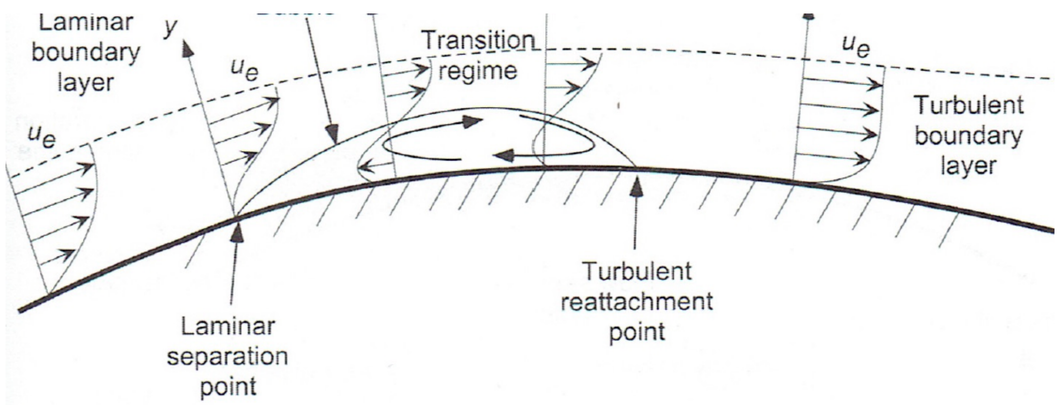
VII. Boundary Layer Theory

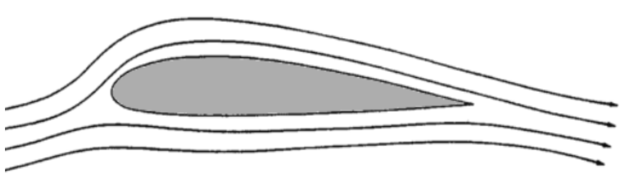
When considering flow over bodies immersed in unbounded fluid, it has been observed that for slender bodies the viscous effects have been confined to a thin layer along the surface of the body. This layer is called the *boundary layer.*

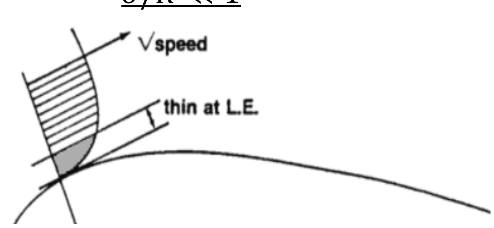
As shown in the figure, the free stream velocity decreases steeply to zero at the boundary within the boundary layer.

Along the surface, the flow eventually is "separated" which means that a region is created next to the boundary with reverse flow. Subsequently, if the boundary is long the flow becomes turbulent with a different velocity profile inside the boundary later.



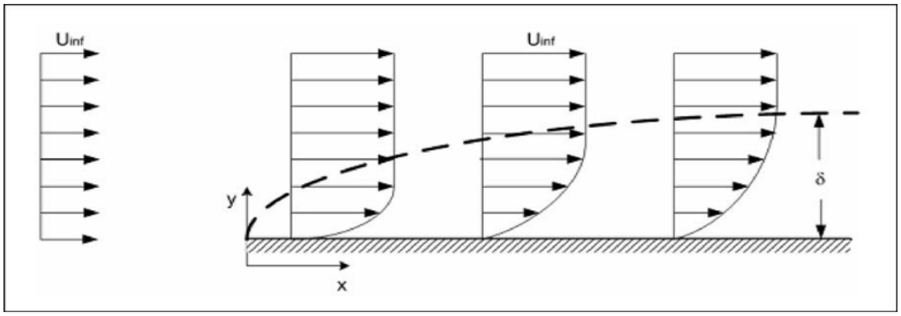
The theory is relevant in part for the study of viscous flow over an airfoil, where near the leading edge the boundary layer theory is applicable.





To simplify this concept, the flow is examined along a semi-infinite flat plate. The flow approaching the plate is uniform at velocity U. Outside the boundary layer, the flow field remains uniform and unchanged. In fact, it is similar to uniform inviscid flow. Inside the boundary layer, where viscous effects are important, the velocity profile decreases to zero at the solid plate. The boundary layer is a mathematical definition, not a real surface, denoted by its thickness δ.

Further along the plate, the flow can become turbulent inside the boundary layer as it continues to grow.



VII.1 Boundary layer Equations

The boundary layer equations which govern the flow inside the thin layer near the solid boundary are a simplification of the two-dimensional steady-state Continuity and the Navier-Stokes equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$
$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+=g_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

In simplifying the N-S equations we use dimensionless analysis (scaling), which leads to the following conclusions,

- The Reynolds number, based on the lateral length scale L, is large. $Re_L \gg 1$. But to maintain laminar flow it is smaller than 500,000 (usually the value when the boundary layer becomes turbulent)
- Gravitational effects inside the boundary layer are negligible because the boundary layer thickness is small.
- The pressure change in the vertical direction is negligible which means

$\frac{\partial p}{\partial y} = 0$. Therefore, only the pressure gradient along the plate can influence the flow field.

Dimensionless variables are scaled for the two-dimensional steady-state Continuity equation and Navier-Stokes equations as follows.

$$(x, y)^* = \left(\frac{x}{L}, \frac{y}{\delta}\right)$$
$$(u, v)^* = \left(\frac{u}{U}, \frac{v}{V}\right)$$

where V is yet unknow, and the pressure can be scaled by the free steam characteristics pressure p_∞.

Substituting these dimensionless variables into the governing equations results in the following Continuity equation.

$$\frac{U}{L}\left(\frac{\partial u}{\partial x}\right)^* + \frac{V}{\delta}\left(\frac{\partial v}{\partial x}\right)^* = 0$$

The flow inside the boundary layer has both velocity components. Therefore, to retain both velocities we need

$$V = \frac{\delta}{L}U$$

Let also denote:

$$\epsilon = \frac{\delta}{L} \ll 1$$

Continuing with the Navier Stokes Equations:

$$\frac{U^2}{L}\left(u\frac{\partial u}{\partial x}\right)^* + \epsilon\frac{U^2}{\delta}\left(v\frac{\partial u}{\partial y}\right)^* = -\frac{p_\infty}{\rho L}\left(\frac{\partial p}{\partial x}\right)^* + \nu\frac{U}{L^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2 u}{\partial y^2}\right)^*$$

$$\epsilon\frac{U^2}{L}\left(u\frac{\partial v}{\partial x}\right)^* + \epsilon^2\frac{U^2}{\delta}\left(v\frac{\partial v}{\partial y}\right)^* = -\frac{p_\infty}{\rho\delta}\left(\frac{\partial p}{\partial y}\right)^* + \nu\frac{\epsilon U}{L^2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2 v}{\partial y^2}\right)^*$$

Rearranging

$$\left(u\frac{\partial u}{\partial x}\right)^* + \left(v\frac{\partial u}{\partial y}\right)^* = -\frac{p_\infty}{\rho U^2}\left(\frac{\partial p}{\partial x}\right)^* + \frac{\nu}{UL}\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2 u}{\partial y^2}\right)^*$$

$$\epsilon\left(u\frac{\partial v}{\partial x}\right)^* + \epsilon\left(v\frac{\partial v}{\partial y}\right)^* = -\frac{1}{\epsilon}\frac{p_\infty}{\rho U^2}\left(\frac{\partial p}{\partial y}\right)^* + \epsilon\frac{\nu}{UL}\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2 v}{\partial y^2}\right)^*$$

Since $\epsilon \ll 1$, the Re is assumed to be large so that

$$Re = \frac{UL}{\nu} \sim \frac{1}{\epsilon^2}$$

and

$$\frac{p_\infty}{\rho U^2} \sim 1$$

The Boundary Layer Equations then become:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(VII-1)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$0 = \frac{\partial p}{\partial y}$$

The equations are valid for a flat plate with a uniform free stream velocity which can change in the $x -$ direction.

Since the pressure is not a function of y inside the boundary layer, the equations can be simplified further:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \end{aligned}\right\} \qquad \text{(VII-2)}$$

Outside the boundary later, where the flow behaves like frictionless fluid, the governing equation is Bernoulli's equation (ignoring gravitational effects).

$$\frac{P_\infty}{\rho} + \frac{U_\infty^2}{2} = const \tag{VII-3}$$

where both the ambient pressure and free stream velocity can depend on x. Since the pressure does not change across the boundary layer, the pressure gradients can be obtained directly from Bernoulli's equation:

$$\frac{1}{\rho}\frac{dP}{dx} = -\frac{d}{dx}\left(\frac{U_\infty^2}{2}\right) = -U_\infty\frac{dU_\infty}{dx}$$

The momentum equation can be written then as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty\frac{dU_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{VII-4}$$

VI. 2 <u>Flow over a Fla Plate – Blasius Solution</u>

For flow over a flat plate, the free stream velocity and the pressure are constants so that Equ. (VII-2) becomes:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2} \end{aligned}\right\} \tag{VII-5}$$

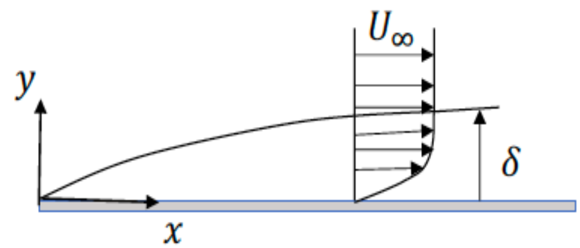
Boundary Conditions:

The boundary layer flow has to satisfy the following boundary conditions:

(i) $u(x,0) = 0$

(ii) $u(x, y \to \infty) = U_\infty$

(iii) $v(x,0) = 0$



The solution to the boundary layer equations with the corresponding boundary conditions can be obtained by using a *similarity variable:*

$$\eta(x,y) = y\sqrt{\frac{U_\infty}{\nu x}} \tag{VII-6}$$

which is a combination of independent variable x and y. The similarity solution is a unique approach which is feasible in some unbounded systems.

We also assume that the horizontal velocity component can be represented as ($'$ denotes derivative)

$$\frac{u(x,y)}{U_\infty} = \frac{df(\eta)}{d\eta} = f'(\eta) \tag{VII-7}$$

where $f(\eta)$ is a function to be determined from the solution.

Using the chain rule with equations (VI-6) and (VI-7) the boundary layer equations can be rewritten as follows:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial \eta}(U_\infty f')\frac{\partial \eta}{\partial x} = -U_\infty f''\frac{y}{2x}\sqrt{\frac{U_\infty}{\nu x}} = -\frac{U_\infty}{2x}\eta f''$$

$$\frac{\partial v}{\partial y} = \frac{U_\infty}{2x}\eta f''$$

therefore:

$$v = \int \frac{U_\infty}{2x}\eta f'' dy = \int \frac{U_\infty}{2x}\sqrt{\frac{\nu x}{U_\infty}}\eta f'' d\eta = \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}(\eta f' - f)$$

$$\frac{\partial u}{\partial y} = U_\infty f'' \sqrt{\frac{U_\infty}{\nu x}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}\right)\frac{\partial \eta}{\partial y} = \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} = U_\infty f''' \frac{U_\infty}{\nu x}$$

Substituting these derivatives into the momentum equation yields the following ordinary differential equation for $f(\eta)$:

$$-U_\infty f' \frac{U_\infty}{2x}\eta f'' + \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}(\eta f' - f)U_\infty f'' \sqrt{\frac{U_\infty}{\nu x}} = \nu\, U_\infty f''' \frac{U_\infty}{\nu x}$$

which simplifies into the final form, with the corresponding boundary conditions:

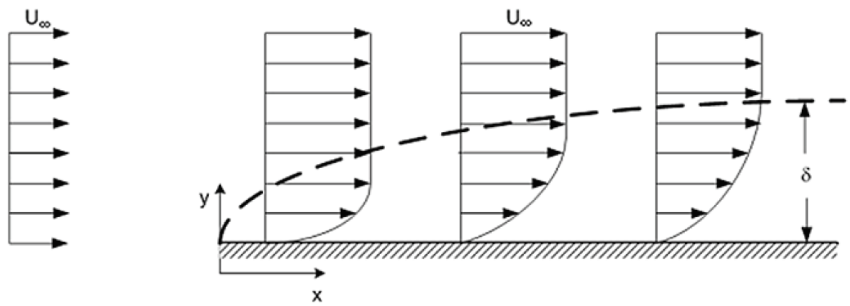
$$2f''' + ff'' = 0 \qquad \text{(VII-8)}$$

(i) $f'(0) = 0$
(ii) $f(0) = 0$
(iii) $f'(\infty) = 1$

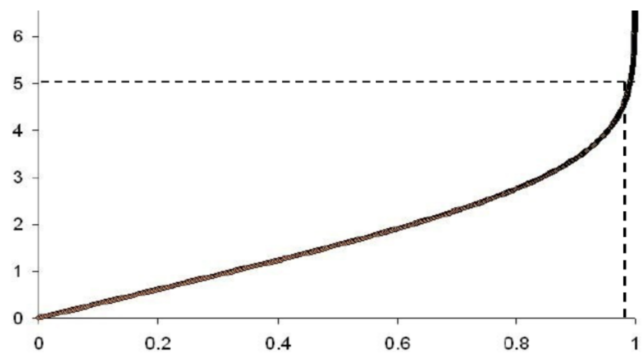
The solution of the differential equation and boundary conditions were obtained numerically by Blasius (1908) and is shown in the table for selected values of η.

$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$	$f(\eta)$	$f'(\eta) = \frac{u}{U_\infty}$	$f''(\eta)$
0.0	0.0	0.0	0.33206
0.8	0.10611	0.26471	0.32739
1.6	0.42032	0.51676	0.29667
2.4	0.92230	0.72899	0.22809
3.2	1.56911	0.87609	0.13913
4.0	2.30576	0.95552	0.06424
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591
5.6	3.88031	0.99748	0.00543

The solution for the velocity profile is represented by a single curve in terms of the similarity variable.



$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$$



$$\frac{u}{U_\infty} = f'(\eta)$$

The boundary layer thickness, $\delta,$ is defined as the value of η were the velocity inside the boundary layer reaches 99% of the free steam velocity. From the numerical solution in the table this value is $\eta = 5$.
Therefore

$$5 = \delta\sqrt{\frac{U_\infty}{\nu x}}$$

or

$$\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}} \qquad \text{(VII-9)}$$

where $Re_x = \frac{U_\infty x}{\nu}$

Note that the dimensionless scaling indicated that,

$$Re = \frac{UL}{\nu} \sim \frac{1}{\epsilon^2} \qquad \text{or} \qquad \frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}}$$

which is a correct order of magnitude representation of (VI-9). As Equ. (VII-9) indicates, the boundary layer thickness grows as $\sqrt{x}$ along the plate.

The solution for the velocity profile makes it now possible to determine the force acting on the plate by first evaluating the tangential shear stress at the wall:

$$\tau_w = \tau_{yx} = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu \frac{\partial u}{\partial \eta}\bigg|_{\eta=0} \sqrt{\frac{U_\infty}{\nu x}} = \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}} f''(0) \qquad \text{(VII-10)}$$

$f''(0)$ is obtained from the numerical solution as 0.332. Therefore,

$$\tau_w = 0.332 \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}} = 0.332 \frac{\rho U_\infty^2}{\sqrt{Re_x}} \qquad \text{(VII-11)}$$

The shear stress decreases along the wall as $x^{-1/2}$ and is singular (infinite) at $x = 0$. The singular value of the stress at the leading edge is not surprising because the flow field is discontinuous decreasing abruptly from U_∞ to zero. Still, the stress is integrable.

In dimensionless form, the shear stress at the wall is represented by the *skin friction coefficien*t defined by

$$C_f = \frac{\tau_w}{\rho U_\infty^2 / 2} = \frac{0.664}{\sqrt{Re_x}} \qquad \text{(VII-12)}$$

Finally, the drag force acting on the plate per unit width is obtained by integrating the shear stress at the wall over the length of the plate,

$$F_D = \int_0^L \tau_w dx = \int_0^L 0.332 \mu U_\infty \sqrt{\frac{U_\infty}{\nu x}} dx$$

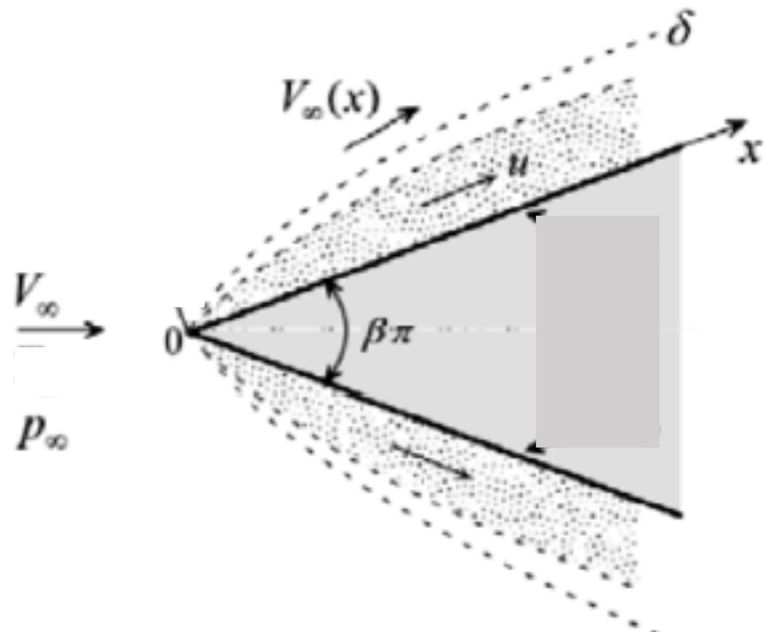
$$F_D = 0.664\mu U_\infty \sqrt{Re_L} = 0.664\rho U_\infty^2 \frac{L}{\sqrt{Re_L}} \quad \text{(VII-13)}$$

or in terms of the dimensionless drag coefficient

$$C_D = \frac{F_D}{\rho U^2 L/2} = \frac{1.328}{\sqrt{Re_L}} \quad \text{(VII-14)}$$

VII.3 <u>Boundary Layer Flow Past a Wedge</u>

A similarity solution of boundary layer problems is limited to certain forms of the free stream velocity that can be used to construct a similarity variable.



A specific form of the velocity profile which makes similarity solution possible is:

$$U(x) = u_1 x^m \quad \text{(VII-15)}$$

with x being measured along one side of the wedge.

This allows the introduction of a similarity variable in the form:

$$\eta = y\sqrt{\frac{m+1}{2}\frac{U}{\nu x}} = y\sqrt{\frac{m+1}{2}\frac{u_1}{\nu}}\,x^{\frac{m-1}{2}} \quad \text{(VII-16)}$$

and the velocity components (v obtained from the continuity equation):

$$\left.\begin{aligned} u &= u_1 x^m f'(\eta) = U f'(\eta) \\ v &= -\sqrt{\frac{m+1}{2}\nu u_1 x^{m-1}}\left[f(\eta) + \frac{m-1}{m+1}\eta f'(\eta)\right] \end{aligned}\right\} \qquad \text{(VII-17)}$$

For a flow over a wedge the value of m is determined by the wedge angle $\pi\beta$ in the form

$$m = \frac{\beta}{2-\beta} \qquad \text{(VII-18)}$$

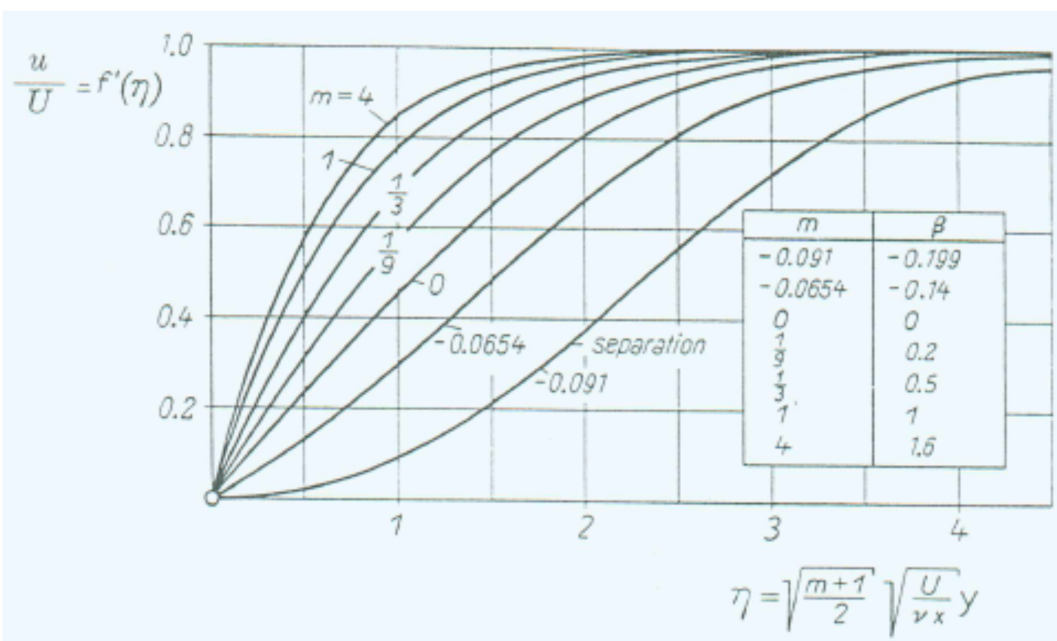
Substituting these expressions into the boundary layer equations results in the ordinary differential equation:

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0 \qquad \text{(VII-19)}$$

with the boundary conditions

$$f(0) = f'(0) = 0; \quad f'(\infty) = 1$$

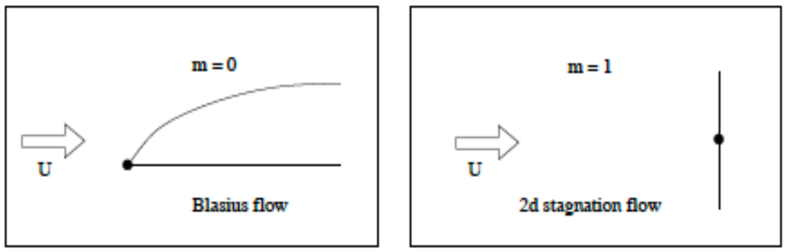
The solution (Falkner-Skan, 1930) of the equation is shown in the graph.



m	β
-0.091	-0.199
-0.0654	-0.14
0	0
1/9	0.2
1/3	0.5
1	1
4	1.6

For $\beta > 0$, the free stream velocity accelerates ($m > 0$) and the profile inside the boundary layer has no point of inflexion. For decelerating flow ($m < 0,\ \beta < 0$) the flow direction is in the opposite direction, and it shows a point of inflexion with separation at $\beta = -0.091$.

The Falkner-Skan solution reduces into the flow over a flat plate with $\beta = 1$, and flow over a flat plate with $\beta = 0$.

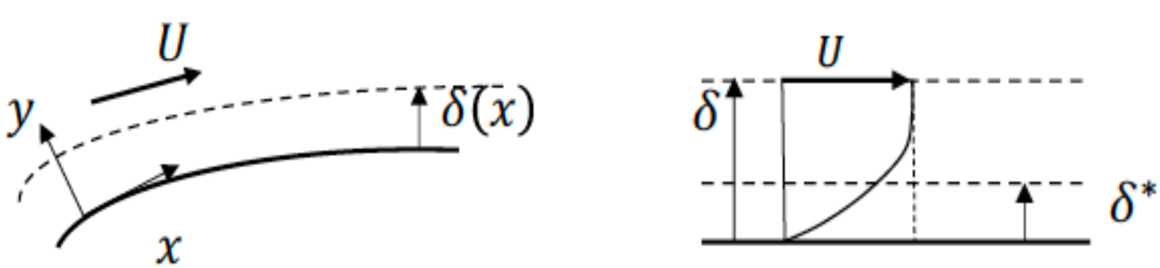


The Falkner-Skan solution reduces into the flow over a flat plate with $\beta = 1$, and flow over a flat plate with $\beta = 0$.

A similar solution exists for a flow with rotational symmetry over a cone.

VII.4 Integral Method

The Integral Method, also called *Von Karman-Pohlhausen* method, is an approximate solution for boundary layer flow. This method can be used for flow over a body of arbitrary shape with pressure gradient $\frac{dp}{dx}$ or $U = U(x)$, and is not limited to laminar flow.



We first introduce two definitions:

Displacement Thickness $\delta^*(x)$

$$U\delta^* = \int_0^{\delta}(U-u)dy \tag{VII-20}$$

The left-hand-side represents flow rate through the displacement thickness δ^*, while the right-hand-side represents the reduction in the actual flowrate compared with a boundary layer with uniform velocity U. Therefore, δ^* represents the distance the boundary has to be displaced to account for the reduction in flow rate.

Momentum Thickness, $\Theta(x)$

$$U^2\Theta = \int_0^{\delta} u(U-u)dy \tag{VI-21}$$

The momentum thickness is a measure of the momentum deficiency due to the presence of the boundary layer.

Starting with the general boundary layer equations, with a free steam pressure gradient

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

The equations are integrated across the boundary layer thickness, namely $\int_o^h dy$ where h is outside the boundary layer.

$$\int_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)dy = \int_0^h U\frac{dU}{dx}dy + \nu\int_0^h \frac{\partial^2 u}{\partial y^2}dy$$

Rearranging and integrating the shear stress,

$$\int_0^h \left(u\frac{\partial u}{\partial x} - U\frac{dU}{dx} + v\frac{\partial u}{\partial y}\right)dy = \nu\frac{\partial u}{\partial y}\bigg|_0^h = -\frac{\tau_w}{\rho}$$

where τ_w is the shear stress at the boundary.

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} \tag{VI-22}$$

Using the continuity equation with $v = 0$ at the boundary

$$v = \int_0^y \frac{\partial v}{\partial y} dy = -\int_0^y \frac{\partial u}{\partial x} dy$$

Therefore,

$$\int_0^h \left(v \frac{\partial u}{\partial y}\right) dy = \int_0^h \left(-\int_0^y \frac{\partial u}{\partial x} dy'\right) \frac{\partial u}{\partial y} dx$$

Integrating by parts. ($\int f dg = fg - \int g df$):

$$\int_0^h \left(v \frac{\partial u}{\partial y}\right) dy = -U \int_0^h \frac{\partial u}{\partial x} dy + \int_0^h u \frac{\partial u}{\partial x} dy$$

Combing this result with the original equation:

$$\int_0^h \left(u \frac{\partial u}{\partial x} - U \frac{dU}{dx} + U \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x}\right) dy = -\frac{\tau_w}{\rho}$$

Adding and subtracting $u \frac{dU}{dx}$ gives

$$-\int_0^h \left[u \left(\frac{dU}{dx} - \frac{\partial u}{\partial x}\right) + (U - u) \frac{\partial u}{\partial x}\right] dy - \int_0^h (U - u) \frac{\partial U}{\partial x} dy = -\frac{\tau_w}{\rho}$$

and finally

$$\frac{d}{dx}\left[U^2 \int_0^h \frac{u}{U}\left(1 - \frac{u}{U}\right) dy\right] + U \frac{dU}{dx} \int_0^h \left(1 - \frac{u}{U}\right) dy = \frac{\tau_w}{\rho}$$

which becomes

$$\frac{d}{dx}(U^2\Theta) + U\frac{dU}{dx}\delta^* = \frac{\tau_w}{\rho}$$

or

$$\frac{\tau_w}{\rho} = U^2\frac{d\Theta}{dx} + (2\Theta + \delta^*)U\frac{dU}{dx} \qquad \text{(VI-23)}$$

The final result (VI-23) is the *Von-Karman Integral Equation.*

VI.5 Pohlhausen Solution

The von-Karman equation can be used by calculating first the displacement thickness, the momentum thickness, and the shear stress at the wall, and by selecting a suitable approximate velocity profile inside the boundary layer.

The *Pohlhausen solution* is a general approach for approximating the velocity profile $u(y)$ inside the boundary layer and by using a fourth order polynomial,

$$\frac{u}{U} = c_1 + c_2\eta + c_3\eta^2 + c_4\eta^3 + c_5\eta^4 \qquad \text{(VI-24)}$$

where,

$$\eta = \frac{y}{\delta(x)} \qquad \text{(VI-25)}$$

and c_i are unknown constant coefficients. The polynomial applies to $0 \leq \eta \leq 1$, whereas for $\eta > 1$ the velocity is $u=U$.

The 5 constant coefficients are determined form the following boundary conditions:

(i) at $y = 0$ $\quad u = 0$
(ii) at $y = \delta$ $\quad u = U$
(iii) at $y = \delta$ $\quad \frac{\partial u}{\partial y} = 0$
(iv) at $y = 0$ both $u = v = 0$ therefore from the B.L equ. $\frac{\partial^2 u}{\partial y^2} = -\frac{U}{\nu}\frac{dU}{dx}$
(v) at $y = \delta,\ \ u = U$ and $v = 0$ therefore from the B.L. equ. $\frac{\partial^2 u}{\partial y^2} = 0$

Applying the five boundary conditions to the assumed profile (V-24) provides solutions for the unknown constant coefficients. Introducing the definition

$$\Lambda = \frac{\delta^2}{\nu}\frac{dU}{dX} \tag{VI-26}$$

The constant coefficient are:

$$c_1 = 0, \quad c_2 = 2 + \frac{\Lambda}{6}, \; c_3 = -\frac{\Lambda}{2}, \quad c_4 = -2 + \frac{\Lambda}{2}, \; c_5 = 1 - \frac{\Lambda}{6}$$

Substituting the constants into the assumed polynomial gives the following solution for the velocity:

$$\frac{u}{U} = 2\eta - 2\eta^3 + \eta^4 + \frac{\Lambda}{6}(\eta - 3\eta^2 + 3\eta^3 - \eta^4) \tag{VI-27}$$

or it can be represented in terms of two functions:

$$\frac{u}{U} = F(\eta) + \Lambda G(\eta)$$

where,

$$F = 2\eta - 2\eta^3 + \eta^4 = 1 - (1-\eta^3)(1+\eta)$$

$$G = \frac{1}{6}\eta(1-\eta)^3$$

Using the velocity profile, the displacement thickness, the momentum thickness and the wall shear stress are evaluated.

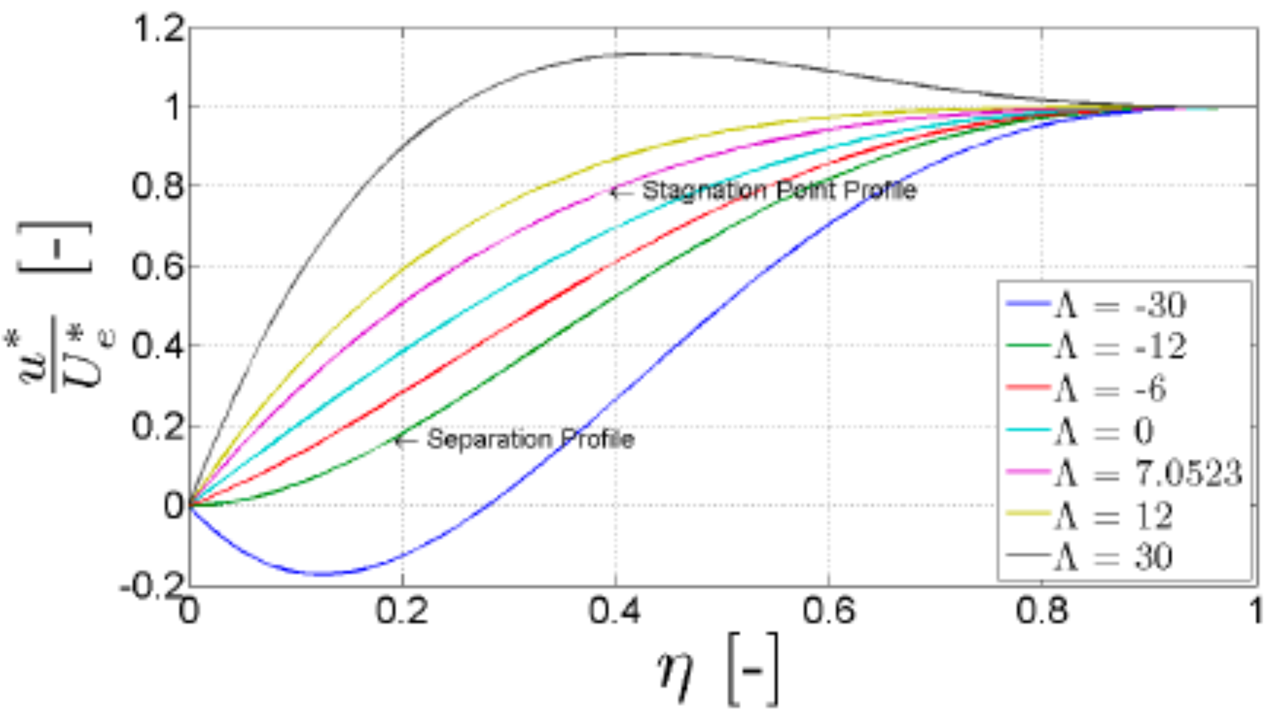
Use $\eta = \frac{y}{\delta}$ to rewrite the integral definitions (they can be written in terms of F and G, skipping the derivation):

$$\frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right) d\eta = \frac{3}{10} - \frac{\Lambda}{120} \tag{VI-28}$$

$$\frac{\Theta}{\delta} = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta = \frac{1}{63}\left(\frac{37}{5} - \frac{\Lambda}{15} - \frac{\Lambda^2}{144}\right) \tag{VI-29}$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\mu}{\delta}\left.\frac{\partial u}{\partial \eta}\right|_{\eta=0} \left.\frac{\partial \eta}{\partial y}\right|_{y=0} = \frac{\mu U}{\delta}\left(2 + \frac{\Lambda}{6}\right) \tag{VI-30}$$

These results are now substituted into the Von-Karman integral equation. This generates a non-linear differential equation for the boundary layer thickness $\delta(x)$, which can be solved numerically for different values of Λ.



The solution shows that realistic velocity profile can exist only for $-12 \leq \Lambda \leq 12$. For $\Lambda = -12$ the stress at the wall is zero, indicating the onset on of a separation point. For lower values the flow will be reversed near the wall. For $\Lambda = 12$, the profile merges with the free stream profile with zero slop. For larger values, the flow will exceed the free stream velocity which is not realistic. For $\Lambda = 0$, the solution corresponds to Blasius solution.

This solution procedure using the Von-Karmen integral equation can be generalized is as follows:

1. Assume a velocity profile for $u(y)$. It can be a polynomial or other functions that can well represent the velocity inside the boundary layer.
2. Apply boundary conditions to the velocity profile in the order listed in the Pohlhausen solution. Use as many conditions as necessary to determine the velocity profile in terms of δ.

(i) at $y = 0$ $u = 0$

(ii) at $y = \delta$ $u = U$

(iii) at $y = \delta$ $\frac{\partial u}{\partial y} = 0$

(iv) at $y = 0$ $\frac{\partial^2 u}{\partial y^2} = -\frac{U}{\nu}\frac{dU}{dx}$

3. Determine δ^*, Θ, and τ_w

$$\frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{u}{U}\right) d\eta, \quad \frac{\Theta}{\delta} = \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) d\eta, \quad \tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\mu}{\delta} \left.\frac{\partial u}{\partial \eta}\right|_{\eta=0}$$

4. Substitute δ^*, Θ, and τ_w into the Von-Karman integral equation. This will result in an ordinary differential equation for $\delta(x)$.

$$\frac{\tau_w}{\rho} = U^2 \frac{d\Theta}{dx} + (2\Theta + \delta^*) U \frac{dU}{dx}$$

5. Solve the differential equation for $\delta(x)$, and substitute back into the assumed velocity profile $u(y)$.

6. Use the velocity profile to determine τ_w and the drag force acting on the plate by integrating the shear stress over the length of the plate.

$$F_D = \int_o^L \tau_w dx$$

<u>Example</u>: For a flow over a flat plate with constant free stream velocity assume the following velocity profile

$$\frac{u}{U} = \sin\left(\frac{\pi y}{2\delta}\right)$$

This profile satisfies the first three boundary conditions in the Pohlhausen procedure.

Determine the momentum thickness and the stress at the wall. The displacement thickness is not needed because the free stream velocity is constant. $\left(\frac{dU}{dx} = 0\right)$.

$$\Theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(sin\frac{\pi y}{2\delta} - sin^2\frac{\pi y}{2\delta}\right) dy = \delta\left(\frac{2}{\pi} - \frac{1}{2}\right)$$

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu U \frac{\pi}{2\delta} \cos\frac{\pi y}{2\delta}\bigg|_{y=o} = \frac{\mu U \pi}{2\delta}$$

The Von-Karmen equation with constant free steam velocity is

$$\frac{\tau_w}{\rho} = U^2 \frac{d\Theta}{dx}$$

Substitute Θ and τ_w into the integral equation:

$$\left(\frac{2}{\pi} - \frac{1}{2}\right)\frac{d\delta}{dx} = \frac{\mu U}{2\delta\rho U}$$

Rearrange the differential equation and solve for $\delta(x)$

$$\int_0^\delta \delta d\delta = \int_0^x \frac{\nu\pi}{\left(\frac{4}{\pi} - 1\right) U} dx$$

$$\delta = \sqrt{\frac{2\pi\nu x}{\left(\frac{4}{\pi} - 1\right) U}}$$

or

$$\frac{\delta}{x} = \frac{4.8}{\sqrt{Re_x}}$$

This result is only 4% smaller than the exact Blasius solution.

Using the assumed velocity profile with the solution for δ, the shear stress at the wall and subsequently the drag force can be determined:

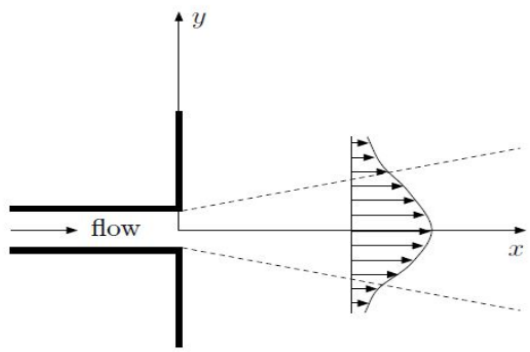
$$\tau_w = \mu U \frac{\pi}{9.6}\sqrt{\frac{U}{\nu x}} = 0.327\mu U\sqrt{\frac{U}{\nu x}}$$

VI.6 Two-Dimensional Jet

Consider a two-dimensional jet of width h and uniform exit velocity u_0 issuing into the same fluid. Following the boundary layer approximation, the jet Reynolds number is sufficiently high to create a long slender jet (Schlichting, 1933).

Assuming the pressure is uniform, the momentum balance in the x direction over an arbitrary control volume requires that

$$M = \rho u_0^2 h = \rho \int_{-\infty}^{\infty} u^2(y) dy$$



Using the boundary layer equation with constant pressure, and ignoring gravity:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial y^2}$$

Introducing the two-dimensional stream function

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

a solution is assumed in the form:

$$\psi = A x^p f(\eta)$$

$$\eta = \frac{y x^{-q}}{B}$$

where the powers of x, and the constants $A,\ B$ are selected to make the resulting equation dimensionless.

The boundary conditions are:

(i) at $y = 0,\ u = 0,\quad f'(0) = 0$
(ii) at $y = 0,\ v = 0,\quad f(0) = 0$
(iii) $at\ y \to \infty,\ u = 0\quad f'(\infty) = 0$

Substituting the similarity solution into the boundary layer equation yields

$$Ax^p f' \frac{1}{Bx^q}\left[\frac{A}{B}(p-q)x^{p-q-1}f'^2 + \frac{A}{B}x^{p-q}f''\left(-\frac{1}{x}\eta\right)\right] - \left[Apx^{p-1}f + A\frac{x^p}{x}f'\eta\right]\frac{Ax^{p-2q}}{B^2}f'' = \nu\frac{Ax^{p-3q}}{B^3}f'''$$

or after simplification

$$\frac{AB}{\nu}x^{p+q-1}[(p-q)f'^2 - pff''] = f'''$$

To eliminate the dependence on x in this equation, we set $p + q = 1$.

Writing the momentum balance in terms of the stream function we get,

$$M = \rho A^2 B^{-1} x^{2p-q} \int_{-\infty}^{\infty} f'^2 d\eta$$

And since M must be a constant, it is required to set $2p = q$. The two conditions generate the result $p = \frac{1}{3}$, $q = \frac{2}{3}$.

Therefore, the differential equation becomes

$$f''' + \frac{AB}{3\nu}(f'^2 + ff'') = 0$$

The solution for this equation, when $\frac{AB}{3\nu} = 2$ is

$$f(\eta) = \tanh \eta$$

This can be achieved by setting

$$A = \left(\frac{9\nu M}{2\rho}\right)^{1/3} \qquad B = \left(\frac{48\nu^2\rho}{M}\right)^{1/3}$$

The resulting velocity profile is

$$u = \left(\frac{3M^2}{32\rho^2\nu x}\right)^{1/3} \operatorname{sech}^2\left[\left(\frac{M}{48\rho\nu^2}\right)^{1/3} \frac{y}{x^{2/3}}\right]$$

Since $\operatorname{sech} 0 = 1,$ the maximum velocity on the x axis decays as $x^{-1/3}$.

Therefore,

$$\frac{u}{u_{max}} = f' = \operatorname{sech}^2\eta$$

The jet thickness can now be defined as the point where the jet velocity is a small fraction of the maximum velocity at the center. This corresponds to the locus of the points $yx^{-2/3} = constant,$ which means that the jet grows as $x^{2/3}$.

Finally, the volumetric flow rate at any location x can be determined

$$Q = \int_{-\infty}^{\infty} u\, dy = \left(\frac{36M\nu}{\varrho} x\right)^{1/3}$$

which indicates that the flow rate increases as $x^{1/3}$ due ambient fluid entrained into the jet.

A similar solution is available for a round jets in cylindrical coordinates.

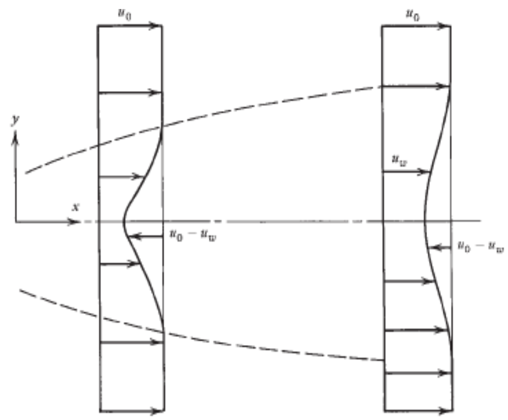
VI.7 <u>Two-Dimensional Wake</u>

The wake behind a two-dimensional symmetric body can also be treated with the boundary layer approximation (Tollmien, 1931).

Rather than solving the problem for the wake velocity profile u_w, the boundary layer equations are rewritten for the velocity defect,

$$u = u_o - u_w$$

which resemble the similarity profile of a jet.



Therefore, the boundary layer equation becomes,

$$(u_0 - u)\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

Using the similarity variable

$$\psi = \frac{f(\eta)}{Ax^p}$$

$$\eta = \frac{yx^{-q}}{B}$$

The drag force associated with the generated wake can be represented in terms of the constant momentum thickness (proof no shown),

$$F_D = \rho u_0^2 \Theta = \int \left[\frac{u_w}{u_0} - \left(\frac{u_w}{u_0}\right)^2\right] dy$$

The analysis follows the derivation for the two-dimensional wake, with the results:

$$q = \frac{1}{2}, \qquad p = 0$$

$$A = \frac{F_D}{\rho u_0 \pi} \qquad\qquad B = \left(\frac{4\nu}{u_0}\right)^{1/2}$$

So that the differential equation becomes:

$$f''' + 2(\eta f'' + f') = 0$$

and the final expressions are:

$$\psi = \frac{F_D}{2\rho u_0} \operatorname{erf}(\eta)$$

$$\eta = y\sqrt{\frac{u_0}{4\nu x}}$$

$$u_0 - u_w = \frac{F_D}{2\rho\sqrt{\pi u_{0x}}} e^{-\eta^2}$$

The result shows that the velocity defect decreases as $x^{-1/2}$ as the wake decays along the x axis. The thickness of the wake can be obtained form the similarity variable which can be set to an arbitrary vale of η when the wake velocity is close to the free stream velocity, so that the wake thickness increases as $\sqrt{x}$.

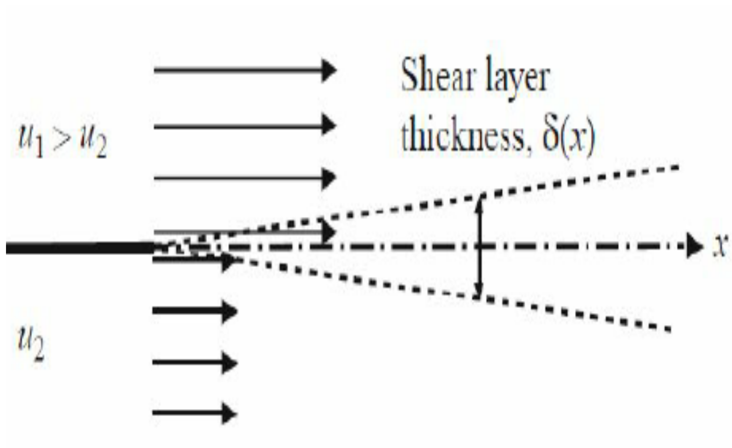
The wake is dependent on the drag force acting on the body generating the wake. This force is defined by the constant momentum thickness. In the far wake, it can be shown that this becomes,

$$F_D = \rho u_0^2 \Theta = \rho u_0 Q$$

where Q is the defect-velocity flow rate. It represents the flow generated in an otherwise quiescent fluid after a moving body had passed by.

VI.8 Shear Layer

A shear layer is created when two fluids are moving at different velocities along a thin plate. The two streams merge where the plate ends at $x = 0$ creating a shear later. The merged region can be treated with the boundary layer equations.



Introducing the Blasius similarity variables for each region with k indicating the upper or lower fluid

$$\eta_k = y\sqrt{\frac{u_1}{\nu_k x}}$$

and the corresponding stream function

$$\psi_k = \sqrt{\nu_k u_1 x} f_k(\eta_k)$$

Then, the velocities are

$$u_k = \frac{u_k}{u_1} = f'_k$$

$$v_k = \frac{1}{2}\sqrt{\frac{u_1 \nu_k}{x}}(\eta_k f'_k - f_k)$$

which leads to the Blasius differential equation

$$2f'''_k + f_k f''_k = 0$$

with the boundary conditions

(i) Continuity of velocity at the interface $f'_1(0) = f'_2(0)$

(ii) Continuity of shear stress at the interface $\sqrt{\rho_1\mu_1} f''_1(0) = \sqrt{\rho_2\mu_2} f''_2(0)$

(iii) In the far stream $f'_1(\infty) = 1, f'_2(\infty) = \frac{u_2}{u_1}$

The numerical solutions of these equations and boundary conditions are shown below (Lock, 1951).

Here

$$A = \frac{\rho_1\mu_1}{\rho_2\mu_2}$$

The case $A = 5.97 \times 10^4$ represents air flow over water at rest with $u_2 = 0$.

The smaller graph shows the interface velocity u_0.