Duality in chip-firing and parking functions on graphs

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Chip-firing on graphs

- The classical theory of *chip-firing* involves a game played on the vertices of a graph, with connections to statistical physics, commutative algebra, algebraic geometry, etc.
- Recent books (and invited AMS talks!):
  - S. Corry and D. Perkinson, *Divisors and Sandpiles: An Intro to Chip-Firing*, AMS, 2017.
  - C. Klivans, *The Mathematics of Chip-Firing*, CRC press, 2018.

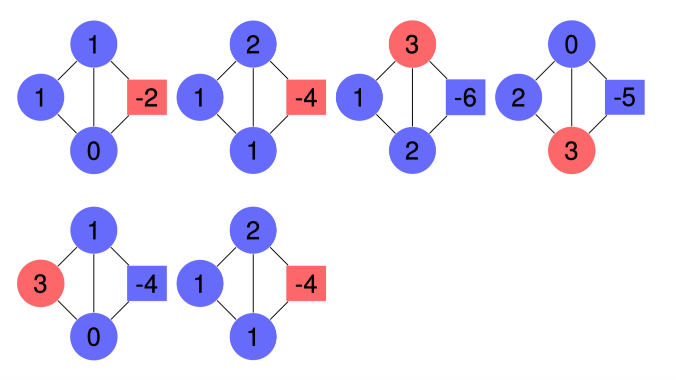
- Goal for this talk: Chip-firing on a graph G (and the resulting G-parking functions) say something about $\mathcal{M}(G)^*$, the *dual* matroid of the underlying graphic matroid. What about *dual chip-firing* and *coparking functions*? Use planar graphs for inspiration.

Chip-firing with a sink

Suppose G is a finite simple graph on vertex set $\{0, 1, \dots, n\}$ with specified sink vertex 0, and edge set E.

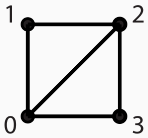
- A *configuration* of chips is a vector $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{N}^n$.
- A vertex i can *fire* when $c_i \geq \deg(i)$.
- The vertex passes chips to each of its neighbors (one for each edge connecting it to i), resulting in a new configuration $\mathbf{c}'$.
- A configuration $\mathbf{c}$ is *stable* if no vertex can fire, say $\mathbf{c} = \mathrm{stab}(\mathbf{d})$.
- The sink can fire only when the configuration is stable, passing a chip to each of its neighbors.

Example



Matrices

- The dynamics of chip-firing are encoded by $\tilde{\mathcal{L}}(G)$, the *(reduced) Laplacian* matrix of G.



$$\tilde{\mathcal{L}}(G) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

- Firing a vertex i corresponds to subtracting the ith row of $\tilde{\mathcal{L}}(G)$.

- $\mathcal{L}(G)$ defines an equivalence relation on the set $\mathbb{Z}^n$: $\mathbf{c} \sim \mathbf{d}$ iff $\mathbf{d} - \mathbf{c} \in \operatorname{im} \tilde{\mathcal{L}}(G)$.

- Define the *critical* group of G as

$$\kappa(G) \cong \mathbb{Z}^n / \operatorname{im} \tilde{\mathcal{L}}(G)$$

- Note that $|\kappa(G)| = \det \tilde{\mathcal{L}}(G) = \#\{\textit{spanning trees} \text{ of } G\}$.

Special configurations

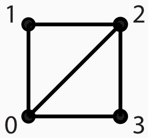
One is interested in finding special representatives of each equivalence class $[\mathbf{c}]$.

- A configuration $\mathbf{c} \geq \mathbf{0}$ is *recurrent* if $\mathbf{c} = \mathrm{stab}(\mathbf{d})$, for 'large enough' $\mathbf{d}$.
  - Another way to define $\kappa(G)$: if $\mathbf{c}$ and $\mathbf{d}$ both critical then $\mathrm{stab}(\mathbf{c} + \mathbf{d})$ is critical.
- A configuration $\mathbf{c} \geq \mathbf{0}$ is *superstable* if no *set* of nonsink vertices can fire simultaneously.
  - Superstables also called G-parking functions, can be thought of as 'energy minimizing'
  - Relate to Tutte polynomial and Stanley's conjecture on h-vector of $\mathcal{M}(G)^*$ [Merino].

Theorem (Biggs, Dhar, ...)

Each equivalence class $[\mathbf{c}]$ *contains exactly one critical and one superstable configuration.*

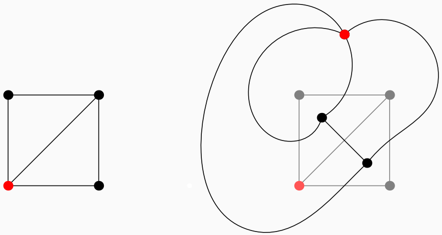
Example



- Critical configurations are $\{121, 021, 111, 120, 101, 111, 020, 110\}$
- Superstable configurations (G-parking functions) are $\{000, 100, 010, 001, 020, 110, 101, 011\}$
- h-vector of $\mathcal{M}(G)^*$ is $\{1, 3, 4\}$

Planarity and dual chip-firing

- If G is a *planar* graph (with an embedding) we have a dual graph G^*.
  Can we interpret chip-firing on G^* in terms of G?



- Note that superstable configurations of G^* relate to the h-vector of $\mathcal{M}(G)$.
- In this context we now fire *cycles*, passing chips according to a rule determined by how these circuits intersect (depends on an *orientation*).
- Critical groups agree [Corri-Rossin]: $\kappa(G^*) \cong \kappa(G)$.

Generalizing to all graphs

- What happens if G is not planar? We no longer have a 'dual graph' but can we still perform 'dual chip-firing'? Do we have 'dual parking functions'?
- Yes! We will develop this theory in two different ways, both approaches interpret (usual) chip-firing as 'vertex cut' chip-firing and will apply ideas to flows/cycles.
    1. Think of vertex cuts as forming a basis for the integral cut space, and dualize to flows.
    2. Mimic intersection properties of vertex cuts, define dual parking functions accordingly.

First approach, integral flows

- The reduced Laplacian $\tilde{\mathcal{L}}(G)$ can be computed as

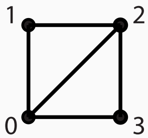
$$\tilde{\mathcal{L}}(G) = \tilde{\partial}\tilde{\partial}^T$$

where $\tilde{\partial}$ is the (reduced) *incidence matrix* of the graph G.

- Integer row space of $\tilde{\partial}$ is called $\mathrm{Cut}(G)$, the *lattice of integral cuts* of G.
- Classical chip-firing is governed by the integral basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for $\mathrm{Cut}(G)$ given by the vertex cuts of the nonsink vertices.
- For the *dual* picture we seek an integer basis for the *kernel* of $\tilde{\partial}$ (called $\mathrm{Flow}(G)$, the *lattice of integral flows* of G).
- For any such basis $\iota = (\mathbf{f}_1 \ldots \mathbf{f}_g)$ let

$$\mathcal{L}^*(G) = \iota^T \iota$$

denote the *dual Laplacian* (w/ respect to this choice of basis).



$$\tilde{\partial} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad \tilde{\mathcal{L}} = \tilde{\partial}\tilde{\partial}^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$\ker \tilde{\partial}$ has basis $\iota = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$ so that $\mathcal{L}^*(G) = \iota^T \iota = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

Integral flow chip-firing

- After fixing a basis $\iota = \{\mathbf{f}_1, \ldots, \mathbf{f}_g\}$, the matrix $\mathcal{L}^*$ determines a potential 'chip-firing' rule for these elements.
- Guzmán and Klivans [GM] have developed a chip-firing theory for any $n \times n$ matrix A that is *avalanche finite* (equivalent to A being an *M-matrix*).

Definition

For a graph G we say a basis $\iota = \{\mathbf{f}_1, \ldots, \mathbf{f}_g\}$ is an integral flow M-basis if the matrix $\mathcal{L}^*$ is an avalanche finite redistribution matrix.

- If G admits an integral flow M-basis then from [GM] we have a good notion of 'chip-firing' on the elements of ι, with notions of z-superstable configurations etc.

Main result

Theorem (D-, Li, Meyers, Samavedam)

Any graph G admits an integral flow M-basis.

- Follows from a purely lattice theoretic result: For any latttice $\Lambda \subset \mathbb{R}^d$ there exists an integral basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_g\}$ for Λ with the property that $\mathbf{f}_i \cdot \mathbf{f}_j \leq 0$ for all $i \neq j$.
- From [GK] we know each equivalence class defined by $\mathcal{L}^*$ contains a unique z-superstable configuration and hence:

Proposition

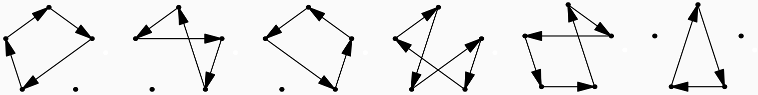
Suppose G has a cycle M-basis with associated dual Laplacian $\mathcal{L}^*$. Then the number of z-superstable configurations of G is given by $|\tau(G)|$, the number of spanning trees of G.

Cycle M-bases

- Want to find an M-basis for G consisting only of *cycles*: each entry should be $0, -1, 1$, with the the nonzero entries corresponding to some cycle (simple closed path) of G.
- Such a basis will be called a *cycle M-basis*.

Proposition (DLMS)

Planar graphs as well as the graphs K_5 and $K_{3,3}$ admit cycle M-bases.



$$\mathcal{L}^* = \begin{bmatrix} 4 & -1 & -2 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & 0 & -1 \\ -2 & 0 & 4 & -3 & -1 & 0 \\ 0 & 0 & -3 & 5 & 0 & -1 \\ 0 & 0 & -1 & 0 & 5 & -2 \\ 0 & -1 & 0 & -1 & -2 & 3 \end{bmatrix}$$

Main open question: Does any graph admit *cycle M*-basis? Perhaps for a class of graphs?

Second approach: Cycle systems

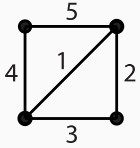
For our second approach to dualizing, we note the following property of vertex cuts:

- In any collection of vertex cuts, the edges that appear *exactly once* form a cut (namely the cut defined by that collection of vertices).
- Suppose G is a graph with cycles $\mathcal{C} = \{C_1, \ldots, C_g\}$, where $g = |E| - |V| + 1$.
- For $I \subset [g]$, let C_I denote the elements of $\cup_{i \in I} C_i$ that belong to exactly one of the C_i.

Definition

A collection $\mathcal{C}$ is a *cycle system* if for all nonempty $I \subset [k]$, the set C_I contains a cycle.

- For our graph G, $g = 2$ and we can take $\mathcal{C} = (123, 145)$.



$C_1 = 123, \; C_2 = 145$ (so that $C_{12} = 2345$)

Which graphs admit cycle systems?

Proposition

The following classes of graph admit cycle systems.

1. *If G is planar graph, the set of bounded faces form a cycle system.*
2. *If G has a cone vertex v then $\mathcal{C} = \{(vu, vw, uw) : uw$ an edge of $G\}$ is a cycle system.*

- The graph $K_{3,3}$ does *not* admit a cycle system.
- Of the 853 connected graphs on 7 vertices, at least 776 admit cycle systems.
- Notion of cycle system generalizes to other matroids.

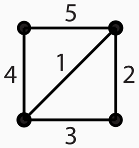
Coparking functions

If G admits a cycle system $\mathcal{C}$, one has a notion of a *coparking function* relative to $\mathcal{C}$.

Definition

Given a cycle system $\mathcal{C} = \{C_1, C_2, \ldots, C_g\}$, a sequence $(a_1, a_2, \ldots, a_g)$ is a $\mathcal{C}$*-coparking function* if for every nonempty $I \subset [k]$ there exists $i \in I$ such that

$$a_i < |C_i \cap C_I|.$$



- Recall for $\mathcal{C} = (123, 145)$
  $C_1 = 123,\ C_2 = 145$, and $C_{12} = 2345$.

  The coparking functions for this cycle covering are $\{(0,0),(1,0),(0,1),(1,1),(2,0),(0,2),(2,1),(1,2)\}$.

Main result

Theorem (Corry, D-, Perkinson)

Suppose $\mathcal{C}$ is a cycle system for G. Then the coparking functions for $\mathcal{C}$ are in bijection with the spanning trees of G.

- Proof uses a dual version of a depth first Dhar's *burning algorithm*. See also S-parking functions of [Dong].

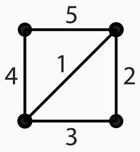
Conjecture

The bijection can be chose so that the set of spanning trees with j internally passive edges is in bijection with the set of $\mathcal{C}$-coparking functions of degree k.

- This would prove Stanley's conjecture for matroids $\mathcal{M}(G)$, where G admits a cycle system.

Example

For G with the cycle system $\mathcal{C} = \{123, 145\}$ we have $h(\mathcal{M}(G)) = (1, 2, 3, 2)$.



T	$IP(B)$
124	$\emptyset$
125	5
134	3
135	35
234	23
235	235
245	45
345	345

Coparking functions

$\{(0, 0),$

$(1, 0), (0, 1)$

$(1, 1), (2, 0), (0, 2),$

$(2, 1), (1, 2)\}.$

Thank you!