Math 821 Problem Set #4
Posted: Friday 3/11/11
Due date: Monday 3/28/11

Problem #1 (Hatcher, p.52, #1) Show that the free product $G * H$ of nontrivial groups G, H has trivial center, and that the only element of $G * H$ of finite order are the conjugates of finite-order elements of G and H.

Solution: First, any non-identity element $w \in G * H$ can be written uniquely as a product $w = w_1 \cdots w_n$ of non-identity elements of G and H, with letters w_i alternating between G and H (p.42). If $w_1 \in H$ then w does not commute with any non-identity element of G, while if $w_1 \in G$ then w does not commute with any non-identity element of H.

Suppose that $w \in G * H$ and $w^n = e$ Write w in reduced form: $w = g_1 \cdots g_k$ where the letters alternate between G and H.

$$w^n = (g_1 \cdots g_k)(g_1 \cdots g_k) \ \cdots \ (g_1 \cdots g_k) = e.$$

We need to be able to somehow cancel this expression using only relations within G and H. The only possibility is that g_k and g_1 belong to the same factor group, WLOG G (so in particular k is odd, say $k = 2K + 1$) and $g_k = g_1^{-1}$, giving

$$w^n = (g_1 \cdots g_{k-1})(g_2 \cdots g_{k-1}) \ \cdots \ (g_2 \cdots g_k) = e.$$

Now the only possibility for cancellation is that $g_{k-1} = g_2^{-1}$. Cancelling and repeating, we find

$$g_k = g_1^{-1}, \quad g_{k-1} = g_2^{-1}, \quad \ldots, \quad g_{K+2} = g_K^{-1}.$$

But this says that $w = xyx^{-1}$, where $x = g_1 \cdots g_K$ and $y = g_{K+1}$. Moreover, y belongs to either G or H because it is a single letter), and

$$y^n = (x^{-1}wx)^n = x^{-1}w^n x = x^{-1}x = e$$

so y has finite order. So we have shown that every finite-order element of $G*H$ is a conjugate of a finite-order element of one of G or H.

Problem #2 The *dunce hat* is the space D obtained from a triangle by identifying all three edges with each other, with the orientations indicated below. Give two separate proofs that D is simply-connected. (There are at least three: (a) show that D is in fact contractible; (b) use Van Kampen's theorem; (c) a slick one-line proof using something we did in class.)

Solution:

(b) Decompose D into two pieces A, B as follows: A is the interior of the 2-cell, and $B = D \setminus \{p\}$, where $p \in A$.

Then:

- A is an open disk, hence contractible.
- B deformation-retracts onto, hence is homotopy-equivalent to, the boundary triangle, which is just a circle (the edge a becomes one loop around the circle).
- $A \cap B$ is an (open) annulus, whose fundamental group is generated by a path γ winding once around p. Note that $\gamma \simeq aaa^{-1}$ in B.

Now, since A is contractible, Van Kampen's Theorem says that

$$\pi_1(D) = \pi_1(B)/i_*\pi_1(A \cap B)$$

and

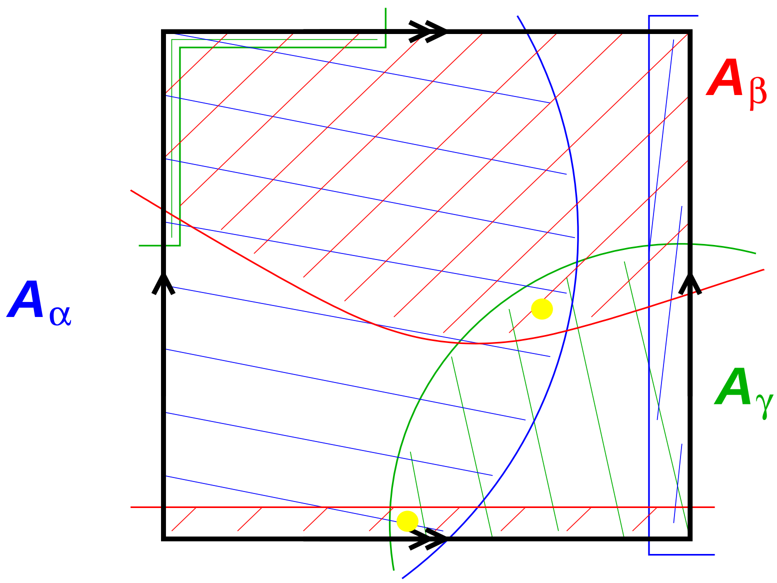
$$i_*\gamma = aaa^{-1} = a$$

so this quotient is in fact trivial.

(c) D is the Cayley complex of the group $G = \langle g \mid ggg^{-1} \rangle$, which is in fact the trivial group, so $\pi_1(D) = G = 0$.

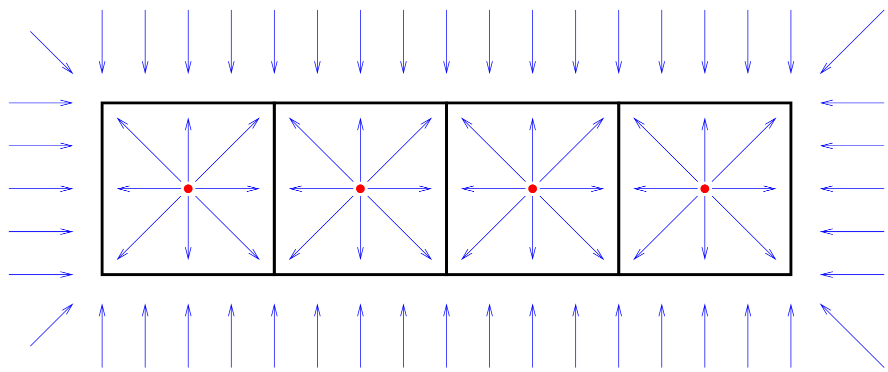
Problem #3 Consider the standard picture of the torus $T = S^1 \times S^1$ as a quotient space of the square. What is wrong with Why does the decomposition $T = A_\alpha \cup A_\beta \cup A_\gamma$ shown below, together with Van Kampen's theorem, *not* imply that T is simply-connected?

Solution: It is true that the sets $A_\alpha, A_\beta, A_\gamma$ are all simply-connected. However, the intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is not path-connected. The picture is misleading (which was the idea of the problem); it actually must look something like this, and the two yellow splotches denote different components of $A_\alpha \cap A_\beta \cap A_\gamma$.



Problem #4 (Hatcher, p.53, #4, modified) Let $n \geq 1$ be an integer, and let $X \subset \mathbb{R}^3$ be the union of n distinct rays emanating from the origin. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.

Solution: The map $f_t(\mathbf{x}) = (1-t)\mathbf{x} + t\frac{\mathbf{x}}{\|\mathbf{x}\|}$ gives a deformation retraction from $\mathbb{R}^3 \setminus X$ to the unit sphere minus n points. We can regard the deleting the first point as giving a copy of $\mathbb{R}^2$, so we now have $\mathbb{R}^2$ minus $n-1$ points. This space deformation-retracts to the wedge of $n-1$ squares, whose fundamental group we know is free on $n-1$ generators.



Problem #5 Let $a_1, \ldots, a_n$ be nonzero integers. Construct a cell complex X from S^1 as follows: For each $j = 1, \ldots, n$, attach a 2-cell to S^1 by wrapping it around the circle a_j times. Compute $\pi_1(X)$.

Solution: This is an example of a Cayley complex for the group with one generator g and relations g^{a_j} — that is, a cyclic group of order $\gcd(a_1, \ldots, a_n)$.

Problem #6 (Hatcher, p.53, #6, modified) Let X be a path-connected cell complex, and let Y be a cell complex obtained from X by attaching an n-cell for some $n \geq 3$. Show that the inclusion $X \hookrightarrow Y$ induces an isomorphism $\pi_1(X) \cong \pi_1(Y)$.

Solution: The proof of Prop. 1.26 goes through, changing e^2_α to 3^n_α. At the very end, we have that A_α deformation-retracts onto a circle in $e^n_\alpha \setminus \{y_\alpha\}$, i.e., an n-ball minus a point. But such a thing is simply-connected (as we know, it is homotopy-equivalent to S^{n-1}) and therefore $\pi_1(A_\alpha) = 0$, and the group N in the statement of the proposition is trivial.

Another argument uses Van Kampen's theorem. Let e be the n-cell that gets attached (so $e \cong D^n$) and let $f : \partial e = S^{n-1} \to X$ be the attaching map. Write $Y = X \cup Z$ where Z is obtained by fattening ∂e slightly into an open set that contains, and deformation-retracts onto, it. (This is a mapping cylinder neighborhood in the sense of Example 0.15.) Then Z is contractible, hence simply-connected (since it deformation-retracts onto an n-ball) and $X \cap Z$ is simply-connected (because it deformation-retracts onto the simply-connected $(n-1)$-sphere ∂e). Now applying Van Kampen's theorem to the decomposition $Y = X \cup Z$ gives a surjection $\pi_1(X) \to \pi_1(Y)$ whose kernel is zero.