Math 821 Problem Set #5
Posted: Friday 4/1/11
Due date: Wednesday 4/13/11

Problem #1 In class on Friday, I asserted that if G is a graph with n vertices and c connected components, and M is the signed vertex-edge incidence matrix of G, then $\operatorname{rank} M = n - c$. Prove this statement (over any ground field).

Solution: Let v_e denote the column of M corresponding to an edge e.

We induct on the number of edges of G. If G has no edges, then $n = c$ and $\operatorname{rank} M = 0$ (because M has no columns). Now, suppose that the statement is true for G. Let G' be the graph formed by adding an edge ij to G, and let M' be its signed vertex-edge incidence matrix (obtained from M by adding a single column).

— If G already has a path P from i to j, then v_{ij} is in the linear span of $\{v_e \mid e \in P\}$ (for example, if all these edges are oriented consistently, then these columns sum to zero). In this case G and G' have the same number of connected components, and $\operatorname{rank} M' = \operatorname{rank} M$.

— If G has no path from i to j, then v_{ij} is linearly independent of the columns of M, for the following reason. Let H be the set of rows corresponding to the vertices of the connected component C of G containing i. Then for each column v_e of M, the entries in rows H are either all zero (if $e \notin C$) or include one $+1$ and one -1 (if $e \in C$). In either case, the sum of these entries is zero. That property does not hold for v_{ij}. which has a single nonzero entry in the rows H (specifically in row i). Therefore, v_{ij} is not in the column span of M. It follows that $\operatorname{rank} M' = \operatorname{rank} M + 1$, and meanwhile G' has one fewer connected component than G. $\square$

Problem #2 Fix a ground field $\mathbb{F}$ and a nonnegative integer n. Let V_k be the vector space with basis $\{\sigma_A\}$, where A ranges over all k-element subsets of $\{1, 2, \ldots, n\}$. Define a linear transformation $\partial_k : V_k \to V_{k-1}$ as follows: if $A = \{a_1, \ldots, a_k\}$ with $a_1 < \cdots < a_k$, then

$$\partial_k(\sigma_A) = \sum_{i=1}^{k} (-1)^{i+1} \sigma_{A \setminus \{a_i\}}.$$

(Having defined ∂_k on the basis elements, it extends uniquely to all of V_k by linearity.)

(#2a) Prove that $\partial_k \circ \partial_{k+1} = 0$ for all k. (Note: I know this calculation is done explicitly in Hatcher, but it is so important that everyone should do it for themselves at least once!) Conclude that

$$\operatorname{im} \partial_k \subseteq \ker \partial_{k+1}.$$

Solution: Let $A = \{a_1 < \cdots < a_{k+1}\}$ be a basis element for V_{k+1}. Then $\partial_k(\partial_{k+1}(\sigma_A))$ will be a sum of elements σ_B, where

$$B = \{a_1, \ \ldots, \ \widehat{a_i}, \ \ldots, \ \widehat{a_j}, \ \ldots, \ a_k\}$$

with $1 \leq i < j \leq k + 1$. This term arises twice:

- once from deleting a_j when applying ∂_{k+1} (with sign $(-1)^{j+1}$) and from deleting a_i when applying ∂_k (with sign $(-1)^{i+1}$);
- once from deleting a_i when applying ∂_{k+1} (with sign $(-1)^{i+1}$) and from deleting a_j when applying ∂_k (with sign $(-1)^j$, since a_j is now the $(j - 1)$st smallest element)

Therefore, the coefficient with which σ_B appears is

$$(-1)^{j+1}(-1)^{i+1} + (-1)^{i+1}(-1)^j = 0$$

and this proves that $\partial^2 = 0$.

(#2b) For $n = 3$, write out the maps ∂_k as explicit matrices.

Solution:

$$\partial_3 = \begin{array}{c} \\ 12 \\ 13 \\ 23 \end{array}\!\!\begin{array}{c} 123 \\ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \end{array}, \qquad \partial_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 12 & 13 & 23 \end{array} \\ \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \end{array}, \qquad \partial_1 = \emptyset \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \end{array}.$$

(#2c) Prove that for every k, the set $\{\partial_k(\sigma_A) : 1 \in A\}$ is a basis for the vector space $\operatorname{im} \partial_k$.

Solution: To show linear independence: Let $A_1, \dots, A_s$ denote the k-element subsets of $[n]$ containing 1. Then for each i, $B_i := A_i \setminus \{1\}$ is a subset of no other A_j. Therefore, in the row corresponding to B_i, the column vector $\partial_k(\sigma_{A_i})$ has a ± 1, but every other $\partial_k(\sigma_{A_j})$ has a zero. It follows that no $\partial_k(\sigma_{A_i})$ is in the span of any other.

To show that the given set spans: Suppose that $C = \{a_1 < \cdots < a_k\}$ does not contain 1. Then

$$\partial_{k+1}(\sigma_{\{1,a_1,\dots,a_k\}} = \sigma_C + \sum_{i=1}^{k} (-1)^i \sigma_{\{1,a_1,\dots,\widehat{a_i},\dots,a_k\}}$$

and hitting both sides with ∂_k gives

$$0 = \partial_k \sigma_C + \sum_{i=1}^{k} (-1)^i \partial_k \sigma_{\{1,a_1,\dots,\widehat{a_i},\dots,a_k\}}$$

or

$$\partial_k \sigma_C = -\sum_{i=1}^{k} (-1)^i \partial_k \sigma_{\{1,a_1,\dots,\widehat{a_i},\dots,a_k\}}$$

which verifies that $\partial_k \sigma_C$ is in the span of $\{\partial_k(\sigma_A) : 1 \in A\}$.

(#2d) Use (2c) to prove that in fact $\operatorname{im} \partial_k = \ker \partial_{k-1}$. (Hint: By (1), all you have to show is that these vector spaces have the same dimension.)

Solution: By (2c), we have

$$\dim \operatorname{im} \partial_k = |\{\partial_k(\sigma_A) : 1 \in A\}| = \binom{n-1}{k-1}$$

and

$$\dim \ker \partial_{k-1} = \dim V_{k-1} - \operatorname{rank} \partial_{k-1} = \binom{n}{k-1} - \binom{n-1}{k-2} = \binom{n-1}{k-1}$$

as desired. (Note: This calculation amounts to proving that a simplex is acyclic.)

Problem #3 Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Describe $\operatorname{coker} A$ (i) if A is regarded as a linear transformation over $\mathbb{Q}$; (ii) if A is regarded as a linear transformation over $\mathbb{Z}$; (iii) if A is regarded as a linear transformation over $\mathbb{F}_q$ (the finite field with q elements).

Solution: (i) $\det A = -2$, so A is invertible over $\mathbb{Q}$. Therefore A defines an isomorphism $\mathbb{Q}^3 \to \mathbb{Q}^3$ and $\operatorname{coker} A = 0$.

(ii) Over $\mathbb{Z}$, $\det A = -2$ means that $\operatorname{coker} A \cong \mathbb{Z}/2\mathbb{Z}$. In fact, $\operatorname{im} A$ consists exactly of those vectors in $\mathbb{Z}^3$ whose entries sum to an even number. Therefore, any vector whose entries sum to an odd number can be taken as a generator for $\operatorname{coker} A$.

(iii) If q is odd, then A is nonsingular and $\operatorname{coker} A = 0$. If q is even, then A has rank 2 and $\operatorname{coker} A \cong \mathbb{F}_q$. (Note: Recall that $\mathbb{F}_2$ is not the only field with an even number of elements — in general, any power of a prime can be the order of a finite field. So the latter case, $\operatorname{coker} A \cong \mathbb{F}_q$, holds whenever q is a power of 2.)

Problem #4 Let $R = \mathbb{F}[x_1, \dots, x_n]$ be the ring of polynomials in n variables over a field $\mathbb{F}$. A *squarefree monomial* in R is a product of distinct indeterminates (e.g., $x_1x_4x_5$, but not $x_1x_5^2$). Let I be an ideal generated by squarefree monomials of degree ≥ 2.

(#4a) Show that the set

$$\Delta = \{\sigma \subset [n] \mid \prod_{i \in \sigma} x_i \notin I\}$$

is an abstract simplicial complex on n vertices. (This is called the *Stanley-Reisner complex* of I — or, alternately, I is the Stanley-Reisner ideal of Δ.)

(Note: I chose not to assign points for this problem because I don't think it was clear that I was looking for a topological description in (b). I hope to come back to this topic later.)

Solution: Neither 1 nor any single variable belong to the ideal. Since every multiple of an element in I is an element of I, it follows that every divisor of a non-element of I is also not an element of I — which corresponds to the condition that every subset of an element of Δ is also an element of Δ.

(#4b) Describe Δ in the case that I is (i) the zero ideal; (ii) generated by a single monomial of degree d; (iii) generated by all monomials of degree d for some $d \leq n$; (iv) (assuming $n = 2m$ is even) generated by the degree-2 monomials x_1x_2, x_3x_4, $\dots$, $x_{2m-1}x_{2m}$.

(i) Δ is a simplex with n vertices.
(ii) If $d = n$, then Δ is the interior of a d-vertex simplex, hence homeomorphic to S^{n-2}. If $d < n$, then Δ is an iterated $(n-d)$-fold cone over S^{n-2}, so homeomorphic to B^{n-2} (i.e., a closed $(n-2)$-ball).
(iii) The maximal faces of Δ are exactly the $(d-1)$-element subsets of $[n]$. That is, Δ is the $(d-2)$-dimensional skeleton of the n-vertex simplex.
(iv) Δ can be realized as the simplicial sphere whose vertices are $\pm e_1$, $\pm e_2$, $\dots$, $\pm e_m$, where $e_1, \dots, e_m$ are the standard basis vectors in $\mathbb{R}^m$. (E.g., if $m = 2$ then Δ is a diamond; if $m = 3$ then Δ is an octahedron.) In general $\Delta \cong S^{m-1}$.