On the Rank Partition of a Matroid

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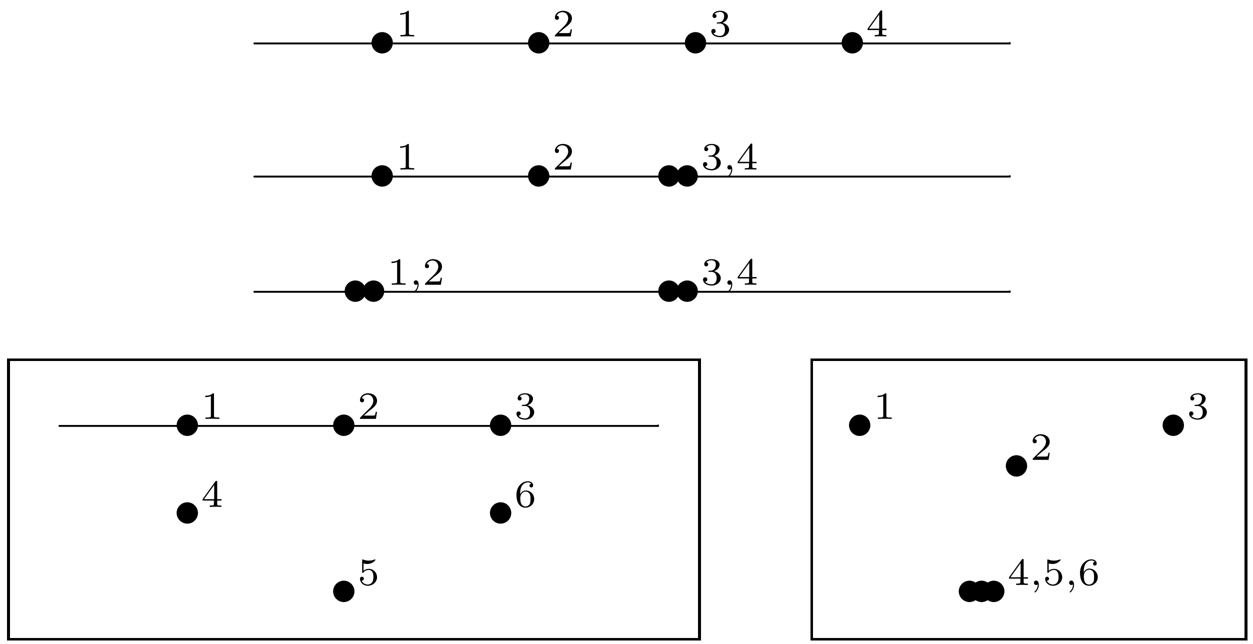
Affine point configurations



give rise to matroids: A subset I of the indexing set is *independent* if and only if the points indexed by I are affinely independent. A maximal independent set is called a *base*.

In general, M will be a matroid with ground set E. Define a sequence $\rho(M) = (\rho_1, \rho_2, \rho_3, \ldots)$ by the condition that

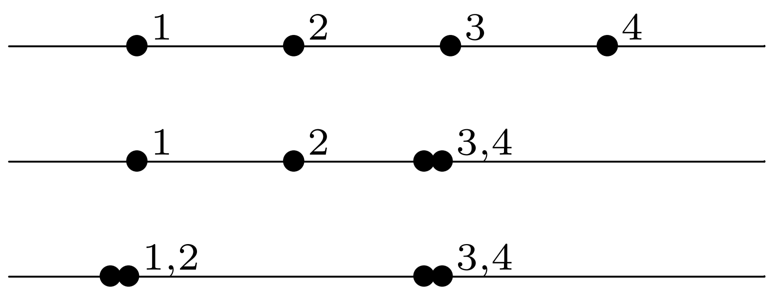
$$\rho_1 + \rho_2 + \cdots + \rho_k = \#(\text{largest union of } k \text{ indep. sets from } M)$$

This is the *rank partition* of M. *A priori* any sequence could occur as $\rho(M)$.

Theorem (Dias da Silva)**.** *For any matroid, $\rho(M)$ is a partition, i.e., $\rho_1 \geq \rho_2 \geq \rho_3 \ldots$.*

Suppose M has no loops. There is a set partition of E into independent sets of M whose sizes are $\lambda_1 \geq \lambda_2 \geq \ldots$ if and only if $\lambda \leq \rho(M)$.

We'll always assume no loops.



$$\overset{\rho}{\rightsquigarrow} (2,2)$$

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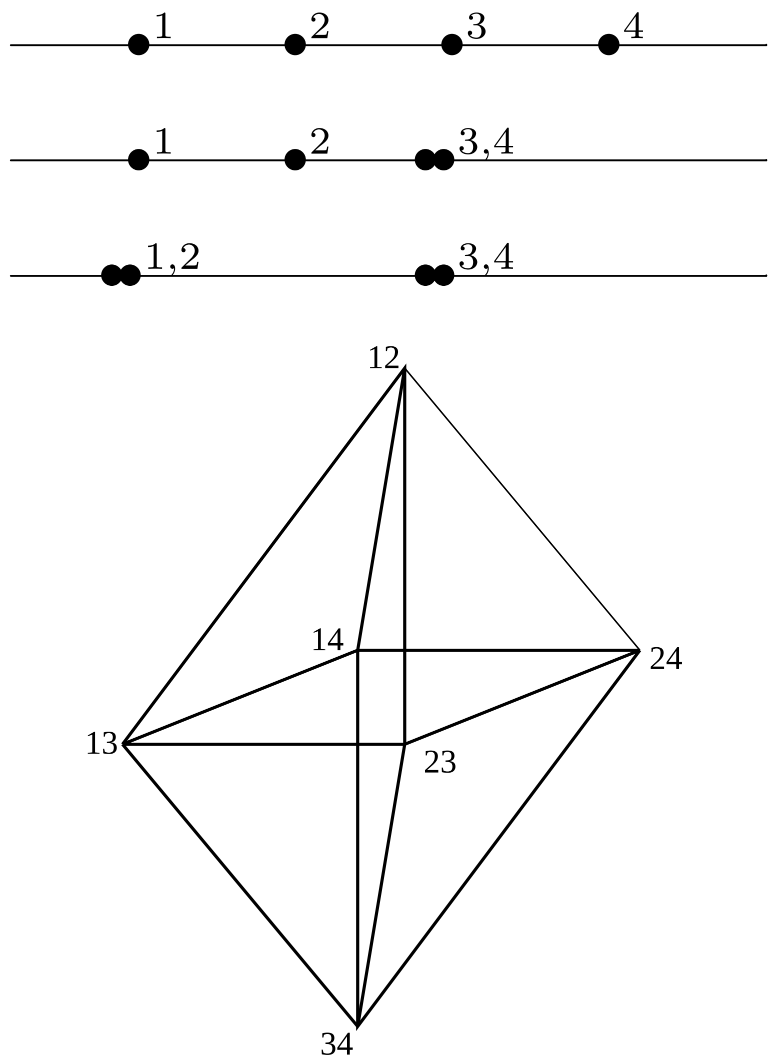
Taking parallel extensions of uniform matroids, any partition can appear as $\rho(M)$.

The rank partition is poorly behaved under nearly all matroid constructions. For example, duality is nearly impossible to nicely describe.

Goal of the talk is to show that matroid base polytopes “see” $\rho(M)$ in a nice way, and explain why this is interesting. What is a matroid base polytope? Two answers:

1. It is the moment polytope of the closure a torus orbit of a point in a Grassmannian representing M.
2. It is $P(M) := \mathrm{conv}\{\sum_{i \in B} e_i : B \in \mathrm{bases}(M)\} \subset \mathbb{R}^E$.

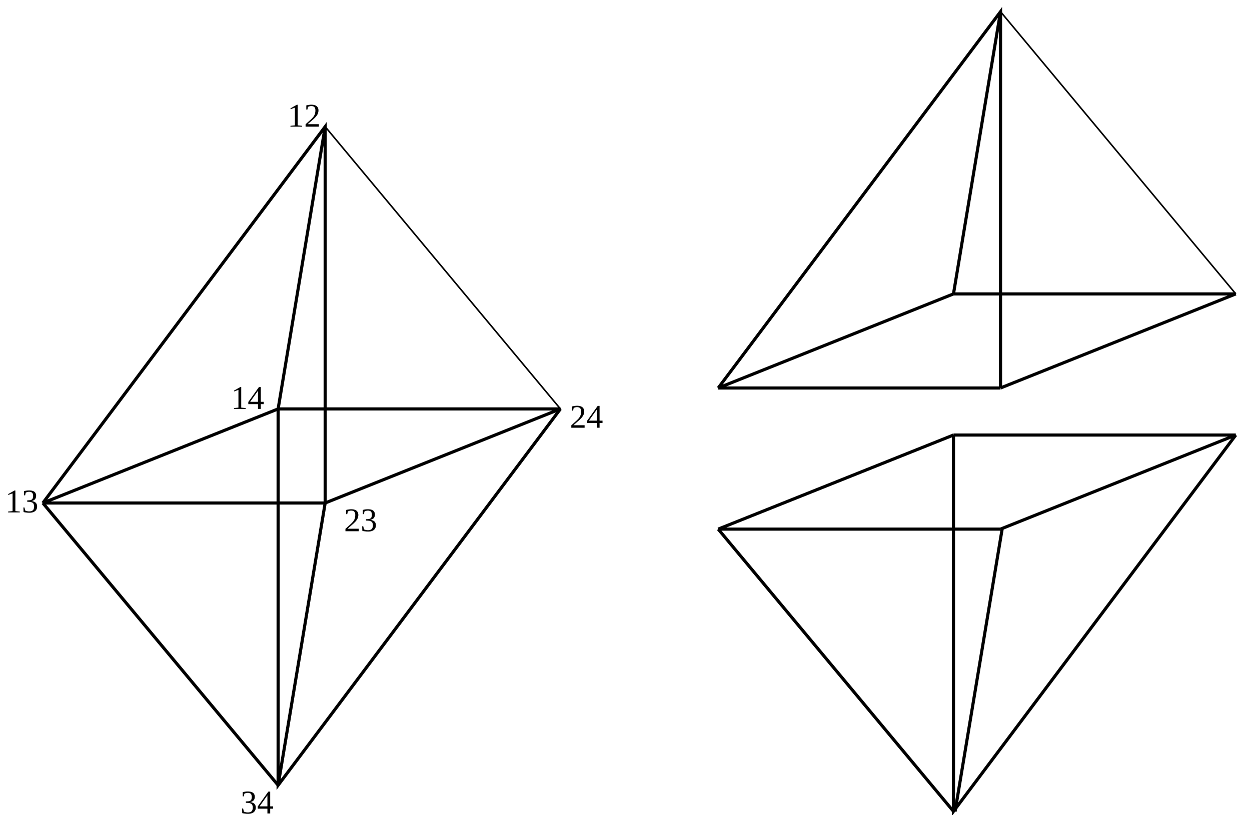

1
2
3
4
1
2
3,4
1,2
3,4
12
14
24
13
23
34

A Matroid base Polytope Decomposition (MPD) is a way of writing

$$P(M) = \bigcup_i P(M^{(i)}), \quad \text{s.t. } P(M^{(i)}) \cap P(M^{(j)}) = \text{face of both.}$$

Such decompositions arise in the work of Kapranov, Lafforgue, Hacking–Keel–Tevelev, Speyer.

Lafforgue's work implies that lack of a MPD for M implies that M has at most finitely many realizations, up to projective equivalence.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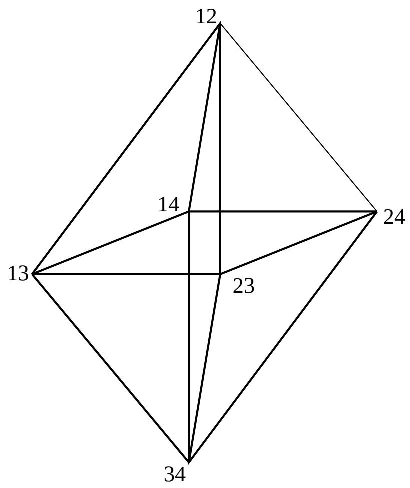
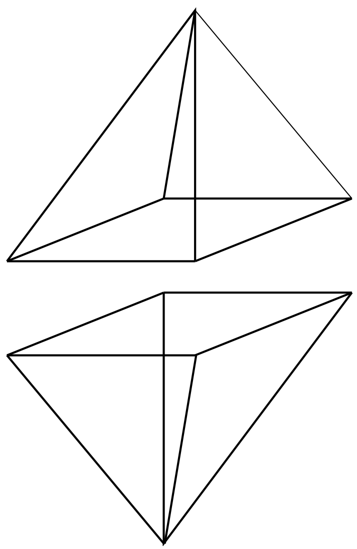
Theorem (B–Omar)**.** *Suppose we have a MPD,*

$$P(M) = \bigcup_i P(M^{(i)}).$$

Then there is an index j such that $\rho(M) = \rho(M^{(j)})$.





Let $v = (v_1, \ldots, v_n)$ be a full rank vector configuration in K^r realizing a matroid M. Package all projectively equivalent configurations together in the $GL_r(K)$ orbit

$$\{gv_1 \otimes gv_2 \otimes \cdots \otimes gv_n\} \subset (K^r)^{\otimes n}.$$

(Why do we get all?)

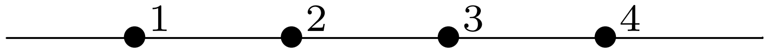
This orbit spans a representation of $GL_r(K)$. It is the smallest representation in $(K^r)^{\otimes n}$ that contains $v_1 \otimes \cdots \otimes v_n$.

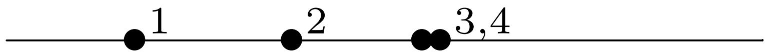
Denote this representation by $G(v)$.

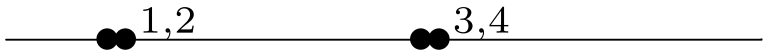
Recall that irreducible polynomial representations of $GL_r(K)$ are paramaterized their highest weights, which are in bijection with partitions that have $\leq r$ parts.

Theorem. $G(v)$ *contains a highest weight vector of weight* λ *if and only if* $\lambda \geq \rho(v)^t$.

Write the latter suggestively as *"if and only if* $K_{\lambda,\rho(v)^t} > 0$*"*. I am very interested in the multiplicity of λ, which appears to be *hard* to compute. These are not known to be matroid invariants.

$G($  $) = \ydiagram{4} + 3\,\ydiagram{3,1} + \ydiagram{2,2}$

$G($  $) = \ydiagram{4} + 2\,\ydiagram{3,1} + \ydiagram{2,2}$

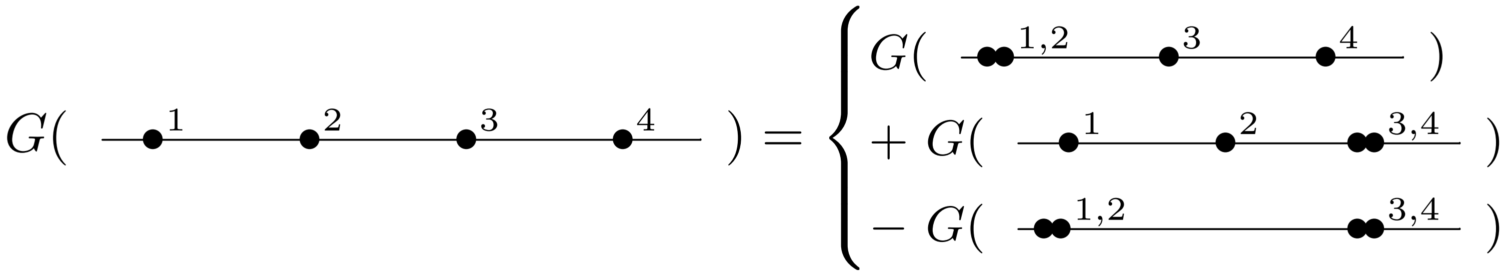
$G($  $) = \ydiagram{4} + \ydiagram{3,1} + \ydiagram{2,2}$

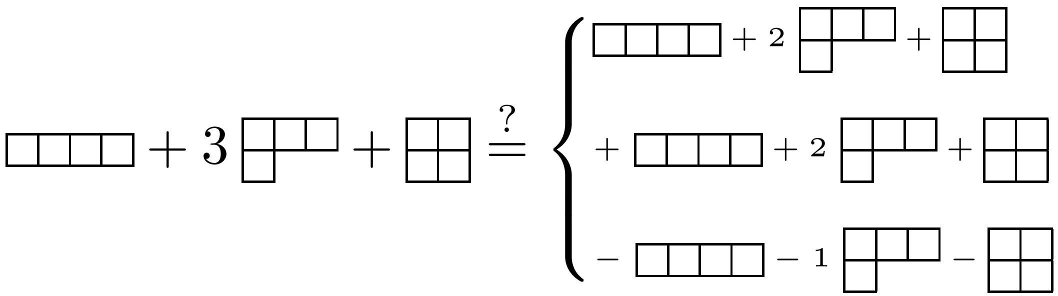
cf.

$$(K^2)^{\otimes 4} = \ydiagram{4} + 3\,\ydiagram{3,1} + 2\,\ydiagram{2,2}$$

For a general configuration v, $G(v)$ is hard to decompose explicitly. You should imagine these numbers as being mysterious and interesting invariants, although in the case of "points on a line" the answer is simple (and beautiful).

The theorem on MPDs and the rank partition was suggested by a conjectural relationship between $G(v)$ and certain MPDs. Namely, that multiplicities behave valuatively:





This should only works on certain MPDs.

One more idea, motivated by the irreducible decomposition of $G(v)$.

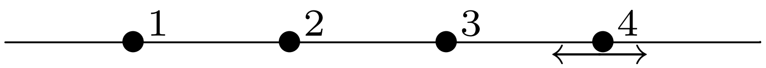
I hope that the multiplicity of λ is at least $K_{\lambda,\rho(v)'}$. A combinatorial version of this conjecture is:

Let M be a labelled matroid on $[n]$. There are at least $K_{\lambda',\rho(v)'}$ SYT of shape λ whose rows index independent sets.

This idea promises to have interesting interactions with vector partition functions.

Thanks for listening!

The idea behind why this will work: Consider a continuous "family" of projectively equivalent configurations



$$v = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & t \end{bmatrix}$$

Here $t \in K = \overline{k((t))}^{alg}$. We want to degenerate this by taking $t \to 0$, but what happens is more subtle then just making points 3 and 4 collide.

$$G\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & t \end{bmatrix}\right) \ni \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & t \end{bmatrix}^{\otimes}$$

$$\ni \begin{bmatrix} t & t & t & t \\ -2 & -1 & 0 & t \end{bmatrix}^{\otimes}$$

$$\ni \begin{bmatrix} t & t & 1 & 1 \\ -2 & -1 & 0 & 1 \end{bmatrix}^{\otimes}$$

In a precise sense,

$$\lim_{t\to 0}$$