Math 724, Fall 2013
Take-Home Test #2

Instructions: Write up your solutions using LaTeX. You may use books and notes, but you are not allowed to collaborate — you may not consult any human other than the instructor. Solutions are due at the start of class on **Friday, November 15.**

Problem #1 Let $n > 0$ be an integer.

(#1a) [10 pts] How many labeled trees T on vertex set $[n]$ have the property that the degree of every vertex is either 1 or 3? Your answer should be a function of n expressed without summation notation.

Answer: First, n has to be even, otherwise the degree sum is odd, which is impossible. Call a degree-3 vertex a *branch*. The Prüfer code must include each branch exactly twice (so in particular n must be even and $b = (n-2)/2$). There are $\binom{n}{b}$ ways of choosing which vertices should be branches, and once we have made that choice, there are $\binom{n-2}{2,2,\ldots,2} = (n-2)!/2^{(n-2)/2}$ ways of distributing the branches among the slots in the Prüfer code. So the number of trees is

$$\boxed{\binom{n}{(n-2)/2}\frac{(n-2)!}{2^{(n-2)/2}}}$$

(or 0 if n is odd).

(#1b) [10 pts] Let k be an integer with $0 \leq k \leq n$. How many labeled trees on vertex set $[n]$ have the property that vertices $1, 2, 3, \ldots, k$ are all leaves (i.e., each shares an edge with exactly one other vertex)? (The tree can have other leaves as well.) Your answer should be a function of n and k expressed without summation notation.

Answer: The Prüfer codes of such trees are exactly those whose digits are all elements of $\{k+1, \ldots, n\}$. Therefore, the answer is

$$\boxed{(n-k)^{n-2}.}$$

Problem #2 [20 pts] Let $S(k,n)$ denote Stirling numbers of the second kind. Give a combinatorial proof that

$$S(k,n) = \sum_i \binom{k-1}{i-1} S(k-i, n-1)$$

for all positive integers k, n. (By "combinatorial," I mean "explain why both sides of the equation count the same set of objects" — do not give a purely algebraic proof using, say, induction.)

Answer: I claim that the summand on the right-hand side counts the number of set partitions of $X = \{x_1, \ldots, x_k\}$ in which the block B containing x_1 has size i. Indeed, given i, there are $\binom{k-1}{i-1}$ choices for B (since it is determined by choosing $i-1$ elements from the set $\{x_2, \ldots, x_k\}$), and given B, the remaining blocks form a set partition Q of the other $k-i$ elements into $n-1$ blocks; by definition, there are $S(k-i, n-1)$ choices for Q.

Therefore, summing over all i counts all set partitions of X into k blocks (whatever the size of B), and therefore equals $S(k,n)$.

Problem #3 Give combinatorial interpretations for the following numbers. (In other words, describe what they count.)

(#3a) [10 pts] The coefficient of x^k in the infinite product

$$\prod_{n=1}^{\infty}(1+x^n+x^{2n}+\cdots+x^{n^2}).$$

Answer: The coefficient is the number of partitions of k with at most one part of size 1, at most two parts of size 2, ..., at most p parts of size p, ...

(#3b) [10 pts] The coefficient of $q^\ell x^k$ in the infinite product

$$\prod_{n=1}^{\infty}\frac{1}{1-qx^n}.$$

Answer: The coefficient is the number of partitions of k with exactly ℓ parts. Each term $q^r x^{nr}$ in one of the geometric series in the infinite product corresponds to using r parts of size n; the power of q keeps track of the number of parts used (without regard to what the parts actually are).

Problem #4 [20 pts] Let p, q be positive integers and let $C(p, q)$ denote the set of weak compositions of p with q parts. Give an explicit bijection $C(p, q) \to C(q-1, p+1)$.

Answer: Remember that there is a bijection between $C(p, q)$ and ordered lists of p 1's and $q-1$ +'s. If we interchange the two symbols 1 and +, we will get an ordered list of $q-1$ 1's and p +'s. This interchanging is certainly a bijection (it is its own inverse!), and the result corresponds to a weak composition of $q-1$ into $p+1$ parts.

Problem #5 [20 pts] Recall that 1 Galleon is worth 17 Sickles and 1 Sickle is worth 29 Knuts. Suppose that the Ministry introduces a 3-Sickle and a 6-Knut piece (known respectively as a Trickle and a Hexknut). With the new coinage, how many ways are there of making change for a Galleon? (If you are not an expert at Arithmancy, I recommend that you use Sage or another computer algebra system to do the calculation.)

Answer: A Galleon is worth $17 \times 29 = 493$ Knuts. The four available coins are worth 87, 29, 6 and 1 Knuts. Therefore, the answer is the coefficient of x^{493} in the series expansion of

$$\frac{1}{(1-x)(1-x^6)(1-x^{29})(1-x^{87})}.$$

My preferred way to calculate the answer is to have Sage do it:

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sage: F = 1/((1-x)*(1-x^6)*(1-x^29)*(1-x^87))
sage: T = F.taylor(x,0,500)
sage: T.coeff(x^493)
1863
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(If you considered a Galleon itself to be change for a Galleon, you would have gotten 1864.)