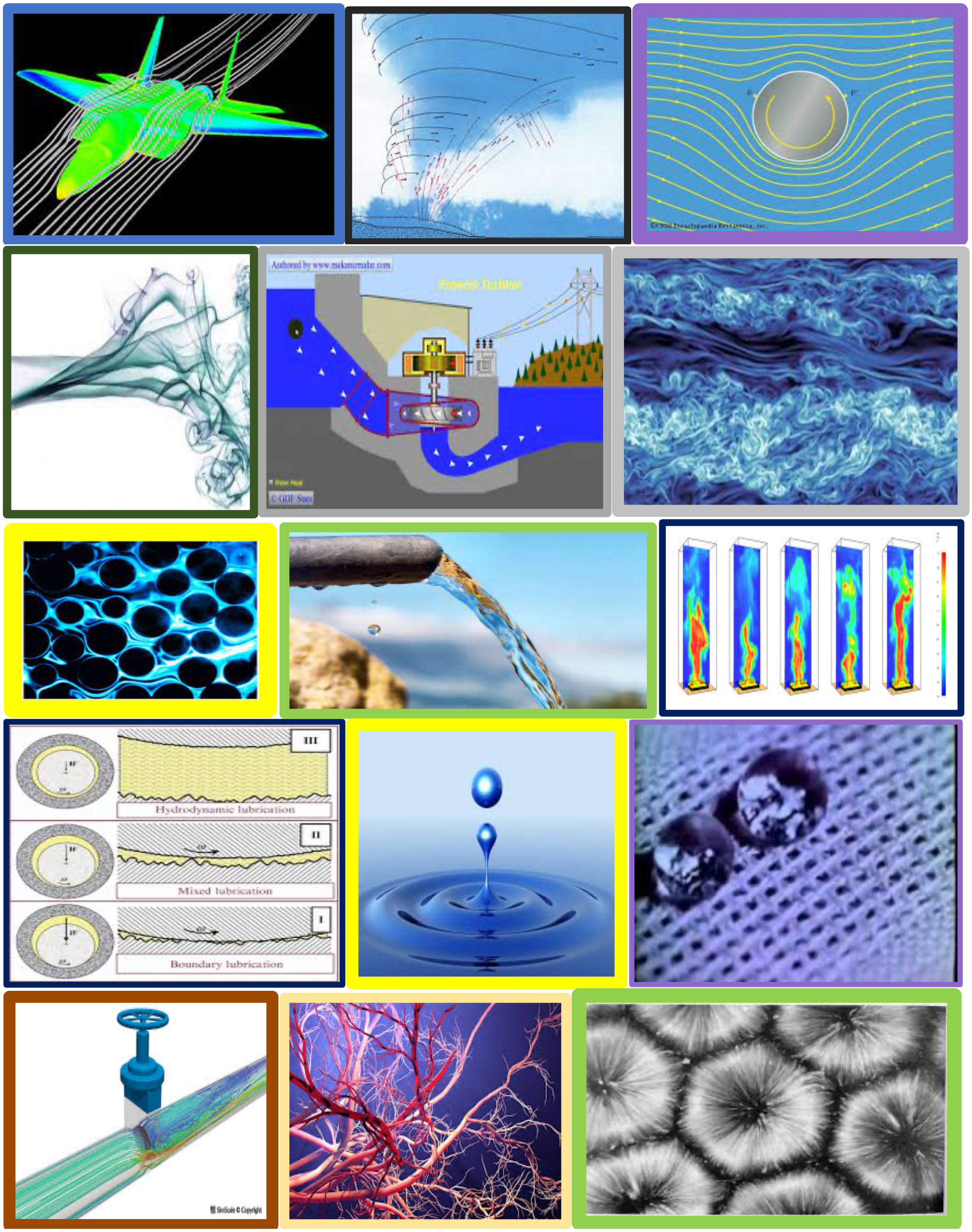
Advanced Fluid Mechanics



I. Introduction

These notes are suitable for an advanced undergraduate course and a first graduate course in Fluid Mechanics. It will focus on several topic and will use more advanced mathematical analyses to study viscous and potential incompressible fluid flow.

Specifically, the course will cover the following topics:

- The Navier Stokes Equations
- Exact Solutions of Navier Stokes Equation
- Scaling & Hydrodynamic Parameters
- Lubrication Theory
- Low Reynolds Number Hydrodynamics
- Boundary Layer Theory
- Jets, Wakes and Shear Layers
- Potential Flow
- Surface Gravity Waves
- Turbulence

The introductory chapter is devoted to a brief mathematical review and basic definitions.

Fluid motion can be described in two different ways:

Statistical Approach **Continuum Approach**

Lagrangian **Eulerian**

Statistical Approach. Macroscopic behavior is determined from molecular motion by using the laws of mechanics and probability applied to large number of molecules.

Continuum Approach. Individual molecules are ignored. The fluid is assumed to be a continuous matter with a unique value at any point for velocity, density, and other flow variable.

Lagrangian Description. (after Louise Lagrange 1736-1813) Attention is fixed on a distinct mass of fluid as it moves (like a motion of a particle). The position of the fluid element is followed in terms of its velocity, e.g., for constant flow $x = x_o + Ut$ where x_o is the initial position, U is the velocity, and t denotes time.

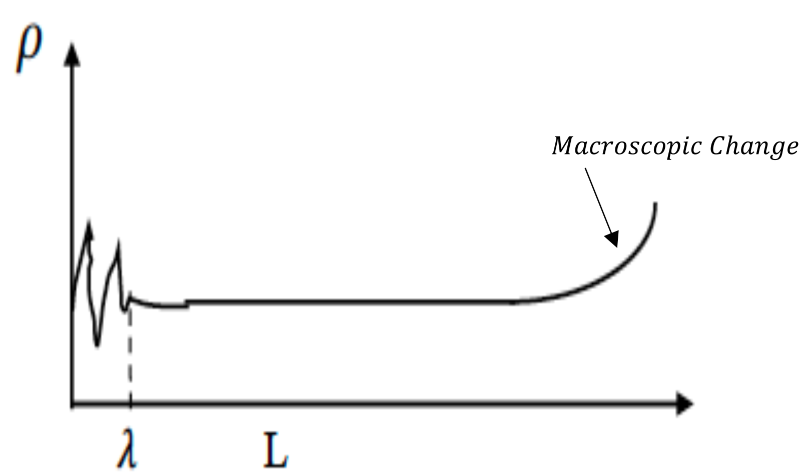
Eulerian Description. (after Leonard Euler 1707-1783) Attention is placed on the entire flow field, and the flow pattern is described at any point by specific flow variables. For example, the fluid velocity at any point can be given by the velocity vector $\vec{V}(x, y, z, t)$.

In this course, we will use the continuum approach and the Eulerian description. The limit of the continuum approach can be illustrated by considering the meaning of value of a flow variable or property at a point.

For example, fluid density in general can be defined as

$$\rho = \lim_{\Delta V \to \varepsilon^3} \frac{\Delta m}{\Delta V}$$

Where ε is a length much smaller than the characteristic length of the flow field (it's typical geometry), but much larger than the mean free path λ (about $10^{-3} mm$ for liquids), which is the average distance a molecule will travel before colliding with another molecule. Therefore, $\lambda \ll \varepsilon \ll L$.



Basic Laws Governing Fluid Flow

The laws governing fluid flow are:

- Conservation of Mass – Continuity Equation
- Conservation of Momentum – Newton's 2nd Law of Motion

The governing equations use empirical relations involving fluid properties and transport processes, like stress-strain relations or the Equation of State for compressible fluid.

The governing equations are formulated mathematically, but their analytical solution can be obtained for limited cases. Computational Fluid Dynamics (CFD) is a field in which numerical solutions are generated for problems that cannot be solved analytically. Still, they are limited to moderate Reynolds numbers at best.

The solution of the governing equations is subject to boundary conditions on the boundary of the fluid system under consideration. Typically, these conditions can be prescribed on the interface between two fluids or at a solid boundary.

I.1 Index Notation

Index notation (also called Cartesian notation) is a useful method for representing and manipulating multidimensional equations involving vectors and tensors. This introduction will present the basic operations using the index notation and the corresponding vector notation.

In the index notation a vector is represented by A_i and a tensor by A_{ij} where the indices usually represent coordinate directions $(1,2,3)$ equivalent to (x, y, z).

An index which is repeated in the same term is called a *dummy index* and it implies summation, therefore,

$$A_j B_{ij} = A_1 B_{i1} + A_2 B_{i2} + A_3 B_{i3}$$

A dummy index can be named by any letter different than all other indices in the same term.

The index i in this equation is called a *free index,* and it appears only once in an additive term. A free index also implies three equations in three coordinate directions, for example,

$$A_i = B_i + C_i$$

means

$$A_1 = B_1 + C_1$$
$$A_2 = B_2 + C_2$$
$$A_3 = B_3 + C_3$$

Terms can include more than one free index but each one must have its own notation such as A_{ij} which represents a tensor with nine terms. The number of free indices in a term defines its *rank*: 0- for a scalar, 1- for a vector, 2- for a tensor. Terms of equal rank can be summed.

The first index in a term represents a raw and the second a column. Therefore, a vector and a tensor are represented as follows,

$$\vec{V} = u_i = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\vec{\vec{\tau}} = \tau_{ij} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{12} & \tau_{13} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

An example of a rank-2 symmetric tensor is the *Kronecker Delta* (symmetric means $\delta_{ij} = \delta_{ji}$)

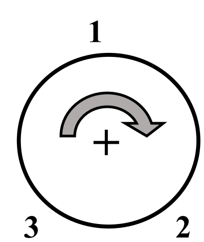
$$\delta_{ij} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

or

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The *alternative unit tensor* is a rank-3 antisymmetric tensor defined by

$$\epsilon_{ijk} = \begin{cases} 1 & if\ ijk = 123,\ 231,\ or\ 312 \\ 0 & if\ two\ indices\ are\ equal \\ -1 & if\ ijk = 132,\ 213,\ or\ 321 \end{cases}$$



Another way to view the definition is that the tensor is positive when the indices are sequential in a clockwise direction, negative in the counter-clockwise direction and zero otherwise.

The relations between the alternative unit tensor and the Kronecker delta are:

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

Individual terms in index notations are scalars so that both the commutative and associative properties are valid.

$$A_i B_j = B_j A_i$$

$$(A_i B_j) C_k = A_i (B_j C_k)$$

Vector operations with index notation

- Multiplying a vector by a scalar

$$a\vec{B} = \vec{C} \qquad\qquad aB_i = C_i$$

- Dot Product

$$\vec{A} \cdot \vec{B} = c \qquad\qquad A_i B_i = c$$

- Cross Product

$$\vec{A} \times \vec{B} = \vec{C} \qquad\qquad \epsilon_{ijk} A_j B_k = C_i$$

To illustrate this operation consider

$$(A_1, A_2, A_3) \times (B_1, B_2, B) = (A_2 B_3 - A_3 B_2, A_3 B_1 - A_1 B_3, A_1 B_2 - A_2 B_1)$$

$$\epsilon_{1jk} A_j B_k = \epsilon_{11k} A_1 B_k + \epsilon_{12k} A_2 B_k + \epsilon_{13k} A_3 B_k$$

$$\begin{gathered} = \epsilon_{111} A_1 B_1 + \epsilon_{112} A_1 B_2 + \epsilon_{113} A_1 B_3 + \\ \epsilon_{121} A_2 B_1 + \epsilon_{122} A_2 B_2 + \epsilon_{123} A_2 B_3 + \\ \epsilon_{131} A_3 B_1 + \epsilon_{132} A_3 B_2 + \epsilon_{133} A_3 B_3 \end{gathered}$$

$$= \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2 = A_2 B_3 - A_3 B_2$$

- Gradient

$$\nabla f \qquad\qquad \frac{\partial f}{\partial x_i} e_i$$

- Divergent

$$\nabla \cdot \vec{V} \qquad\qquad \frac{\partial u_i}{\partial x_i}$$

- Curl

$$\nabla \times \vec{V} \qquad\qquad \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

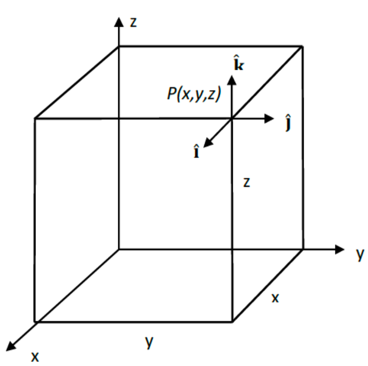
Sometimes shorter notation is used where the derivative is truncated as follows

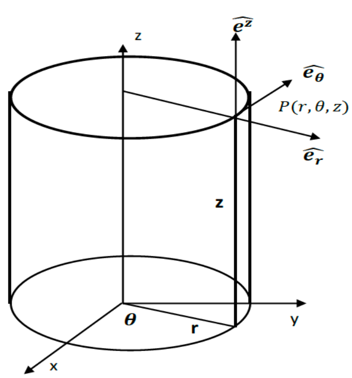
$$\frac{\partial f}{\partial x_i} = f_{,i} = \partial_i f$$

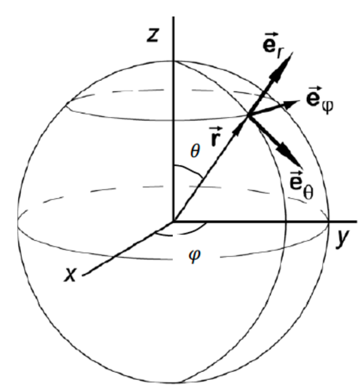
$$\frac{\partial u_i}{\partial x_i} = u_{i,i} = \partial_i u_i$$

I.2 Mathematical Background

We define the velocity vector in Cartesian coordinates as $\vec{V}(x, y, z, t)$, cylindrical (polar) coordinates as $\vec{V}(r, \theta, z, t)$, and spherical coordinate system $\vec{V}(r, \theta, \varphi, t)$. The velocity components are shown below.







$$\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k} \qquad \vec{V} = u\hat{e}_r + v\hat{e}_\theta + w\hat{e}_z \qquad \vec{V} = u\hat{e}_r + v\hat{e}_\theta + w\hat{e}_\varphi$$

Note that the same notation (u, v, w) is used for the velocity components in each coordinate system but they represent different velocity directions.

The velocity of a fluid particle in the Lagrangian approach (also called material description) is defined by

$$u_i = \frac{\partial x_i}{\partial t} \qquad i = 1,2,3$$

The notation used here denoted x_i as the coordinate system (x_1, x_2, x_3) or in Cartesian coordinated (x, y, z).

The acceleration is then

$$a_i = \frac{\partial^2 x_i}{\partial t^2}$$

In the Eulerian description, the velocity and acceleration are a function of time and position, and the Cartesian coordinates x_i are independent of time. To find the velocity and acceleration in this approach we must follow a particle for the time interval dt during which its coordinates have changed by dx_i. The corresponding change in velocity consists of two parts; a local change in time, and a change due to change in position of the particle. The for the velocity component u_i in the x_i direction, then $u_i(t, x_i)$ and

$$du_i = \frac{\partial u_i}{\partial t} dt + \frac{\partial u_i}{\partial x_j} dx_j$$

The repeated index j implies summation over $j = 1,2,3$.

Since dx_j represents the displacement then

$$dx_j = du_j dt$$

and by dividing the previous equation by $dt,$ the acceleration is:

$$a_i = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \tag{I-1}$$

Again, here the repeated index implies summation. The operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \tag{I-2}$$

is called the *material derivative* and the acceleration is therefore,

$$a_i = \frac{Du_i}{Dt} \qquad (i = 1,2,3) \qquad \text{(I-3)}$$

The transient term is called the local or Lagrangian derivative, and the last three terms are called the convective or the Eulerian derivative.

I.3 Orthogonal Curvilinear Coordinates

Depending on the geometry it would be convenient to use coordinate systems which can readily define the system boundaries. Curvilinear coordinates can be expressed in terms of the orthogonal Cartesian system. Changing from the Cartesian coordinates (x_1, x_2, x_3) to a curvilinear coordinates denoted by u_i, which are a function of x_1 so that,

$$u_1 = u_1(x_1, x_2, x_3)$$
$$u_2 = u_2(x_1, x_2, x_3)$$
$$u_3 = u_3(x_1, x_2, x_3)$$

and the reverse transformation can be generalized as

$$x_i = x_i(u_1, u_2, u_3)$$

Thus, the same point in space can be defined by the Cartesian coordinates x_i or the curvilinear coordinates u_i. For example, in the Cartesian system $x_i = constant$ defines planes with unit normal vectors e_1 which are perpendicular to the planes. In the curvilinear system on the other hand $u_i = contant$ does not necessarily define a plane and the unit normal vectors are not necessarily perpendicular. Curvilinear coordinates in which the unit normal is perpendicular to the surface $u_i = contant$ are called *Orthogonal Curvilinear Coordinates*.

In a spherical system the coordinates are defined in terms of the Cartesian coordinates as follows:

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \varphi = \tan^{-1}\left(\frac{y}{x}\right) \qquad \theta = \cos^{-1}\left\{\frac{z}{\sqrt{x^2+y^2+z^2}}\right\}$$

Here $r = contant$ represents a sphere, $\theta = constant$ is a cone along the z axis, and $\varphi = constant$ is a vertical plane. The surfaces are not planar, but the unit vectors are perpendicular to them.

In general, any point P can be defined by the position vector $\vec{R} = \vec{R}(u_1, u_2, u_3)$, then a small change in u_1 by du_1 generates a change in $\vec{R}$ to $\vec{R} + d\vec{R}$ so that

$$d\vec{R} = \frac{\partial \vec{R}}{\partial u_1} du_1$$

We can now define a *scale factor* h_1 and the *unit vector* $\hat{e}_1$ *as follows:*

$$h_1 = \left|\frac{\partial \vec{R}}{\partial u_1}\right| \qquad\qquad \hat{e}_1 = \frac{1}{h_1}\frac{\partial \vec{R}}{\partial u_1}$$

$\hat{e}_1$ is a unit vector in the direction of increasing u_1 and the scale factor h_1 represents the magnitude of $d\vec{R}$ when u_1 changes to $u_1 + du_1$. Therefore, for infinitesimal change in u_1

$$\left|d\vec{R}\right| = h_1 du_1$$

Similarly, $h_1$and $\hat{e}_i$ can be defined for $i = 2{,}3$.

Note that the unit vectors $\hat{e}_i$ are not constant like in the Cartesian system. They depend on the position vector $\vec{r}$ and their direction can change as u_i change. For an orthogonal system

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}.$$

For example, in cylindrical coordinates $u_1 = r,\ u_2 = \theta$, and $u_3 = z$, then

$$\vec{R} = r\cos\theta\hat{\imath} + r\sin\theta\hat{\jmath} + z\hat{k}$$

$$\frac{\partial \vec{R}}{\partial r} = \cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath} \qquad \frac{\partial \vec{R}}{\partial \theta} = -r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath} \qquad \frac{\partial \vec{R}}{\partial z} = \hat{k}$$

Therefore,

$$h_r = 1, \qquad h_\theta = r, \qquad h_z = 1$$

and

$$\hat{e}_r = \cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath} \qquad \hat{e}_\theta = -\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath} \qquad \hat{e}_z = \hat{k}$$

Here the unit vectors are normal to the surfaces defined by cylindrical coordinates, and are orthogonal, namely,

$$\hat{e}_r \cdot \hat{e}_\theta = \hat{e}_r \cdot \hat{e}_z = \hat{e}_\theta \cdot \hat{e}_z = 0$$

$$\hat{e}_r \times \hat{e}_z = \hat{e}_\theta \qquad \hat{e}_z \times \hat{e}_\theta = \hat{e}_r \qquad \hat{e}_\theta \times \hat{e}_r = \hat{e}_z$$

Arc Length

The arc length ds can be defined in terms of Cartesian coordinates as

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = d\vec{R} \cdot d\vec{R}$$

$$d\vec{R} = \frac{\partial \vec{R}}{\partial u_1} du_1 + \frac{\partial \vec{R}}{\partial u_2} du_2 + \frac{\partial \vec{R}}{\partial u_3} du_3 = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

For orthogonal curvilinear coordinates, because the unit vectors are orthogonal, therefore

$$(ds)^2 = h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2$$

In cylindrical coordinates $h_1 = 1, \ h_2 = r, \quad h_3 = 1$

$$(ds)^2 = (dr)^2 + r^2 (d\theta)^2 + (dz)^2$$

This approach can be applied to derive area and volume elements, and subsequently the expressions for a gradient, divergent, curl, and Laplacian. The results are given here without derivation.

Gradient

$$\nabla f = \sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial f}{\partial u_i} \hat{e}_i$$

Divergent

$$\nabla \cdot \vec{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial (V_1 h_2 h_3)}{\partial u_1} + \frac{\partial (V_2 h_3 h_1)}{\partial u_2} + \frac{\partial (V_3 h_1 h_2)}{\partial u_3} \right]$$

Curl

$$\nabla \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

Laplacian

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial f}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial u_3} \right) \right]$$

Cartesian (x, y, z)	Cylindrical (r, θ, z)	Spherical (r, θ, φ)

The Del Operator ∇

$\hat{\imath} \frac{\partial}{\partial x} + \hat{\jmath} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$	$\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial r} + \hat{e}_z \frac{\partial}{\partial z}$	$\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\varphi \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi}$

The divergent of a vector $\nabla \cdot \vec{V}$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$	$\frac{1}{r} \frac{\partial r u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}$	$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (v \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial w}{\partial \varphi}$

Laplacian $\nabla^2 f$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial x^2}$	$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 f) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2\, sin^2\,\theta} \frac{\partial^2 f}{\partial \varphi^2}$

I.4 Acceleration

In Cartesian coordinates the total derivative of a scalar function $f(x, y, z, t)$ is

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} \tag{I-4}$$

The acceleration can be represented as the total derivative of the velocity vector

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} \tag{I-5}$$

and the components of the acceleration vector are:

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k} = \frac{Du}{Dt}\hat{\imath} + \frac{Dv}{Dt}\hat{\jmath} + \frac{Dw}{Dt}\hat{k}$$

$$\left.\begin{aligned} a_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right\} \tag{I-6}$$

Cylindrical Coordinates

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial r} + \frac{v}{r}\frac{\partial \vec{V}}{\partial \theta} + w\frac{\partial \vec{V}}{\partial z} \tag{I-7}$$

$$\left.\begin{aligned} a_r &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} \\ a_\theta &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} \\ a_z &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} \end{aligned}\right\} \tag{I-8}$$

Spherical Coordinates

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial r} + \frac{v}{r}\frac{\partial \vec{V}}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial \vec{V}}{\partial \varphi} \tag{I-9}$$

$$\left.\begin{aligned}
a_r &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial u}{\partial \varphi} - \frac{v^2}{r} - \frac{w^2}{r} \\
a_\theta &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial v}{\partial \varphi} + \frac{uv}{r} - \frac{w^2}{r}\cot\theta \\
a_\varphi &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial w}{\partial \varphi} + \frac{uw}{r} + \frac{vw}{r}\cot\theta
\end{aligned}\right\} \tag{I-10}$$

I.5 <u>The Stream Function</u>

In the Lagrangian description, the path of a fluid particle can be described by a parametric equation with the space coordinates being a function of time. Namely,

$$x_i = F_i(t)$$

This equation represents a *path line.*

The Eulerian description is not convenient for the definition of path lines, but is much better in describing a *streamline,* which is a line to which the velocity vectors are tangent at every point. Therefore, there is no flow crossing the streamline. In Cartesian coordinates the equation for a streamline is given by

$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \frac{dx_3}{u_3} \tag{I-11}$$

This differential equation equates the slope of the streamline at any point to the ratio between the corresponding velocity components of the tangent velocity vector.

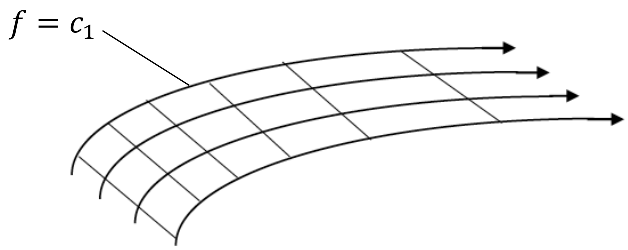
For steady flows, the path lines and streamline coincide.

Streamlines which form a closed tubular surface define a *stream tube.*

The function that satisfies the differential relations (I-11) can be written in the form

$$f(x_1, x_2, x_3) = c$$

Where c is a constant. For different values of c the function f defines a stream surface formed by a family of streamlines

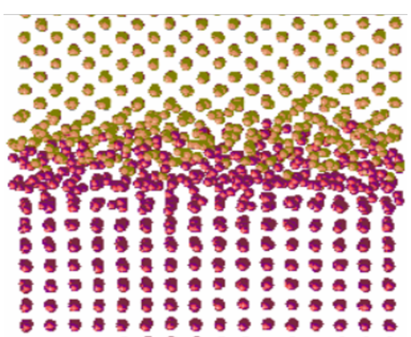


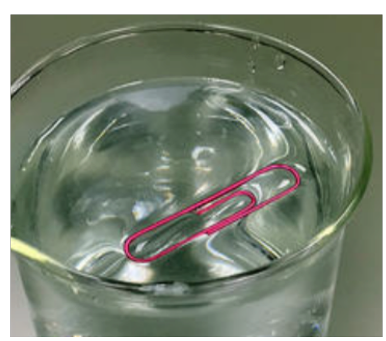
Another way to visualize a flow field can be done by following all the particles that pass through a fixed point, like bubbles or smoke injected into the flow. The lines formed by these particles are called *Streak lines.* Pathlines, streamlines and streak lines coincide in steady state flow.

I.6 Surface Tension

The interface between two fluids is of special interest in fluid mechanics. This interface consists of a very thin layer in which molecules of both fluid in contact are present. The analysis of these thin layers is greatly simplified in fluid mechanics by assuming that it is a two-dimensional surface of zero thickness and zero mass.

In each fluid, the forces acting by the same molecules on each other are balanced. In the thin layer where the molecules of both fluids are present, the net interacting force between them is
manifested by a tangential force per unit length acting on the interface called *Surface Tension,* γ, $[F/L]$ which is a property of the two fluids in contact like oil and water or water and air.

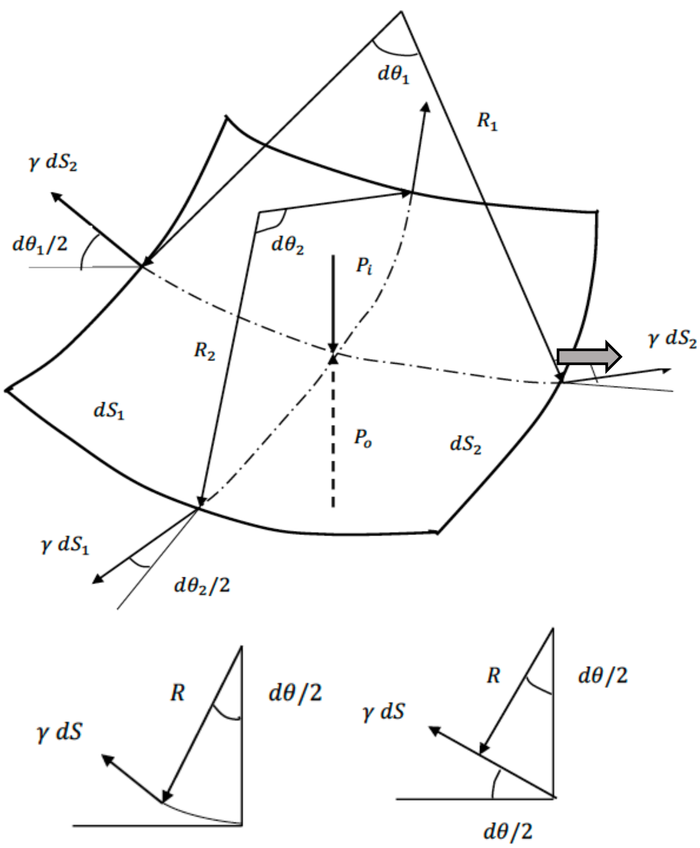




The interface between two fluids is not a flat geometrical surface. In the continuum approach, we reduce the interface between the two fluids into a mathematical surface of zero thickness. The tangential velocity along the interface has to be continuous (no-slip boundary conditions), but the velocity perpendicular to the interface can be discontinuous when mass is transferred across the interface (e.g., evaporation). For two immiscible fluids the normal velocity is continuous.

We consider a small surface element $dS_1 \times dS_2$, and the forces acting on this element are shown in the figure below.

Apply Newton's second law for the interface in the vertical direction. Keeping in mind that the interface has zero mass, we need only to consider the balance of forces acting in the vertical direction.



$$P_o dS_1 dS_2 - P_1 dS_1 dS_2 + 2\gamma dS_2 \sin\frac{d\theta_1}{2} + 2\gamma dS_1 \sin\frac{d\theta_2}{2} = 0$$

For small θ, $\sin\frac{d\theta}{2} = \frac{d\theta}{2}$, then

$$P_o dS_1 dS_2 - P_1 dS_1 dS_2 + \gamma dS_2\, d\theta_1 + \gamma dS_1 d\theta_2 = 0$$

substituting the relations $dS = Rd\theta$, the equation becomes

$$P_o dS_1 dS_2 - P_1 dS_1 dS_2 + \gamma dS_2 \frac{dS_1}{R_1} + \gamma dS_1 \frac{dS_2}{R_2} = 0$$

or

$$P_i - P_o = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \qquad \text{(I-13)}$$

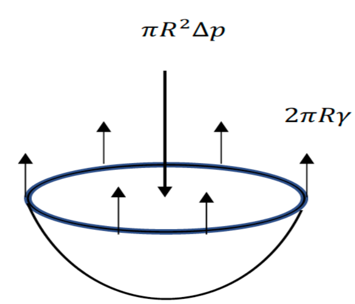
Note that the pressure inside (on the concave side) is always larger than the pressure on the outside.

For a special case when both $R_1$and $R_2 \to \infty$, the surface is a flat and $P_i = P_o$.

For example, consider a spherical droplet in air. The force balance on the ring of radius R is given by

$2\pi R\gamma = \pi R^2 \Delta p$

or



$$\Delta p = \frac{2\gamma}{R} \qquad \text{(I-14)}$$

Where Δp is the difference between the inside and outside pressure.

Equation (I-13) is the *Young-Laplace Equation.* Skipping the mathematical derivation, the curvature of a one-dimensional curve $h(x)$ is given by,

$$\frac{1}{R} = k = \frac{\frac{d^2h}{dx^2}}{\left[1 + \left(\frac{dh}{dx}\right)^2\right]^{3/2}}$$

If the shape of a two-dimensional surface is given by a function $h(x, y)$ which specifies the local position of the interface above a reference plane, the pressure difference can be approximated in terms of the radii of curvature for *small slope* approximation.

For small slopes $\frac{\partial h}{\partial x} \ll 1$, $\frac{\partial h}{\partial y} \ll 1$ then,

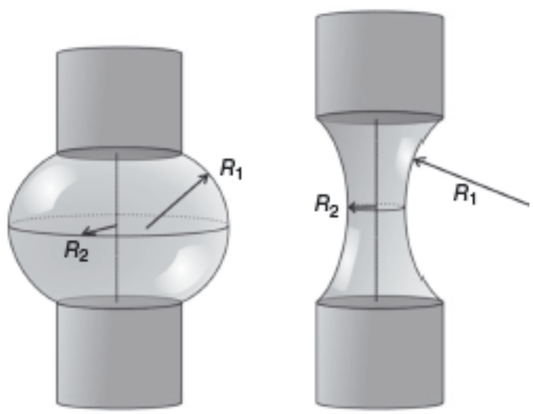
$$\Delta p = \gamma \nabla^2 h$$

Another relevant application is the capillary bridge with cylindrical symmetry as shown in the figure. In this case the full Young-Laplace equation reduces to an ordinary differential equation in the form

$$\Delta p = 2\gamma \left(\frac{1}{rS} - \frac{1}{S^3} \frac{\partial^2 r}{\partial z^2} \right)$$

where $r(z)$ represents the shape of the interface (z is the axial direction) and

$$S = \sqrt{1 + \left(\frac{\partial r}{\partial z} \right)^2}$$



For small interfacial deflection the equation can be simplified to

$$\Delta p = 2\gamma \left(\frac{1}{r} - \frac{\partial^2 r}{\partial z^2} \right)$$

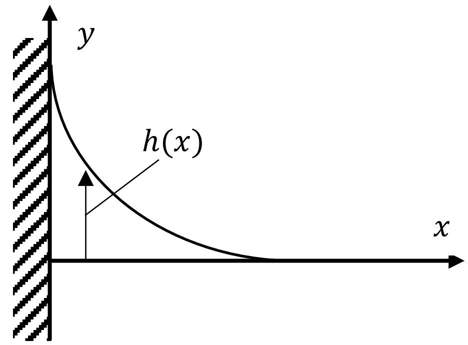
For a static system, such as liquid meniscus against a vertical wall, the height of the meniscus can be written as $h(x)$, and the pressure in the liquid due to gravity is

$$p = p_o - \rho g y$$

where p_o is the constant pressure in the gas phase above.

Here only one of the radii of curvature is significant and it can be expressed by

$$\frac{1}{R} = -\frac{\frac{d^2h}{dx^2}}{\left[1 + \left(\frac{dh}{dx}\right)^2\right]^{3/2}}$$



Assuming the p_o represents the atmospheric pressure in the gas, the pressure across the interface can be expressed in terms of the curvature of the meniscus by:

$$\rho g h = \frac{\gamma \frac{d^2h}{dx^2}}{\left[1 + \left(\frac{dh}{dx}\right)^2\right]^{3/2}}$$

The analytical solution can be obtained in explicit form of $x,$ which requires numerical calculations to obtain h:

$$\frac{x}{a} = \frac{1}{\sqrt{2}} \ln\left[\frac{\sqrt{2} + \sqrt{2 - \left(\frac{h}{a}\right)^2}}{\frac{h}{a}}\right] - \sqrt{2 - \left(\frac{h}{a}\right)^2} + C$$

where $a = \sqrt{\frac{2\gamma}{\rho g}}$ is called the *capillary length,* and the constant C depends on the contact angle at the wall θ

$$C = \sqrt{1 + \sin\theta} + \frac{1}{\sqrt{2}} \ln\left(\frac{\sqrt{1 - \sin\theta}}{\sqrt{2} + \sqrt{1 + \sin\theta}}\right)$$

h/a