A Zonotopal Interpretation of Power in Weighted Voting Systems

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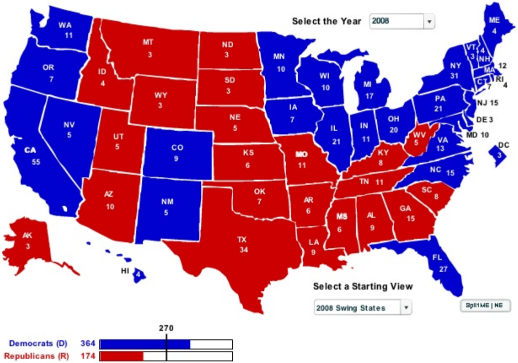

Party	No. of seats
Kadima	28
Likud - Ahi	27
Yisrael Beytenu	15
Labor	13
Shas	11
United Torah Judaism	5
(6 other parties with 4 or fewer seats)	

Table: Current seats in the Israeli Knesset. A coalition of 60 is needed to form a government.

Definition

A *weighted voting system* $[q; v_1, v_2, \ldots, v_n]$ has n players $P_1, P_2, \ldots, P_n$. Player P_i has v_i votes $(v_1 \geq v_2 \geq \cdots \geq v_n \geq 0)$, and the number of votes needed to pass a measure is q, the *quota*.

In addition,

- $q > \frac{v_1+v_2+\cdots+v_n}{2}$
- $q \leq v_2 + v_3 + \cdots + v_n$ (no veto power.)

As a consequence, no player can pass a measure alone.

A set of players that can pass a measure is a *winning coalition*; a set of players that cannot pass a measure is a *losing coalition*.

Definition

A player P_i is *critical* to a winning coalition if that coalition would lose without P_i.

Example

$[103; 76, 51, 42, 36]$

$\{P_1, P_3, P_4\}$ is a winning coalition since $76 + 42 + 36 = 154 \geq 103$.

- P_1 is critical, since $\{P_3, P_4\}$ is a losing coalition.
- P_3 is not critical, since $\{P_1, P_4\}$ is a winning coalition.
- P_4 is not critical, since $\{P_1, P_3\}$ is a winning coalition.

Definition

The *Banzhaf Power Index* of a player P_i is

$$BPI(P_i) = \frac{\#\text{ of times } P_i \text{ is critical}}{\text{total } \#\text{ of critical instances for all players}}.$$

Example

$[103; 76, 51, 42, 36]$

Winning Coalitions	Critical Players
$\{P_1, P_2, P_3, P_4\}$	None
$\{P_1, P_2, P_3\}$	P_1
$\{P_1, P_2, P_4\}$	P_1
$\{P_1, P_3, P_4\}$	P_1
$\{P_2, P_3, P_4\}$	P_2, P_3, P_4
$\{P_1, P_2\}$	P_1, P_2
$\{P_1, P_3\}$	P_1, P_3
$\{P_1, P_4\}$	P_1, P_4

There are 12 critical instances in all, 6 for player P_1 and 2 each for the other players. The Banzhaf Power Distribution is $(1/2, 1/6, 1/6, 1/6)$.

Slices of Cubes

Let $C \subseteq \{P_1, P_2, \ldots P_n\}$ be a coalition of players. Define a vector $x_C = [x_1, x_2, \ldots, x_n]$ by $x_i = 1$ if $P_i \in C$ and $x_i = 0$ if $P_i \notin C$. (This is a vertex of the n-cube.)

Let $v = [v_1, v_2, \ldots, v_n]$ be the vector of the players' votes. Then C is a winning coalition if and only if $x_C \cdot v \geq q$, where q is the quota.

The hyperplane $x \cdot v = q$ slices the cube, separating the winning coalitions from the losing coalitions.

Loosening the Quota Restrictions

If some voters have already made a commitment, we're on a face of the cube. The quota restrictions may no longer hold. (Example: Sandra Day O'Connor)

Definition

A function f is a *positive threshold function* defined on the vertices of the n-cube if there is a vector v with $v_i \geq 0$ for $1 \leq i \leq n$ and a $q \geq 0$ such that $f(x) = 1$ if $x \cdot v \geq q$ and $f(x) = 0$ if $x \cdot v < q$. If we allow $v_i < 0$, we just have a *threshold function*.

Two (positive) threshold functions f and g are *equivalent* if $f(x) = g(x)$ for all x.

Example

[3;2,2,2] and [4;2,2,2] are equivalent.

Counting Weighted Voting Systems

- There is only one three-player weighted voting system. Its Banzhaf Power Distribution is $(1/3, 1/3, 1/3)$.
- There are five different four-player weighted voting systems. (Tolle)
- There are 36 different five-player weighted voting systems, two of which have the same Banzhaf Power Distribution. (Gay, Harris, Tolle)
- There are 446 different six-player weighted voting systems. (Cuttler, DeGuire, and Rowell)

Different (but not original) Approach: Count threshold functions by counting regions in a hyperplane arrangement.

Recall: Two threshold functions are equivalent if they have the same winning and losing coalitions.

Example

In a five-player game, $\{P_1, P_3, P_4\}$ wins $\iff v_1 + v_3 + v_4 \geq q$ $\iff [q, v_1, v_2, v_3, v_4, v_5]$ is on the positive side of the hyperplane $-q + v_1 + v_3 + v_4 = 0$.

Plan: Count the regions in the hyperplane arrangement with the 2^n hyperplanes $-q + \sum_{P_i \in C} v_i = 0$, where C is a coalition of players.

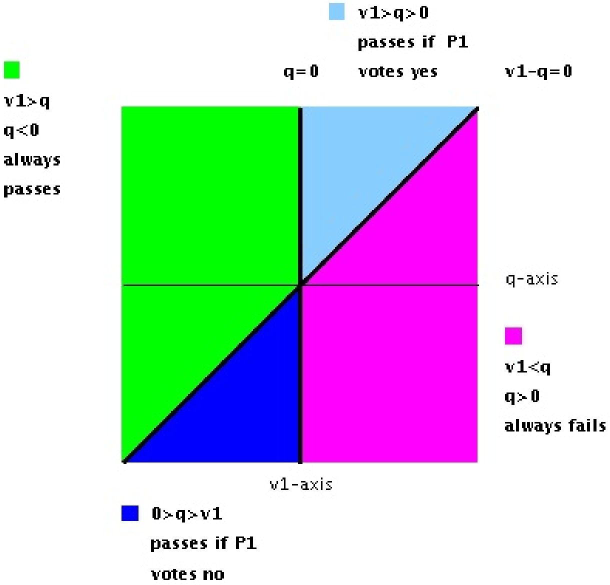

Figure: Hyperplane arrangement for 1-player threshold functions.

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To count these regions, we need to know the affine dependencies of the vertices of the n-cube.

Good News: The Dual Zonotope and the BPI

Vertex Coordinates	Banzhaf Power Distribution
$n = 3$	
$(0, 2, 2, 2)$ $(0, 4, 0, 0)$	$\frac{1}{6}(2, 2, 2)$ $\frac{1}{4}(4, 0, 0)$
$n = 4$	
$(6, 0, 0, 0, 0)$ $(0, 4, 4, 4, 0)$ $(0, 6, 2, 2, 2)$ $(4, 2, 2, 0, 0)$ $(2, 4, 2, 2, 0)$	$\frac{1}{12}(3, 3, 3, 3)$ $\frac{1}{12}(4, 4, 4, 0)$ $\frac{1}{12}(6, 2, 2, 2)$ $\frac{1}{12}(4, 4, 2, 2)$ $\frac{1}{12}(5, 3, 3, 1)$

Theorem

Let Z be the zonotope dual to the hyperplane arrangement of threshold functions. If $(y_0, y_1, y_2, \ldots, y_n)$ is a vertex of Z corresponding to a positive threshold function, then the vector of critical instances for $P_1, P_2, \ldots P_n$ is $(y_1 + \frac{y_0}{2}, y_2 + \frac{y_0}{2}, \ldots, y_n + \frac{y_0}{2})$.

New Question: What happens to the Banzhaf Power Distribution when you fix the v_i but change the quota?

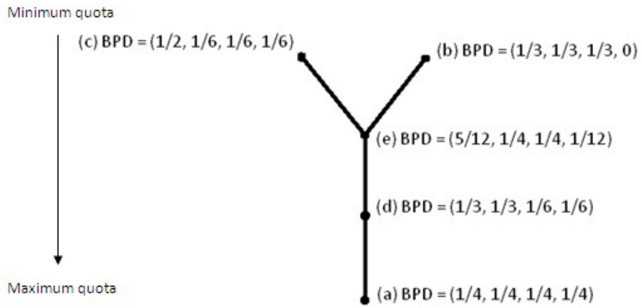

Figure: The relationship of the 5 different 4-player WVS (Buckley).

Theorem

Let $p_1, p_2, \ldots, p_k$ be points in n-space. The vertices of the projection of the k-permutahedron by the matrix $[p_1, p_2, \ldots, p_k]$ correspond to the orderings of these points by sweeping a hyperplane through them.

Application: Let the points be the vertices x_C of the n-cube. Let v be the vector of voting weights. Order the points according to the value of $x_C \cdot v$. We get all possible orderings by projecting the 2^n-permutahedron. Pick out the orderings that start with $[0, 0, \ldots, 0]$ and end with $[1, 1, \ldots, 1]$.

Current and Future Work

- Cutting down the size of some computations in polymake.
- Interpretation of WVS and BPI in terms of slices of cubes.
- Connect the vertex orderings back to the BPI.
- Ojha: Zaslavsky and symmetry.
- Other measures of power, especially Shapley-Shubik.

Some sources:

- John Banzhaf, "Weighted Voting Doesn't Work," *Rutgers Law Review* (1965).
- Elise Buckley, "An Exploration of the Application of the Banzhaf Power Index to Weighted Voting Systems," submitted to the *Rose-Hulman Undergraduate Mathematics Journal*.
- John Tolle, "Power Distribution in Four-Player Weighted Voting Systems," *Mathematics Magazine* (2003).