Math 724, Fall 2013
Homework #5

Instructions: Write up your solutions in LaTeX and hand in a hard copy in class on **Friday, November 8.** Collaboration is allowed (and in fact encouraged), but each student must write up his or her solutions independently and acknowledge all collaborators.

(#1) Problem #168.

Let λ^* denote the conjugate of λ (i.e., what you get by interchanging rows and columns).

(a) I'm going to use the following definition of a Young diagram: it is a collection of boxes in the fourth quadrant so that if x is a box in λ, then every box north, west, or northwest of x is also in λ. If y is a box *not* in λ, then every box south, east or southeast of y is also not in λ. Rotating by 180° flips north/south and east/west, so the complement of λ is a partition.

Alternately, if $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $m = \lambda_1 \geq \cdots \geq \lambda_n > 0$, then the complementation operation produces the sequence of integers

$$\underbrace{m', \ \ldots, \ m'}_{n'-n \text{ times}}, \ m' - \lambda_n, \ m' - \lambda_{n-1}, \ \ldots, \ m' - \lambda_2, \ m' - \lambda_1$$

and the inequalities on the λ_i's imply that this sequence is weakly decreasing, so it is a partition (after removing any 0's at the end).

(b) $m'n' - k$.

(c) This can be read off from the formula for the complement above. Either $m' > m$ and $n' = n$, or $m' = m$ and $n' = n + \lambda^*_m$ (because λ^*_m is the number of largest parts of λ).

(d) This condition is the conjugate to (c): either $n' > n$ and $m' = m$, or else $n' = n$ and $m' = m + \lambda_1$.

(e) Sure, it's possible. λ can even equal its complement, for example if $\lambda = (m, m-1, \ldots, 2, 1)$ and $m' = m$, $n' = m + 1$.

(f) Yes. If $T \subseteq S$, then $S \setminus (S \setminus T) = T$. Take S to be the set of all squares in the $n' \times m'$ rectangle and T to be the set of squares in the original Young diagram.

(#2) Problem #170.

Every composition of k into n parts — i.e., every ordered n-tuple $(a_1, \ldots, a_n)$ such that $\sum_{i=1}^n a_i = k$ — gives rise to a partition of k with n parts, by ignoring the order of the a_i's. The number of such compositions that give rise to a particular partition is at most $n!$ — in fact, it is the binomial coefficient $\binom{n}{z_1, \ldots, z_n}$, where z_j denotes the number of occurrences of j among the a_i's. Therefore,

$$P(k, n) \geq \frac{\text{number of compositions of } k \text{ into } n \text{ parts}}{n!} = \frac{\binom{k-1}{n-1}}{n!}$$

(by, e.g., Chapter 1, Supplemental Problem #1).

(#3) Problem #181. $(1 + a + a^2 + a^3)(1 + b + b^2 + b^3)(1 + p + p^2 + p^3)$.

(#4) Problem #195.

$$\begin{aligned}\frac{1}{(1-x)^n} &= \sum_{k=0}^{\infty} (\text{number of weak compositions of } k \text{ into } n \text{ parts})\, x^k\\ &= \sum_{k=0}^{\infty} \binom{k+n-1}{k} x^k.\end{aligned}$$

Therefore,

$$(1+x)^{-n} = \frac{1}{(1+x)^n} = \frac{1}{1-(-x)^n} = \sum_{k=0}^{\infty} (-1)^k \binom{k+n-1}{k} x^k. \tag{1}$$

Recall that the binomial theorem says that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

By analogy (1) says that for $n > 0$, we should define

$$\binom{-n}{k} = \begin{cases} (-1)^k \binom{k+n-1}{k} & \text{if } k \geq 0,\\ 0 & \text{if } k < 0.\end{cases} \tag{2}$$

(#5) Problem #196. Having defined extended binomial coefficients in the previous problem, the general binomial theorem ought to state that

$$(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k}$$

Observe that this apparently infinite sum is actually finite when $n \geq 0$, provided we adopt the convention that $\binom{n}{k} = 0$ whenever $n \geq 0$ and $k \notin [0, n]$. (It makes perfect sense to do so — after all, no subset of a set S has cardinality less than 0 or greater than $|S|$.)

In particular, for $n = -3$ we have

$$\begin{aligned}(x+y)^{-3} &= \sum_{k=0}^{\infty} \binom{-3}{k} x^k\\ &= \sum_{k=0}^{\infty} (-1)^k \binom{k+2}{k} x^k y^{-3-k}\\ &= \frac{1}{y^3} - \frac{3x}{y^4} + \frac{6x^2}{y^5} - \frac{10x^3}{y^6} + \frac{15x^4}{y^7} - \cdots\end{aligned}$$

Gee, you'd expect the expression for $(x+y)^{-n}$ to be symmetric in x and y, wouldn't you? This means that there is another equally valid power series expansion for $(x+y)^{-3}$ obtained by interchanging x and y — so it doesn't make sense to talk about "the" coefficient of a particular Laurent monomial[1], since it depends on the choice of expansion. There's a reason we don't work with doubly infinite power series: try multiplying $\sum_{n=-\infty}^{\infty} x^n$ by $1 - x$ and observe what happens.

[1]A Laurent monomial is something like $x^4 y^{-7}$: a product of integer powers of variables — but not necessarily nonnegative integer powers, as in an ordinary monomial.

(#6) Problem #197. If there were only one child then the generating function would be $1+x+x^2+x^3+x^4$; for three children it is

$$(1+x+x^2+x^3+x^4)^3 = \frac{(1-x^5)^3}{(1-x)^3}.$$

Write this as $(1-x^5)^3(1-x)^{-3}$ and expand, using the classical and extended binomial theorems:

$$\begin{aligned}(1-x^5)^3(1-x)^{-3} &= \left[1-3x^5+3x^{10}-x^{15}\right]\left[\sum_{k=0}^{\infty}\binom{-3}{k}(-x)^k\right]\\ &= \left[1-3x^5+3x^{10}-x^{15}\right]\left[\sum_{k=0}^{\infty}\binom{k+2}{k}x^k\right]\end{aligned}$$

of which the x^{10} coefficient is

$$1\binom{12}{10} - 3\binom{7}{5} + 3\binom{2}{0} \;=\; 66-3\cdot 21+3 \;=\; \boxed{6.}$$

Indeed, there are 6 ways to pass out 10 pieces of candy so that no child gets more than 4 pieces:

$$(4,4,2),\quad (4,2,4),\quad (2,4,4),\quad (4,3,3),\quad (3,4,3),\quad (3,3,4).$$

(#7) Problem #205. For every number k, the picture enumerator for choosing any nonnegative number of parts of size k is $1+q^k+q^{2k}+q^{3k}+\cdots$ Therefore, the generating function we are looking for is the product of these terms for all even k. If we let $j=k/2$ then j ranges over all positive integers and we can write the generating function as

$$\prod_{j=0}^{\infty}\frac{1}{1+q^{2j}} \;=\; (1+q^2+q^4+q^6+\cdots)(1+q^4+q^8+q^{12}+\cdots)(1+q^6+q^{12}+q^{18}+\cdots)\cdots. \tag{3}$$

(#8) Problem #206. We did this in class; it's $\prod_{i=1}^{\infty}(1+q^i)$.

(#9) Problem #207. For every number j, the picture enumerator for choosing any nonnegative even number of parts of size j is $1+q^{2j}+q^{4j}+q^{6j}+\cdots$ Therefore, the generating function we are looking for is the product of these terms for all j — which is the same expression as (3).

In fact, there is a bijective proof of this fact: to say that all rows of the Ferrers diagram of λ have even length is to say that every possibly column length occurs an even number of times. So conjugation is a bijection between the two sets of partitions we are counting.

Extra credit. A monomial in n variables has the form $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$. Think of this as choosing a_1 copies of x_1, a_2 copies of x_2, etc. So the Hilbert series of $R=\mathbb{C}[x_1,\dots,x_n]$ is the generating function for weak compositions:

$$H_R(q) = \sum_{k\geq 0}(\dim R_k)q^k = \sum_{k\geq 0}\binom{k+n-1}{n-1}q^k = \frac{1}{(1-q)^n}.$$