We study configuration varieties parametrizing plane pictures $\mathbf{P}$ of a given graph G, with vertices v and edges e represented respectively by points $\mathbf{P}(v) \in \mathbb{P}^2$ and lines $\mathbf{P}(e)$ connecting them in pairs. Three such varieties naturally arise: the *picture space* $\mathcal{X}(G)$ of all pictures of G; the *picture variety* $\mathcal{V}(G)$, an irreducible component of $\mathcal{X}(G)$; and the *slope variety* $\mathcal{S}(G)$, essentially the projection of $\mathcal{V}(G)$ on coordinates m_e giving the slopes of the lines $\mathbf{P}(e)$. In practice, we most often work with affine open subvarieties $\tilde{\mathcal{X}}(G)$, $\tilde{\mathcal{V}}(G)$, $\tilde{\mathcal{S}}(G)$, in which the points $\mathbf{P}(v)$ lie in an affine plane and the lines $\mathbf{P}(e)$ are nonvertical.

We prove that the algebraic dependence matroid of the slopes is in fact the *generic rigidity matroid* $\mathcal{M}(G)$ studied by Laman *et. al.* [2], [1]. For each set of edges forming a circuit in $\mathcal{M}(G)$, we give an explicit determinantal formula for the polynomial relation among the corresponding slopes m_e. This polynomial enumerates decompositions of the given circuit into complementary spanning trees. We prove that precisely these "tree polynomials" cut out $\mathcal{V}(G)$ in $\mathcal{X}(G)$ set-theoretically. We also show how the full component structure of $\mathcal{X}(G)$ can be economically described in terms of the rigidity matroid, and show that when $\mathcal{X}(G) = \mathcal{V}(G)$, this variety has Cohen-Macaulay singularities.

We study intensively the case that G is the complete graph K_n. Describing $\mathcal{S}(K_n)$ corresponds to the classical problem of determining all relations among the slopes of the $\binom{n}{2}$ lines connecting n general points in the plane. We prove that the tree polynomials form a Gröbner basis for the affine variety $\tilde{\mathcal{S}}(K_n)$ (with respect to a particular term order). Moreover, the facets of the associated Stanley-Reisner simplicial complex $\Delta(n)$ can be described explicitly in terms of the combinatorics of decreasing planar trees. Using this description, we prove that $\Delta(n)$ is shellable, implying that $\mathcal{S}(K_n)$ is Cohen-Macaulay for all n. Moreover, the Hilbert series of $\tilde{\mathcal{S}}(K_n)$ appears to have a combinatorial interpretation in terms of perfect matchings.

References

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