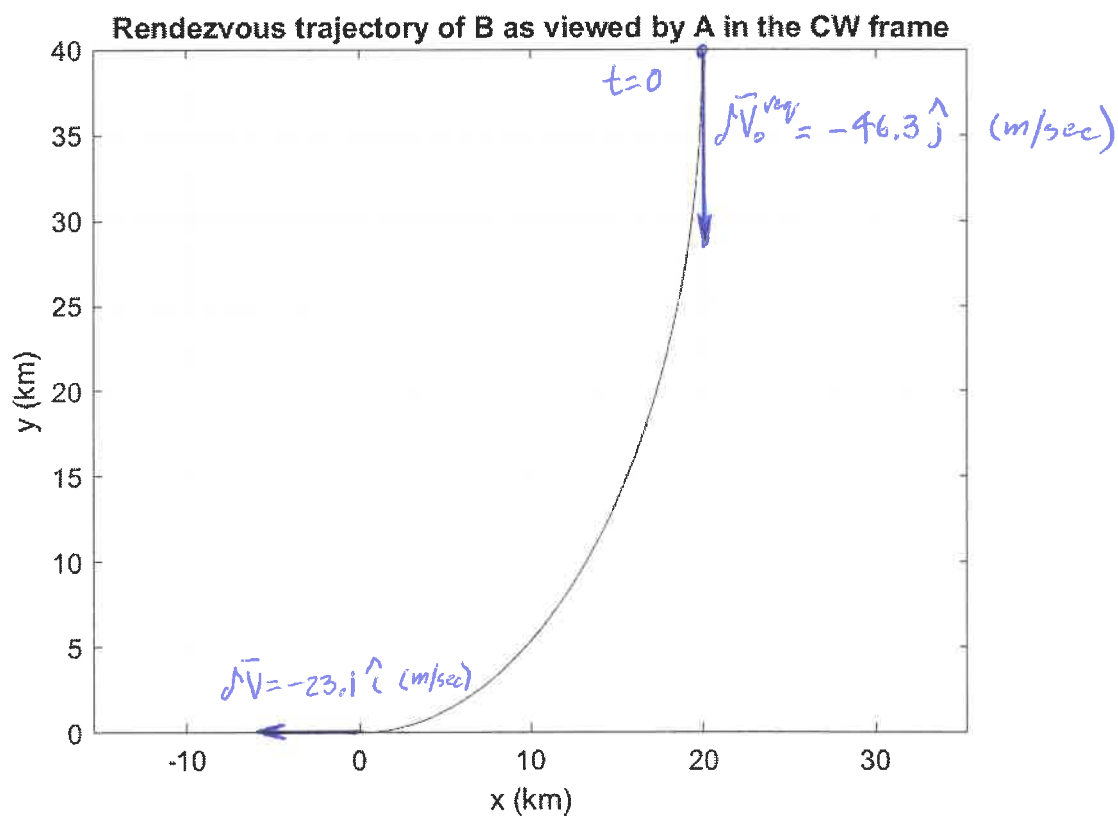
$$= \begin{bmatrix} 3.4707\times10^{-3} & 0 \\ -6.9415\times10^{-3} & 0 \end{bmatrix}\begin{bmatrix} 20 \\ 40 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ -0.0463 \end{bmatrix}$$

$$= \begin{bmatrix} -0.0231 \\ 0 \end{bmatrix} \text{ (km/sec)}$$

or

$$\boxed{\delta\bar{V} = \begin{bmatrix} -23.1 \\ 0 \end{bmatrix} \text{ (m/sec)}}$$

The trajectory of the chaser can be found from (9.117a) as t is varied from $t=0$ to $t=T/4$ and is plotted on the following page.


Rendezvous trajectory of B as viewed by A in the CW frame
t=0
$\delta\bar{V}_0^{req} = -46.3\,\hat{j}$ (m/sec)
$\delta\bar{V} = -23.1\,\hat{i}$ (m/sec)
y (km)
x (km)
40
35
30
25
20
15
10
5
0
-10
0
10
20
30