The Hodge Structure of the Coloring Complex of a Hypergraph

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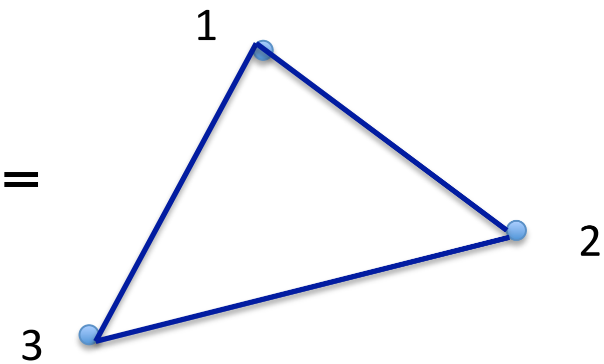
Outline:

I. Coloring Complex of a Graph
II. The Chromatic Polynomial of a Graph
III. The Eulerian Idempotents
IV. The Hodge Structure of the Coloring Complex of a Hypergraph

Let G be a graph on n vertices.

$$\Delta_r = \left\{ (B_1, B_2, ..., B_{r+2}) \;\middle|\; \begin{array}{c} (B_1, B_2, ..., B_{r+2}) \;\; is \;\; an \;\; ordered \;\; partition \\ of \;\; \{1, 2, ..., n\} \;\; and \;\; B_i \;\; contains \\ an \;\; edge \;\; of \;\; G \;\; for \;\; some \;\; i \end{array} \right\}$$

Ex.

$G =$ 

Δ_0	Δ_{-1}
$(12,3)$	(123)
$(3,12)$	
$(13,2)$	
$(2,13)$	
$(1,23)$	
$(23,1)$	

Boundary maps

Let C_r denote the vector space with basis Δ_r .

Define a map $\partial_r : C_r \to C_{r-1}$ by:

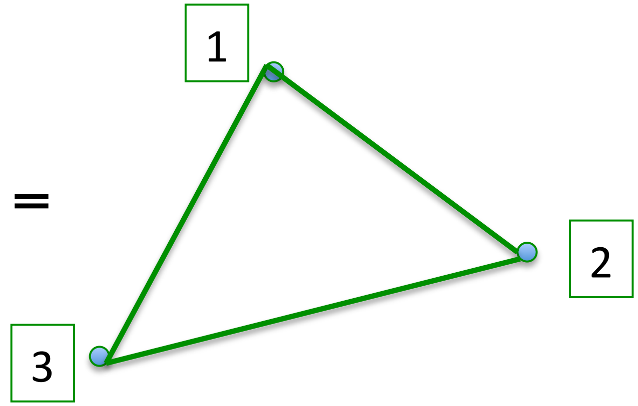
$$\partial_r\left(B_1,\ldots,B_{r+2}\right) = \sum_{i=1}^{r+1} (-1)^{i+1}\left(B_1,B_2,\ldots,B_i \cup B_{i+1},\ldots,B_{r+2}\right)$$

Homology of the Coloring Complex

$$H_r(\Delta(G)) = {}^{\ker(\partial_r)}\!\big/_{\mathrm{Im}(\partial_{r+1})}$$

$$\dim(H_r(\Delta(G))) = \dim(\ker(\partial_r)) - \dim(\mathrm{Im}(\partial_{r+1}))$$

Ex.

$G =$ 

$\dim\left(\mathrm{Im}(\partial_1)\right) = 0$

$\dim\left(\mathrm{Im}(\partial_0)\right) = 1$

$\dim\left(\ker(\partial_{-1})\right) = 1$

$\dim\left(\ker(\partial_0)\right) = 5$

$\dim\left(H_0(\Delta(G))\right) = 5 - 0 = 5$

$\dim\left(H_{-1}(\Delta(G))\right) = 1 - 1 = 0$

Acyclic Orientation

Let G(V,E) be a graph. G is a *directed graph* if each edge *e* in E has an orientation from one of its vertices to the other vertex.

An *acyclic orientation* of G is an orientation of the edges of G so that G has no cycles.

Theorem [Jonsson]

Let G be a simple graph with at least one edge. Then

$$\dim\left(H_{n-3}(\Delta(G))\right) = a_G - 1$$

where a_G denotes the number of acyclic orientations of G, and

$$\dim\left(H_r(\Delta(G))\right) = 0, \qquad -1 \le r \le n-4.$$

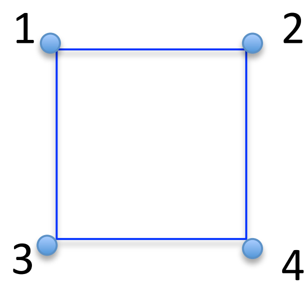
Graph Coloring

A *proper k-coloring* of a graph G(V,E) is a function $f : V \to \{1,\ldots,k\}$ such that $f(u) \neq f(v)$ if uv is an edge of G.

Chromatic polynomial

The *chromatic polynomial,* $\chi_G(\lambda)$, evaluated at λ , gives the number of ways to properly color a graph G, with at most λ colors.

Ex.

G = 

$$\chi_G(\lambda) = \lambda(\lambda-1)^2 + \lambda(\lambda-1)(\lambda-2)^2$$

$$= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda$$

Def. The *Eulerian idempotents,* $e_n^{(1)}, e_n^{(2)}, \cdots, e_n^{(n)}$, are given by

$$\sum_{j=1}^{n} r^j e_n^{(j)} = \sum_{\pi \in S_n} \binom{n+r-des(\pi)-1}{n} \operatorname{sgn}(\pi)\, \pi$$

Ex.

$$e_3^{(1)} = \frac{1}{6}(2 \cdot id + 213 + 132 - 231 - 312 - 2 \cdot 321)$$

$$e_3^{(2)} = \frac{1}{2}(id + 321)$$

$$e_3^{(3)} = \frac{1}{6}(id - 213 - 132 + 231 + 312 - 321)$$

Lemma [Hanlon].

For each $r, j,$

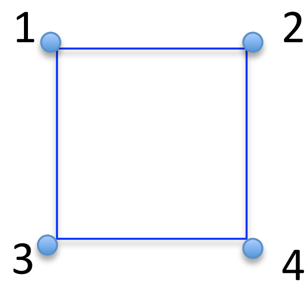
$$\partial_r \circ e_{r+2}^{(j)} = e_{r+1}^{(j)} \circ \partial_r .$$

Punch line…

Fix j. The set of $C_r^{(j)}(\Delta(G)) = e_{r+2}^{(j)} \cdot C_r(\Delta(G))$ form a subcomplex of $(C_*(\Delta(G)), \partial_*)$. Let $H_*^{(j)}(\Delta(G))$ be the homology of this subcomplex. Then

$$H_*(\Delta(G)) = \bigoplus_{j=1}^{n-1} H_*^{(j)}(\Delta(G)).$$

Ex.



$$\chi_G(\lambda) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda$$

$$\dim(\, H_1^{(3)}(\Delta(G))\,) = 4 \qquad \dim(\, H_1^{(2)}(\Delta(G))\,) = 6 \qquad \dim(\, H_1^{(1)}(\Delta(G))\,) = 3$$

Theorem. [Hanlon, 2008]

Let G be a simple graph with n vertices and at least one edge. For each j, the dimension of the j^{th} Hodge piece in the homology of $\Delta(G)$ is equal to the absolute value of the j^{th} coefficient of the chromatic polynomial of G.

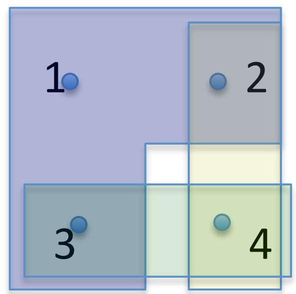
i.e.

$$\dim(H_{n-3}^{(j)}(\Delta(G))) = \left|\left[\lambda^j\right]\chi_G(\lambda)\right|$$

$$= \left[\lambda^j\right](-1)^n(\chi_G(-\lambda) - (-\lambda)^n)$$

Def. A *hypergraph,* H, is a pair (X,E), where X is a set of vertices and E is a set of non-empty subsets of X. A hypergraph is *uniform of rank r* if the edges of H have size r.

Ex.



$$H = (\{1,2,3,4\}, E)$$
$$E = \{123, 34, 24\}$$

Δ_1	Δ_0	Δ_{-1}
(1, 2, 34)	(3, 124)	(1234)
(1, 34, 2)	(124, 3)	
(2, 1, 34)	(13, 24)	
(2, 34, 1)	(24, 13)	
(34, 1, 2)	(12, 34)	
(34, 2, 1)	(34, 12)	
(1, 24, 3)	(123, 4)	
(1, 3, 24)	(4, 123)	
(3, 1, 24)	(134, 2)	
(3, 24, 1)	(2, 134)	
(24, 3, 1)	(1, 234)	
(24, 1, 3)	(234, 1)	

Ex.

$$\dim(\, C_1^{(3)}(\Delta(H))\,) = 2$$

$$\dim(\, C_1^{(2)}(\Delta(H))\,) = 6 \quad \dim(\, C_0^{(2)}(\Delta(H))\,) = 6$$

$$\dim(\, C_1^{(1)}(\Delta(H))\,) = 4 \quad \dim(\, C_0^{(1)}(\Delta(H))\,) = 6 \quad \dim(\, C_{-1}^{(1)}(\Delta(H))\,) = 1$$

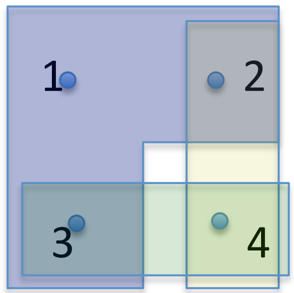
$$\dim(\, H_1^{(3)}(\Delta(H))\,) = 2$$

$$\dim(\, H_1^{(2)}(\Delta(H))\,) = 1 \quad \dim(\, H_0^{(2)}(\Delta(H))\,) = 1$$

$$\dim(\, H_1^{(1)}(\Delta(H))\,) = 0 \quad \dim(\, H_0^{(1)}(\Delta(H))\,) = 1 \quad \dim(\, H_{-1}^{(1)}(\Delta(H))\,) = 0$$

Chromatic Polynomial of a Hypergraph

The *chromatic polynomial,* $\chi_H(\lambda)$, evaluated at λ, gives the number of ways to color a hypergraph H, with at most λ colors, so that the vertices of each edge are colored with at least 2 colors.



$$\chi_H(\lambda) = \lambda(\lambda-1)^2 + \lambda^2(\lambda-1)(\lambda-2)$$
$$= \lambda^4 - 2\lambda^3 + \lambda$$

Euler Characteristic:

$$X^{(d)}(\Delta(H)) = \sum_{i=-1}^{n-r-1}(-1)^i \dim(\, C_i^{(d)}(\Delta(H)\,))$$

$$X^{(3)}(\Delta(H)) = -2$$

$$X^{(2)}((H)) = 0$$

$$X^{(1)}(\Delta(H)) = 1$$

Theorem [Long and R.]

Let H be a hypergraph with n vertices and at least one edge. For each j,

$$X^{(j)}(\Delta(H)) = -\left[\lambda^{j}\right](\chi_{H}(-\lambda) - (-\lambda)^{n}).$$

Question:

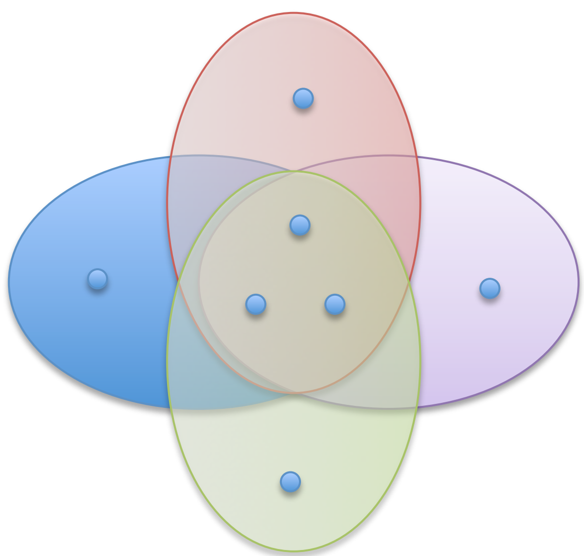
For which hypergraphs H is it the case that, for each j, there exists at most one k for which $\dim(H_k^{(j)}(\Delta(H)))$ is nonzero, and therefore

$$\dim(H_k^{(j)}(\Delta(H))) = (-1)^{k+1}\left[\lambda^j\right](\chi_H(-\lambda) - (-\lambda)^n)\,?$$

Def.

Let H be a uniform hypergraph of rank r. H is a *star hypergraph* if all of its edges intersect in a common set of size r-1.

Ex.



Theorem [Long and R.].

If H is a star hypergraph, then the homology of $\Delta(H)$ is nonzero only in dimension n-r-1. In particular,

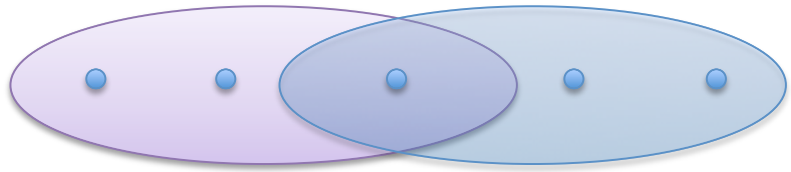
$$\begin{aligned}\dim(H^{(j)}_{n-r-1}(\Delta(H))) &= (-1)^{n-r}\left[\lambda^j\right](\chi_H(-\lambda)-(-\lambda)^n) \\ &= (-1)^{n-r}\left[\lambda^j\right](-\lambda(-\lambda-1)^{n-(r-1)}-(-\lambda)^{n-(r-1)})\end{aligned}$$

Theorem [Long and R.].

Let H be a uniform hypergraph of rank $r \neq 2$ with $n \leq 5$ vertices. If each edge of H intersects at least one other hyperedge of H in a set of size r-1, then the homology of $\Delta(H)$ is nonzero only in dimension n-r-1. In particular,

$$\dim(H_{n-r-1}^{(j)}(\Delta(H))) = (-1)^{n-r}\left[\lambda^j\right](\chi_H(-\lambda) - (-\lambda)^n).$$

Ex.

H = 

$$\dim(\ H_1^{(3)}(\Delta(H))\ ) = 2$$

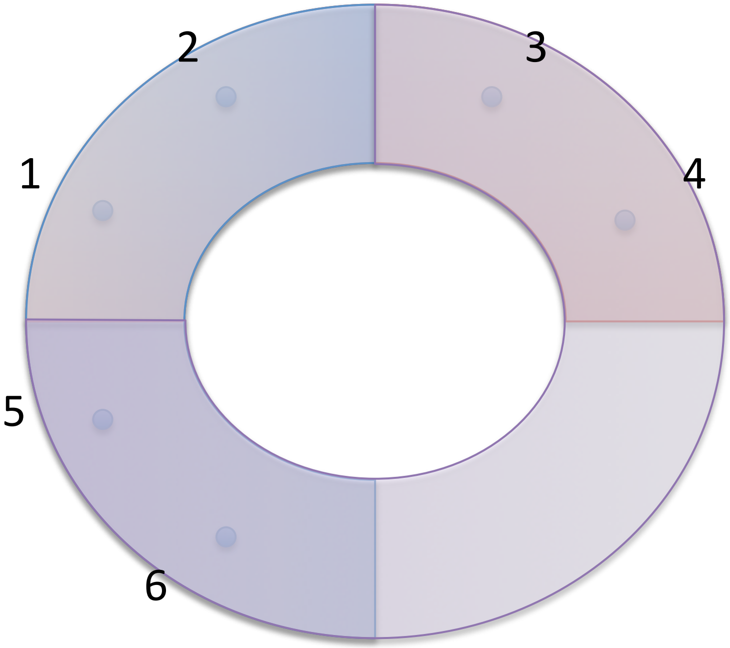
$$\dim(\ H_1^{(2)}(\Delta(H))\ ) = 0 \quad \dim(\ H_0^{(2)}(\Delta(H))\ ) = 0$$

$$\dim(\ H_1^{(1)}(\Delta(H))\ ) = 0 \quad \dim(\ H_0^{(1)}(\Delta(H))\ ) = 1 \quad \dim(\ H_{-1}^{(1)}(\Delta(H))\ ) = 0$$

Theorem [Long and R.].

Let H be a uniform hypergraph of rank $r \neq 2$ having $n \leq 5$ vertices. Then all Hodge subcomplexes of $\Delta(H)$ have at most one nonzero homology group.

Ex.



$$H = (\{1,2,3,4,5,6\}, E)$$

$$E = \{1234, 1256, 3456\}$$

$$\dim(\, H_1^{(3)}(\Delta(H))\,) = 3$$

$$\dim(\, H_1^{(2)}(\Delta(H))\,) = 3 \quad \dim(\, H_0^{(2)}(\Delta(H))\,) = 3$$

$$\dim(\, H_1^{(1)}(\Delta(H))\,) = 0 \quad \dim(\, H_0^{(1)}(\Delta(H))\,) = 2 \quad \dim(\, H_{-1}^{(1)}(\Delta(H))\,) = 0$$

Open Questions

- Is it possible to determine formulas for the dimensions of the homology groups of the coloring complex of an arbitrary uniform hypergraph?
- Is there a condition that can be placed on a uniform hypergraph with 6 or more vertices that will make each of the Hodge subcomplexes have homology concentrated in one dimension?