Math 821 Problem Set #2
Posted: Friday 2/11/11
Due date: Monday 2/21/11

Problem #1 (Hatcher, p.18, #2) Construct an explicit deformation retraction of $X = \mathbb{R}^n \setminus \{0\}$ onto S^{n-1}. ("Explicit" means that you should write down an actual formula for the map $f_t : X \to X$, and check that the family of maps you have defined satisfies the conditions of a deformation retraction.)

Solution: One such homotopy is

$$f_t(\mathbf{x}) = (1-t)\mathbf{x} + t\frac{\mathbf{x}}{\|\mathbf{x}\|}$$

for $\mathbf{x} \in X$, $t \in I$. Note that

- f_t is well-defined provided that $\|\mathbf{x}\| \neq 0$, i.e., on X;
- f_t is continuous in $\mathbf{x}$ and t, so the family $\{f_t\}$ defines a homotopy $X \times I \to X$;
- $f_0(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$;
- $f_1(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|$, so f_1 maps X to S^{n-1};
- if $\|\mathbf{x}\| = 1$, then $f_t(\mathbf{x}) = \mathbf{x}$ for all t.

This verifies that f_t is a deformation retraction.

(There are many other possible solutions.)

Problem #2 (Hatcher, p.19, #12) Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y, and that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y. Prove also the corresponding statements with components instead of path-components. Deduce that if the components of a space X coincide with its path-components, then the same holds for any space Y homotopy equivalent to X.

Solution: In general, if p is a point in a topological space, let's write $\langle\langle p\rangle\rangle$ for the component of that space containing p, and $\langle p\rangle$ for the path-component of that space containing p.

First, we want to prove that $\langle x\rangle = \langle x'\rangle \iff \langle f(x)\rangle = \langle f(x')\rangle$ The $\implies$ direction is easy: if ϕ is an x,x'-path in X, then $f \circ \phi$ is an $f(x), f(x')$-path in Y.

For the reverse direction, if ψ is an $f(x), f(x')$-path in Y, then $g \circ \psi$ is a $g(f(x)), g(f(x'))$-path in X. On the other hand, $\langle x\rangle = \langle g(f(x))\rangle$ because, by definition of homotopy equivalence, there is a homotopy $h_t : X \to X$ with $h_0 = \mathbb{1}$ and $h_1 = g \circ f$; the function $\gamma : I \to X$ given by $\gamma(t) = h_t(x)$ therefore defines a path from x to $g(f(x))$. Similarly, we can construct a path from $g(f(x'))$ to x'. Concatenating these paths with $g \circ \psi$ gives an x,x'-path in X and establishes the $\impliedby$ direction.

Second, we want to prove that $\langle\langle x\rangle\rangle = \langle\langle x'\rangle\rangle \iff \langle\langle f(x)\rangle\rangle = \langle\langle f(x')\rangle\rangle$. The $\implies$ direction follows from the fact that the continuous image of a connected space is connected.

We now want to show that if $\langle\langle f(x)\rangle\rangle = \langle\langle f(x')\rangle\rangle$, then $\langle\langle x\rangle\rangle = \langle\langle x'\rangle\rangle$. By the $\implies$ direction, the hypothesis implies that $\langle\langle g(f(x))\rangle\rangle = \langle\langle g(f(x'))\rangle\rangle$. Again, consider the homotopy $H : X \times I \to X$ with $H(z,0) = z$ and $H(z,1) = g(f(z))$. Let $U = \langle\langle H(x,1)\rangle\rangle$ and $V = X \setminus U$; then the clopen decomposition $X = U \dot\cup V$ pulls back to a clopen decomposition

$$X \times I = H^{-1}(U) \dot\cup H^{-1}(V)$$

but both $(x,1)$ and $(x',1)$ lie in the same piece of this decomposition because H maps them into the same component of X. That piece must $H^{-1}(U)$, On the other hand, path-components are contained in connected components, and X has paths from $(x,0)$ to $(x,1)$ and from $(x',0)$ to $(x',1)$, namely

$$\phi(t) = H(x,t), \qquad \phi'(t) = H(x',t),$$

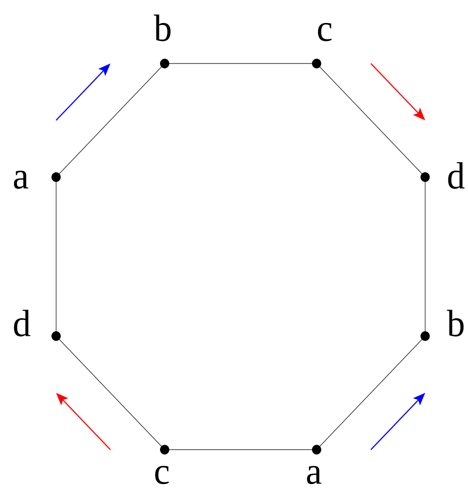
so $(x,0)$ and $(x',0)$ belong to $H^{-1}(U)$ as well, which is to say that $x, x' \in U$ as desired.

The "deduce that..." part is immediate. This problem is another justification that homotopy equivalence is a sensible thing to consider.

Note: Some of you argued that since the continuous image of a (path-)connected space is (path-)connected, the maps f and g induce surjections $P_X \to P_Y$ and $P_Y \to P_X$ respectively, where P_X means the set of (path-)connected components of X; therefore, $|P_X| \geq |P_Y| \geq |P(X)|$ and equality holds throughout. However, these numbers may be infinite, when the statement "$a \leq b \leq a \implies a = b$" is the (highly nontrivial) Bernstein-Schröder-Cantor theorem of set theory. In the context of topology, I think it's more natural to show directly that f and g induce bijections.

Problem #3 (Hatcher, p.19, #17) Construct a 2-dimensional cell complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts. (This implies that these two spaces are homotopy-equivalent.)

Solution: Here's one way to do it: start with a solid octagon, attach the two blue edges to each other with the orientation shown, and attach the two red edges to each other with the orientation shown.



Now contracting the blue edge (which is a deformation retraction — you're collapsing a contractible subspace, namely the edge ab) produces a cell complex homeomorphic to a Möbius strip, while contracting the red edge cd yields an annulus.

Note: We know that both the annulus A and the Möbius band $\tilde{A}$ deformation-retract onto S^1. If we call the deformation retractions F and G respectively, then we can construct the mapping cylinders M_F and M_G and attach them along their respective copies of S^1. This produces a space X that has both A and $\tilde{A}$ as deformation retracts (because deformation-retracting one of these things produces the other); however, X is actually 3-dimensional as a cell complex and so doesn't qualify. (Actually, there's a simpler 3-dimensional construction: Just take $\tilde{A} \times I$. Squashing the "$\times I$" part gives $\tilde{A}$, while squashing $\tilde{A}$ onto its central circle gives $S^1 \times I = A$.)

OTOH, given deformation retracts of A and $\tilde{A}$ respectively to subcomplexes isomorphic to S^1, we can glue A to $\tilde{A}$ along those S^1's. This is arguably less pretty than the octagon construction, but more general.

Problem #4 Let X, Y be cell complexes of finite type. Recall that the *f-polynomial* of X is

$$f(X;q) = \sum_{\text{cells } e^i_\alpha \in X} q^i = \sum_{i=0}^{\dim X} f_i q^i$$

where f_i is the number of i-dimensional cells (and recall that "of finite type" means that $f_i < \infty$ for each i). In terms of $f(X;q)$ and $f(Y;q)$, find formulas for

(a) $f(X \times Y;q)$;
(b) $f(X/Y;q)$ (assuming that (X,Y) is a CW-pair);
(c) $f(CX;q)$;
(d) $f(X * Y;q)$.

You don't have to give detailed proofs. (See pp. 8–10 of Hatcher for the definitions of these operations.)

Solution: (a) $f(X \times Y;q) = f(X;q)f(Y;q)$. Each cell $e \in X \times Y$ is a product of a pair of cells e_X, e_Y in X and Y, and $\dim e = \dim e_X + \dim e_Y$.

(b) $f(X/Y;q) = f(X;q) - f(Y;q) + 1$. Start with X, throw away Y, and toss in a single point.

(c) $f(CX;q) = 1 + (1+q)f(X;q)$. The initial 1 is for the cone point.

(d) $f(X * Y;q) = f(X;q) + f(Y;q) + qf(X;q)f(Y;q)$. This is because $X * Y$ contains a copy of each of X and Y, and the join of an i-cell and a j-cell is an $(i+j+1)$-cell.

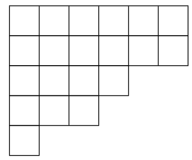
Note: It is possible to do this by writing out the coefficients of $f(X;q)$ and $f(Y;q)$, but it is much more convenient to avoid this if you possibly can!

Problem #5 Let $0 \leq k \leq n$. Recall from class that the *Grassmannian* $G(k,n)$ is defined as the space of k-dimensional subspaces $V \subset \mathbb{R}^n$, so that in particular, $G(1,\mathbb{R}^n) = \mathbb{R}P^{n-1}$. (Fact: Everything in this problem works the same way if you change $\mathbb{R}$ to $\mathbb{C}$, except that the dimensions of all the cells get doubled.)

(#5a) Work out an explicit cell decomposition for $G(2,4)$ as a finite CW-complex. That is, describe how to decompose the set $G(2,4)$ into pieces, each of which is isomorphic to a $\mathbb{R}$-vector space. If you do this correctly (hint: row-reduced echelon form), then the isomorphisms should be straightforward from the construction.

(#5b) Describe the attaching poset of $G(2,4)$. (Recall that this is the partially ordered set whose elements are the cells e_α, and whose order relation is given by $e_\alpha \geq e_\beta$ if $\overline{e_\alpha} \supseteq e_\beta$).

(#5c) A *Ferrers diagram* is a collection of square boxes that are top- and right-justified: for instance,



Write out the poset $P(2,2)$ of all Ferrers diagrams with at most two rows and at most two columns, ordered by containment (as sets of squares). Compare it to your previous answer.

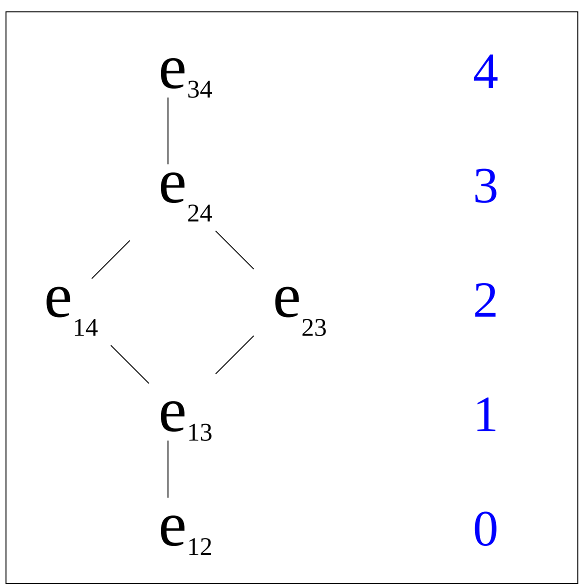
(#5d) Make as many conjectures as you dare about the cell structure of $G(k,n)$. (In particular, what happens to the attaching poset if you reverse all the relations?)

Solution: (a) Any $V \in G(2,4)$ can be expressed as the column span of a $k \times n$ matrix M. Performing elementary column operations on the matrix doesn't change the span, and we know that we can eventually put M into a unique reduced column-echelon form, i.e., one of the following things:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ * & 0 \\ 0 & 1 \\ * & * \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ * & 0 \\ * & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ * & * \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ * & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This gives a cell structure with f-polynomial $q^4 + q^3 + 2q^2 + q + 1$. Specifically, the cells classify points in $G(2,4)$ by the locations of the pivots in its reduced column-echelon form. Moving the pivot 1's up gives you a bigger cell; specifically, if we write e_{ij} for the cell whose pivots are in rows i, j with $i < j$, then $e_{ij} \subseteq \overline{e_{i'j'}}$ iff $i \leq i'$ and $j \leq j'$.

(b,c) These two posets are isomorphic. They look like this, where the blue numbers indicate dimensions.



(d) In general, $G(k,n)$ can be decomposed into cells corresponding to the Ferrers diagrams with at most $n-k$ rows and at most k columns (i.e., "fitting inside a $k \times (n-k)$ rectangle"). There are exactly $\binom{n}{k}$ such Ferrers diagrams (why?) and the cell-closure relationship is exactly containment of Ferrers diagrams. This poset is self-dual (i.e., reversing all the order relations gives an isomorphic poset); on the combinatorial side, taking the complement of a Ferrers diagram within the $k \times (n-k)$ rectangle reverses the containment relation. The self-duality is not nearly as easy to see on the topological side of things, but is connected to Poincaré duality for manifolds.

Problem #6 (Extra credit; Hatcher, p.19, #16) Show that S^∞ is contractible.

Solution: For each $i \geq 0$, pick an i-cell e_k, and let $D_k = \overline{e_k}$. The notation is justified because $\overline{e_k}$ is indeed homeomorphic to a k-dimensional disk. Moreover, we have $D_0 \subset D_1 \subset D_2 \subset \cdots$, and also $S^\infty = \bigcup_k D_k$.

The idea is:

during the time interval $I_1 = [0, 1/2]$, collapse D_1 onto e_0 by a homotopy h_1,
during the time interval $I_2 = [1/2, 2/3]$, collapse D_2 onto e_0 by a homotopy h_2,
$\ldots$,
during the time interval $I_k = [(k-1)/k, k/(k+1)]$, collapse D_k onto e_0 by a homotopy h_k,
$\ldots$.

Technically, the thing we are collapsing during the time interval I_k is not D_k, but rather $E_k = D_k/D_{k-1}$, i.e., the image of D_k after the first $k-1$ contractions. Note that E_k is itself contractible because D_k ad D_{k-1} are.

Also technically, the map h_k is a function $S^\infty \times I_k \to S^\infty$; it is the homotopy whose existence asserts that the quotient map $S^\infty \to S^\infty/D_k$ has a homotopy inverse.

Taking the union of the maps h_i gives a map $h : S^\infty \times [0, 1) \to S^\infty$. This is well-defined because every point $p \in S^\infty$ belongs to some e_k, hence is mapped to e_0 by all h_ℓ for $\ell \geq k$. But this says exactly that h can be extended continuously to a homotopy $S^\infty \times [0, 1] \to S^\infty$ such that $h(p, 1) = e_0$ for all p, and the existence of such a homotopy says that S^∞ is contractible.