How Helical Can a Closed, Twisted Space Curve be?

Author(s): Joel L. Weiner

Source: *The American Mathematical Monthly*, Vol. 107, No. 4 (Apr., 2000), pp. 327-333
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2589177

How Helical Can a Closed, Twisted Space Curve Be?

Joel L. Weiner

1. THE QUESTION. A helix is a space curve (or the set traced by a curve) $\mathbf{h}: \mathbb{R} \to \mathbb{R}^3$ defined by

$$\mathbf{h}(t) = (a \cos t, a \sin t, bt),$$

where a and b are nonzero constants. These conditions on a and b do not permit a straight line or a circle to be a helix. See Figure 1 for the graph of a helix where the z-axis is pointing more or less to the right. The velocity of $\mathbf{h}$ is given by

$$\mathbf{h}'(t) = (-a \sin t, a \cos t, b)$$

and has the notable property that it makes a constant angle with the unit vector $\hat{\mathbf{k}} = (0, 0, 1)$. For if θ is the angle between $\mathbf{h}'(t)$ and $\hat{\mathbf{k}}$, then

$$\cos \theta = \frac{\hat{\mathbf{k}} \cdot \mathbf{h}'(t)}{|\mathbf{h}'(t)|} = \frac{b}{\sqrt{a^2 + b^2}},$$

where $|\mathbf{h}'(t)|$ denotes the length of $\mathbf{h}'(t)$.

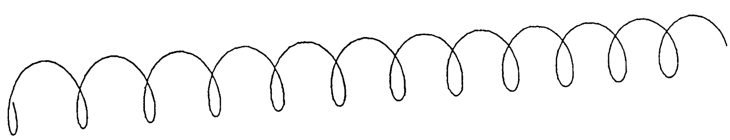

Figure 1

We consider only space curves $\mathbf{x}: \mathbb{R} \to \mathbb{R}^3$ that are regular, that is, have a parametrization whose derivative is never 0. This being so, we can always assume the parametrization used to define $\mathbf{x}$ has this property. In fact, we can go one step further and assume $\mathbf{x}$ is parametrized by arc length s, i.e.,

$$|\mathbf{x}'(s)| = 1, \quad \text{for all } s.$$

For this parametrization we call $\mathbf{x}'$ the (unit) *tangent vector field* along the curve.

Using our observation about helices as motivation, one calls a regular space curve $\mathbf{x}: \mathbb{R} \to \mathbb{R}^3$ a *generalized helix* if its tangent vector field $\mathbf{x}'$ makes a constant angle θ with a fixed unit vector $\hat{\mathbf{u}}$, where $0 < \theta < \pi/2$. Such curves can never be closed; this means that $\mathbf{x}$ cannot be a periodic function of its arc length parameter. This is so since the tangent vector field always has a positive component in the direction of $\hat{\mathbf{u}}$. The question we ask is how "close" to a generalized helix can a closed space curve be. To make this question more precise we need to consider the differential geometry of space curves. In particular, we first introduce the curvature and torsion of a space curve and then use them to describe what it means for a space curve to be a generalized helix.

From here on we deal only with regular space curves $\mathbf{x}: \mathbb{R} \to \mathbb{R}^3$ that have at least four continuous derivatives. Let $\mathbf{t}$ denote $\mathbf{x}'$. It is possible that $\mathbf{t}'(s) = \mathbf{0}$ for some s, however, we assume that this never happens. Then we can introduce a unique unit vector field $\mathbf{n}$ and positive function κ so that $\mathbf{t}' = \kappa\mathbf{n}$. We call $\mathbf{t}'$ the *curvature vector field*, $\mathbf{n}$ the *principal normal vector field*, and κ the *curvature*. Since $\mathbf{t}$ is a constant length vector field one may show that $\mathbf{n}$ is orthogonal to $\mathbf{t}$. The *binormal vector field* $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is a unit vector field that is orthogonal to both $\mathbf{t}$ and $\mathbf{n}$. One defines the *torsion* τ by the equation $\mathbf{b}' = -\tau\mathbf{n}$. Of course one has to show that $\mathbf{b}'$ is a scalar multiple of $\mathbf{n}$. This is done in some detail in most calculus texts that treat functions of several variables [**1**, pp. 206–208].

We have computed the derivative of every vector field introduced except $\mathbf{n}$. That can be done easily; again see [**1**]. We present all these derivatives in a set of equations known as the Serret-Frenet equations:

$$\begin{aligned} \mathbf{t}' &= \phantom{-\kappa\mathbf{t}} \quad \kappa\mathbf{n} \\ \mathbf{n}' &= -\kappa\mathbf{t} \qquad\qquad + \tau\mathbf{b} \\ \mathbf{b}' &= \phantom{-\kappa\mathbf{t}} \quad -\tau\mathbf{n} \end{aligned}$$

We now use the Serret-Frenet equations to prove a well known characterization of generalized helices involving their curvature and torsion. The proof is easy and identifies a vector field that is important in the proof of the main theorem.

Proposition 1. *A space curve* $\mathbf{x}$ *is a generalized helix if and only if the ratio* τ/κ *is a non-zero constant.*

Proof: First, let's assume that τ/κ is constant. Let $\mathbf{a} = (\tau/\kappa)\mathbf{t} + \mathbf{b}$. Differentiating $\mathbf{a}$ with respect to arclength and employing the Serret-Frenet equations shows that $\mathbf{a}' = \mathbf{0}$. Thus we may regard $\mathbf{a}$ as a nonzero vector. Then $(\mathbf{a} \cdot \mathbf{t})' = \mathbf{a} \cdot (\kappa\mathbf{n}) = 0$. This, of course, means that $\mathbf{t}$ makes a constant angle with $\mathbf{a}$. It is clear from the form of $\mathbf{a}$ that this angle is not 0 or $\pi/2$. Hence $\mathbf{x}$ is a generalized helix.

Now let's assume that $\mathbf{x}$ is a generalized helix. Let $\mathbf{u}$ be a vector such that $\mathbf{u} \cdot \mathbf{t} = c$, a constant. Differentiating this equation with respect to arc length gives $\mathbf{u} \cdot (\kappa\mathbf{n}) = 0$; thus the principal normal vector field $\mathbf{n}$ is everywhere orthogonal to $\mathbf{u}$. Regard $\mathbf{u}$ as a (constant) vector field along $\mathbf{x}$. Its expression as a unique linear combination of the orthonormal basis $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ takes the form $\mathbf{u} = f\mathbf{t} + g\mathbf{b}$ for appropriate functions f and g. Neither f nor g is zero, since $\mathbf{t}$ is never orthogonal or parallel to $\mathbf{u}$. We differentiate and obtain

$$\mathbf{0} = f'\mathbf{t} + (\kappa f - \tau g)\mathbf{n} + g'\mathbf{b}.$$

Thus f and g are non-zero constants and $\tau/\kappa = f/g$ is a non-zero constant as well. ∎

From the proof we observe that a generalized helix makes a constant angle with $\mathbf{a} = (\tau/\kappa)\mathbf{t} + \mathbf{b}$.

We say a space curve $\mathbf{x}$ is *twisted* if its torsion is nonzero somewhere; of course, because the torsion is continuous this condition implies that it is nonzero on a interval. If a curve lies in a plane then its torsion must be identically zero, so a twisted space curve cannot lie in a plane. We are now ready to state our question in a somewhat more precise form. How constant is the ratio τ/κ for a closed twisted space curve? Actually we examine the question of how constant the ratio κ/τ is, since it is for this ratio that we intend to prove a theorem. It is possible for κ/τ to be undefined since τ may be zero. If, however, τ is zero somewhere then κ/τ blows up so κ/τ cannot in any reasonable sense be close to a constant function. Until we get to the main theorems, we consider only curves for which, say, $\tau > 0$. Finally let's denote κ/τ by ϕ.

2. SOME EXPERIMENTS. A closed space curve $\mathbf{x}: \mathbb{R} \to \mathbb{R}^3$ parametrized by arc length s is a periodic function of period L, where L is the length of the curve. It is often convenient to restrict the domain of $\mathbf{x}$ and all related functions of geometric interest to $[0, L]$. Clearly, the values and derivatives of all such functions agree at 0 and L.

How should we measure how constant ϕ is? The most natural quantity to consider first is the difference $B[\phi] = \max_{[0,L]}\phi - \min_{[0,L]}\phi$, which measures the breadth of the graph of ϕ. To see how small we can make $B[\phi]$, notice that a curve that winds in a uniform fashion around an anchor ring seems sort of helical; see Figure 2. Recall that an anchor ring is the surface one gets by revolving about the z-axis a circle in the x,z-plane that does not meet the z-axis. Such a curve looks more helical, close up, the smaller the radius of the circle and the larger the number of times it winds around the anchor ring. Consider, for example, the closed curves $\mathbf{x}_n : [0, 2\pi n] \to \mathbb{R}^3$ defined by

$$\mathbf{x}_n(t) = \left(\left(1 - \frac{\cos(t)}{n}\right)\cos\left(\frac{t}{n}\right), \left(1 - \frac{\cos(t)}{n}\right)\sin\left(\frac{t}{n}\right), \frac{\sin(t)}{n}\right).$$

For each n, $\mathbf{x}_n$ is a curve that winds n times about the anchor ring obtained by revolving the circle with center $(1, 0)$ and radius $1/n$ about the z-axis. In fact, Figure 2 is the graph of $\mathbf{x}_{50}$.

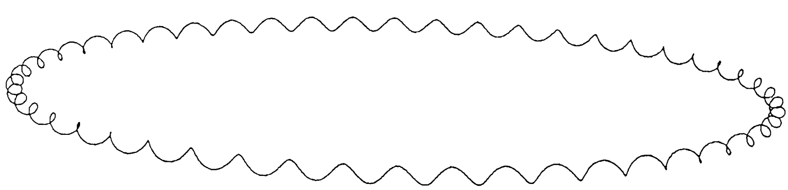

Figure 2

For the curves $\mathbf{x}_n$, the variable t is not the arclength, but one can use the following formula for ϕ:

$$\phi = \frac{|\mathbf{x}' \times \mathbf{x}''|^3}{(\mathbf{x}' \cdot \mathbf{x}'' \times \mathbf{x}''')\,|\mathbf{x}'|^3},$$

where the prime, as always, denotes differentiation with respect to the argument of the given function $\mathbf{x}(t)$.

Let ϕ_n denote the ratio of curvature to torsion for $\mathbf{x}_n$. Using *Mathematica* one can find a formula for ϕ_n and by means of simple estimates obtain upper and lower bounds of the following form on the values of ϕ_n:

$$\frac{(16n^6 + \cdots)^{\frac{3}{2}}}{2n(4n^2 + \cdots)^{\frac{3}{2}}(4n^5 + \cdots)} \le \phi_n(t) \le \frac{(16n^6 + \cdots)^{\frac{3}{2}}}{2n(4n^2 + \cdots)^{\frac{3}{2}}(4n^5 + \cdots)},$$

where $\cdots$ denotes lower order terms in n. Since the limits of the upper bound and lower bound are 1 as $n \to \infty$, we see that $\lim_{n\to\infty} B[\phi_n] = 0$. From this point of view there are closed twisted space curves arbitrarily close to a generalized helix.

The curves $\mathbf{x}_n$ that make $B[\phi_n]$ small do so at a cost. Their ϕ_n wiggle a lot. A rotation of space by $2\pi/n$ about the z-axis transforms the trace of $\mathbf{x}_n$ onto itself. This leads to the observation that $\phi_n(t + 2\pi) = \phi_n(t)$. Thus the values $\phi_n(t)$ rise and fall between their extremes at least $2n$ times as one makes a complete circuit of the curve $\mathbf{x}_n$. The total variation of ϕ on [0, L] measures the extent to which a function $\phi : [0, L] \to \mathbb{R}$ wiggles. It is defined as follows: A partition of $[0, L]$ is a finite ordered set $P = \{0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = L\}$. Let

$$V(\phi, P) = \sum_{k=1}^{n} |\phi(s_k) - \phi(s_{k-1})|.$$

Then the *total variation* of ϕ, $TV[\phi]$, is defined by

$$TV[\phi] = \sup_P V(\phi, P).$$

If you imagine that $\phi(s)$ represents the position of a point at time s, then $TV[\phi]$ gives the total distance traveled by the point over the time interval $[0, L]$. This observation ought to make the following proposition believable if you think of $|\phi'|$ as giving the speed of the point. The proof of this proposition is a direct application of the Mean Value Theorem and the definition of the Riemann integral, if $\mathbf{x}$ has four continuous derivatives.

Proposition 2.

$$TV[\phi] = \int_0^L |\phi'(s)|\, ds.$$

The total variation of ϕ is independent of parametrization. If the substitution $s = s(t)$, $a \le t \le b$, satisfies the conditions that $s([a, b]) = [0, L]$ and ds/dt is never zero, then

$$\int_a^b \left| \frac{d\phi}{dt} \right| dt = \int_0^L \left| \frac{d\phi}{ds} \right| ds.$$

Because of this one may write $TV[\phi] = \int |d\phi|$.

It is clear that the total variation of a function is 0 if and only if the function is constant. Thus we can use total variation as a measure of the extent to which a function is not constant. Once again we restate our question: How close to zero is $TV[\phi]$?

To see how close to zero we can make $TV[\phi]$ we experiment with curves like the $\mathbf{x}_n$ but we allow the radius of the anchor ring and the number of windings to be independent of each other. Define $\mathbf{x}_{n,k} : [0, 2\pi n] \to \mathbb{R}^3$ by

$$\mathbf{x}_{n,k}(t) = \left(\left(1 - \frac{\cos(t)}{kn}\right) \cos\left(\frac{t}{n}\right), \left(1 - \frac{\cos(t)}{kn}\right) \sin\left(\frac{t}{n}\right), \frac{\sin(t)}{kn} \right).$$

Note that $\mathbf{x}_{n,1} = \mathbf{x}_n$. Let $\phi_{n,k}$ denote the ratio of curvature to torsion for $\mathbf{x}_{n,k}$. Again we compute $\phi_{n,k}$ with the help of *Mathematica* and obtain the following formula where $\cdots$ stands for lower order terms in n and k, which also may contain a single factor of the form $\cos(mt)$, where $m = 1$, 2, 3, or 4.

$$\phi_{n,k}(t) = \frac{(8k^6n^2 + 8k^4n^4 - 16k^3n^5\cos(t) + \cdots)^{\frac{3}{2}}}{2n(4k^2n^2 + \cdots)^{\frac{3}{2}}(4kn^5 - 4k^2n^4\cos(t) + \cdots)}.$$

For the moment regard $\mathbf{x}_{n,k}$ and $\phi_{n,k}$ as functions with domain $\mathbb{R}$ by extending them to be a periodic of period $2\pi n$. Let R_θ^z denote the rotation of θ about the

z-axis. Then one checks that $R^z_{2\pi/n}(\mathbf{x}_{n,k}(t)) = \mathbf{x}_{n,k}(t + 2\pi)$; since curvature and torsion are invariant under Euclidean motions, $\phi_{n,k}$ has period 2π. If R^x_θ is the rotation of θ about the x-axis, then $R^x_\pi(\mathbf{x}_{n,k}(t)) = \mathbf{x}_{n,k}(-t)$; hence $\phi_{n,k}$ is an even function. Thus $\phi_{n,k}$ is completely determined by its restriction to the interval $[0, \pi]$. Because of the symmetries in $\phi_{n,k}$ just mentioned both 0 and π are critical points of $\phi_{n,k}$. Viewing the graphs of some examples of $\phi_{n,k}$ indicates that 0 is a local minimum and π is a local maximum of $\phi_{n,k}$. (In fact it appears that when $n \geq k$ and both are large that these are the only critical points in $[0, \pi]$ and so are global extremes.) It is thus reasonable to consider the partition P of $[0, 2\pi n]$ consisting of all the multiples of π ranging from 0 to $2\pi n$. Since $\phi_{n,k}(\pi) > \phi_{n,k}(0)$ it follows that $V(\phi_{n,k}, P) = 2n(\phi_{n,k}(\pi) - \phi_{n,k}(0))$.

Figure 3 shows the graph of $V(\phi_{n,k}, P)$ as a function of (continuous variables) n and k over the set $[1000, 10000] \times [1000, 10000]$. One also finds that $\lim_{n\to\infty} V(\phi_{n,k}, P) = 8(1 + k^{-2})$. For these results we availed ourselves of *Mathematica* again. This suggests that $\inf_{n>0,\,k>0} V(\phi_{n,k}, P) = 8$, and we have other evidence to support this assertion. This implies that $TV[\phi_{n,k}] > 8$. Examining $\phi_{n,n}$ in particular, one can show for sufficiently large n that 0 and π are its only critical points in $[0, \pi]$. Thus $TV[\phi_{n,n}] = V[\phi_{n,n}, P]$. Finally one may check, again using *Mathematica*, that $\lim_{n\to\infty} TV[\phi_{n,n}] = 8$. Thus we are led to the following:

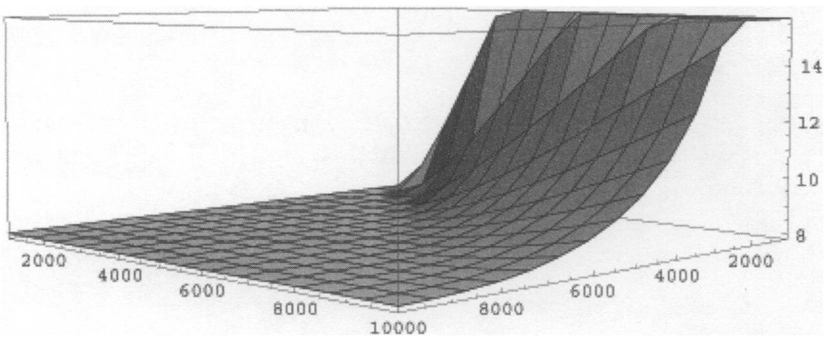

Figure 3

Conjecture. $TV[\phi] > 8$ *for every closed twisted space curve.*

3. THE THEOREMS. We know that the lower bound in the Conjecture can be no higher. The best we can say so far is in Theorem 1. In it we agree that $TV[\phi] = \infty$ if the torsion is zero somewhere. This is consistent with the fact that ϕ blows up as we approach a point where the torsion is zero from points where the torsion is not zero.

Theorem 1. $TV[\phi] > 4$ *for every closed twisted space curve.*

Proof: We assume from the outset that the torsion is positive. We introduce the vector field $\tilde{\mathbf{a}} = \phi\mathbf{a} = \mathbf{t} + \phi\mathbf{b}$. Then $\tilde{\mathbf{a}} \cdot \mathbf{x} = \mathbf{t} \cdot \mathbf{x} + \phi(\mathbf{b} \cdot \mathbf{x})$. Differentiating with respect to s and taking into account the Serret-Frenet equations and the definition of ϕ gives

$$(\tilde{\mathbf{a}} \cdot \mathbf{x})' = 1 + \phi'(\mathbf{b} \cdot \mathbf{x}).$$

We restate this equation using differentials:

$$d(\tilde{\mathbf{a}} \cdot \mathbf{x}) = ds + d\phi(\mathbf{b} \cdot \mathbf{x}).$$

Since $(\tilde{\mathbf{a}} \cdot \mathbf{x})(0) = (\tilde{\mathbf{a}} \cdot \mathbf{x})(L)$, integration over the interval $[0, L]$ gives

$$-\int_0^L (\mathbf{b} \cdot \mathbf{x})\, d\phi = L.$$

In [**3**] it is shown by a simple but elegant argument that any smooth closed curve of length L is contained within a sphere of radius $L/4$. We place the origin of our coordinate system at the center of such a sphere. Since $|\mathbf{x}| < L/4$, it's the case that $|\mathbf{b} \cdot \mathbf{x}| < L/4$. Then, using this upper bound for $|\mathbf{b} \cdot \mathbf{x}|$ and the fact that the absolute value of an integral is less than the integral of the absolute value, the last equation gives

$$\frac{L}{4}\int_0^L |d\phi| > L,$$

i.e., $TV[\phi] > 4$. ■

The literature on space curves contains a couple of examples of theorems of the sort proved here. The contrapositive of a theorem by Oliker and Volkov [**2**] can be interpreted as giving an answer to the question of how constant individually the curvature and torsion can be. However, it applies to a restricted class of curves. The curve's torsion must be positive and the curve's length is restricted from above by the magnitude of the curvature and torsion.

An earlier theorem of the author [**4**] can be interpreted as another answer to the question posed here. We rephrase "how constant is the ratio κ/τ" as "to what extent is κ a scalar multiple of τ". We now think of κ and τ as members of the vector space of all continuous real-valued functions on the interval $[0, L]$. By means of the inner product

$$\langle f, g \rangle = \int_0^L f(s) g(s)\, ds$$

defined on that space one can measure the extent to which one function is a scalar multiple of another. The Cauchy-Schwarz inequality ensures that

$$1 - \frac{\left(\int_0^L \kappa\tau\, ds\right)^2}{\int_0^L \kappa^2\, ds \int_0^L \tau^2\, ds} \geq 0 \tag{1}$$

with equality if and only if κ and τ are scalar multiples of one another. We may use the left side of (1) as a measure of how far from constant the ratio κ/τ is. The author's theorem alluded to [**4**] may be restated so that the left side of (1) appears in its formulation.

Theorem 2. *If* $\mathbf{x}$ *is a closed twisted space curve, then*

$$1 - \frac{\left(\int_0^L \kappa\tau\, ds\right)^2}{\int_0^L \kappa^2\, ds \int_0^L \tau^2\, ds} > \frac{16}{L\int_0^L \kappa^2\, ds}.$$

REFERENCES

1. S. J. Colley, *Vector Calculus*, Prentice-Hall, Upper Saddle River, NJ, 1998.
2. V. I. Oliker and Y. A. Volkov, A stability theorem for unbounded curves in $\mathbb{R}^3$ with periodic curvature and torsion, 303–310, in: *Pitman Monographs Surveys Pure Appl. Math.* **52**, Longman Sci. Tech., Harlow, 1991.
3. H. Rutishauser and H. Samelson, Sur le rayon d'une sphere dont la surface contient une courbe fermée, *C. R. Acad. Sci. Paris* **227** (1948) 755–757.
4. J. L. Weiner, An inequality involving the length, curvature, and torsions of a curve in Euclidean n-space, *Pacific J. Math.* **74** (1978) 531–534.

JOEL L. WEINER is Professor of Mathematics at the University of Hawaii at Manoa. He received his Ph.D. in mathematics from UCLA in 1971. In 1995 the University of Hawaii at Manoa awarded him a Presidential Citation for Meritorius Teaching. His research is concerned primarily with the study of curves and surfaces in spaces of constant curvature. More recently, he is also interested in affine differential geometry.
University of Hawaii at Manoa, Honolulu, HI 96816
joel@math.hawaii.edu

From *The New York Times*

It has been some time since I bothered to read anything about the Information Revolution. In common with most people I know, I have more pressing matters to think about, like reading the latest issues of American Mathematical Monthly. I keep up as best I can, of course; I have an E-mail account and shall get a personal Web page, oh, sometime soon....

From "A Library That's All Too Free", an Op-Ed piece by John Derbyshire
The New York Times, Tuesday, August 24, 1999, p. A15