4) Derive the expression (8.26) [in the notes] for t_p, the transfer time on a parabolic orbit between points P_1 and P_2. Start with Eq. (8.15) for an elliptic orbit, proceed to the limit as $a \to \infty$. Be sure to account for the two cases $\theta < \pi$ and $\theta > \pi$.

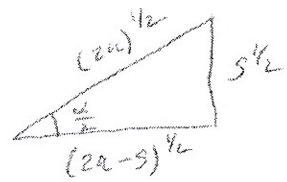
Let $\varepsilon = \frac{1}{a}$

As $a \to \infty$, $\varepsilon \to 0$.

Using (8.16) in the notes

$$\sin\left(\frac{\alpha}{2}\right) = \left(\frac{s}{2a}\right)^{1/2}$$

$$\cos\left(\frac{\alpha}{2}\right) = \left(\frac{2a-s}{2a}\right)^{1/2}$$



$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 2\left(\frac{s}{2a}\right)^{1/2}\left(\frac{2a-s}{2a}\right)^{1/2} = \left(\frac{2s}{a} - \frac{s^2}{a^2}\right)^{1/2}$$

$$= \left(2s\varepsilon - s^2\varepsilon^2\right)^{1/2} = (2s\varepsilon)^{1/2}\left(1 - \frac{s\varepsilon}{2}\right)^{1/2}$$

Using the Taylor series expansion

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

$$\sin\alpha = (2s\varepsilon)^{1/2}\left[1 - \frac{s\varepsilon}{4} - \frac{s^2\varepsilon^2}{16} - \cdots\right] \qquad (1)$$

Also from (8.16) in the notes

$$\alpha = 2\sin^{-1}\left(\frac{s}{2a}\right)^{1/2} = 2\sin^{-1}\left(\frac{s\varepsilon}{2}\right)^{1/2}$$

Using the Taylor series expansion

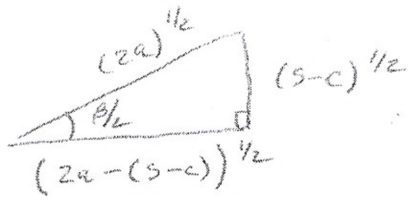
$$\sin^{-1}x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots$$

$$\alpha = 2\left[\left(\frac{s\varepsilon}{2}\right)^{1/2} + \frac{1}{6}\left(\frac{s\varepsilon}{2}\right)^{3/2} + \frac{3}{40}\left(\frac{s\varepsilon}{2}\right)^{5/2} + \cdots\right]$$

$$\alpha = (2s\varepsilon)^{1/2}\left[1 + \frac{1}{6}\left(\frac{s\varepsilon}{2}\right) + \frac{3}{40}\left(\frac{s\varepsilon}{2}\right)^{2} + \cdots\right] \qquad (2)$$

For $0<\theta<\pi$, using (8.18) in the notes, $\beta = \beta_0$ where $0 \le \beta_0 \le \pi$. Therefore (8.17) is written as

$$\sin\left(\frac{\beta}{2}\right) = \left(\frac{s-c}{2a}\right)^{1/2}$$



$$\cos\left(\frac{\beta}{2}\right) = \left(\frac{2a-(s-c)}{2a}\right)^{1/2}$$

$$\sin\beta = 2\sin\frac{\beta}{2}\cos\frac{\beta}{2} = 2\left(\frac{s-c}{2a}\right)^{1/2}\left(\frac{2a-(s-c)}{2a}\right)^{1/2}$$

$$= \left(\frac{2(s-c)}{a} - \frac{(s-c)^2}{a^2}\right)^{1/2} = \left(2(s-c)\varepsilon - (s-c)^2\varepsilon^2\right)^{1/2}$$

$$= \left[2(s-c)\varepsilon\right]^{1/2}\left(1 - \frac{(s-c)\varepsilon}{2}\right)^{1/2}$$

$$\sin\beta = \left[2(s-c)\varepsilon\right]^{1/2}\left[1 - \frac{(s-c)\varepsilon}{4} - \frac{(s-c)^2\varepsilon^2}{16} - \cdots\right] \qquad (3)$$

Also from (8.17) in the notes

$$\beta = 2\sin^{-1}\left(\frac{s-c}{2a}\right)^{1/2} = 2\sin^{-1}\left(\frac{(s-c)\varepsilon}{2}\right)^{1/2}$$

$$\beta = \left[2(s-c)\varepsilon\right]^{1/2}\left[1 + \frac{1}{6}\left(\frac{(s-c)\varepsilon}{2}\right) + \frac{3}{40}\left(\frac{(s-c)\varepsilon}{2}\right)^{2} + \cdots\right] \qquad (4)$$

Using (8.15) in the notes

$$\sqrt{\mu}\,(t_2 - t_1) = a^{3/2}\left[\alpha - \beta - (\sin\alpha - \sin\beta)\right]$$

$$\sqrt{\mu}\, t_p = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{3/2}}\left[\alpha - \beta - (\sin\alpha - \sin\beta)\right] \quad (5)$$

Substituting (1), (2), (3) & (4) into (5)

$$\begin{aligned}
\sqrt{\mu}\, t_p = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{3/2}} \Bigg\{ & (2s\varepsilon)^{1/2}\left[\cancel{1} + \frac{1}{6}\left(\frac{s\varepsilon}{2}\right) + \frac{3}{40}\left(\frac{s\varepsilon}{2}\right)^2 + \cdots\right] \\
& - \left(2(s-c)\varepsilon\right)^{1/2}\left[\cancel{1} + \frac{1}{6}\left(\frac{(s-c)\varepsilon}{2}\right) + \frac{3}{40}\left(\frac{(s-c)\varepsilon}{2}\right)^2 + \cdots\right] \\
& - (2s\varepsilon)^{1/2}\left[\cancel{1} - \frac{s\varepsilon}{4} - \frac{s^2\varepsilon^2}{16} - \cdots\right] \\
& + \left(2(s-c)\varepsilon\right)^{1/2}\left[\cancel{1} - \frac{(s-c)\varepsilon}{4} - \frac{(s-c)^2\varepsilon^2}{16} - \cdots\right]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\sqrt{\mu}\, t_p = \lim_{\varepsilon \to 0} \Bigg\{ & (2s)^{1/2}\left[\frac{1}{6}\left(\frac{s}{2}\right) + \frac{3}{40}\left(\frac{s}{2}\right)^2 \varepsilon + \cdots\right] \\
& - \left(2(s-c)\right)^{1/2}\left[\frac{1}{6}\left(\frac{s-c}{2}\right) + \frac{3}{40}\left(\frac{s-c}{2}\right)^2 \varepsilon + \cdots\right] \\
& - (2s)^{1/2}\left[-\left(\frac{s}{4}\right) - \left(\frac{s^2}{16}\right)\varepsilon - \cdots\right] \\
& + \left(2(s-c)\right)^{1/2}\left[-\frac{(s-c)}{4} - \frac{(s-c)^2}{16}\varepsilon - \cdots\right]\Bigg\}
\end{aligned}$$

$$\sqrt{\mu}\, t_p = (2s)^{1/2}\frac{1}{6}\left(\frac{s}{2}\right) - \left(2(s-c)\right)^{1/2}\frac{1}{6}\left(\frac{s-c}{2}\right)$$

$$+(2s)^{1/2}\left(\frac{s}{4}\right) - \left(2(s-c)\right)^{1/2}\left(\frac{s-c}{4}\right)$$

$$= \sqrt{2}\,\frac{s^{3/2}}{12} - \sqrt{2}\,\frac{(s-c)^{3/2}}{12} + \sqrt{2}\,\frac{s^{3/2}}{4} - \sqrt{2}\,\frac{(s-c)^{3/2}}{4}$$

$$= \sqrt{2}\,\frac{s^{3/2}}{3} - \sqrt{2}\,\frac{(s-c)^{3/2}}{3}$$

$$\sqrt{\mu}\, t_p = \frac{\sqrt{2}}{3}\left[s^{3/2} - (s-c)^{3/2}\right] \quad (6) \qquad \text{for } 0<\Delta\theta<\pi$$

For $\pi<\Delta\theta<2\pi$, using (8.18) in the notes, $\beta = -\beta_0$ where $0\le\beta_0\le\pi$ so that $\sin\beta = -\sin\beta_0$. Therefore the sign of eqs. (3) and (4) will change. This will lead to

$$\sqrt{\mu}\, t_p = \frac{\sqrt{2}}{3}\left[s^{3/2} + (s-c)^{3/2}\right] \quad (7) \qquad \text{for } \pi<\Delta\theta<2\pi$$

Since $\operatorname{sgn}(\sin\Delta\theta) = \begin{cases} +1 & \text{for } 0<\Delta\theta<\pi \\ -1 & \text{for } \pi<\Delta\theta<2\pi \end{cases}$

Eqs. (6) and (7) can be combined into a single equation as

$$\boxed{\sqrt{\mu}\, t_p = \frac{\sqrt{2}}{3}\left[s^{3/2} - \operatorname{sgn}(\sin\Delta\theta)(s-c)^{3/2}\right]}$$