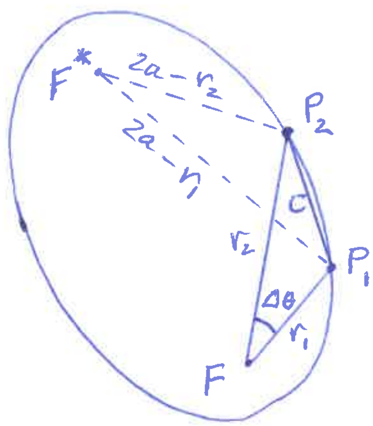
8. Lambert's Problem

Lambert's problem deals with the determination of transfer trajectories between 2 specified points P_1, P_2. The trajectories may be elliptic, parabolic or hyperbolic but for simplicity only elliptic transfer orbits will be considered



F = focus

F* = vacant focus

Figure 1

The triangle FP_1P_2 is called the <u>space triangle</u>

For a given space triangle, consider the effect of varying the semi-major axis of the transfer ellipse

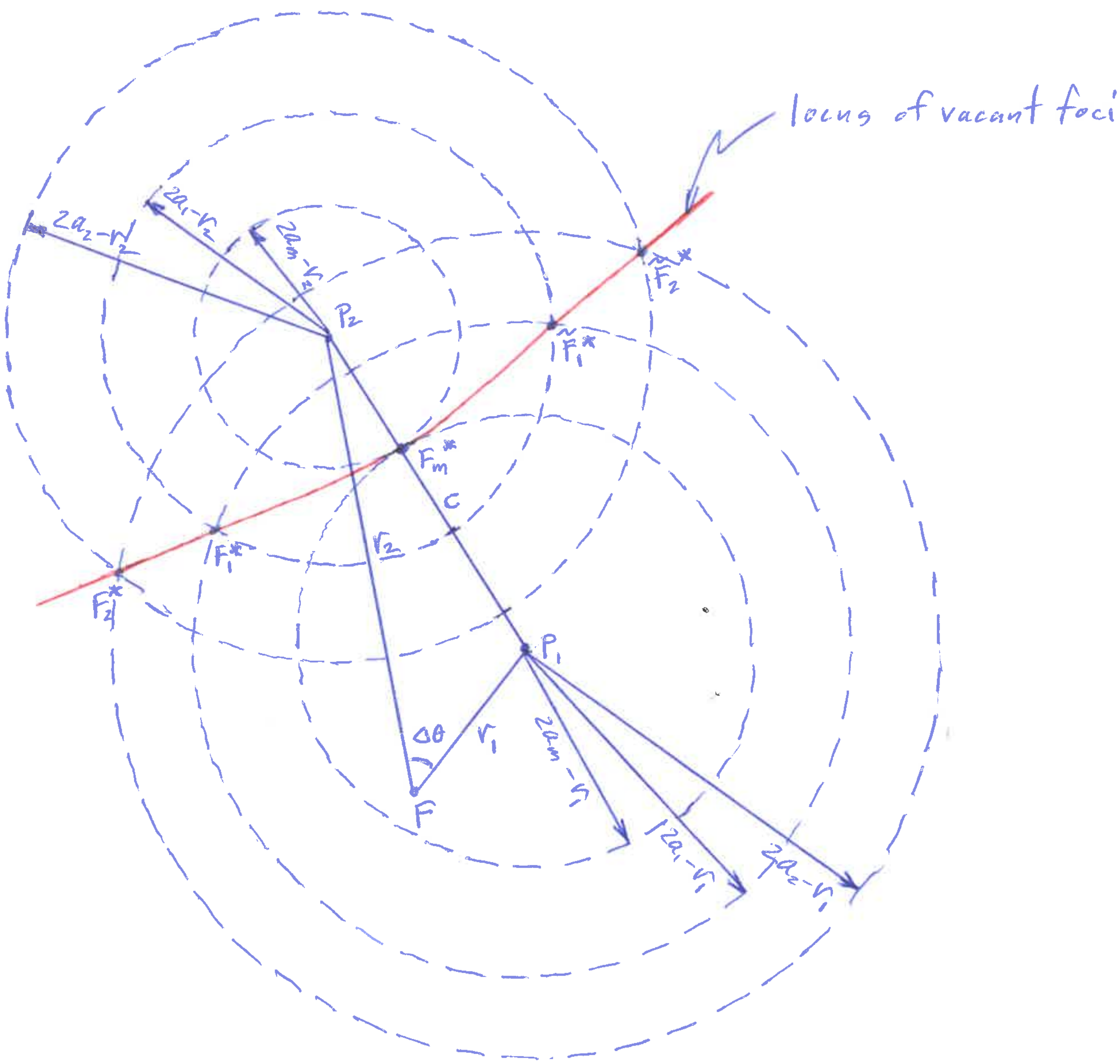
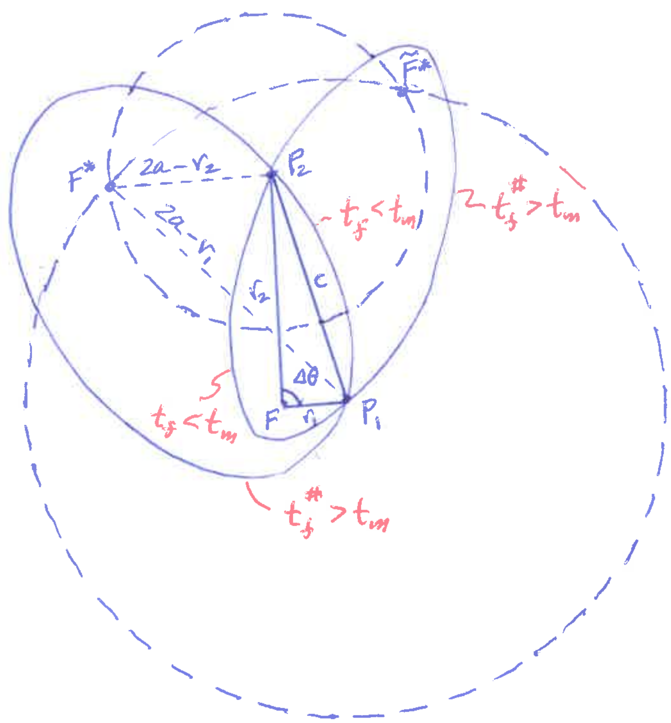


Figure 2

For a given value of $a = a_k > a_m$, there are 2 possible locations of the vacant focus of the transfer ellipse denoted by F_k^* and $\tilde{F}_k^*$. Thus there are 2 possible transfer ellipses between points P_1 and P_2. The two ellipses for the same value of a have different eccentricities and transfer times but the same total energy $-\frac{\mu}{2a}$



t_f = time of flight (from P_1 to P_2)

t_m = t_f along MET

Figure 3

For $a = a_m$ there is only one transfer ellipse. Since the total energy of the transfer ellipse is $-\frac{\mu}{2a}$, this represents the minimum energy trajectory.

For $a < a_m$, no transfer ellipse exists between P_1 & P_2.

To calculate a_m, from Figure 2

$$(2a_m - r_2) + (2a_m - r_1) = c$$

or

$$a_m = \frac{s}{2} \qquad (8.1)$$

where

$$s = (r_1 + r_2 + c)/2 \qquad (8.2)$$

is the semi-perimeter of the space triangle FP_1P_2

Lambert's Theorem

The time required to transverse an elliptic arc between 2 specified points depends only on the semi-major axis of the ellipse, the chord length and the sum of the radii from the focus to the 2 points, i.e.

$$t_2 - t_1 = f(a, c, r_1 + r_2) \qquad (8.3)$$

To find this equation for the transfer time, use Kepler's equation

$$n(t_2 - t_1) = E_2 - E_1 - e(\sin E_2 - \sin E_1) \qquad (8.4)$$

where

$$n = \sqrt{\frac{\mu}{a^3}}$$

This equation contains e and the eccentric anomalies of the two points which are unknown.

To obtain a more convenient form, define

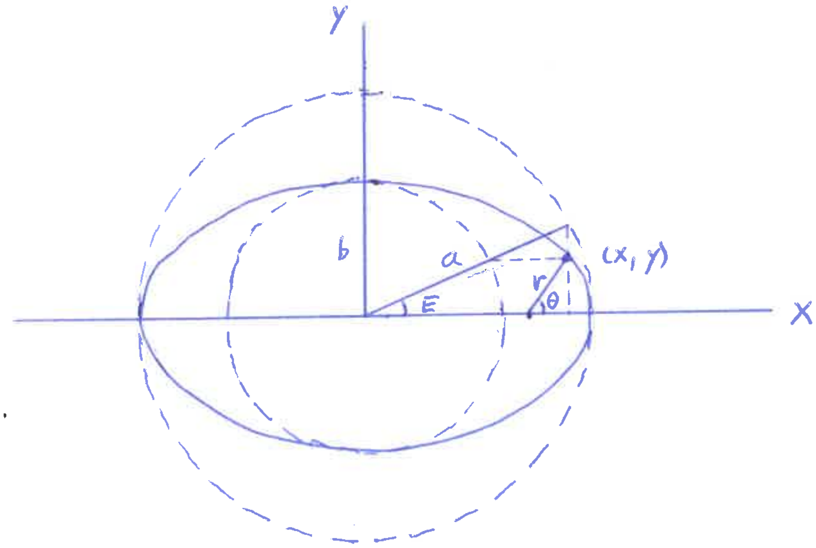
$$E_p = \frac{1}{2}(E_2 + E_1) \qquad E_m = \frac{1}{2}(E_2 - E_1) \qquad (8.5a, b)$$

Using $r = a(1 - e\cos E)$

$$r_1 + r_2 = a\left[2 - e(\cos E_1 + \cos E_2)\right]$$

$$= 2a\left[1 - e\cos E_p \cos E_m\right] \qquad (8.6)$$

Using a cartesian coordinate system with origin at the center of the ellipse



$$x = a\cos E \qquad y = b\sin E \qquad b = a\sqrt{1-e^2}$$

The chord distance can be obtained from

$$c^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

where x_1, y_1 and x_2, y_2 are the cartesian coordinates of P_1, P_2.

Can write

$$c^2 = a^2\left[(\cos E_2 - \cos E_1)^2 + (1-e^2)(\sin E_2 - \sin E_1)^2\right]$$

$$= 4a^2 \sin^2 E_m \,(1 - e^2\cos^2 E_p) \qquad (8.7)$$

Since $e<1$, $-1 < e\cos E_p < 1$. Therefore let

$$\cos \xi = e \cos E_p \qquad (8.8)$$

This makes the right hand side of (8.7) a perfect square, resulting in

$$c = 2a \sin E_m \sin \xi \qquad (8.9)$$

Eq. (8.6) can be rewritten as

$$r_1 + r_2 = 2a\left(1 - \cos E_m \cos \xi\right) \qquad (8.10)$$

Define

$$\alpha = \xi + E_m \qquad \beta = \xi - E_m \qquad (8.11a, b)$$

Combine (8.9) (8.10) and (8.11) to give

$$r_1 + r_2 + c = 2a\left(1 - \cos\alpha\right) = 4a \sin^2\left(\frac{\alpha}{2}\right) \qquad (8.12)$$

$$r_1 + r_2 - c = 2a\left(1 - \cos\beta\right) = 4a \sin^2\left(\frac{\beta}{2}\right) \qquad (8.13)$$

Eq. (8.4) for the transfer time becomes

$$n(t_2 - t_1) = E_m - \cos\xi \sin E_m \qquad (8.14)$$

or

$$\boxed{n(t_2 - t_1) = \alpha - \beta - (\sin\alpha - \sin\beta)} \qquad (8.15)$$

where

$$\sin\left(\frac{\alpha}{2}\right) = \left(\frac{s}{2a}\right)^{1/2} \qquad (8.16)$$

$$\sin\left(\frac{\beta}{2}\right) = \left(\frac{s-c}{2a}\right)^{1/2} \qquad (8.17)$$

and $s = \dfrac{r_1 + r_2 + c}{2}$

The following table summarizes how to properly choose the correct quadrant for α, β.

In all cases where $0 \le \Delta\theta \le 2\pi$, α_0 and β_0 are solutions of (8.16) and (8.17) for which

$0 \le \beta_0 \le \alpha_0 \le \pi$ (principal values)

$$\begin{aligned}
&\text{If } 0 \le \Delta\theta < \pi && \beta = \beta_0 \\
&\text{If } \pi \le \Delta\theta < 2\pi && \beta = -\beta_0 \\
&\text{If } t_2 - t_1 = t_f \le t_m && \alpha = \alpha_0 \\
&\text{If } t_2 - t_1 = t_f^{\#} > t_m && \alpha = 2\pi - \alpha_0
\end{aligned} \qquad (8.18)$$

where t_m is the time it takes to go from P_1 to P_2 along the minimum energy trajectory. (will show how to find t_m later).

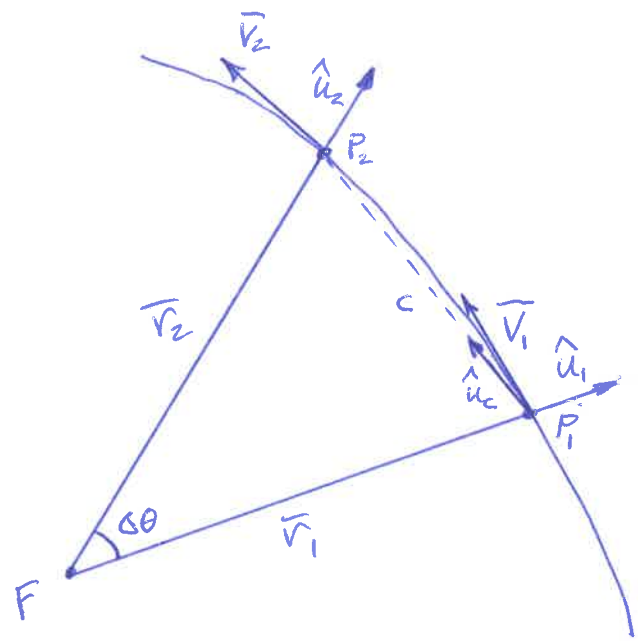
The parameter p and the eccentricity e of the elliptic transfer orbit are determined from

$$p = \frac{4a(s-r_1)(s-r_2)}{c^2} \sin^2\left(\frac{\alpha+\beta}{2}\right) \qquad (8.19)$$

and $p = a(1-e^2) \Rightarrow e = \left(1 - \frac{p}{a}\right)^{1/2}$

Velocities

Define the unit vectors as follows



$$\hat{u}_1 = \frac{\bar{r}_1}{r_1} \qquad (8.20a)$$

$$\hat{u}_2 = \frac{\bar{r}_2}{r_2} \qquad (8.20b)$$

$$\bar{u}_c = \frac{\bar{r}_2 - \bar{r}_1}{c} \qquad (8.20c)$$

The velocities $\bar{V}_1$ and $\bar{V}_2$ at P_1 and P_2 can be found from

$$\bar{V}_1 = (B+A)\,\hat{u}_c + (B-A)\,\hat{u}_1 \qquad (8.21)$$

$$\bar{V}_2 = (B+A)\,\hat{u}_c - (B-A)\,\hat{u}_2 \qquad (8.22)$$

where

$$A = \left(\frac{\mu}{4a}\right)^{1/2} \cot\left(\frac{\alpha}{2}\right) \qquad (8.23a)$$

$$B = \left(\frac{\mu}{4a}\right)^{1/2} \cot\left(\frac{\beta}{2}\right) \qquad (8.23b)$$

Appropriate values of α, β are given by (8.16) & (8.17) with quadrant adjustments given by (8.18)

For the minimum energy trajectory, using (8.1)

$$a = a_m = \frac{s}{2}$$

reduces (8.16) & (8.17) to

$$\left.\begin{array}{l} \sin\left(\frac{\alpha_m}{2}\right) = \left(\frac{s}{2\left(\frac{s}{2}\right)}\right)^{1/2} = 1 \quad \Rightarrow \quad \alpha_m = \pi \\ \sin\left(\frac{\beta_m}{2}\right) = \left(\frac{s-c}{2\left(\frac{s}{2}\right)}\right)^{1/2} \quad \Rightarrow \quad \sin\left(\frac{\beta_m}{2}\right) = \left(\frac{s-c}{s}\right)^{1/2} \end{array}\right\} \quad (8.24)$$

and from (8.15) the transfer time is given by

$$\sqrt{\frac{\mu}{\left(\frac{s}{2}\right)^3}}\, t_m = \pi - \beta_m + \sin\beta_m$$

$$t_m = \sqrt{\frac{s^3}{8\mu}}\left(\pi - \beta_m + \sin\beta_m\right) \qquad (8.25)$$

This value is needed to determine the correct quadrant for α (see eq. (8.18)).

Taking the limit of (8.15) as $a \to \infty$ gives

$$t_p = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left[s^{3/2} - \operatorname{sgn}(\sin\Delta\theta)(s-c)^{3/2}\right] \qquad (8.26)$$

(Euler's equation)

where $\operatorname{sgn}(\sin\Delta\theta) = \begin{cases} +1 & \text{for } 0 < \Delta\theta \leq \pi \\ -1 & \text{for } \pi < \Delta\theta < 2\pi \end{cases}$

t_p represents the transfer time from P_1 to P_2 along a parabolic trajectory.

Note:

Elliptic transfer trajectories exist only for $t_2 - t_1 > t_p$

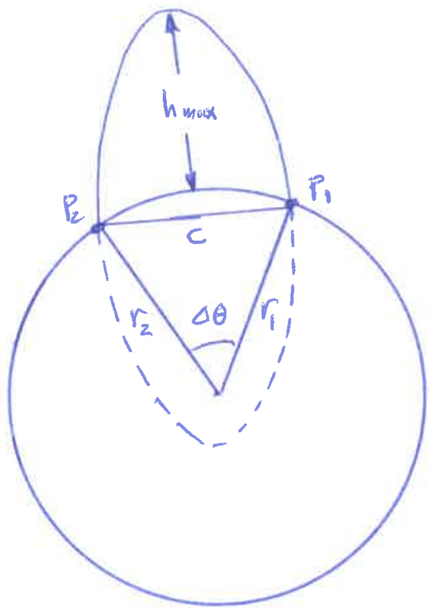
If $t_2 - t_1 < t_p$, the transfer trajectory must be hyperbolic

EXAMPLE (see HW4, prob. 1)

a) What is the apogee altitude of a ballistic missile fired on an optimum trajectory across an 6000 km range?

b) What is the apogee altitude of the missile when fired with the same velocity to a range of 3000 km?

a)



range $R = 6000$ km

$r_1 = r_2 = r_0 = 6368$ km

$$\Delta\theta = \frac{R}{r_0} = \frac{6000}{6368} = 0.942211 \text{ radians}$$

$$c^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\Delta\theta = r_0^2 + r_0^2 - 2r_0^2 \cos\Delta\theta$$

$$= 2r_0^2(1-\cos\Delta\theta) = 2(6368)^2[1-\cos(0.942211)]$$

$$= 3.34143 \times 10^7$$

$$c = 5780.5 \text{ km}$$

$$s = \frac{r_1 + r_2 + c}{2} = \frac{6368 + 6368 + 5780.5}{2} = 0.9258.3 \text{ km}$$

$$a_m = \frac{s}{2} = \frac{9258.3}{2} = 4629.13 \text{ km}$$

From (8.24)

$$\alpha_m = \pi$$

$$\sin\left(\frac{\beta_m}{2}\right) = \left(\frac{s-c}{s}\right)^{1/2} = \left(\frac{9258.3 - 5780.5}{9258.3}\right)^{1/2} = 0.612896$$

$$\Rightarrow \beta_m = 1.31944 \text{ radians}$$

Using (8.19)

$$p_m = \frac{4a_m(s-r_1)(s-r_2)}{c^2} \sin^2\left(\frac{\alpha_m+\beta_m}{2}\right)$$

$$= \frac{4(4629.13)(9258.3-6368)^2}{(5780.5)^2} \sin^2\left(\frac{\pi + 1.31944}{2}\right)$$

$$= 2890.34 \text{ km}$$

$$e_m = \left(1 - \frac{p_m}{a_m}\right)^{1/2} = \left(1 - \frac{2890.34}{4629.13}\right)^{1/2} = 0.612878$$

At apogee

$$r_{a_m} = \frac{p_m}{1+e_m\cos\pi} = \frac{2890.34}{1+(0.612878)(-1)} = 7466.23 \text{ km}$$

$$h_{max} = r_{a_m} - r_0 = 7466.23 - 6368$$

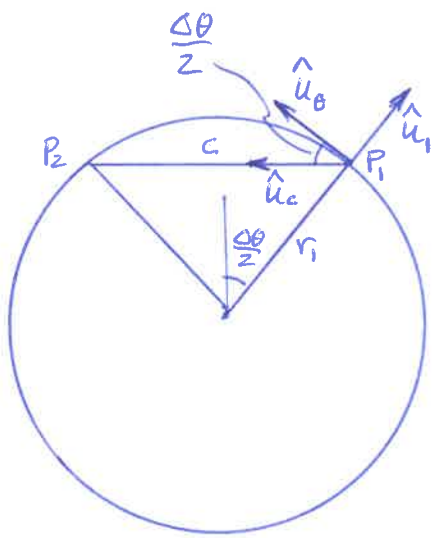
$$\boxed{h_{max} = 1098 \text{ km}}$$ (Agrees with HW 4 prob. 1a)

To find velocity at launch point, use (8.21) & (8.23)

$$A = \left(\frac{\mu}{4a_m}\right)^{1/2} \cot\left(\frac{\alpha_m}{2}\right) = \left(\frac{3.986\times10^5}{4(4629.13)}\right)^{1/2} \cot\left(\frac{\pi}{2}\right) = 0$$

$$B = \left(\frac{\mu}{4a_m}\right)^{1/2} \cot\left(\frac{\beta_m}{2}\right) = \left(\frac{3.986\times10^5}{4(4629.13)}\right)^{1/2} \cot\left(\frac{1.31944}{2}\right) = 5.98163 \ \frac{km}{sec}$$

$$\bar{V}_1 = (B+A)\,\hat{u}_c + (B-A)\,\hat{u}_1$$



$$\hat{u}_1 = \hat{u}_r$$

$$\hat{u}_c = -\sin\frac{\Delta\theta}{2}\,\hat{u}_r + \cos\frac{\Delta\theta}{2}\,\hat{u}_\theta$$

With $A = 0$

$$\bar{V}_1 = B(\hat{u}_c + \hat{u}_1) = B\left[\left(1-\sin\frac{\Delta\theta}{2}\right)\hat{u}_r + \cos\frac{\Delta\theta}{2}\,\hat{u}_\theta\right]$$

$$= 5.98163\left[\left(1-\sin\frac{0.942211}{2}\right)\hat{u}_r + \cos\frac{0.942211}{2}\,\hat{u}_\theta\right]$$

$$= 3.26674\ \hat{u}_r + 5.33003\ \hat{u}_\theta \qquad (km/sec)$$

$$V_1 = |\bar{V}_1| = \left[(3.26674)^2 + (5.33003)^2\right]^{1/2} = 6.25146 \ \frac{km}{sec}$$

$$V_s = \sqrt{\frac{\mu}{r_0}} = \sqrt{\frac{3.986\times10^5}{6368}} = 7.91165 \ km/sec$$

$$\nu_1 = \frac{V_1}{V_s} = \frac{6.25146}{7.91165} = 0.7902$$ (Agrees with HW 4 sol'n. for ν_0)

b) $V_1 = 6.25146$ km/sec

$R = 3000$ km

$$\Delta\theta = \frac{R}{r_0} = \frac{3000}{6368} = 0.471106 \ \text{radians}$$

$$c^2 = 2r_0^2(1-\cos\Delta\theta) = 2(6368)^2(1-\cos(0.471106))$$
$$= 8.834796\times10^6$$

$C = 2972.34$ km

$$S = \frac{r_1+r_2+c}{2} = \frac{6368+6368+2972.34}{2} = 7854.17 \ \text{km}$$

$$a_m = \frac{s}{2} = \frac{7854.17}{2} = 3927.08 \ \text{km}$$

Using the energy equation for an elliptic trajectory

$$\frac{V_1^2}{2} - \frac{\mu}{r_1} = -\frac{\mu}{2a}$$

Solve for a

$$a = \frac{r_1}{2 - \frac{r_1 V_1^2}{\mu}} = \frac{6368}{2 - \frac{(6368)(6.25146)^2}{3.986 \times 10^5}} = 4629.09 \text{ km}$$

Note: $a > a_m \Rightarrow$ 2 solutions are possible

Using (8.16) & (8.17)

$$\sin\left(\frac{\alpha_0}{2}\right) = \left(\frac{s}{2a}\right)^{1/2} = \left(\frac{7854.17}{2(4629.09)}\right)^{1/2} = 0.921059$$

$$\Rightarrow \alpha_0 = 2.34158 \text{ radians} \Rightarrow \alpha = \begin{cases} \alpha_0 = 2.34158 \text{ rad} & (t_f \le t_m) \\ 2\pi - \alpha_0 = 3.94161 \text{ rad} & (t_f^{\#} > t_m) \end{cases}$$

$$\sin\left(\frac{\beta_0}{2}\right) = \left(\frac{s-c}{2a}\right)^{1/2} = \left(\frac{7854.17 - 2972.34}{2(4629.09)}\right)^{1/2} = 0.726154$$

$$\Rightarrow \beta_0 = 1.62542 \text{ radians}$$

Since $0 \le \Delta\theta \le \pi$, $\beta = \beta_0 = 1.62542$ radians

<u>Solution 1</u> $(t_f \le t_m)$

$\alpha = \alpha_0 = 2.34158$ radians $\quad \beta = 1.62542$ radians

Using (8.19)

$$p = \frac{4a(s-r_1)(s-r_2)}{c^2} \sin^2\left(\frac{\alpha+\beta}{2}\right)$$

$$p = \frac{4(4629.09)(7854.17 - 6368)^2}{(2972.34)^2} \sin^2\left(\frac{2.34158 + 1.62542}{2}\right)$$

$$= 3884.40 \text{ km}$$

$$e = \left(1 - \frac{p}{a}\right)^{1/2} = \left(1 - \frac{3884.40}{4629.09}\right)^{1/2} = 0.401088$$

$$r_a = \frac{p}{1-e} = \frac{3884.40}{1 - 0.401088} = 6485.76 \text{ km}$$

$$h_a = r_a - r_o = 6485.76 - 6368$$

$\boxed{h_a = 118 \text{ km}}$ (Agrees with HW 4 prob 1b for h_{a2})

<u>Solution 2</u> $(t_f^{\#} > t_m)$

$\alpha = 2\pi - \alpha_o = 3.94161$ radians $\quad \beta = 1.62542$ radians

Using (8.19)

$$\tilde{p} = \frac{4a(s - r_1)(s - r_2)}{c^2} \sin^2\left(\frac{\alpha + \beta}{2}\right)$$

$$\tilde{p} = \frac{4(4629.09)(7854.17 - 6368)^2}{(2972.34)^2} \sin^2\left(\frac{3.94161 + 1.62542}{2}\right)$$

$$= 568.602 \text{ km}$$

$$\tilde{e} = \left(1 - \frac{p}{a}\right)^{1/2} = \left(1 - \frac{568.602}{4629.09}\right)^{1/2} = 0.936572$$

$$\tilde{r}_a = \frac{\tilde{p}}{1-\tilde{e}} = \frac{568.602}{1 - 0.936572} = 8964.53 \text{ km}$$

$$\tilde{h}_a = r_a - r_o = 8964.53 - 6368$$

$$\boxed{\tilde{h}_a = 2597 \text{ km}}$$

(Agrees with HW 4 prob 1b for h_{a1})