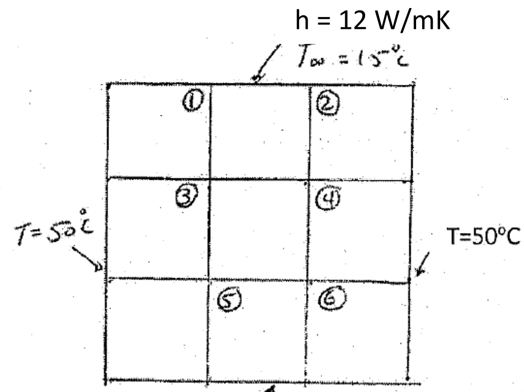
Solutions for Assignment #2

Q1

For the square solid without any heat generation shown in Fig. 2, numerically solve the steady state 2-D heat conduction equation for the temperatures $T_1 - T_4$. Thermal conductivity is k = 15 W/mK and each square mesh is 25 cm wide. The upper surface is subjected to convection to a fluid at T_f = 15 °C with a heat transfer coefficient of h = 12 W/mK. To simplify the analysis, lateral symmetry in the temperature profile can be assumed.

$\boxed{\text{12 marks}}$



For nodes 3, 4, 5 and 6 :

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0$$

Node 3 : $T_4 + 50 + T_5 + T_1 - 4T_3 = 0$

Node 4 : $T_3 + T_2 + 50 + T_6 - 4T_4 = 0$

Node 5 : $50 + T_6 + T_3 + 50 - 4T_5 = 0$

Node 6 : $T_4 + T_5 + 50 + 50 - 4T_6 = 0$

For nodes 1 and 2 :

$$2T_{m-1,n} + T_{m,n+1} + T_{m,n-1} + \frac{2h\Delta x}{k}T_\infty - 2\left(\frac{h\Delta x}{k} + 2\right)T_{m,n} = 0$$

$T = 50°C$
$\Delta x = \Delta y = 25cm$
$k = 15$ W/m k

See the derivation below.

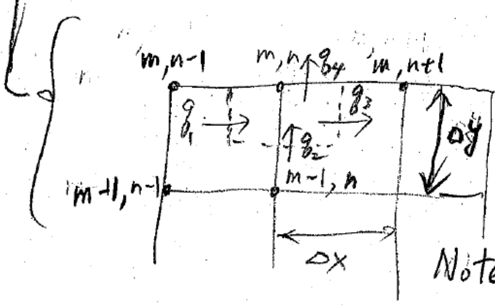
Node 1: $2T_3 + T_2 + 50 + 6 - 4.4T_1 = 0$

Node 2: $2T_4 + 50 + T_1 + 6 - 4.4T_2 = 0$

Solving the above 6 eqn's or 3 eqn's after using the symmetry,

$T_1 = T_2 = 44.7$ °C, $T_3 = T_4 = 48.0$ °C and $T_5 = T_6 = 49.3$ °C

$\boxed{\text{2 marks each for } T_1\text{-}T_6}$



Perform a heat balance on the control volume indicated by a dashed line. $q_1 + q_2 - q_3 - q_4 = 0$

$\boxed{\text{As discussed in class}}$

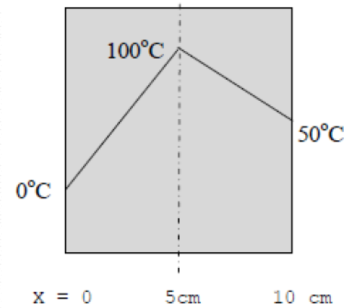
where $q_1 = -k\frac{\Delta y}{2}\left(\frac{T_{m,n} - T_{m,n-1}}{\Delta x}\right)$; $q_3 = -k\frac{\Delta y}{2}\left(\frac{T_{m,n+1} - T_{m,n}}{\Delta x}\right)$

$q_2 = -k\Delta x\left(\frac{T_{m,n} - T_{m-1,n}}{\Delta y}\right)$; $q_4 = h\Delta x(T_{m,n} - T_\infty)$

Note: $q_i = -k\frac{\Delta T}{\Delta X}A_x$ where A_x = width × unit depth $= \Delta X$ or $\frac{\Delta y}{2}$

Q2 Consider a slab with an initial temperature profile as shown on right. Calculate the temperature profile numerically as done in class, using $\alpha\Delta t/(\Delta x)^2 = 0.1$ and 0.6. Show all the temperature profiles up to 10 time steps for each case.

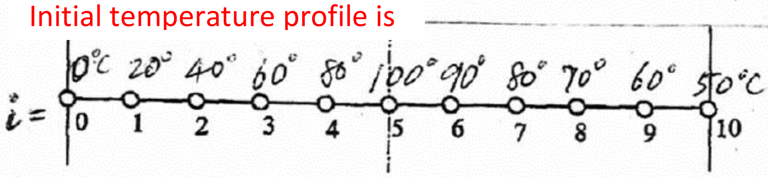
14 marks



Solution

1. This solution is based on 11 nodes. ⇒ $\Delta x = 1$ cm.
2. At all times nodes 0 and 10 are kept at 0° and 50° C, respectively. ⇒ $T(0,t) = 0$°C and $T(10,t) = 50$°C
3. At t= 0 s temperature in node 5 is $T(5,0) = 100$°C. Temperatures in other nodes are: $T(0,0) = 0$°C, $T(1,0) = 20$°C, $T(2,0) = 40$°C, $T(3,0) = 60$°C, $T(4,0) = 80$°C, $T(6,0) = 90$°C, $T(7,0) = 80$°C, $T(8,0) = 70$°C, $T(9,0) = 60$°C and $T(10,0) = 50$°C.

Initial temperature profile is



Set the equation used in class for nodes 1 to 9.

$$T_{i,j+1} = T_{i,j} + \underbrace{\frac{\alpha * \Delta t}{(\Delta x)^2}}_{=r} * (T_{i+1,j} - 2 * T_{i,j} + T_{i-1,j}) \quad , \quad j = \text{time step}$$

For r = 0.1

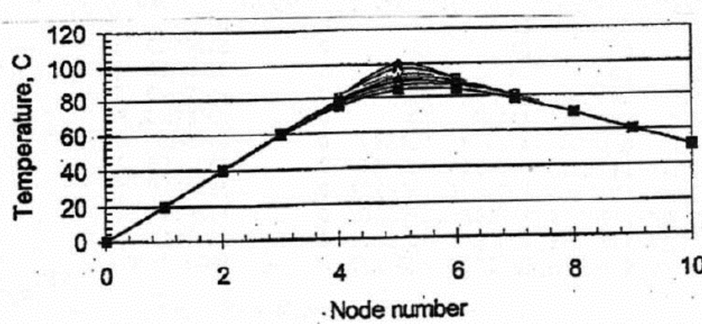
j	t, s	T(0), C	T(1), C	T(2), C	T(3), C	T(4), C	T(5), C	T(6), C	T(7), C	T(8), C	T(9), C	T(10), C
j=0	0	0	20	40	60	80	100	90	80	70	60	50
1	0.119048	0	20	40	60	80	97	90	80	70	60	50
2	0.238096	0	20	40	60	79.7	94.6	89.7	80	70	60	50
3	0.357144	0	20	40	59.97	79.22	92.62	89.22	79.97	70	60	50
4	0.476192	0	20	39.997	59.898	78.635	90.94	88.635	79.898	69.997	60	50
5	0.59524	0	19.9997	39.9874	59.7816	77.9918	89.479	87.9918	79.7816	69.9874	59.9997	50
6	0.714288	0	19.9985	39.96805	59.6232	77.3195	88.18156	87.3195	79.6232	69.96805	59.9985	50
7	0.833336	0	19.99561	39.93661	59.42732	76.63608	87.00915	86.63608	79.42732	69.93661	59.99561	50
8	0.952384	0	19.99015	39.89158	59.19912	75.95251	85.93453	85.95251	79.19912	69.89158	59.99015	50
9	1.071432	0	19.98127	39.83219	58.94371	75.27537	84.93813	85.27537	78.94371	69.83219	59.98127	50
10	1.19048	0	19.96824	39.75825	58.66572	74.60848	84.00558	84.60848	78.66572	69.75825	59.96824	50

5 marks for all correct answers

For r = 0.6

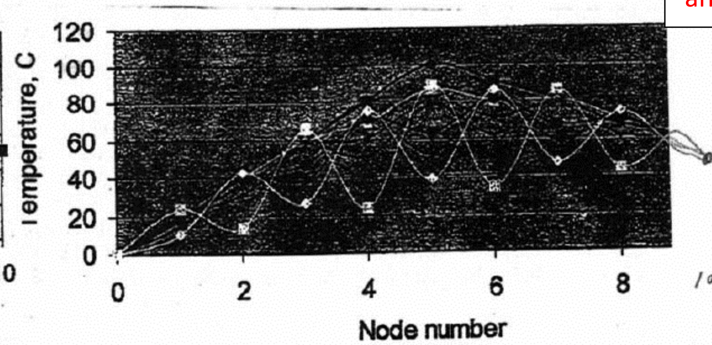
j	t, s	T(0), C	T(1), C	T(2), C	T(3), C	T(4), C	T(5), C	T(6), C	T(7), C	T(8), C	T(9), C	T(10), C
j=0	0	0	20	40	60	80	100	90	80	70	60	50
1	0.714286	0	20	40	60	80	82	90	80	70	60	
2	1.428572	0	20	40	60	69.2	85.6	79.2	80	70	60	
3	2.142858	0	20	40	53.52	73.52	71.92	83.52	73.52	70	60	
4	2.857144	0	20	36.112	57.408	60.56	79.84	70.56	77.408	66.112	60	
5	3.57143	0	17.6672	39.2224	46.5216	70.2368	62.704	80.2368	66.5216	69.2224	57.6672	
6	4.285716	0	20	30.6688	56.3712	51.488	77.74336	61.488	76.3712	60.6688	60	
7	5.000002	0	14.40128	39.68896	38.01984	70.17114	52.23693	80.17114	58.01984	69.68896	54.40128	
8	5.714288	0	20.93312	23.51488	58.31209	40.11983	79.75798	50.11983	78.31209	53.51488	60.93312	
9	6.428574	0	9.922304	42.84415	26.51841	74.81807	38.1922	84.81807	46.51841	72.84415	49.9223	
10	7.14286	0	23.72203	13.2956	65.29365	23.86275	88.14325	33.86275	85.29365	43.2956	63.72203	50

5 marks for all correct answers



For r = 0.1

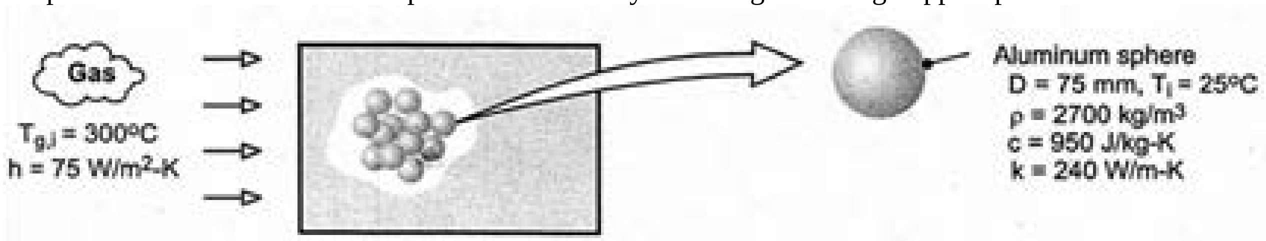
2 marks for correct figure



For r = 0.6

2 marks for correct figure

Q3. Thermal energy storage systems for a high temperature gas reactor would involve a packed bed of solid spheres, through which a hot gas flows if the system is being charged to store heat, or a cold gas if heat is being discharged. In a charging process, heat transfer from the hot gas increases the thermal energy stored within the colder spheres; during discharge, the stored energy decreases as heat is transferred from the warmer spheres to the cooler gas. Consider a packed bed of 75-mm-diameter aluminum spheres (ρ=2,700 kg/m^3, c_p= 950 J/kgK, k= 240 W/mK) and a charging process for which a hot gas enters the storage unit at a temperature of T_g=300°C. If the initial temperature of the spheres is T_i=25°C and the convection heat transfer coefficient is h = 75 W/m^{2}K, how long does it take the sphere near the inlet of the system to accumulate 90% of the maximum possible thermal energy? What is the corresponding temperature at the center of the sphere? Is there any advantage to using copper spheres instead of aluminum?



12 marks

Answer: Calculate Bi= h(r_o/3)/k = 75 W/m^{2}Kx0.0125m/240W/mK = 0.0039 < 0.1, so the lumped heat capacity method can be used.

$$\frac{Q}{\rho c V \theta_i} = 0.90 = 1 - \exp(-t/\tau_t)$$

where $\tau_t = \rho V c / h A_s = \rho D c / 6h = 2700\,\text{kg/m}^3 \times 0.075\,\text{m} \times 950\,\text{J/kg}\cdot\text{K} / 6 \times 75\,\text{W/m}^2\cdot\text{K} = 427\text{s}$. Hence,

$$t = -\tau_t \ln(0.1) = 427\text{s} \times 2.30 = 984\text{s}$$ 5 marks

From Eq. (5.6), the corresponding temperature at any location in the sphere is

$$T(984\text{s}) = T_{g,i} + (T_i - T_{g,i})\exp(-6ht/\rho D c)$$

$$T(984\text{s}) = 300°\text{C} - 275°\text{C}\exp\left(-6 \times 75\,\text{W/m}^2\cdot\text{K} \times 984\text{s} / 2700\,\text{kg/m}^3 \times 0.075\,\text{m} \times 950\,\text{J/kg}\cdot\text{K}\right)$$

$$T(984)\text{s} = 272.5°\text{C}$$ 5 marks

Obtaining the density and specific heat of copper , we see that $(\rho c)_{Cu} \approx 8900\,\text{kg/m}^3 \times 400\,\text{J/kg}\cdot\text{K} = 3.56 \times 10^6\,\text{J/m}^3\cdot\text{K} > (\rho c)_{Al} = 2.57 \times 10^6\,\text{J/m}^3\cdot\text{K}$. Hence, for an equivalent sphere diameter, the copper can store approximately 38% more thermal energy than the aluminum.

2 marks for stating copper can store more energy than aluminum

Q4. A 0.6 cm diameter steel rod at 38 °C is suddenly immersed in a hot liquid at 93 °C with a heat transfer coefficient of h = 11 W/m^2°C. The length of the rod is L and it has the following thermophysical properties: k = 43 W/m°C, C_p = 473 J/kg°C, ρ = 7,801 kg/m^3 and $\alpha = k/\rho C_p = 1.172\times10^{-5}$ m^2/s. Determine the time required for the rod to warm to 88 °C.

10 marks total

Solution

Calculate the Biot number

$$B_i = \frac{\bar{h}_c D}{4k} = \frac{11\ \frac{W}{m^2K}(0.006\,m)}{4\left(43\ \frac{W}{mK}\right)} = 0.00038 << 0.1$$

2 marks

Therefore, the internal resistance of the rod is negligible.

The temperature-time history of the rod, from *equation (2.84)* is:

$$\frac{T-T_\infty}{T_o-T_\infty} = \exp\left(-\frac{\bar{h}_c A_s}{c\rho V}t\right)$$

$$\frac{\bar{h}_c A_s}{c\rho V} = \frac{\bar{h}_c \pi D L}{c\rho \frac{\pi}{4} D^2 L} = \frac{4\bar{h}_c}{c\rho D} = \frac{4\left(110\ \frac{W}{m^2K}\right)\left(\frac{J}{Ws}\right)}{473\ \frac{J}{kgK}\left(7801\ \frac{kg}{m^3}\right)(0.006\,m)} = 0.001987\ s^{-1}$$

$$\frac{T-T_\infty}{T_o-T_\infty} = e^{-0.001987t}$$

Solve for t by taking ln of both sides and noting that 1/0.001987 = 503.3 sec,

$$t = -503.3\ \ln\left(\frac{T-T_\infty}{T_o-T_\infty}\right)$$

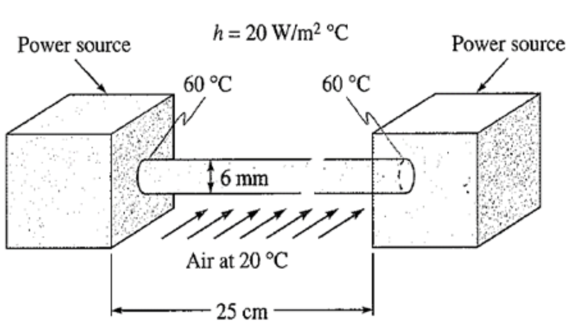
The time required to reach 88°C is:

$$t = -503.3\ \ln\left(\frac{88-93}{38-93}\right) = 1206.8\ \text{sec}$$

8 marks

Q5. Two power sources contain radioisotopes that generate heat. The power sources are at 60 °C and connected by a cylindrical metal rod with k = 60.5 W/m°C. The convective heat transfer coefficient between the rod and air that flows over it is 20 W/m^2°C. The air temperature is 20 °C. Estimate the heat transfer rate from the rod to the surrounding air if the diameter of the rod is 6 mm and the length is 25 cm.

10 marks total

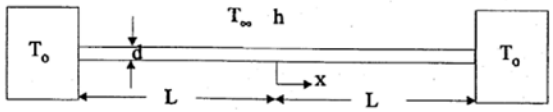


Hint for Q5.

Hint: Measure x from the midpoint of the fin, so the temperature profile in the fin is given by,

$$\theta = C_1 \cosh mx + C_2 \sinh mx$$

The midpoint of the fin at x = 0 can be treated as an insulated fin tip.



Then, apply the BCs: T(x = -L) = T(x = L) = 60 °C.

The two ends of the fin at x = +/- L can be considered as the fin base.

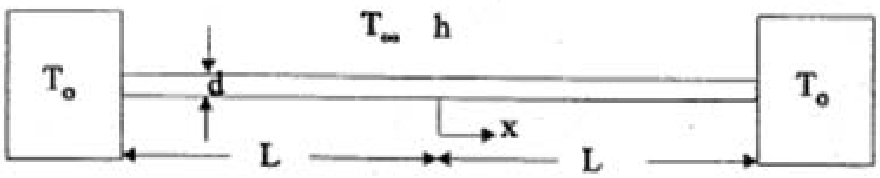
$T_o = 60\ ^\circ C$

$T_\infty = 20\ ^\circ C$

$h = 20\ W/m^2\ ^\circ C$

$2L = 0.25\ m$

$d = 0.006\ m$



Plain-carbon steel

Assumptions: 1. Steady state.

2. Constant properties.

3. One-dimensional temperature distribution.

From Table A1, for plain carbon steel, k = 60.5 W/m K

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad \theta\ (x = \pm L) = \theta_o = T_o - T_\infty \qquad \theta = C_1 \cosh mx + C_2 \sinh mx$$

$$\theta\ (x = -L) = \theta_o = C_1 \cosh mL - C_2 \sinh mL \quad (1) \qquad \theta\ (x = L) = \theta_o = C_1 \cosh mL + C_2 \sinh mL \quad (2)$$

Subtract (1) from (2) to get

$$C_2 = 0 \qquad C_1 = \frac{\theta_o}{\cosh mL} \qquad \theta = \theta_o \frac{\cosh mx}{\cosh mL}$$

Note: Eq (2) – Eq. (1) gives $\Theta_o - \Theta_o = 0 = 2C_2 \sinh mL$

Total heat transfer rate from the fin is given by

$$q^c = \int_{-L}^{L} hP\ \theta\ dx = \int_{-L}^{L} hP\ \theta_o \frac{\cosh mx}{\cosh mL}\ dx = \frac{2hP\theta_o}{m} \tanh mL$$

4 marks

$$\theta_o = T_o - T_\infty = 60 - 20 = 40\ ^\circ C \qquad m = \sqrt{\frac{4h}{kd}} = \sqrt{\frac{4 \times 20}{60.5 \times 0.006}} = 14.85\ m^{-1}$$

$$mL = 14.85 \times 0.125 = 1.856$$

$$q^c = \frac{2 \times 20 \times \pi \times 0.006 \times 40}{14.85} \tanh (1.856) = \underline{1.93\ W}$$

6 marks