Creating and Shelling Cut Complexes

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What is a Simplicial Complex?

Definition

An abstract simplicial complex, Δ is a collection of subsets closed under inclusion. If $f \in \Delta$, we call f a face of Δ.

Examples of simplicial complexes: All subsets of $[n]$ with size at most 4. All sets of edges that do not contain a cycle for a particular graph. All independent sets of vertices for a graph.

Definition

The facets of a simplical complex are the maximal faces, faces not contained in any other face.

A simplicial complex is completely defined by its facets.

What is a Simplicial Complex?

We can visualize abstract simplicial complexes as geometric simplicial complexes.

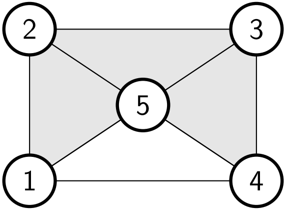


Figure: $\Delta = \langle \{1,2,5\}, \{2,3,5\}, \{3,4,5\}, \{1,4\} \rangle = \langle 125, 235, 345, 14 \rangle$

Shelling Order

Definition

An ordering $F_1, F_2, \ldots, F_t$ of the facets of a simplicial complex Δ is a shelling if, for every $1 < j \leq t$,

$$\left(\bigcup_{i=1}^{j-1} \langle F_i \rangle\right) \bigcap \langle F_j \rangle$$

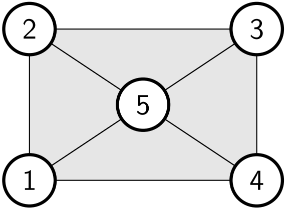
is a simplicial complex whose facets all have cardinality $|F_j| - 1$, where $\langle F_i \rangle$ is the simplex generated by the face. If Δ has a shelling, Δ is called shellable.

Think of shellings as ways to construct a complex such that every piece fits tightly with the previous pieces.

Example and Non-example

The simplicial complex $\langle 125, 235, 345, 145 \rangle$ is shellable, however not every order of facets is a shelling.

Example



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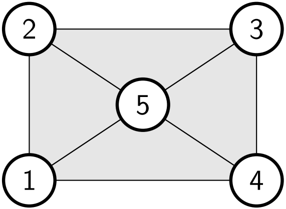

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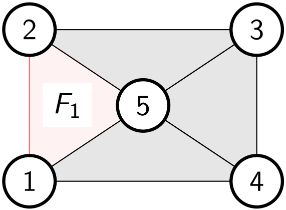

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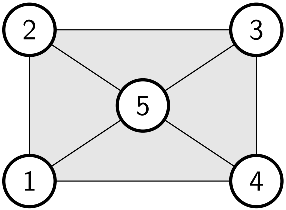

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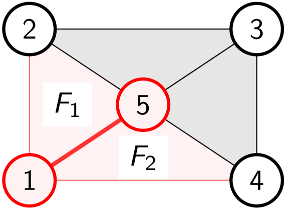

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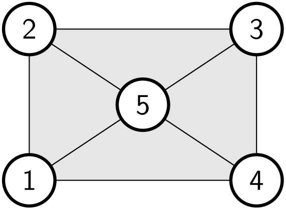

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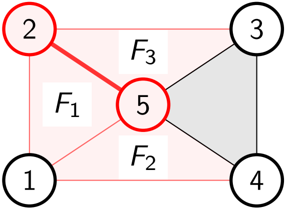

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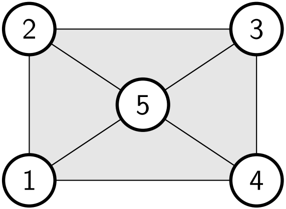

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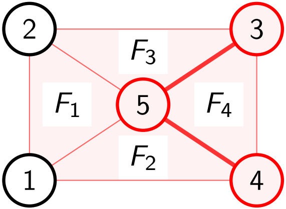

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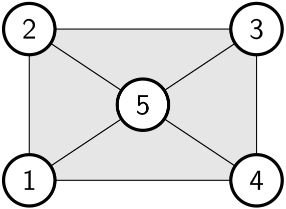

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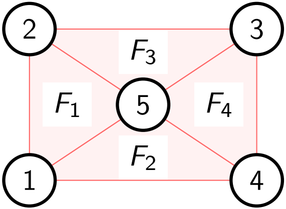

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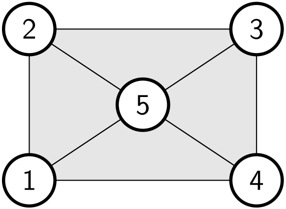

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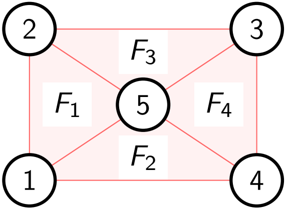

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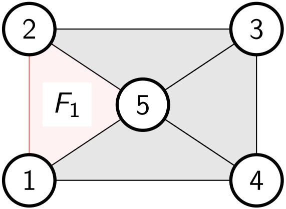

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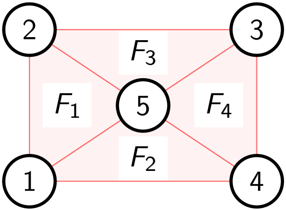

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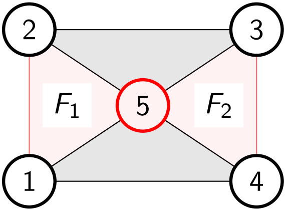

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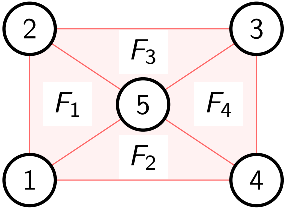

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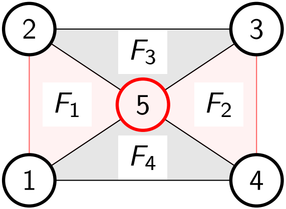
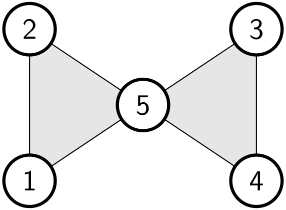


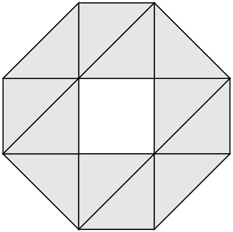
Nonshellable Complexes

The bow-tie complex $\langle 125, 345\rangle$ is not shellable because the only two facets only intersect at the codimension 2 central vertex.



Nonshellable Complexes

Any triangulation of the annulus is not shellable.



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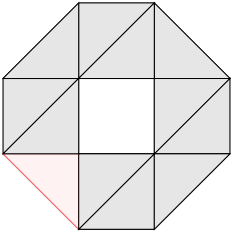

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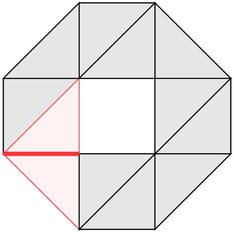

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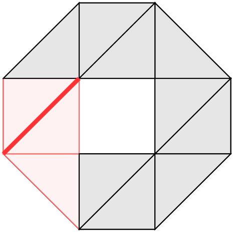

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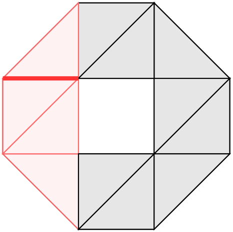

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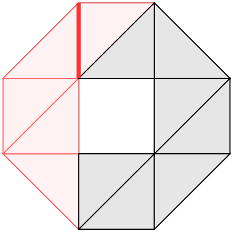

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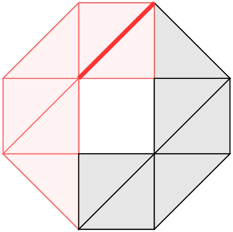

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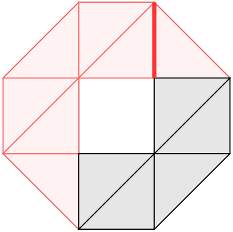

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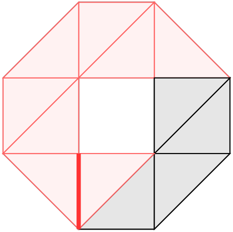

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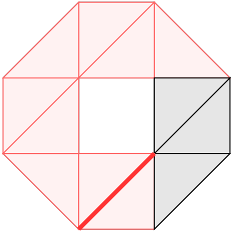

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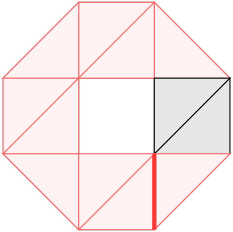

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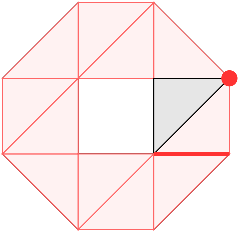


If we try to add in either final facet, it has a codimension 2 portion of the intersection.

Shelling Criteria

When actually verifying that a particular facet order is a shelling order, the following remark is often useful

Remark

An ordering $F_1, F_2, \dots, F_t$ of the facets of a simplicial complex Δ is a shelling if and only if for all i, j such that $1 \le i < j \le t$, there exists $k < j$ such that

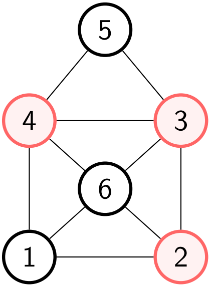
$$F_i \cap F_j \subset F_k \cap F_j \text{ and } |F_k \cap F_j| = |F_j| - 1$$

This is condition we will be using to verify shellability for all proofs in this talk.

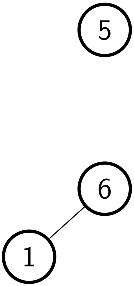
Cut Sets

Definition

Given a Graph $G = (V, E)$, a separating set, or cut set, S is a subset of the vertices such that $G[V \setminus S]$ is disconnected.



$S = \{2, 3, 4\}$



$G[V \setminus S]$

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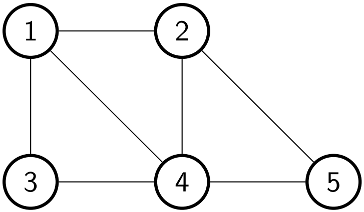
The Cut Complex

Definition

Given a graph $G = (V, E)$ and $k \geq 2$, the k-cut complex of G, $\Delta_k(G)$, is the simplicial complex whose facets are $|V| - k$ sized separating sets of G.
$\Delta_k(G) = \langle S \subseteq V \mid |S| = |V| - k, G[V \setminus S] \text{ is disconnected}\rangle$

The Graph G



The complex $\Delta_2(G)$

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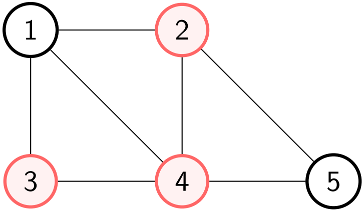

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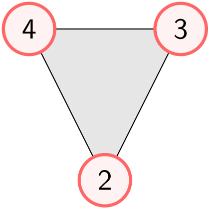

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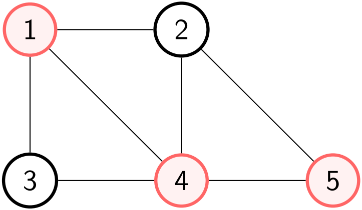

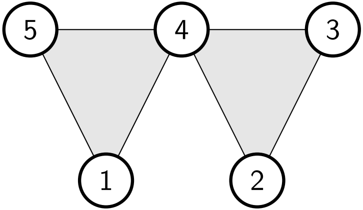

$\Delta_2(G) = \langle 234, 145 \rangle$

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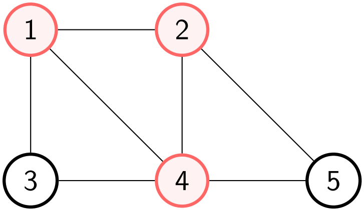

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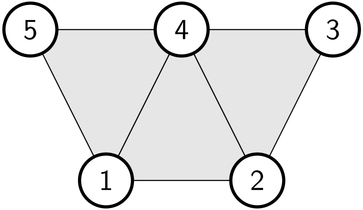


$\Delta_2(G) = \langle 234, 145, 124\rangle$

Fröberg's Theorem

Theorem (Fröberg 1990, Eagon and Reiner 1996)

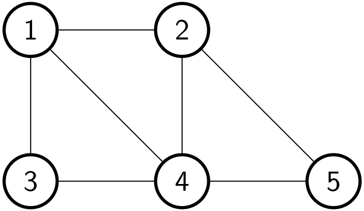
$\Delta_2(G)$ *is shellable if and only if* G *is chordal.*

- The original goal of this project was to extend Fröberg's theorem, which was originally stated in terms of the linear resolutions of edge ideals of graphs, but later reinterpreted by Eagon and Reiner to statements about particular simplical complexes constructed from graphs.
- Cut complexes are an extension of those complexes, and our goal is to understand how the properties of graphs affect the shellability of these complexes.

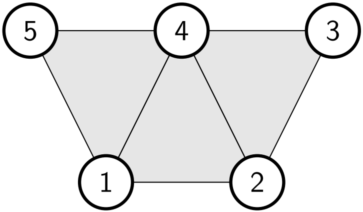
Example of Fröberg's Theorem

We see the graph from the previous example was in fact chordal, so our resulting 2-cut complex was guaranteed to be shellable.

The Graph G



$\Delta_2(G)$



$\Delta_2(G) = \langle 145, 124, 234 \rangle$

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Constructing Graphs with Shellable Complexes

Our first collection of results takes graphs with shellable cut complexes and uses them to construct new graphs with shellable cut complexes. We have results for three graph operations:

- Disjoint Union
- Join
- Wedge Product

Cut Complexes of Disjoint Unions

Definition

Given $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ graphs on disjoint vertex sets, then their disjoint union is $G_1 + G_2 = (V_1 \sqcup V_2, E_1 \sqcup E_2)$.

Theorem

$\Delta_k(G_1 + G_2)$ is shellable if and only if $\Delta_k(G_1)$ and $\Delta_k(G_2)$ are shellable.

This theorem means that the shellability criterion only needs to be checked for the connected components of a graph.

Cut Complexes of Joins

Definition

Given $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ graphs on disjoint vertex sets, then their join is their disjoint union with set of all edges between G_1 and G_2 added as well.

Theorem

$\Delta_k(G_1 * G_2)$ is shellable if and only if either $\Delta_k(G_1)$ or $\Delta_k(G_2)$ is the void complex (it has no faces) and the other is shellable.

In many ways the join of graphs behaves opposite of the union of graphs. It destroys cut complex shellability except in very specific boundary conditions. However, the special case of a single dominating vertex is worth highlighting.

Corollary

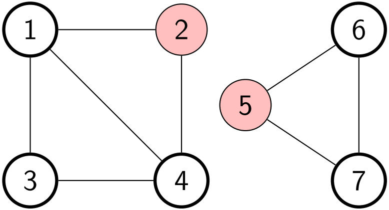
Given a graph G, $\Delta_k(G * 1)$ is shellable if and only if $\Delta_k(G)$ is shellable.

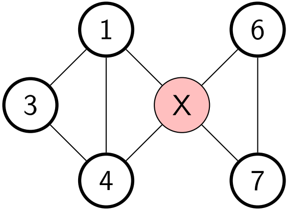
Cut Complexes of Wedge Products

Definition

Given G_1 and G_2, a wedge product of G_1 and G_2, $G_1 \vee G_2$ is formed by taking a vertex from G_1 and a vertex from G_2 and identifying them.

Note: in general the wedge product of two graphs is not unique.





Theorem

$\Delta_k(G_1 \vee G_2)$ is shellable if and only if $\Delta_k(G_1)$ and $\Delta_k(G_2)$ are shellable.

Constructive Corollaries - Threshold Graphs

These three constructive theorems actually gave us our two previous major results as corollaries.

Definition

A threshold graph is a graph constructed from a single vertex by adding a sequence of isolated and dominating vertices.

Corollary

If G is a threshold graph, $\Delta_k(G)$ is shellable for all $k \geq 2$.

The cut complex the graph on one vertex is the void complex for all $k > 2$, so is trivially shellable. Then we just disjoint union and join more of these vertices with trivially shellable complexes onto our graph until we have the full threshold graph, preserving shellability the entire way.

Constructive Corollaries - Trees

Fact

A tree on n vertices is the wedge product of $n-1$ copies of K_2.

Corollary

If G is a tree, $\Delta_k(G)$ is shellable for all $k \geq 2$.

The cut complex for any complete graph is the void complex for all $k > 2$, and so is trivially shellable, so the the wedge product of these shellable graphs is shellable.

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Forbidden Subgraph Characterization

Theorem

Given a graph G, $\Delta_k(G)$ is shellable if and only if $\Delta_k(H)$ is shellable for all induced subgraphs H of G.

Because induced subgraphs preserve cut complex shellability, it means we can describe shellable graphs by avoiding some family of forbidden induced subgraphs.

Recall (Fröberg's Theorem)

Given a graph G, $\Delta_2(G)$ is shellable if and only if G is chordal.

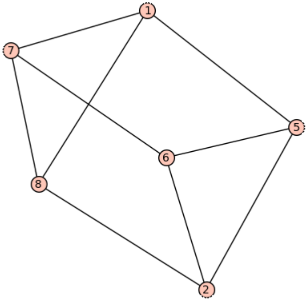
This fits with the forbidden induced subgraph characterization, because a graph is chordal if and only if it has no induced cycle of length greater than 3.

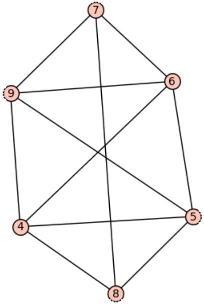
Forbidden Subgraph Characterization

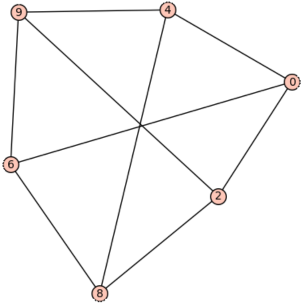
Question

Can we characterize the family of minimal forbidden induced subgraphs for some k?

This question in general seems very hard, but the $k = 3$ case has unique properties that make it more promising.







What Complexes are Cut Complexes?

A natural question is whether these cut complexes have any properties that are shared generally. They are necessarily pure as a consequence of how they are defined, but that is the only guaranteed property.

Theorem

Given a pure simplicial complex Δ, there exists a graph G and a $k \geq 2$ such that $\Delta = \Delta_k(G)$.

The general idea of the construction is you start with a clique whose vertices correspond to the ground set of your complex. Then for each facet you add a vertex attached the the vertices the facet contains. If n is the size of the ground set of Δ, l is the number of facets of Δ and t is the size of the facets of Δ, then $\Delta_{n+l-t}(G) = \Delta$.

Thank You!

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- I also give thanks to my Master's advisor Jay Schweig for initially starting me on this project.