ME 57200 Aerodynamic Design

Lecture #21: Shock Waves and Speed of Sound

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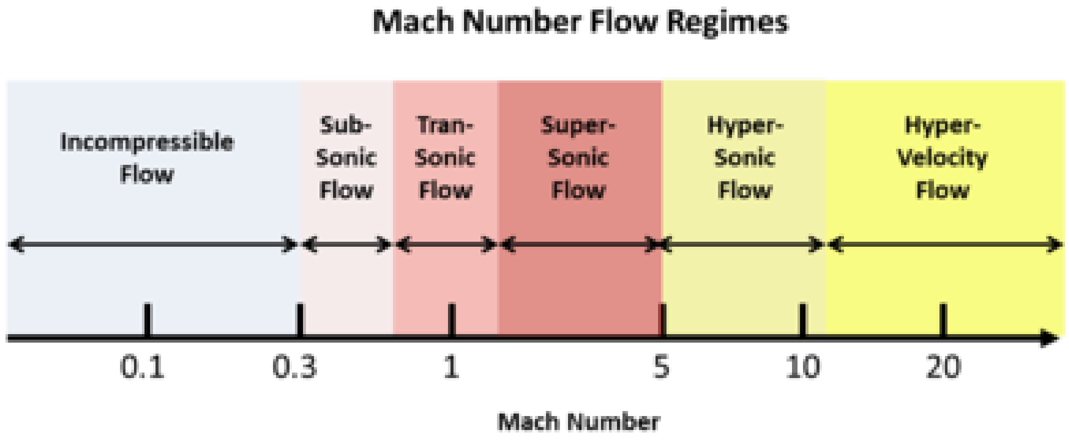
Compressible Flow

Mach number

- The ratio of local flow velocity V to the local speed of sound a:

$$M \equiv \frac{V}{a}$$

- When M > 0.3, the gas flow should be considered compressible.



Compressible Flow

Total (Stagnation) Conditions

- If the flow is steady,

$$\rho \frac{D(h + V^2/2)}{Dt} = \frac{\partial p}{\partial t} \Longrightarrow \rho \frac{D(h + V^2/2)}{Dt} = 0$$

$$\boxed{h + \frac{V^2}{2} = \text{const}} \Longrightarrow \boxed{h + \frac{V^2}{2} = h_0}$$

- At any point in a flow, the total enthalpy is given by the sum of the static enthalpy plus the kinetic energy.

- For a steady, adiabatic, inviscid flow, the total enthalpy is constant along a streamline.

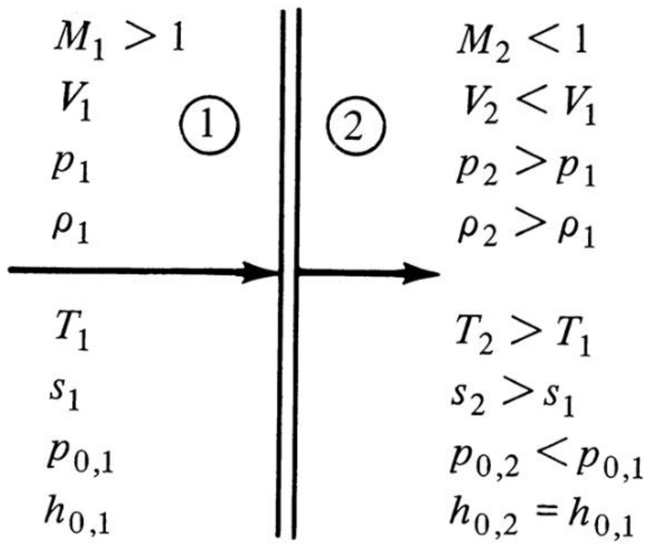
- *Total temperature* $h_0 = c_p T_0$ $\boxed{T_0 = \text{const}}$

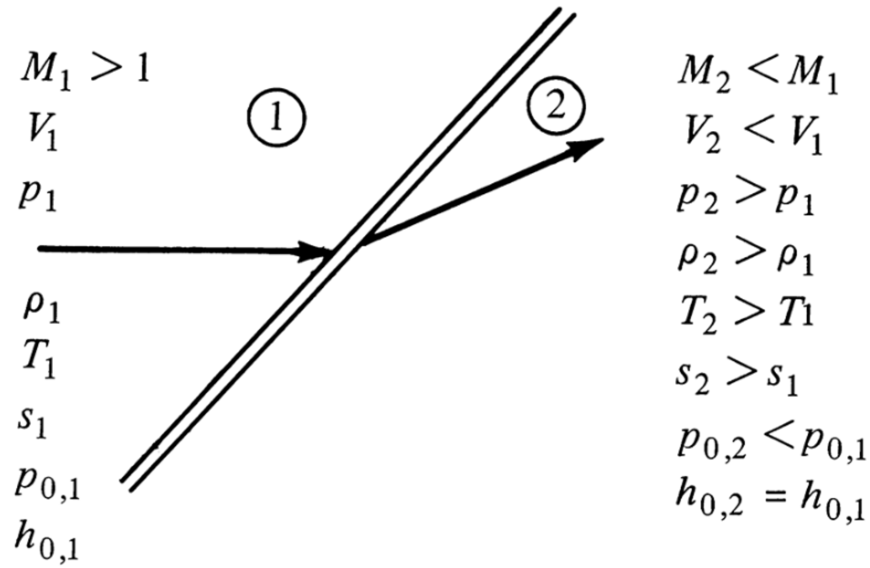
- The total temperature is constant throughout the steady, adiabatic, inviscid flow of a calorically perfect gas.

Compressible Flow

Shock Waves

- A shock wave is an extremely thin region, typically on the order of 10^{-5} cm, across which the flow properties can change drastically.
  - Normal shock
  - Oblique shock

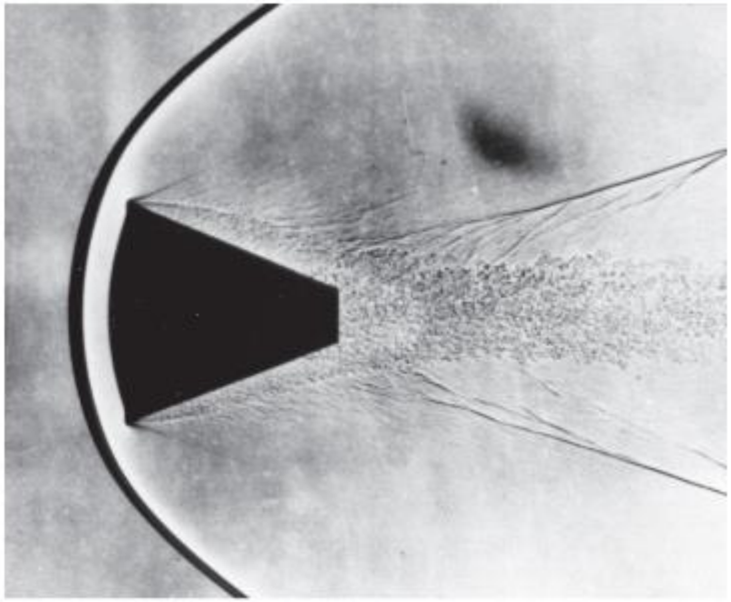


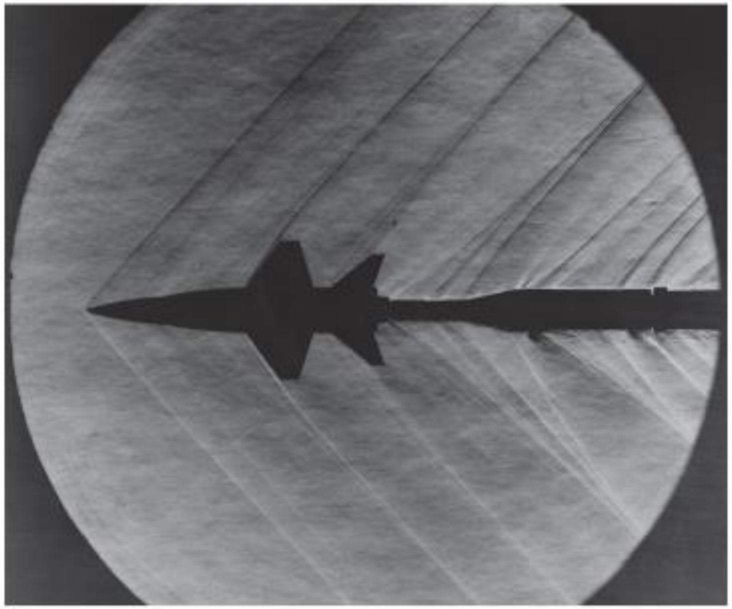


- Physically, the flow across a shock wave is adiabatic – the total enthalpy is constant across the wave

Compressible Flow

Shock Waves

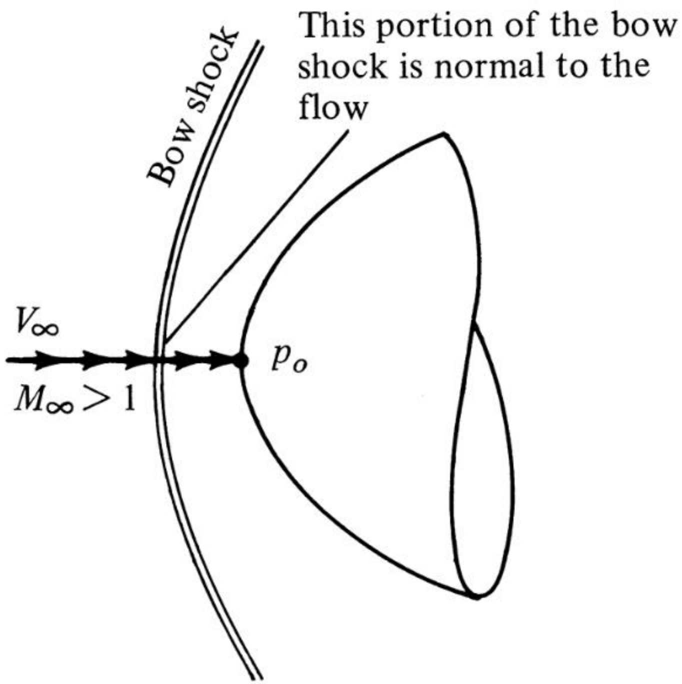




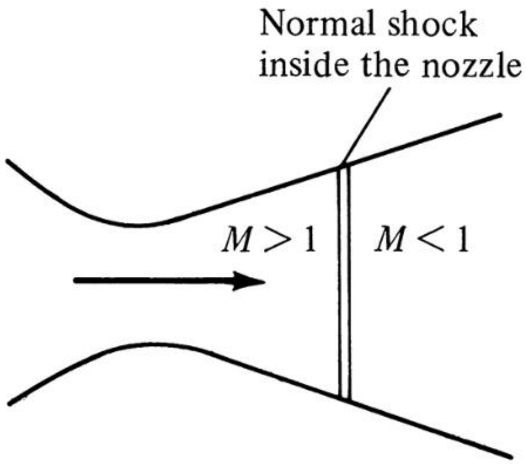
- Air is transparent, we cannot usually see shock waves with the naked eye.
- But, due to the density changes across the shock wave, light rays propagating through the flow can be refracted across the shock
    - Schlieren and Shadowgraphs (ME 59911 Experimental Method in Fluid Mechanics)

Shock Waves

<u>Normal Shock Waves</u>



Flow over a blunt body

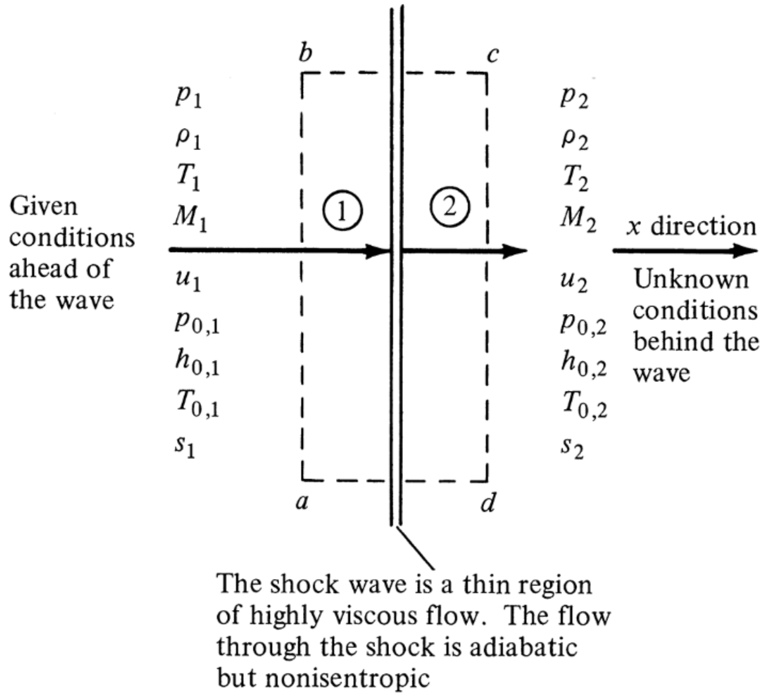


Overexpanded flow through a nozzle

Shock Waves

Normal Shock Waves

- Given the flow properties upstream of the wave, how to calculate the flow properties downstream of the wave?



- The flow is steady
- The flow is adiabatic
- There are no viscous effects on the sides of the control volume
- There are no body forces

Shock Waves

<u>Normal Shock Waves</u>

- Continuity equation

$$\oiint_S \rho \mathbf{V} \cdot \mathbf{dS} = 0$$

$$-\rho_1 u_1 A + \rho_2 u_2 A = 0$$

$$\boxed{\rho_1 u_1 = \rho_2 u_2}$$

- Momentum equation

$$\oiint_S (\rho \mathbf{V} \cdot \mathbf{dS})\mathbf{V} = -\oiint_S p\mathbf{dS}$$

$$\rho_1(-u_1 A)u_1 + \rho_2(u_2 A)u_2 = -(-p_1 A + p_2 A)$$

$$\boxed{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2}$$

Shock Waves

Normal Shock Waves

- Energy equation $$\oiint_S \rho \left(e + \frac{V^2}{2}\right)\mathbf{V}\cdot\mathbf{dS} = -\oiint_S p\mathbf{V}\cdot\mathbf{dS}$$

$$-\rho_1\left(e_1 + \frac{u_1^2}{2}\right)u_1 A + \rho_2\left(e_2 + \frac{u_2^2}{2}\right)u_2 A = -(-p_1 u_1 A + p_2 u_2 A)$$

$$p_1 u_1 + \rho_1\left(e_1 + \frac{u_1^2}{2}\right)u_1 = p_2 u_2 + \rho_2\left(e_2 + \frac{u_2^2}{2}\right)u_2$$

$$\frac{p_1}{\rho_1} + e_1 + \frac{u_1^2}{2} = \frac{p_2}{\rho_2} + e_2 + \frac{u_2^2}{2}$$

$$\boxed{h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}}$$

Shock Waves

Normal Shock Waves

- Continuity equation $$\boxed{\rho_1 u_1 = \rho_2 u_2}$$
- Momentum equation $$\boxed{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2}$$
- Energy equation $$\boxed{h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}}$$

Given that all conditions upstream of the wave, ρ_1, u_1, p_1, etc., are known.

Three equations, four unknowns!

Shock Waves

Normal Shock Waves

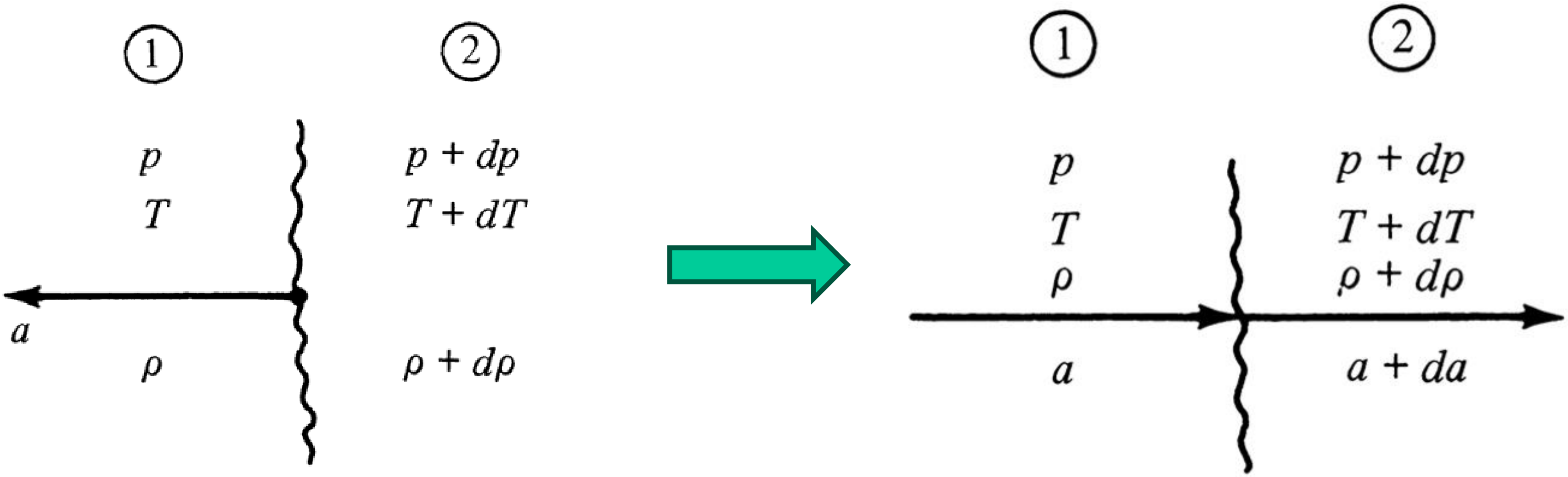
- Continuity equation $$\boxed{\rho_1 u_1 = \rho_2 u_2}$$
- Momentum equation $$\boxed{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2}$$
- Energy equation $$\boxed{h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}}$$
- Enthalpy $$h_2 = c_p T_2$$
- Equation of State $$p_2 = \rho_2 R T_2$$

Speed of Sound

- **Sound: the propagation of the energy wave through the gas**
- *The physical mechanism of sound propagation in a gas is based on molecular motion.*
- *Energized molecules collide with some of their neighboring molecules and transfer their high energy to the neighbors.*
- *“Domino” effect*
- *T, p, ρ are macroscopic averages of the detailed microscopic molecular motion, the regions of energized molecules are regions of slight variations in the local temperature, pressure, and density.*

Speed of Sound

Consider a sound wave propagating through a gas with velocity a. (from right to left)



Speed of Sound

Consider a sound wave propagating through a gas with velocity a. (from right to left)

- Continuity Equation

$$\rho a = (\rho + d\rho)(a + da)$$

$$\rho a = \rho a + a d\rho + \rho da + d\rho da$$

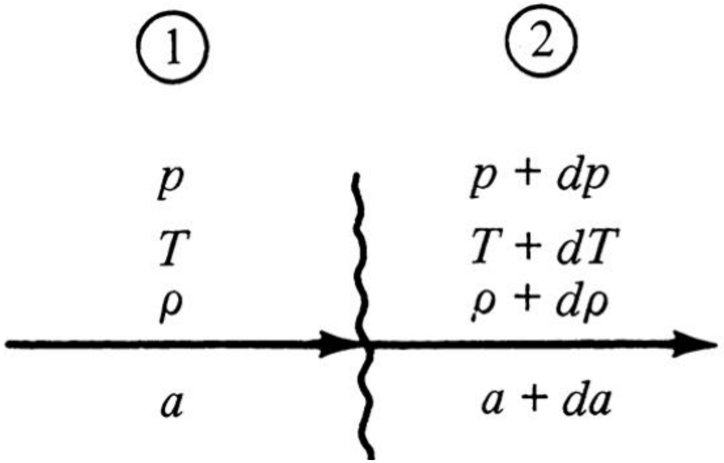
$$a = -\rho \frac{da}{d\rho}$$

- Momentum Equation

$$p + \rho a^2 = (p + dp) + (\rho + d\rho)(a + da)^2$$

$$dp = -2a\rho\, da - a^2 d\rho$$

$$da = \frac{dp + a^2\, d\rho}{-2a\rho}$$



$$a = -\rho \frac{dp/d\rho + a^2}{-2a\rho}$$

$$\Longrightarrow \quad a^2 = \frac{dp}{d\rho} \qquad \boxed{a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}}$$

Speed of sound in a gas

Speed of Sound

- Assume the gas is calorically perfect, isentropic relation can be applied

$$\frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\gamma} \Longrightarrow p = c\rho^{\gamma}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = c\gamma\rho^{\gamma-1}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \left(\frac{p}{\rho^{\gamma}}\right)\gamma\rho^{\gamma-1} = \frac{\gamma p}{\rho}$$

$$\boxed{a = \sqrt{\frac{\gamma p}{\rho}}} \xrightarrow{\frac{p}{\rho} = RT} \boxed{a = \sqrt{\gamma RT}}$$

- The speed of sound in a calorically perfect gas is a function of temperature only

Speed of Sound

- Compressibility

$$\tau_s = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_s$$

$$\tau_s = -\rho\left[-\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial p}\right)_s\right] = \frac{1}{\rho(\partial p/\partial \rho)_s}$$

$$\tau_s = \frac{1}{\rho a^2}$$

$$a = \sqrt{\frac{1}{\rho \tau_s}}$$

Relation between the speed of sound and the compressibility of a gas

Speed of Sound

Relation between the speed of sound and the compressibility of a gas

$$a = \sqrt{\frac{1}{\rho \tau_s}}$$

- The lower the compressibility, the higher the speed of sound.
  - *The speed of sound in a theoretically incompressible fluid is infinite*
  - *In turn, for an incompressible flow with finite velocity, V, the Mach number, M = V/a, is zero.*

Speed of Sound

Mach Number: consider a fluid element moving along a streamline, the ratio between the kinetic and internal energies is

$$\frac{V^2/2}{e} = \frac{V^2/2}{c_v T} = \frac{V^2/2}{RT/(\gamma - 1)} = \frac{(\gamma/2)V^2}{a^2/(\gamma - 1)} = \frac{\gamma(\gamma - 1)}{2} M^2$$

- The square of Mach number is proportional to the ratio of kinetic energy to internal energy of a gas flow.
- The Mach number is a measure of the directed motion of the gas compared with the random thermal motion of the molecules.

Speed of Sound

Example Practice:

Consider an airplane flying at a velocity of 250 m/s. Calculate its Mach number if it is flying at a standard altitude of (a) sea level, (b) 5 km, and (c) 10 km

Speed of Sound

Example Practice:

Consider an airplane flying at a velocity of 250 m/s. Calculate its Mach number if it is flying at a standard altitude of (a) sea level and (b) 5 km

- Solution

at sea level, $T_\infty = 288$ K.

$$a_\infty = \sqrt{\gamma R T} = \sqrt{(1.4)(287)(288)} = 340.2\text{m/s}$$

$$M_\infty = \frac{V_\infty}{a_\infty} = \frac{250}{340.2} = \boxed{0.735}$$

At 5 km, $T_\infty = 255.7$.

$$a_\infty = \sqrt{(1.4)(287)(255.7)} = 320.5\text{m/s}$$

$$M_\infty = \frac{V_\infty}{a_\infty} = \frac{250}{320.2} = \boxed{0.78}$$

Speed of Sound

Example Practice:

Calculate the ratio of kinetic energy to internal energy at a point in a point in an airflow where the Mach number is: (a) M = 2, and (b) M = 20

Speed of Sound

Example Practice:

Calculate the ratio of kinetic energy to internal energy at a point in a point in an airflow where the Mach number is: (a) M = 2, and (b) M = 20

■ Solution

$$(a)\qquad \frac{V^2/2}{e} = \frac{\gamma(\gamma-1)}{2}M^2 = \frac{(1.4)(0.4)}{2}(2)^2 = \boxed{1.12}$$

$$(b)\qquad \frac{V^2/2}{e} = \frac{\gamma(\gamma-1)}{2}M^2 = \frac{(1.4)(0.4)}{2}(20)^2 = \boxed{112}$$

Spatial Forms of Energy Equation

Energy equation for adiabatic flow

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

For calorically perfect gas

$$\boxed{c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2}}$$

$$\Downarrow$$

$$\frac{\gamma R T_1}{\gamma - 1} + \frac{u_1^2}{2} = \frac{\gamma R T_2}{\gamma - 1} + \frac{u_2^2}{2}$$

Spatial Forms of Energy Equation

$$\frac{\gamma R T_1}{\gamma - 1} + \frac{u_1^2}{2} = \frac{\gamma R T_2}{\gamma - 1} + \frac{u_2^2}{2}$$

$$a = \sqrt{\gamma R T},$$

$$\boxed{\frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma - 1} + \frac{u_2^2}{2}}$$

At the stagnation point, the stagnation speed of sound is a_0

$$\boxed{\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a_0^2}{\gamma - 1}}$$

Spatial Forms of Energy Equation

$$\boxed{\frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma - 1} + \frac{u_2^2}{2}}$$

In a sonic flow, where, $u = a^*$

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a^{*2}}{\gamma - 1} + \frac{a^{*2}}{2}$$

$$\boxed{\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2}}$$

$$\Downarrow$$

$$\boxed{\frac{\gamma + 1}{2(\gamma - 1)} a^{*2} = \frac{a_0^2}{\gamma - 1} = \text{const}}$$

Spatial Forms of Energy Equation

$$\boxed{c_p T + \frac{u^2}{2} = c_p T_0}$$

$$\frac{T_0}{T} = 1 + \frac{u^2}{2c_p T}$$

$$\frac{T_0}{T} = 1 + \frac{u^2}{2\gamma RT/(\gamma - 1)} = 1 + \frac{u^2}{2a^2/(\gamma - 1)}$$

$$\boxed{\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2}$$

Only the Mach number dictates the ratio of total temperature to static temperature.

Spatial Forms of Energy Equation

For isentropic compression of the flow to zero velocity

$$\frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\gamma/(\gamma-1)}$$

$$\boxed{\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)}}$$

$$\boxed{\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)}}$$

Hence, for a given gas (i.e., given γ), the ratios T_0/T, p_0/p, and ρ_0/ρ depend only on Mach number.

Spatial Forms of Energy Equation

For a sonic flow, M = 1

$$\boxed{\frac{T^*}{T_0} = \frac{2}{\gamma + 1}}$$

$$\boxed{\frac{p^*}{p_0} = \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)}}$$

$$\boxed{\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma + 1}\right)^{1/(\gamma-1)}}$$

For $\gamma = 1.4$, these ratios are $\frac{T^*}{T_0} = 0.833$ $\quad \frac{p^*}{p_0} = 0.528$ $\quad \frac{\rho^*}{\rho_0} = 0.634$