Math 725, Spring 2006
Solution Set #2

#1. [West 1.2.40] *Let P and Q be paths of maximum length in a connected graph G. Prove that P and Q have a common vertex.*

Let $v_0, v_1, \ldots, v_n$ and $w_0, w_1, \ldots, w_n$ be the vertices of P and Q respectively. Suppose that $V(P) \cap V(Q) = \emptyset$. Let R be a shortest path of minimum length from a vertex of P to a vertex of Q; by the previous hypothesis, R has at least one edge. Suppose that the endpoints of R are v_j and w_k. We may assume without loss of generality that $j, k \geq n/2$. (Otherwise, we can just reverse the labeling of the vertices of P and/or Q.)

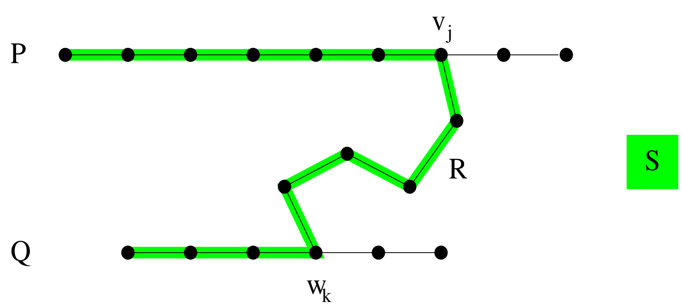
Label the vertices of R as

$$v_j,\ z_1,\ \ldots,\ z_s,\ w_k$$

where $s = e(R) - 1 \geq 0$. Note that no internal vertex of R belongs to $V(P)$ or to $V(Q)$; if it did, then R would contain a proper subpath from P to Q, which would contradict its minimality. That means that the walk S given by

$$w_0,\ v_1,\ \ldots,\ v_j,\ z_1,\ \ldots,\ z_s,\ w_k,\ w_{k-1},\ \ldots,\ w_0$$

is a path (see figure), and by construction it contains $j + k + e(R) > n$ edges, a contradiction.



#2. [West 1.3.24] *Prove that $K_{3,2}$ is not a subgraph of any hypercube Q_n.*

Recall that $V(Q_n) = \{0,1\}^n$, the set of bit strings of length n. Write $\bar{b}$ for the complement of a bit b; that is, $\bar{0} = 1$ and $\bar{1} = 0$.

Finding a copy of $K_{3,2}$ inside Q_n is identical with finding two distinct vertices v, w with three distinct common neighbors. In particular, v, w must be in the same partite set of Q_n, and they must be at mutual distance 2. By permuting the bits, we can assume without loss of generality that

$$\begin{aligned} v &= b_1 b_2 b_3 \cdots b_n, \\ w &= \bar{b}_1 \bar{b}_2 b_3 \cdots b_n. \end{aligned}$$

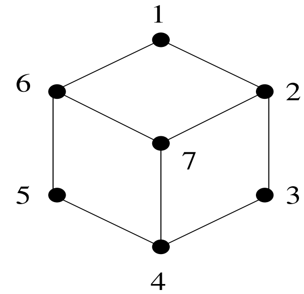
Then the only bit strings that have $n-1$ bits in common with both v and w are

$$\bar{b}_1 b_2 b_3 \cdots b_n, \qquad b_1 \bar{b}_2 b_3 \cdots b_n.$$

That is, $|N(v) \cap N(w)| = 2$, and we are done.

#3.
emphDoes every connected graph G with $\delta(G) \geq 2$ have a connected Eulerian spanning subgraph? Either prove that it does, or give a counterexample.

No. For example, any graph G with a cut-edge and $\delta(G) \geq 2$ is a counterexample. So is $K_{2,3}$, and so is the following graph G (formed by deleting any one vertex from Q_3):



Note that $\delta(G) = 2$. If $H \subseteq G$ is a connected Eulerian spanning subgraph, then $\delta(H) \geq 2$; in particular, H must contain all the edges incident to vertices 1, 3, and 5 (the outer hexagon in the figure above). In addition, it must contain at least one edge incident to 7, and the other endpoint of that edge (2, 4 or 6) will have degree 3, so H cannot be Eulerian.

#4. [West 1.4.10] *Prove that a digraph D is strongly connected if and only if for each partition of its vertex set $V(D) = S \sqcup T$, with S, T nonempty, there is an edge whose tail is in S and whose head is in T.*

If D is strongly connected and $V = S \sqcup T$, then let P be a minimal S,T-path. By the same reasoning as in problem 1.2.22 (#10 on HW#1), P must consist of a single edge.

Now suppose that D satisfies the given condition; we want to show that D is strongly connected. For variety, here is an argument based on a standard trick in topology. Fix $s \in V$ and let R be the set of vertices reachable from s; that is, R is the set of vertices for which D contains an s,t-walk. Notice that D cannot contain any edge e whose tail is in R and whose head is in $V - R$. (If t is the tail and h is the head, then there is an s,t-walk in D (because $t \in R$) and appending e to this path produces an s,h-walk, which says that $h \in R$, but by hypothesis $h \notin R$.) Doesn't this contradict the given condition? Not quite—it means that either $R = V$ or $R = \emptyset$. Well, certainly $s \in R$, so it must be the case that $R = V$. That is, every vertex is reachable from s. But s was chosen arbitrarily, so D must be strongly connected.

#5. [West 2.1.29]

(a) Prove that every tree is bipartite.
(b) Let X, Y be a bipartition of a tree T, and suppose that $|X| \geq |Y|$. Prove that X contains a leaf of T.

(a.) This is immediate from Theorem 1.2.18; a tree is acyclic, so in particular contains no odd cycles.

(b.) Let $n = n(T)$, so $|X| \geq n/2$. Notice that

$$\sum_{x \in X} d_T(x) \;=\; e(T) \;=\; n - 1, \tag{$*$}$$

because each edge of T has exactly one endpoint in X, hence contributes 1 to the sum on the left-hand side. (Of course, the same is true for Y; this fact might be called "bipartite handshaking".) On the other hand, if $d_T(x) \geq 2$ for every $x \in X$, then

$$\sum_{x \in X} d_T(x) \;\geq\; 2|X| \;\geq\; n,$$

which contradicts $(*)$.

#6. *Let T be a tree with ℓ leaves. Prove that T is a* caterpillar *(that is, there is some path in T that either contains or is incident to every edge) if and only if its diameter is $\ell - 2$.*

Oops.

#7. [West 2.1.37] *Let T, T' be two spanning trees of a connected graph G. For every $e \in E(T) - E(T')$, prove that there exists an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ and $T - e + e'$ are both spanning trees of G.*

Let x, y be the endpoints of e. The forest $T - e$ has two components, whose vertex sets partition $V(G)$. Let the vertex sets of these components be X and Y, where $x \in X$ and $y \in Y$.

The tree T' has a unique u, v-path P, whose length r is at least 2 (since $e \notin T'$). Label the vertices of P as

$$x = z_0,\ z_1,\ z_2,\ \ldots,\ z_{r-1}, z_r = y.$$

Notice that $z_0 \in X$ and $z_r \in Y$. Therefore, there is some index i such that $z_{i-1} \in X$ and $z_i \in Y$. (For example, we could take i to be the maximum index such that $z_i \in Y$.) Let $e' = z_{i-1}z_i$.

Since $T - e$ is a forest with two components (namely X, Y) and the edge e' has one endpoint in each of X, Y, it follows that $T - e + e'$ is a tree.

Since $T' + e$ is connected and has a cycle containing e' (namely $P \cup \{e\}$), it follows that $T' + e - e'$ is also connected. Moreover, $|T' + e - e'| = |T'|$. So $T' + e - e'$ is a tree.

Notice that it is *not* sufficient to require only that e' have one endpoint in each of X and Y. That condition does imply that $T - e + e'$ is a tree, but not that $T' + e - e'$ is. Similarly, it is not enough to pick e' to be a member of the unique cycle of $T' + e$; this guarantees that $T' + e - e'$ is a tree but not that $T - e + e'$ is.

Bonus problem: *Recall that an* orientation *of a graph G is a digraph whose underlying graph is G. Let G be connected. Prove that G has a strong orientation if and only if it has no cut-edge.*

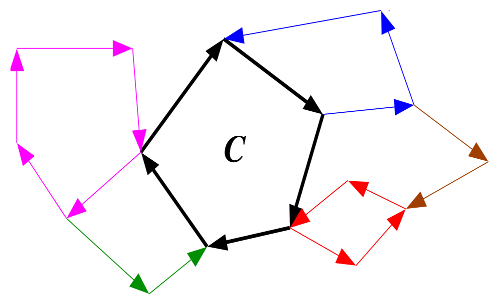
Suppose that G has a cut-edge $e = xy$. Let X, Y be the components of $G - e$ containing x, y respectively. If D is any orientation of G, then either e is oriented as there is $\vec{xy}$, when D has no edge from X to Y, or as $\vec{xy}$, when D has no edge from Y to X. In either case D is not strong by problem #4 above.

For the converse, here is an algorithm to orient the edges of G. I'll show that the algorithm works and produces a strong orientation D. Let's assume for simplicity that G has no loops.

Let C be a cycle in G. Orient the edges of C consistently (i.e., in one of the two ways that make C into a directed cycle). Let $S = V(C)$.

`while` $S \neq V(G)$ `do`
{
    `Let` $e = v_0v_1$ `be an edge with` $v_0 \in S$ `and` $v_1 \notin S$
    `Find a path` $P = v_1, v_2, \ldots, v_n = v_0$ `in the graph` $G - e$
    `Let` k `be the smallest index` > 0 `such that` $v_k \in S$
    `Orient the edges of` P `as` $\overrightarrow{v_1v_2}, \ldots, \overrightarrow{v_{k-1}v_k}$
    `Let` $S := S \cup \{v_1, \ldots, v_{k-1}\}$
} `Orient the remaining edges of` G `arbitrarily`

In this figure, C is the original cycle. The paths P oriented in each iteration are shown in different colors.



At the start of the algorithm, $D[S]$ (the vertex-induced subdigraph of D with vertex set S) is strongly connected, since it contains the cycle C. The same is true after each iteration. Also, there will always exist an edge e while $S \neq V(G)$, for if no such edge exists then $G[S]$ would be a maximal connected subgraph, which contradicts the condition that G is connected. Since e is not a cut-edge, we can always choose a v_1, v_0-path in $G - e$. Moreover, the set S increases in cardinality after each iteration (because at least one vertex, namely v_1, has been added to it). Since $V(G)$ is finite, this implies that the loop eventually terminates, and when it does, the oriented edges form a strong orientation of a subgraph of G, which we then extend to a strong orientation of G.