IX. Potential Flow

IX.1 Flow of Inviscid Fluid

An *inviscid fluid* is defined as fluid with zero viscosity. Therefore, the viscous terms in the Navier-Stokes equations are neglected. Consequently, some conditions on the fluid-solid boundary are not relevant. Specifically, the no-slip boundary condition, and conditions related to stresses.

The governing equation of motion for inviscid fluids is the *Euler Equation:*

$$\frac{Du_i}{Dt} = g_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \qquad \text{(IX-1)}$$

For irrotational flow (the vorticity is zero everywhere) the velocity components can be represented by the *velocity potential* ϕ,

$$u_i = -\frac{\partial \phi}{\partial x_i} \qquad \text{(IX-2)}$$

$[\vec{\omega} = \nabla \times \vec{V} = \nabla \times (\nabla\phi) = 0]$. The negative sign is introduced for convenience but is not essential.

For steady state, Euler equations become

$$u_j \frac{\partial u_i}{\partial x_j} = g_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i}$$

Multiplying the equation by u_i, and introducing the elevation H, the equation can be written as

$$u_j \frac{\partial}{\partial x_j}\left(\frac{u_i u_i}{2}\right) = -u_i\left(\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_0\frac{\partial H}{\partial x_i}\right) = -u_j\frac{\partial}{\partial x_j}\left(\frac{p}{\rho} + g_0 H\right)$$

Therefore,

$$\frac{D}{Dt}\left(\frac{u_i u_i}{2} + \frac{p}{\rho} + g_0 H\right) = 0$$

which means that along a streamline:

$$\left(\frac{u_i u_i}{2} + \frac{p}{\rho} + g_0 H\right) = Constant \qquad \text{(IX-3)}$$

This is Bernoulli's equation for inviscid fluid.

For irrotational flow equations (IX-1) and (IX-2) can be written following the same procedure as

$$\frac{\partial \phi}{\partial t} + \frac{u_i u_i}{2} + \frac{p}{\rho} + g_0 H = Constant$$

IX.2 Irrotational Flow

Potential flow is an idealized flow in which fluid viscosity and shear stresses are ignored. The flow can be treated mathematically and provide solutions for flow with complicated geometry but away from the boundary. It is especially applicable in aerodynamics in studying flows around airfoils.

The main assumptions used in potential flow are:

- Incompressibility ρ =constant
- Irrotationality $\nabla \times \vec{V} = 0$
- Steady State $\frac{\partial}{\partial t} = 0$ (not required)

Since vorticity is given by

$$\vec{\omega} = \nabla \times \vec{V} = 0$$

Consider the Navier Stokes Equations

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{V}$$

Since

$$\nabla^2 \vec{V} = \nabla \times \left(\nabla \times \vec{V}\right) - \nabla\left(\nabla \cdot \vec{V}\right) = 0$$

The viscous terms in the Navier Stokes equation are zero due to irrotationality and incompressibility, even if the fluid is not necessarily inviscid, and the momentum equations are reduced to Euler's equations

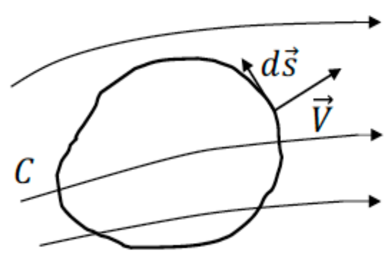
$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} = -\frac{1}{\rho} \nabla p$$

The physical representation of zero vorticity is that all fluid elements maintain their orientation - do not rotate – during their motion.

IX.3 Circulation

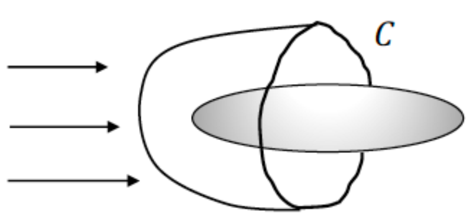
Circulation is defined as follows:

$$\Gamma = \oint \vec{V} \cdot d\vec{s}$$

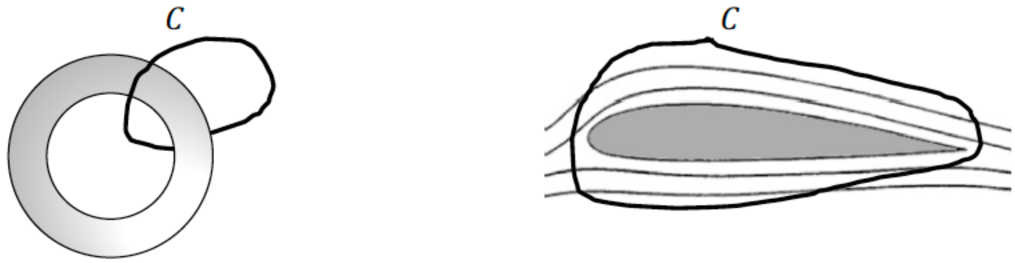


It is a contour integral over a closed path C, of the tangential velocity component.

The definition applies to a *Simply Connected Region,* which is defined as one where every closed path forms the edge of a family of surfaces which do not cut through the boundaries of the flow domain.



Otherwise, the region is called *Multiconnected Region.*



Using Stokes' theorem for a simply connected region

$$\oint \vec{V} \cdot d\vec{s} = \iint \nabla \times \vec{V} \cdot d\vec{A}$$

where $d\vec{A} = \hat{n} dA$ ($\hat{n}$ being the unit normal to the surface A), and the area integral represents total vorticity over the surface.

Therefore, for simply connected region in irrotational flow,

$$\Gamma = 0 \tag{IX-6}$$

Since $\nabla \times \vec{V}$=0 for potential flow, a velocity potential ϕ can be defined so that

$$\vec{V} = -\nabla\phi$$

The definition satisfies the condition of irrotationality because of the following vector identity for any scalar function

$$\nabla \times \nabla\phi = 0$$

The velocity potential in Cartesian coordinates is

$$u = -\frac{\partial\phi}{\partial x} \qquad v = -\frac{\partial\phi}{\partial y} \qquad w = -\frac{\partial\phi}{\partial z} \tag{IX-7}$$

Cylindrical coordinates

$$u = -\frac{\partial\phi}{\partial r} \qquad v = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} \qquad w = -\frac{\partial\phi}{\partial z} \tag{IX-8}$$

Spherical coordinates

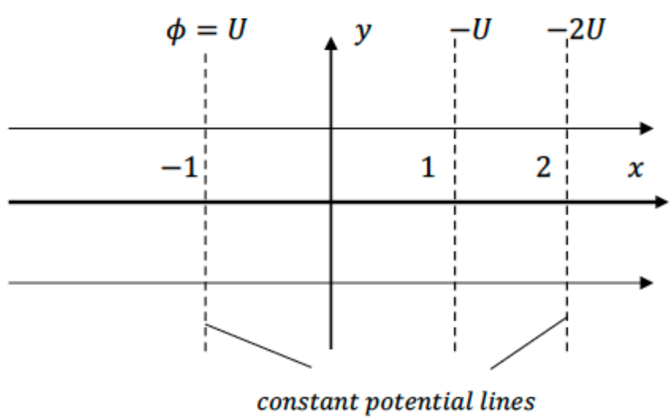
$$u = -\frac{\partial \phi}{\partial r} \qquad v = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} \qquad w = -\frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \varphi} \tag{VIII-9}$$

To illustrate the meaning of the velocity potential, consider uniform flow.
Therefore,

$$u = U \qquad U = -\frac{\partial \phi}{\partial x}$$

$$v = 0 \qquad 0 = -\frac{\partial \phi}{\partial y} \qquad\qquad \phi = -Ux + C \quad (C \text{ can be set to zero})$$

$$w = 0 \qquad 0 = -\frac{\partial \phi}{\partial z}$$



The negative sign in the potential definition is used to construct velocity direction from high to low potential lines, but this sign is arbitrary.

IX.4 The Stream Function

The *stream function* ψ can be defined for two-dimensional flow as follows

$$u = -\frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x} \tag{IX-10}$$

and in cylindrical coordinates

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v = \frac{\partial \psi}{\partial r} \tag{IX-11}$$

The definition satisfies the continuity equation for incompressible fluid. For example, in Cartesian coordinates

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substituting the definition (IX-6) gives

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = 0$$

In addition, since $\psi = \psi(x, y)$ then

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -vdx + udy$$

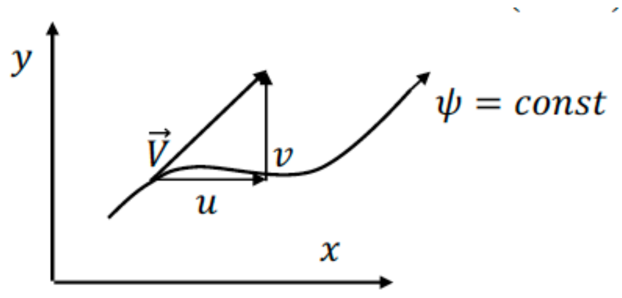
and if $\psi(x, y) = cont$ defines a curve (surface), then

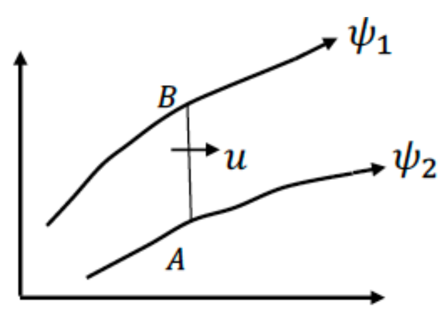
$$d\psi = 0 \qquad \text{and} \qquad -vdx + udy = 0.$$

or

$$\frac{dy}{dx} = \frac{v}{u} \qquad \text{(VIII-12)}$$

which defines a streamline.





Since $\psi = const$ along a streamline, the function can be integrated between two streamlines:

$$\int_{\psi_1}^{\psi_2} d\psi = \psi_1 - \psi_2$$

Consider the flowrate between two streamline surfaces. Since no flow is crossing the streamlines, an arbitrary path $\overline{AB}$ can be selected to evaluate the flowrate, Q

$$Q = \int_{y_1}^{y_2} u\, dy = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y}\, dy = \psi_2 - \psi_1$$

or in general

$$Q_{12} = |\psi_2 - \psi_1| \qquad \text{(IX-13)}$$

For two-dimensional flow, using the stream function reduces the number of dependent variables (unknows) from 3 (u, v, p) to 2 (ψ, p); since the continuity equation is already satisfied, the remaining governing equations are two momentum equations.

In two-dimensional flow, constant potential lines $\phi(x, y) = const$ and constant stream function lines (streamlines) $\psi(x, y) = const.$ are perpendicular to each other (intersect at right angles) forming a *flownet.*

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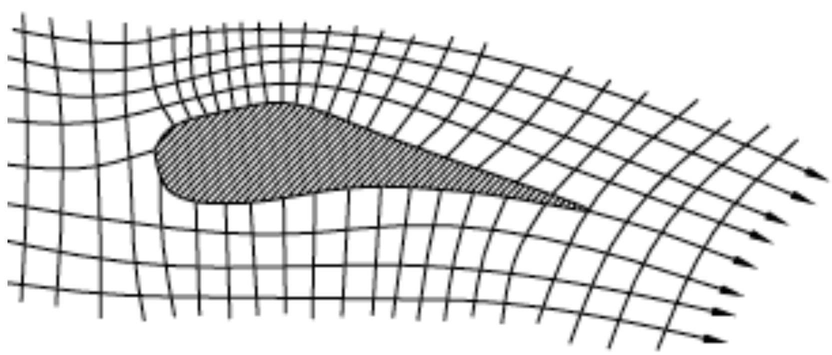


IX.5 Governing Equations for Potential Flow

Continuity Equation

$$\nabla \cdot \vec{V} = 0$$

Using the definition of the velocity potential $\vec{V} = -\nabla\phi$ the continuity equation becomes

$$\nabla^2\phi = 0$$

and in Cartesian coordinates

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \text{(IX-10)}$$

Any solution of Laplace's Equation (IX-10) is called a *harmonic function.*

Cylindrical Coordinate

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \text{(IX-11)}$$

Spherical Coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\,sin^2\,\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0 \qquad \text{(IX-12)}$$

For two-dimensional flow, the stream function satisfies the continuity equation, and using the definition of the potential and the stream function the following relations are stablished:

$$u = -\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y}$$

$$v = -\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x}$$

These relations are call the *Cauchy-Riemann Condition*

$$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &= \frac{\partial\psi}{\partial y} \\ \frac{\partial\phi}{\partial y} &= -\frac{\partial\psi}{\partial x} \end{aligned}\right\} \qquad \text{(IX-13)}$$

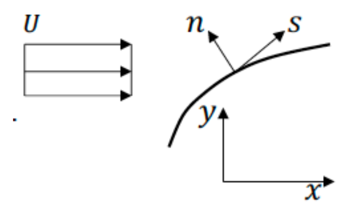
Differentiating the Cauchy-Riemann condition with respect to y and x respectively, and subtracting gives an equation for the stream function, for two-dimensional flow

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \text{(IX-14)}$$

The solution of the governing equation is subject to boundary conditions which in potential flow are as follows

At a solid boundary

$$\frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial s}$$

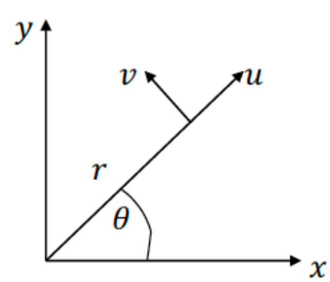


Tangential velocity is not zero, because the no-slip condition is not satisfied for potential flow.

The free stream velocity far from the boundary can be written as

$$u = -\frac{\partial \phi}{\partial x} = U \quad \text{or} \quad -\frac{\partial \psi}{\partial y} = U$$

These equations in cylindrical coordinates are as follows:



$$x = r\cos\theta. \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial r} & v &= -\frac{1}{r}\frac{\partial \phi}{\partial \theta} & \frac{\partial \phi}{\partial r} &= \frac{1}{r}\frac{\partial \psi}{\partial \theta} \\ u &= -\frac{1}{r}\frac{\partial \psi}{\partial \theta} & v &= \frac{\partial \psi}{\partial r} & \frac{1}{r}\frac{\partial \phi}{\partial \theta} &= -\frac{\partial \psi}{\partial r} \end{aligned}\right\} \text{(VIII-15)}$$

IX.6 <u>Simple Potential Flows</u>

Since potential flow satisfies Laplace's equation, the sum of harmonic functions is also a harmonic function. Therefore, complex flows can be constructed by superposition of simple flows. We will consider only two-dimensional flows which represent multi-connected regions. Therefore, Γ is not necessarily zero.

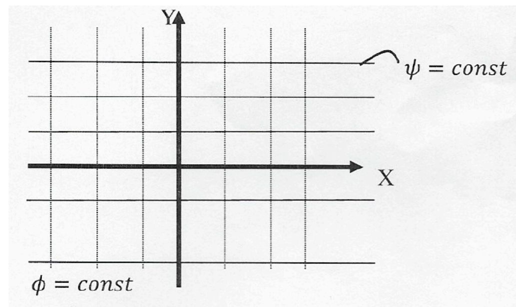
Uniform Flow

$$u = U \qquad -\frac{\partial \phi}{\partial x} = U \qquad \phi = -Ux + const$$

$$v = 0 \qquad -\frac{\partial \phi}{\partial y} = 0$$

$$u = -\frac{\partial \psi}{\partial y} = U \qquad \psi = -Uy + const$$

$$v = \frac{\partial \psi}{\partial x} = 0$$



The velocity potential is determined from its definition, and the stream function from the definition or the Cauchy- Riemann conditions. Constants of integration can be ignored.

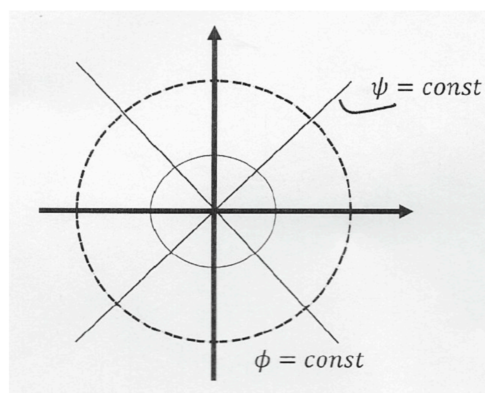
Source (Sink)

The flow describes a two-dimensional line source located at $r = 0$.

Consider the potential,

$\phi = -\frac{\Lambda}{2\pi} \ln r \qquad$ (Λ- positive constant)

This potential satisfies the continuity equation in cylindrical coordinates.



$$\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \qquad \left[-\frac{\Lambda}{2\pi}\frac{\partial}{\partial r}\left(r\frac{1}{r}\right) = 0\right]$$

The stream function can be determined from the Cauchy-Riemann condition:

$$\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = -\frac{\Lambda}{2\pi}\frac{1}{r} \qquad \psi = -\frac{\Lambda}{2\pi}\theta$$

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = 0$$

The streamlines are defined by $\theta = const$, and the potential lines by $r = const$.

The radial velocity is given by

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\Lambda}{2\pi}$$

and $v = 0$.

Note that when $r \to 0$, $u \to \infty$, which is a singularity, and as $r \to \infty$, $u \sim \frac{1}{r}$

The flow rate can be determined by integrating the radial velocity:

$$Q = \int_0^{2\pi} u r d\theta = \int_0^{2\pi} \frac{\Lambda}{2\pi r} r d\theta = \Lambda$$

or

$$Q = |\psi(0) - \psi(2\pi)| = \Lambda$$

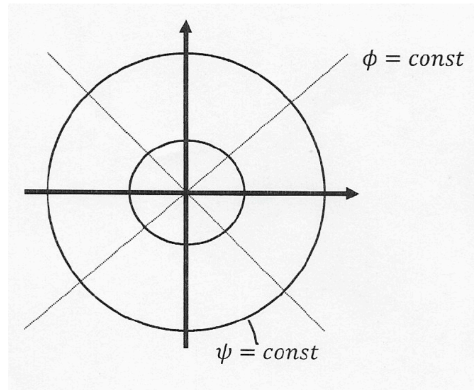
Λ is the volumetric flow rate, also called *the strength of the sources*. If $\Lambda < 0$ potential represents a sink.

The circulation has to be determined by integration because this is not a simply connected region. Selecting a circle of radius r as the path around the source the circulation is

$$\Gamma = \int_0^{2\pi} v r d\theta = 0$$

Vortex

Consider the potential:



$$\phi = -\frac{\Lambda}{2\pi}\theta \qquad (\Lambda\text{- positive constant})$$

This potential is harmonic. The stream function is determined from the Cauchy-Riemann condition.

$$\psi = \frac{\Lambda}{2\pi}\ln r$$

and the velocity components are:

$$v = \frac{\Lambda}{2\pi r} \qquad u = 0$$

Circulation: $$\Gamma = \int_0^{2\pi} vr\, d\theta = \int_0^{2\pi} \frac{\Lambda}{2\pi r} r\, d\theta = \Lambda$$

The circulation is called the *strength of the vortex.* Change in sign will change the direction of rotation.

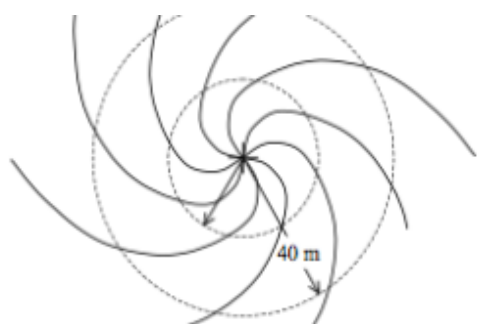
Example: Consider a sink and a vortex at the same origin (whirlpool)

$$\phi = \frac{\Lambda_2}{2\pi}\ln r - \frac{\Lambda_1}{2\pi}\theta \qquad \psi = \frac{\Lambda_1}{2\pi}\ln r + \frac{\Lambda_2}{2\pi}\theta$$

$$u = -\frac{\Lambda_2}{2\pi r} \qquad v = -\frac{\Lambda_2}{2\pi}$$

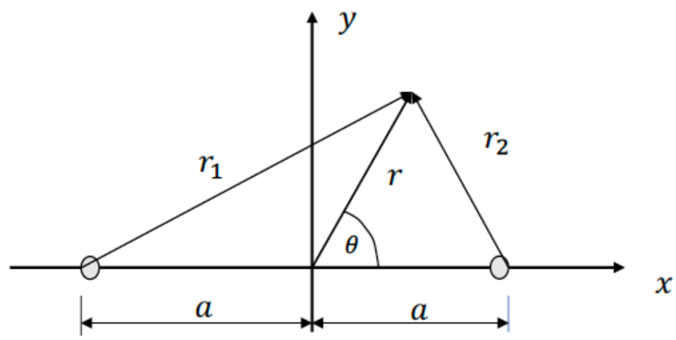
$$\phi = const \quad \Rightarrow \quad \theta \sim \frac{\Lambda_2}{\Lambda_1}\ln r$$

$$\psi = const \quad \Rightarrow \quad \theta \sim \frac{\Lambda_1}{\Lambda_2}\ln r$$



Doublet

The doublet is formed by combining a source and a sink of equal strength at vanishing distance ($a \to 0$)



For the source and sink

$$\phi_{source} = -\frac{\Lambda}{2\pi}\ln r_1 \qquad \phi_{sink} = \frac{\Lambda}{2\pi}\ln r_2$$

Combining the two potentials

$$\phi = \phi_{source} + \phi_{sink} = -\frac{\Lambda}{2\pi}\ln r_1 + \frac{\Lambda}{2\pi}\ln r_2$$

To introduce common cylindrical coordinates, we use the law of cosines:

$$r_2 = \sqrt{r^2 + a^2 - 2ra\cos\theta}$$

$$r_1 = \sqrt{r^2 + a^2 - 2ra\cos(180^o - \theta)} = \sqrt{r^2 + a^2 + 2ra\cos\theta}$$

Substitute into the potential to get

$$\phi = \frac{\Lambda}{4\pi}[\ln(r^2 + a^2 - 2ra\cos\theta) - \ln(r^2 + a^2 + 2ra\cos\theta)]$$
$$= \frac{\Lambda}{4\pi}\left[\ln\left(1 - \frac{2ra\cos\theta}{r^2 + a^2}\right)(r^2 + a^2) - \ln\left(1 + \frac{2ra\cos\theta}{r^2 + a^2}\right)(r^2 + a^2)\right]$$

Then

$$\phi = \frac{\Lambda}{4\pi}\left[\ln\left(1 - \frac{2ra\cos\theta}{r^2+a^2}\right) - \ln\left(1 + \frac{2ra\cos\theta}{r^2+a^2}\right)\right]$$

One can show that the term $\left|\frac{2ra\cos\theta}{r^2+a^2}\right| < 1$ and therefore $\ln(1 \pm x)$ can be expanded as a series as follows:

$$\ln(1 \pm x) = \mp x - \frac{x^2}{2} \mp \frac{x^3}{3} - \cdots$$

Thus, only the negative even terms remain, and only the first term is largest as $a \to 0$.

Therefore, the potential can be reduced into

$$\phi = -\frac{\Lambda}{\pi}\frac{ra\cos\theta}{r^2+a^2}$$

In the limit of vanishing distance $a \to 0$, let the strength increase so that $\frac{a\Lambda}{\pi} = \chi$ (constant). Then the potential is

$$\phi = -\chi\frac{\cos\theta}{r}$$

and the stream function

$$\psi = \chi\frac{\sin\theta}{r}$$

Constant potential lines and streamlines are defined by setting the above expressions equal to a constant.

Using Cartesian coordinates with $\sin\theta = \frac{y}{r} = \frac{y}{x^2+y^2}$ and $\cos\theta = \frac{x}{r} = \frac{x}{x^2+y^2}$

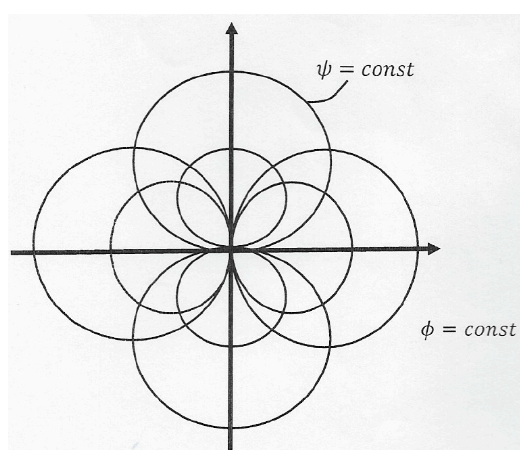
The streamlines are given by

$$x^2 + y^2 - c_1 y = 0$$

And the potential lines

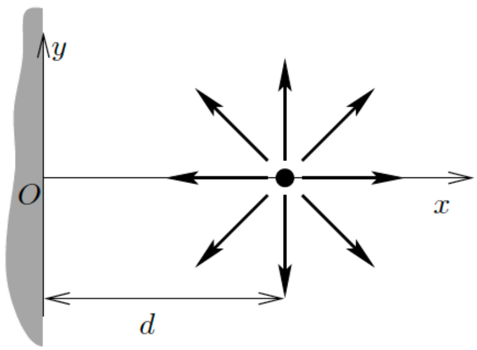
$$x^2 + y^2 - c_2 x = 0$$

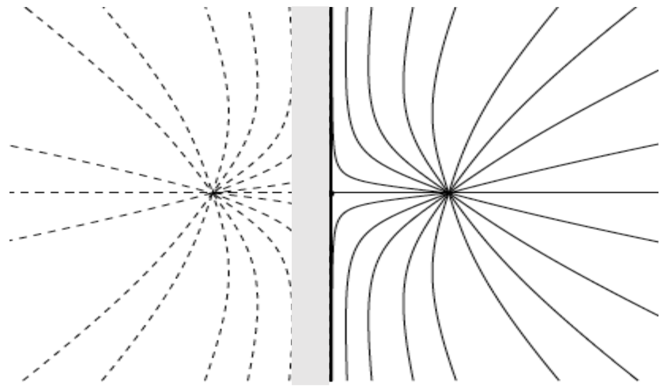
It can also be shown that $\Gamma = 0$.



IX.7 Method of Images

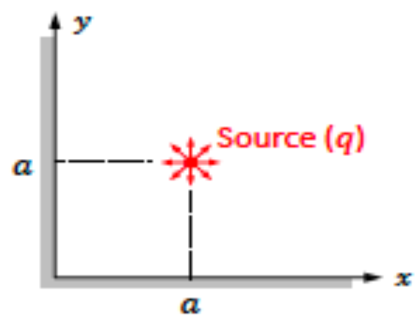
The simple potential flows introduced in the previous section IX.5 are all harmonic functions, and any linear combination of harmonic functions is also harmonic. This fact can be exploited in constructing more complex flows. Foe example a source located near an infinite wall:





The same approach can be used to construct a vortex near a wall by placing a mirror image on the other side.

How can this method be used to construct potential flow of a source near a corner?



IX.8 Principle of Superposition

As mentioned before, because the governing equations for potential flow are linear, specific solutions can be combined in terms of the velocity components, the potential and the steam function to create more complex flow fields. For the velocity potential the governing equation is

$$\nabla^2\phi = 0$$

Assume ϕ_1, ϕ_2, ϕ_3…. Are solutions of the governing equation, then.

$$\phi = \phi_1 + \phi_2 + \phi_3 \dots$$

is also a solution of the governing equation as is shown by the substitution:

$$\nabla^2\phi_1 + \nabla^2\phi_2 + \nabla^2\phi_3 \ldots = 0$$

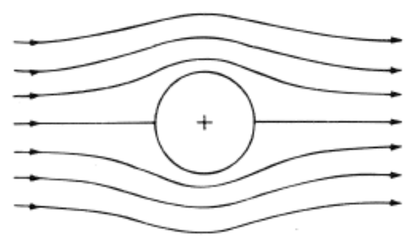
The same principle applies to the stream function in two-dimensional flows because it also satisfied Laplace's equation.

IX.9 Flow over a Cylinder

To describe potential flow over a cylinder the solution is constructed by combining two simple solutions: uniform flow and a doublet.

$$\phi = -Ux - \chi\frac{\cos\theta}{r}$$

$$\psi = -Uy + \chi\frac{\sin\theta}{r}$$



Consider the equation for a streamline namely $\psi = const$

$$-Uy + \chi\frac{\sin\theta}{r} = const$$

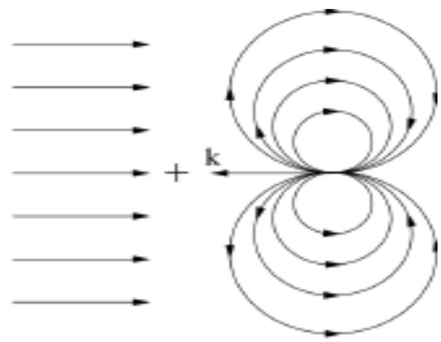
If the constant is zero, and with $y = r\sin\theta$ it becomes

$$\sin\theta\left(Ur - \frac{\chi}{r}\right) = 0$$

which can be satisfied when $\theta = 0, \pi$ and

$$r^2 = \frac{\chi}{U}$$

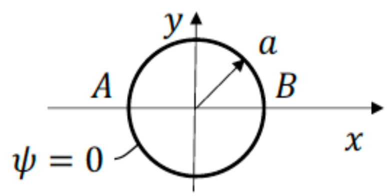
The superposition of the two flows is illustrated in the figure.



The zero streamline as defined by the previous equation is a circle of radius

$$a = \sqrt{\frac{\chi}{U}}$$

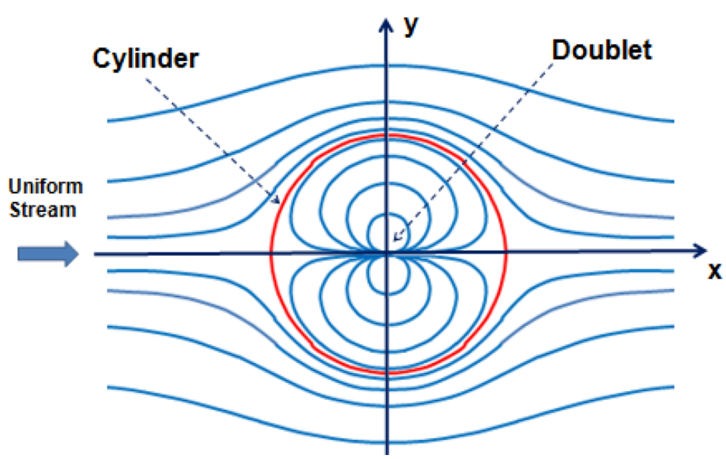
with two stagnation points at $A(a, \pi)$ and $B(a, 0)$



The velocity components are:

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

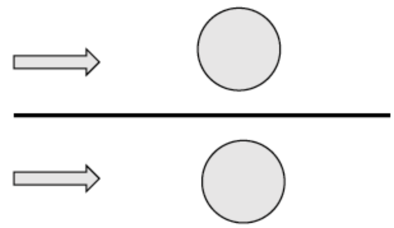
$$v = \frac{\partial \psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$



The streamline defining the circle can be considered a solid boundary in potential flow, with the streamlines inside being of no importance.

Questions:

Can we describe flow over half a cylinder?
Can we use the method of images to describe uniform flow past two cylinders?



Pressure distribution in the flow past a cylinder can be obtained from Bernoulli's equation (neglecting gravity)

$$\frac{|V|^2}{2} + \frac{p}{\rho} = \frac{U^2}{2} + \frac{p_o}{\rho}$$

where p_o is a uniform pressure far from the cylinder

On the surface of the cylinder at $r = a, \;\; u = 0$, and

$$v = -2U \sin\theta$$

Therefore

$$p|_{r=a} = p_o + \frac{\rho U^2}{2} - 2\rho U^2 sin^2\theta$$

Since there are no shear stresses in potential flow, the drag force on the cylinder is only due to pressure acting on the boundary in the direction of the flow (x direction)

$$F_D = -\int_0^{2\pi} p|_{r=a} \cos\theta \, r d\theta = 0$$

This result referred to as *D'Alembert Paradox* is not surprising (why?)

Similarly, the lift force is zero

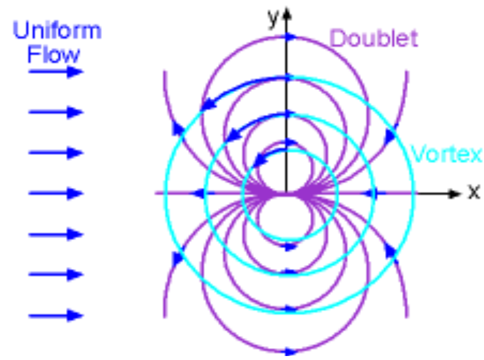
$$F_L = -\int_0^{2\pi} p|_{r=a} \sin\theta \;\, r d\theta = 0$$

IX.10 Flow over a Rotating Cylinder

The flow over a rotating cylinder can be constructed by combining the potential for uniform flow, a doublet and a vortex.

$$\phi = -Ur\cos\theta - \chi\frac{\cos\theta}{2\pi} + \frac{\Lambda}{2\pi}\theta$$

$$\psi = -Ur\sin\theta + \chi\frac{\sin\theta}{r} - \frac{\Lambda}{2\pi}\theta$$



To find the stagnation points where both u and v are zero

$$u = \cos\theta\left(U - \frac{\chi}{r^2}\right) = 0$$

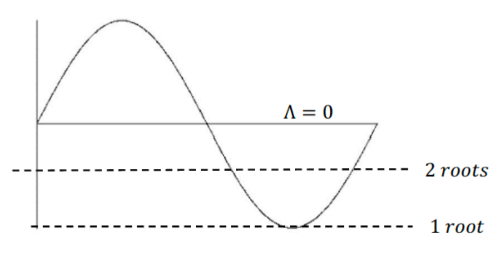
$$v = -\sin\theta\left(U + \frac{\chi}{r^2}\right) - \frac{\Lambda}{2\pi r} = 0$$

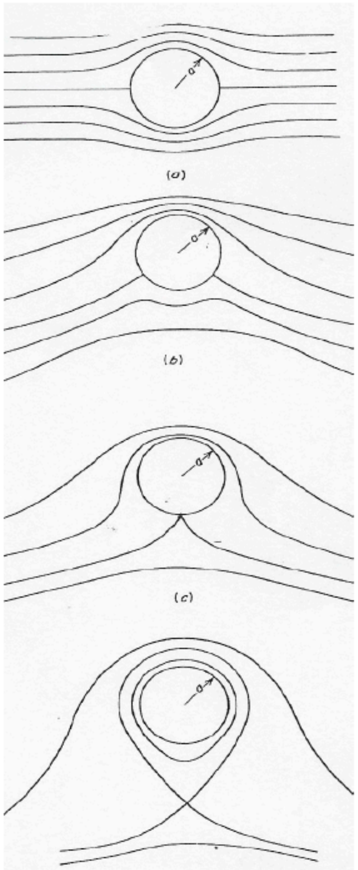
The first equation gives: $\theta = \pm\frac{\pi}{2}$ and $r = \sqrt{\chi/U}$

and the second equation give $\sin\theta = -\frac{\Lambda}{4\pi\sqrt{\chi U}}$ $(\Lambda > 0)$

If $\Lambda = 0$, there are two stagnation points without rotation, which corresponds to the previous solution for flow past a fixed cylinder. Since the radius of the cylinder is $a = \sqrt{\chi/U}$ then $\chi = a^2U$ and the equation for the stagnation points with rotation becomes:

$$\sin\theta = -\frac{\Lambda}{4\pi aU}$$





$\Lambda = 0$

$\frac{\Lambda}{4\pi a U} < 1$

$\frac{\Lambda}{4\pi a U} = 1$

$\frac{\Lambda}{4\pi a U} > 1$

Circulation can be determined from the definition

$$\Gamma = \oint \vec{V} \cdot d\vec{s} = \Lambda$$

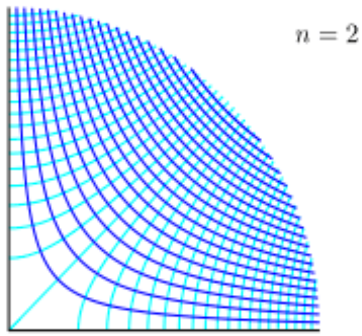
Also, it can be shown that $F_D = 0$, but the lift force is:

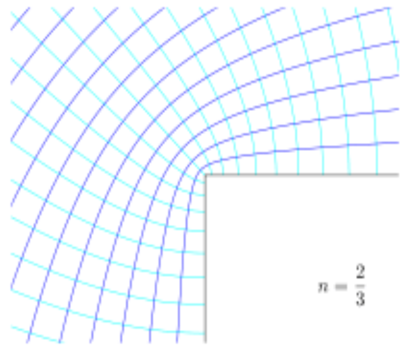
$$F_L = \rho U \Lambda = \rho U \Gamma$$

This equation is valid for any two-dimensional body and is called the *Magnus Effect.*

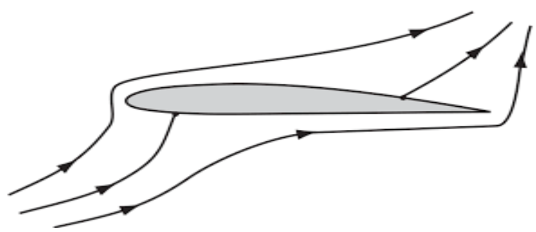
Many two-dimensional flows can be constructed by using conformal mapping.

Flow around a corner $\psi = Ar^n \sin n\theta$





Airfoil
(Joukowski Transformation changes an airfoil into an off-axis circle)



IX.11 Three-Dimensional Potential Flow

Three-dimensional potential flow is best described by using spherical coordinates. For example, uniform flow can be described by the stream function:

$$\psi = \frac{1}{2} U r^2 \sin^2\theta$$

Point Source (sink)

A point source point with volumetric flow rate $4\pi m$ in spherical coordinates can be defined by the potential

$$\phi = -\frac{m}{r}$$

with the radial velocity

$$u = \frac{m}{r^2}$$

From the definition of the stream function:

$$u = \frac{m}{r^2} = \frac{1}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta} \qquad\qquad v = 0 = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

Hence

$$\psi = -m\cos\theta$$

If the source is located at another point (x_0, y_0, z_0) the coordinate r is replaced with

$$r^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

<u>3-D Doublet</u>

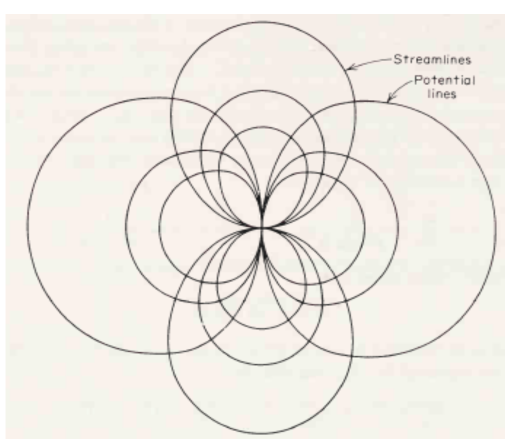
A three-dimensional doublet is located at the origin on the x axis, the velocity potential is given by

$$\phi = -\frac{kx}{r^3}$$

where k is a constant. x can be replaced by y or z, if the doublet is located on the respective axis.

The stream function for the doublet is

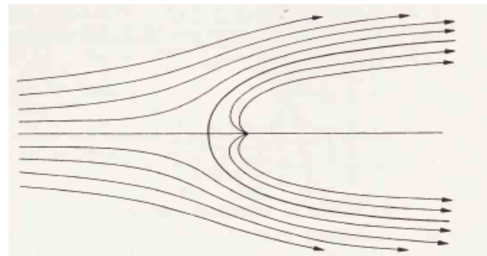
$$\psi = \frac{k\sin^2\theta}{r}$$



IX.12 Flow Past a Half Body

Using the principle of superposition, we can combine uniform flow with a three-dimensional source of strength m gives the combined potential

$$\phi = Ux - \frac{m}{r}$$



and the stream function

$$\psi = \frac{1}{2}Ur^2\sin^2\theta - m\cos\theta$$

Here, for $\theta = 0, \pi$ a streamline is defined by $\psi = -m, m$. But for $\psi = m$ the following streamline also exists

$$r = \sqrt{\frac{m}{U}}\csc\frac{\theta}{2}$$

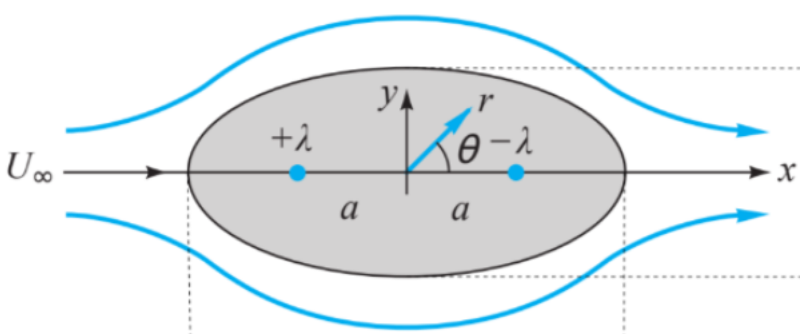
which defines a boundary symmetric with respect to the polar axis with a cross sectional radius at infinity equal to $2\sqrt{\frac{m}{U}}$.

This result can be deduced from the fact that the flow rate at infinity must be equal to flow rate from the source. Namely

$$4\pi m = \pi r_0^2 U$$

IX.13 Rankine Body

A flow over a closed axisymmetric body called the Rankine Body is a source and a sink are placed apart on the x axis.



For any axisymmetric body, the flow can be approximated by superimposing a uniform flow with several sources or sinks at different locations along the axis and of various strengths (Von-Karman, 1927).

IX.14 <u>Flow Past a Sphere</u>

Uniform flow past a sphere can be constructed by adding the stream function for uniform flow and a doublet in the negative x direction.

$$\psi = \frac{1}{2}Ur^2\sin^2\theta - \frac{k}{r}\sin^2\theta = \left(\frac{1}{2}\mathrm{Ur}^2 - \frac{\mathrm{k}}{\mathrm{r}}\right)\sin^2\theta$$

The stream function $\psi = 0$ defines a streamline at $\theta = 0, \pi$ and the radius

$$r = \left(\frac{2k}{U}\right)^{1/3}$$

If the radius of the sphere is $a,$ then $k = \frac{1}{2}Ua^3$.

The same result can be obtained by adding the uniform flow for the stream function to the solution the governing equation directly with the transformation $\beta = \cos\theta$, then

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}(1-\beta^2)\frac{\partial^2}{\partial\beta^2}\right]\psi = 0$$

The solution can be obtained with the substitution

$$\psi = (1-\beta^2)f(r)$$

Then,

$$f'' - \frac{2}{r^2}f = 0$$

and

$$f = C_1r^2 + \frac{C_2}{r}$$

The first solution generates the uniform flow far from the sphere and the second term represents the doublet.

A solution can also be obtained for a sphere moving at a constant speed in an otherwise quiescent fluid, the uniform flow part should be deleted from the previous solution. Therefore,

$$\psi = -\frac{Ua^3}{2r}\sin^2\theta$$

IX.15 Added Mass

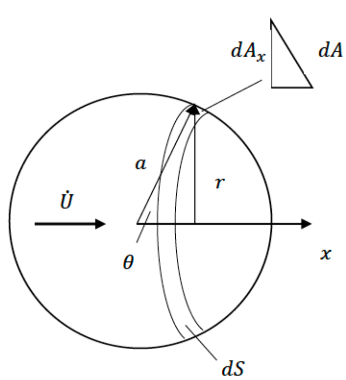
When a body moves in fluid it induces fluid motion around it. If the body accelerates, then this moving fluid can accelerate as well. Therefore, additional force is required to accelerate the body in fluid than in vacuum. This additional force, which in general is mass times acceleration, can be represented as an imaginary *added mass* of the body in the fluid.

The added mass can be derived by considering the hydrodynamic force acting on the body as it accelerates. For example, for a sphere of radius $a,$ accelerating at a rate $\dot{U},$ the force in the flow direction x, can be obtained by integrating the pressure over the surface area projected in the flow direction.

$$\vec{F}_x = \int p\, d\vec{A}_x = \int p\cos\theta\, dA$$

Here

$$dA = 2\pi r dS$$
$$r = a\sin\theta$$
$$dS = ad\theta$$



From the unsteady Bernoulli's equation

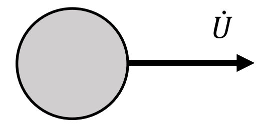
$$p = -\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2\right)$$

For axisymmetric flow around the sphere

$$\phi = U\frac{a^3}{2r^2}\cos\theta$$

Therefore,

$$\left.\frac{\partial\phi}{\partial t}\right|_a = \dot{U}\cos\theta\,\frac{a}{2}$$



Similarly, the added mass can be determined for a cylinder of radius a and length L accelerating in the x direction.

$$m_a = \rho\pi a^2 L$$