Math 724, Fall 2013
Homework #4 Solutions

(#1) Problem #128.

Since the rules say that each recipient gets at least m objects, let's do that first. We are now faced with the problem of distributing $k-mn$ identical objects to n distinct recipients with no further restrictions. This is equivalent to counting the number of solutions in nonnegative integers to the equation $x_1+\cdots+x_n = k-mn$. By Problem #127, this is equivalent to the number of ways of arranging $k-mn$ dots and $n-1$ bars from left to right, which can be written in several equivalent ways:

$$\boxed{\binom{k-n(m-1)-1}{n-1} = \binom{k-mn+n-1}{n-1} = \binom{k-mn+n-1}{k-mn}}$$

To put it another way, there is a bijection

$$\{(y_1,\dots,y_n)\in\mathbb{Z}^n\colon\ y_i\geq m\ \forall i,\quad y_1+\cdots+y_n=k\} \xrightarrow{\phi}$$
$$\{(x_1,\dots,x_n)\in\mathbb{Z}^n\colon\ x_i\geq 0\ \forall i,\quad x_1+\cdots+x_n=k-mn\}$$

given by $\phi(y_1,\dots,y_n)=(y_1-m,\dots,y_n-m)$. The domain is the set we are trying to count, and the range is the set counted in Problem #127 (and equivalently in the previous problems).

(#2) Problem #129.

This is a special case of #128 with $m=1$. The answer:

$$\boxed{\binom{k-1}{n-1} = \binom{k-1}{k-n}}$$

Note that this is zero if $k<n$, which makes sense — if there are more shelves than books than some shelf must remain empty.

(#3) Problem #136.

The problem asks for the number of partitions of a $(k=9)$-element set (the sandwiches) into $n=3$ nonempty subsets (the lunch bags); this is the Stirling number $S(9,3)=\boxed{3025.}$ Here is Sage code to calculate Stirling numbers recursively and print out a table of all $S(k,n)$ for $1\leq k\leq 10$, $1\leq n\leq k$:

```
def S(k,n):
    if k==0 and n==0:
        result = 1
    elif n < 1 or n > k:
        result = 0
    elif n == 1 or k == n:
        result = 1
    else:
        result = S(k-1,n-1) + n * S(k-1,n)
    return result

for k in range(1,11):
    print [S(k,n) for n in range(1,k+1)]
```

Note that $S(0,n)=1$ if $n=0$ and $S(0,n)=0$ for $n>0$ — there is one set partition of the empty set, and its number of blocks is zero. Also, in Sage, `range(a,b)` means $\{x\in\mathbb{Z}\colon\ a\leq x<b\}$.

(#4) Problem #137.

The answer is

$$\boxed{\frac{9!}{(3!)^4} = 280.}$$

Imagine numbering the sandwiches $1, \dots, 9$ and putting $1, 2, 3$ in one bag, $4, 5, 6$ in the second bag, and $7, 8, 9$ in the last bag. There are 9! ways to number the sandwiches. However, the order of each triple doesn't matter (so we have overcounted by $(3!)^3$) and in addition the order of the triples themselves doesn't matter (so we have overcounted by an additional factor of 3!).

This is a special case of the general formula for #139, which says that the number of partitions of k distinct elements into k_1 blocks of size 1, k_2 blocks of size 2, …, is

$$\frac{k!}{\prod_i i!^{k_i}\ k_i!}$$

(#5) Problem #142.

(a) 123, 1|23, 2|13, 3|12, 1|2|3. See? Five of them. In general

$$B(k) = \sum_{n=0}^{k} S(k, n). \tag{0.1}$$

(b,c) We can construct a set partition of $[k]$ recursively by first choosing the block X that 1 belongs to, then choosing a set partition of the remaining numbers. Say $|X| = r$; then there are $\binom{n-1}{r-1}$ possibilities for X (since we need to choose the blockmates of 1) and then $n - r$ remaining numbers to partition. Therefore, the Bell numbers satisfy the recurrence

$$B(0) = 1, \qquad B(k) = \sum_{r=1}^{k} \binom{k-1}{r-1} B(k - r) \quad \text{for } k \geq 1. \tag{0.2}$$

Here is a Sage implementation and a list of the first few Bell numbers (in particular the output for (c)):

```
sage: def B(k):
     if k==0:
         bell = 1
     else:
         bell = sum( b(k-1,r-1)*B(k-r) for r in range(1,k+1) )
     return bell
sage: [B(k) for k in range(10)]
[1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147]
sage: [B(k) for k in range(4,7)]      # answer to (c)
[15, 52, 203]
```

(#6) Problem #149.

A function $f : K \to N$ is equivalent to a weak set partition $K = B_1 \cup \cdots \cup B_n$, where $B_y = \{x \in K \colon f(x) = y\}$. (By "weak," I mean that the blocks are allowed to be empty.) Observe that

$$\#\{f : K \to N \colon |B_y| = j_y \ \ (\forall y \in N)\} \ = \ \binom{k}{j_1, \dots, j_n}.$$

Since the number of functions $K \to N$ is just n^k, we obtain the formula

$$n^k \;=\; \sum_{\substack{j_1+\cdots+j_n=k\\ j_1,\ldots,j_n\geq 0}} \binom{k}{j_1,\ldots,j_n}.$$

For example, if $n=3$ and $k=2$, this says that

$$\begin{aligned} 3^2 &= \binom{2}{2,0,0} + \binom{2}{0,2,0} + \binom{2}{0,0,2} + \binom{2}{1,1,0} + \binom{2}{1,0,1} + \binom{2}{0,1,1} \\ &= 3\left(\frac{2!}{2!\,0!\,0!}\right) + 3\left(\frac{2!}{1!\,1!\,0!}\right) \;=\; 3\cdot 1 + 3\cdot 2 \;=\; 9. \end{aligned}$$

(#7) Problem #150.

An onto function $f : K \twoheadrightarrow N$ is equivalent to a set partition $K = B_1 \cup\!\!\!\cdot\, \cdots \cup\!\!\!\cdot\, B_n$, where $B_y = \{x \in K \colon f(x) = y\}$ and $B_y \neq \emptyset$ (unlike the previous problem, the blocks are not allowed to be empty.) Since the number of onto functions $K \twoheadrightarrow N$ is $n!S(k,n)$ (we proved this in #143), we obtain the formula

$$n!S(k,n) \;=\; \sum_{\substack{j_1+\cdots+j_n=k\\ j_1,\ldots,j_n> 0}} \binom{k}{j_1,\ldots,j_n}.$$

For example, if $n=3$ and $k=5$, this says that

$$\begin{aligned} 3!S(5,3) &= \binom{5}{3,1,1} + \binom{5}{1,3,1} + \binom{5}{1,1,3} + \binom{5}{2,2,1} + \binom{5}{2,1,2} + \binom{5}{2,2,1} \\ &= 3\left(\frac{5!}{3!\,1!\,1!}\right) + 3\left(\frac{5!}{2!\,2!\,1!}\right) \;=\; 3\cdot 20 + 3\cdot 30 \;=\; 150. \end{aligned}$$

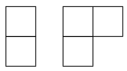
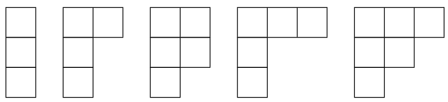
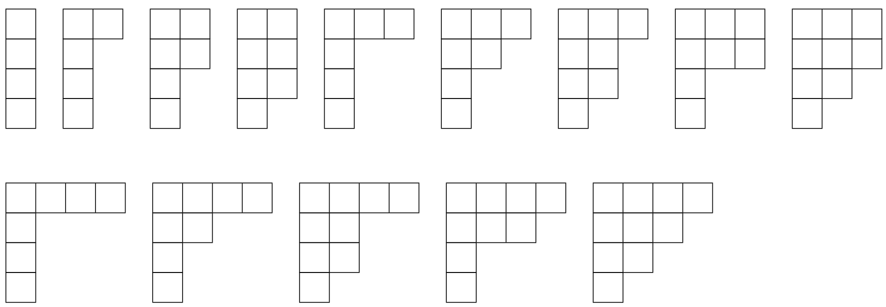
Indeed, $S(5,3) = 25$ (at least, so says the Sage code I wrote for problem #136) and $3! = 6$.

(#8) How many partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ with n parts have the property that $1 \leq \lambda_k \leq n+1-k$ for all k?

Here is a list. (I'm dropping parentheses and commas, so, e.g., 3221 is short for the partition $(3,2,2,1)$.)

n	Partitions	Number of partitions
$n=1$	1	1
$n=2$	11, 21	2
$n=3$	111, 211, 221, 311, 321	5
$n=4$	1111, 2111, 2211, 2221, 3111, 3211, 3221, 3311, 3321, 4111, 4211, 4221, 4311, 4321	14

In fact, these are the Catalan numbers. The bijection is easier to see if we draw the Young diagrams of the partitions:

n	Partitions
$n = 1$	
$n = 2$	
$n = 3$	
$n = 4$	

Think of the Ferrers diagram as sitting inside a rectangle with n rows and $n + 1$ columns. Then the **southeastern edge** of the diagram (ignoring the bottom of the first column, and augmented with east steps at the top as necessary) is a lattice path from $(0, 0)$ to (n, n) that does not go below the line $y = x$. A partition can be reconstructed from its southeastern edge, so this is a bijection from the partitions we are interested in to Catalan paths. Here is the bijection for $n = 3$: