11.2 The Restricted Three-Body Problem

(Earth-Moon-Spaceship Problem)

Assumptions

- Two masses are of finite size and move in coplanar circular orbits about the center of mass of the system.
- The third mass is of infinitesimal size.

Define $\mu = \dfrac{m_2}{m_1 + m_2}$

If $m_2 \leq m_1$, then $\mu \leq \frac{1}{2}$

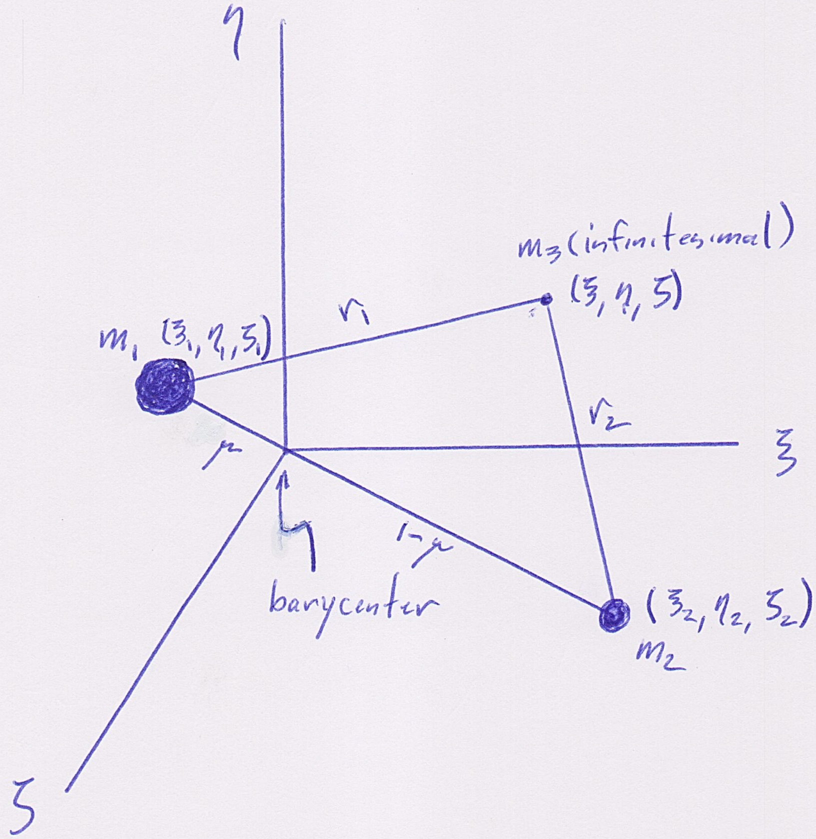
Non-dimensionalize distances by d (distance between m_1 and m_2)

Non-dimensionalize time t by $\left[\dfrac{d^3}{G(m_1+m_2)}\right]^{1/2}$

The dimensionless angular velocity about the center of mass is $\dfrac{d\theta}{dt} = 1$.

Coordinates of m_1 are ξ_1, η_1, ζ_1
Coordinates of m_2 are ξ_2, η_2, ζ_2
Coordinates of m_3 are ξ, η, ζ

measured from non-rotating coordinate system at center of mass.



The equations of motion of m_3 are

$$\ddot{\xi} = (1-\mu)\frac{\xi_1 - \xi}{r_1^3} + \mu \frac{\xi_2 - \xi}{r_2^3} \qquad (11.26a)$$

$$\ddot{\eta} = (1-\mu)\frac{\eta_1 - \eta}{r_1^3} + \mu \frac{\eta_2 - \eta}{r_2^3} \qquad (11.26b)$$

$$\ddot{\zeta} = (1-\mu)\frac{\zeta_1 - \zeta}{r_1^3} + \mu \frac{\zeta_2 - \zeta}{r_2^3} \qquad (11.26c)$$

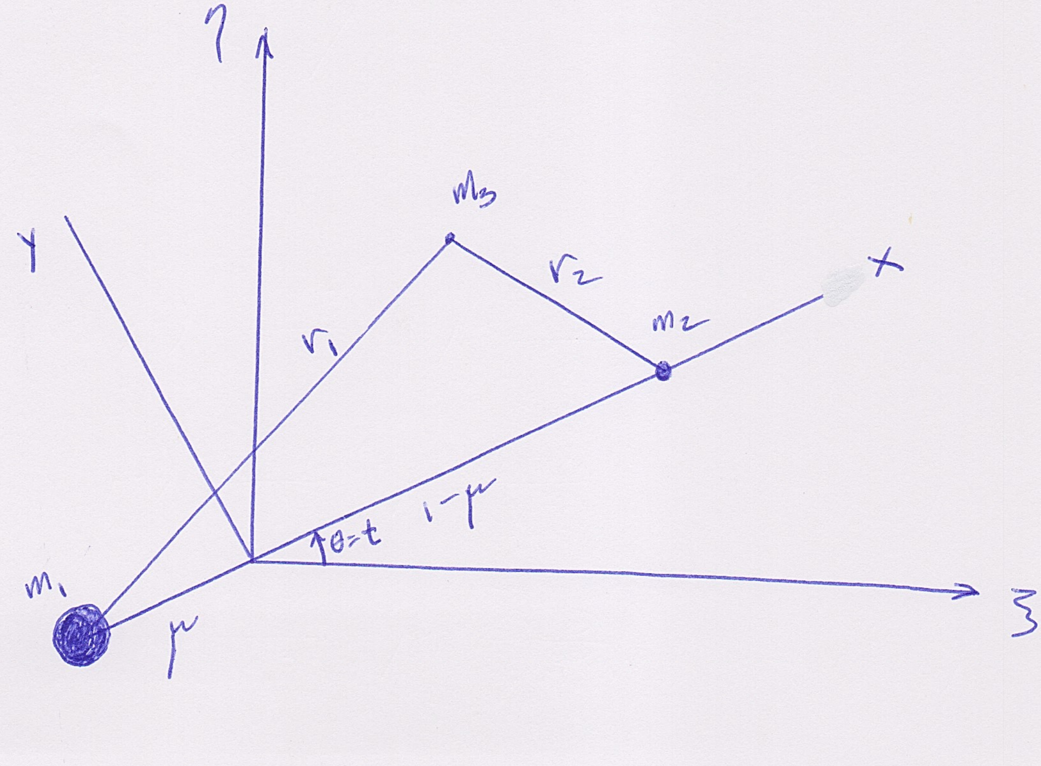
where

$$r_1^2 = (\xi_1 - \xi)^2 + (\eta_1 - \eta)^2 + (\zeta_1 - \zeta)^2$$

$$r_2^2 = (\xi_2 - \xi)^2 + (\eta_2 - \eta)^2 + (\zeta_2 - \zeta)^2$$

Without loss of generality, let the ζ-axis be perpendicular to the plane of rotation of the two massive particles so that $\zeta_1 = \zeta_2 = 0$

Introduce a rotating coordinate system (x, y, z) having origin at the barycenter with the z-axis coinciding with the ζ-axis (perpendicular to the plane of the paper).



In the rotating frame

Coordinates of m_1 are $(X_1, 0, 0)$

Coordinates of m_2 are $(X_2, 0, 0)$

Coordinates of m_3 are (x, y, z)

$$X_1 = -\mu \qquad X_2 = 1-\mu$$

and $X_2 - X_1 = 1$.

Can write

$$r_1^2 = (X_1 - x)^2 + y^2 + z^2$$

$$r_2^2 = (X_2 - x)^2 + y^2 + z^2$$

Coordinate transformation

$$\left.\begin{aligned} \xi &= x\cos t - y\sin t \\ \eta &= x\sin t + y\cos t \\ \zeta &= z \end{aligned}\right\} \qquad (11.27)$$

Differentiate (11.27) twice and sub. into (11.26)

$$(\ddot{x}-2\dot{y}-x)\cos t-(\ddot{y}+2\dot{x}-y)\sin t=$$

$$\left[(1-\mu)\frac{x_1-x}{r_1^3}+\mu\frac{x_2-x}{r_2^3}\right]\cos t+\left[\frac{1-\mu}{r_1^3}+\frac{\mu}{r_2^3}\right]y\sin t \qquad (11.28a)$$

$$(\ddot{x}-2\dot{y}-x)\sin t+(\ddot{y}+2\dot{x}-y)\cos t=$$

$$\left[(1-\mu)\frac{x_1-x}{r_1^3}+\mu\frac{x_2-x}{r_2^3}\right]\sin t-\left[\frac{1-\mu}{r_1^3}+\frac{\mu}{r_2^3}\right]y\cos t \qquad (11.28b)$$

$$\ddot{z}=-\left[\frac{1-\mu}{r_1^3}+\frac{\mu}{r_2^3}\right]z \qquad (11.28c)$$

Mult. (11.28a) by cos t, (11.28b) by sin t and add, then mult. (11.28a) by −sin t, (11.28b) by cos t and add

$$\left.\begin{aligned}\ddot{x}-2\dot{y}-x&=-(1-\mu)\frac{x-x_1}{r_1^3}-\mu\frac{x-x_2}{r_2^3}\\ \ddot{y}+2\dot{x}-y&=-\left(\frac{1-\mu}{r_1^3}+\frac{\mu}{r_2^3}\right)y\\ \ddot{z}&=-\left(\frac{1-\mu}{r_1^3}+\frac{\mu}{r_2^3}\right)z\end{aligned}\right\}\quad(11.29)$$

Eqs. (11.29) are the equations of motion of the infinitesimal body m_3 with respect to the rotating coordinates.

Define $U = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$

Eqs. (11.29) may be written as

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} \qquad (11.30a)$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y} \qquad (11.30b)$$

$$\ddot{z} = \frac{\partial U}{\partial z} \qquad (11.30c)$$

Mult. (11.30a) by $\dot{x}$, (11.30b) by $\dot{y}$, (11.30c) by $\dot{z}$ and add

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \frac{\partial U}{\partial x}\dot{x} + \frac{\partial U}{\partial y}\dot{y} + \frac{\partial U}{\partial z}\dot{z}$$

which is an exact differential since U depends only on x, y, z.

Integrating

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2U - C \qquad (11.31)$$

where C is a constant of integration called
<u>Jacobi's constant</u>

or

$$V^2 = 2U - C \qquad (11.32)$$

where V is the velocity of the infinitesimal mass m_3
in the rotating frame.

or

$$V^2 = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C \qquad (11.33)$$

Jacobi's constant C is determined from initial
conditions.

For zero particle velocity in the rotating frame,
set V=0 in (11.32)

$$2U = C$$

or

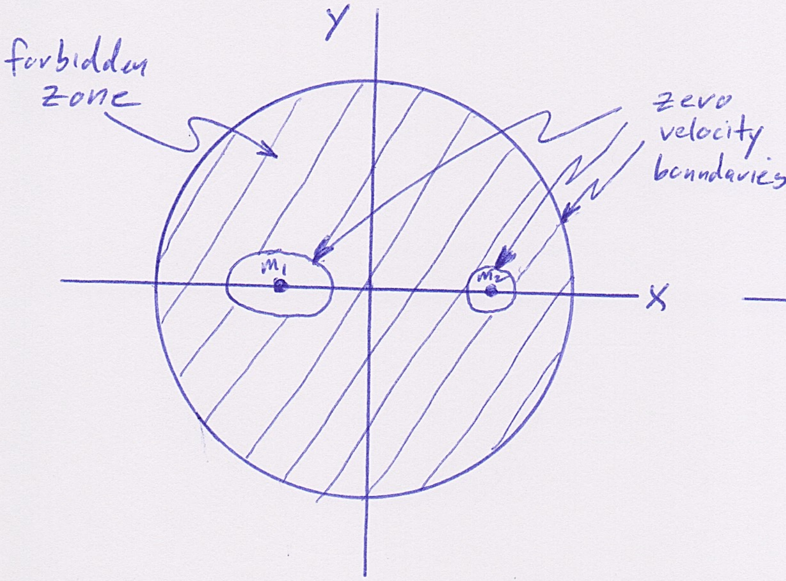
$$x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} = C \qquad (11.34)$$

For a given value of C, (11.34) defines the boundaries of regions in which the particle m_3 must be found.

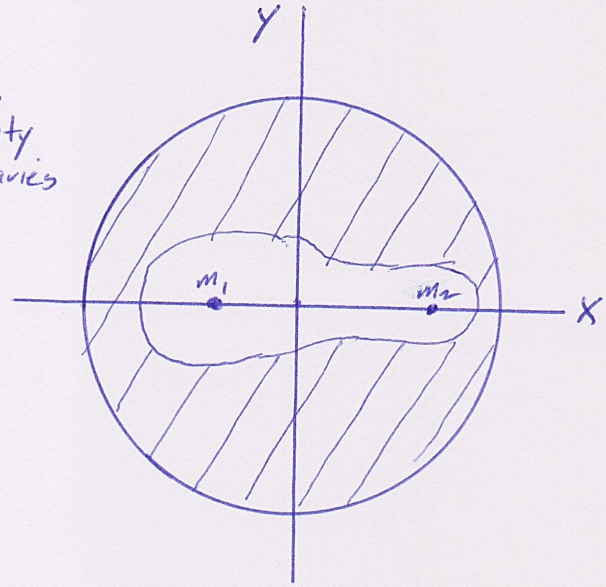
In these regions, require $2U > C$ otherwise $V^2 < 0$
$\Rightarrow V$ is imaginary (see eq. (11.32))

Suppose both C and x^2+y^2 are large.
Then from (11.34) $x^2+y^2 \sim C$,
The region outside this circle is accessible.

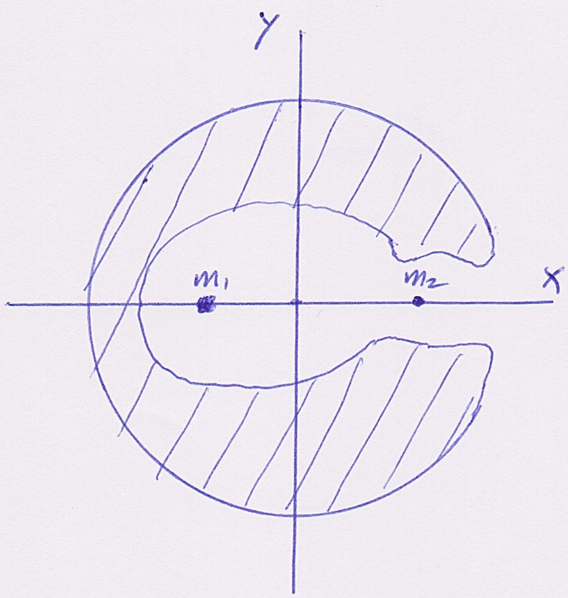
If C is large and r_1 or r_2 are small
Then oval regions enclosing m_1 and m_2 are also accessible


forbidden
zone
zero
velocity
boundaries
Y
X
m_1
m_2

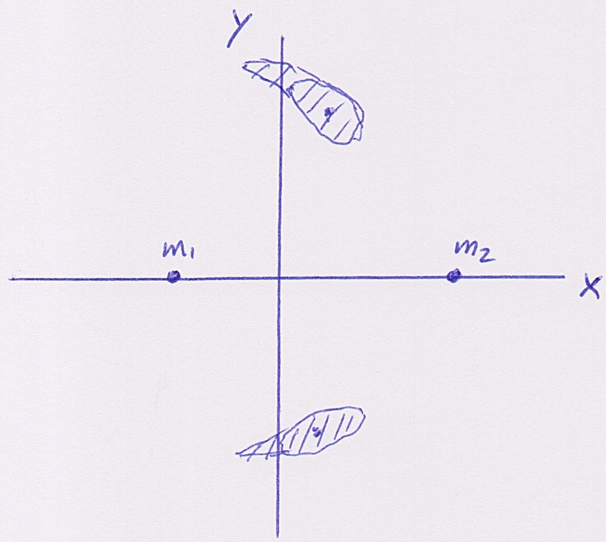
C_1


Y
X
m_1
m_2

C_3

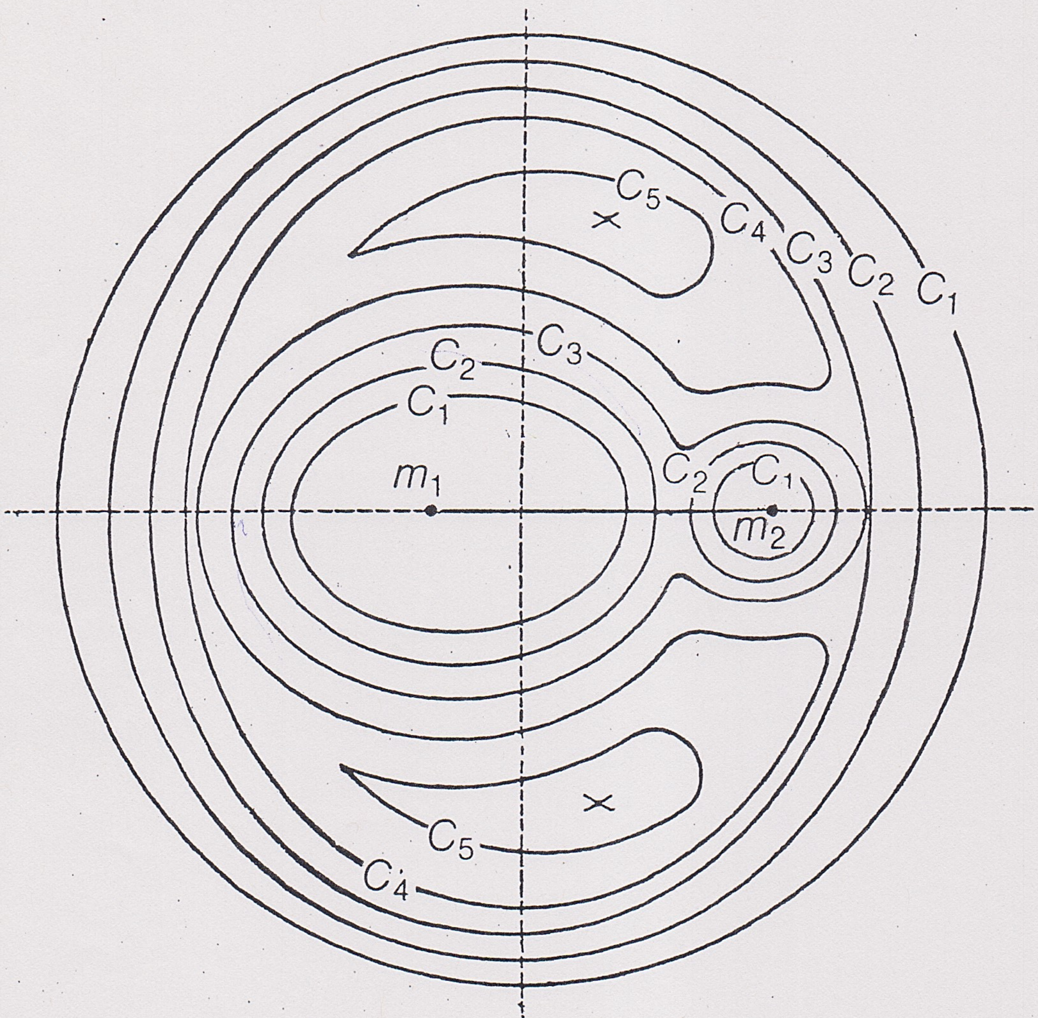

Y
X
m_1
m_2

C_4


Y
X
m_1
m_2

C_5

The Restricted Problem of Three Bodies
Contours of zero relative velocity in the plane z=0
($m_1>m_2$) ($m_3<<m_1+m_2$) ($C_1>C_2>C_3>C_4>C_5$)



Determination of the Location of the Lagrange Points

For stationary configurations, relative velocity and relative acceleration must be zero. From eq. (11.30) this requires

$$\frac{\partial U}{\partial x} = 0 \qquad \frac{\partial U}{\partial y} = 0 \qquad \frac{\partial U}{\partial z} = 0$$

$$U = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

$$\frac{\partial U}{\partial x} = x - \frac{1-\mu}{r_1^2}\frac{\partial r_1}{\partial x} - \frac{\mu}{r_2^2}\frac{\partial r_2}{\partial x}$$

$$\frac{\partial U}{\partial y} = y - \frac{1-\mu}{r_1^2}\frac{\partial r_1}{\partial y} - \frac{\mu}{r_2^2}\frac{\partial r_2}{\partial y}$$

$$\frac{\partial U}{\partial z} = - \frac{1-\mu}{r_1^2}\frac{\partial r_1}{\partial z} - \frac{\mu}{r_2^2}\frac{\partial r_2}{\partial z}$$

$$r_1^2 = (\overset{-\mu}{X_1} - x)^2 + y^2 + z^2 = (x+\mu)^2 + y^2 + z^2$$

$$r_2^2 = (\underset{1-\mu}{X_2} - x)^2 + y^2 + z^2 = (x+\mu-1)^2 + y^2 + z^2$$

$$\frac{\partial r_1}{\partial x} = \frac{x+\mu}{r_1} \qquad \frac{\partial r_2}{\partial x} = \frac{x+\mu-1}{r_2}$$

$$\frac{\partial r_1}{\partial y} = \frac{y}{r_1} \qquad \frac{\partial r_2}{\partial y} = \frac{y}{r_2}$$

$$\frac{\partial r_1}{\partial z} = \frac{z}{r_1} \qquad \frac{\partial r_2}{\partial z} = \frac{z}{r_2}$$

$$\left.\begin{aligned} x - \frac{(1-\mu)}{r_1^3}(x+\mu) - \frac{\mu}{r_2^3}(x+\mu-1) &= 0 \\ y - \frac{(1-\mu)}{r_1^3}\, y - \frac{\mu}{r_2^3}\, y &= 0 \\ -\frac{(1-\mu)}{r_1^3}\, z - \frac{\mu}{r_2^3}\, z &= 0 \end{aligned}\right\} \quad (11.35)$$

The solution of (11.35) gives the location of the Lagrange points.

For L_1, L_2, L_3, set $y = z = 0$ in (11.35)

$$x - \frac{(1-\mu)}{r_1^3}(x+\mu) - \frac{\mu}{r_2^3}(x+\mu-1) = 0 \qquad (11.36)$$

where $r_1 = |x+\mu|$ $\quad r_2 = |x+\mu-1|$

The roots of (11.36) give the location of L_1, L_2, L_3.

For L_4, L_5, setting $z=0$ in (11.35) shows

$$r_1 = r_2 = 1$$

is a solution resulting in the equilateral triangle configuration.

EXAMPLE

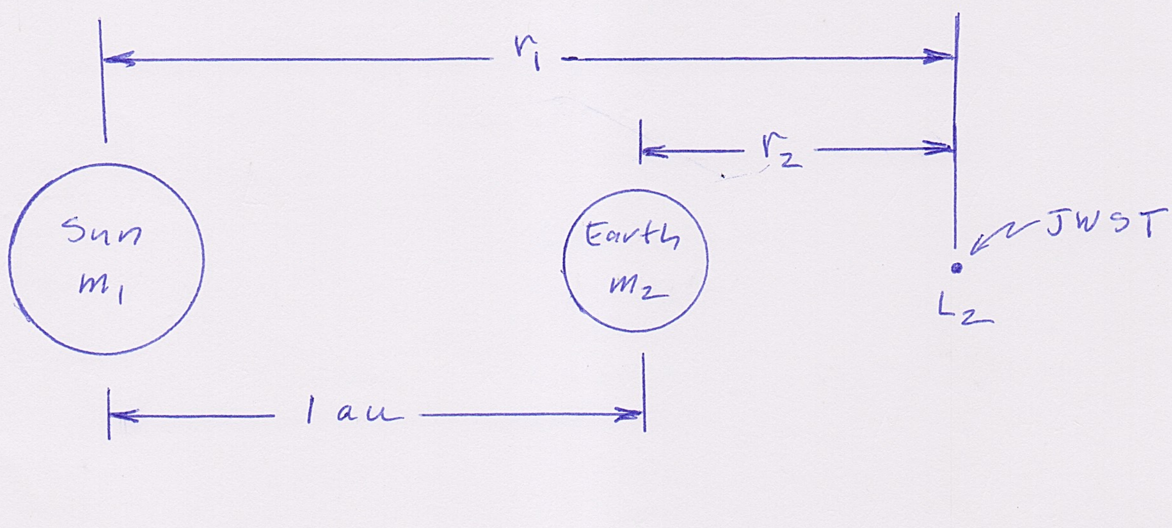
In 2021, the James Webb Space Telescope is scheduled to be launched and positioned at the earth-sun L_2 Lagrangian point, about 1.50×10^6 km from earth. (The moon's distance from the earth is 3.844×10^5 km) ($\sim 4\times$)

a) Calculate the distance from the center of the sun in au of the earth-sun barycenter.

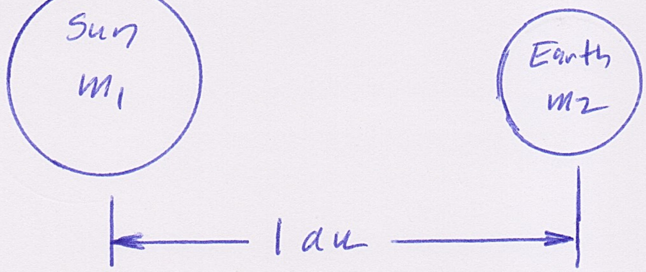
b) Determine the value of Jacobi's constant needed to reach L_2 with zero relative velocity.

c) Determine the minimum burnout velocity of the Ariane 5 launch vehicle at an altitude of 300 km required for the JWST to reach L_2.

d) Compare the value of velocity obtained in part (c) with the burnout velocity required for a spacecraft to (i) escape the earth and (ii) escape the solar system.



a)



$$\mu = \frac{m_2}{m_1 + m_2} = \frac{5.974 \times 10^{24}}{1.989 \times 10^{30} + 5.974 \times 10^{24}} = 0.00000300351$$

Origin at center of sun

$$r_{cm} = \frac{\sum m_i r_i}{\sum m_i} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} = \frac{m_1 (0) + m_2 (1)}{m_1 + m_2} = \mu$$

$$\boxed{r_{cm} = 0.0000030051 \text{ au}}$$

Sun's radius

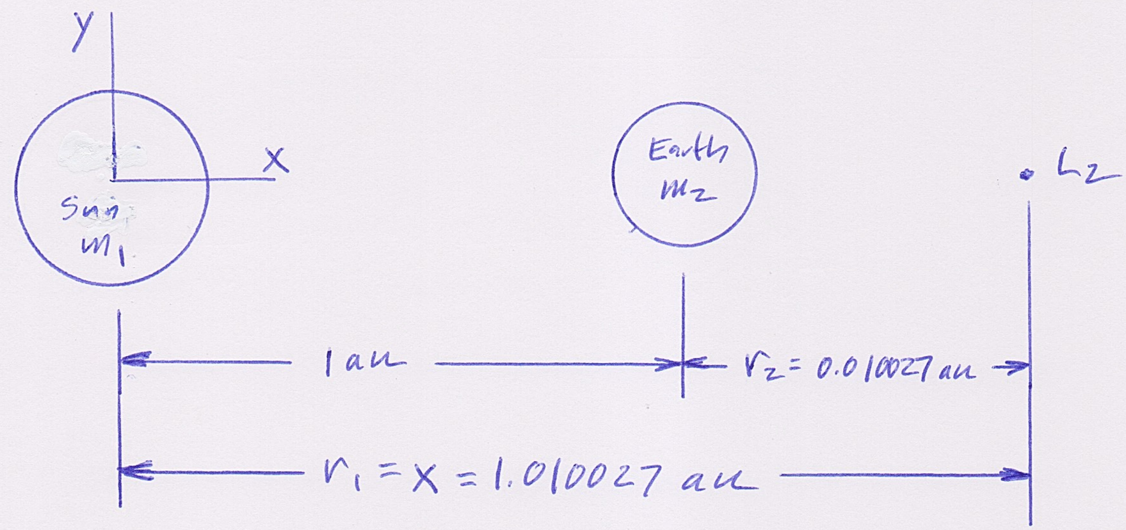
$$r_{sun} = \frac{6.9599 \times 10^5 \text{ km}}{1.495978 \times 10^8 \frac{\text{km}}{\text{au}}} = 0.004652 \text{ au}$$

$$\frac{r_{cm}}{r_{sun}} = \frac{0.0000030051}{0.004652} = 0.000646$$

b) From (11.34) for $V=0$ at L_2

$$C = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2}$$

Distance of L_2 from earth $= \dfrac{1.50 \times 10^6 \text{ km}}{1.495978 \times 10^8 \frac{\text{km}}{\text{au}}} = 0.010027 \text{ au}$



Origin at barycenter (center of sun)

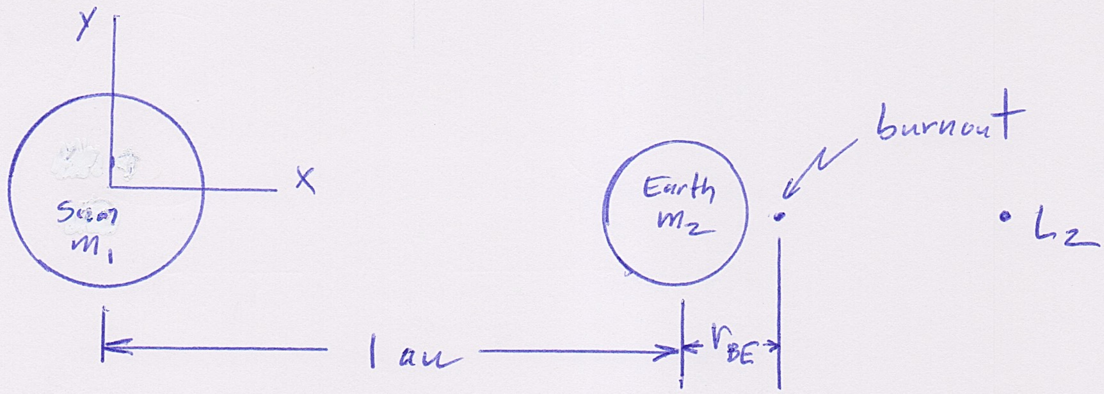
$$C = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2}$$

$$= (1.010027)^2 + (0)^2 + \frac{2(1 - 3.00351 \times 10^{-6})}{1.010027} + \frac{2(3.00351 \times 10^{-6})}{0.010027}$$

$$C = \underline{\underline{3.00089}}$$

c) Distance of burnout from earth

$$r_{BE} = \frac{6368 + 300}{1.495978 \times 10^8} = 4.65782 \times 10^{-5} \text{ au}$$



At burnout

$$X = 1 + r_{BE} = 1 + 4.65782 \times 10^{-5} = 1$$

$$y = 0$$

$$r_1 = 1 + r_{BE} = 1$$

$$r_2 = r_{BE} = 4.65782 \times 10^{-5}$$

Applying (11.33) at burnout

$$V_B^2 = X^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C$$

$$= (1)^2 + 0 + \frac{2(1 - 3.00351 \times 10^{-6})}{1} + \frac{2(3.00351 \times 10^{-6})}{4.65782 \times 10^{-5}}$$

$$- 3.00089$$

$$= 1 + 2 + 0.128966 - 3.00089$$

$V_B^2 = 0.128076$

$V_B = 0.357877$ (dimensionless)

$$\tau = \left[\frac{d^3}{G(m_1+m_2)}\right]^{1/2} = \left[\frac{(1.495978 \times 10^8 \text{ km})^3}{1.327 \times 10^{11} \text{ km}^3/\text{sec}^2}\right]^{1/2}$$

$$= 5.02287 \times 10^6 \text{ sec}$$

$$V_B = 0.357877 \left(\frac{1.495978 \times 10^8 \text{ km}}{5.02287 \times 10^6 \text{ sec}}\right)$$

$$\boxed{V_B = 10.6588 \ \frac{\text{km}}{\text{sec}}}$$

d) i) $V_{esc\,earth} = \sqrt{2}\sqrt{\dfrac{\mu_e}{V_{BE}}} = \sqrt{2}\sqrt{\dfrac{3.986 \times 10^5 \text{ km}^3/\text{sec}^2}{6368 + 300 \text{ km}}}$

$$\boxed{V_{esc\,earth} = 10.9097 \ \frac{\text{km}}{\text{sec}}}$$

ii) $V_{esc\,sun} = \sqrt{2}\sqrt{\dfrac{\mu_{sun}}{r_c}} = \sqrt{2}\sqrt{\dfrac{1.327 \times 10^{11} \text{ km}^3/\text{sec}^2}{1.495978 \times 10^8 \text{ km}}}$

$$\boxed{V_{esc\,sun} = 42.12 \ \frac{\text{km}}{\text{sec}}}$$