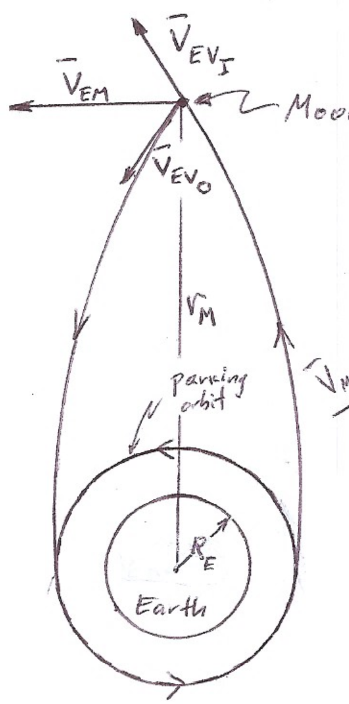
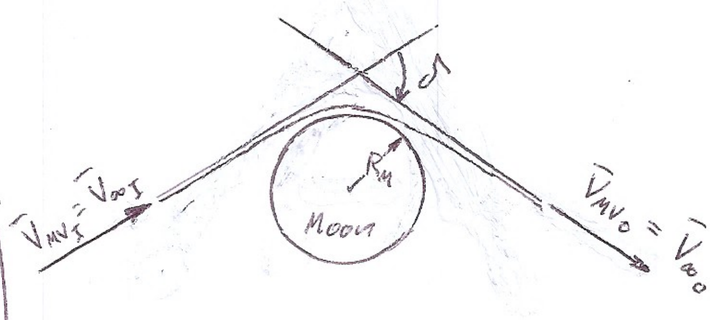
EXAMPLE (Prussing & Conway)

Apollos 8, 10, 11 & 12 were launched into "free return" trajectories to the moon meaning that the spacecraft could return to the earth without the need to perform any powered maneuvers. This trajectory requires the spacecraft to arrive in the vicinity of the moon with a non-zero radial component of velocity. Using lunar gravity to reverse the direction of this radial component (with the tangential component remaining the same) results in a return trajectory which is a mirror image of the outward bound trajectory. If the radial component of the geocentric spacecraft velocity at lunar arrival is 0.75 km/sec (typical value for the Apollo missions) determine the perilun altitude which will yield a free return trajectory.

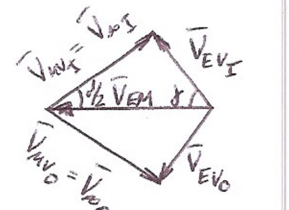


View from Earth



View from Moon

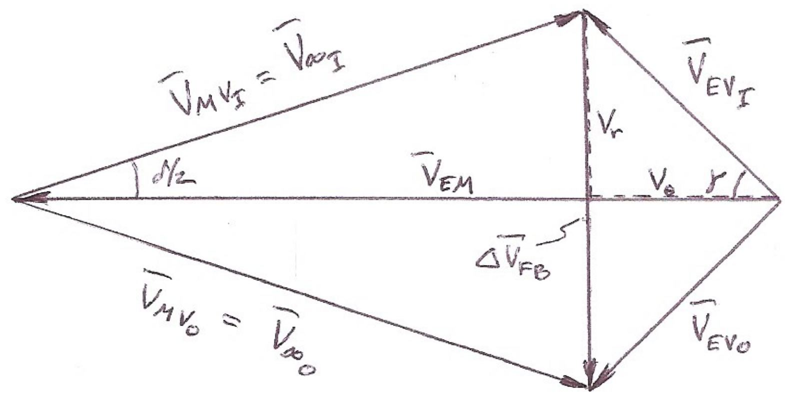
(Arrives at Moon from left because speed of Moon is larger than speed of Vehicle)



Velocity Diagram

<u>Subscripts</u>

E = Earth
M = Moon
V = Vehicle



The magnitude of the incoming velocity is

$$V_{EV_I} = \sqrt{V_r^2 + V_\theta^2} \qquad (1)$$

with $V_r = 0.75$ km/sec. The tangential component is given by

$$V_\theta = \frac{h}{r} = \frac{\sqrt{\mu a(1-e^2)}}{r} = \frac{\sqrt{\mu_E\, a(1-e)(1+e)}}{r_M} \qquad (2)$$

The departure point from the parking orbit is close to being the perigee of the transfer orbit so

$$R_E = a(1-e) \qquad (3)$$

Since the spacecraft arrives at the moon with a radially outward component of velocity the semi-major axis of the transfer orbit must be larger than the radius (semi-major axis) of the moon's orbit which is 384,400 km. Since $R_E = a(1-e)$ with $a > 384,400$, the eccentricity of the transfer orbit must have a lower bound of 0.983.

Thus eq (2) becomes

$$V_\theta = \frac{\sqrt{\mu_E\, a(1-e)(1+e)}}{r_M} \approx \frac{\sqrt{\mu_E R_E (1+1)}}{r_M} = \frac{\sqrt{(3.986\times10^5)(6378)(2)}}{384,400}$$

$$= 0.1855 \ \frac{km}{sec}$$

Thus from eq (1)

$$V_{EV_I} = \sqrt{V_r^2 + V_\theta^2} = \sqrt{(0.75)^2 + (0.1855)^2} = 0.7726 \ \frac{km}{sec}$$

The flight path angle at lunar arrival is found as

$$\cos\gamma = \frac{V_\theta}{V_{EV_I}} = \frac{0.1855}{0.7726} = 0.2401$$

$$\gamma = 76.11°$$

$$V_{EM} = \sqrt{\frac{\mu_E}{r_M}} = \sqrt{\frac{3.986\times10^5}{384,400}} = 1.018 \ \frac{km}{sec}$$

From the velocity diagram

$$V_\infty = \sqrt{V_{EV_I}^2 + V_{EM}^2 - 2V_{EV_I}V_{EM}\cos\gamma}$$

$$= \sqrt{(0.7726)^2 + (1.018)^2 - 2(0.7726)(1.018)\cos 76.11°}$$

$$V_\infty = 1.121 \ \frac{km}{sec}$$

Also from the velocity diagram

$$\frac{V_\infty}{\sin \gamma} = \frac{V_{EV_I}}{\sin \frac{\delta}{2}}$$

$$\sin \frac{\delta}{2} = \frac{V_{EV_I}}{V_\infty} \sin \gamma = \frac{0.7726}{1.121} \sin 76.11^\circ = 0.6691$$

$$\frac{\delta}{2} = 42.00^\circ \quad \text{or} \quad \delta = 84.00^\circ$$

For the hyperbolic trajectory as viewed from the moon

$$\sin \frac{\delta}{2} = \frac{1}{e}$$

$$e = \frac{1}{\sin \frac{\delta}{2}} = \frac{1}{0.6691} = 1.495$$

Using

$$e = 1 + \psi = 1 + \left(\frac{V_\infty}{V_s}\right)^2 \left(\frac{r_p}{r_s}\right)$$

and

$$\mu = r_s V_s^2$$

get

$$e = 1 + \frac{r_p V_\infty^2}{\mu_M}$$

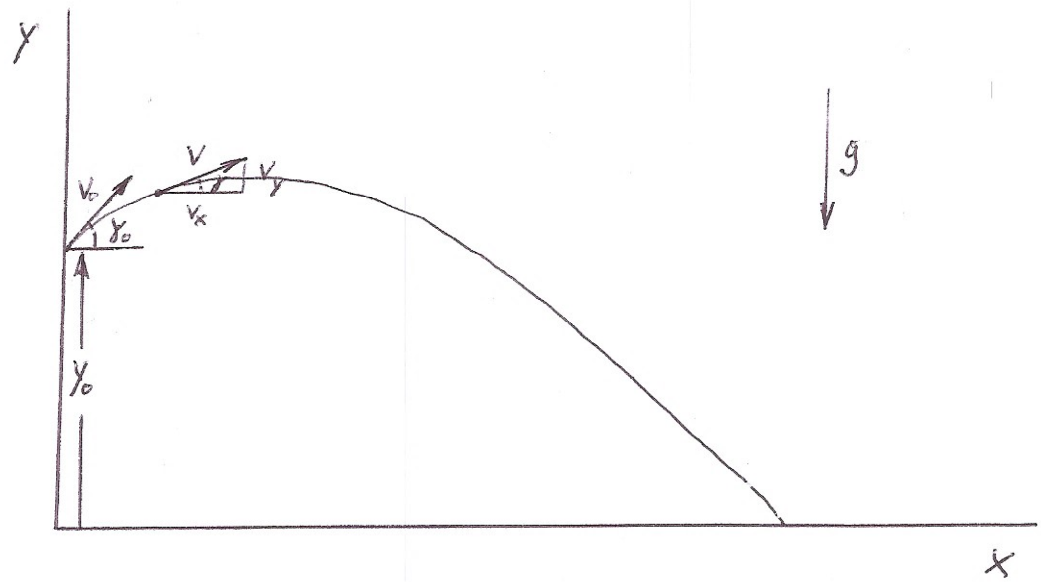
Solve for the perilun radius

$$r_p = \frac{\mu_M (e-1)}{V_\infty^2} = \frac{(4.903 \times 10^3)(1.495-1)}{(1.1215)^2} = 1931 \text{ km}$$

The perilun altitude for this free-return trajectory is

$$H_p = r_p - R_M = 1931 - 1738 = \underline{\underline{193 \text{ km}}}$$

Flat Planet Model for Atmospheric Entry



$$V^2 = V_x^2 + V_y^2$$

$$\cos\gamma = \frac{V_x}{V} = \frac{V_x}{\sqrt{V_x^2+V_y^2}} \qquad \sin\gamma = \frac{V_y}{V} = \frac{V_y}{\sqrt{V_x^2+V_y^2}}$$

$\Sigma F_x = m a_x$

$$-\frac{1}{2} C_D \rho V^2 A \cos\gamma = m \frac{dV_x}{dt}$$

$$-\frac{1}{2} C_D \rho (V_x^2+V_y^2) A \frac{V_x}{\sqrt{V_x^2+V_y^2}} = m \frac{dV_x}{dt}$$

$$-\frac{1}{2} C_D \rho \sqrt{V_x^2+V_y^2} A V_x = m \frac{dV_x}{dt}$$

$$\boxed{\frac{dV_x}{dt} = -\frac{C_D A}{2m} \rho \sqrt{V_x^2+V_y^2}\, V_x} \quad (1) \qquad \boxed{V_x(0) = V_0 \cos\gamma_0}$$

$\Sigma F_y = m a_y$

$$-\frac{1}{2} C_D \rho V^2 A \sin\gamma - mg = m \frac{dV_y}{dt}$$

$$-\frac{1}{2} C_D \rho (V_x^2 + V_y^2) A \frac{V_y}{\sqrt{V_x^2+V_y^2}} - mg = m \frac{dV_y}{dt}$$

$$-\frac{1}{2} C_D \rho \sqrt{V_x^2+V_y^2}\, A\, V_y - mg = m \frac{dV_y}{dt}$$

$$\boxed{\frac{dV_y}{dt} = -\frac{C_D A}{2m} \rho \sqrt{V_x^2+V_y^2}\, V_y - g} \quad (2) \qquad \boxed{V_y(0) = V_0 \sin\gamma_0}$$

Also

$$\boxed{\frac{dx}{dt} = V_x} \quad (3) \qquad \boxed{x(0) = 0}$$

$$\boxed{\frac{dy}{dt} = V_y} \quad (4) \qquad \boxed{y(0) = y_0}$$

Eqs. (1) - (4) may be solved numerically to determine the position and velocity of the spacecraft

For re-entry of Orion capsule into earth's atmosphere

Heat shield diameter $D = 5.03\ m$ $\quad A = \frac{\pi D^2}{4} = 19.87\ m^2$

$= 19.87 \times 10^{-6}\ km^2$

Re-entry mass $m = 9,300\ kg$

Drag coefficient (Heat shield forward) $C_D \approx 1.5$

For atmospheric density distribution

$$\rho = \rho_o e^{-\frac{y}{H}}$$

curve fit to get ρ_o & H

$\rho_o = 1.28\ \frac{kg}{m^3}$ $\quad H \approx 9\ km$

$= 1.28 \times 10^9\ \frac{kg}{km^3}$

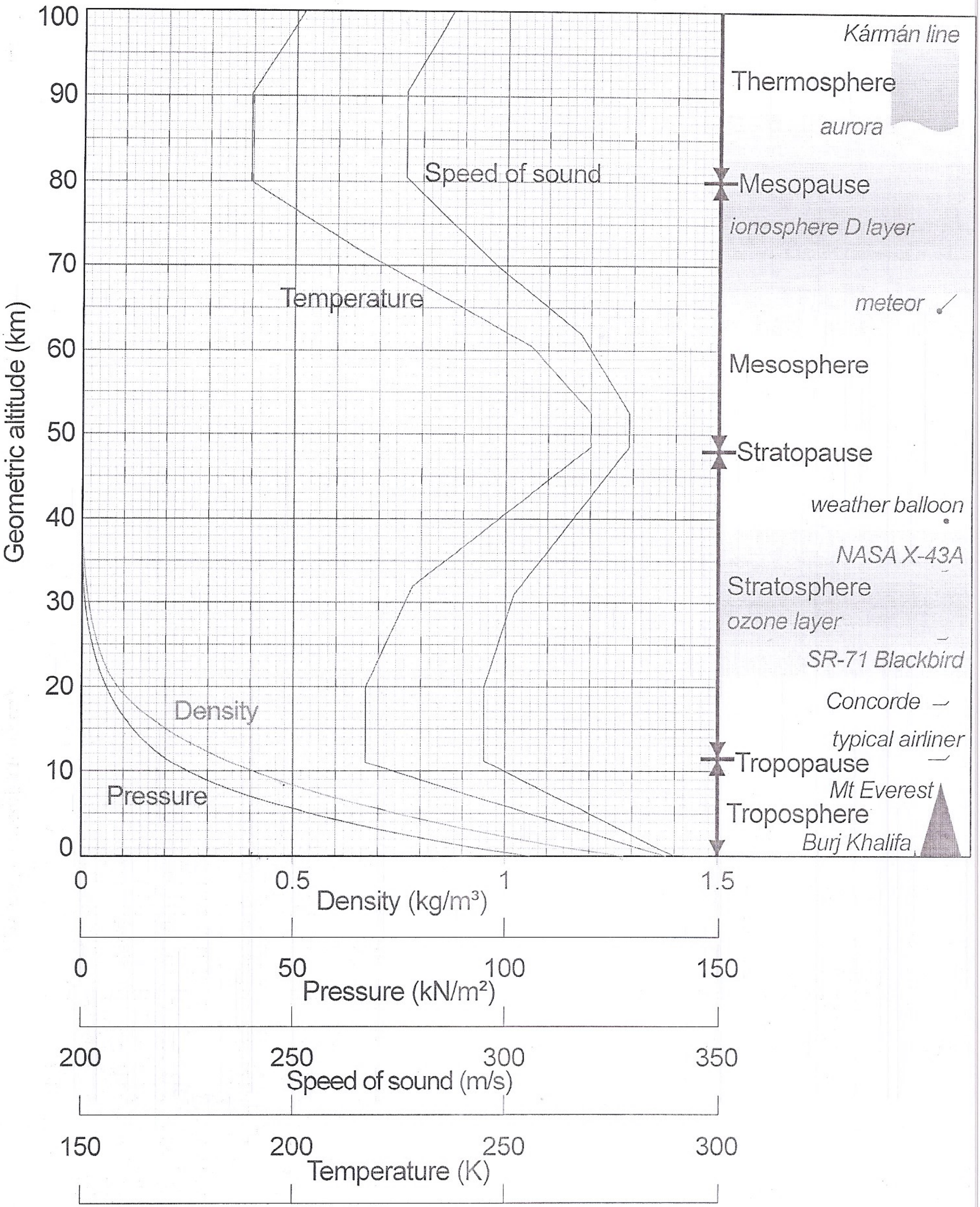
For this example

$y_o = 100\ km$

$\gamma_o = 0°$

$V_o = 11\ \frac{km}{sec}$

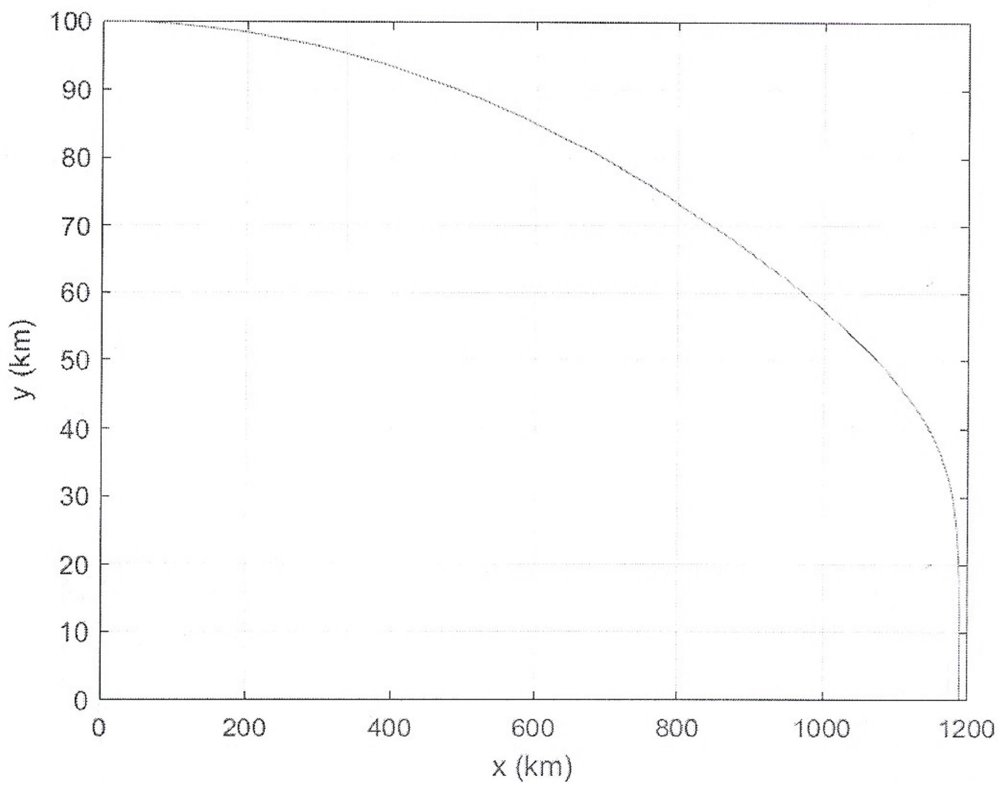
$g = 9.81\ \frac{m}{sec^2} = 9.81 \times 10^{-3}\ \frac{km}{sec^2}$

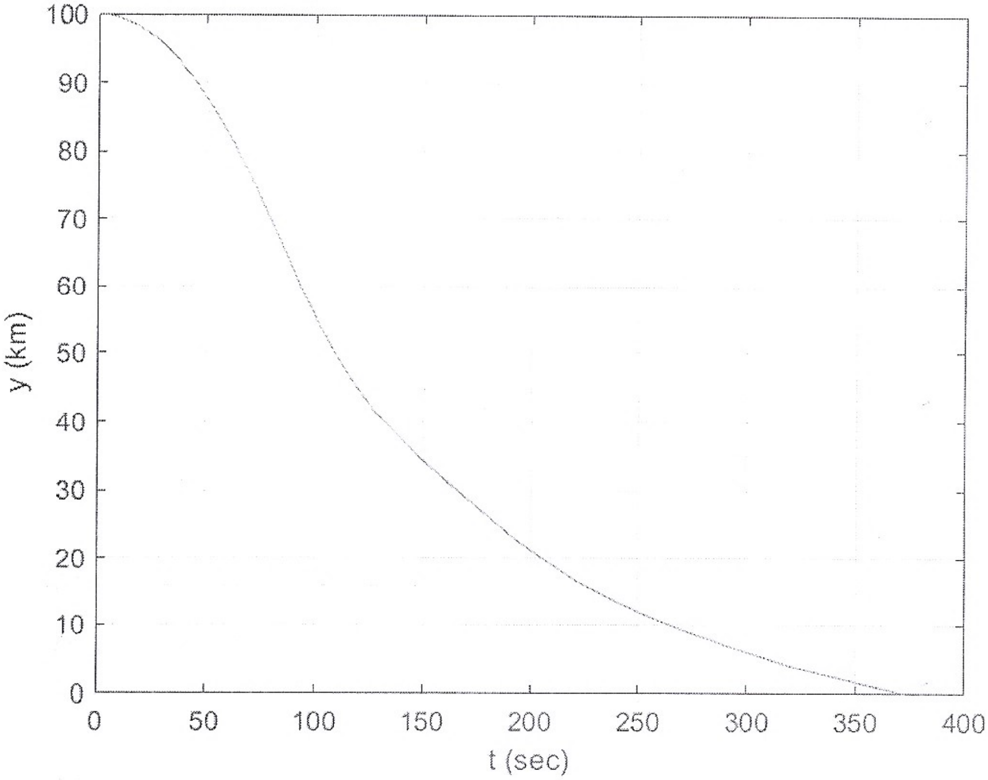


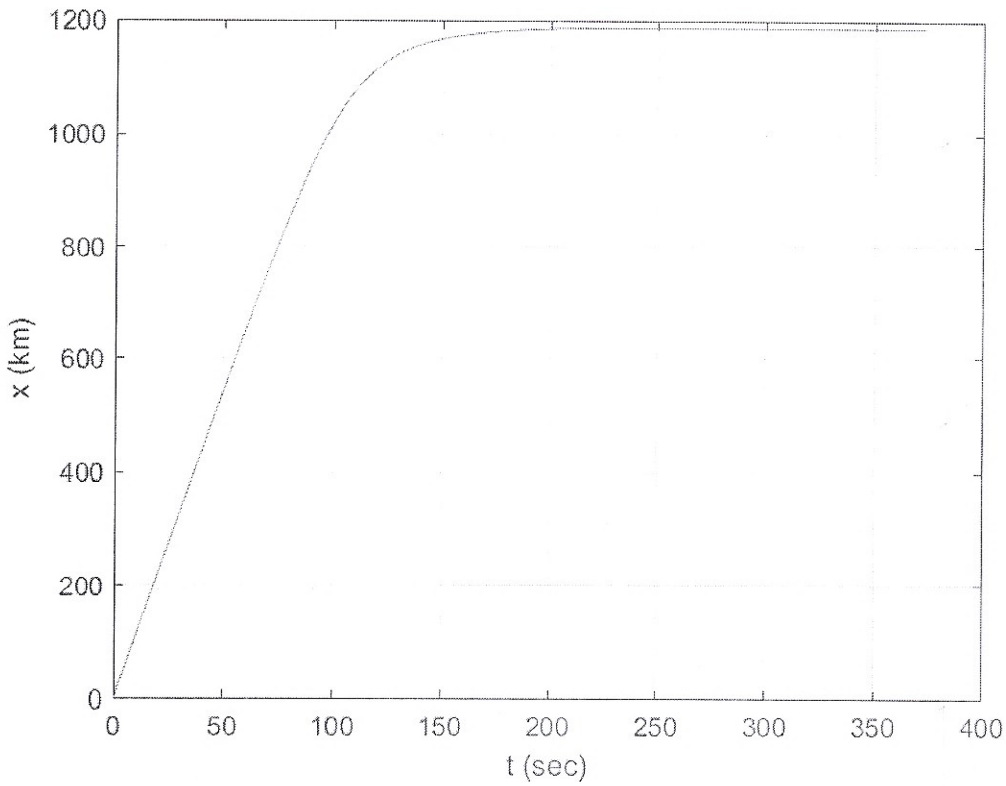
(from Wikipedia)

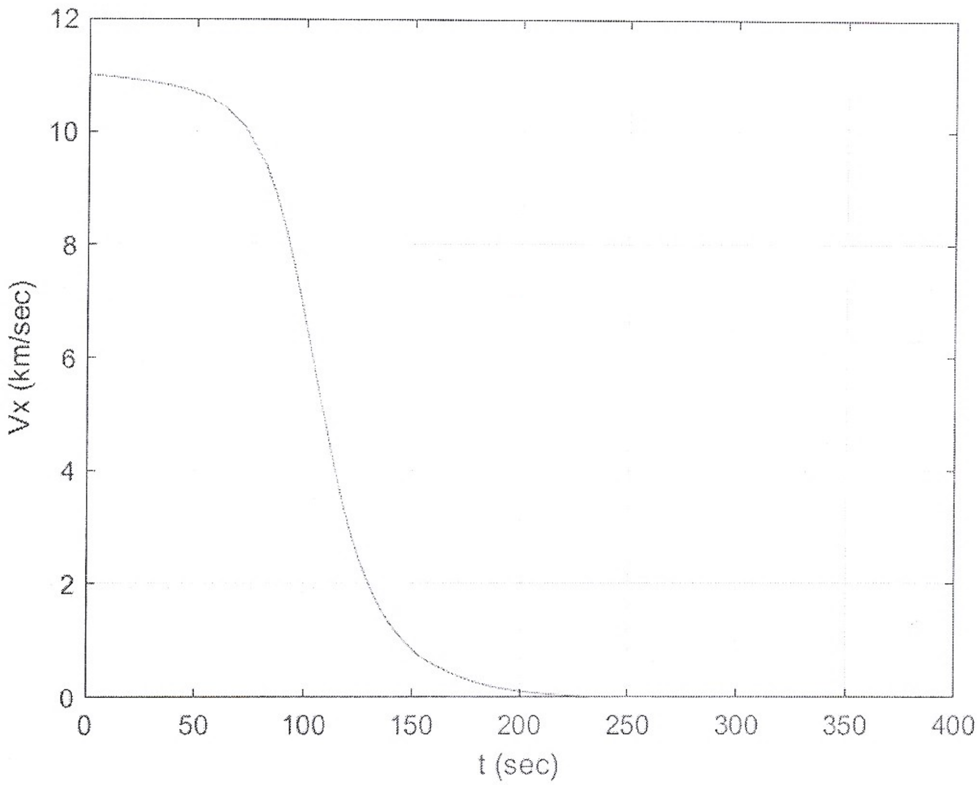
```
global beta rho0 H g
A=19.87e-6;
m=9300;
Cd=1.5;
rho0=1.28e9;
H=9;
y0=100;
gamma=0;
V0=11;
g=9.81e-3;
beta=Cd*A/(2*m);
tspan=[0 373];
yi=[V0*cos(gamma) V0*sin(gamma) 0 y0]';
[t,y]=ode45(@yprime,tspan,yi)
plot(y(:,3),y(:,4))
xlabel('x (km)')
ylabel('y (km)')
grid
```

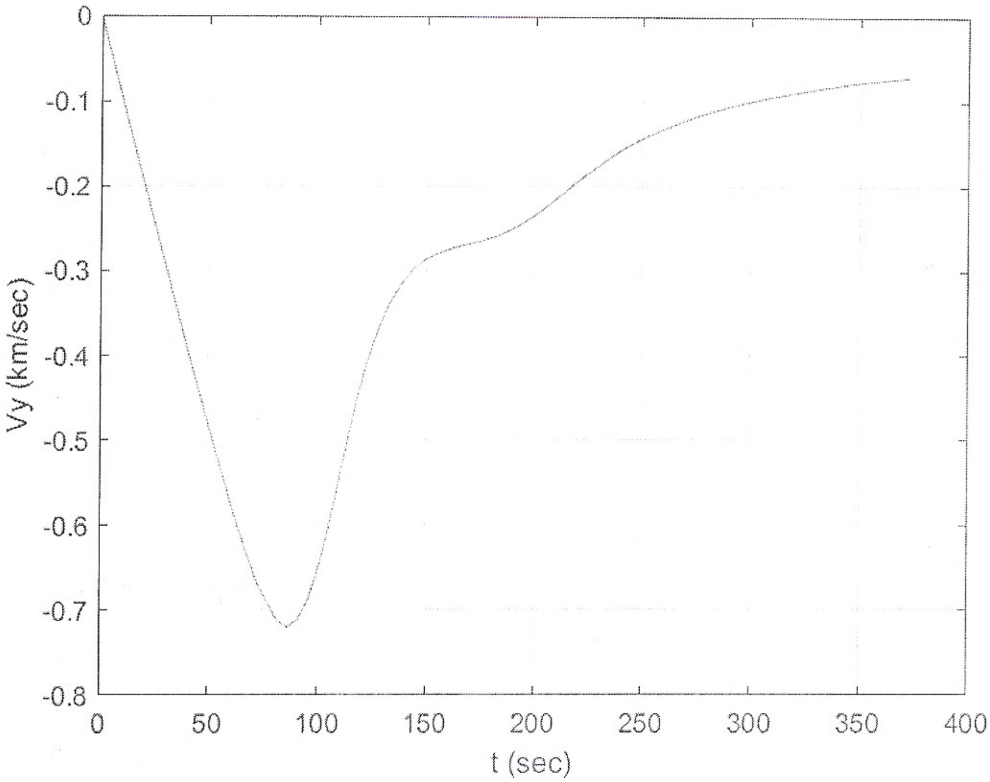
```
function yp=yprime(t,y)
global beta rho0 H g
rho=rho0*exp(-y(4)/H);
V=sqrt(y(1)^2+y(2)^2);
yp=[-beta*rho*V*y(1); -beta*rho*V*y(2)-g; y(1); y(2)];
end
```


100
90
80
70
60
50
40
30
20
10
0
y (km)
0
200
400
600
800
1000
1200
x (km)


100
90
80
70
60
50
40
30
20
10
0
y (km)
0
50
100
150
200
250
300
350
400
t (sec)


1200
1000
800
600
400
200
0
x (km)
0
50
100
150
200
250
300
350
400
t (sec)


12
10
8
6
4
2
0
Vx (km/sec)
0
50
100
150
200
250
300
350
400
t (sec)


0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
Vy (km/sec)
0
50
100
150
200
250
300
350
400
t (sec)