IX. Turbulent Flow

IX.1 <u>General Description</u>

Turbulent flow is complex and appears chaotic. The velocity fluctuates at any point. Mixing is very fast dues to fluctuations and eddies.

In principle, if the solutions of the Navier Stokes equations could be obtained, they should be exact for any Reynolds number. But not every solution of the equation of motion, even if it is exact, can exist in nature. To exist the solution must be *stable.* In a stable solution, small perturbation will tend to decrease with time. On the other hand, when they increase with time, the solution becomes unstable and leads to a different flow pattern.

The study of hydrodynamic instability is as area which attempts to predict the onset of turbulence or conditions required for the transition from laminar to turbulent flow.

Once the flow becomes turbulent characterized by large Reynolds number, the flow is called *fully developed turbulence.* The velocity fluctuates about a mean value at any point in the flow. The fluctuating velocity can be defined as the difference between the velocity vector and its time average $u' = u - \bar{u}$.

In general, turbulent flow can be described by many interacting *turbulent eddies* of different sizes. The size of an eddy can be described as the magnitude of a distance (length scale) over which the velocity varies appreciably.

As the Reynolds number increases, large eddies appear first, and smaller eddies appear later. The smaller the eddy the later it appears. The large eddies have the largest amplitude and have the scales corresponding to the entire size flow region, L. The corresponding velocity variation in these eddies is $\Delta\bar{u}$. The time scale for these eddies is $L/\bar{u}$, or their frequency $\bar{u}/L$ which is typically small. For the smaller eddies the frequency is much larger (or smaller time scale) with much smaller amplitude.

For any eddy of size λ the corresponding Reynolds number is $Re_\lambda = u_\lambda\lambda/\nu$, and it decreases with decreasing eddy size. For large Re_λ the eddy is large but also the viscosity can be small. Therefore, viscosity is not significant in large eddy structure, and the motion is dominated by kinetic energy. Viscosity becomes important when the Reynolds number decreases and the eddy is much smaller, say

λ_0 when $Re \sim 1$. Thus, fully developed turbulence is described by energy transfer from large eddies to smaller eddies where it is dissipated due to viscosity into heat. For the large eddies the kinetic energy dissipation per unit time is

$$\varepsilon \sim \frac{(\Delta \bar{u})^2}{t} \sim \frac{(\Delta \bar{u})^2}{L/\Delta \bar{u}} \sim \frac{(\Delta \bar{u})^3}{L}$$

Using scaling considerations, large scale turbulence can have a corresponding "turbulent viscosity" estimated as

$$\nu_{turb} \sim \frac{L^2}{t} \sim L\Delta \bar{u}$$

With this definition, the ratio between turbulent viscosity and the dynamic viscosity is:

$$\frac{\nu_{turb}}{\nu} \sim \frac{L\Delta \bar{u}}{\nu} \sim Re$$

The energy dissipation expressed in terms of ν_{turb} is

$$\varepsilon \sim L\Delta \bar{u} \frac{(\Delta \bar{u})^2}{L^2} \sim \nu_{turb} \left(\frac{\Delta \bar{u}}{L}\right)^2$$

With the same energy dissipation in small eddies, we can conclude that

$$\varepsilon \sim \frac{u_\lambda^3}{\lambda}$$

Here, u_λ is the characteristic velocity of small eddies.

To determine when fluid viscosity becomes important in the smaller eddies of size λ_0, the local Reynolds number is

$$Re_\lambda \sim \frac{u_\lambda \lambda}{\nu} \sim \frac{(\varepsilon\lambda)^{1/3}\lambda}{\nu} \sim \frac{\Delta \bar{u}\lambda}{\nu}\left(\frac{\lambda}{L}\right)^{1/3} \sim \frac{\Delta \bar{u} L}{\nu}\left(\frac{\lambda}{L}\right)^{4/3} \sim Re\left(\frac{\lambda}{L}\right)^{4/3}$$

where Re is the Reynolds number for the flow. Thus, for viscosity to be important when $Re \sim 1,$ the corresponding eddy length scale is (Kolmogorov, 1941)

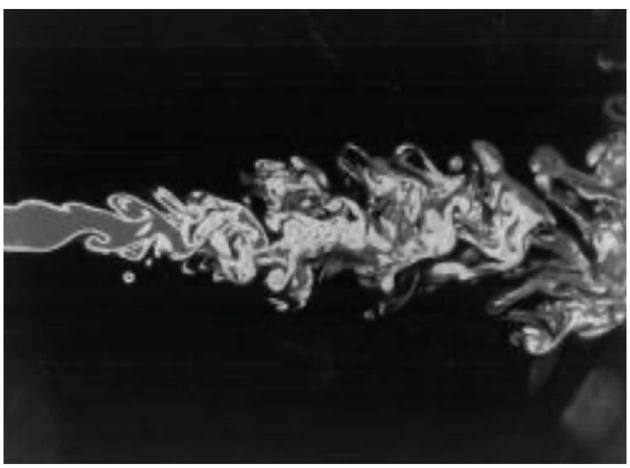
$$\frac{\lambda_0}{L} \sim {Re_\lambda}^{-3/4}$$

or in terms of the viscous dissipation, the smallest eddy size and its velocity are

$$\lambda_0 \sim L\left(\frac{u_\lambda L}{\nu}\right)^{-3/4} \sim L\left(\frac{\varepsilon^{1/3}L^{1/3}L}{\nu}\right)^{-3/4} \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

$$u_{\lambda_0} \sim (\varepsilon\lambda_0)^{1/3} \sim \left(\varepsilon\frac{\nu^{3/4}}{\varepsilon^{1/4}}\right)^{1/3} \sim (\varepsilon\nu)^{1/4}$$

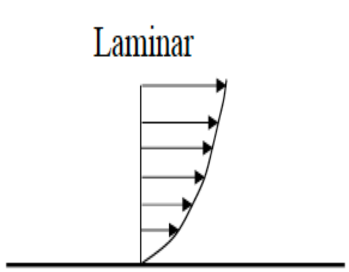
Thus, turbulence is characterized many multiple characteristic scales which the energy cascading from large eddies to small eddies where it is dissipated by viscosity.

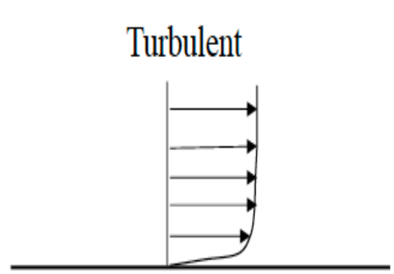


Turbulent Jet: Transition from laminar to turbulent flow

Eruption of a volcano

It what follows we shall concentrate on some practical understanding of turbulent flow near a boundary and in flows over blunt objects.





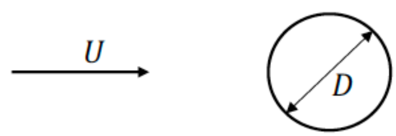
In general, when turbulent flow is described in terms of an average velocity, it appears more uniform near a boundary due to larger momentum, but it also generates larger shear stresses.

The drag acting on a body in uniform flow was first studies experimentally (Reynolds, 1882) for a sphere.

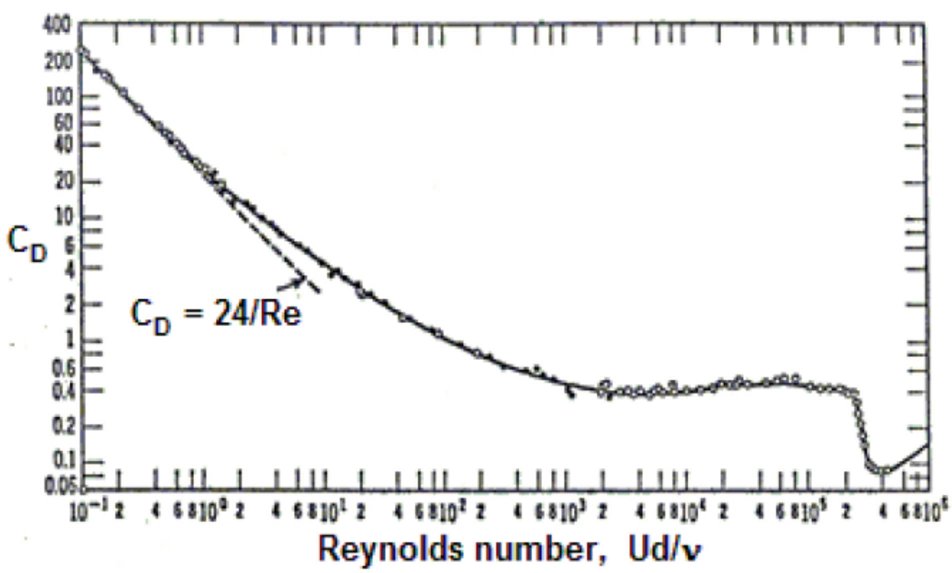
The drag force acting on a sphere, F_D is shown in terms of the dimensionless *Drag Coefficient* defined by

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 A_D}$$

where $A_D = \frac{\pi D^2}{4}$ is the projected area normal to the flow based on the sphere diameter.



The graph below shows the drag coefficient as function of Re_D.



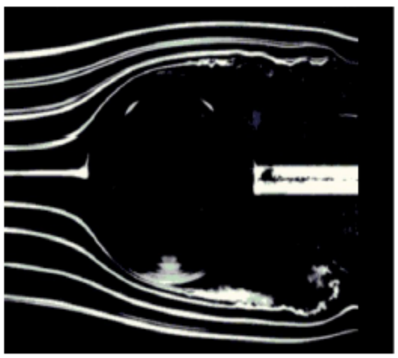
At low Reynolds number the solution is available analytically in the form

$$F_D = 3\pi\mu UD$$

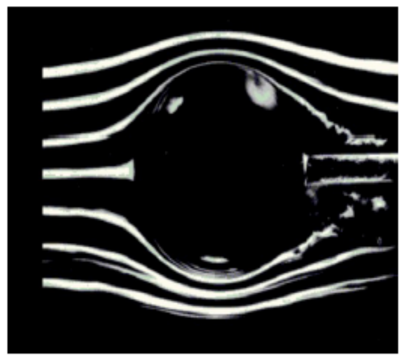
The flow is laminar up to $Re_D \sim 2 \times 10^5$. Still the flow around the sphere is not symmetric with increasing Reynolds number. At $Re_D \sim 1000$, a boundary layer appears along the front surface of the sphere, while separation takes place at at angle of about 84^o. Transition to turbulence occurs at about 2×10^5 with the point of separation pushed towards the back of the sphere at an angle of 118^o, and at the same time the drag drops sharply. This sharp decrease in the drag coefficient is call the *Drag Crisis*.

This behavior is shown in the pictures and is related to the pressure distribution on the surface of the sphere.

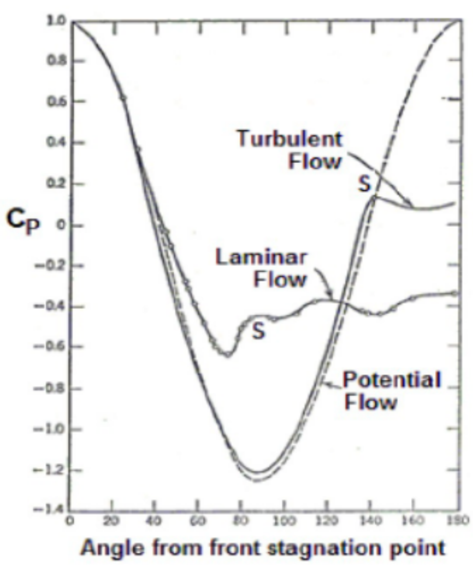
Laminar Separation

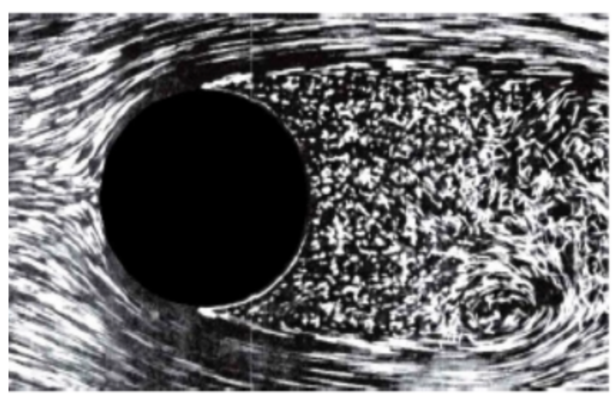


Turbulent Separation



The pressure distribution is shown in terms of the Pressure Coefficient for uniform flow past a cylinder, which closely resembles the behavior for a sphere. After separation the pressure behind the sphere increases due to a region of low velocity circulating motion, and this region is smaller after the transition to turbulence.





IX.2 Reynolds Equations

Introduce the decomposition of the total velocity $\tilde{u}_j$ into the sum of the average velocity U_j and the fluctuations u_j. A similar decomposition is defined for the total pressure.

$$\tilde{u}_j = U_j + u_j \qquad \tilde{p} = P + p \qquad \text{(IX-1,2)}$$

The Navier Stokes equations are (neglecting gravity):

$$\frac{\partial \tilde{u}_j}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_j} + \nu \frac{\partial^2 \tilde{u}_j}{\partial x_j \partial x_j}$$

The time average of the dependent variable is defined by the following integrals:

$$U_i = \frac{1}{T}\int_{t_0}^{t_0+T} \tilde{u}_i dt \qquad \text{(IX-3)}$$

So that the average of the fluctuating velocity is

$$\bar{u}_i = \frac{1}{T}\int_{t_0}^{t_0+T} (\tilde{u}_i - U_i) dt = 0 \qquad \text{(IX-4)}$$

The time average is relevant only if the average quantities are independent of time. The mean value of the spatial derivatives is then

$$\overline{\frac{\partial \tilde{u}_i}{\partial x_j}} = \frac{\partial U_i}{\partial x_j} \quad \overline{\frac{\partial u_i}{\partial x_j}} = \frac{\partial}{\partial x_j}\bar{u}_i = 0$$

The average of products follows the procedure:

$$\overline{\tilde{u}_i \tilde{u}_j} = \overline{(U_i + u_i)(U_j + u_j)} = U_i U_j + \overline{u_i u_j} + \overline{U_i u_j} + \overline{U_j u_i} = U_i U_j + \overline{u_i u_j}$$

The mean value of a products consisting of an average quantity and a fluctuation vanish when averaged, because the mean value is a constant coefficient in averaging the product, while the average of the fluctuating quantity is zero.
The quantity $\overline{u_i u_j} \neq 0$ means that u_i and u_j are *correlated*. If the average of the product is zero, the quantities are uncorrelated.

The equations for the mean flow can now be obtained by averaging the Continuity Navier Stokes equations. The average of the Continuity equation is

$$\frac{\partial \tilde{u}_i}{\partial x_i} = \frac{\partial}{\partial x_i}(U_i + u_i)$$

$$\overline{\frac{\partial \tilde{u}_i}{\partial x_i}} = \overline{\frac{\partial}{\partial x_i}(U_i + u_i)}$$

Then

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{IX-6}$$

Subtracting this result for the Continuity Equation provides the continuity equation for the fluctuating velocity,

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{IX-7}$$

The Navier Stokes equations for the mean flow is obtained in a similar way by decomposing the total velocity and taking the temporal average. Here we assume the average quantities are independent of time.

$$\overline{\frac{\partial (U_i + u_i)}{\partial t} + (U_j + u_j)\frac{\partial (U_i + u_i)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial (P + p)}{\partial x_j} + \nu \frac{\partial^2 (U_i + u_i)}{\partial x_j \partial x_j}}$$

Then

$$U_j \frac{\partial U_i}{\partial x_j} + \overline{u_j \frac{\partial u_i}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j}$$

Using the Continuity equation, the second term can be replaced as follows

$$\overline{u_j \frac{\partial u_i}{\partial x_j}} = \overline{\frac{\partial u_i u_j}{\partial x_j}} - \overline{u_i \frac{\partial u_j}{\partial x_j}} = \overline{\frac{\partial u_i u_j}{\partial x_j}}$$

So that the equation of motion for the average velocity becomes,

$$U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} - \overline{\frac{\partial u_i u_j}{\partial x_j}}$$

When written in terms of the stress tensor, they are

$$U_j \frac{\partial U_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left(\tau_{ij} - \rho \overline{u_i u_j} \right) \tag{IX-8}$$

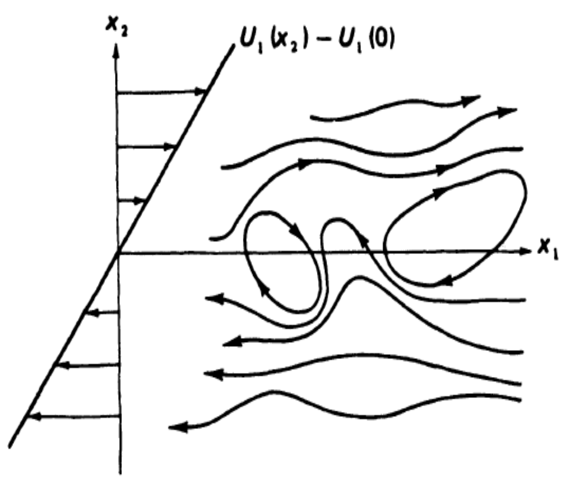
and the stress tensor can be written in terms of the rate of deformation tensor as follows:

$$\tau_{ij} = -P\delta_{is} + 2\mu\varepsilon_{ij} - \rho\overline{u_i u_j}$$

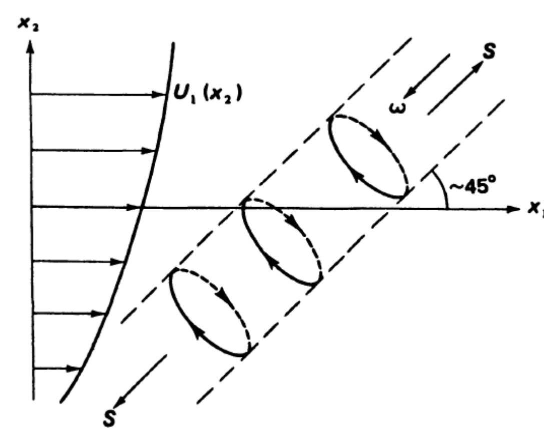
The contribution of the turbulent motion to the mean stress is called the *Reynolds Stress*

$$\tau_{ij}^R = -\rho\overline{u_i u_j} \tag{IX-9}$$

The equations for the mean flow contain now six additional independent quantities generated by the Reynolds stress components. If one tries to continue this process to obtain equations for the fluctuating quantities new terms are generated such as $\overline{u_i u_j u_j}$ from the non-linear convective terms. This is called the closure problem of turbulence and is typical of stochastic non-linear systems.



Shear flow with one mean velocity $U_1(x_2)$. Instantaneous streamlines are relative to coordinates moving with velocity U_1



3-D eddies with vorticity ω are stretched by the rate os strain S. The fluctuations are strong in the plane normal to the vorticity vector

The estimate of the Reynolds stress has been one of the most perplexing tasks in turbulent flow. For example, in a simple shear flow the fluctuating velocities are correlated and when $\frac{\partial U_1}{\partial x_2} > 0$ negative values of u_1 should be more frequent than positive values of u_2, and vice versa. The interaction between eddies and the mean flow is three-dimensional.

For pure shear flow, a mixing-model model has been introduced. This approach has limited applications but has some historical value.

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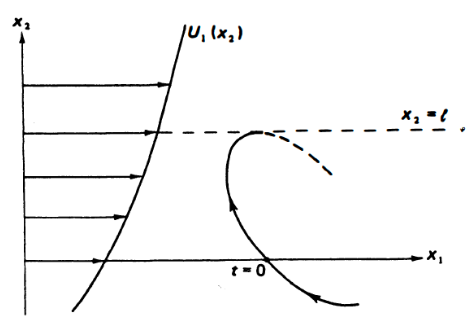
Assume a fluid moves at $x_2 = 0$ at $t = 0$. Its momentum in the x_1 direction per unit volume is $\rho\tilde{u}_1(0,0)$. If the fluid moves upward to a position x_2 at time t its momentum change is

$$\Delta M = \rho\tilde{u}_1(x_2, t) - \rho\tilde{u}_1(0,0)$$

Then

$$\Delta M = \rho[U_1(x_2) - U_1(0)] + \rho[u_1(x_2, t) - \rho u_1(0,0)]$$



If the change of momentum due to fluctuations can be neglected, and the change in momentum due to the average velocity can be approximated by the gradient, namely,

$$\Delta M = \rho x_2 \frac{\partial U_1}{\partial x_2}$$

This approximate derivation leads to the representation of the shear stress as follows:

$$\tau_{12} = \rho \nu_e \frac{\partial U_1}{\partial x_2}$$

Where ν_e is called the eddy viscosity, which can be estimated by

$$\nu_e = c\rho\left(\overline{u_2^2}\right)^{1/2}\ell$$

Here c is an unknown coefficienct and ℓ is called the *mixing length*. The mixing length model favors large scale motions or represents the largest eddies.

Von Karman (1930) introduced the approximation for the mixing length in the form'

$$\ell \sim \frac{\frac{\partial U_1}{\partial x_2}}{\frac{\partial^2 U_1}{\partial x_2^2}}$$

Prandtl introduced the approximation.

$$\tau_{12} = \rho \ell^2 \frac{\partial U_1}{\partial x_2}\left|\frac{\partial U_1}{\partial x_2}\right|$$

For turbulent shear flow near a rigid wall $U_1(x_2)$ with negligible pressure gradient, the momentum in the x_1 direction is:

$$0 = \frac{1}{\rho}\frac{\partial}{\partial x_2}\left(\tau_{12} - \rho\overline{u_1 u_2}\right)$$

If integrated from the boundary $x_2 = 0$ to a value where viscous effects are negligible

$$\tau_{12}(0) = -\rho\overline{u_1 u_2}$$

Define the *friction velocity* as

$$u_* = \sqrt{\frac{\tau_{12}(0)}{\rho}}$$

Then

$$-\overline{u_1 u_2} = u_*^2$$

Using the definition of the stress τ_{12} and the eddy viscosity the Reynolds stress can be expressed with the mixing length $\ell = kx_2$ as:

$$-\overline{u_1 u_2} = u_*^2 = k u_* x_2 \frac{\partial U_1}{\partial x_2}$$

which is integrated to yield a logarithmic profile:

$$U_1 = \frac{u_*}{k} x_2 \ln x_2 + const.$$

k- is the Von Karman constant.

To make the mixing length approach zero more quickly in the viscous sublayer, the *Van-Driest model* gives ℓ in the form:

$$\ell = ky\left[1 - \exp\left(\frac{yu^*}{A\nu}\right)\right]$$

Here k is a constant and $A = 26$ for zero pressure gradient, and $u^* = \sqrt{\frac{\tau_w}{\rho}}$.

The constant A has another representation when the flow includes a pressure gradient near the wall. This model is based on experimental data, and on the idea that the Reynolds stress approaches zero near the wall at a rate proportional to y^3. This has also been verified by Direct Numerical Simulations (DNS).

In recent years the mixing length approach has been discredited, but it seems to work only when there is a single length scale such as in a flow near a wall.

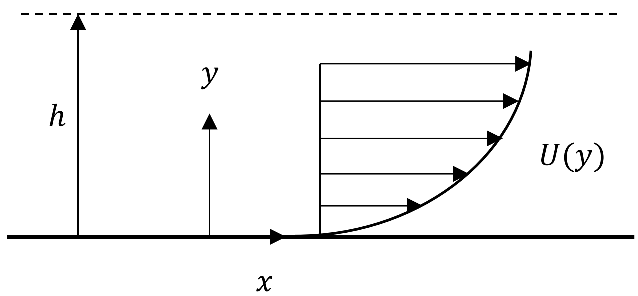
IX.3 Turbulent Flow Parallel to a wall

The Reynolds equations developed for turbulent flow, with the assumption that the velocity can be decomposed into the sum of the temporal mean velocity and its fluctuations, introduce a new tensor called the Reynolds Stress, with 6 additional unknown quantities.

Consider a simple turbulent flow near a solid boundary, or channel flow. Assuming the flow is parallel, we retain only the average velocity component in the x direction

$$\vec{\bar{V}} = U\hat{\imath}. \qquad \bar{v} = \bar{w} = 0 \qquad \frac{\partial U}{\partial x} = 0$$

Also $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = 0,$



$$0 = -\frac{1}{\rho}\frac{d\bar{p}}{dx} - \frac{d\overline{u'v'}}{dx} + \nu\frac{d^2U}{dy^2} \qquad \text{(IX} - 10)$$

Integrating from the wall at $y = 0,$ to an arbitrary distance

$$0 = -\frac{y}{\rho}\frac{d\bar{p}}{dx} - \overline{u'v'} + \nu\frac{dU}{dy} - \frac{\tau_w}{\rho} \qquad \text{(IX} - 11)$$

At the centerline of the channel $y = h$ (pipe) all stresses are zero due to symmetry which provides the relations between the shear stress at the wall and the pressure gradient

$$-\frac{h}{\rho}\frac{d\bar{p}}{dx} = \frac{\tau_w}{\rho} \qquad \text{(IX} - 12)$$

The stress at the surface is defined in terms of the *friction velocity* by

$$u_* = \sqrt{\frac{\tau_w}{\rho}} \qquad \text{(IX} - 13\text{)}$$

Then substituting the pressure gradient into the equation becomes

$$-\overline{u'v'} + \nu \frac{dU}{dy} = u_*^2 \left(1 - \frac{y}{h}\right) \qquad \text{(IX} - 14\text{))}$$

To solve this problem, the flow near a boundary is divided into three regions:

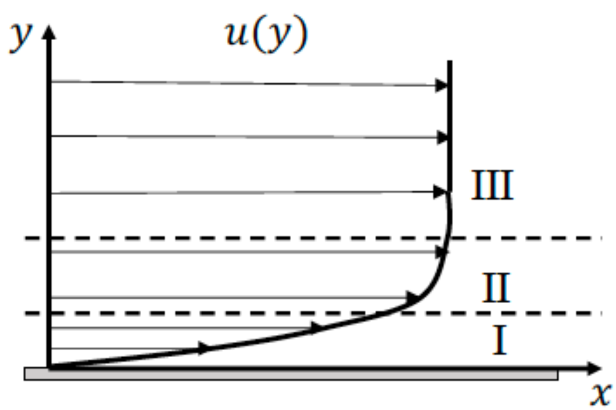
Region I: a *Viscous Sublayer* adjacent to the boundary dominated by viscous effects

Region II: a *Buffer Zone* between Region I and Region III where the two solutions have to be matched

Region III: an *Inertial Sublayer* far from the wall with fully developed turbulent flow.

The three regions are shown in the figure below in terms of a typical average velocity profile.

The solutions will be obtained first for regions I and III, and then matched in the buffer zone.



The non-dimensional scaling of this equation shows that the friction velocity u_*^2 is the proper characteristic velocity for the Reynolds stress component $\overline{u'v'}$ because the viscous stress is expected to be small at large Reynolds numbers. On the other hand, the average shear stress $\frac{dU}{dy}$ should be scaled by u_*/h which represent the characteristic velocity and length scale of the channel.

Thus, the equation can be written in dimensionless form:

$$-\frac{\overline{u'v'}}{u_*^2}+\frac{\nu}{u_*h}\frac{d\left(\frac{U}{u_*}\right)}{d\left(\frac{y}{h}\right)}=1-\frac{y}{h}$$

If we define the Reynolds number as

$$Re^*=\frac{u_*h}{\nu}$$

then for large Reynolds numbers this equation ignores the viscous stress, and therefore it cannot be valid near the boundary where the viscous stress is important.

Near the boundary other scales must be selected such that the viscous stress is retained at large Reynolds numbers. This can be done by replacing the length scale in the definition of the Reynolds number by y. Then,

$$-\frac{\overline{u'v'}}{u_*^2}+\frac{d\left(\frac{U}{u_*}\right)}{d\left(\frac{yu_*}{\nu}\right)}=1-\frac{\nu}{u_*h}\frac{yu_*}{\nu}$$

In this form the change of stress in the y direction is diminished at large Reynolds numbers.

Introduce the dimensionless variable:

$$Y=\frac{yu_*}{\nu}\qquad \eta=\frac{y}{h}$$

Then two equations become:

$$-\frac{\overline{u'v'}}{u_*^2}+\frac{1}{Re^*}\frac{d\left(\frac{U}{u_*}\right)}{d\eta}=1-\eta \qquad \text{(IX} - 15)$$

$$-\frac{\overline{u'v'}}{u_*^2}+\frac{d\left(\frac{U}{u_*}\right)}{dY}=1-\frac{1}{Re^*}Y \qquad \text{(IX} - 16)$$

The solutions for these equations at large Reynolds numbers are asymptotic solutions far from the wall (large η) when the viscous stress is negligible, and a finite value of Y (small η) when the viscous stress is important. The layer of viscous effects is limited to the vicinity of the wall, where both the Reynolds number Uy/ν and Y are too small for turbulence to be sustained. The two solutions must be matched at an intermediate distance from the wall.

Near the centerline where $\eta \sim O(1)$ and large Reynolds number Equ. (IX-15) is reduced to:

$$-\frac{\overline{u'v'}}{u_*^2}=1-\eta \qquad \text{(IX} - 17)$$

This equation is valid far from the boundary, but does not apply to η which corresponds to finite value of Y. For large Reynolds number, Equ (5) is applicable to the near boundary region with $Y \sim O(1)$:

$$-\frac{\overline{u'v'}}{u_*^2}+\frac{d\left(\frac{U}{u_*}\right)}{dY}=1 \qquad \text{(IX} - 18)$$

Therefore, we can assume that

$$\frac{U}{u_*}=F(y) \quad -\frac{\overline{u'v'}}{u_*^2}=G(Y) \qquad \text{(XI} - 20)$$

With the boundary conditions

$$F(0)=0 \qquad G(0)=0$$

In the core region, we can assume that the appropriate scales can be

$$\frac{dU}{dy} = \frac{u_*}{h}\frac{df}{d\eta} \qquad \text{(IX} - 21)$$

Where $\frac{df}{d\eta} \sim O(1)$ and f is an unknown function.

Integrating this equation for the center of the channel $\eta = 1$, to the boundary

$$\frac{U - U_0}{u_*} = f(\eta)$$

This result is not valid near the wall for small η.

We have now a two-region description for the flow near a wall. The two regions overlap at some intermediate distance from the wall.

The matching of the two solutions can be done by matching the velocity gradients. Near the surface the velocity gradient is

$$\frac{dU}{dy} = u_* \frac{dF}{dy} = \frac{u_*^2}{\nu}\frac{dF}{dY}$$

In the core region the velocity gradient is given by Equ. (XI-21). Equating the two velocity gradients:

$$\frac{u_*}{h}\frac{df}{d\eta} = \frac{u_*^2}{\nu}\frac{dF}{dY}$$

or after multiplying by y/u_*

$$\eta\frac{df}{d\eta} = Y\frac{dF}{dY} = \frac{1}{k}$$

Where k is a constant. Therefore, the functions become

$$f(\eta) = \frac{1}{k}\ln\eta + C_1$$

$$F(Y) = \frac{1}{k}\ln Y + C_2$$

These results are valid for $\eta \ll 1,$ and $Y \gg 1.$

The logarithmic velocity profile in the inertial layer is an important result in turbulence theory. If we match the Reynolds stress in this flow according to Equ. (IX-17)

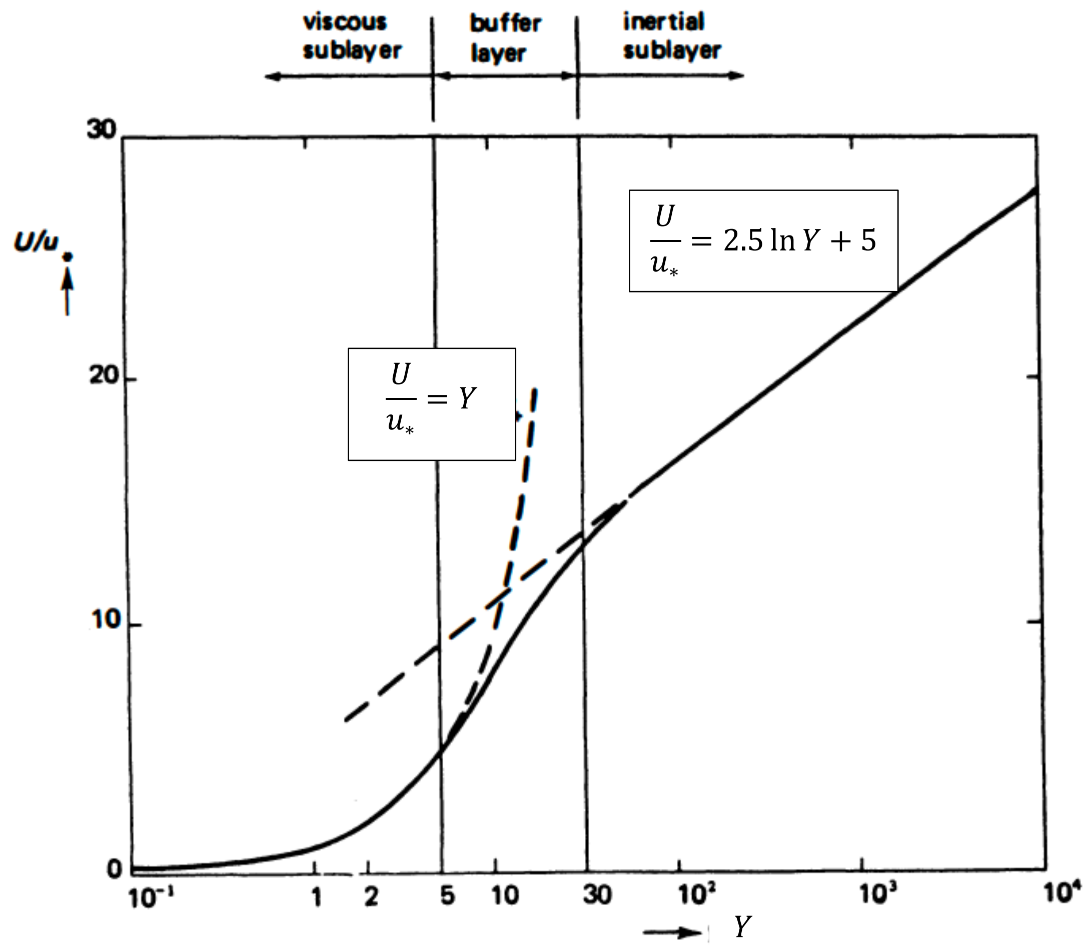
$$\frac{\overline{u'v'}}{u_*^2} \to 1, \qquad \eta \to 0$$

and according to Equ. (IX-18);

$$-\frac{\overline{u'v'}}{u_*^2} = 1 - \frac{d\left(\frac{U}{u_*}\right)}{dY} = 1 - \frac{1}{kY}$$

So as $Y \to \infty$ the same result is obtained.

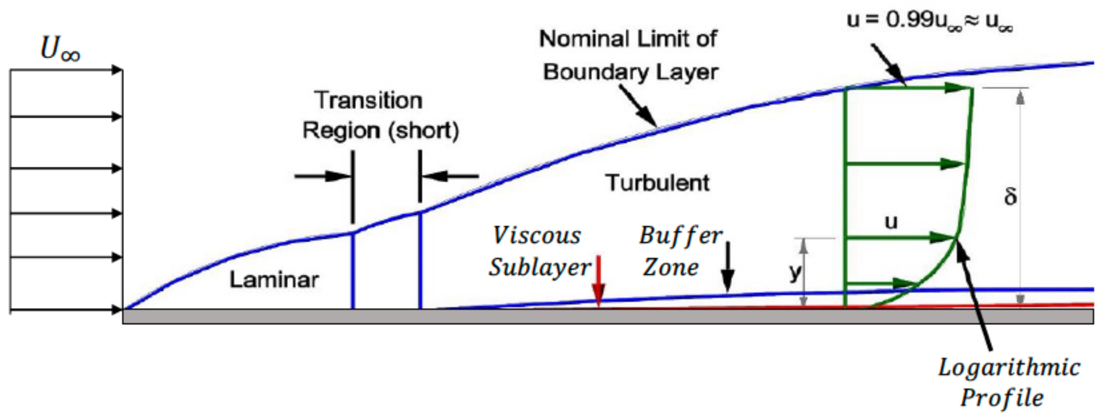
The inertial layer is therefore a region of approximately constant Reynolds stress and negligible viscos stress. Experimental results are used to determine the constant coefficients, and the velocity profile is shown below.



Similar results are available for pipe flow with different numerical coefficients

IX.4 Turbulent Boundary Layer

Turbulent boundary layer over a flat plate is shown schematically below. The flow begins as laminar near the leading edge, and at a critical location along the plate transition to turbulent boundary layer begins. Subsequently, the boundary layer remains turbulent.



Hansen (1930) has shown that transition to turbulence is manifested by a sudden increase of the boundary layer thickness, and that it can happen at a critical Reynolds number based on the distance from the leading edge at about 3.2×10^5. But there is no unique value for the transition to turbulence. Depending on the flow conditions it can happen for $300{,}000 \leq Re_{cr} \leq 10^6$.

The free stream turbulence can influence the transition value as well. Define the % of free stream turbulence as follows:

$$\frac{\overline{|u'|}}{\bar{u}} \times 100$$

Blasius offered an empirical result for turbulent boundary layer for smooth plates as follows:

$$\tau_w = 0.0225\rho U^2 \left(\frac{\nu}{U\delta}\right)^{1/4} \qquad\qquad 5 \times 10^5 \leq Re_x \leq 10^7$$

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$$

Using these empirical results with the Von-Karmen integral equation and constant free stream velocity

$$\frac{d\Theta}{dx} = \frac{\tau_w}{\rho U^2}$$

Then

$$-\frac{\tau_w}{\rho} = \frac{d}{dx}\int_0^{\delta} u(u-U)dy$$

Substituting the shear stress and velocity profile from Blasius empirical velocity profile gives:

$$0.225\,\left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}} = \frac{7}{72}\frac{d\delta}{dx}$$

Integrating

$$\left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}} x = 3.456\delta^{5/4} + c$$

The constant of integration cannot be determined directly because there is no boundary condition. A reasonable approximation is to assume that the boundary layer thickness extends to the leading edge so that $\delta(0) = 0$. Therefore, for $5 \times 10^5 \le Re_x \le 10^7$

$$\frac{\delta}{x} = \frac{0.37}{Re_x^{1/5}} \qquad \text{(IX-22)}$$

IX.5 Skin Friction Coefficient

The *Local Skin Friction Coefficient* (dimensionless shear stress) is given by

$$C'_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.0225\rho U^2\left(\frac{\nu}{U\delta}\right)^{1/4}}{\frac{1}{2}\rho U^2}$$

and with the result for δ it becomes

$$C'_f = \frac{0.0577}{{Re_x}^{1/5}} \qquad \text{for } 5 \times 10^5 < Re_x < 10^7 \qquad \text{(XI-23)}$$

The drag force

$$F_D = \int \tau_w dA = \int_0^L C'_f \frac{1}{2}\rho U^2$$

and by substituting (XI-23) the total force is:

$$F_D = 0.00361 \rho U^2 L Re_L^{-1/5}$$

Total Skin Friction Coefficient (dimensionless drag force/unit width)

$$C_f = \frac{F_D}{\frac{1}{2}\rho U^2 L} = \frac{0.072}{Re_L^{1/5}}$$

This result was corrected by experiments.

$$C_f = \frac{F_D}{\frac{1}{2}\rho U^2 L} = \frac{0.074}{Re_L^{1/5}} \qquad 5 \times 10^5 < Re_x < 10^7 \qquad \text{(IX-24)}$$

The above equation applies to turbulent boundary layer flow over the entire plate. Since near the leading edge the boundary layer flow can be laminar, and therefore has lower skin drag, the equation is adjusted to take the laminar region into account (*Prandtl-Schlichting* skin friction):

$$C_f = \frac{0.074}{Re_L^{1/5}} - \frac{A}{Re_L} \qquad 5 \times 10^5 < Re_x < 10^7 \qquad \text{(IX-25)}$$

Where the constant A is given in the following table

Re_{cr}	300,000	500,000	10^6	3×10^6
A	1050	1700	3300	8700

For $Re_L > 10^7$, the following empirical equations are used

$$C_f = \frac{0.455}{(\log Re_L)^{2.58}} \qquad \text{(IX-26)}$$

when the turbulent boundary layer begins at the leading edge, or

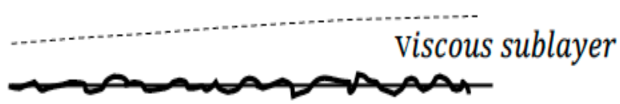
$$C_f = \frac{0.455}{(\log Re_L)^{2.58}} - \frac{A}{Re_L} \tag{IX-27}$$

which is adjusted to include the laminar region near the leading edge. The constant A is given by the same table above.

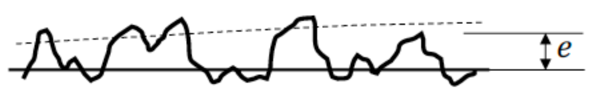
XI.6 Effect of Surface Roughness

The roughness of the surface of a plate is defined as L/e where L is the length of the plate and e is the average height of surface protuberances. The magnitude of the surface roughness L/e is used to define three distinct flow conditions:

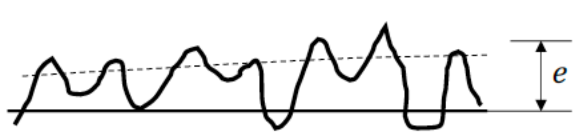
1. *Hydraulically Smooth Zone:* surface roughness is too small to influence the flow; the protuberances are small and covered entirely by the viscous sublayer.



2. *Transition Zone*: protuberances are partially outside the viscous sublayer. (This transition zone should not be confused with the transition from laminar to turbulent flow).



3. *Rough Zone*: protuberances are large and exposed to the main flow outside the viscous sublayer. In this zone the skin-friction coefficient is independent of the Reynolds number. The surface roughness dominates the flow characteristics near the solid wall, and the Reynolds number has little influence on the drag force



Since the skin-friction is larger for rougher plates, it is important sometimes to polish the surface in order to reduce the drag force. When the boundary layer is turbulent from the leading edge, the effect of roughness will diminish with distance along the plate as the viscous sublayer grows and the protuberances become more submerged in the sublayer and less important. Therefore, it is sufficient to polish the region near the leading edge only.

A plate is considered smooth if e is smaller than the *admissible roughness* e_{adm}. The admissible roughness is given by

$$e_{adm} = L\frac{100}{Re_L} \qquad \text{(IX-28)}$$

Thus, if $e < e_{adm}$ the plate is smooth and the previous equations for C_f can be used.

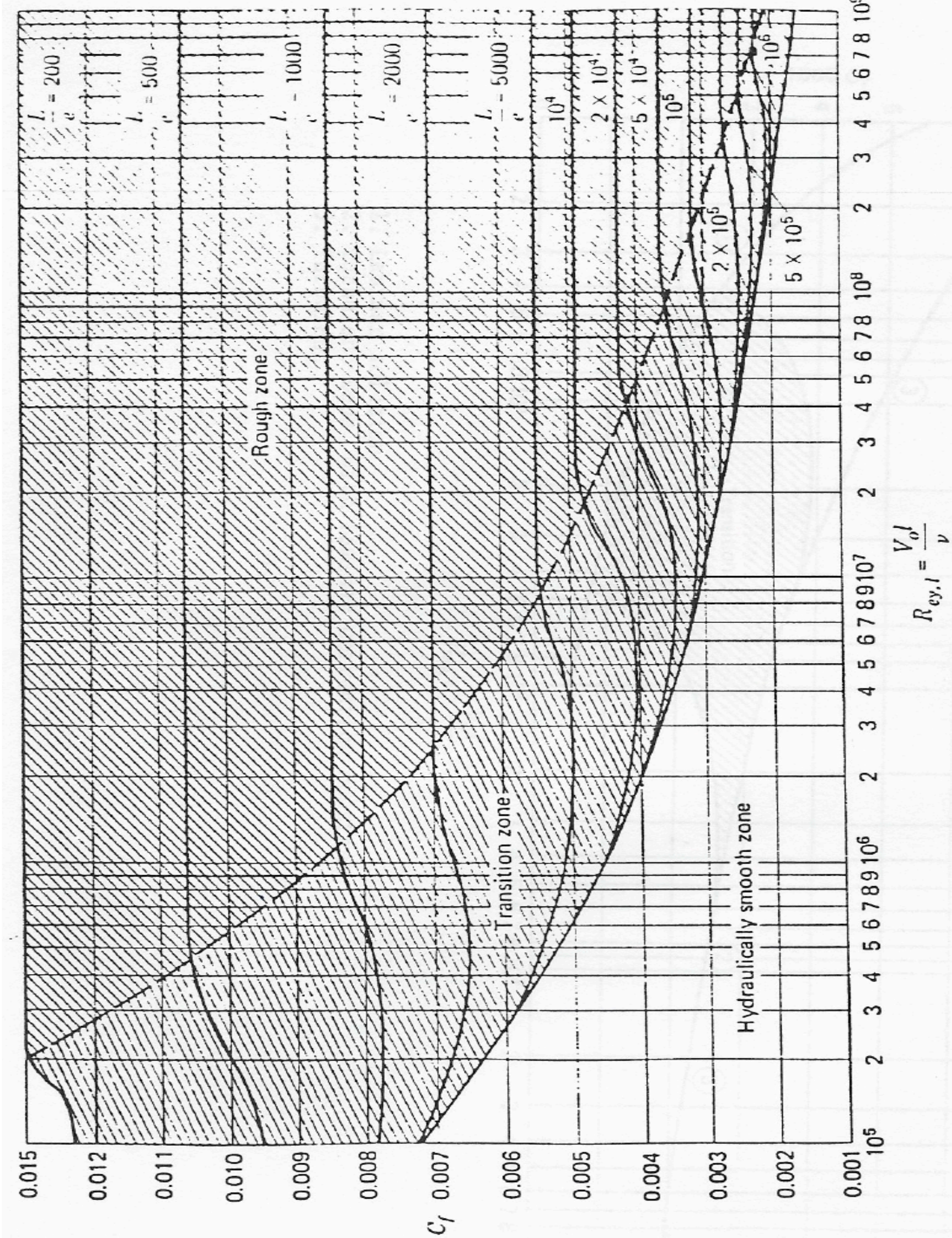
For rough plates $e > e_{adm}$, the skin friction coefficient can be obtained from the chart or the following equations.

In the *Transition Zone* (A is given by the previous table)

$$C_f = \frac{0.031}{Re_L^{1/7}} - \frac{A}{Re_L} \qquad \text{(IX-29)}$$

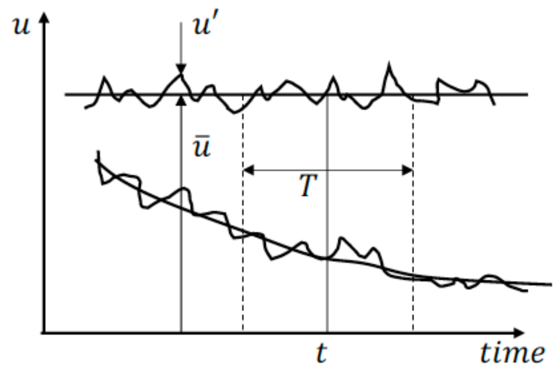
In the *Rough Zone*

$$C_f = \left(1.89 + 1.62\log\frac{L}{e}\right)^{-2.5} \qquad \text{(XI-30)}$$


C_f
0.015
0.012
0.011
0.010
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
10^5 2 3 4 5 6 7 8 9 10^6 2 3 4 5 6 7 8 9 10^7 2 3 4 5 6 7 8 10^8 2 3 4 5 6 7 8 10^9
$R_{ey,l} = \frac{V_o l}{\nu}$
Rough zone
Transition zone
Hydraulically smooth zone
$\frac{L}{e} = 200$
$\frac{L}{e} = 500$
$\frac{L}{e} = 1000$
$\frac{L}{e} = 2000$
$\frac{L}{e} = 5000$
10^4
2×10^4
5×10^4
10^5
2×10^5
5×10^5
10^6

Appendix: The Reynolds Equations in Cartesian Coordinates

There is no comprehensive theory of turbulence. The Reynolds equations are equations derived for turbulent flow based on the assumption the velocity can be represented as the sum of a temporal mean (time average) and a fluctuating component. For one velocity component u at a point we can write



$$u = \bar{u} + u' \tag{A-1}$$

$\bar{u}$ - average velocity
u' - fluctuating velocity

and the temporal mean is defined by

$$\bar{u} = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} u\, dt \tag{A-2}$$

In general, the average velocity can also depend on time, and on the size of the time interval T. Quasi-steady turbulence is defined as one where $\bar{u}$ is independent of t, (T can be very large).

In deriving the Reynolds equations, we need first to introduce the properties of a temporal mean.

The average of u'

$$\overline{u'} = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} u' dt = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} (u - \bar{u}) dt = 0 \tag{A-3}$$

$$\overline{a+b} = \bar{a} + \bar{b}$$

$$\overline{ka} = k\bar{a} \quad (k \text{ is a constant})$$

$$\overline{\frac{\partial a}{\partial x}} = \frac{\partial \bar{a}}{\partial x} \qquad \overline{\frac{da}{dt}} = \frac{d\bar{a}}{dt} \quad \text{(can be shown by Leibnitz rule)}$$

$$\overline{ab} \neq \bar{a}\bar{b}$$

For three-dimensional flow we can introduce the following decomposition,

$$\vec{V} = \overline{\vec{V}} + \overrightarrow{V'} \qquad \text{(A-4)}$$
$$p = \bar{p} + p'$$

This decomposition doubles the number of dependent variables from 4 to 8 in the Navier-Stokes equations, which will require 4 additional differential equation.

Starting with the continuity equation, we take the time average of the equation resulting in a continuity equation for the average velocities

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \qquad \text{(A-5)}$$

By subtracting this equation from the original continuity equation, we obtain a continuity equation for the fluctuating components

$$\frac{\partial u\prime}{\partial x} + \frac{\partial v\prime}{\partial y} + \frac{\partial w\prime}{\partial z} = 0 \qquad \text{(A-6)}$$

The same approach can be followed with the Navier Stokes equations. First, we take the temporal mean,

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} + \overline{u'\frac{\partial u'}{\partial x}} + \overline{v'\frac{\partial u'}{\partial y}} + \overline{w'\frac{\partial u'}{\partial z}} = g_x - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right)$$

$$\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + \bar{w}\frac{\partial \bar{v}}{\partial z} + \overline{u'\frac{\partial v'}{\partial x}} + \overline{v'\frac{\partial v'}{\partial y}} + \overline{w'\frac{\partial v'}{\partial z}} = g_y - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} + \nu\left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} + \frac{\partial^2 \bar{v}}{\partial z^2}\right)$$

$$\bar{u}\frac{\partial \bar{w}}{\partial x} + \bar{v}\frac{\partial \bar{w}}{\partial y} + \bar{w}\frac{\partial \bar{w}}{\partial z} + \overline{u'\frac{\partial w'}{\partial x}} + \overline{v'\frac{\partial w'}{\partial y}} + \overline{w'\frac{\partial w'}{\partial z}} = g_z - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} + \nu\left(\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial y^2} + \frac{\partial^2 \bar{w}}{\partial z^2}\right)$$

The following terms can be rewritten in the first equation

$$
\begin{aligned}
\overline{u'\frac{\partial u'}{\partial x}} &= \overline{\frac{\partial u'^2}{\partial x}} - \overline{u'\frac{\partial u'}{\partial x}} \\
\overline{v'\frac{\partial u'}{\partial y}} &= \overline{\frac{\partial u'v'}{\partial y}} - \overline{u'\frac{\partial v'}{\partial y}} \\
\overline{w'\frac{\partial u'}{\partial z}} &= \overline{\frac{\partial u'w'}{\partial z}} - \overline{u'\frac{\partial w'}{\partial z}}
\end{aligned}
$$

Adding the three terms

$$
\overline{u'\frac{\partial u'}{\partial x}} + \overline{v'\frac{\partial u'}{\partial y}} + \overline{w'\frac{\partial u'}{\partial z}} = \overline{\frac{\partial u'^2}{\partial x}} + \overline{\frac{\partial u'v'}{\partial y}} + \overline{\frac{\partial u'w'}{\partial z}} - \overline{u'\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w}{\partial x}\right)}
$$

where the last term is zero due to the continuity equation for the fluctuating velocity components.

Similarly

$$
\begin{aligned}
\overline{u'\frac{\partial v'}{\partial x}} + \overline{v'\frac{\partial v'}{\partial y}} + \overline{w'\frac{\partial v'}{\partial z}} &= \overline{\frac{\partial v'u'}{\partial x}} + \overline{\frac{\partial v'^2}{\partial y}} + \overline{\frac{\partial v'w'}{\partial z}} - \overline{v'\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w}{\partial x}\right)} \\
\overline{u'\frac{\partial w'}{\partial x}} + \overline{v'\frac{\partial w'}{\partial y}} + \overline{w'\frac{\partial w'}{\partial z}} &= \overline{\frac{\partial w'u'}{\partial x}} + \overline{\frac{\partial w'v'}{\partial y}} + \overline{\frac{\partial w'^2}{\partial z}} - \overline{w'\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w}{\partial x}\right)}
\end{aligned}
$$

The resulting quasi-steady momentum equations for the average velocity are: (A-7)

$$
\begin{aligned}
\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} + \frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} &= g_x - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right) \\
\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + \bar{w}\frac{\partial \bar{v}}{\partial z} + \frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'^2}}{\partial y} + \frac{\partial \overline{v'w'}}{\partial z} &= g_y - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} + \nu\left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} + \frac{\partial^2 \bar{v}}{\partial z^2}\right) \\
\bar{u}\frac{\partial \bar{w}}{\partial x} + \bar{v}\frac{\partial \bar{w}}{\partial y} + \bar{w}\frac{\partial \bar{w}}{\partial z} + \frac{\partial \overline{u'w'}}{\partial x} + \frac{\partial \overline{v'w'}}{\partial y} + \frac{\partial \overline{w'^2}}{\partial z} &= g_z - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} + \nu\left(\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial y^2} + \frac{\partial^2 \bar{w}}{\partial z^2}\right)
\end{aligned}
$$

The next step would be to subtract these equations from the full Navier-Stokes equations to obtain a momentum equations for the fluctuating quantities.

Equs. (A-7) resemble the Navier-Stokes equations for the average velocity components, and average pressure, but have 9 additional terms (6 unknown) involving average of products of the fluctuating velocity components (in the box).

The newly derived terms are defined as the *Reynolds Stress Tensor:*

$$\overrightarrow{\overrightarrow{\tau_R}} = -\rho \begin{pmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'v'} \\ \overline{v'u'} & \overline{v'^2} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'^2} \end{pmatrix} \tag{A-8}$$

and the general quasi-steady Reynolds Equations in vector form become

$$\rho\vec{\overline{V}} \cdot \nabla\vec{\overline{V}} = \vec{g} - \nabla\bar{p} + \mu\nabla^2\vec{\overline{V}} + \nabla \cdot \overrightarrow{\overrightarrow{\tau_R}} \tag{A-9}$$

Attempting to continue and derive momentum equations for the fluctuating components will introduce additional unknown average quantities of velocity products. This process requires "closure" which will replace the newly derived unknowns with expressions that will close the process. Namely, establish a set of equations equal to the number of dependent variables.