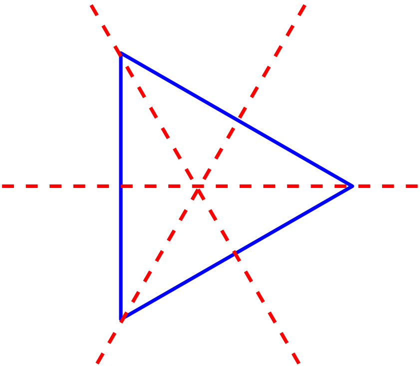
Rigid Motions and Symmetries

A **rigid motion** is the action of taking an object and moving it to a different location without altering its shape or size.

A **symmetry** is a rigid motion that moves the object back onto itself.

- **What sets of symmetries can an object have?**
- **When do two objects have the same set of symmetries?**

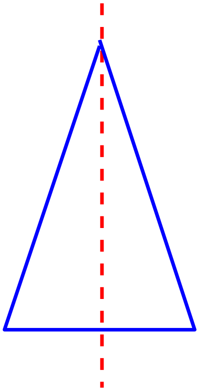
Equilateral: Six symmetries



Three reflections

Three rotations

Isosceles: Two symmetries

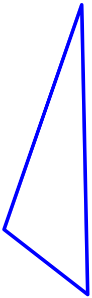


One reflection

One rotation

Symmetries

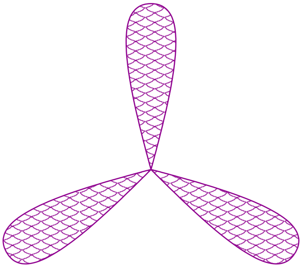
Scalene: One symmetry



No reflections

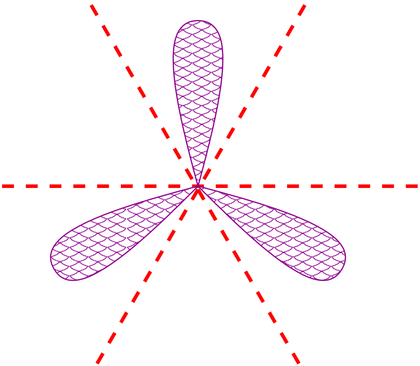
One rotation

Symmetries



Symmetries

Six symmetries

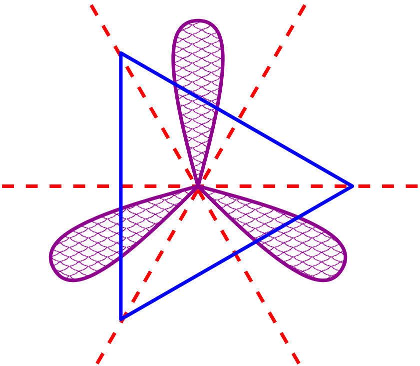


Three reflections

Three rotations

Symmetries

Same symmetry type!



Three reflections

Three rotations

Symmetries

An equilateral triangle and a 3-bladed propeller each have **three rotational symmetries** (by 0°, 120°, and 360°) and **three reflection symmetries**.

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Symmetries

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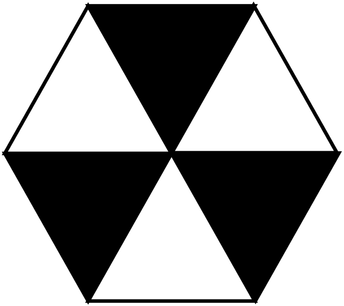
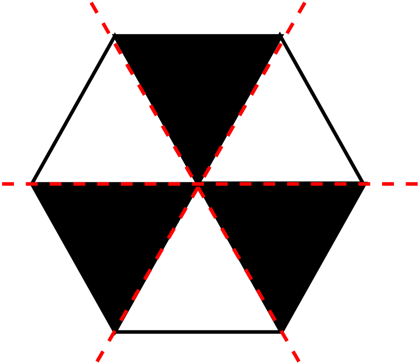
We say that the triangle and the propeller **have the same symmetry type.**
Any object with exactly this set of symmetries is said to have **symmetry type D_3.**

D Symmetry Types



D Symmetry Types

Symmetry type D_3



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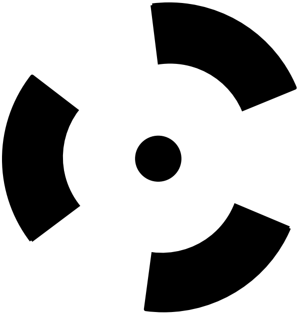
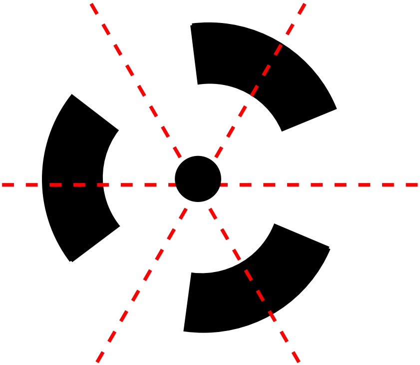
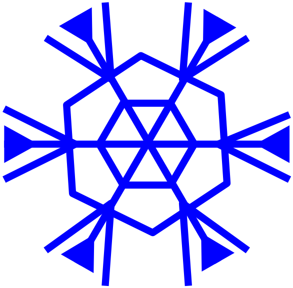


D Symmetry Types

Symmetry type D_3

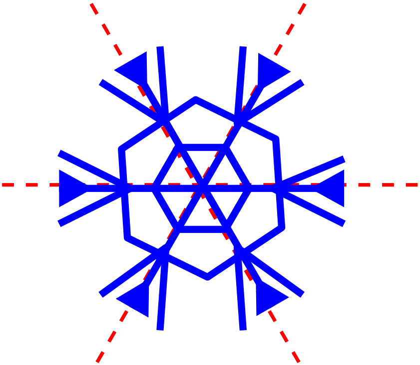


Symmetry type?



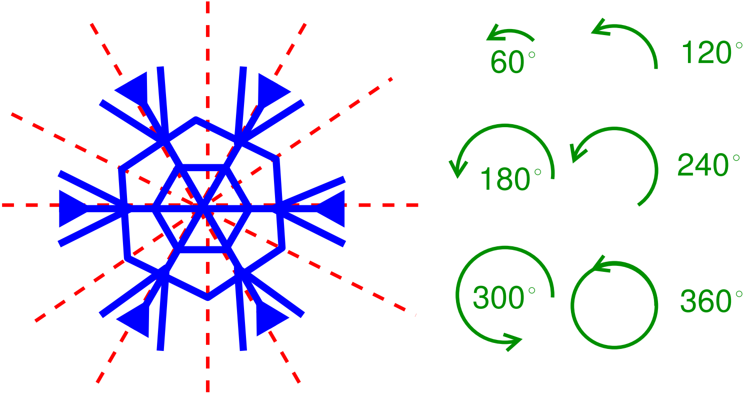
D Symmetry Types

Symmetry type?



D Symmetry Types

Symmetry type D_6



D Symmetry Types

An object has **symmetry type** D_N if has N reflection symmetries and N rotation symmetries (and no others).

D Symmetry Types

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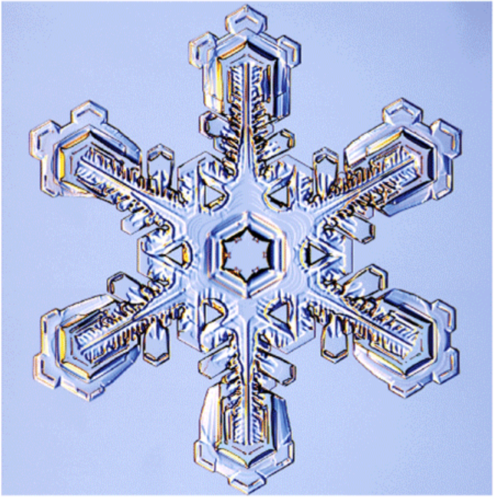

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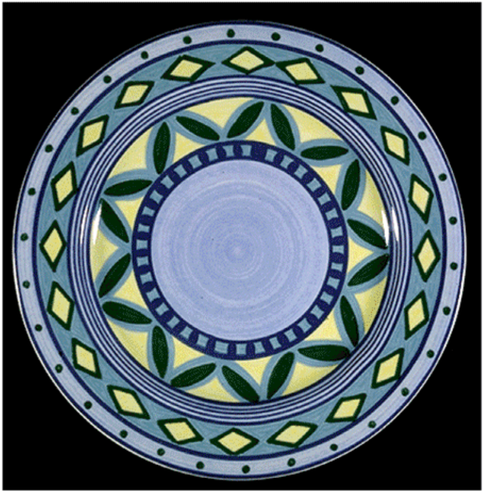

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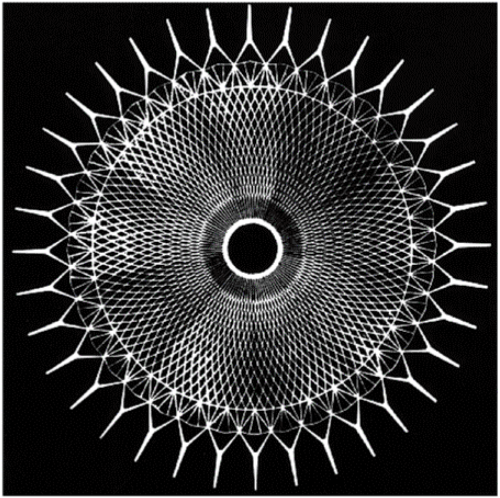
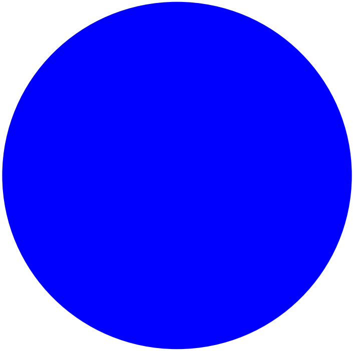


D Symmetry Types

Symmetry type?



Infinitely Many Symmetries!

If we have a circle with center O...

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If we have a circle with center O...

- **Every** line through the point O is an axis of reflection for a symmetry of the circle.

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- So, the circle has **infinitely many symmetries**.

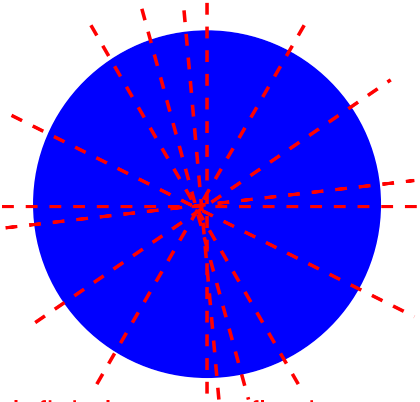
Infinitely Many Symmetries!

If we have a circle with center O...

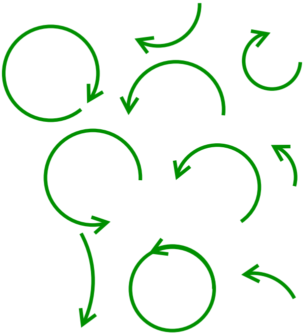
- **Every** line through the point O is an axis of reflection for a symmetry of the circle.
- **Every** rotation with rotocenter O is a symmetry
- So, the circle has **infinitely many symmetries**.

We say that the circle has symmetry type D_∞.

Symmetry type D_∞



Infinitely many reflections



Infinitely many rotations

D Symmetry Types

An object has **symmetry type** D_N if has N reflection symmetries and N rotation symmetries (and no others).

N can be a positive integer, or it can be infinity (∞).

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(Why "D"?)

Symmetry Type D_1

An object with **symmetry type $\mathbf{D_1}$** has:

- one rotation symmetry, which must be the identity motion;
- one reflection symmetry.

D Symmetry Types

- Officially, the “D” stands for “dihedral”.
- The prefix “di” (two) reminds you that an object of symmetry type D_N has **two kinds of symmetries**: reflections and rotations.
- The *number* of symmetries for an object of symmetry type D_N is therefore $2N$.

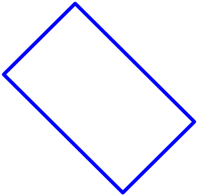
Symmetry Type D_2

An object with **symmetry type $\mathbf{D_2}$** has:

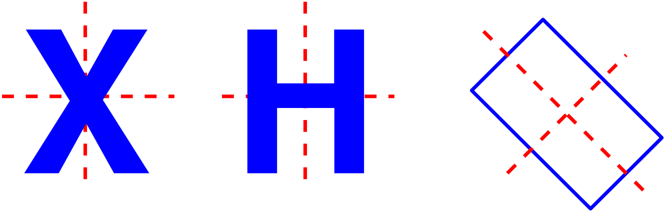
- two rotation symmetries, which must be the identity motion (360°) and a “half-turn” (180°);
- two reflection symmetries, whose axes must meet at a right angle.

Symmetry type D_2

X H



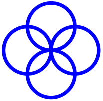
Symmetry type D_2

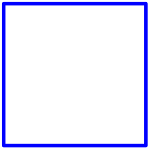


Two reflections
(axes meet at
a right angle)

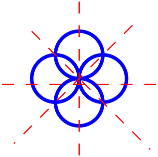
Two rotations
(same rotocenter,
180° and 360°)

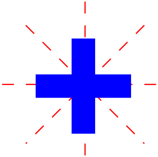
Symmetry type D_4

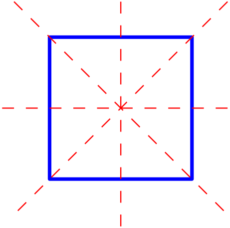




Symmetry type D_4







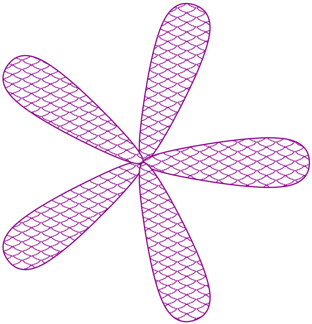
Four reflections
(axes evenly spaced)

Four rotations
(90, 180, 270, 360)

Z Symmetry Types

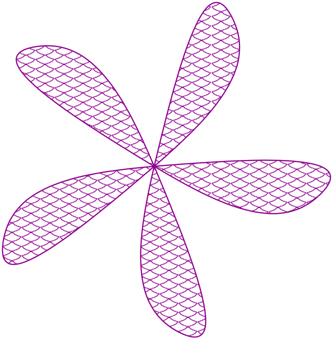
An object can have rotational symmetry but no reflection symmetry.

Typically, this happens when there is something "clockwise" about the object that would be reversed by a reflection.



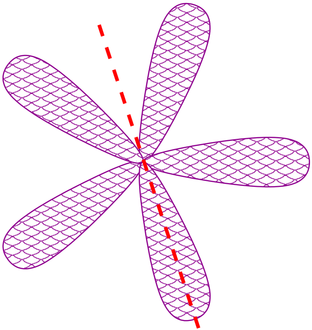
Propeller #1

(has reflection symmetry)



Propeller #2

(no reflection symmetry)



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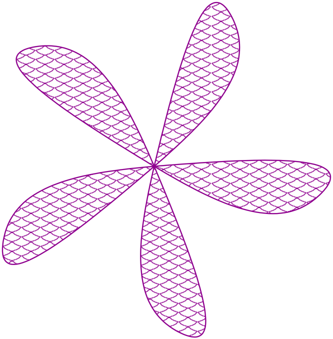


Propeller #2
(no reflection symmetry)

Z Symmetry Types

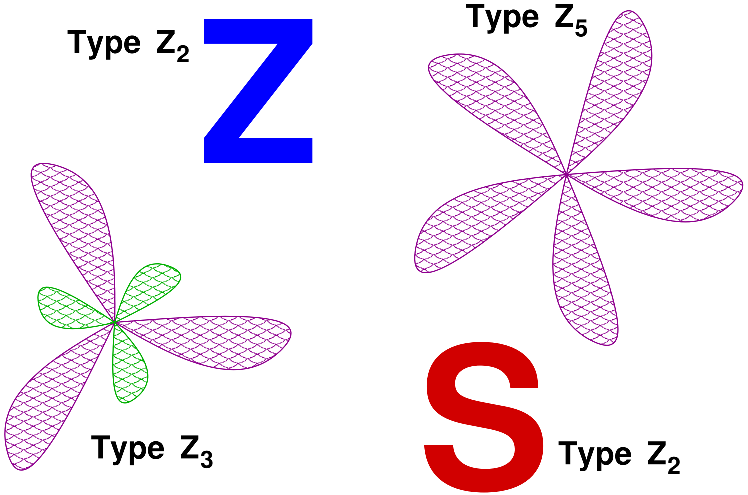
An object with **symmetry type $\mathbf{Z_N}$** has:

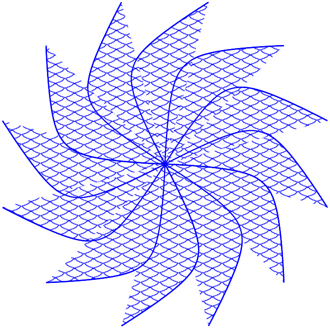
- N rotational symmetries with the same rotocenter, and angles

$$1 \times \frac{360^\circ}{N}, \;\; 2 \times \frac{360^\circ}{N}, \;\; \ldots, \;\; N \times \frac{360^\circ}{N}.$$

- No reflection symmetries.

For example, an object with symmetry type Z_4 has rotational symmetries of 90°, 180°, 270° and 360°.


Type Z_2
Z
Type Z_5
Type Z_3
S
Type Z_2



Type Z_{12}

Type Z_2

96

Type Z_1

R

Summary: D and Z Symmetry Types

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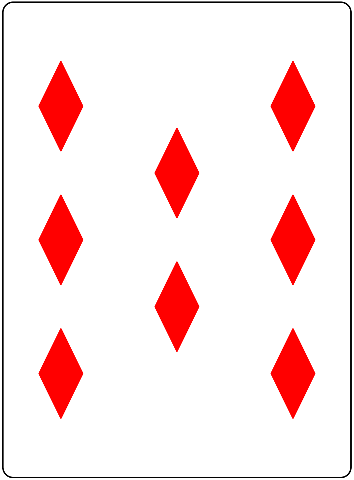
Summary: D and Z Symmetry Types

- An object has **symmetry type $\mathbf{D_N}$** if it has N rotation symmetries and N reflection symmetries (for a total of $2N$ symmetries).
- An object can also have symmetry type D_∞ (e.g., a circle or a disk).
- An object has **symmetry type $\mathbf{Z_N}$** if it has N rotation symmetries and no reflection symmetries.

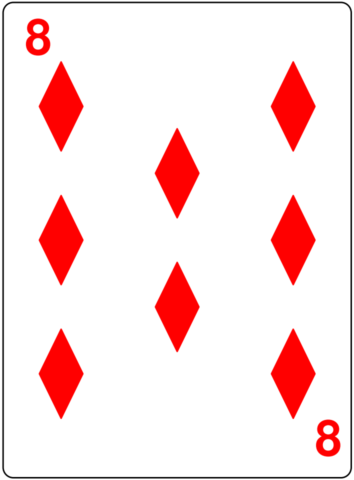
(Mnemonic: The letter D has symmetry type D_1, while the letter Z has symmetry type Z_2.)

Symmetry Types

Symmetry Types



Symmetry Types



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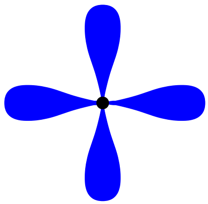

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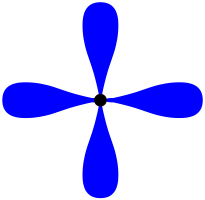
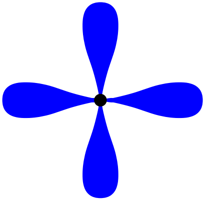


$\mathbf{D_4}$

Symmetry Types



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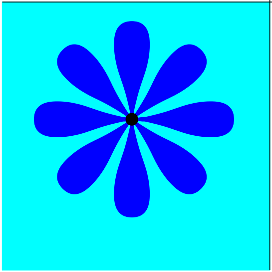

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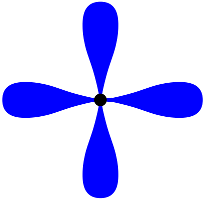
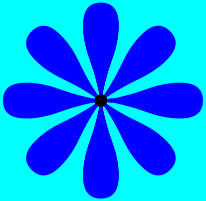
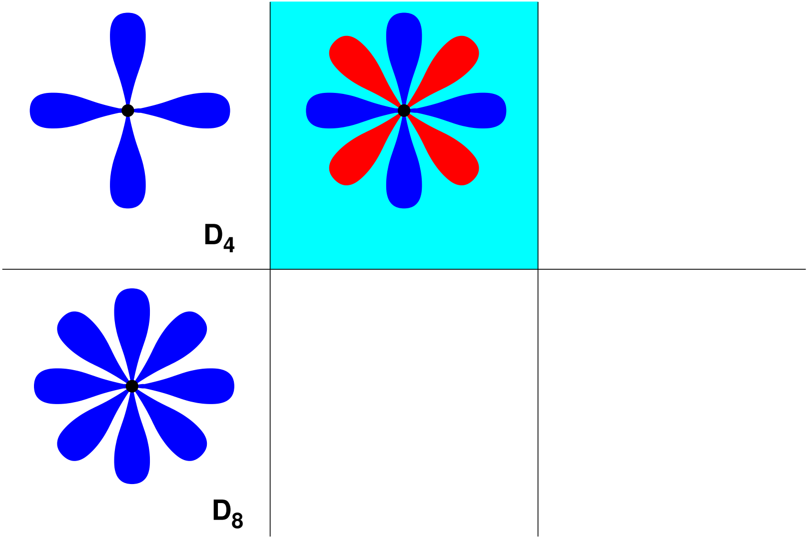


$\mathbf{D_4}$

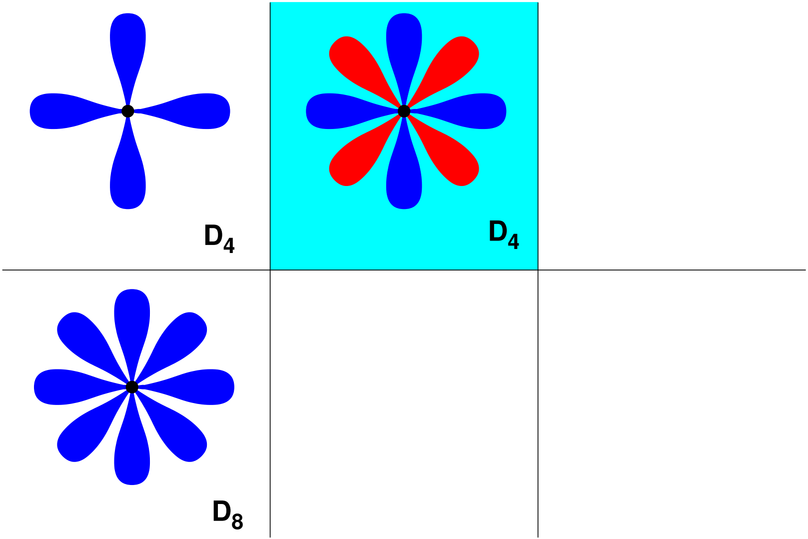


$\mathbf{D_8}$

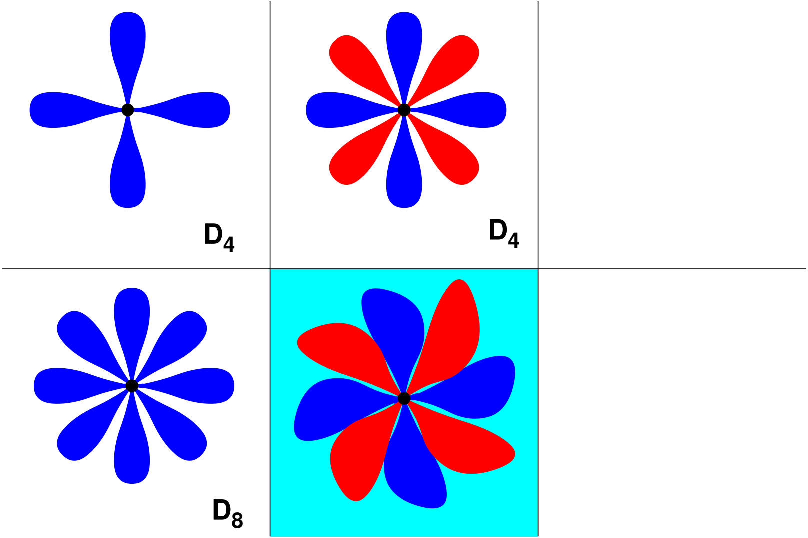
Symmetry Types



Symmetry Types



Symmetry Types



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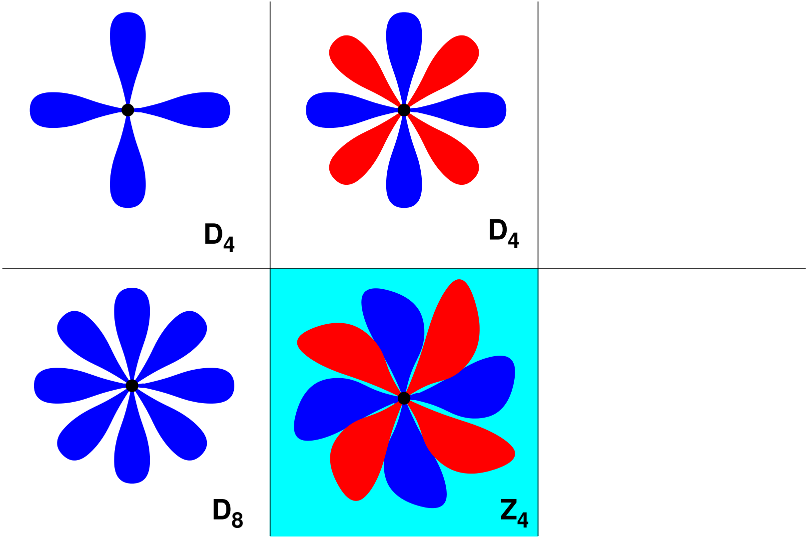

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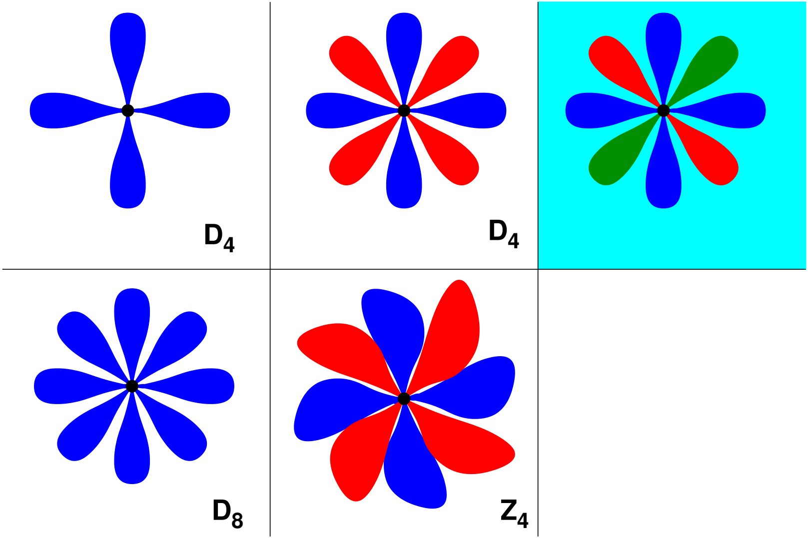

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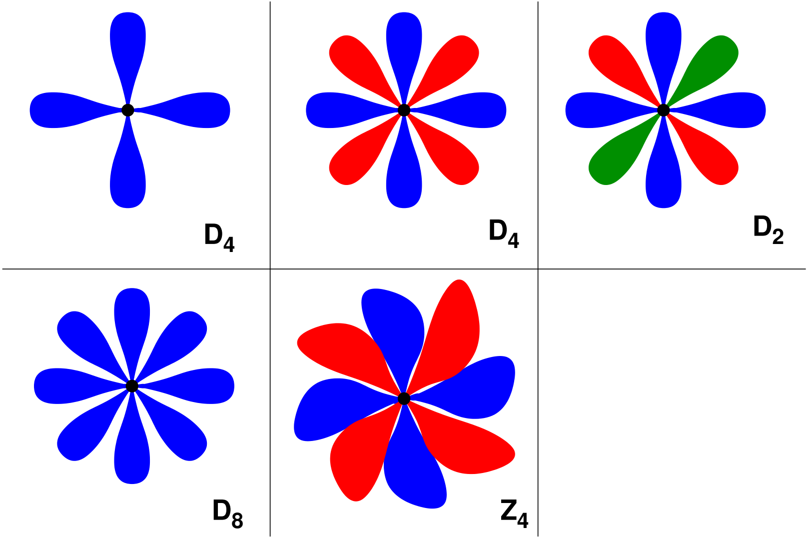

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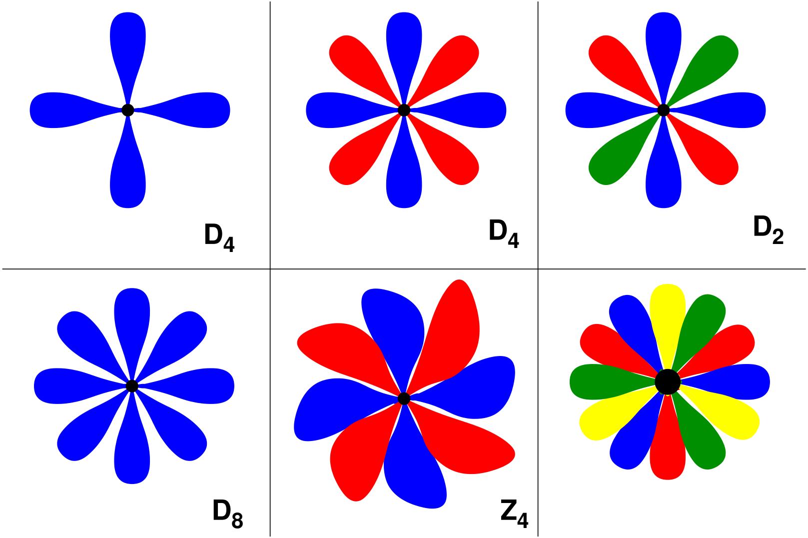

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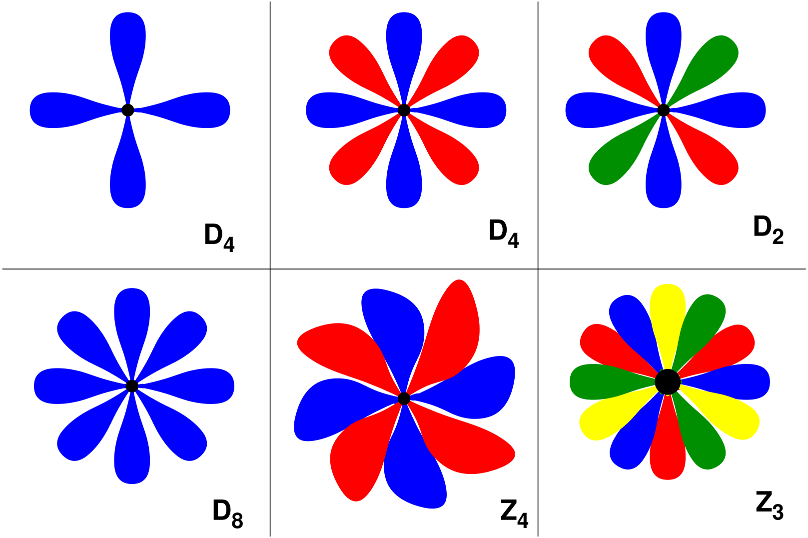

Other Symmetry Types?

- Are there any other symmetry types for 2-dimensional figures?

- In particular, if an object has N rotational symmetries, can its number of reflections be **anything other than** N or 0?

Composition

Definition: If $\mathcal{M}$ and $\mathcal{P}$ are rigid motions, then the **composition** of $\mathcal{M}$ and $\mathcal{P}$ is the rigid motion you get by first doing $\mathcal{M}$, then doing $\mathcal{P}$.

(Notation: $\mathcal{M} \star \mathcal{P}$.)

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- If $\mathcal{M}$ and $\mathcal{P}$ are symmetries of an object, then so is $\mathcal{M} \star \mathcal{P}$.
- $\mathcal{M} \star \mathcal{P}$ is not always the same as $\mathcal{P} \star \mathcal{M}$ (although they can be the same).

Examples of Composition

(All rotations on this page have the same rotocenter.)

- $\mathcal{M} = 30^\circ$ clockwise; $\mathcal{P} = 45^\circ$ clockwise

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- $\mathcal{M} = 30^\circ$ clockwise; $\mathcal{P} = 45^\circ$ clockwise
- $\boldsymbol{\mathcal{M} \star \mathcal{P} = 75^\circ}$ **clockwise**

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- **$\mathcal{M} \star \mathcal{P} = 75^\circ$ clockwise**

- $\mathcal{M} = 30^\circ$ clockwise; $\mathcal{P} = 45^\circ$ counterclockwise

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- $\mathcal{M} = 30^\circ$ clockwise; $\mathcal{P} = 30^\circ$ counterclockwise

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(All rotations on this page have the same rotocenter.)

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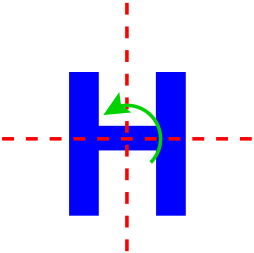
- $\mathcal{M} = 30^\circ$ clockwise; $\mathcal{P} = 30^\circ$ counterclockwise
- $\boldsymbol{\mathcal{M} \star \mathcal{P} =}$ **identity**

Composition of Symmetries of Type D_2

Symmetry type $\mathbf{D_2}$



$\mathcal{A}$: identity

$\mathcal{B}$: rotation 180°

$\mathcal{C}$: vertical reflection

$\mathcal{D}$: horizontal reflection

Composition of Symmetries of Type D_2

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$
$\mathcal{A}$				
$\mathcal{B}$				
$\mathcal{C}$				
$\mathcal{D}$				

(Entry in $\mathcal{M}$th row and $\mathcal{P}$th column is $\mathcal{M} \star \mathcal{P}$)

Composition of Symmetries of Type D_2

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$
$\mathcal{A}$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$
$\mathcal{B}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{D}$	$\mathcal{C}$
$\mathcal{C}$	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{B}$
$\mathcal{D}$	$\mathcal{D}$	$\mathcal{C}$	$\mathcal{B}$	$\mathcal{A}$

(Entry in $\mathcal{M}$th row and $\mathcal{P}$th column is $\mathcal{M} \star \mathcal{P}$)

Each symmetry occurs once in each row and each column.

Composition of Symmetries

The composition of a rotation and a rotation is a rotation.
The composition of a rotation and a reflection is a reflection.
The composition of a rotation and a reflection is a reflection.
The composition of a reflection and a reflection is a rotation.

Composition of Symmetries

A even number plus a even number is even.
A even number plus a odd number is odd.
A even number plus a odd number is odd.
A odd number plus a odd number is even.

Composition of Symmetries

A positive number times a positive number is positive.
A positive number times a negative number is negative.
A positive number times a negative number is negative.
A negative number times a negative number is positive.

Composition of Symmetries

Composition

	Rotation	Reflection
Rotation	Rotation	Reflection
Reflection	Reflection	Rotation

Addition

	Even	Odd
Even	Even	Odd
Odd	Odd	Even

Multiplication

	Positive	Negative
Positive	Positive	Negative
Negative	Negative	Positive

Composition of Symmetries

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$
$\mathcal{A}$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$
$\mathcal{B}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{D}$	$\mathcal{C}$
$\mathcal{C}$	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{B}$
$\mathcal{D}$	$\mathcal{D}$	$\mathcal{C}$	$\mathcal{B}$	$\mathcal{A}$

Composition of Symmetries

	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$
$\mathcal{A}$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$
$\mathcal{B}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{D}$	$\mathcal{C}$
$\mathcal{C}$	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{B}$
$\mathcal{D}$	$\mathcal{D}$	$\mathcal{C}$	$\mathcal{B}$	$\mathcal{A}$

If we write down the "multiplication table" of symmetries of any object with reflection symmetry, then half the entries will be blue and half red.

Possible Symmetry Types

- In the “multiplication table” of symmetries of any object **with** reflection symmetry, half the entries are blue and half are red.
- In the “multiplication table” of symmetries of any object **without** reflection symmetry, all the entries are blue.

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Punchline: Any two-dimensional object with N rotation symmetries has either N or 0 reflection symmetries.

Therefore, the only two-dimensional symmetry types are D_N and Z_N.