Critical Groups of Simplicial Complexes

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Graphs

Let $G = (V, E)$ be a finite, undirected, loopless **graph**:

- V is a finite set of *vertices*
- E is a finite set of *edges*
- Each edge connects two distinct vertices called its *endpoints*
- Parallel edges are allowed

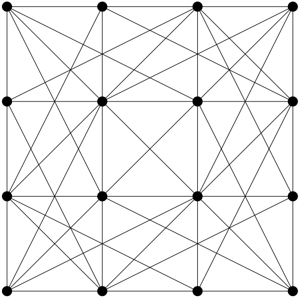
Spanning Trees

Definition A **spanning tree of G** is a set of edges T (or a subgraph (V, T)) such that:

1. (V, T) is **connected**: every pair of vertices is joined by a path
2. (V, T) is **acyclic**: there are no cycles
3. $|T| = |V| - 1$.

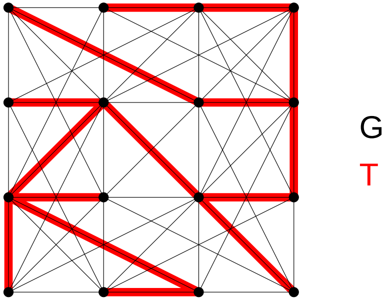
Any two of these conditions together imply the third.

Spanning Trees



G

Spanning Trees



The Matrix-Tree Theorem

Label the vertices of G as $1, 2, \ldots, n$.

Definition The **Laplacian of G** is the $n \times n$ matrix $L = [\ell_{ij}]$:

$$\ell_{ij} = \begin{cases} \deg_G(i) & \text{if } i = j \\ -\#\text{ of edges with endpoints } i, j & \text{otherwise.} \end{cases}$$

- Symmetric $n \times n$ matrix
- $\operatorname{rank} L = n - 1$
- $L = \partial\partial^*$, where $\partial =$ signed vertex-edge incidence matrix $=$ simplicial boundary map

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(1) Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then the number of spanning trees of G is

$$\tau(G) = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}\ .$$

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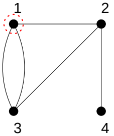
(2) Pick any $1 \leq i \leq n$, and form the *reduced Laplacian* $\tilde{L}$ by deleting the i^{th} row and i^{th} column of L. Then

$$\tau(G) = \det \tilde{L} .$$

The Matrix-Tree Theorem

Example

$G =$



$$L = \begin{bmatrix} 3 & -1 & -2 & 0 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \tilde{L} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Eigenvalues: $0,\ 1,\ 4,\ 5$

$$(1 \cdot 4 \cdot 5)/4 = 5$$

$$\det \tilde{L} = 5$$

Chip-Firing and Sandpiles

- Discrete dynamical system on graphs discovered independently by many: Biggs, Dhar, Merino Lopez, . . .

- Combinatorics: *chip-firing game*
- Theoretical economics: *dollar game*
- Theoretical physics: *abelian sandpile model*
- Related constructions: *rotor-router model*, . . .

- See L. Levine and J. Propp, "What Is. . . A Sandpile?", Notices of the AMS 57, no. 8 (2010)

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- Firing vertex i $=$ subtracting i^{th} column of L from state $\mathbf{c}$

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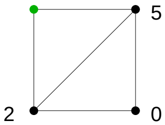

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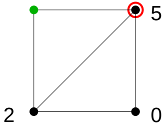

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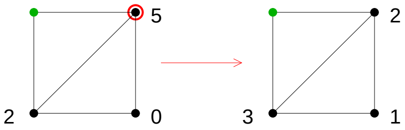

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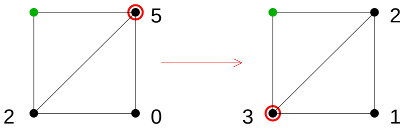

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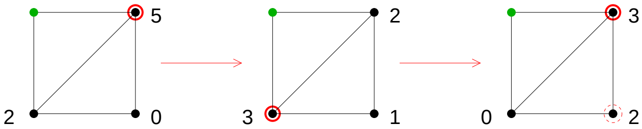

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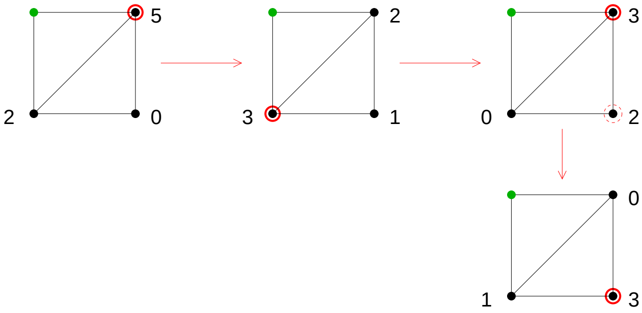

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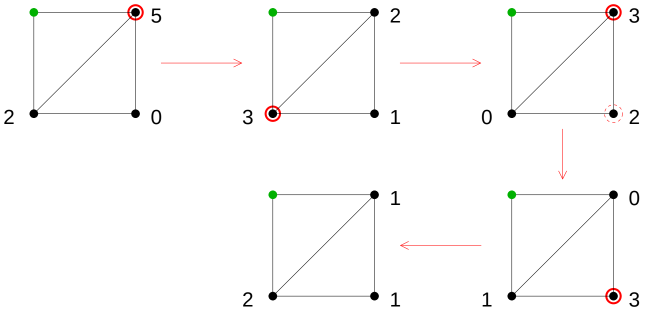

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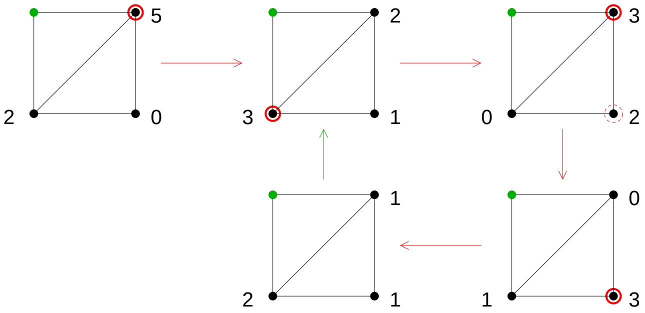

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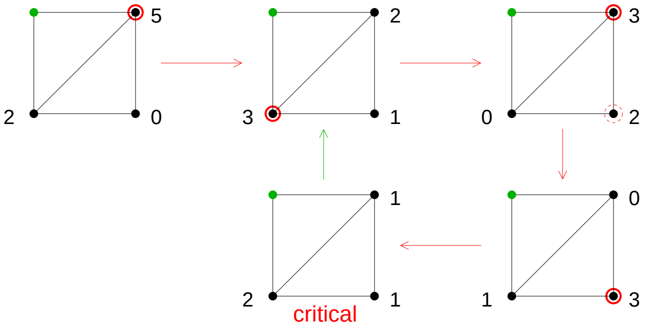

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- The set of critical configurations forms a group $K(G)$ under addition-and-firing.
- Algebraically: $\mathbb{Z}^n/\,\text{colspace}(L) \;\cong\; \mathbb{Z} \oplus K(G)$.
- $K(G)$ is the **critical group** of G.

Simplicial Complexes

Simplicial complex: a set family $\Delta \subseteq 2^{\{1,2,\dots,n\}}$ such that $\sigma \in \Delta,\ \sigma' \subseteq \sigma \implies \sigma' \in \Delta$.

Faces = elements of Δ
Dimension of a face: $\dim \sigma = |\sigma| - 1$
Facet = maximal face
Pure complex: all facets have the same dimension
Ridge = face of codimension 1
i-skeleton: $\Delta_{(i)} = \{\sigma \in \Delta \mid \dim \sigma \leq i\}$

$\tilde{H}_i(\Delta) = i^{th}$ reduced simplicial homology group of Δ over $\mathbb{Z}$

Simplicial Spanning Trees

Henceforth, let Δ be a pure simplicial complex of dimension d.

Definition A **simplicial spanning tree** (SST) is a subcomplex $\Upsilon \subset \Delta$ such that:

1. $\Upsilon \supseteq \Delta_{(d-1)}$;
2. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$;
3. $|\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})| < \infty$.

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  - Contractible complexes $\approx$ acyclic graphs
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  - Some noncontractible complexes also qualify, notably $\mathbb{RP}^2$
- Δ is a simplicial sphere $\implies$ SSTs are $\Delta \setminus \{\text{facet}\}$
  - Simplicial spheres $\approx$ cycle graphs

Combinatorial Laplacians

$\partial : C_d(\Delta) \to C_{d-1}(\Delta)$ simplicial boundary map

$L = \partial\partial^* : C_{d-1}(\Delta) \to C_{d-1}(\Delta)$: (updown) Laplacian

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- τ_1 = number of spanning trees of 1-skeleton

Counting Simplicial Spanning Trees

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- *Problem:* Construct a simplicial analogue of the critical group.

Simplicial Critical Groups

Consider the simplicial chain complex

$$0 \to C_d(\Delta) \xrightarrow{\partial_d} C_{d-1}(\Delta) \xrightarrow{\partial_{d-1}} \cdots$$

Definition The $\mathbf{i^{th}}$ **critical group** of Δ is