Higher integrality conditions and volumes of slices

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Basic definitions

A *(convex) polytope* P in the d-dimensional Euclidean space $\mathbb{R}^d$ is the convex hull of finitely many points $V = \{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^d$. In other words,

$$P = \mathrm{conv}(V) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \ : \ \text{all } \lambda_i \geq 0, \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

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For any $0 \leq \ell \leq d$, let $\pi^{(\ell)} : \mathbb{R}^d \to \mathbb{R}^{d-\ell}$ be the projection which drops the last ℓ coordinates.

Lattice points of a polytope

Definition 1. For any polytope $P \subset \mathbb{R}^d$ and positive integer $m \in \mathbb{N}$, the *mth dilated polytope* of P is $mP = \{m\mathbf{x} \ : \ \mathbf{x} \in P\}$. We denote by

$$i(P, m) = |mP \cap \mathbb{Z}^d|$$

the number of lattice points in mP.

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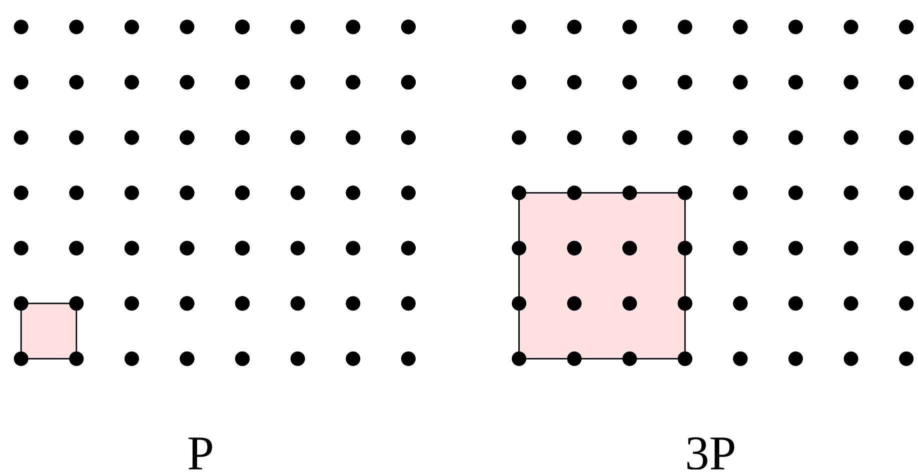
Example: For any d, P be the convex hull of the set $\{(x_1, x_2, \dots, x_d) \in \mathbb{R}^d : x_i = 0 \text{ or } 1\}$, i.e. P is the *unit cube* in $\mathbb{R}^d$. Then $i(P, m) = (m + 1)^d$.



$$i(P, 3) = (3 + 1)^2$$

Theorem of Ehrhart (on integral polytopes)

Theorem 2 (Ehrhart)**.** *Let P be a d-dimensional integral polytope. Then $i(P, m)$ is a polynomial in m of degree d.*

Therefore, we call $i(P, m)$ the *Ehrhart polynomial* of P.

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Question: When are the coefficients positive? In particular, when can they be described by volumes?

Ehrhart polynomials of integral cyclic polytopes

Theorem 3. *Suppose P is a d-dimensional integral cyclic polytope. Then*

$$i(P, m) = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P)) m^i.$$

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- $\pi^{(3)}(P) = \mathbb{R}^0 : \mathrm{Vol}(\pi^{(3)}(P)) = 1.$
- The theorem says:

$$i(P, m) = 2m^3 + 4m^2 + 3m + 1.$$

Higher integrality conditions for affine spaces

Definition 4. An ℓ-dimensional affine space $U \subset \mathbb{R}^d$ is *integral* if

$$\pi^{(d-\ell)}(U \cap \mathbb{Z}^d) = \mathbb{Z}^\ell.$$

Or equivalently, the projection $\pi^{(d-\ell)}$ induces a bijection between $U \cap \mathbb{Z}^d$ and $\mathbb{Z}^\ell$.

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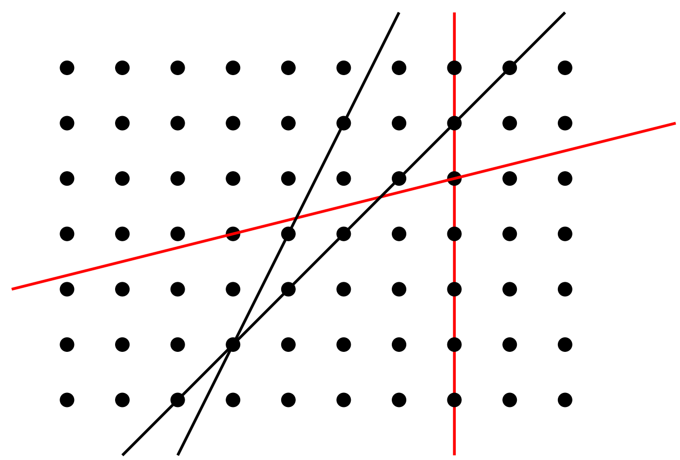

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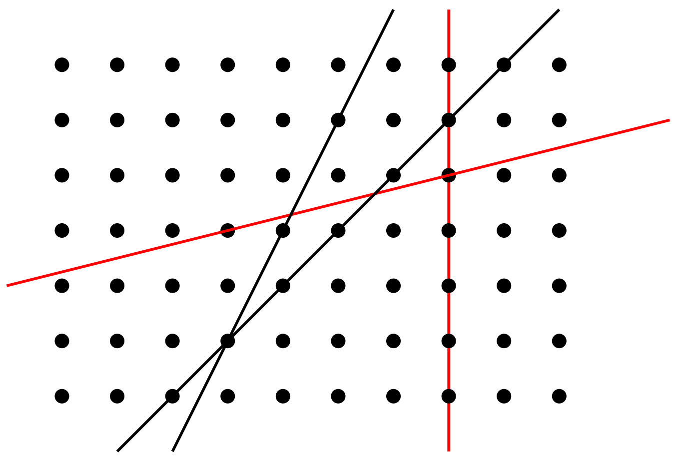

In general, U is integral if and only if U contains a lattice point and $\mathrm{dir}(U) = (1, z_2, \ldots, z_d) \in \mathbb{Z}^d$.

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Definition 5. A polytope P is *k-integral* if for any $0 \leq \ell \leq k$, we have that $\operatorname{aff}(F)$ is integral for any ℓ-dimensional face F of P.

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Fact: An integral cyclic polytope is fully integral.

An example of 1-integral polytope

Example: $P = \text{conv}\{(0,0,0),(4,0,0),(3,6,0),(2,2,2)\}$. One can check that P is 1-integral.

$$i(P,m) = 8m^3 + 10m^2 + 4m + 1,$$

and $\text{Vol}(\pi^{(d-1)}(P)) = \text{Vol}([0,4]) = 4$ and $\text{Vol}(\pi^{(d-0)}(P)) = \text{Vol}(\mathbb{R}^0) = 1$.

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Question: Is there any interesting way to calculate the higher degree coefficients?

Slices of a polytope

Definition 8. For any $\mathbf{y} \in \pi^{(d-k)}(P)$, we define *the slice of P over $\mathbf{y}$*, denoted by $\pi_{d-k}(\mathbf{y}, P)$, to be the intersection of P with the inverse image of $\mathbf{y}$ under $\pi^{(d-k)}$.

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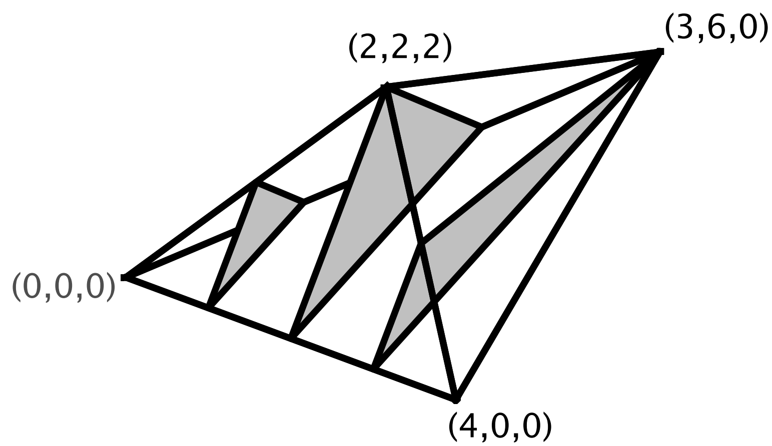
Recall $\pi^{(d-1)}(P) = [0,4]$ and $i(P,m) = 8m^3 + 10m^2 + 4m + 1$.



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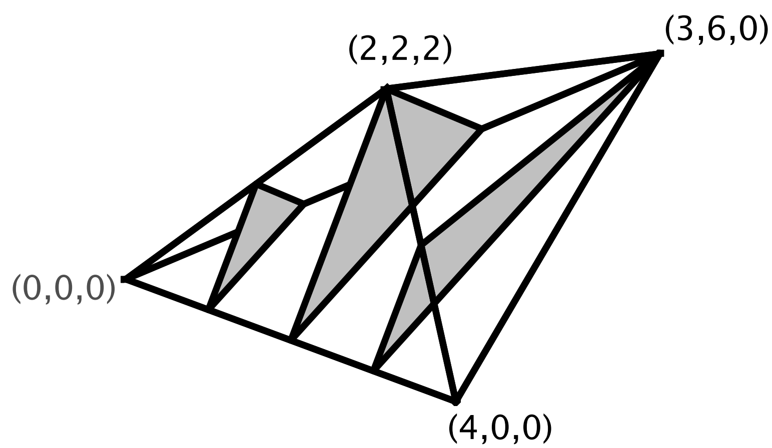

$i(\pi_2(0,P),m) = 1, i(\pi_2(1,P),m) = m^2 + 2m + 1, i(\pi_2(2,P),m) = 4m^2 + 4m + 1, i(\pi_2(3,P),m) = 3m^2 + 4m + 1$ and $i(\pi_2(4,P),m) = 1$. Their sum is

$$8m^2 + 10m + 5.$$

Main theorems

Theorem 9. *If P is k-integral, then the coefficient of m^ℓ in $i(P,m)$ is*

$$\begin{cases} \mathrm{Vol}(\pi^{d-\ell}(P)) & \text{if } 0 \le \ell \le k, \\ \sum_{\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \text{coefficient of } m^{\ell-k} \text{ in } i(\pi_{d-k}(\mathbf{y}, P), m) & \text{if } k+1 \le \ell \le d \end{cases}$$

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Theorem 10. *Suppose $k < d$. If P is k-integral, then*

$$\mathrm{Vol}(P) = \sum_{\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \mathrm{Vol}_{d-k}(\pi_{d-k}(\mathbf{y}, P)),$$

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Definition 11. We define the *kth S-volume of P* to be

$$\mathrm{SVol}^k(P) = \sum_{\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \mathrm{Vol}_{d-k}(\pi_{d-k}(\mathbf{y}, P)).$$

Remarks

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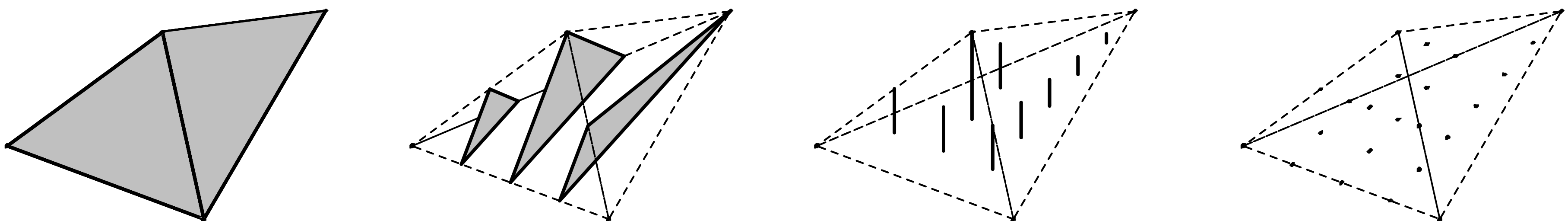

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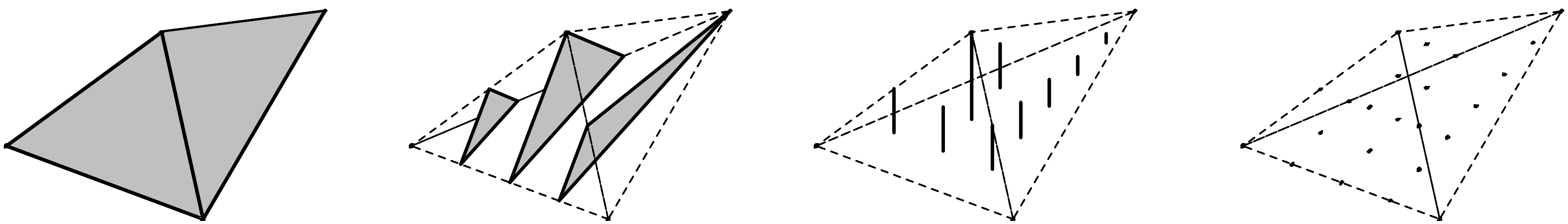

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Remark 13. Theorem 10 says if $k < d$ and P is k-integral, then $\mathrm{Vol}(P) = \mathrm{SVol}^k(P)$. Note that P is ℓ-integral for any $\ell \leq k$, so we have

$$\mathrm{Vol}(P) = \mathrm{SVol}^0(P) = \mathrm{SVol}^1(P) = \cdots = \mathrm{SVol}^k(P).$$

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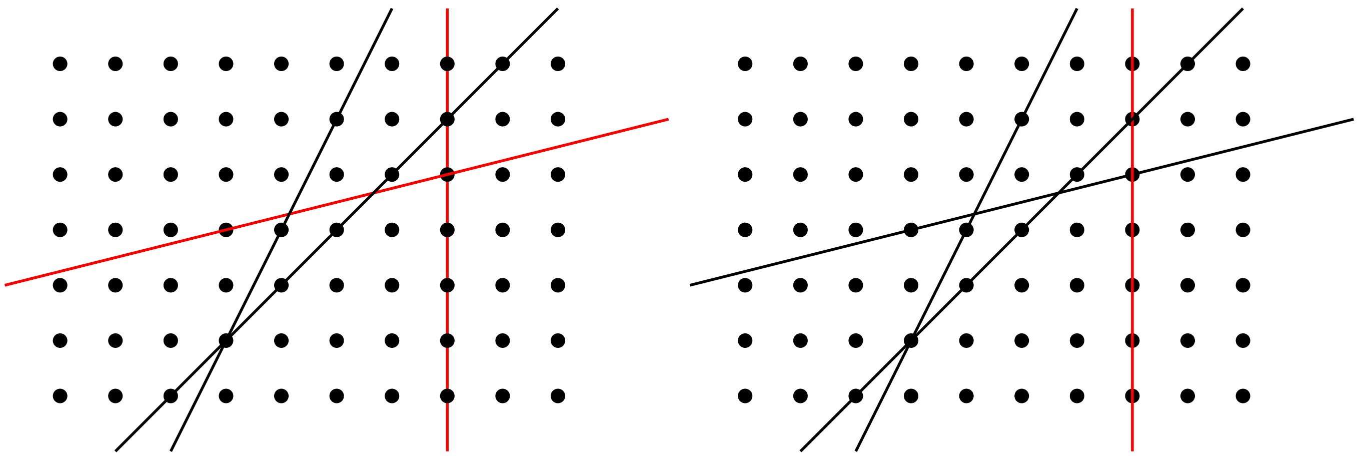

In general, U is in general position if and only if $\mathrm{dir}(U) = (1, y_1, \dots, y_d) \in \mathbb{R}^d$, or equivalently U is not perpendicular to the x_1-axis.

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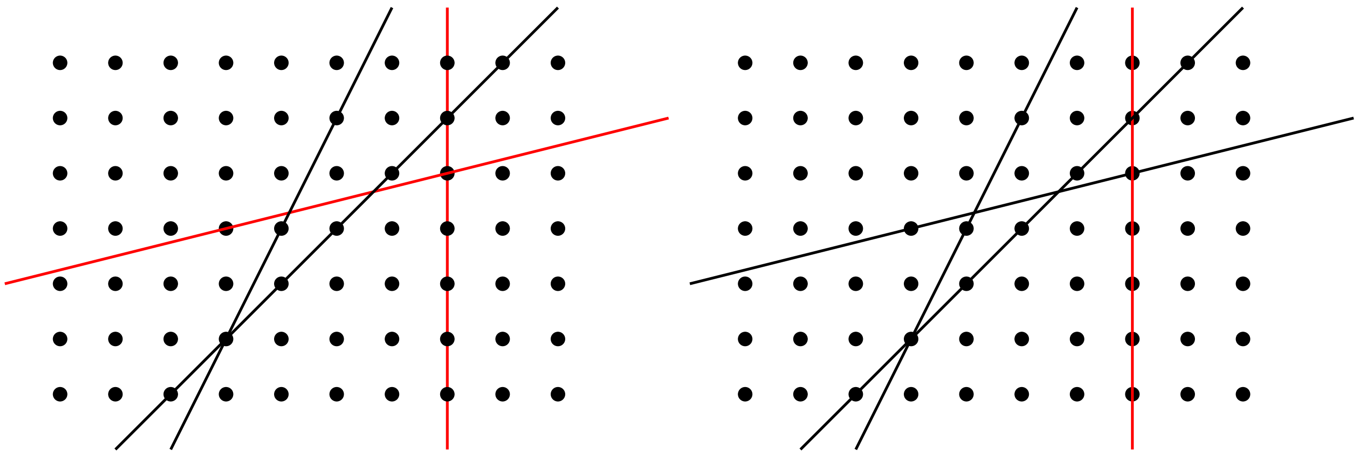
Example: 1-dimensional affine space in $\mathbb{R}^2$. integral vs in general position:



In general, U is in general position if and only if $\mathrm{dir}(U) = (1, y_1, \dots, y_d) \in \mathbb{R}^d$, or equivalently U is not perpendicular to the x_1-axis.

Theorem 9 can be reduced to Theorem 10.

Possible Applications

- Prove positivity conjectures of special families of polytopes: Birkhoff polytopes, matroid polytopes.

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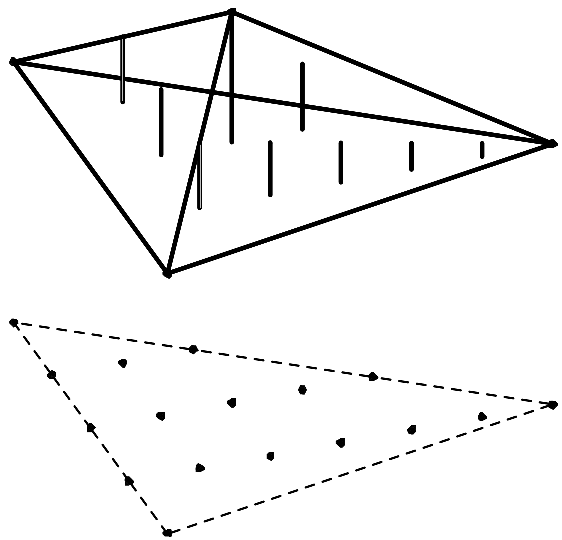
- Algorithm to calculate volumes of rational polytopes: for any rational polytope P, one can always choose a coordinate system such that P is in 1-general position. Choose $D \in \mathbb{N}$ such that DP is integral. Then

$$\mathrm{Vol}(P) = \frac{1}{D} \sum_{\mathbf{y} \in \pi^{(d-1)}(P) \cap \frac{1}{D}\mathbb{Z}^1} \mathrm{Vol}_{d-1}(\pi_{d-1}(\mathbf{y}, P)).$$

Reduction to volume formula

i. If P is fully integral, then in particular P is $(d-1)$-integral. For any $\mathbf{y} \in \pi^{(1)}(P) \cap \mathbb{Z}^{d-1}$, the slice $\pi_1(\mathbf{y}, P)$ is either a 1-dimensional integral polytope, or a lattice point. In either case, we have that $|\pi_1(\mathbf{y}, P) \cap \mathbb{Z}^d| = 1 + \mathrm{Vol}_1(\pi_1(\mathbf{y}, P))$.



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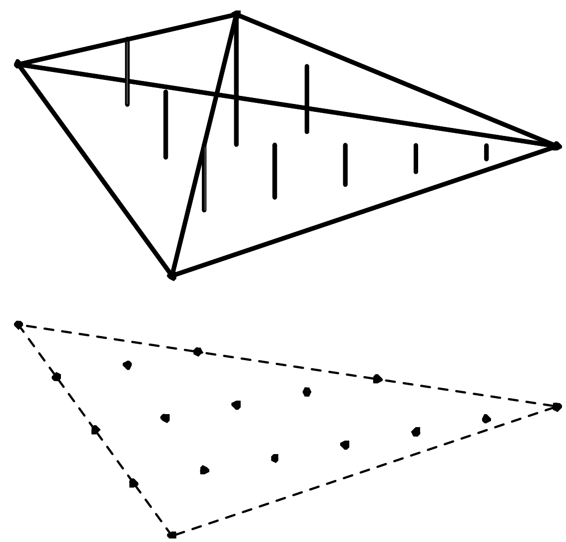

Then

$$|P \cap \mathbb{Z}^d| = \sum_{\mathbf{y} \in \pi^{(1)}(P) \cap \mathbb{Z}^{d-1}} |\pi_1(\mathbf{y}, P) \cap \mathbb{Z}^d|$$

$$= \sum_{\mathbf{y} \in \pi^{(1)}(P) \cap \mathbb{Z}^{d-1}} (1 + \mathrm{Vol}_1(\pi_1(\mathbf{y}, P))) = |\pi^{(1)}(P) \cap \mathbb{Z}^{d-1}| + \mathrm{Vol}(P).$$

Reduction to volume formula

However, $\pi^{(1)}(P)$ is fully integral. Hence,

$$|P \cap \mathbb{Z}^d| = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P)).$$

Note that P is k-integral $\Rightarrow mP$ is k-integral as well. Therefore,

$$i(P,m) = |mP \cap \mathbb{Z}^d| = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(mP)) = \sum_{i=0}^{d} \mathrm{Vol}(\pi^{(d-i)}(P))m^i.$$

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We prove Theorem 9 by using the result on fully integral polytopes as well as a local formula relating the number of lattice points to volumes of faces for integral polytopes obtained by Morelli, McMullen, Pommersheim-Thomas, Berline-Vergne.

Sketch of the proof of Theorem 10

Recall the theorem: If P is k-integral, then

$$\mathrm{Vol}(P) = \sum_{\mathbf{y} \in \pi^{(d-k)}(P) \cap \mathbb{Z}^k} \mathrm{Vol}_{d-k}(\pi_{d-k}(\mathbf{y}, P)) = \mathrm{SVol}^k(P).$$

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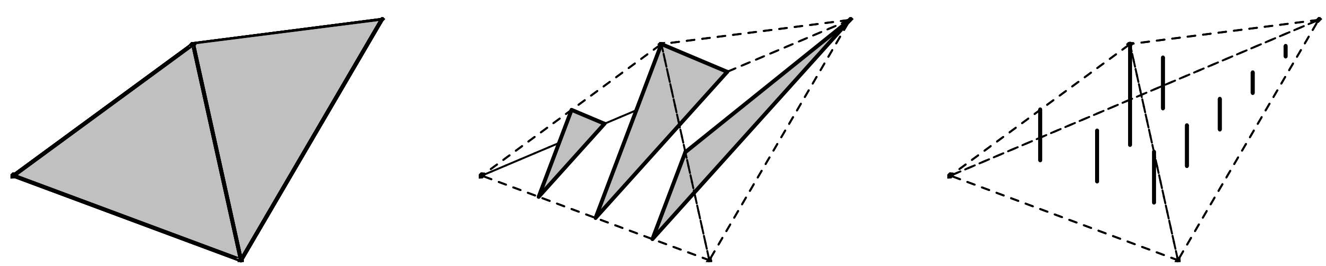
Idea: If P is k-integral (with $k > 1$), then each slice $\pi_{d-1}(\mathbf{y}, P)$ contributing to $\mathrm{SVol}^1(P)$ is $(k-1)$-integral.



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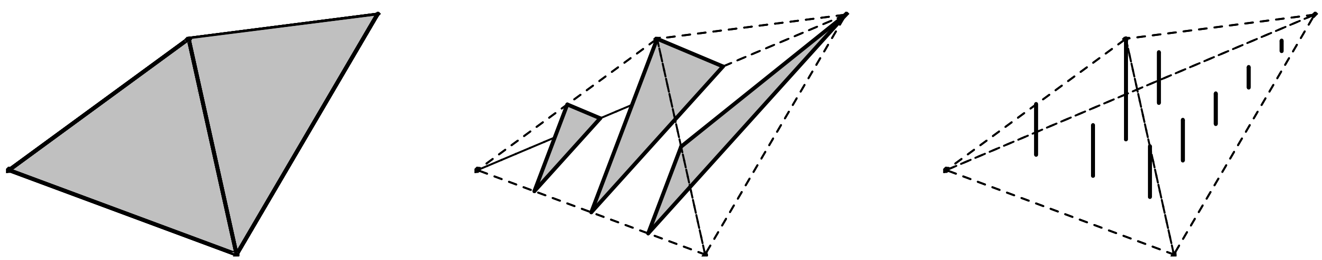

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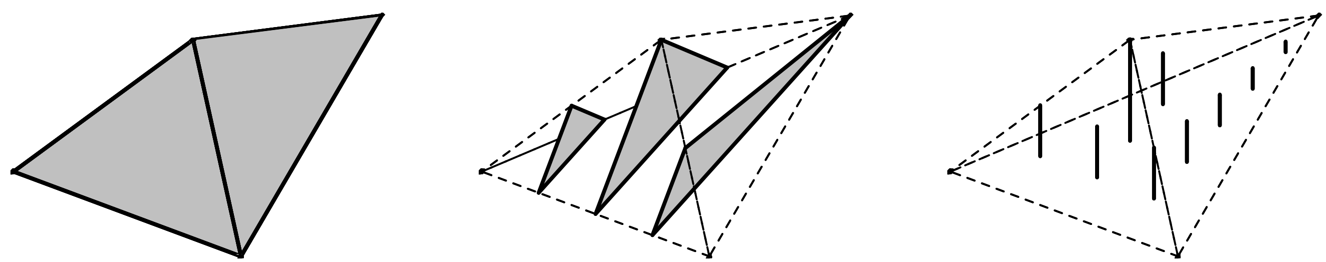

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iii. Prove the case of a simplex with $k = 1$.