ME 572 Aerodynamic Design
HW #2 (Due at 11:59 pm on Friday, Feb 16)

Problem 1 [20 pt]

The shock waves on a vehicle in supersonic flight cause a component of drag called supersonic wave drag D_w. Define the wave-drag coefficient as $C_{D,w} = D_w/q_\infty S$, where S is a suitable reference area for the body. In supersonic flight, the flow is governed in part by its thermodynamic properties, given by the specific heats at constant pressure c_p and at constant volume c_v. Define the ratio $c_p/c_v \equiv \gamma$. Using Buckingham's pi theorem, show that $C_{D,w} = f(M_\infty, \gamma)$. Neglect the influence of friction.

Solution:

Step 1: List all the variables that are involved in the problem

$$D_w = f_1(\rho_\infty, V_\infty, a_\infty, c_p, c_v, l)$$

2 Points

$$k = 7$$

Step 2: Express each of the variables in terms of basic dimensions.

- M = Mass
- K = Temperature
- L = Length
- T = Time

4 Points

$$D_w = [MLT^{-2}];\ \rho_\infty = [ML^{-3}];\ V_\infty = [LT^{-1}];\ a_\infty = [LT^{-1}];\ c_p = [L^2T^{-2}K^{-1}];$$
$$c_v = [L^2T^{-2}K^{-1}];\ l = [L]$$

$$r = 4$$

Step 3: Select repeating variables.

- No dependent variable.
- Should contain all r dimensions (M, K, L and T).
- No dimensionless variable
- Pick simple parameters over complex parameters whenever possible

We need to pick r = 4 repeating variables, as highlighted below

2 Points

$$D_w = [MLT^{-2}];\ \rho_\infty = [ML^{-3}];\ V_\infty = [LT^{-1}];\ a_\infty = [LT^{-1}];\ c_p = [L^2T^{-2}K^{-1}];$$
$$c_v = [L^2T^{-2}K^{-1}];\ l = [L]$$

Step 4: The number of π-parameters is k-r.

1 Points

$$k - r = 3$$

Step 5: **Write the π-terms by combining the repeating variables with each of the remaining variables.**

$$\Pi_1 = \rho_\infty^a V_\infty^b l^c c_p^d D_w$$

$$\Pi_2 = \rho_\infty^a V_\infty^b l^c c_p^d a_\infty$$

$$\Pi_3 = \rho_\infty^a V_\infty^b l^c c_p^d c_v$$

3 Points

Step 6: **Solve the equations from step 5.**

$$\Pi_1 = [ML^{-3}]^a[LT^{-1}]^b[L]^c[L^2T^{-2}K^{-1}]^d[MLT^{-2}] = M^0K^0L^0T^0$$

$$\begin{cases} a+1=0 \\ -d=0 \\ -3a+b+c+2d+1=0 \\ -b-2d-2=0 \end{cases}$$

2 Points

$$\begin{cases} a=-1 \\ b=-2 \\ c=-2 \\ d=0 \end{cases}$$

$\Pi_1 = \rho_\infty^{-1}V_\infty^{-2}l^{-2}D_w \; or \; \Pi_1 = \frac{D_w}{q_\infty S} = C_{D,w}$, where $q_\infty \sim \rho_\infty V_\infty^2, S \sim l^2$

$$\Pi_2 = [ML^{-3}]^a[LT^{-1}]^b[L]^c[L^2T^{-2}K^{-1}]^d[LT^{-1}] = M^0K^0L^0T^0$$

$$\begin{cases} a=0 \\ -d=0 \\ -3a+b+c+2d+1=0 \\ -b-2d-1=0 \end{cases}$$

$$\begin{cases} a=0 \\ b=-1 \\ c=0 \\ d=0 \end{cases}$$

2 Points

$$\Pi_2 = V_\infty^{-1}a_\infty \; or \; \Pi_2 = \frac{a_\infty}{V_\infty} = \frac{1}{M_\infty}$$

$$\Pi_3 = [ML^{-3}]^a[LT^{-1}]^b[L]^c[L^2T^{-2}K^{-1}]^d[L^2T^{-2}K^{-1}] = M^0K^0L^0T^0$$

$$\begin{cases} a=0 \\ -d-1=0 \\ -3a+b+c+2d+2=0 \\ -b-2d-2=0 \end{cases}$$

$$\begin{cases} a = 0 \\ b = 0 \\ c = 0 \\ d = -1 \end{cases}$$

2 Points

$$\Pi_3 = c_p^{-1} c_v \; or \; \Pi_3 = \frac{c_v}{c_p} = \gamma$$

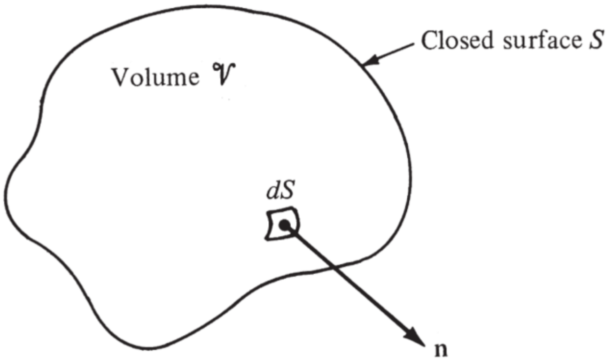
Therefore, $\Pi_1 = f(\Pi_2, \Pi_3)$

$$C_{D,w} = f(M_\infty, \gamma)$$

2 Points

Problem 2 [10 pt]
Consider a body of arbitrary shape. If the pressure distribution over the surface of the body is constant, prove that the resultant pressure force on the body is zero.

Solution:



The resultant pressure force is the integral of the pressure over the entire surface (closed surface of the arbitrary shape body):

$$\vec{F} = \oiint_S P d\vec{S}$$

3 Points

Since the pressure is constant over the surface.

$$\vec{F} = P \oiint_S d\vec{S} = P \oiint_S \vec{n} dS$$

3 Points

Based on the Divergence Theorem.

$$\vec{F} = P \oiint_S \vec{n} dS = P \oiiint_\forall \nabla \vec{n} d\forall$$

$$\nabla \vec{n} = 0$$

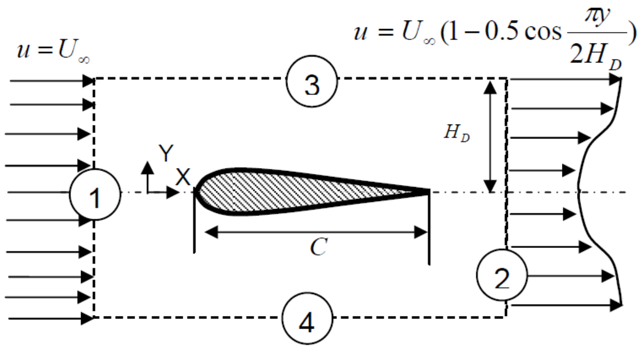
4 Points

Therefore

$$\vec{F} = P \oiint_S \vec{n} dS = P \oiiint_\forall \nabla \vec{n} d\forall = 0$$

Problem 3 [30 pt]

Velocity profiles are measured at the upstream end (surface 1) and at the downstream end (surface 2) of the control volume shown in the figure. The flow is incompressible, two dimensional, and steady. The gage pressure on the surfaces along the dashed line is equal to zero.



(a). What is the total volume flow rate crossing the horizontal surfaces (surface 3 and 4)

(b). If $H_D = 0.025\,c$, where c is the chord length of the airfoil, what is the drag coefficient C_D of the airfoil?

Solution:

a) Take the area enclosed by surfaces ①②③④ to be the Control volume.

From the conservation of the mass, we have

$$\int_{C.V.} \frac{\partial \rho}{\partial t} d\forall + \int_{C.S.} (\rho \vec{v}) \cdot d\vec{A} = 0$$

2 Points

Since the flow is 2D steady incompressible, we have

$$-\int_1 u\,dy + \int_2 u\,dy - \int_{3+4} v\,dx = 0$$

2 Points

$$\int_{3+4} v\,dx = \int_2 u\,dy - \int_1 u\,dy$$

$$= \int_{-H_D}^{H_D} U_\infty\left(1 - 0.5\cos\frac{\pi y}{2H_D}\right) dy - \int_{-H_D}^{H_D} U_\infty\, dy$$

$$= -0.5U_\infty \int_{-H_D}^{H_D} \cos\frac{\pi y}{2H_D}\, dy$$

4 Points

$$= -0.5U_\infty \frac{2H_D}{\pi} \int_{-H_D}^{H_D} \cos\frac{\pi y}{2H_D}\, d\left(\frac{\pi y}{2H_D}\right)$$

$$= -0.5U_\infty \frac{2H_D}{\pi} \sin\frac{\pi y}{2H_D}\Big|_{-H_D}^{H_D}$$

$$= -0.5U_\infty \frac{2H_D}{\pi}(1 - (-1))$$

$$= -\frac{2}{\pi} U_\infty H_D$$

Therefore, the total volumetric flow rate out of the control volume is $\frac{2}{\pi} U_\infty H_D$.

2 Points

From the conservation of momentum, we have

$$\int_{C.V.} \frac{\partial}{\partial t} \rho \vec{v}\, d\forall + \int_{C.S.} \rho \vec{v}\, \vec{v} \cdot d\vec{S} = \Sigma \vec{F}$$

2 Points

Since flow is steady, we have

$$\int_{C.V.} \frac{\partial}{\partial t} \rho \vec{v}\, d\forall = 0$$

Since the gage pressure on surfaces ① ② ③ ④ are equal to zero, also neglect the body force, we have

2 Points

$$\Sigma \vec{F} = -D$$

where D represents the drag.

Further, we have in x direction,

$$-\int_{①} \rho u u\, dy + \int_{②} \rho u u\, dy + \int_{③} \rho u v\, dx + \int_{④} \rho u v\, dx = -D$$

2 Points

Specifically, for the incompressible flow in this case

$$\int_{①} \rho u u\, dy = 2\int_0^{H_D} \rho U_\infty^2\, dy = 2\rho U_\infty^2 H_D$$

$$\int_{②} \rho u u\, dy = 2\int_0^{H_D} \rho u^2\, dy = 2\int_0^{H_D} \rho U_\infty^2 \left(1 - 0.5\cos\frac{\pi y}{2H_D}\right)^2 dy$$

$$= 2\rho U_\infty^2 \int_0^{H_D} \left[1 - \cos\frac{\pi y}{2H_D} + 0.25\cos^2\frac{\pi y}{2H_D}\right] dy$$

$$= 2\rho U_\infty^2 \int_0^{H_D} \left[1 - \cos\frac{\pi y}{2H_D} + 0.25\,\frac{1+\cos\frac{\pi y}{H_D}}{2}\right] dy$$

4 Points

$$= 2\rho U_\infty^2 \left[H_D - \frac{2H_D}{\pi}\sin\frac{\pi y}{2H_D}\Big|_0^{H_D} + 0.125 H_D + \frac{0.125 H_D}{\pi}\sin\frac{\pi y}{H_D}\Big|_0^{H_D}\right]$$

$$= 2\rho U_\infty^2 \left[H_D - \frac{2H_D}{\pi} + 0.125 H_D + 0\right]$$

$$= 2\left[1.125 - \frac{2}{\pi}\right]\rho U_\infty^2 H_D$$

2 Points

By symmetry, we have

$$\int_{③} \rho u v\, dx = \int_{④} \rho u v\, dx$$

Since the upstream velocity is uniformly U_∞, and the edge of the downstream control surface also have velocity U_∞ ($u = U_\infty\left(1 - 0.5\cos\frac{\pi H_D}{2H_D}\right) = U_\infty$) Therefore, we can assume that on surface ③ and ④, the velocities in the x direction are U_∞

Also, we know from a) that

$$\int_{3+4} v\,dx = \frac{2}{\pi} U_\infty H_D$$

Thus,

$$\int_{③} \rho u v\,dx + \int_{④} \rho u v\,dx = \rho U_\infty \int_{③+④} v\,dx = \frac{2}{\pi}\rho U_\infty^2 H_D$$

2 Points

$$-\int_{①} \rho u u\,dy + \int_{②} \rho u u\,dy + \int_{③} \rho u v\,dx + \int_{④} \rho u v\,dx$$

$$= -2\rho U_\infty^2 H_D + 2\left[1.125 - \frac{2}{\pi}\right] \rho U_\infty^2 H_D + \frac{2}{\pi}\rho U_\infty^2 H_D$$

4 Points

$$= \left[-2 + 2.25 - \frac{4}{\pi} + \frac{2}{\pi}\right] \rho U_\infty^2 H_D$$

$$= -0.3866 \rho U_\infty^2 H_D$$

Namely, $-D = -0.3866 \rho U_\infty^2 H_D$

$D = 0.3866 \rho U_\infty^2 H_D$

Specifically, the drag force applied on the fluid is in the $-x$ direction, and the drag force imposed on the airfoil is in the x direction

$$C_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 C} = \frac{0.3866 \rho U_\infty^2 H_D}{\frac{1}{2}\rho U_\infty^2 C} = \frac{0.3866 \rho U_\infty^2 \, 0.025C}{\frac{1}{2}\rho U_\infty^2 C}$$

2 Points

$$= 0.01933$$