Deformation Retracts of Neighborhood Complexes of Stable Kneser Graphs

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- Lovász proved this conjecture by modeling the problem as a graph coloring problem: define the *Kneser graph*, denoted $KG_{n,k}$, to be the graph with vertices the n-subsets of $[2n+k]$ and edges between disjoint subsets. Kneser's conjecture is equivalent to the statement that the chromatic number of $KG_{n,k}$ is equal to $k+2$.

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- Later that year, Alexander Schrijver identified a vertex-critical family of subgraphs of $KG_{n,k}$, called the stable Kneser graphs and denoted $SG_{n,k}$, and determined their chromatic numbers using a method developed by Barany that also used the Borsuk-Ulam theorem.

- In 2003, Anders Björner and Mark de Longueville gave a new proof of Schrijver's result by applying Lovász's method to the stable Kneser graphs; in particular, they proved that the neighborhood complex of $SG_{n,k}$ is homotopy equivalent to a k-sphere.

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- In the final section of their paper, Björner and De Longueville showed that when $n = 2$, the neighborhood complex of $SG_{2,k}$ contains the boundary complex of a $(k+1)$-dimensional associahedron as a deformation retract. Their paper concluded with the following open problem:

Question

For all n and k, does the neighborhood complex of $SG_{n,k}$ contain as a deformation retract the boundary complex of a simplicial polytope?

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We show the following:

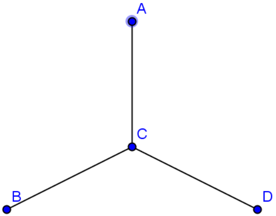
Theorem

There exists a discrete Morse matching that simplicially collapses the neighborhood complex of $SG_{n,2}$ onto a subcomplex homeomorphic to a triangulated 2-sphere.

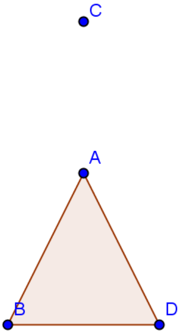
Definition

Given a graph $G = (V, E)$, the *neighborhood complex of* G is the simplicial complex $\mathcal{N}(G)$ with vertex set V and faces given by subsets of V sharing a common neighbor in G, i.e.

$$\{F \subset V : \exists v \in V \text{ s.t. } \forall u \in F, \{u, v\} \in E\}.$$



Becomes

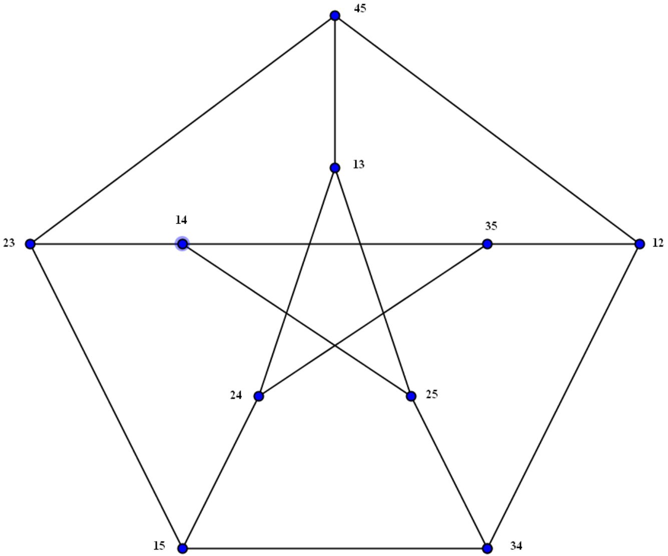


Definition

For $n \geq 1$ and $k \geq 0$ the *Kneser graph*, denoted $KG_{n,k}$, is the graph whose vertices are the subsets of $[2n+k] = \{1, 2, \ldots, 2n+k\}$ of size n. We connect two such vertices with an edge iff they are disjoint as sets.

Let $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ be an n-set of $[2n+2]$. We call α *stable* if α does not contain the subset $\{1, 2n+2\}$ or any of the subsets $\{i, i+1\}$ for $i = 1, \ldots, 2n+1$.

The *stable Kneser graph*, denoted $SG_{n,k}$, is the induced subgraph of $KG_{n,k}$ whose vertices are the stable subsets of $[2n+k]$.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Definition

For α and β stable *n*-sets, we call α and β *immediate neighbors* if $\alpha \pm 1 = \beta$, where

$$\alpha + 1 := \{\alpha_1 + 1, \dots, \alpha_n + 1\}$$

($\alpha - 1$ is defined similarly).
We call α and β *outer neighbors* if

$$\beta = (\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_{i-1} + 1, \alpha_i + 2, \alpha_{i+1} + 1, \dots, \alpha_n + 1)$$

and α and β are neighbors in $SG_{n,2}$.

Definition

We call a stable n-set α *tight* if

$$\alpha = \{\alpha_i, \alpha_i + 2, \alpha_i + 4, \dots, \alpha_i + 2(n-1)\}$$

for some $\alpha_i \in [2n+2]$. Otherwise, we call α a *loose* stable n-set.

By definition, all facets of $\mathcal{N}(SG_{n,2})$ arise from either a loose set or tight set. We handle the two cases separately.

Definition

For γ a vertex of $SG_{n,2}$, let Σ_γ be the facet in $\mathcal{N}(SG_{n,2})$ formed by the neighbors of γ.

- For any loose stable n-set γ, Σ_γ is a 3-simplex in $\mathcal{N}(SG_{n,2})$.

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- The edge consisting of γ's outer neighbors is free in $\mathcal{N}(SG_{n,2})$.
- Why? For a loose stable set $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, there exists $1 \leq r < s \leq n$, such that $\gamma_r + 3 = \gamma_{r+1}$, $\gamma_s + 3 = \gamma_{s+1}$ and for all other i ,$\gamma_i + 2 = \gamma_{i+1}$.

- Why? Note, for a tight stable set, between every element there is a gap of size 1 except for two elements, where there is a gap of size 3.

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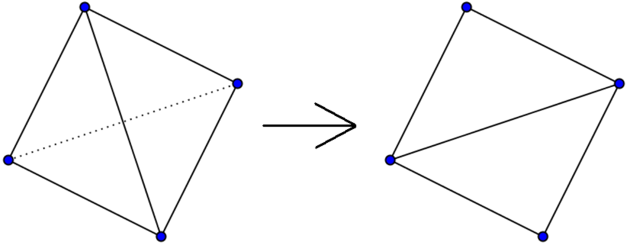
- A gap of size greater than 3 is not possible nor is more than two gaps of size 2.
- Thus, for each 3-simplex in $\mathcal{N}(SG_{n,2})$ we may collapse all faces that contain that edge.



- For a tight stable n-set α of $[2n+2]$, assign its neighbors the labels $1, 2, \ldots, n+2$ as follows: for $i = 1, \ldots, n+1$, let i be the tight stable n-set $\{1, 3, 5, \ldots, 2(n-1)\} + 2(i-1)$ or $\{2, 4, 6, \ldots, 2n\} + 2(i-1)$, whichever is a neighbor of α.

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- The remaining vertex, $n+2$, is α's unique outer neighbor.

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- An outer neighbor of α must contain the one element p of the same parity as the elements of α.
- As the outer neighbor is a stable n-set, it cannot contain $p \pm 1$.
- There are only $n-1$ viable elements left in $[2n+2] \setminus \alpha$, an outer neighbor of α must contain them all. Hence, α has a unique outer neighbor, and our claim is verified.

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- Σ_α collapses to the simplicial complex N_α where N_α consists of the following facets and their subsets:

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for some $j \in [n+1]$.

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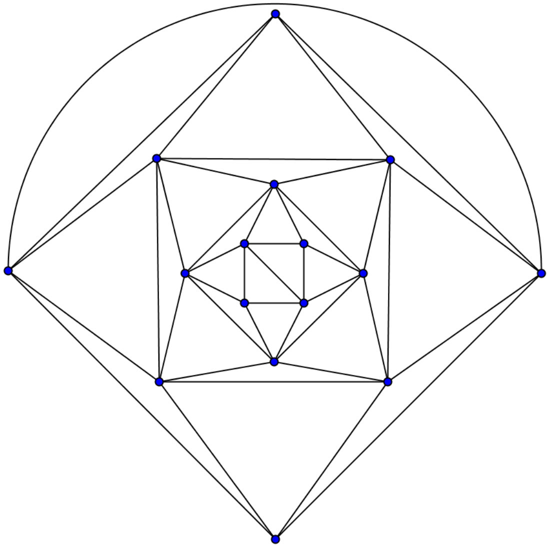
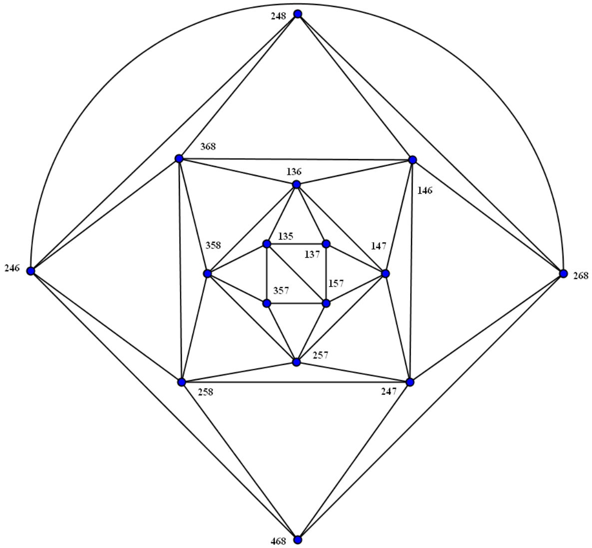
  for some $j \in [n+1]$.
- That is, Σ_α collapses to a triangulated $(n+1)$-gon where all diagonals in the triangulation emanate from the vertex labeled 1 and there is a triangle $\{j,j+1,n+2\}$ attached to the $(n+1)$-gon. Note: if $j = n+1$ then the last set listed above is replaced by $\{1,n+1,n+2\}$.

What does our collapsed complex look like?

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Thank You!

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