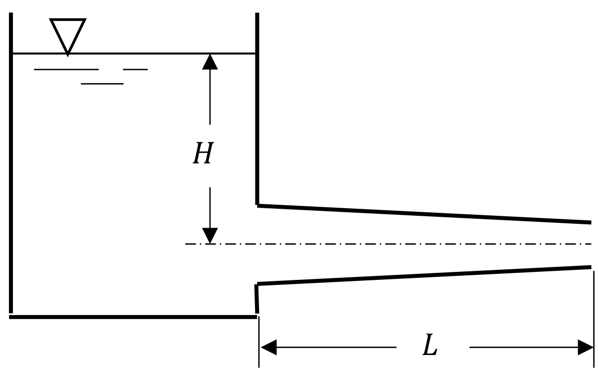
ME 55600/I0200
Homework #5: Lubrication Approximation

Consider viscous incompressible flow through a converging nozzle of length L at the bottom of a large tank (see figure; not to scale). Assume the tank is very large so that the pressure at the entrance to the nozzle is hydrostatic. Use a cylindrical coordinate system at the entrance to the nozzle so that it radius $R(z)$ can be given by

$$R(z) = R_i - (R_i - R_e)\frac{z}{L}$$

where $R(0) = R_i$ and $R(L) = R_e$, and the convergence is very slow so that $\frac{R}{L} \ll 1$.



(a) Write down the governing equations and boundary conditions for the flow through the nozzle using the lubrication approximation.
(b) Solve the governing equation for the velocity profile $w(r)$ in terms of the pressure gradient.
(c) Determine the flow rate, and use the result to determine the pressure distribution $p(z)$.
(d) Use the result for the pressure to show that the flow rate is given by

$$Q = \frac{3\pi\rho g H}{8\mu L}\frac{R_i - R_e}{R_e^{-3} - R_i^{-3}}$$

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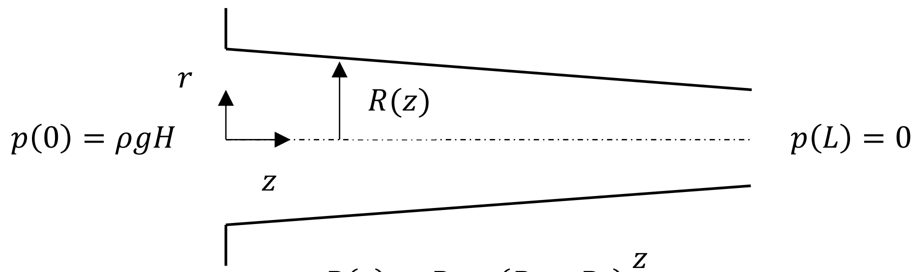

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(a) Differential Equation:

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{dp}{dz}$$

Boundary Conditions:

(i) $w(0)$ is finite

(ii) $w(R) = 0$

(b) Solution of D.E.

$$w(r,z) = \frac{1}{4\mu}\frac{dp}{dz}r^2 + c_1 \ln r + c_2$$

From B.C. (i) $c_1 = 0$; from (ii): $c_2 == -\frac{1}{4\mu}\frac{dp}{dz}R^2$

Therefore, the solution is

$$w(r,z) == -\frac{1}{4\mu}\frac{dp}{dz}(R^2 - r^2)$$

(c) The flow rate:

$$Q = \int_0^R w 2\pi r dr = 2\pi \int_0^R -\frac{r}{4\mu}\frac{dp}{dz}(R^2 - r^2)dr = -\frac{\pi R^4}{8\mu}\frac{dp}{dz}$$

The pressure gradient can be represented in terms of the flow rate:

$$\frac{dp}{dz} = -\frac{8\mu Q}{\pi R^4}$$

(d) Pressure Distribution. Integrate the pressure gradient.

$$p(L) - p(0) = -\frac{8\mu Q}{\pi}\int_0^L \frac{dz}{\left[R_i - (R_i - R_e)\frac{z}{L}\right]^4}$$

$$-\rho g H = -\frac{8\mu Q}{3\pi}\frac{1}{3m(-mz + R_i)^3}\Bigg|_0^L$$

Here $m = \frac{R_i - R_e}{L}$

Solve for Q

$$Q = \frac{3\pi\rho g H}{8\mu L}\frac{R_i - R_e}{R_e^{-3} - R_i^{-3}}$$