Lattice paths and Lagrangian matroids

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Special Session on Geometric Combinatorics

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Joint work with:
Anna Gundert, *ETH Zürich*
Daria Schymura, *Freie Universität Berlin*

The playbill

The cast

$$\text{lattice path matroids} \subset \text{matroids},$$
$$\text{lattice path Lagrangian matroids} \subset \text{Lagrangian matroids}.$$

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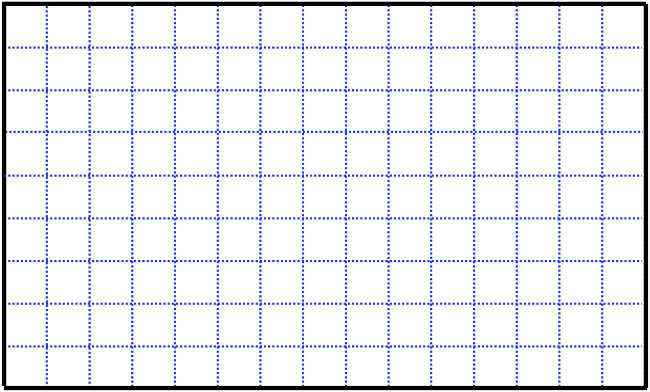
The cast (in order of appearance????)

$$\begin{aligned} \text{lattice path matroids} &\subset \text{matroids}, \\ \text{lattice path Lagrangian matroids} &\subset \text{Lagrangian matroids}. \end{aligned}$$

Lattice path matroids

Fix an $m \times n$ grid.

- Let P be a North/East path from $(0, 0)$ to (m, n).
- Let Q be a North/East path that stays above P.
- Let $\mathcal{B}$ be the collection of all North/East paths B between P and Q. Then $\mathcal{B}$ is the set of bases of a matroid.



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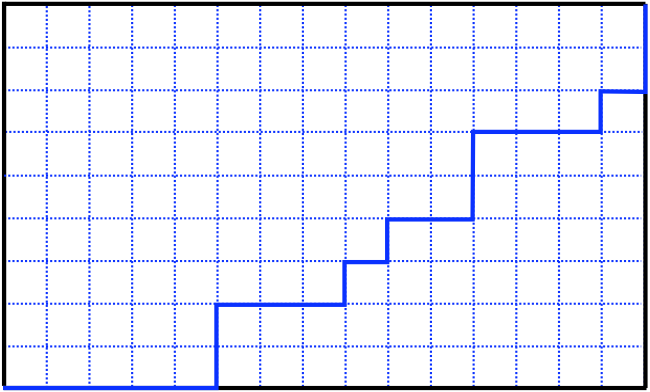

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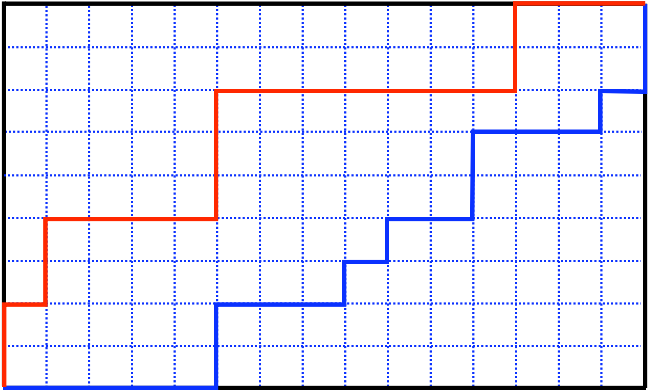

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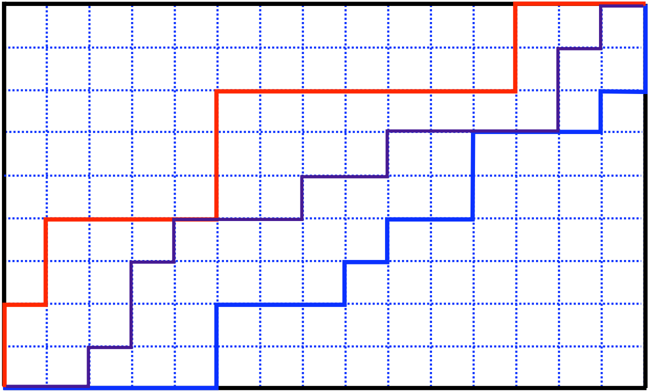

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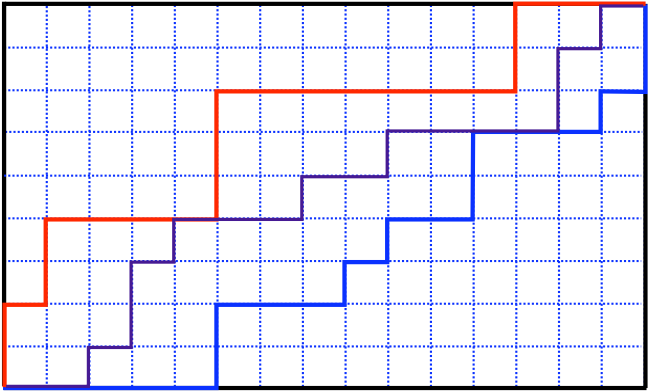

Lagrangian matroids

$[n]^* = \{1^*, \ldots, n^*\}$, $S = [n] \cup [n]^*$

Definition

- $B \subset S$ is a transversal of S if $|B \cap \{i, i^*\}| = 1$ for all $i \in [n]$.
- For transversals X and Y of S, we say i is a divergence if $i \in X \triangle Y$.

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Let $\mathcal{B}$ be a set of transversals of S. $\mathcal{B}$ is the set of bases of a Lagrangian matroid if the following *symmetric exchange axiom* holds:

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Lattice path Lagrangian matroids (LPLM)

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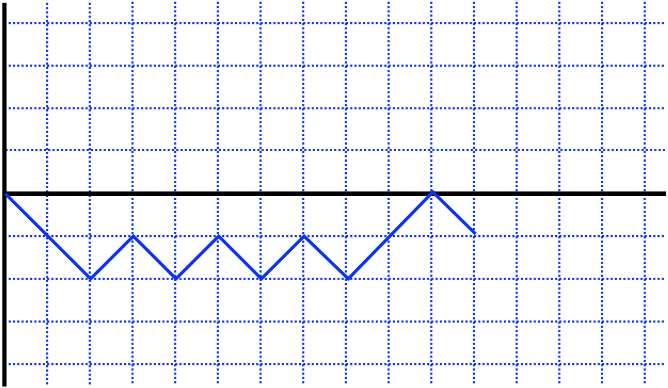

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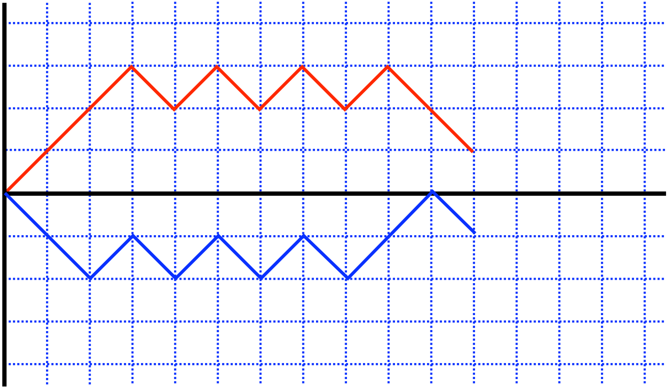

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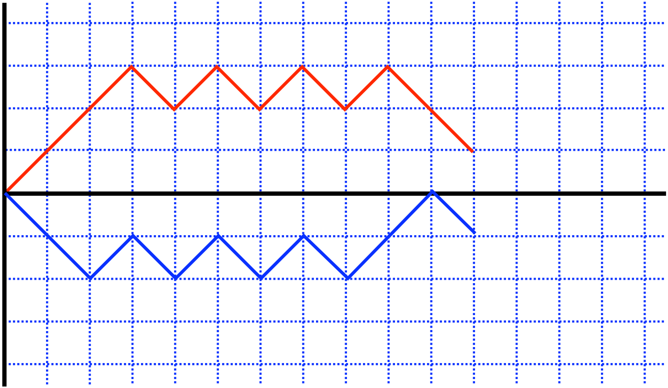

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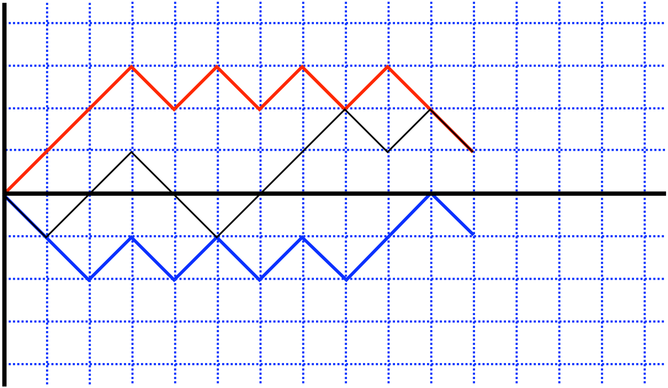

an example path $B \in \mathcal{B}[P, Q]$

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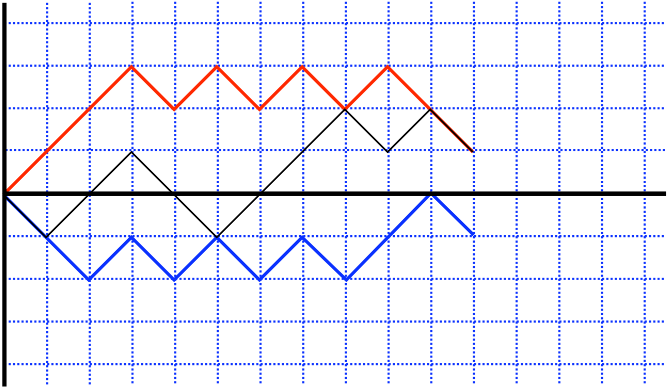

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Outline

$$\text{lattice path matroids} \subset \text{matroids},$$
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Question: de Mier

What is the relationship between lattice path Lagrangian matroids and lattice path matroids?

Theorem

Let $\mathcal{B}$ be a collection of transversals of $S = [n] \cup [n]^*$. For a transversal T of S, define $\mathcal{I}_T^{\mathcal{B}} = \{I : I \subseteq B \cap T \text{ for some } B \in \mathcal{B}\}$.

Then, $\mathcal{B}$ is a Lagrangian matroid $\Leftrightarrow$ $\mathcal{I}_T^{\mathcal{B}}$ is the set of independent sets of a matroid on $T \subset S$.

The plan...

Given a LPLM $\mathcal{B} = \mathcal{B}[P, Q]$ show that (i) $\mathcal{I}_T^{\mathcal{B}}$ is a LPM and (ii) construct it!

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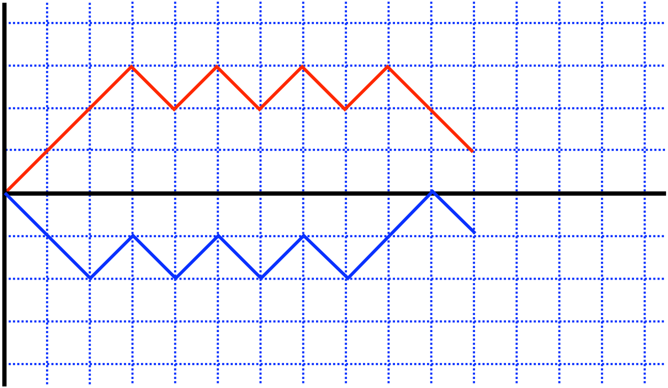

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We call these maximal paths.

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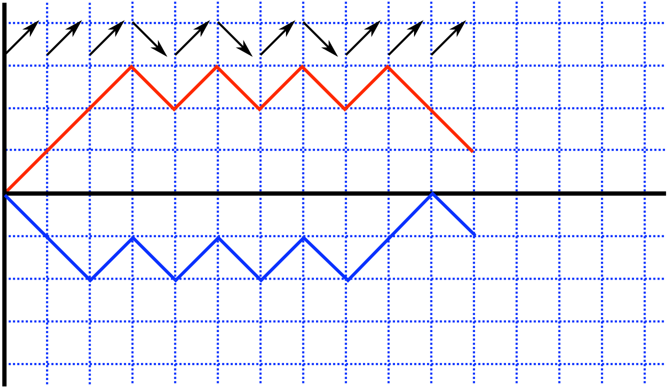

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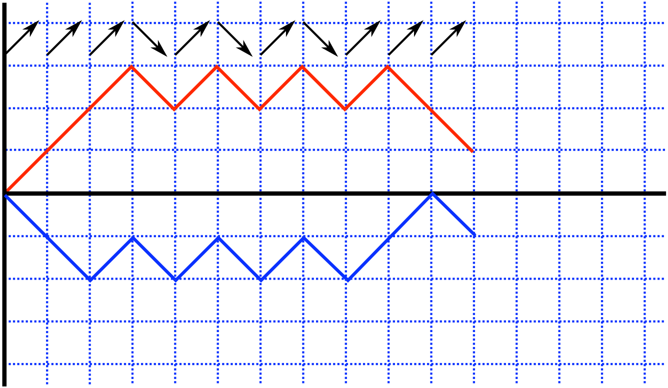

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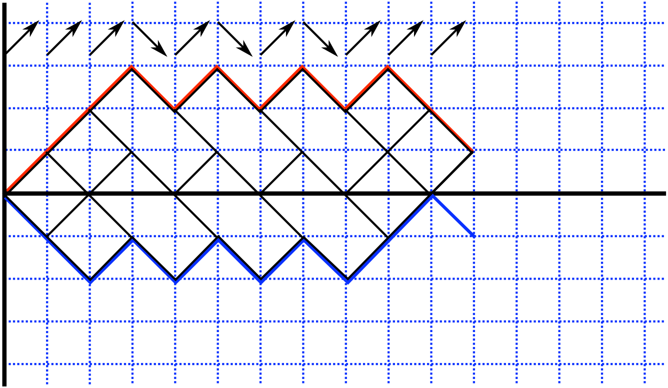

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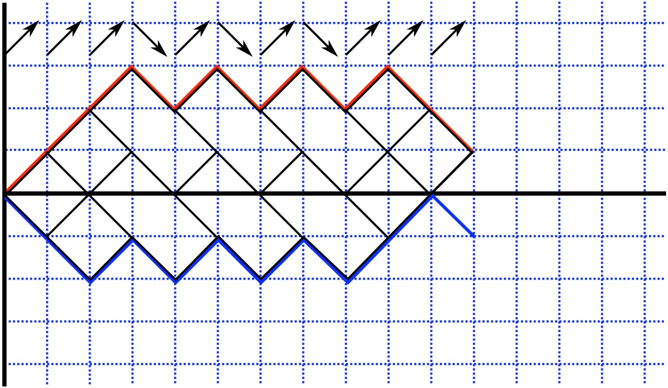

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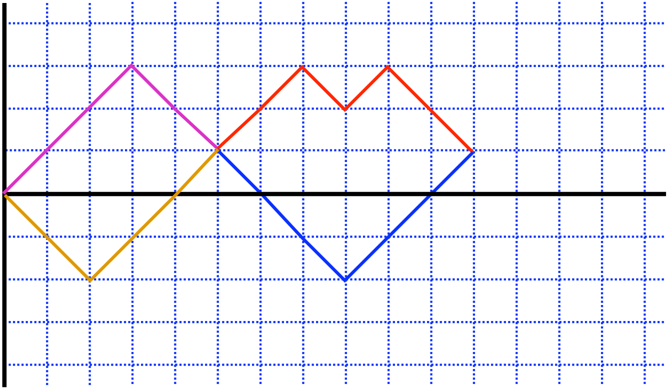
$P^{\max}$ and $Q^{\max}$

If two maximal paths cross,...



So there is a lowest maximal path $P^{\max}$ and a highest maximal path $Q^{\max}$.

We only have to consider paths between $P^{\max}$ and $Q^{\max}$!

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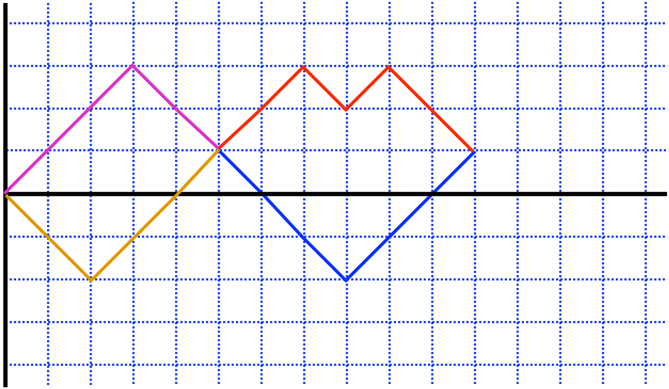

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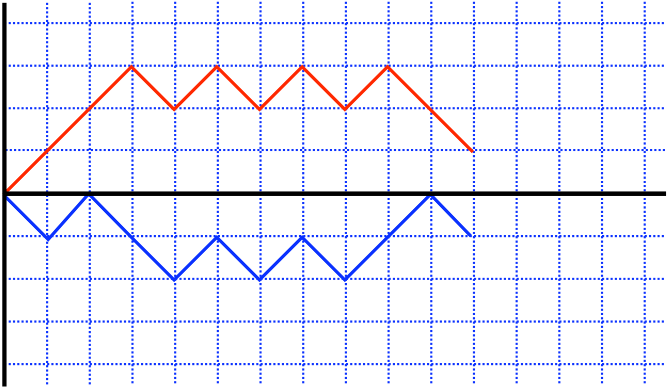
All maximal paths end at the same point.



Suppose S, R maximal paths with $h_S(n) > h_R(n)$.

Among the disagreeing steps there have to be more with

$R(j) = \searrow, S(j) = \nearrow$.

R cannot agree with T in each of these steps.

$\Rightarrow$ W has one more agreement than R.

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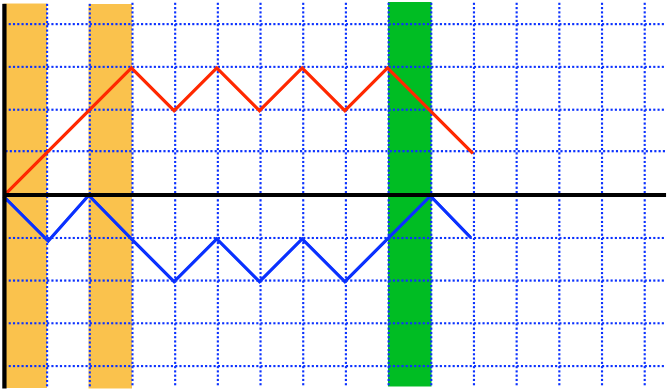

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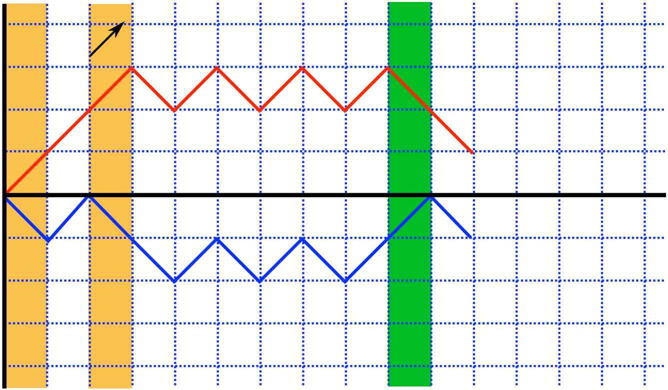

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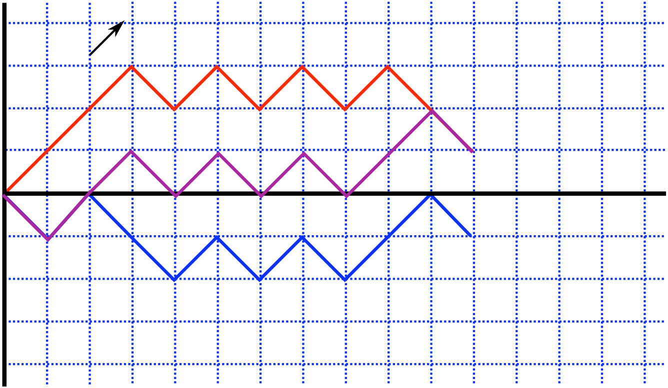

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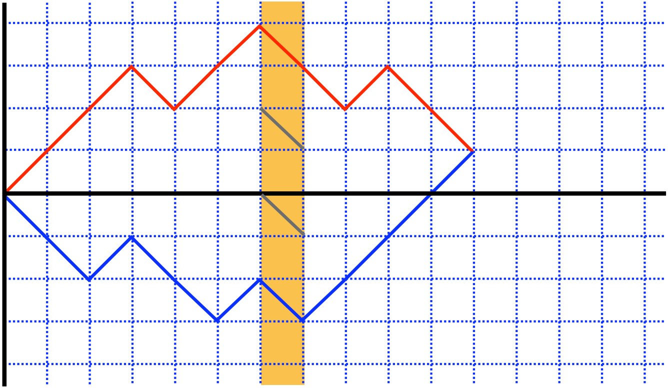
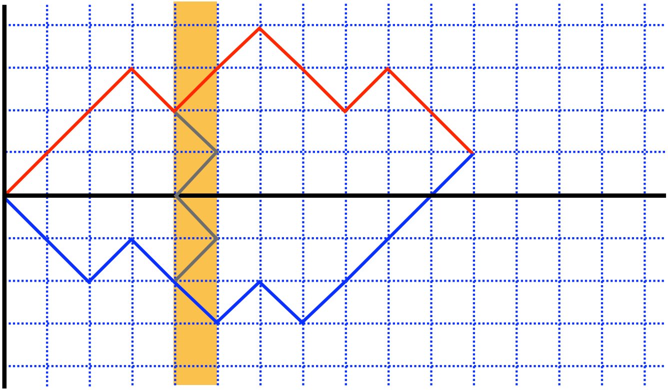
Suppose all maximal paths agree in step *i*...



...then it looks like this (i.e., no gaps)

$\Longrightarrow$ *i* is a "parallel step"

Suppose NOT all maximal paths agree in step i...



...then the maximal paths form a zig-zag-shape!

$\Longrightarrow$ i is a "zig-zag step"

Proof of the theorem

We can assume:

- $P^{\max}$ and $Q^{\max}$ do not meet before the last step.
- There are no parallel steps.

Then:

- **All** paths between $P^{\max}$ and $Q^{\max}$ are maximal. (Only zig-zag steps!)
- T is constant. W.l.o.g. $T = \nearrow\nearrow \ldots \nearrow$.

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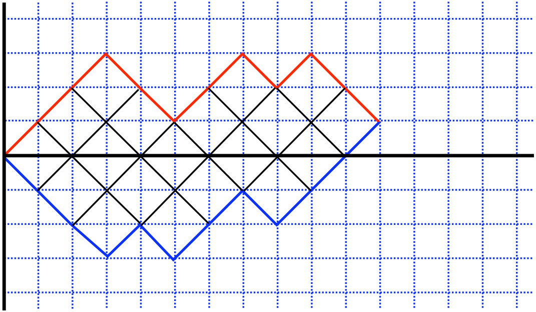

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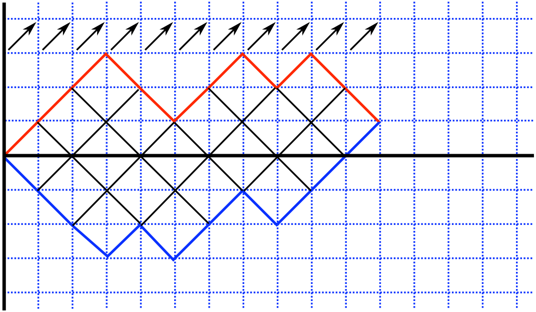

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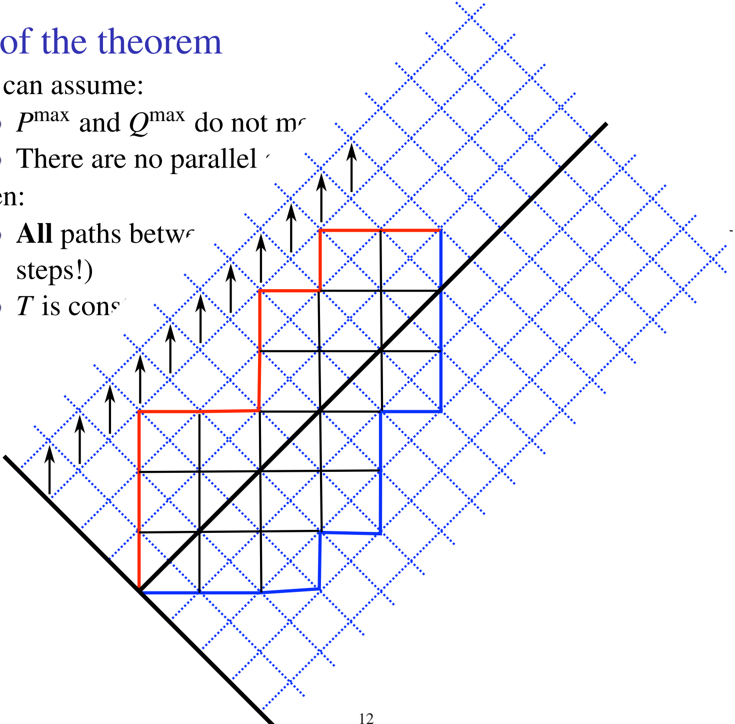

Algorithm

The maximally-agreeing paths correspond to bases of a lattice path matroid.

- TASK: Compute the lattice path matroid explicitly.

Plan of attack

Given T, P and Q of length n, iteratively build the lattice path matroid.

- If $n > 1$, then we take the lattice path matroid corresponding to
  - $T(1, \ldots, n-1)$
  - $P(1, \ldots, n-1)$
  - $Q(1, \ldots, n-1)$

  and modify it.

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Given T, P and Q of length n, iteratively build the lattice path matroid.

- If $n > 1$, then we take the lattice path matroid corresponding to
  - $T(1, \ldots, n-1)$
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  and modify it.

Algorithm

The maximally-agreeing paths correspond to bases of a lattice path matroid.

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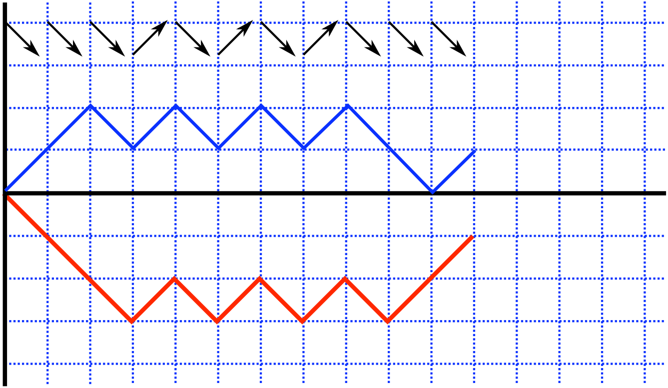

$T(n) = \searrow$, so "flip" everything.

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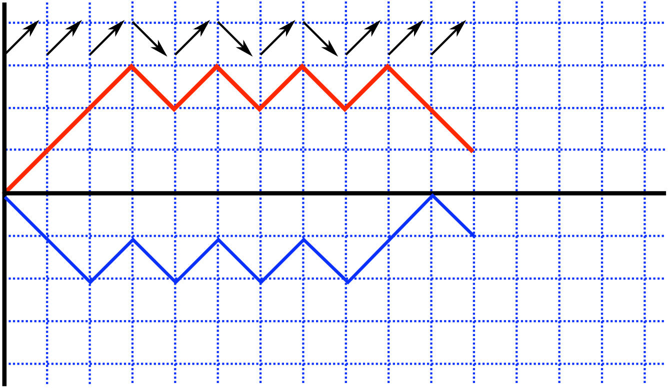

$T(n) = \nearrow$, paths agree on same steps.

Rank remains the same

Harder case: from the endpoint of all maximal paths for $n = 10$, can not agree with T on 11th step.

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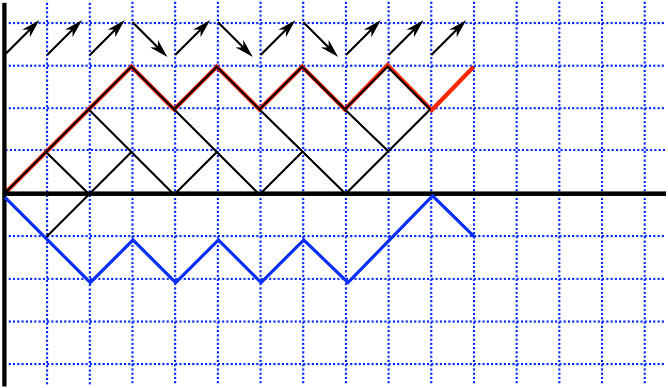

In the easy case, all paths could extend

Rank remains the same

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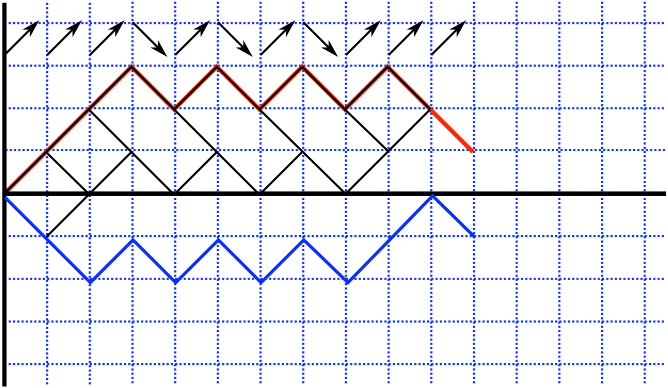
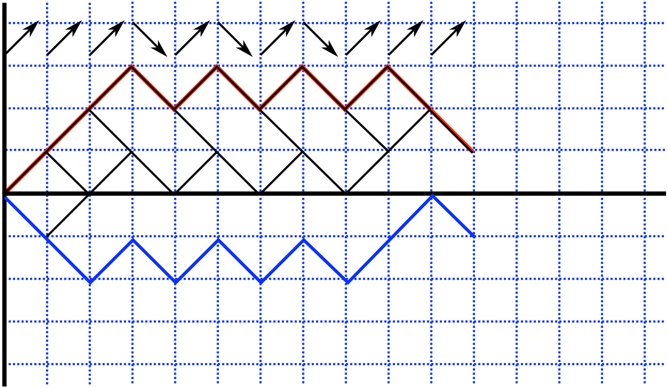


Now, cannot extend to agree with T

Rank remains the same

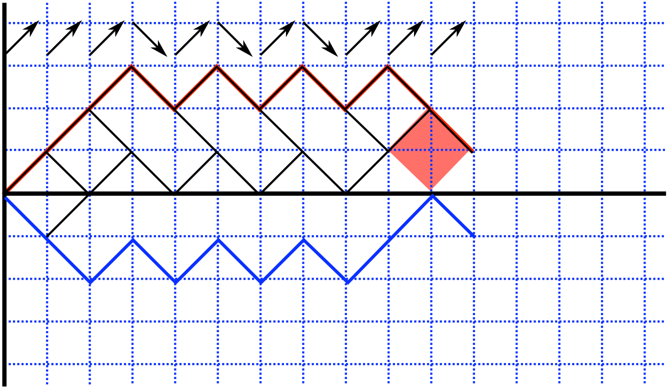
Harder case: from the endpoint of all maximal paths for $n = 10$, can not agree with T on 11th step.



All maximal paths will still agree with T nine times

Rank remains the same

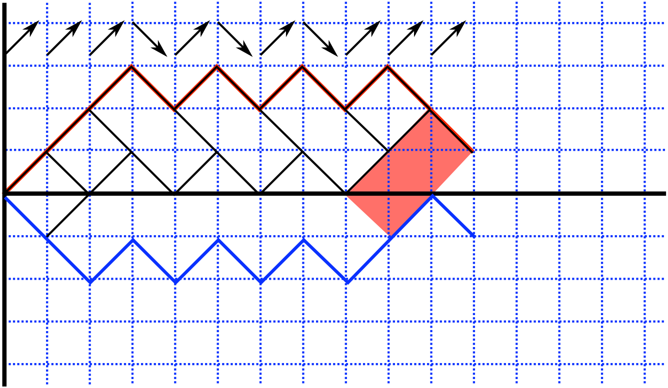
Harder case: from the endpoint of all maximal paths for $n = 10$, can not agree with T on 11th step.



Build space of potential new paths (one under $P^{\max}$)

Rank remains the same

Harder case: from the endpoint of all maximal paths for $n = 10$, can not agree with T on 11th step.



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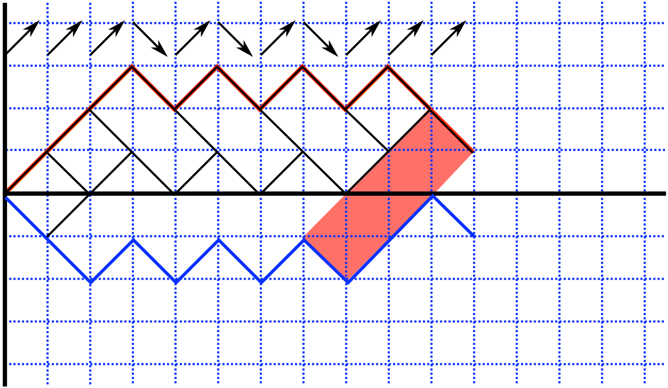

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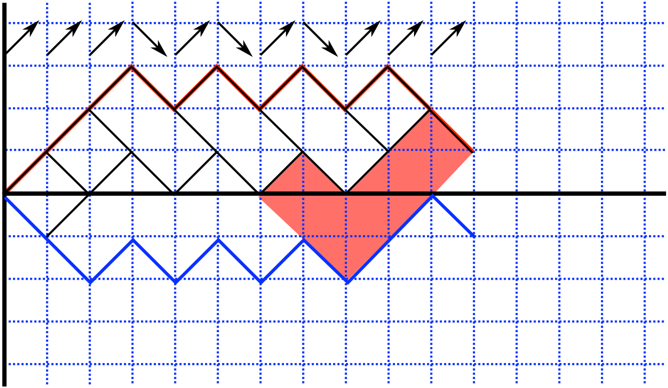

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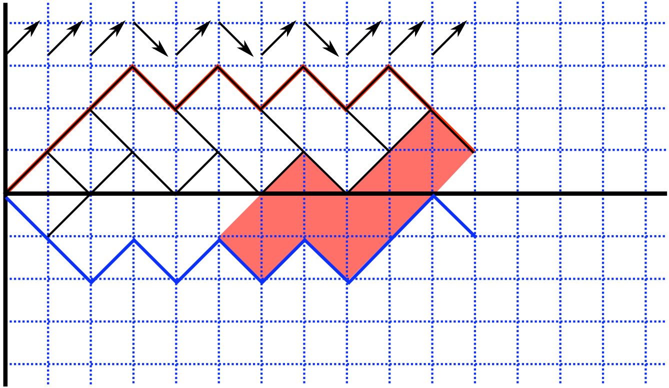

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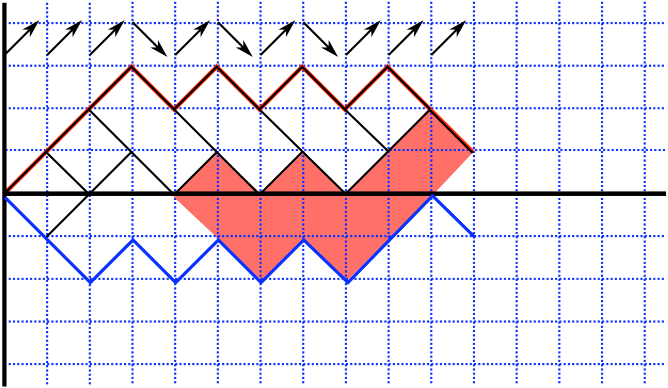

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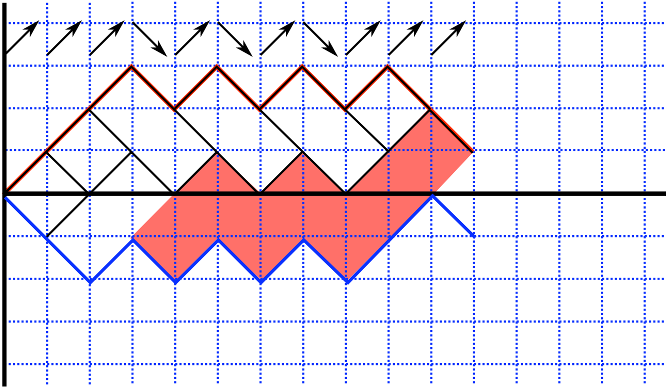
Harder case: from the endpoint of all maximal paths for $n = 10$, can not agree with T on 11th step.



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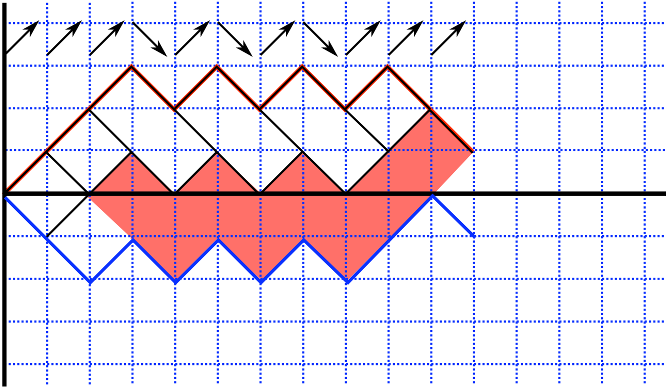

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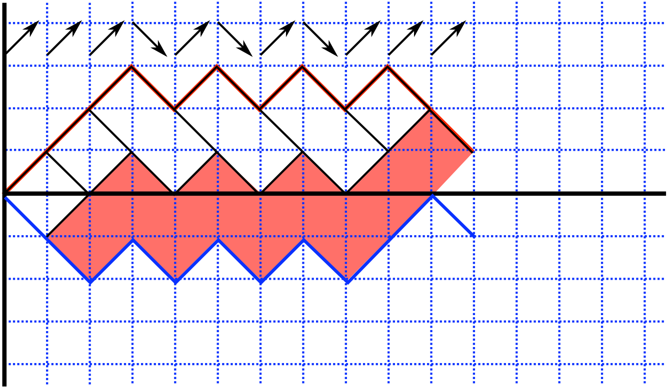

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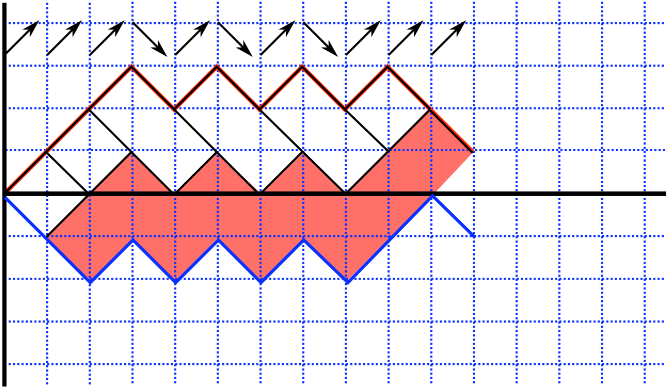
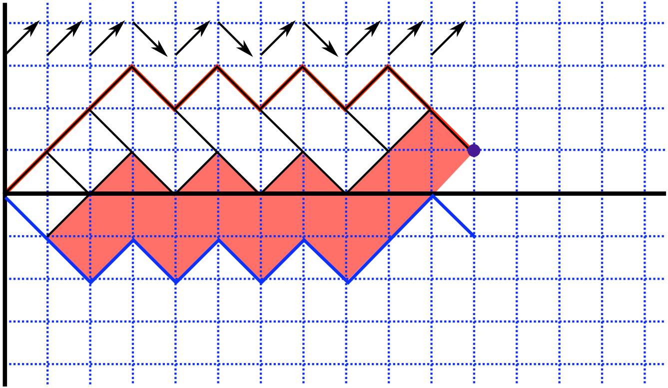


All new maximal paths agree with T exactly 9 times

Rank remains the same

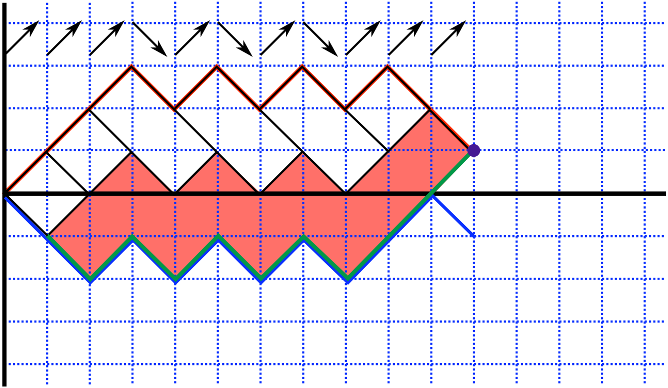
Harder case: from the endpoint of all maximal paths for $n = 10$, can not agree with T on 11th step.



This is the area of **all** potential new paths

Rank remains the same

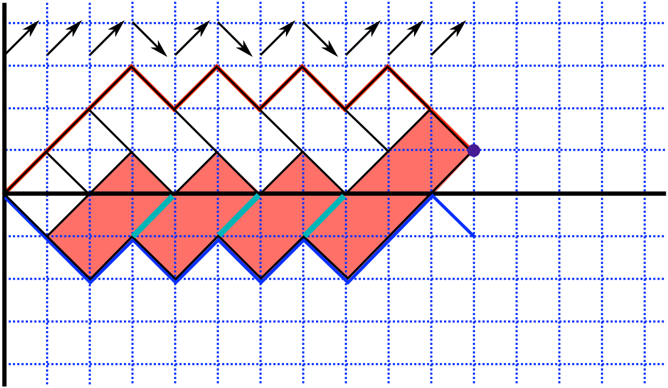
Harder case: from the endpoint of all maximal paths for $n = 10$, can not agree with T on 11th step.



New lowest path $P^{\max}$

Rank remains the same

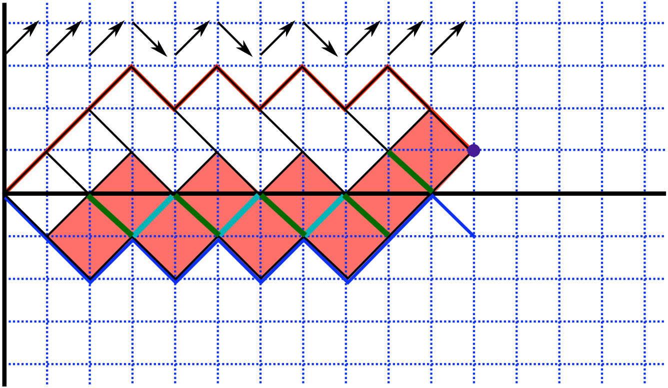
Harder case: from the endpoint of all maximal paths for $n = 10$, can not agree with T on 11th step.



Steps that are not allowed

Rank remains the same

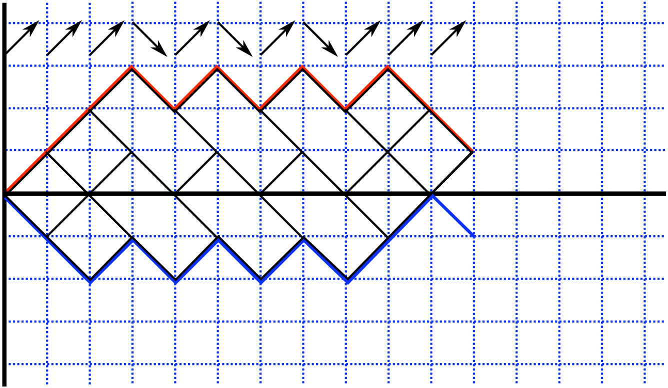
Harder case: from the endpoint of all maximal paths for $n = 10$, can not agree with T on 11th step.



New segments that are allowed

Rank remains the same

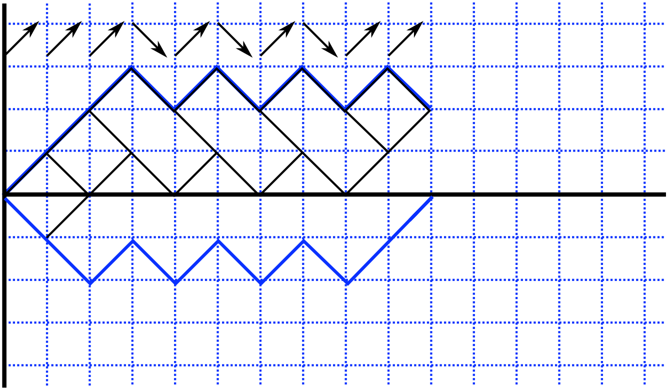
Harder case: from the endpoint of all maximal paths for $n = 10$, can not agree with T on 11th step.



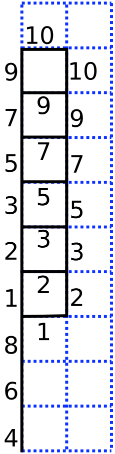
All maximal paths for $n = 11$

Rank remains the same

Modify the lattice path matroid



Maximal Lagrangian lattice paths for $n = 10$



Rank remains the same

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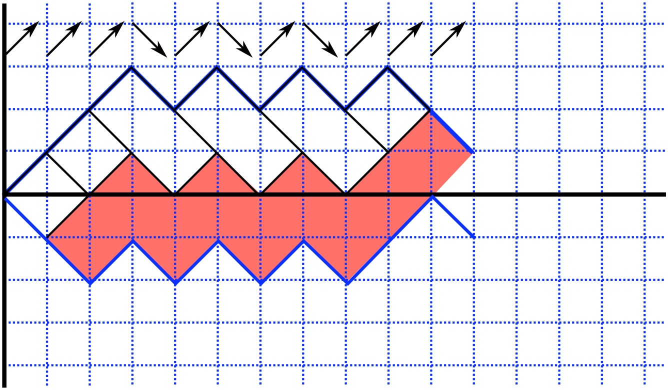
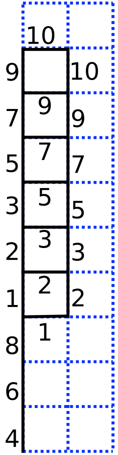
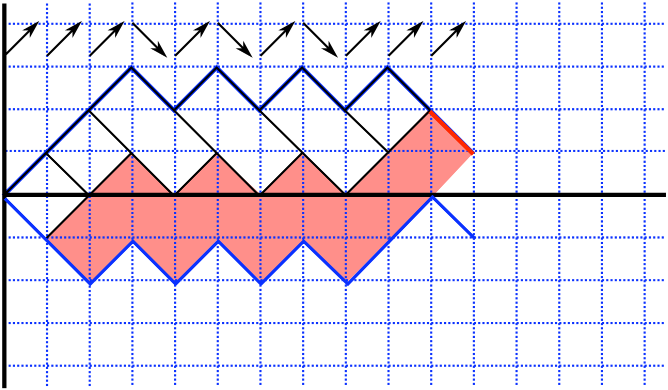


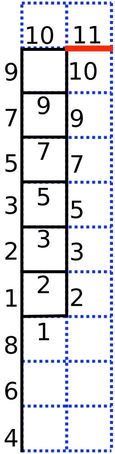


Identify space of new paths

Rank remains the same

Modify the lattice path matroid

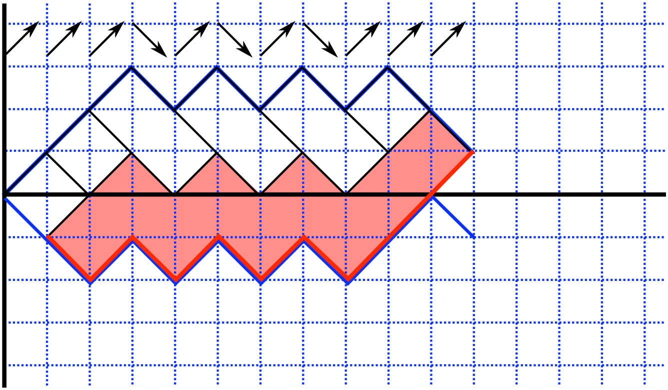


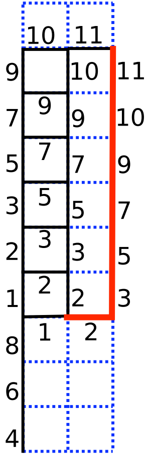


Add last step for old paths

Rank remains the same

Modify the lattice path matroid

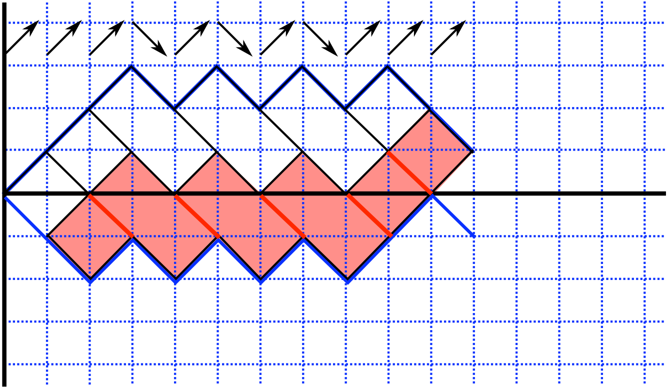


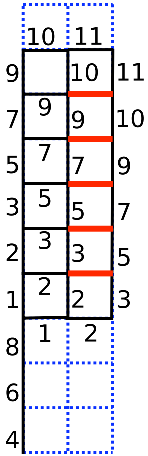


Add new lowest path

Rank remains the same

Modify the lattice path matroid

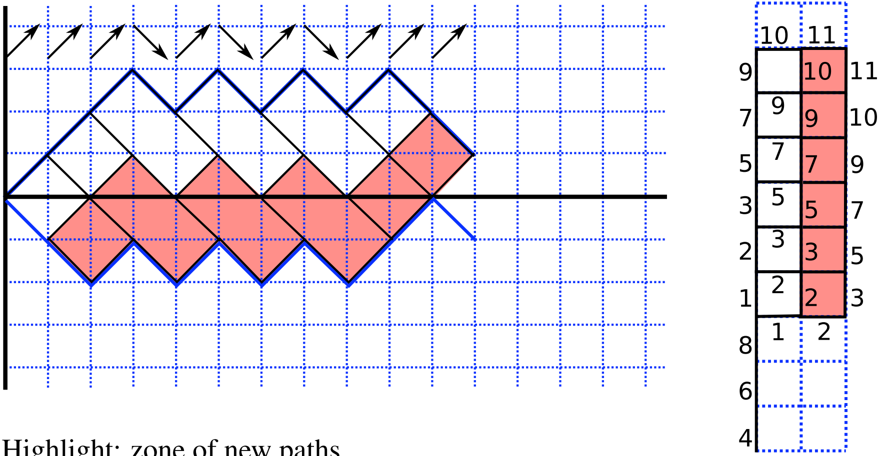




Add new legal inside steps

Rank remains the same

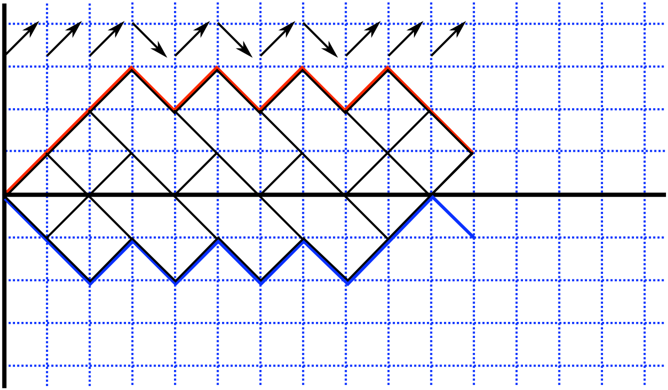
Modify the lattice path matroid

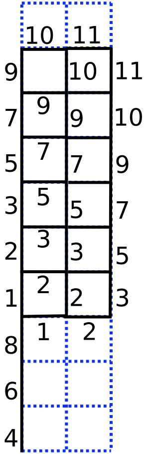


Highlight: zone of new paths

Rank remains the same

Modify the lattice path matroid





Final picture

Thank you!

Thank you for your attention!

Special thanks to: Margaret Bayer, Andrew Berget, Joe Bonin, Jesús De Loera, Anna de Mier, Matthias Köppe, and Jay Schweig. Research supported in part by NSF grant DMS-0608785, NSF VIGRE grants DMS-0135345 and DMS-0636297, the Centre de Recerca Matemàtica at the Universitat Autónoma de Barcelona, and the Deutsche Forschungsgemeinschaft Research Training Group on Methods for Discrete Structures.