Math 290 Final Exam Review Problems

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1. Let A be an arbitrary square matrix. Prove that A and A^T have the same eigenvalues.

2. Explain why every 3×3 matrix must have at least one eigenvalue.

3. Let V be a vector space and let $T : V \to V$ be a linear transformation. Prove that

$$\text{nullity}(T \circ T) \geq \text{nullity}(T).$$

4. Let A be a square matrix such that the sum of the entries in every row is zero. Prove that A is singular. (Hint: Find a nonzero vector in the nullspace of A?)

5. Let A and B be the following sets of vectors in R^3.

$$\begin{aligned} A &= \Big\{(2,-1,6),\ (3,1,-2)\Big\}, \\ B &= \Big\{(2,-1,6),\ (1,2,-8),\ (11,2,0)\Big\}. \end{aligned}$$

Prove that span(A)=span(B). Is A linearly independent? Is B linearly independent?

6. Let V and W be vector spaces. Their *intersection* X (usually denoted by $V \cap W$) is the set of all vectors that belong to *both* V and W. (For example, if V is the x-axis in R^2 and W is the y-axis, then X has only one point, namely the origin.) Prove that X is a vector space.

7. Let V be a vector space, and let S be a spanning set for V. Let T be a subset of V such that every element of S can be written as a linear combination of elements of T. Prove that T is a spanning set.

8. Let V be the vector space of 2×3 matrices. Suppose that B is a subset of V that is linearly independent, but not spanning. What are the possibilities for the size of B?

9. Let M be a $2x2$ matrix whose eigenvalues are 2 and -3. Prove that $M + I$ and $-M$ have the same eigenvalues. (Here I is the 2x2 identity matrix.) (Note: We don't know any rules about what happens to eigenvalues when you apply an operation such as adding I or multiplying by a scalar, so you will have to figure out those rules from scratch.)

10. Consider the $4x4$ matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & 0 \\ \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -1 & -2 & -4 & -7 \\ 0 & -3 & -5 & -8 \\ 0 & 0 & -6 & -9 \\ 0 & 0 & 0 & -10 \end{bmatrix}.$$

Find $\det(ABA^{-1}B^{-1})$ while doing as little work as possible.

11. Consider the matrix

$$M = \begin{bmatrix} 5 & 1 \\ 2+q & -q \end{bmatrix}$$

where q is a real number. Find all values of q so that M has (a) two, (b) one, (c) zero real eigenvalues.

12. Remember that if A is an $n \times n$ square matrix, then the *characteristic equation* of A is the equation

$$\det(\lambda I - A) = 0,$$

where I is the $n \times n$ identity matrix. Explain how to use the characteristic equation to prove that the determinant of A is the product of its eigenvalues.

13. A square matrix A is called *nilpotent* if some power of A is the zero matrix (that is, $A^m = O$ for some m). Prove that if A is nilpotent, then it has no nonzero eigenvalues.

14. Let V be the vector space of all $n \times n$ matrices, and let Q be an invertible $n \times n$ matrix. Define a function $T : V \to V$ by $T(M) = QMQ^{-1}$. First, prove that T is a linear transformation. Then, prove that T is an isomorphism.

15. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that T is onto, but not one-to-one. Prove that $\dim(V) > \dim(W)$.