Infinity

1 Problems involving infinity

What does "infinite" mean? How do you measure the size of an infinite set? Do line segments of different lengths have the same number of points? Are there more points in a big line segment or a small triangle? These questions are hard, and have occupied mathematicians for millennia,

To make things a bit more concrete, consider two line segments of lengths 1 and 2; for example, the intervals $A = [0, 1]$ and $B = [0, 2]$ on the number line. (Strictly speaking, we're thinking of each of A and B as a set of real numbers. For example, $\pi/4 \in A$, $\pi/2 \in B$, and $\pi/2 \notin A$.)[1]

On the one hand, A is a proper subset of B (that is, every point in A is also in B, but not vice versa), so it is plausible that A should have fewer points than B does. On the other hand, both sets are clearly infinite, so they ought to have the same size.

Actually, neither of these arguments is correct. Just because A is a subset of B does <u>not</u> mean that A has fewer points than B. In fact, these two intervals have exactly the same sizes. On the other hand, something even more mind-boggling is true: there are lots of different sizes of infinite sets — in fact, infinitely many different infinities!

2 Greek ideas about infinity

The ancient Greeks regarded infinity not as a number, but as sort of an unattainable concept; Aristotle distinguished between "actual infinity" (which doesn't exist) and "potential infinity" (which does). An example of this is Euclid's prime number theorem (see "Prime numbers" from the course notes). Today, we would phrase the theorem as

"There are infinitely many prime numbers,"

or

"The set of all prime numbers is infinite,"

but Euclid said,

"No finite set of prime numbers can possibly be the complete list of all prime numbers."

Another kind of phrasing you might see in ancient mathematical writings is

"No finite set of prime numbers exhausts all the prime numbers."

The Greeks found ways to do mathematics without resorting to infinity. An example is Archimedes' "quadrature of the parabola", whose ideas anticipate modern integral calculus, but are phrased differently.

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[1] Remember, the symbols $\in$ and $\notin$ mean "is a member of" and "is not a member of" respectively.
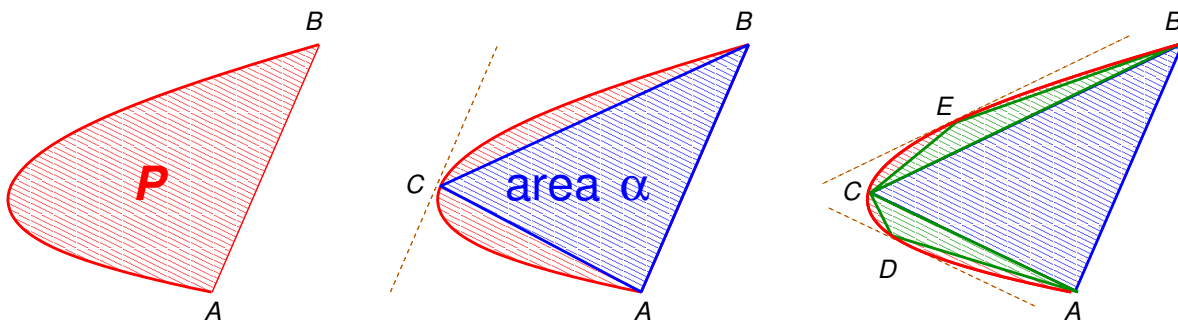
A **parabolic segment** is the region bounded by a piece of a parabola cut off by a line segment, such as the red shaded region P shown above. Archimedes wanted to calculate the area of a parabolic segment. Today, we'd probably use calculus to express the area as a definite integral, but Archimedes didn't have calculus in his toolkit (for that matter, he didn't even have coordinates). Instead, he came up with the following geometric construction.

1. Find the point C on the parabola such that the tangent line at C is parallel to $\overline{AB}$. Draw the triangle ΔABC. (See figure above, center.)

2. Find points D, E on the parabola where the tangent lines are parallel to AC and BC respectively. Draw the triangles ΔADC and ΔCEB. (See figure above, right.)

3. Repeat this process to draw four more triangles, say $\Delta AWD, \Delta DXC, \Delta CYE, \Delta EZB$ (not shown).

4. Repeat this process to draw eight more triangles.

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Next, Archimedes showed (for the purposes of this discussion, never mind how) that each triangle has one-eighth the area of its parent triangle. Let α be the area of ΔABC; then

$$\text{area}(\Delta ADC) = \text{area}(\Delta CEB) = \alpha/8,$$
$$\text{area}(\Delta AWD) = \text{area}(\Delta DXC) = \text{area}(\Delta CYE) = \text{area}(\Delta EZB) = \alpha/64,$$

etc. So the total area covered in the first n steps is

$$\alpha \left(1 + 2 \cdot \frac{1}{8} + 4 \cdot \frac{1}{64} + \cdots + 2^{n-1} \cdot \frac{1}{8^{n-1}} \right) \; = \; \alpha \left(1 + \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^{n-1}} \right).$$

This is a partial sum of a geometric series; it is equal to

$$\frac{4\alpha}{3} \left(1 - \left(\frac{1}{4} \right)^n \right). \tag{1}$$

A modern mathematician would say that the area of the parabolic segment is therefore

$$\text{Area}(P) \; = \; \lim_{n \to \infty} \frac{4\alpha}{3} \left(1 - \left(\frac{1}{4} \right)^n \right) \; = \; \frac{4\alpha}{3}.$$

Archimedes, remember, didn't have tools like limits. Instead, he reasoned as follows:

- Any number $x < 4\alpha/3$ must be less than the area of P, because if you repeat this process enough times, the area covered by the triangles eventually exceeds x.

- Any number $y > 4\alpha/3$ must be greater than the actual area of P, because no matter how many triangles you draw, the area they cover (namely the quantity in formula (1)) is always less than $4\alpha/3$.

- Therefore, the area of P must be exactly $4\alpha/3$.

This reasoning is perfectly correct, and actually contains some fairly deep ideas about limits and sequences and things hidden inside it — but notice how it's phrased so as to avoid any explicit mention of infinity, or limits, or convergence, or infinitesimals.

3 Bijections

Suppose that F and G are two sets. How can we tell if F and G are the same size? For that matter, what does "size" (or "cardinality", which is the technical term) mean?

To say that F has n elements (symbolically, $|F| = n$) is to say that there's a function

$$q : F \to \{1, 2, \ldots, n\}$$

that is one-to-one and onto. Such a function is called a <u>bijection</u>.

So, $|F| = |G|$ if there exist bijections

$$q : F \to \{1, 2, \ldots, n\}, \qquad r : G \to \{1, 2, \ldots, n\}$$

for some nonnegative integer n.

On the other hand, we don't need to know the actual size of two sets to know that they are the same cardinality. We can just say that $|F| = |G|$ if there exists a bijection $b : F \to G$ — in other words, if it is possible to pair the elements of F with the elements of G so that every element of F has exactly one "mate" in G.

This definition of "same size" has the advantage that it applies to infinite sets as well. A good way to think about a bijection between two sets is as a way of labelling the elements of one set with the elements of the other set, using each label exactly once.

Under this definition, the intervals $[0, 1]$ and $[0, 2]$ have the same cardinality, because there is a bijection between them, namely $f : [0, 1] \to [0, 2]$ defined by $f(x) = 2x$. It doesn't matter that the first interval is a proper subset of second — in fact, the rule "If A is a proper subset of B, then $|A| < |B|$" applies **if and only if** A is finite.

This definition, while sensible, has a number of striking consequences. For example, the function $f(x) = 2x$ defines a bijection between the infinite sets

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\},$$
$$\mathbb{E} = \{0, 2, 4, 6, \ldots\}.$$

So $|\mathbb{N}| = |\mathbb{E}|$, even though there are infinitely many numbers that are in $\mathbb{N}$ but not in $\mathbb{E}$!

In addition, the sets $\mathbb{Z}^2$ (ordered pairs of integers), $\mathbb{N}^2$ (ordered pairs of natural numbers), and even $\mathbb{Q}$ (rational numbers) have the same cardinality as $\mathbb{N}$. Sets like this are called **countably infinite**. For example, to label all the elements of $\mathbb{N}^2$ with the elements of $\mathbb{N}$, we can arrange the elements of $\mathbb{N}^2$ in a square grid, draw a zig-zag path that covers all of them, then label the path in order.
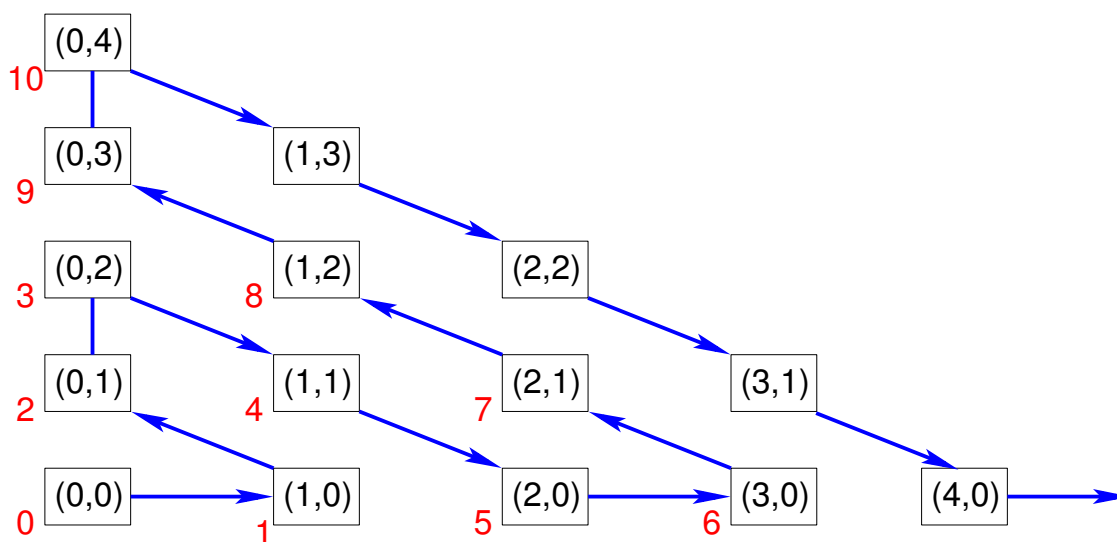
On the other hand, the set $\mathbb{R}$ of real numbers is <u>not</u> countably infinite. The astonishing truth, proved by Georg Cantor, is that there can exist no bijection between $\mathbb{N}$ and $\mathbb{R}$. (Stay tuned!)