On Kostant's Partition Function

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Outline

- Definitions and examples
- A collection of results
    - Representations of Lie algebras
    - Connection to polyhedral geometry
- Throughout I will sprinkle the talk with open problems and conjectures.

Goal: Intrigue you enough so that you'll want to work with me and my friends on these problem or so that you recommend students to come and work with me at UW Milwaukee!

Vector Partition Functions

Let A be an $m \times d$ integral matrix.

Goal: Compute the vector partition function

$$\phi_A(\mathbf{b}) = \#\{\mathbf{x} \in \mathbb{N}^d \ : \ A\mathbf{x} = \mathbf{b}\}$$

defined for $\mathbf{b}$ in the nonnegative linear span of the columns of A.

Applications in

- Number Theory (partitions)
- Representation theory (weight multiplicities)
- Commutative Algebra (Hilbert Series)
- Algebraic Geometry (toric varieties)
- Discrete Geometry (polyhedra)
- Optimization (integer programming)

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Kostant's partition function

Bertram Kostant (1928-2017)

Kostant's partition function is a vector partition function where the columns of the matrix A are the positive roots of a Lie algebra.

Classical Lie algebras

- Type A: $\mathfrak{sl}_{r+1}(\mathbb{C}) = \{X \in M_{r+1}(\mathbb{C}) : Tr(X) = 0\}$.
- Type B: $\mathfrak{so}_{2r+1}(\mathbb{C}) = \{X \in M_{2r+1}(\mathbb{C}) : X^t = -X\}$.
- Type C: $\mathfrak{sp}_{2r}(\mathbb{C}) = \{X \in M_{2r}(\mathbb{C}) : X^t J = -JX\}$, where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ with I the $r \times r$ identity matrix.
- Type D: $\mathfrak{so}_{2r}(\mathbb{C}) = \{X \in M_{2r}(\mathbb{C}) : X^t = -X\}$.

In these cases the Lie bracket is the commutator bracket. That is, if $X, Y \in \mathfrak{g}$, then $[X, Y] = XY - YX$.

Roots of the classical Lie algebra of type A_r

Let e_i be a standard basis element of $\mathbb{R}^{r+1}$.

If

$$\alpha_i = e_i - e_{i+1}$$

then the positive roots are

$$\phi^+ = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \cup \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j : 1 \leq i < j \leq r\}.$$

The vectors in ϕ^+ are the columns of matrix A.

Kostant's partition function in type A_2

The positive roots are

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Thus $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$.

Given a vector $\mathbf{b} \in Col(A)$, Kostant's partition function would return the number of ways to express $\mathbf{b}$ as a linear combination of the vectors in Φ^+ where the coefficients are nonnegative integers.

Denote this count by $\wp(\mathbf{b})$.

Kostant's partition function in type A_2

Lemma

If $\mathbf{b} = n_1\alpha_1 + n_2\alpha_2 = \begin{bmatrix} n_1 \\ n_2 - n_1 \\ -n_2 \end{bmatrix}$ *with* $n_1, n_2 \in \mathbb{N} = \{0, 1, 2, \ldots\}$,

then

$$\wp(\mathbf{b}) = min(n_1, n_2) + 1.$$

Proof.

Suffices to count how many times the root $\alpha_1 + \alpha_2$ can be used. Since it can be used from 0 to $min(n_1, n_2)$ times, the result follows. ☐

Kostant's partition function in type A_3

The positive roots are

$$\begin{aligned}\Phi^+ &= \{\alpha_1,\ \ \alpha_2,\ \ \alpha_3,\ \ \alpha_1+\alpha_2,\ \ \alpha_2+\alpha_3,\ \ \alpha_1+\alpha_2+\alpha_3\}\\ &= \left\{ \begin{bmatrix} 1\\-1\\0\\0\end{bmatrix}, \begin{bmatrix} 0\\1\\-1\\0\end{bmatrix}, \begin{bmatrix} 0\\0\\1\\-1\end{bmatrix}, \begin{bmatrix} 1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix} 0\\1\\0\\-1\end{bmatrix}, \begin{bmatrix} 1\\0\\0\\-1\end{bmatrix} \right\}\end{aligned}$$

Thus $A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1\\ -1 & 1 & 0 & 0 & 1 & 0\\ 0 & -1 & 1 & -1 & 0 & 0\\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}.$

Kostant's partition function in type A_3

Theorem (G. Ngwe, C. Perez, A. Siddiqui, 2017)

If $n_1, n_2, n_3 \in \mathbb{N}$, then the value of $\wp(n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3)$ is given by computing

$$1 + min(n_1, n_2) + min(n_2, n_3) + min(n_1, n_2, n_3) + \sum_{i=1}^{min(n_1,n_2)} min(n_2 - i, n_3)$$

$$+ \sum_{i=1}^{min(n_1,n_2,n_3)} min(n_1 - i, n_2 - i) + \sum_{i=1}^{min(n_1,n_2,n_3)} min(n_2 - i, n_3 - i)$$

$$+ \sum_{i=1}^{min(n_1,n_2,n_3)} \sum_{j=1}^{min(n_1-i,n_2-i)} min(n_2 - i - j, n_3 - i).$$

Computational formula in type A_3

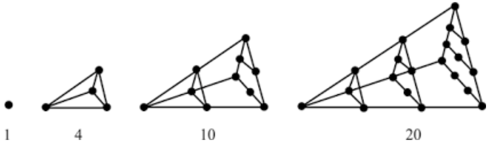
Theorem (G. Ngwe, C. Perez, A. Siddiqui, 2017)

If $\mathbf{b} = n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3$, *where* $n_1, n_2, n_3 \in \mathbb{N}$, *then*

$$\wp(\mathbf{b}) = \begin{cases} \frac{1}{6}(n_2+1)(n_2+2)(n_2+3) & \text{if } n_1, n_3 \geq n_2 \\ 1 + 2n_3 + n_2 + 3\binom{n_3}{2} + \binom{n_3}{3} + 2n_3(n_2 - n_3) & \text{if } n_1 \geq n_2 \geq n_3 \\ 1 + n_1 + 2n_3 + 2n_1n_3 + (-2 - n_1 - n_3 + \frac{2n_3}{3} + \frac{1}{3})(\frac{n_3(n_3+1)}{2}) & \\ +n_3^2(n_1+1) & \text{if } n_2 \geq n_1 \geq n_3 \text{ and } n_2 - n_3 \geq n_1 \\ \frac{1}{6}(6 - n_1^3 + n_2^3 - 2n_3^3 + (3n_3^2 + 3n_1^2)(-1 + n_2) + n_2(2 - 3n_2) & \\ +n_1(4 + 6n_2 - 3n_2^2) + n_3(5 - 3n_1^2 + 6n_2 - 3n_2^2 + n_1(3 + 6n_2)) & \text{if } n_2 \geq n_1 \geq n_3 \text{ and } n_2 - n_3 \leq n_1 \end{cases}$$

Corollary (Tetrahedral numbers)

If $\mathbf{b} = n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3$, *with* $n_1, n_3 \geq n_2$ *and* $n_1, n_2, n_3 \in \mathbb{N}$, *then* $\wp(n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3) = \frac{(n_2+1)(n_2+2)(n_2+3)}{6}$.



Open questions

There are no general known closed formulas for Kostant's partition function.

It was shown by Blakley (1964) that there exists a finite decomposition of $\mathbb{N}^d$ such the partition function ϕ_A associated to the matrix A is a polynomial of degree $n-d$ on each piece.

Finding these polynomials and the decomposition of $\mathbb{N}^d$ is still a very active area of research.

Generalizing Kostant's partition function

The q-analog of Kostant's partition function $\wp_q$ a polynomial valued function in q whose evaluation at $q = 1$ recovers Kostant's partition function.

Namely,

$$\wp_q(\mathbf{b}) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots + a_k q^k$$

where a_i is the number of ways to write the vector $\mathbf{b}$ as a sum of exactly i positive roots.

q-analog of KPF result

Every Lie algebra has a highest root, it is the positive root with largest coefficient sum. In type A_r it is well-known that

$$\wp_q(\tilde{\alpha}) = q(1+q)^{r-1}.$$

Question: What about in other Lie types?

Erik Insko

Mohamed Omar

We q-counted the number of ways to express the highest root of a classical Lie algebra as a nonnegative integral sum of the positive roots for Lie algebras of type B, C, D.

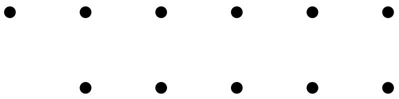
Gravity Diagrams

A *antigravity diagram* is a diagram with n up-adjusted columns of dots representing multiple copies of the simple roots α_1 through α_n and horizontal and vertical lines (under some conditions) representing positive roots.

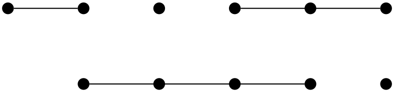
For example the highest root of the Lie algebra of type B_6 is given by

$$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6$$

an antigravity diagram can be



Example



This antigravity diagram corresponds to the partition

$$\{\alpha_1 + \alpha_2,\ \ \alpha_3,\ \ \alpha_4 + \alpha_5 + \alpha_6,\ \ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,\ \ \alpha_6\}$$

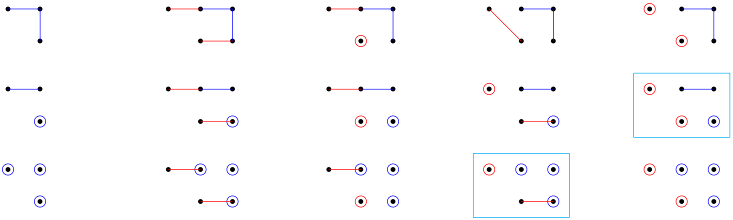
of

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6.$$

Extending gravity diagrams

Using gravity diagrams and *extensions* we were able to find the generating formulas for the number of ways to express the highest root as a sum of positive roots.

B_2 B_3



Main Result

Theorem (H., Insko, Omar - 2016)

The closed formulas for the generating functions $\sum_{r\geq 1}\mathcal{P}_{B_r}(q)x^r$, $\sum_{r\geq 1}\mathcal{P}_{C_r}(q)x^r$, *and* $\sum_{r\geq 4}\mathcal{P}_{D_r}(q)x^r$, *are given by*

$$\begin{aligned}
\sum_{r\geq 1}\mathcal{P}_{B_r}(q)x^r &= \frac{qx+(-q-q^2)x^2+q^2x^3}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2},\\
\sum_{r\geq 1}\mathcal{P}_{C_r}(q)x^r &= \frac{qx+(-q-q^2)x^2}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2},\\
\sum_{r\geq 4}\mathcal{P}_{D_r}(q)x^r &= \frac{(q+4q^2+6q^3+3q^4+q^5)x^4-(q+4q^2+6q^3+5q^4+3q^5+q^6)x^5}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2},
\end{aligned}$$

where $\mathcal{P}_{B_r}(q)$, $\mathcal{P}_{C_r}(q)$, *and* $\mathcal{P}_{D_r}(q)$ *denote* $\wp_q(\tilde{\alpha})$ *when* $\tilde{\alpha}$ *is the highest root of the Lie algebras of type* B_r, C_r, *and* D_r, *respectively.*

Explicit Formulas

$$\begin{aligned}
\text{Type } A_r\ (r \geq 1): \quad \mathcal{P}_{A_r}(q) &= q(1+q)^{r-1}, \\
\text{Type } B_r\ (r \geq 2): \quad \mathcal{P}_{B_r}(q) &= b_1(q) \cdot (f_1(q))^{r-2} + b_2(q) \cdot (f_2(q))^{r-2}, \\
\text{Type } C_r\ (r \geq 3): \quad \mathcal{P}_{C_r}(q) &= c_1(q) \cdot (f_1(q))^{r-1} + c_2(q) \cdot (f_2(q))^{r-1}, \\
\text{Type } D_r\ (r \geq 4): \quad \mathcal{P}_{D_r}(q) &= d_1(q) \cdot (f_1(q))^{r-4} + d_2(q) \cdot (f_2(q))^{r-4},
\end{aligned}$$

where

$$f_1(q) = \frac{(q^2+2q+2) + q\sqrt{q^2+4}}{2}, \quad f_2(q) = \frac{(q^2+2q+2) - q\sqrt{q^2+4}}{2}$$

and

$$b_1(q) = \frac{(q^5+q^4+5q^3+4q^2+4q) + (q^4+q^3+3q^2+2q)\sqrt{q^2+4}}{2(q^2+4)},$$

$$b_2(q) = \frac{(q^5+q^4+5q^3+4q^2+4q) - (q^4+q^3+3q^2+2q)\sqrt{q^2+4}}{2(q^2+4)},$$

$$c_1(q) = \frac{\left(q^3+4q\right) + q^2\sqrt{q^2+4}}{2(q^2+4)}, \qquad c_2(q) = \frac{\left(q^3+4q\right) - q^2\sqrt{q^2+4}}{2(q^2+4)},$$

$$d_1(q) = \frac{(q^7+3q^6+10q^5+16q^4+25q^3+16q^2+4q) + (q^6+3q^5+8q^4+12q^3+9q^2+2q)\sqrt{q^2+4}}{2(q^2+4)},$$

$$d_2(q) = \frac{(q^7+3q^6+10q^5+16q^4+25q^3+16q^2+4q) - (q^6+3q^5+8q^4+12q^3+9q^2+2q)\sqrt{q^2+4}}{2(q^2+4)}.$$

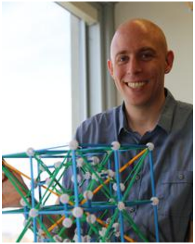
Connection to Juggling!

Setting $q = 1$ in the generating functions for the Lie algebras of type B and C recovers the generating functions used by Butler and Graham to count the number of certain multiplex juggling sequences.

In fact a lot more was true!!!

Math friends

Carolina Benedetti



Christopher Hanusa

Alejandro Morales

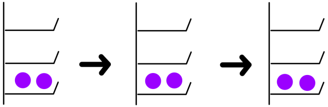
Anthony Simpson

Example

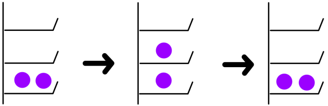
In type A_2: $\tilde{\alpha} = \alpha_1 + \alpha_2$, and $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. The set of partitions of $2\tilde{\alpha}$ are

$$\{2\alpha_1, 2\alpha_2\}, \{2(\alpha_1 + \alpha_2)\}, \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$$

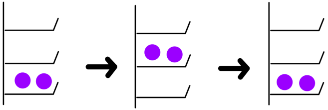
The multiplex juggling sequences with start and end states $\langle 2 \rangle$, hand capacity 2 and length 2 are:



$$\{2\alpha_1, 2\alpha_2\}$$



$$\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$$



$$\{2(\alpha_1 + \alpha_2)\}$$

Juggling take away

The key insight for the results is the equivalence between the throwing of a ball during a juggling sequence at

time i to height j

and the positive root

$$e_i - e_{i+j}$$

appearing in a partition of a weight of a Lie algebra of type A_r.

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Annals of Combinatorics

Kostant's Partition Function and Magic Multiplex Juggling Sequences

Carolina Benedetti, Christopher R. H. Hanusa, Pamela E. Harris, Alejandro H. Morales and Anthony Simpson

Open problems:

- What is the juggling analog of Kostant's partition function for exceptional Lie algebras?
- Study the behavior of the number of throws made in sets js(s, e, m, t).

Kostant's weight multiplicity formula

The multiplicity of a weight μ in $L(\lambda)$ is given by

$$m(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp(\sigma(\lambda + \rho) - (\mu + \rho)),$$

where W denotes the Weyl group, $\wp$ denotes Kostant's partition function and $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

Complications

The following complications arise when using Kostant's weight multiplicity formula and its q-analog:

$$m(\lambda,\mu) = \sum_{\sigma\in W} (-1)^{\ell(\sigma)} \wp(\sigma(\lambda+\rho)-(\mu+\rho))$$

1. Closed formulas for the value of Kostant's partition function are not known in much generality.
2. The order of the Weyl group grows exponentially as the rank $r \to \infty$.

My work has focused on addressing both of these complications.

The support of KWMF

Theorem (P. H. - 2011, K. Chang - 2019)

Let $\tilde{\alpha}$ be the highest root of the Lie algebra of type A_r. Then the number of Weyl group elements contributing to the multiplicity $m(\tilde{\alpha}, 0)$, is given by F_r the r^{th} Fibonacci number.

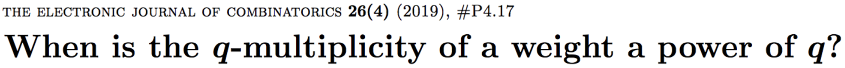
Theorem (P.H., E. Insko, L. K. Williams UWM'13 - 2016)

The number of Weyl group elements contributing nontrivially to the multiplicity of the zero-weight in the adjoint representation of a finite-dimensional classical Lie algebra can be enumerate through linear homogeneous recurrence relations with constant coefficients.

Theorem (P. H., K. Chang - 2019)

Let λ be the sum of the simple roots of the Lie algebras of type B_r and C_r. Then the number of Weyl group elements contributing to the multiplicity $m(\lambda, 0)$, is given by a multiple of the r^{th} Lucas number.

A conjecture



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When is the q-multiplicity of a weight a power of q?

Margaret Rahmoeller

Lisa Schneider

Anthony Simpson

Conjecture

If λ and μ are pair of weights for which $m(\lambda, \mu) = 1$, then $m_q(\lambda, \mu) = q^{f(r)}$, where $f(r)$ is a function in the rank of the Lie algebra.

Kostant's Partition function and polytopes

Integral polytopes

P a polytope in $\mathbb{R}^N$ with integral vertices:

P is the **convex hull** of finitely many vertices $\mathbf{v}$ in $\mathbb{Z}^N$

OR

P is the intersection of finitely **half spaces**

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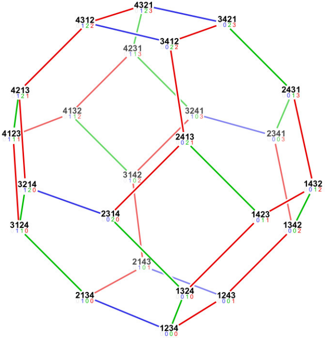

(wikipedia)

Permutahedron: convex hull of permutations $(w_1, w_2, \ldots, w_n)$

Volume of polytopes

- $\mathrm{vol}(P)$ is normalized volume $= \dim(P)! \cdot$ euclidean volume

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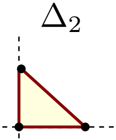
- $\#P \cap \mathbb{Z}^N$ number of lattice points (discrete volume)

Volume of polytopes

- $\mathrm{vol}(P)$ is normalized volume $= \dim(P)!\cdot$ euclidean volume
- $\# P \cap \mathbb{Z}^N$ number of lattice points (discrete volume)

Example:

standard simplex $\Delta_n = \{(x_1, \ldots, x_n) \mid \sum x_i \leq 1,\ x_i \geq 0\}$



euclidean volume $\frac{1}{2}$

(normalized) volume 1

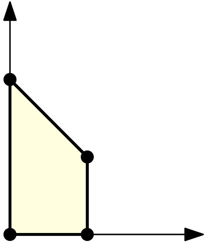
3 lattice points

Pitman-Stanley polytope

$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$

$$\mathrm{PS}_n(\mathbf{a}) = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \,\middle|\, (x_1, \ldots, x_n) \preceq (a_1, \ldots, a_n) \right\}$$

Example



$PS_2(1,1)$

Pitman-Stanley polytope

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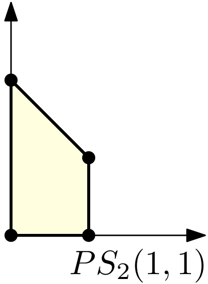
Example



Pitman-Stanley polytope

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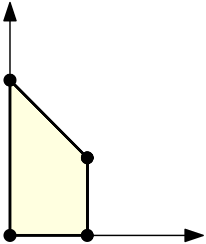
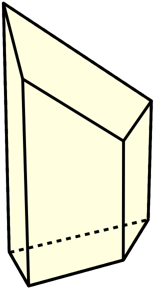
Example



$PS_2(1,1)$



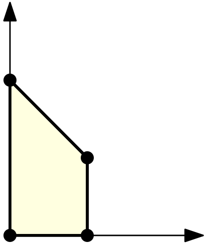
$PS_3(1,1,1)$

Pitman-Stanley polytope

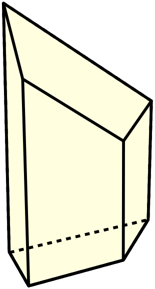
$\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$

$$\mathrm{PS}_n(\mathbf{a}) = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \;\middle|\; \begin{array}{c} x_1 \leq a_1 \\ x_1 + x_2 \leq a_1 + a_2 \\ \vdots \\ x_1 + \cdots + x_n \leq a_1 + \cdots + a_n \end{array} \right\}$$

Example



$PS_2(1,1)$



$PS_3(1,1,1)$

- 2^n vertices and is n dimensional

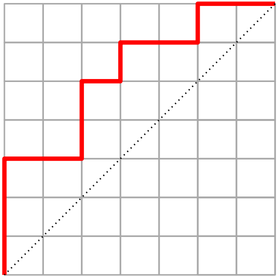
volume Pitman-Stanley polytope

Theorem (Pitman-Stanley 01)

$$\operatorname{vol} \mathrm{PS}_n(\mathbf{a}) = \sum_{\mathbf{j} \succeq (1,\ldots,1)} \binom{n}{j_1, \ldots, j_n} a_1^{j_1} \cdots a_n^{j_n}$$

$$= \sum_{\mathsf{f}\ \text{parking function}} a_{\mathsf{f}(1)} \cdots a_{\mathsf{f}(n)}$$

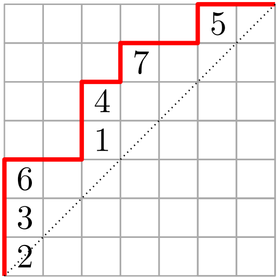
parking functions



A parking function of n is:

- a Dyck path from $(0, 0)$ to (n, n)

parking functions



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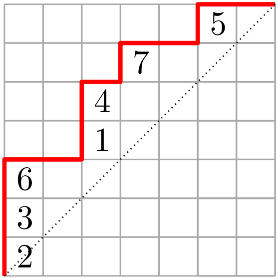

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encodes parking preferences n cars one way street

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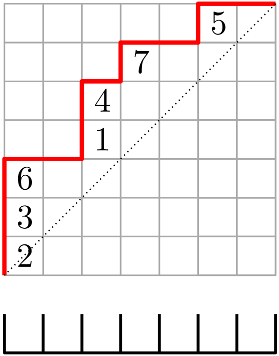

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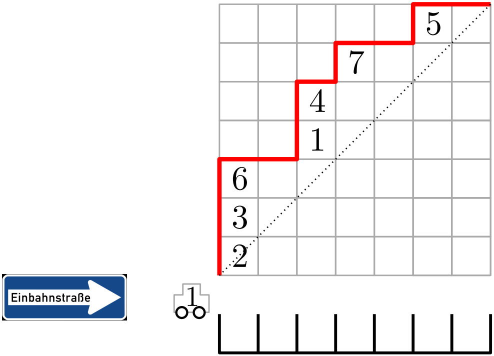

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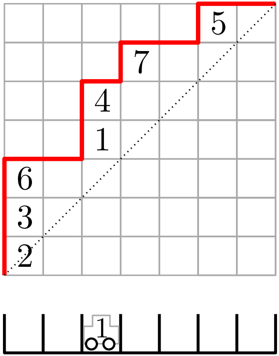

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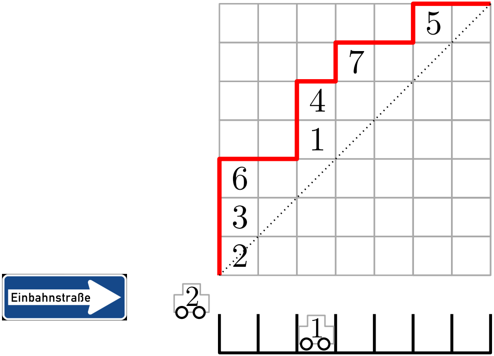

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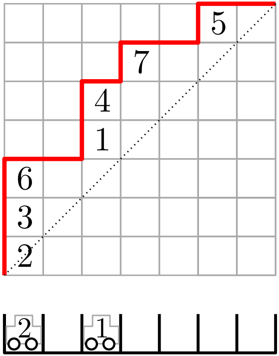

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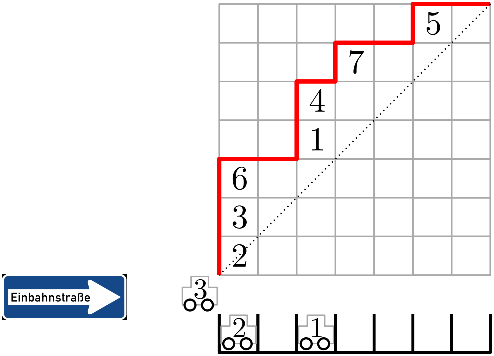

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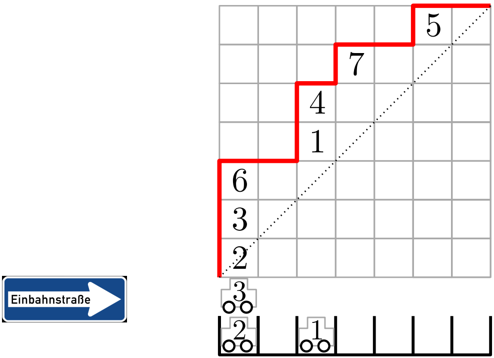

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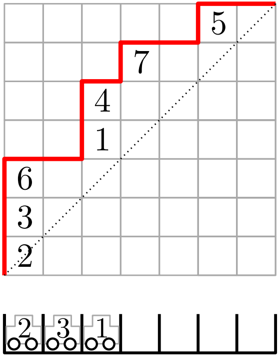

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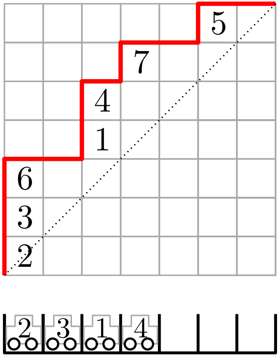

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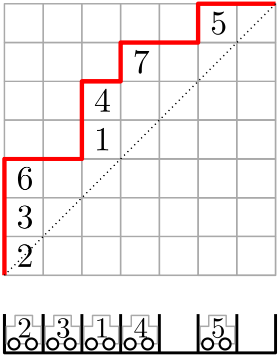

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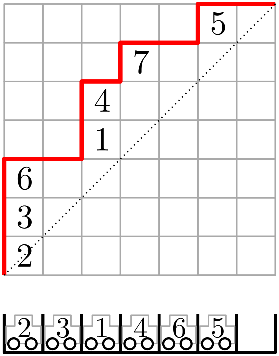

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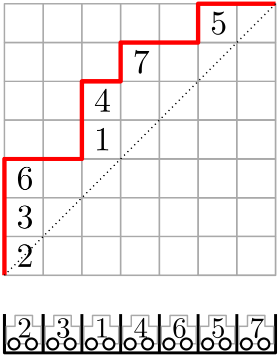

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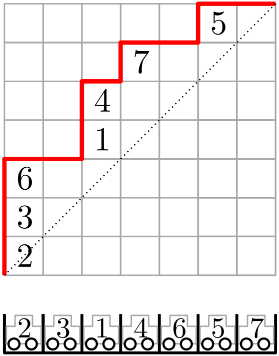

There are $(n+1)^{n-1}$ parking functions of size n.

Flow polytopes

G graph $n+1$ vertices m edges

$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$

$\mathcal{F}_G(\mathbf{a}) = \{$flows $x(\epsilon) \in \mathbb{R}_{\geq 0}$, $\epsilon \in E(G) \mid$ netflow vertex $i = a_i\}$

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Example

$$x_{12} + x_{13} + x_{14} = a_1$$

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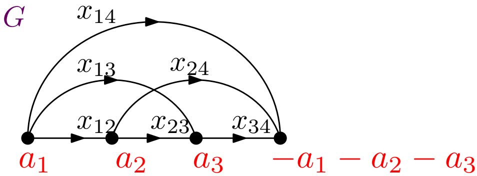

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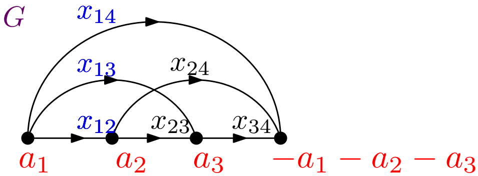

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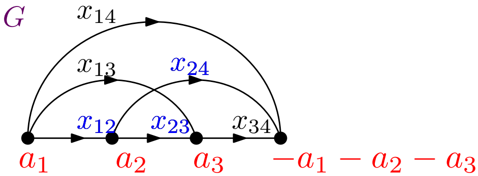

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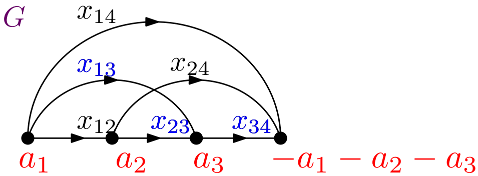

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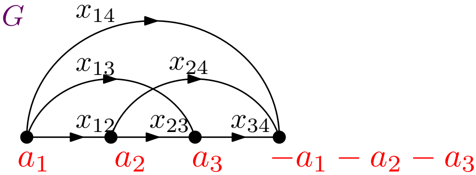

lattice points of $\mathcal{F}_G(\mathbf{a})$ are integral flows on G with netflow $\mathbf{a}$
let $K_G(\mathbf{a}) := \#(\mathcal{F}_G(\mathbf{a}) \cap \mathbb{Z}^{\#E(G)})$

Kostant's vector partition function

When G is complete graph k_{n+1}, $K_{k_{n+1}}(\mathbf{a})$ is called **Kostant's partition function**.

$K_{k_{n+1}}(\mathbf{a}) = \#$ of ways of writing $\mathbf{a}$ as an $\mathbb{N}$-combination of vectors
$e_i - e_j$

Kostant's vector partition function

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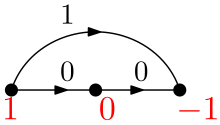
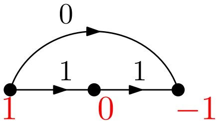


$(1, 0, -1) = e_1 - e_3$

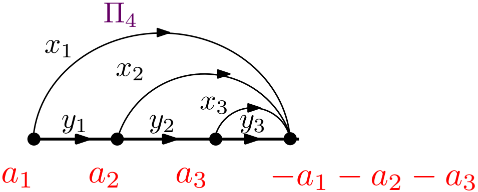


$(1, 0, -1) = (e_1 - e_2) + (e_2 - e_3)$

Examples flow polytopes

$\mathcal{F}_G(\mathbf{a}) = \{$flows $x(\epsilon) \in \mathbb{R}_{\geq 0}$, $\epsilon \in E(G) \mid$ netflow vertex $i = a_i\}$

Example (Baldoni-Vergne 2008)



$$x_1 + y_1 = a_1 \quad \longrightarrow \quad x_1 \leq a_1$$

$$x_2 + y_2 - y_1 = a_2 \quad \longrightarrow \quad x_1 + x_2 \leq a_1 + a_2$$

$$x_3 + y_3 - y_2 = a_3 \quad \longrightarrow \quad x_1 + x_2 + x_3 \leq a_1 + a_2 + a_3$$

$\mathcal{F}_{\Pi_{n+1}}(\mathbf{a})$ is the Pitman-Stanley polytope!

Lidskii volume formula

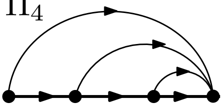
Theorem (Baldoni-Vergne 08, Postnikov-Stanley 08)

G m edges, $n+1$ vertices, $a_i \geq 0$

$$\mathrm{vol}\mathcal{F}_G(a_1,\ldots,a_n) = \sum_{\mathbf{j}\succeq\mathbf{o}} \binom{m-n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n} \times K_G(j_1-o_1,\ldots,j_n-o_n,0)$$

where $\mathbf{o}=(o_1,\ldots,o_n)$, $o_v = outdeg(v)-1$

Pitman-Stanley polytope:



$$\mathrm{vol}\mathcal{F}_{\Pi_{n+1}}(\mathbf{a}) = \sum_{\mathbf{j}\succeq(1,\ldots,1)} \binom{n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n} \cdot 1$$

Parking function model Lidskii formula

$$\mathrm{vol}\mathcal{F}_G(1,\ldots,1) = \sum_{\mathbf{j}\succeq\mathbf{o}} \binom{m-n}{j_1,\ldots,j_n} \times K_G(j_1-o_1,\ldots,j_n-o_n,0)$$

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Unified diagram model (B-G D'L-H-H-K-M-Y 18)

Parking function model Lidskii formula

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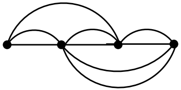
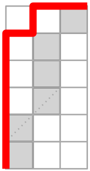
Unified diagram model (B-G D'L-H-H-K-M-Y 18)

- $\displaystyle\sum_{\mathbf{j}\succeq(out_1,\ldots,out_n)}$: lattice paths above diagonal with $(out_1,\ldots,out_n)$ boxes.

Example



$(out_1,out_2,out_3)=(2,3,1)$



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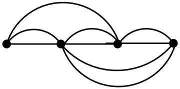

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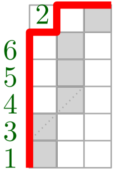

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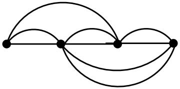
- $K_G(j_1-o_1,\ldots,j_n-o_n,0)$: dot line diagram under the path

Example



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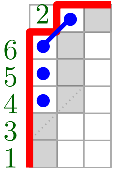
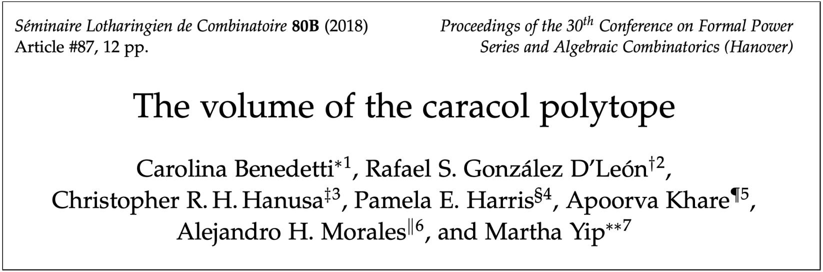


Proof of concept



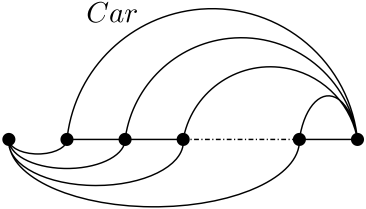
Séminaire Lotharingien de Combinatoire **80B** (2018)
Article #87, 12 pp.

Proceedings of the 30th Conference on Formal Power Series and Algebraic Combinatorics (Hanover)

The volume of the caracol polytope

Carolina Benedetti[*1], Rafael S. González D'León[†2],
Christopher R. H. Hanusa[‡3], Pamela E. Harris[§4], Apoorva Khare[¶5],
Alejandro H. Morales[‖6], and Martha Yip[**7]

Application unified model: Caracol polytope



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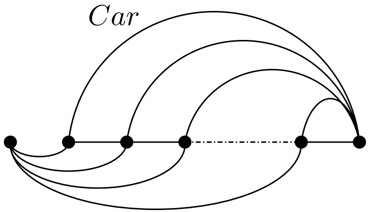

$P := \mathcal{F}_{Car}(1, 0, \ldots, 0, -1)$ $\mathrm{vol}(P) = \mathsf{Cat}_{n-1}$ (Postnikov)

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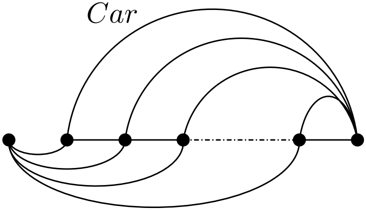

$P := \mathcal{F}_{Car}(1, 0, \ldots, 0, -1) \quad \mathrm{vol}(P) = \mathsf{Cat}_{n-1}$ (Postnikov)

$Q := \mathcal{F}_{Car}(0, 1, \ldots, 1, -n) \quad \mathrm{vol}(Q) = n^{n-2}$ (Pitman-Stanley)

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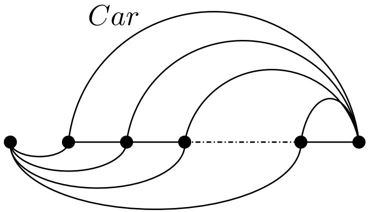

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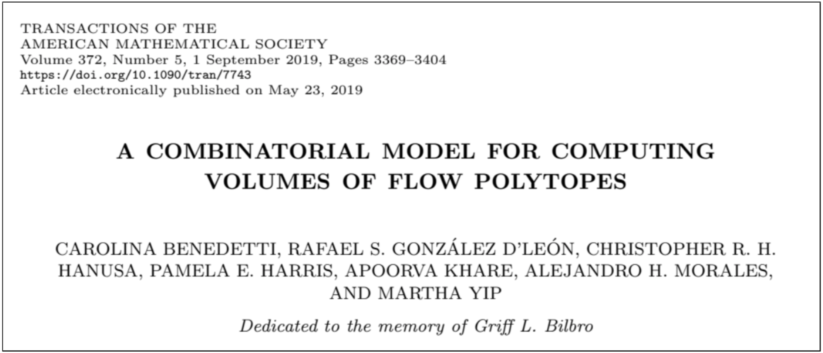
Theorem (B-G D'L-H-H-K-M-Y 18)

$\mathrm{vol}\mathcal{F}_{Car}(1, 1, \ldots, 1, -n-1) = \mathsf{Cat}_{n-1} \cdot (n+1)^{n-1}$

A new model!



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Volume 372, Number 5, 1 September 2019, Pages 3369–3404
https://doi.org/10.1090/tran/7743

A COMBINATORIAL MODEL FOR COMPUTING VOLUMES OF FLOW POLYTOPES

CAROLINA BENEDETTI, RAFAEL S. GONZÁLEZ D'LEÓN, CHRISTOPHER R. H. HANUSA, PAMELA E. HARRIS, APOORVA KHARE, ALEJANDRO H. MORALES, AND MARTHA YIP

Dedicated to the memory of Griff L. Bilbro

Open problem

- The Chan-Robbins-Yuen (CRY) polytope

$$\mathcal{F}_{K_{n+2}}(1, 0, 0, \ldots, -1)$$

has volume the product of consecutive Catalan numbers. Zeilberger proved this by evaluating the Morris constant term identity, but no combinatorial proof is known!

Thank You!

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