Simplicial and Cellular Spanning Trees, III: Critical Groups, Cuts and Flows

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The Critical Group of a Graph

$G = (V, E)$: connected loopless graph with n vertices and e edges
$\tau(G)$ = "complexity" = # of spanning trees
∂ = signed vertex-edge incidence matrix
$L = \partial\partial^* = n \times n$ Laplacian matrix (rank $n-1$)

Definition The **critical group** $K(G)$ is the torsion summand of coker $L = \mathbb{Z}^n / \operatorname{im} L$.

- $K(G) = \operatorname{coker} L_i$ for any reduced Laplacian L_i
- Matrix-Tree Theorem $\implies |K(G)| = \tau(G)$

Shows up in several different contexts: graph theory, algebraic geometry, discrete dynamical systems, number theory...

The Critical Group of a Graph

In general, it is hard to determine the structure of $K(G)$ as an abstract group.
Roughly: G more structured $\implies$ $K(G)$ less likely to be cyclic. (E.g., $K(K_n) \cong (\mathbb{Z}_n)^{n-2}$.)
Even if Laplacian eigenvalues are integers (e.g., threshold graphs), they don't necessarily correspond to cyclic summands of $K(G)$.

- Complete bipartite graphs: Lorenzini 1991
- Complete multipartite graphs, products of complete graphs: Jacobson–Niedermayer–Reiner 2002
- Threshold graphs: Christianson–Reiner 2002
- Line graphs: Berget–Manion–Maxwell–Potechin–Reiner 2009

The Chip-Firing Game

Discrete dynamical system: Each vertex has a stack of poker chips. A state of the system is specified by how many chips each vertex has.

If a chip has at least as many chips as its degree, then it can "fire" by donating one chip to each neighboring vertex.

One vertex, the "bank", is allowed to fire if only if no other vertex can fire.

Long-term behavior: The system will eventually reach a critical state in which no non-bank vertex can fire. Once the bank fires, other firings become possible – and in fact each non-bank vertex will fire once, returning to the critical state.

Each firing adds a column of the Laplacian matrix to the current state of the system, so the long-term behavior of the system corresponds to a coset of L.

The critical states are a system of coset representatives for $K(G)$.

The Flow Space

Let A be an abelian group (typically $\mathbb{R}$ or $\mathbb{Z}$ or $\mathbb{Z}_n$).

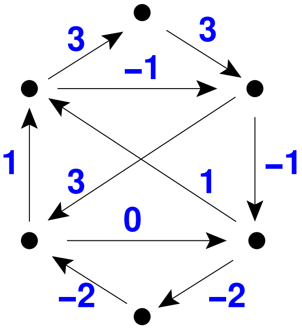
Fix an arbitrary orientation on G so that every edge e has a tail $t(e)$ and a head $h(e)$.

A flow is a function $f : E \to A$ such that

$$\sum_{e:t(e)=v} f(e) = \sum_{e:h(e)=v} f(e)$$

for every vertex v (i.e., inflow = outflow).

The set of flows form a vector space, the *flow space* Flow(G).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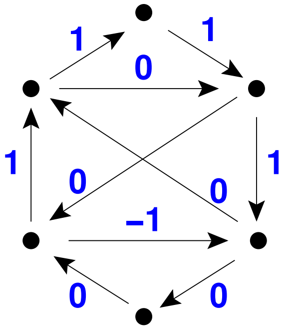
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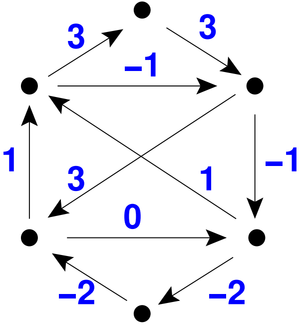
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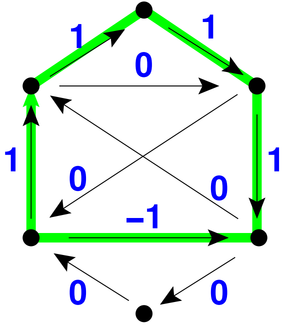
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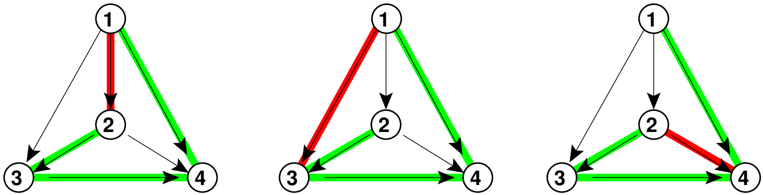
Bases of the Flow Space

Let $T \subseteq G$ be a spanning tree. For each $e \notin G$, let $C(T, e) =$ unique cycle in $T \cup \{e\}$ = *fundamental cycle* of e w.r.t. T.
Theorem: For every T, {signed characteristic vectors of $C(T, e)$: $e \notin T$} is a vector space basis for Flow(G).

Bases of the Flow Space



$$\text{Flow}(G) = \text{rowspace} \begin{array}{cccccc} \color{red}{12} & \color{red}{13} & 14 & 23 & \color{red}{24} & 34 \end{array}$$

$$\begin{pmatrix} \color{red}{1} & \color{red}{0} & -1 & 1 & \color{red}{0} & 1 \\ \color{red}{0} & \color{red}{1} & -1 & 0 & \color{red}{0} & 1 \\ \color{red}{0} & \color{red}{0} & 0 & -1 & \color{red}{1} & -1 \end{pmatrix}$$

Flow lattice: $\ker \partial_G$, considered as a $\mathbb{Z}$-module.

The Flow Space

- $\dim \mathrm{Flow}(G) = e - (n-1) =$ size of the complement of a spanning tree)
- Flows are linear combinations of characteristic vectors of cycles.
- $\mathrm{Flow}(G) = \ker \partial_G =$ space of topological 1-cycles

The Flow Space

$$\mathrm{Flow}(G) = \text{rowspace} \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix}$$

$$\partial_G = \text{rowspace} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

Bases of the Flow Space

Does there exist an everywhere-nonzero flow?

The Cut Space and Cut Lattice

The cut space is $\operatorname{im} \partial^*$.
The cut space is the orthogonal complement of the flow space.
So-called because each bond in G gives rise to a cut vector:
[picture]
$\dim \operatorname{Flow}(G) = e - (n-1)$ (size of the complement of a spanning tree)
For every spanning tree T, we can build a basis of the cut space as follows: for every $e \notin T$, take the signed characteristic vector of the unique bond in $T \cup \{e\}$.

Lattices

A *lattice* is a discrete subgroup $\mathcal{L} \subset \mathbb{R}^n$. (So $\mathcal{L} \cong \mathbb{Z}^n$.) Write $\mathcal{L}_{\mathbb{R}}$ for the vector space spanned by the elements of $\mathcal{L}$.
The *dual lattice* to $\mathcal{L}$ is

$$\mathcal{L}^\sharp = \{v \in \mathcal{L}_{\mathbf{R}} \ \mid \ v \cdot w \in \mathbb{Z} \ \ \forall w \in \mathcal{L}\}.$$

(This is naturally isomorphic to the dual group $\mathcal{L}^* = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{L}, \mathbb{Z})$.)
Ex.: $\mathcal{L} = \mathbb{Z}$, $\mathcal{L}^\sharp = \mathbb{Z}$ Ex.: $\mathcal{L} = 3\mathbb{Z}$, $\mathcal{L}^\sharp = \frac{1}{3}\mathbb{Z}$ Ex.: $\mathcal{L} = \mathbb{Z}\{(1,1)\}$, $\mathcal{L}^\sharp = \mathbb{Z}\{(\frac{1}{2}, \frac{1}{2})\}$
If the rows of M are an integral basis for $\mathcal{L}$, then the rows of $(MM^T)^{-1}M$ are an integral basis for $M^\sharp$.
A lattice LL is *integral* if $\mathcal{L} \subset \mathcal{L}^\sharp$; in this case the *determinant group* is

$$D(\mathcal{L}) = \mathcal{L}^\sharp / \mathcal{L}.$$

In particular, if the rows of M are an integral basis for M, then $|D(\mathcal{L})| = \det(MM^T)$.
E.g., if $\operatorname{rank} \mathcal{L} = 1$ and $\mathcal{L} = \mathbb{Z}v$, then $\mathcal{L}^\sharp = \frac{1}{v \cdot v}\mathcal{L}$.

Cuts, Flows and the Critical Group

Theorem (Bacher, de la Harpe, Nagnibeda):

$$K(G) \cong \mathcal{F}^\sharp/\mathcal{F} \cong \mathcal{C}^\sharp/\mathcal{C} \cong \mathbb{Z}^{E(G)}/(FF \oplus \mathcal{C}).$$

Proof sketch?

Higher-Dimensional Critical Groups

Definition (DKM) Let X be a cell complex of dimension d and let $0 \leq i < d$. The i^{th} critical group of X is

$$K_i(X) = \frac{\ker \partial_i(X)}{\operatorname{im} \partial_{i+1}(X)\partial^*_{i+1}(X)}.$$

- $K_0(X)$ is the (usual) critical group of the 1-skeleton of X.

Theorem (DKM'11): $|K_i(X)| = \tau_{i+1}(X)$.
(Why the funny indexing? Because elements of the i^{th} critical group are naturally represented by cellular i-chains – e.g., in the graph case, configurations of the chip-firing game.)

Higher-Dimensional Cuts and Flows

Definition (DKM; Beck, Godkin, Kemper, ...) Let X be a cell complex of dimension d and let $0 \leq i \leq d$. Flow space: $\mathrm{Flow}_i(X) = \ker \partial_i\,(X; \mathbb{R})$ Cut space: $\mathrm{Cut}_i(X) = \mathrm{im}\, \partial_i^*(X; \mathbb{R})$ Flow lattice: $\mathrm{Flow}_i(X) = \ker \partial_i\,(X; \mathbb{Z})$ Cut lattice: $\mathrm{Cut}_i(X) = \mathrm{im}\, \partial_i^*(X; \mathbb{Z})$

Example: X = hollow tetrahedron $\tau_2(X) = \#$ of spanning trees $=$ 4 (remove a single triangle) Boundary map:

$$\begin{array}{c} \\ 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{array}{c} \begin{array}{cccc} 123 & 124 & 134 & 234 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

$\mathcal{F} = \mathbb{Z}\{(1,-1,1,-1)\}; \mathcal{F}^\sharp = \mathbb{Z}\{(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4})\}$

$$\mathcal{C} = \text{rowspace} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}; \qquad \mathcal{C}^\sharp = \text{rowspace} \begin{bmatrix} 1 & 0 & 0 & \\ 1/4 & -1/4 & 1/4 & 3 \\ 0 & 0 & 1 & \end{bmatrix}$$

Example: $X = \mathbb{RP}^2$ $\partial \ = [2]$ $\mathcal{F} = \mathcal{F}^\sharp = 0$
$\mathcal{C} = 2\mathbb{Z}; \quad \mathcal{C}^\sharp = \frac{1}{2}\mathbb{Z}; \quad |\mathcal{C}^\sharp/\mathcal{C}| = 4$
Meanwhile, the critical group is $K_1(X) = \mathbb{Z}/2\mathbb{Z}$.

More elaborate example: $\partial \; = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$

$\mathcal{F} = \mathbb{Z}\{(1, -2, 1)\}$; $\mathcal{F}^\sharp = \frac{1}{6}\mathcal{F}$

$|\mathcal{C}^\sharp/\mathcal{C}| = \det(\partial \;\; \partial^*) = \det \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix} = 24.$

Meanwhile, $/Zz^3/(\mathcal{F} \oplus \mathcal{C}) = \text{coker} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & -2 & 1 \end{bmatrix} = \mathbb{Z}_{12}.$

Theorem: $K_{i-1}(X) \cong \mathcal{C}^\sharp/\mathcal{C}$.
The cocritical group, defined as $K^*_{i+1}(X) = \operatorname{im} \partial_i^* / \operatorname{im}(\partial_i^* \partial_i)$, is isomorphic to $\mathcal{F}^\sharp/\mathcal{F}$.
The flowcut group $\mathbb{Z}^n/(\mathcal{F} \oplus \mathcal{C})$ interpolates between these via the short exact sequences

$$0 \to \mathbb{Z}^n/(\mathcal{F} \oplus \mathcal{C}) \to \mathcal{C}^\sharp/\mathcal{C} \to T \to 0$$

$$0 \to T \to \mathbb{Z}^n/(\mathcal{F} \oplus \mathcal{C}) \to \mathcal{F}^\sharp/\mathcal{F} \to 0$$

where T is the torsion subgroup of $\tilde{H}_{d-1}(X;\mathbb{Z})$.
(In the graph case, T is trivial and everything in sight is isomorphic.)

Combinatorial Inequalities from Discrete Geometry

(Kotani-Sunada)