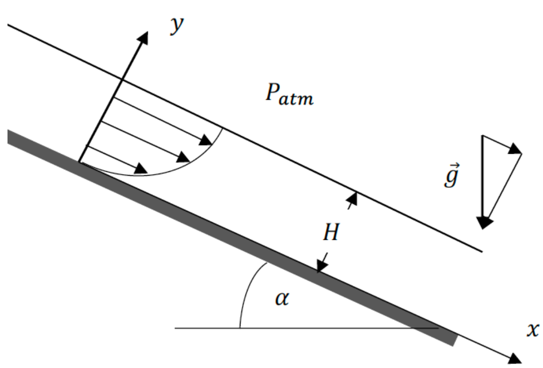
III. Exact Solutions

Exact solutions of Navier Stokes equations are limited to relatively simple laminar flows. Typically, the flow is unidirectional. Namely it has only one velocity component, with the other components being zero. Solutions are also available to some two-dimensional and transient flows. We will consider three basic examples with steady unidirectional flow.

III.1 Flow down an inclined plane

Consider laminar unidirectional laminar flow of a liquid layer of thickness H, down a plane inclined by an angle α. The free surface of the liquid layer is exposed to air at atmospheric pressure. Determine the velocity profile and the volumetric flowrate.



In this example, we will show how the assumptions we make simplify the equations considerably leading to an analytic solution.

The governing equations for two-dimensional (no change in z direction) viscous incompressible fluid flow are the Continuity and Navier Stokes equation.

Assumptions:
Steady state flow: $\frac{\partial}{\partial t} = 0$
Unidirectional flow: $v = w = 0$

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} = 0$$

$$\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$
$$\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}}\right)$$

The simplified equations become:

$$\frac{\partial u}{\partial x} = 0 \qquad \text{(a)}$$

$$0 = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \text{(b)}$$

$$0 = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{(c)}$$

From (a) u is not a function of x, therefore $u = u(y)$ only. This eliminates the first viscous term in (b).

The gravitational components can be represented by

$$g_x = g\sin\alpha\,, \qquad\qquad g_y = -g\cos\alpha$$

Therefore, the final form of the governing equations become:

$$0 = g\sin\alpha - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right) \qquad \text{(d)}$$

$$0 = -g\cos\alpha - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{(e)}$$

Note that the flow is driven by gravity only.

Equations (d) and (e) have to solved subject to the following boundary condition:

(i) at the solid plate, a no-slip boundary condition: $y = 0,\ u = 0$ or

$$u(0) = 0$$

(ii) at the free surface $y = H,\ \tau_{xy} = 0,\ \sigma_{yy} = -p_{atm}$

or

$$\frac{\partial u}{\partial y} = 0, \quad p = p_{atm}$$

Solution of governing equation (e) is obtained by direct integration. From (e)

$$p(x, y) = -gy\cos\alpha + f(x) \qquad \text{(f)}$$

where $f(x)$is an unknown function (or constant) of integration

Apply B.C. (ii) $p(x, H) = p_{atm}$

therefore, $\quad -gH\cos\alpha + f(x) = p_{atm}$

which can be solved for $f(x)$:

$$f(x) = p_{atm} + gH\cos\alpha \qquad \text{(g)}$$

Substitute (f) back into (g) to get the solution for the pressure distribution:

$$p(y) = p_{atm} + g\cos\alpha(H - y) \qquad \text{(h)}$$

which is hydrostatic pressure.

Now we can substitute the solution for the pressure (h) into (d) and solve for the velocity profile:

$$\mu\frac{\partial u}{\partial y} = -\rho g\sin\alpha \qquad \text{(j)}$$

Integrating (j) twice gives:

$$u(y) = -\frac{g}{2\nu}\sin\alpha\ y^2 + C_1 y + C_2 \qquad \text{(k)}$$

where C_1 and C_2 are constants of integration.

To determine the constants of integration, apply boundary conditions (i) and (ii).

From (i) $u(0) = 0$, therefore $C_2 = 0$

From (ii): at. $y = H,\ \frac{\partial u}{\partial y} = 0$.

Therefore: $-\frac{g}{\nu}\sin\alpha\, H\ + C_1 = 0$. or $C_1\ = \frac{g}{\nu}\sin\alpha\, H$

Substitute C_1 and C_2 into the solution (k) to get the velocity profile:

$$u(y) = \frac{g}{\nu}\sin\alpha\, y\left(H - \frac{y}{2}\right) \qquad \text{(m)}$$

Thus, the velocity profile is parabolic with zero velocity at the plate, and zero stress at the free surface.

The flowrate per unit width can now be determined by integrating the velocity profile across the thickness of the liquid layer, namely,

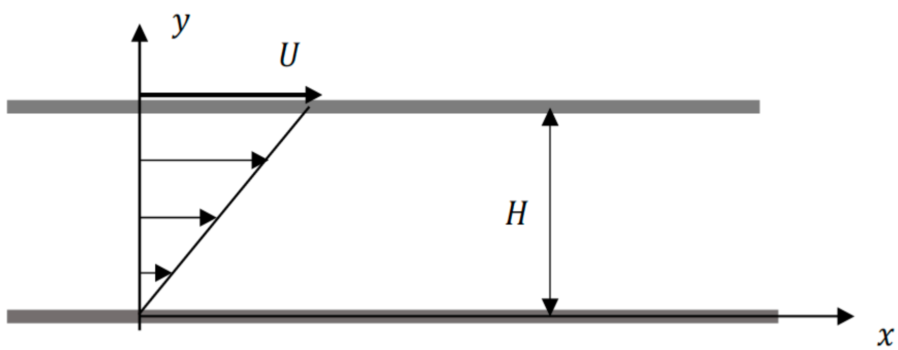
$$Q = \int_0^H u\, dy = \frac{g}{\nu}\sin\alpha \int_0^H y\left(H - \frac{y}{2}\right) dy = \frac{gH^3\sin\alpha}{3\nu} \qquad \text{(n)}$$

The solution shows that flow rate is proportional to the fluid density and inversely proportional to the viscosity.

Increasing the layer thickness increases the flow rate significantly ($\propto H^3$). The solution is valid for $0 \le \alpha \le 90^o$.

III.2 <u>Couette Flow</u>

Couette flow is flow between two parallel plates. We consider the example of flow between two horizontal plates separated by distance H, with the lower plate at the rest, and the upper plate moving at a constant speed U. For simplicity we ignore gravitational forces which do not drive the motion and only result in a hydrostatic pressure distribution. This example is a good representation of flow in narrow gaps between two surfaces like in a journal bearing.



Following the same procedure as in III.1 the governing equations become

$$0 = \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2}\right) \quad \text{(a)}$$

$$0 = \frac{\partial p}{\partial y} \quad \text{(b)}$$

The equations indicate that the pressure is not a function of y because the gravitational force is ignored. In general, it can depend on x if the fluid is pushed by a pump. Without a pressure induced motion, the fluid motion is a result of the motion of the upper plate and therefore in this example $\frac{\partial p}{\partial x} = 0$.

Consequently, the governing equation becomes

$$\frac{d^2 u}{dy^2} = 0 \quad \text{(c)}$$

with the boundary conditions:

(i) $u(0) = 0$
(ii) $u(H) = U$

Integrating (c) and applying the boundary conditions we obtain the velocity profile

$$u(y) = C_1 y + C_2$$

From (i) $C_2 = 0$

From (ii) $C_1 = \frac{U}{H}$

Then

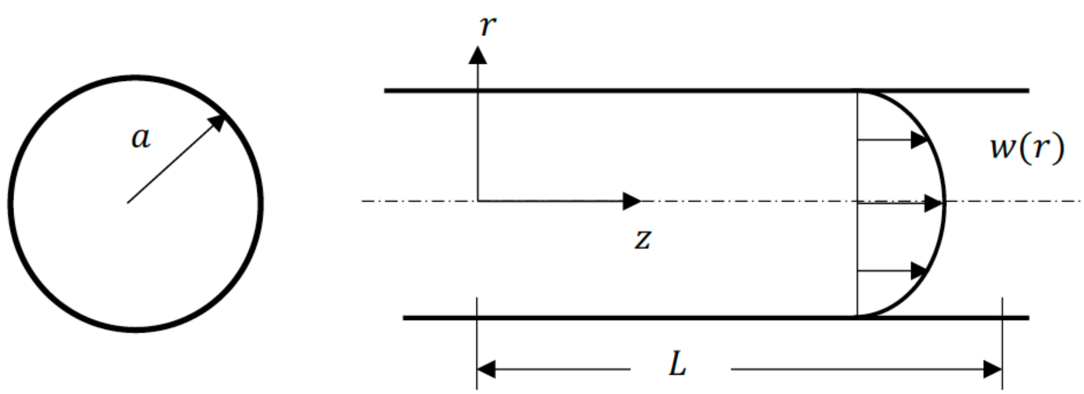
$$u(y) = U\frac{y}{H}$$

and

$$Q = \frac{UH}{2}$$

III.3 <u>Poiseuille Flow</u>

Poiseuille flow describes laminar flow in a pipe. We consider a horizontal pipe with the flow driven by a pressure gradient. As in the previous section we ignore the gravitational force because it has not influence on the flow (it should be included if the pipe is not horizontal). The flow geometry is presented in cylindrical coordinated



Using the Continuity and N-S equations in cylindrical coordinates, the governing equations are:

$$\mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{\partial p}{\partial z} = C_1 \qquad \text{(a)}$$

where the pressure is only a function of z. The corresponding boundary conditions are:

(i) $p(0) = p_1$
(ii) $p(L) = p_2$
(iii) $w(a) = 0$

Since the LHS of (a) is only a function of r, and the RHS a function of z, both sides must be equal to the same constant C_1.

Then

$$p(z) = C_1 z + C_2 \tag{b}$$

and the solution for $w(r)$ becomes,

$$r\frac{\partial w}{\partial r} = \frac{C_1 r^2}{2\mu} + C_2$$

$$w(r) = \frac{C_1 r^2}{4\mu} + C_2 \ln r + C_3 \tag{c}$$

Applying the two conditions on the pressure determines C_1 and C_2.

$$C_1 = \frac{p_2 - p_1}{L} \qquad C_2 = p_1$$

and the solution for the pressure becomes,

$$p(z) = p_1 + (p_2 - p_1)\frac{z}{L} \tag{d}$$

Note that the pressure is linear, and the pressure gradient is negative, with the pressure larger upstream.

Consider now the velocity profile (c). Note that the term $\ln r$ is not finite at $r = 0$. Therefore, it has to be eliminated by setting $C_2 = 0$.

Mathematically, this condition can be replaced by a symmetry condition at the center of the pipe which is the correct boundary condition for the domain $0 \leq r \leq a$.

With the remaining boundary condition at $r = a$, C_3 can be determined

$$C_3 = -\frac{C_1 a^2}{4\mu}$$

and the velocity profile becomes:

$$w(r) = \frac{p_2 - p_1}{4\mu L}(r^2 - a^2) = \frac{1}{4\mu}\frac{\partial p}{\partial z}(r^2 - a^2) \qquad \text{(e)}$$

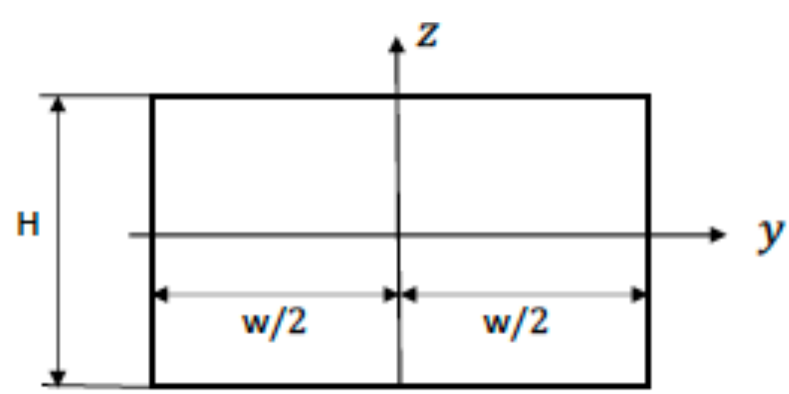
The solution for the velocity profile (e) can be written in term of the pressure gradient, which is a negative constant that can be measured at two points along the pipe separated by the distance L.

The volumetric flow rate is obtained by integrating the velocity over the cross-sectional area of the tube:

$$Q = \int_0^a w(r) 2\pi r dr = -\frac{\partial p}{\partial z}\frac{\pi a^4}{8\mu} \qquad \text{(f)}$$

III.4 <u>Flow Through a Rectangular Duct</u>

Consider steady unidirectional flow in the x direction through an inclined rectangular channel. The inclination angle is α and the cross-sectional geometry is shown below.



Let the constant pressure gradient be defined by

$$-\frac{\partial p}{\partial x} = K$$

So that the governing equation becomes,

$$\mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = -K - \rho g \sin\alpha$$

with the boundary conditions

$u = 0$ at $z = \pm H/2$
$u = 0$ at $y = \pm w/2$

The solution for the non-homogeneous differential equation can be written as the sum of a particular solution and the homogeneous solution, namely

$$u = u_h + u_p$$

Where the particular solution satisfies the equation

$$\mu\frac{\partial^2 u_p}{\partial z^2} = -K - \rho g \sin\alpha$$

and the homogeneous solution

$$\mu\left(\frac{\partial^2 u_h}{\partial y^2} + \frac{\partial^2 u_h}{\partial z^2}\right) = 0$$

This decomposition makes it possible for u_p to satisfy the no-slip boundary condition at to satisfy the oslip boundary conditions at $z = \pm H/2$ and then the homogeneous solution will be used to satisfy all boundary conditions. Therefore,

$$u_p = -\frac{1}{2\mu}(K + \rho g \sin\alpha)z^2 + C_1 z + C_2$$

The boundary conditions at $z = \pm H/2$ provides the solutions for the constant coefficients:

$$C_1 = 0, C_2 = GH^2/4 \qquad \text{where} \qquad G = \frac{1}{2\mu}(K + \rho g \sin\alpha)$$

The particular solution is then,

$$u_p = G\left(\frac{H^2}{4} - z^2\right)$$

The homogeneous solution is obtained by separation of variable (product solution):

Let $u_h = Y(y)Z(z)$, substitute into differential equation and separate variables to get

$$\frac{Y''}{Y} = -\frac{Z''}{Z} = \lambda_n^2$$

where λ_n^2 is the characteristic value so that the solution generates orthogonal function of z. Then

$$u_h = \sum_{n=0}^{\infty} (A_n \sin \lambda_n z + B_n \cos \lambda_n z)(C_n \sinh \lambda_n y + D_n \cosh \lambda_n y)$$

The boundary conditions at $z = \pm H/2$ require than $A_n = 0$ and $\lambda_n = \frac{(2n-1)\pi}{H}$, and the symmetry condition at $y = 0$ requires that $C_n = 0$. Thus, the solution now is

$$u_h = \sum_{n=0}^{\infty} a_n \cos \lambda_n z \cosh \lambda_n y$$

It can be used now to satisfy the boundary conditions at $\pm w/2$ as follows

$$\sum_{n=0}^{\infty} a_n \cos \lambda_n z \cosh \lambda_n w/2 + G\left(\frac{H^2}{4} - z^2\right) = 0$$

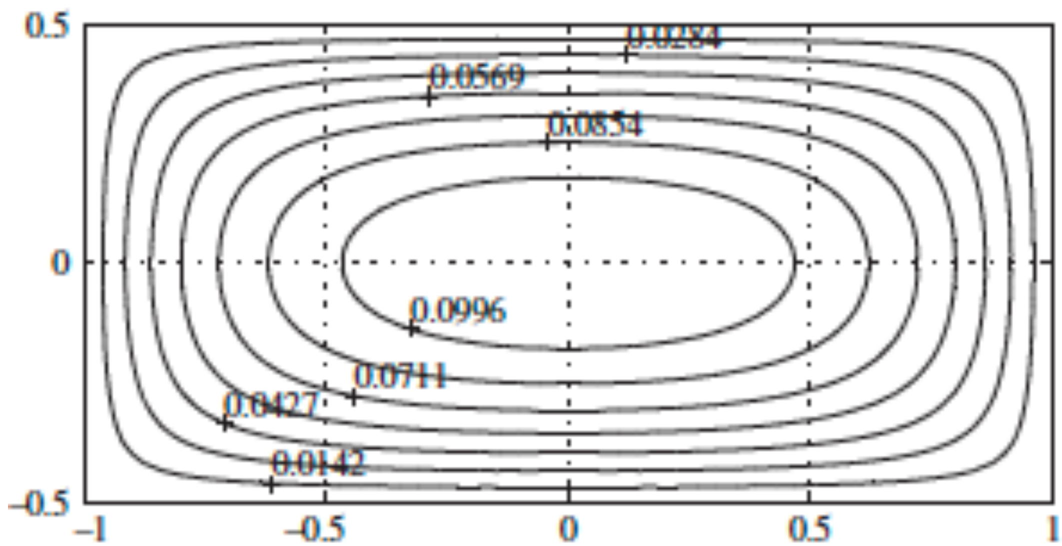
which can be solved for a_n

$$a_n = \frac{1}{\cosh \lambda_n w/2} \frac{\int_0^{H/2} G\left(\frac{H^2}{4} - z^2\right) \cos \lambda_n z \; dz}{\int_0^{H/2} \cos^2 \lambda_n z dz}$$
$$= \frac{4}{\cosh \lambda_n w/2} \frac{(-1)^n}{\lambda_n^3} \frac{(K + \rho g \sin \alpha)}{\mu H}$$

Substituting a_n into the homogeneous part of the solution and adding the particular solution provide the complete solution for this problem.

$$u(y,z) = \frac{1}{2\mu}(K + \rho g \sin\alpha)\left[\left(\frac{H^2}{4} - z^2\right) + \frac{8}{H}\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n^3}\frac{\cos\lambda_n z\ \cosh\lambda_n z}{\cosh\lambda_n \frac{W}{2}}\right]$$

The plot for the constant velocity lines in a cross-section is shown below where the width is twice the height, and the numerical values represent u/u_{max} which is unity at the center point $(0,0)$.



III. 5 Transient Unidirectional Flows

We consider two transient unidirectional flow problems for viscous incompressible fluid. The fluid is semi-infinite on top of a flat plate which is accelerated suddenly or oscillating. These are

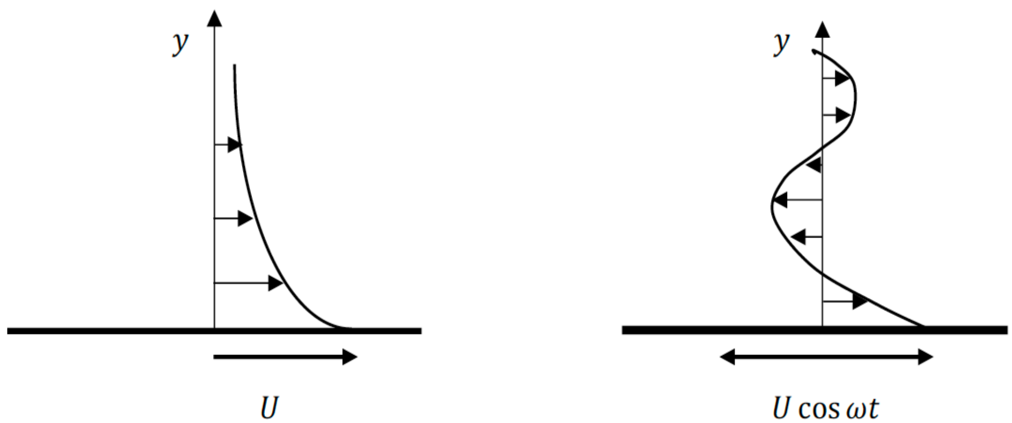
1. Stokes 1st Problem (Rayleigh Problem) – a suddenly accelerating plate
2. Stokes 2nd Problem – oscillating plate

In both problems the fluid motion is induced by the motion of the plates so that gravitational effects and the pressure gradient can be neglected, and the velocity $\vec{V} = u(t,y)\hat{\imath}$ is a function of time and distance from the plate.

Based on the stated assumptions the governing equation is:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

The boundary conditions and initial condition for these problems are



I.C. $u(0, y) = 0$

B.C. $u(t, 0) = U$ $\qquad\qquad$ $u(t, 0) = U \cos \omega t$

$u(t, \infty) = 0$ $\qquad\qquad$ $u(t, \infty) = 0$

The solution for the first problem is obtained by using the Laplace Transform defined as

$$\mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st} dt = \hat{f}(s)$$

Using this definition, the governing equation and boundary conditions are transformed into:

$$\mathcal{L}\left(\frac{\partial u}{\partial t}\right) = \int_0^\infty \frac{\partial u}{\partial t} e^{-st} dt$$

$$= u(t, y)\Big|_0^\infty + s\int_0^\infty u(t, y) e^{-st} dt = -u(0, y) + s\hat{u}(s, y)$$

The transformed governing equation is now

$$s\hat{u}(s,y) = \nu \frac{\partial^2 u}{\partial y^2}$$

and the initial condition has been satisfied through the transformation.

The general solution of the transformed equation is:

$$\hat{u}(s,y) = C_1 e^{-\sqrt{\frac{s}{\nu}}y} + C_2 e^{\sqrt{\frac{s}{\nu}}y}$$

and the transformed B.C. become:

$$\hat{u}(s,0) = \int_0^\infty U e^{-st} dt = \frac{U}{s}$$

$$\hat{u}(s,\infty) = 0$$

Using the two transformed boundary condition, the constants of integrations can be determined.

$$C_1 = \frac{U}{s} \quad \text{and} \quad C_2 = 0$$

The transformed solution then is:

$$\hat{u}(s,y) = \frac{U}{s} e^{-\sqrt{\frac{s}{\nu}}y}$$

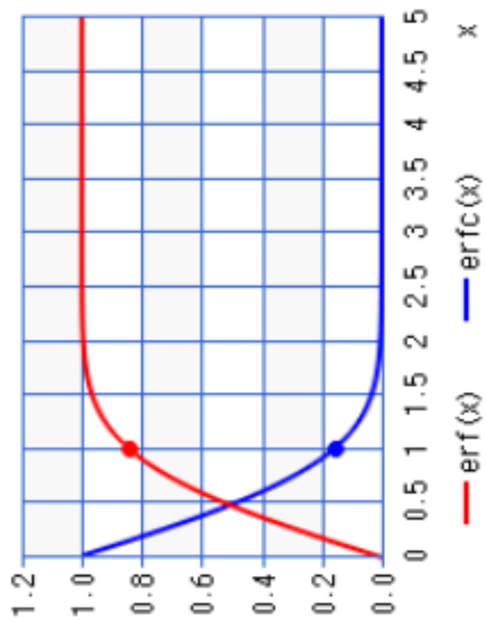
The inverse Laplace transform of this equation will result in the solution for $u(t,y)$.

It can be obtained analytically or from Laplace Transform tables:

$$u(t,y) = U\left[1 - erf\left(\frac{y}{2\sqrt{\nu t}}\right)\right]$$

where $erf(\eta)$ is the *Error Function* defined by

$$erf(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\zeta^2} d\zeta$$



The second problem is solved by using a specific product solution. First, the boundary condition is rewritten as

$$u(t, 0) = U \cos \omega t = \mathcal{R}\{Ue^{-iwt}\}$$

The solution can then be introduced in the form

$$u(t, y) = f(y)e^{-iwt}$$

Substituting the product solution into the governing equation generates a differential equation for the function $f(y)$:

$$f''(y) + \frac{i\omega}{\nu} f(y) = 0$$

and the solution for $f(y)$

$$f(y) = C_1 e^{i\sqrt{\frac{i\omega}{\nu}}} + C_2 e^{i\sqrt{\frac{i\omega}{\nu}}}$$

Since $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$
Then $\sqrt{i} = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}}$

Thus, the solution can be written in the form:

$$f(y) = C_1 e^{iy\sqrt{\frac{\omega}{2\nu}}(1+i)} + C_2 e^{-iy\sqrt{\frac{\omega}{2\nu}}(1+i)}$$

Since the second term becomes infinite as $y \to \infty$ we set $C_2 = 0$, and the full solution now has the form

$$u(t,y) = C_1 e^{(i-1)y\sqrt{\frac{\omega}{2\nu}} - i\omega t}$$

The boundary condition at $y = 0$ determines $C_1 = U$, and the final expression is obtained by taking the real part of the above equation

$$u(t,y) = Ue^{-y\sqrt{\frac{\omega}{2\nu}}} \cos\left(y\sqrt{\frac{\omega}{2\nu}} - \omega t\right)$$

The solution represents oscillatory motion with phase shift exponentially decaying away from the wall.

III.6 <u>Transient Couette Flow</u>

Consider transient Couette flow between two concentric cylinders of radius a and b $(b > a)$.

The governing equation in cylindrical coordinates is

$$\frac{\partial v}{\partial t} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right)$$

The outer cylinder is stationary, and the inner cylinder ω at $t = 0$. Therefore,

(i) $v(0,r) = 0$
(ii) $v(t,a) = \omega a$
(ii) $v(t,b) = 0$

The solution can be generalized by defining the dimensionless variable

$$\tau = \frac{t\nu}{a^2}, \qquad \xi = \frac{r}{a}$$

The transformed differential equation becomes

$$\frac{\partial v}{\partial \tau} = \left(\frac{\partial^2 v}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial v}{\partial \xi} - \frac{v}{\xi^2}\right)$$

with the boundary conditions

(i) $v\,(0, \xi) = 0$
(ii) $v(\tau, 1) = \omega a$
(ii) $v\,(\tau, \alpha) = 0$

where $\alpha = a/b.$ The solution to this problem can be constructed by the sum of two solutions: a steady state solution as $\tau \to \infty$ and a transient part.

The steady state solution is obtained from the solution of the same steady-state problem with the outer cylinder stationary and the inner rotating at constant angular velocity ω, namely

$$v_s(\xi) = -\frac{\omega a \xi}{\alpha^2 - 1} + \frac{wa\alpha^2}{\alpha^2 - 1}\frac{1}{\xi}$$

Therefore, the unsteady solution, which must satisfy the unsteady governing equation has to satisfy the following conditions:

(i) $v(\tau \to \infty, \xi) = 0$
(ii) $v(0,\xi) = -v_s(\xi)$
(iii) $v(\tau, 1) = 0$
(iv) $v\,(t, \alpha) = 0$

Using a product solution

$$v_u = T(\tau)R(\xi)$$

The differential equation can be rewritten as

$$\frac{T'}{T} = \frac{R'' + \frac{1}{\xi}R' - \frac{1}{\xi^2}R}{R} = -\lambda_n^2$$

or the two differential equations

$$T' + \lambda_n^2 T = 0$$

$$\xi^2 R'' + \xi R' + (\lambda_n^2 \xi^2 - 1)R = 0$$

The solution for the differential equation for $R(\xi)$ is a Bessel function shown in Appendix A.

$$v_u = \sum_{n=0}^{\infty} e^{-\lambda_n^2 \tau}[a_n J_1(\lambda_n \xi) + b_n Y_n(\lambda_n \xi)]$$

where $J_1(\lambda_n \xi)$ and $Y_n(\lambda_n \xi)$ are the Bessel function and the modified Bessel function of the first kind, respectively.

One coefficient and λ are determined from the boundary conditions at $\xi = 1, \alpha$.

Therefore, the solution can be rewritten as

$$v_u = \sum_{n=0}^{\infty} A_n e^{-\lambda_n^2 \tau}[J_1(\lambda_n \xi) + \beta_n Y_n(\lambda_n \xi)]$$

Where $\beta_n = -J_1(\lambda_n)/Y_1(\lambda_n)$ and λ_n and is obtained numerically from the equations

$$J_1(\lambda_n \alpha) + \beta_n Y_n(\lambda_n \alpha) = 0$$

To satisfy condition (ii) it is required that,

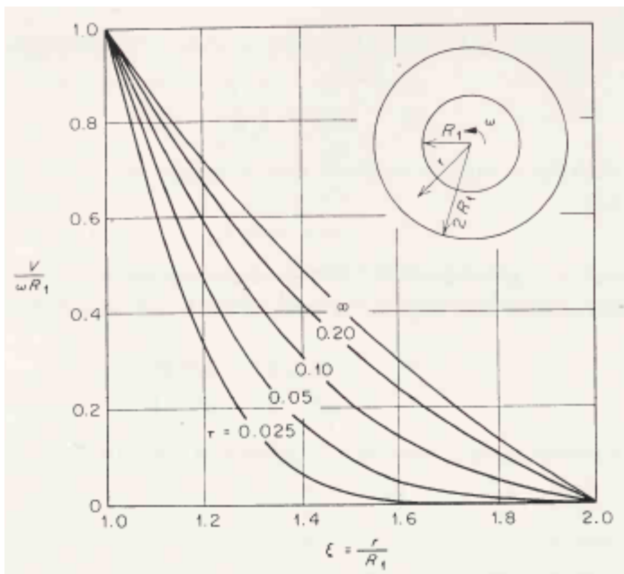
$$\sum_{n=0}^{\infty} A_n [J_1(\lambda_n \xi) + \beta_n Y_n(\lambda_n \xi)] = -u_s$$

Using the orthogonality relations

$$\int_1^{\alpha} \xi [J_1(\lambda_n \xi) + \beta_n Y_n(\lambda_n \xi)][J_1(\lambda_m \xi) + \beta_m Y_n(\lambda_m \xi)] d\xi = 0 \quad \text{for } m \neq n$$

The solution for A_n is:

$$A_n = -\frac{\int_1^{\alpha} v_s \xi [J_1(\lambda_n \xi) + \beta_n Y_n(\lambda_n \xi)] d\xi}{\int_1^{\alpha} \xi [J_1(\lambda_n \xi) + \beta_n Y_1(\lambda_n \xi)]^2 d\xi}$$
$$= -\frac{2\omega a [J_0(\lambda_n) + \beta_n Y_0(\lambda_n)]}{\lambda_n^2 \{\alpha^2 [J_0(\lambda_n \alpha) + \beta_n Y_0(\lambda_n \alpha)]^2 - [J_0(\lambda_n \alpha) - \beta_n Y_0(\lambda_n)]^2\}}$$



III.7 <u>Two-Dimensional Stagnation Flow</u>

Several exact solutions of Navier-Stokes equations exist for two-dimensional flows. One example is a two-dimensional viscous flow impinging on an infinite plate as shown below.

The velocity field far from the boundary is obtained from potential flow theory with

$$u = kx \text{ and } v = -ky$$

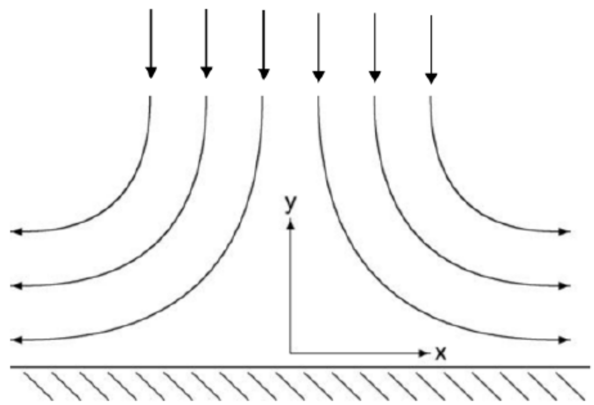
where k is a constant.

The corresponding pressure distribution obtained from Bernoulli's equation is:

$$p_o - p = \frac{1}{2}\rho(u^2 + v^2) = \frac{1}{2}\rho k^2(x^2 + y^2)$$

For the viscous region near the stagnation point the pressure distribution is assumed to be:

$$p_o - p = \frac{1}{2}\rho k^2[x^2 + F(y)]$$



The Navier Stokes equation near the stagnation region becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = k^2 x + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

The continuity equation can be satisfied by introducing the stream function

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

Using the following representation for the stream function

$$\eta = y\sqrt{\frac{k}{\nu}}$$

$$\psi = \sqrt{k\nu}\, x f(\eta)$$

So that

$$u = kxf'(\eta) \qquad v = -\sqrt{k\nu} f(\eta)$$

The governing equation can be reduced into the following ordinary differential equation:

$$f'^2 - ff'' = 1 + f'''$$

with the boundary conditions

$$u = v = 0 \text{ at } y = 0$$
$$u = kx \text{ at } y \to \infty$$

and in terms of $f(\eta)$

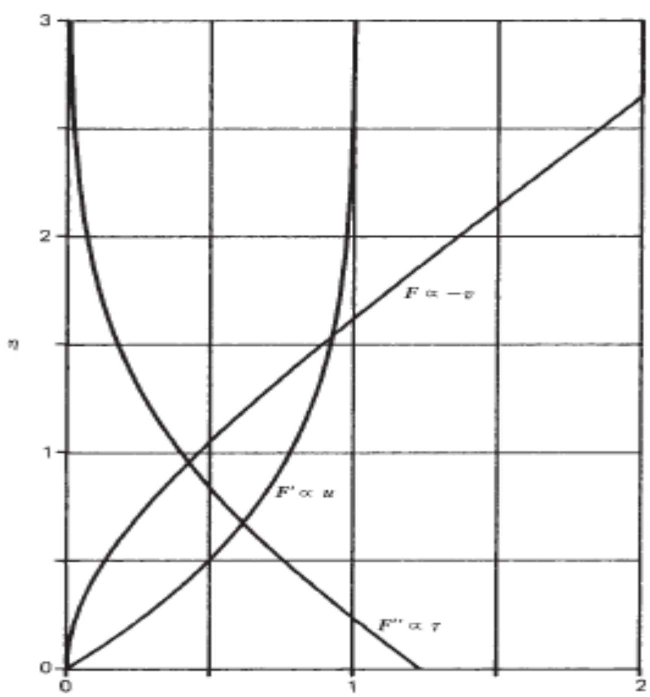
$$f'(0) = f(0) = 0 \qquad\qquad f'(\infty) = 1$$

The solution was obtained numerically by Hiemenz (1911) and is shown in the graph (here the function F for f).

The second momentum equation can then be used to determine the pressure distribution function F.

$$ff' = \frac{1}{2}F' - f''$$

where the values of f, f', and f'' are already available.



Two-dimensional solutions also exist for the following viscous flows:

- Axisymmetric stagnation point flow
- Axisymmetric flow near a rotating plate
- Two-dimensional spiral flow
- Flow between two rotating parallel plates
- Two-dimensional converging channel

<u>References</u>:
Fluid Mechanics by Chia-Shun Yih, West River Press 1979
Boundary Layer Theory by Hermann Schlichting , McGraw Hill. 1979

Appendix: **Orthogonality and Bessel Functions**

Two functions $\varphi_n(x)$ and $\varphi_m(x)$ are orthogonal with respect to a weighting function $w(x)$ in the interval [a,b] if

$$\int_a^b w(x)\varphi_n(x)\,\varphi_m(x)dx = 0 \quad \text{for } m \neq n$$

A function $f(x)$ can be represented by a series of orthogonal functions:

$$f(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x)$$

The constant coefficient in this series, a_n, can be determined by multiplying both sides by $w(x)\varphi_m(x)$ and integrating from a to b. Then

$$\int_a^b w(x)\,f(x)\varphi_m(x)dx = \sum_{n=0}^{\infty} a_n \int_a^b w(x)\,\varphi_n(x)\varphi_m(x)dx$$

Using the orthogonality relations, a solution for a_n is:

$$a_n = \frac{\int_a^b w(x)\,f(x)\varphi_m(x)dx}{\int_a^b w(x)\varphi_n^2(x)\,dx}$$

Trigonometric functions $\sin\lambda_n x$ and $\cos\lambda_n x$ are orthogonal with $w(x) = 1$ and λ_n is a characteristic value determined from a boundary condition

Bessel Differential Equations and Bessel Functions

A class of linear, second order ordinary differential equations with variable coefficients can be written in a general form:

$$x^2\frac{d^2y}{dx^2}+[(1-2A)x-2Bx^2)]\frac{dy}{dx}$$
$$+[C^2D^2x^{2C}+B^2x^2-B(1-2A)x+A^2-C^2n^2]y=0$$

where *A,B,C,D* and n are constants; *D* can be real or imaginary; n is the *order* of the differential equation.

The equation has four types of solutions depending on the values of *D* and n. In this solutions C_1 and C_2 are constants of integration.

n is zero or positive integer, *D* is real

$$y(x)=x^Ae^{Bx}[C_1J_n(Dx^C)+C_2Y_n(Dx^C)]$$

$J_n(Dx^C)$ - Bessel function of order n of the first kind
$Y_n(Dx^C)$ - Bessel function of order n of the second kind

n is neither zero nor a positive integer, *D* is real

$$y(x)=x^Ae^{Bx}[C_1J_n(Dx^C)+C_2J_{-n}(Dx^C)]$$

n is zero or positive integer, *D* is imaginary

$$y(x)=x^Ae^{Bx}[C_1I_n(px^C)+C_2K_n(px^C)]$$

$p=D/i$, $i=\sqrt{-1}$
$I_n(px^C)$ - modified Bessel function of the first kind
$K_n(px^C)$ - modified Bessel function of the second kind

n is neither zero nor a positive integer, *D* is imaginary

$$y(x)=x^Ae^{Bx}[C_1I_n(px^C)+C_2I_{-n}(px^C)]$$

Special Relations for Bessel Functions

All Bessel functions represent different infinite power series. In some cases, they have a simpler representation.

$n = integer/2$ $\qquad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x \qquad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cosh x$$

Bessel functions of higher orders of the same type, can be determined from the recurrence formula

$$J_{k+\frac{1}{2}}(x) = \frac{2k-1}{x} J_{k-\frac{1}{2}}(x) - J_{k-\frac{3}{2}}(x)$$

$$I_{k+\frac{1}{2}}(x) = -\frac{2k-1}{x} I_{k-\frac{1}{2}}(x) - I_{k-\frac{3}{2}}(x)$$

$n = integer$

$$J_{-n}(x) = (-1)^n J_n(x) \qquad Y_{-n}(x) = (-1)^n Y_n(x)$$

$$I_{-n}(x) = I_n(x) \qquad K_{-n}(x) = K_n(x)$$

Derivatives and integrals of Bessel Functions

$$\frac{d}{dx}[x^n Z_n(mx)] = \begin{cases} mx^n Z_{n-1}(mx) & Z = J, Y, I \\ -mx^n Z_{n-1}(mx) & Z = K \end{cases}$$

$$\frac{d}{dx}[x^{-n} Z_n(mx)] = \begin{cases} -mx^{-n} Z_{n+1}(mx) & Z = J, Y, K \\ mx^{-n} Z_{n+1}(mx) & Z = I \end{cases}$$

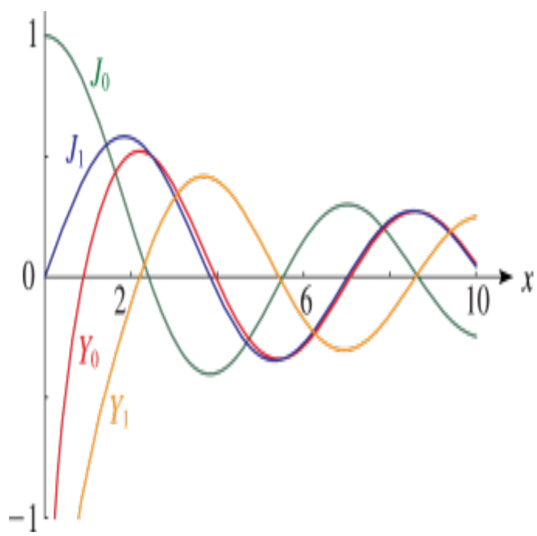
$$\frac{d}{dx}[Z_n(mx)] = \begin{cases} mZ_{n-1}(mx) - \frac{n}{x} Z_n(\mathrm{mx}) & Z = J, Y, I \\ -mZ_{n-1}(mx) - \frac{n}{x} Z_n(mx) & Z = K \end{cases}$$

$$\frac{d}{dx}[Z_n(mx)] = \begin{cases} -mZ_{n-1}(mx) + \frac{n}{x}Z_n(\mathrm{mx}) & Z = J, Y, K \\ mZ_{n-1}(mx) + \frac{n}{x}Z_n(mx) & Z = I \end{cases}$$

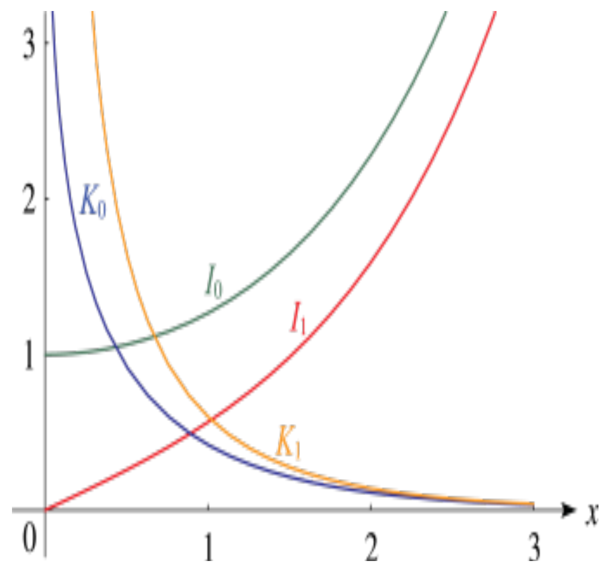
$$\int x^n Z_{n-1}(mx)dx = \frac{x^n}{m} Z_n(mx) \quad Z{=}J,Y,I$$

$$\int x^{-n} Z_{n+1}(mx)dx = -\frac{x^{-n}}{m} Z_n(mx) \quad Z{=}K$$

Graphs of Bessel Functions



Modified Bessel Functions



Behavior at $x = 0$ and $x \to \infty$

x	$J_0(x)$	$J_n(x)$	$I_0(x)$	$I_n(x)$	$Y_n(x)$	$K_n(x)$
0	1	0	1	0	$-\infty$	∞
∞	0	0	∞	∞	0	0

Orthogonality of Bessel Function

The Bessel function $J_n(\lambda_k x)$ is orthogonal with respect to a weighting factor $w(r) = r$. Applying orthogonality leads to integrals of the form

$$N_n = \int_0^a r J_n^2\,(\lambda_k r) dr$$

where N_n is called the normalization integral. The value of the integral depends on the form of the homogeneous boundary condition that leads to the characteristic equation which defines λ_k. Below are values for N_n corresponding to three different boundary conditions at $r = a$.

Boundary condition at $r = a$	$N_n = \int_0^a r J_n^2\,(\lambda_k r) dr$
$J_n(\lambda_k a) = 0$	$\frac{a^2}{2\lambda_k^2}\left[\frac{dJ_n(\lambda_k a)}{dr}\right]^2$
$\frac{dJ_n(\lambda_k a)}{dr} = 0$	$\frac{1}{2\lambda_k^2}\left[(\lambda_k a)^2 - n^2\right] J_n^2(\lambda_k a)$
$-k\frac{dJ_n(\lambda_k a)}{dr} = \mathrm{h}J_n(\lambda_k a)$	$\frac{1}{2\lambda_k^2}\left[(Bi)^2 + (\lambda_k a)^2 - n^2\right] J_n^2(\lambda_k a)$

[1] Hildebrand, F.B., *Advanced Calculus for Application,* 2nd edition, Prentice-Hall, Englewood Cliffs, New Jersey, 1976