Odd covers of graphs

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Let $G = (V, E)$ be a simple graph. An odd cover of G is a collection of bicliques (complete bipartite graphs) on subsets of V with the property that $uv \in E$ if and only if uv is in an odd number of bicliques.
An odd cover always exists: Trivially, the collection of bicliques with partite sets $\{u\}$ and $\{v\}$ for each pair of adjacent vertices u and v in G constitutes one such cover.
Let $b_2(G)$ denote the minimum cardinality of an odd cover of G.

The "odd cover problem," a variation of the Graham-Pollak problem, was posed by Babai and Frankl (1992): What is the minimum number of bicliques which cover every edge of K_n an odd number of times?

Radhakrishnan, Sen, and Vishwanathan (2000) determined $b_2(K_n)$ for an infinite but density zero subset of positive integers n.

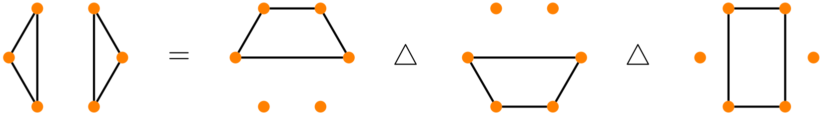
We will significantly sharpen their result.

There are two notions closely related to odd cover: biclique partition and biclique covering.
A biclique partition of G is a collection of edge-disjoint complete bipartite subgraphs of G whose edges partition the edge set of G.
The minimum cardinality of a biclique partition of G is denoted $\mathrm{bp}(G)$.
A biclique covering of G is a collection of complete bipartite subgraphs of G such that every edge of G appears at least once.
The minimum cardinality of a biclique covering of G is denoted $\mathrm{bc}(G)$.

Every biclique partition is both a biclique covering and an odd cover, which implies that $\mathrm{bc}(G) \leq \mathrm{bp}(G)$ and $b_2(G) \leq \mathrm{bp}(G)$, but there are biclique coverings and odd covers which are not biclique partitions.

Note the important distinction between each of these ideas and odd covers: We can include non-edges of G in an odd cover but not in a biclique partition or a biclique covering.

Somewhat counterintuitive: $b_2(mG) \neq mb_2(G)$, where mG denotes the disjoint union of m copies of G.



A minimum odd cover of $2K_3$.

Generalizing further, $b_2(mK_3) = m + 1$ vs. $mb_2(K_3) = 2m$.

General lower bound: For any graph G,

$$b_2(G) \geq \mathrm{rank}_{\mathbb{F}_2}(A(G))/2.$$

Idea of proof: If $G_1, \ldots, G_k$ are bicliques (along with isolated vertices) which form an odd cover of G, then $A(G) = \sum_{i=1}^{k} A(G_i) \pmod 2$. Recall that matrix rank is subadditive.

Theorem: If G is bipartite, then

$$b_2(G) = \mathrm{rank}_{\mathbb{F}_2}(A(G))/2.$$

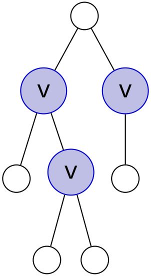
Furthermore, there exists a minimum odd cover of G that respects its bipartition.

Proof uses induction on the number of vertices. We show some easy consequences of this theorem.

Let $G = (V, E)$ be a simple graph. A vertex cover of G is a subset S of V with the property that every edge of G has at least one member of S as an endpoint.
A vertex cover always exists: Trivially, V constitutes one such cover.
Let $\tau(G)$ denote the minimum cardinality of a vertex cover of G.
For any forest F, utilizing a result of König that relates the maximum matching problem to the minimum vertex cover problem, it is known that

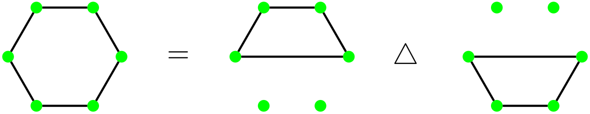
$$\tau(F) = \mathrm{rank}_{\mathbb{F}_2}(A(F))/2.$$

Corollary: For any forest F, we have $b_2(F) = \tau(F)$.

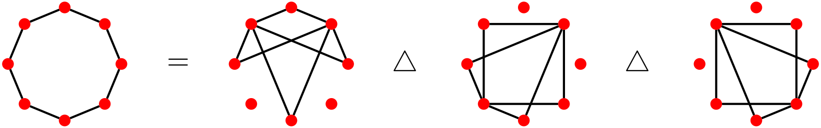


A minimum vertex cover of a forest induces a minimum odd cover.

Corollary: For $n \geq 2$, $b_2(C_{2n}) = n - 1$.

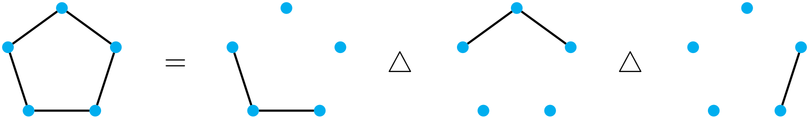


A minimum odd cover of C_6.
Note that $b_2(C_6) < \mathrm{bc}(C_6) = \mathrm{bp}(C_6) = 3$.



A minimum odd cover of C_8.
Note that $b_2(C_8) < \mathrm{bc}(C_8) = \mathrm{bp}(C_8) = 4$.

Theorem: For $n \geq 2$, $b_2(C_{2n-1}) = n$.

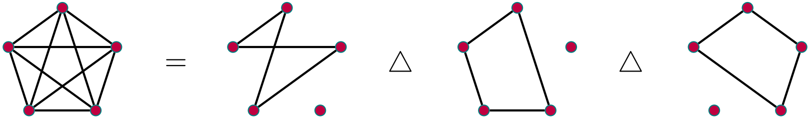


A minimum odd cover of C_5.
Note that $b_2(C_5) = \mathrm{bp}(C_5) = \mathrm{bc}(C_5)$.

For any positive integer n,

$$\left\lceil \frac{n}{2} \right\rceil \leq b_2(K_n) \leq \left\lceil \frac{n}{2} \right\rceil + 1.$$

In particular, $b_2(K_n) = \lceil n/2 \rceil$ when $8 | n$ or $n \equiv \pm 1, 3 \mod 8$.
We have thus determined $b_2(K_n)$ for a density 5/8 subset of the positive integers.



A minimum odd cover of K_5.
Note that $b_2(K_5) = \mathrm{bc}(K_5) < \mathrm{bp}(K_5) = 4$.
Asymptotically $\mathrm{bc}(K_n) = \lceil \log_2 n \rceil$ and $\mathrm{bp}(K_n) = n - 1$.

Proof utilizes an alternative interpretation of odd covers.

$a^{(1)}$	ε	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$a^{(2)}$	1	ε	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$a^{(3)}$	1	0	ε	1	0	0	0	0	0	0	0	0	0	0	0	0
$a^{(4)}$	1	1	0	ε	0	0	0	0	0	0	0	0	0	0	0	0
$a^{(5)}$	1	1	1	1	ε	0	0	0	0	0	0	0	0	0	0	0
$a^{(6)}$	1	1	1	1	1	ε	1	0	0	0	0	0	0	0	0	0
$a^{(7)}$	1	1	1	1	1	0	ε	1	0	0	0	0	0	0	0	0
$a^{(8)}$	1	1	1	1	1	1	0	ε	0	0	0	0	0	0	0	0
$a^{(9)}$	1	1	1	1	1	1	1	1	ε	0	0	0	0	0	0	0
$a^{(10)}$	1	1	1	1	1	1	1	1	1	ε	1	0	0	0	0	0
$a^{(11)}$	1	1	1	1	1	1	1	1	1	0	ε	1	0	0	0	0
$a^{(12)}$	1	1	1	1	1	1	1	1	1	1	0	ε	0	0	0	0
$a^{(13)}$	1	1	1	1	1	1	1	1	1	1	1	1	ε	0	0	0
$a^{(14)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	ε	1	0
$a^{(15)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	0	ε	1
$a^{(16)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	ε

Vectors $a^{(i)}$ for K_{32}, $b^{(i)}$ is the complement of $a^{(i)}$.

Going from biclique to triclique: For any integer k, the graph T_k has vertex set consisting of all strings of length k with entries in $\{0, 1, 2, \epsilon\}$, where vertex v and vertex u are adjacent if and only if the number of places where they differ and neither is ϵ is odd.
We have $b_2(T_k) \geq \log_3(4) \cdot k$ and $\mathrm{rank}_{\mathbb{F}_2}(A(T_k)) = 2k$.
This implies the existence of an infinite family of graphs where the general lower bound $\mathrm{rank}_{\mathbb{F}_2}(A(G))/2$ is arbitrarily far away from $b_2(G)$.

Thank You! Questions?

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