γ-VECTORS:

ON FACE ENUMERATION FOR FLAG SPHERES

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Reflects joint work with Kyle Petersen and Bridget Eileen Tenner

BIG question:

Characterize the face numbers of

simplicial flag spheres.

f- and h-vectors

Δ $=$ a $(d-1)$-dimensional finite simplicial complex.

($\Delta \subseteq 2^{[n]}, T \subseteq F \in \Delta \Rightarrow T \in \Delta$.)

f-vector:

$f_i(\Delta) =$ number of i-dimensional faces of Δ

$= \#\{F \in \Delta : \ \#F = i+1\}$

$f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \cdots, f_{d-1}(\Delta))$

$f_\Delta(t) = 1 + f_0 t + f_1 t^2 + \cdots + f_{d-1} t^d$

h-vector:

$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}$

$h_\Delta(t) = 1 + h_1 t + h_2 t^2 + \cdots + h_d t^d$

Example:

$\Delta = \partial(\mathrm{triangle}) \Rightarrow \ \mathrm{f}(\Delta) = (1,3,3), \ \ \mathrm{h}_\Delta(\mathrm{t}) = 1 + \mathrm{t} + \mathrm{t}^2.$

g-vectors

Dehn-Sommerville relations:

If $\|\Delta\| \cong S^{d-1}$, or more generally if Δ is a homology sphere, then

$$h_i(\Delta) = h_{d-i}(\Delta)$$

for any $0 \leq i \leq d$.

g-vector:

$g(\Delta) = (g_0, g_1, \cdots, g_{\lfloor \frac{d}{2} \rfloor})$,

$g_0 = h_0 = 1$,

$g_i = h_i - h_{i-1}$ for $i = 1, 2, \cdots, \lfloor \frac{d}{2} \rfloor$.

Example:

$\Delta = \partial(\text{triangle}) \Rightarrow \ \mathrm{g}(\Delta) = (1, 0)$

g-Theorem: (Billera-Lee, Stanley)

An integer vector g is the g-vector for some boundary complex of a simplicial polytope iff g is an M-sequence.

In particular: $g \geq 0$ componentwise in this case.

g-Conjecture: (asked by McMullen)

If Δ is a simplicial (or homology) sphere, then $g(\Delta)$ is an M-sequence, in particular:

$$g(\Delta) \geq 0.$$

Flag spheres

Definition: A simplicial complex Δ is flag if its faces are the cliques of its one skeleton.

Equivalently, Δ is flag iff all its minimal non-faces have size two.

Examples:

- $\Delta = \partial(\mathrm{octaheder})$ is flag.
- $\Delta = \partial(\mathrm{bipyramid})$ is not flag.
- $\Delta = \partial(\mathrm{tetraheder})$ is not flag.

Question: What (more) can be said on f-vectors of flag (homology) spheres?

γ-vectors

$$B_d := \{(1+t)^d, t(1+t)^{d-2}, \cdots, t^{\lfloor \frac{d}{2} \rfloor}(1+t)^{d-2\lfloor \frac{d}{2} \rfloor}\}$$

is a basis for

$$S_d := \{\text{polynomials over } \mathbb{Q} \text{ with symmetry axis at } \frac{\mathrm{d}}{2}\}.$$

If Δ is a homology sphere of dimension $d-1$, then $h_\Delta(t) \in S_d$.

$$\gamma(\Delta) = (\gamma_0, \gamma_1, \cdots, \gamma_{\lfloor \frac{d}{2} \rfloor})$$

is the expansion of $h_\Delta(t)$ w.r.t. B_d.

Example: $\gamma(\partial(\text{cross polytope})) = (1, 0, 0, \cdots)$

Gal's conjecture:

If Δ is a flag homology sphere then $\gamma(\Delta) \geq 0$. ($\Rightarrow g(\Delta) \geq 0$)

The importance of Gal's conjecture:

Conjecture(Hopf):

Let M be a Riemanian orientable closed $2k$-dimensional manifold with nonpositive sectional curvature at every point.

Then $(-1)^k\chi(M) \geq 0$.

Case $k = 1$ is true by Gauss-Bonnet: $\int_M \mathrm{curv} = 2\pi\chi(\mathrm{M})$.

Theorem(Gromov):

If M is cubical (i.e. cells are geometric cubes) then:

$\sec(\mathrm{M}) \leq 0$ iff $\mathrm{lk}_M(v)$ is a flag sphere for all vertices $v \in M$.

Conjecture(Charney-Davis):

If Δ is a $(2k-1)$-dim flag sphere then $(-1)^k h_\Delta(-1) \geq 0$.

Charney-Davis, continued:

Conjecture(Charney-Davis):

If Δ is a $(2k-1)$-dim flag sphere then $(-1)^k h_\Delta(-1) \geq 0$.

Charney-Davis conjecture implies the cubical case of Hopf conjecture:

any i-cube has 2^i vertices, hence

$\chi(M) = -f_{M-\{\emptyset\}}(-1) = \sum_{v \in M} f_{\mathrm{lk_M(v)}}(-\frac{1}{2}) = \frac{1}{2^{2k}} \sum_{v \in M} h_{\mathrm{lk_M(v)}}(-1)$.

Note: $\gamma_k(\Delta) \geq 0$ is equivalent to the Charney-Davis conjecture.

Question: What about upper bounds on $\gamma(\Delta)$?

Upper bounds, and more, for flag spheres

A simplicial complex Γ is balanced if its vertices can be colored by $1 + \dim(\Gamma)$ colors s.t. no edge is monochromatic.

Equivalently, s.t. restricting to any face, no color repeats twice.

Conjecture:(N.-Petersen)

If Δ is a flag homology sphere then $\gamma(\Delta)$ is the f-vector of a balanced complex.

Equivalently, $\gamma(\Delta)$ satisfies the Frankl-Füredi-Kalai inequalities.

Evidence:

1. This conjecture holds if $\dim(\Delta) \leq 4$ (Gal, based on Davis-Okun). Namely: $0 \leq \gamma_2 \leq \frac{\gamma_1^2}{4}$.

2. Theorem:(N.-Petersen)

This conjecture holds if:

- Δ is a Coxeter complex.
- $\gamma_1(\Delta) \leq 3$.
- Some other infinite families, but also where $f_0(\Delta) \leq \mathrm{func}(\dim(\Delta))$.

The following theorem gives evidence with this limitation removed:

3. Theorem:(N.-Petersen-Tenner)

This conjecture holds if Δ is a barycentric subdivision of a homology sphere.

h-vector of barycentric subdivision

S_n denotes the symmetric group on $[n]$.

For $w \in S_n$, the descent number of w is

$d(w) = \#\{i : w(i) > w(i+1)\}$.

$A(n,i,j) := \#\{w \in S_n : \ w(1) = j \text{ and } \mathrm{d(w) = i}\}$.

Theorem:(Brenti-Welker)

Let Δ be a $(n-1)$-dimensional boolean complex. Then for $0 \leq i \leq n$,

$$h_i(\mathrm{sd}(\Delta)) = \sum_{j=0}^{n} A(n+1, i, j+1) h_j(\Delta).$$

γ-vector of barycentric subdivision

$S_{n,j} := \{w \in S_n : w(1) = j\}$.

The restricted Eulerian polynomials are

$A_{n,j}(t) := \sum_{w \in S_{n,j}} t^{d(w)} = \sum_{i=0}^{n-1} A(n,i,j)t^i$.

The symmetric restricted Eulerian polynomials are

$\mathbf{A}_{n,j}(t) := \sum_{w \in S_{n,j} \cup S_{n,n+1-j}} t^{d(w)}$.

Expressing in the basis B_{n-1} defines a corresponding γ-vector:

$\mathbf{A}_{n,j}(t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_i^{(n,j)} t^i (1+t)^{n-1-2i}$

$\gamma^{(n,j)} := (\gamma_0^{(n,j)}, \gamma_1^{(n,j)}, \ldots, \gamma_{\lfloor (n-1)/2 \rfloor}^{(n,j)})$.

Corollary:If Δ is a simplicial $(d-1)$-sphere then

$$\gamma(\mathrm{sd}(\Delta)) = \sum_{i=0}^{\lfloor d/2 \rfloor} h_i(\Delta) \gamma^{(d+1,i+1)}.$$

Combinatorial interpretation for $\gamma^{(n+1,1)}$

Eulerian polynomials: Let Δ be the boundary of the $(n-1)$-simplex. The Eulerian polynomial is $A_n(t) = \sum_{w \in S_{n+1}} t^{d(w)} = h_{\mathrm{sd}(\Delta)}(t)$. Its γ-vector is $\gamma^{(n+1,1)}$.

$\widehat{S}_n := \{w \in S_n : w_{n-1} < w_n, \text{ if } \mathrm{w_{i-1} > w_i \text{ then } w_i < w_{i+1}}\}$

Theorem(Foata-Schützenberger):

$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)}$, i.e.

$\gamma_i^{(n+1,1)} = |\{w \in \widehat{S}_{n+1} : d(w) = i\}|$.

w	$d(w)$	$t^{d(w)}$
123	0	1
132	1	t
213	1	t
231	1	t
312	1	t
321	2	t^2

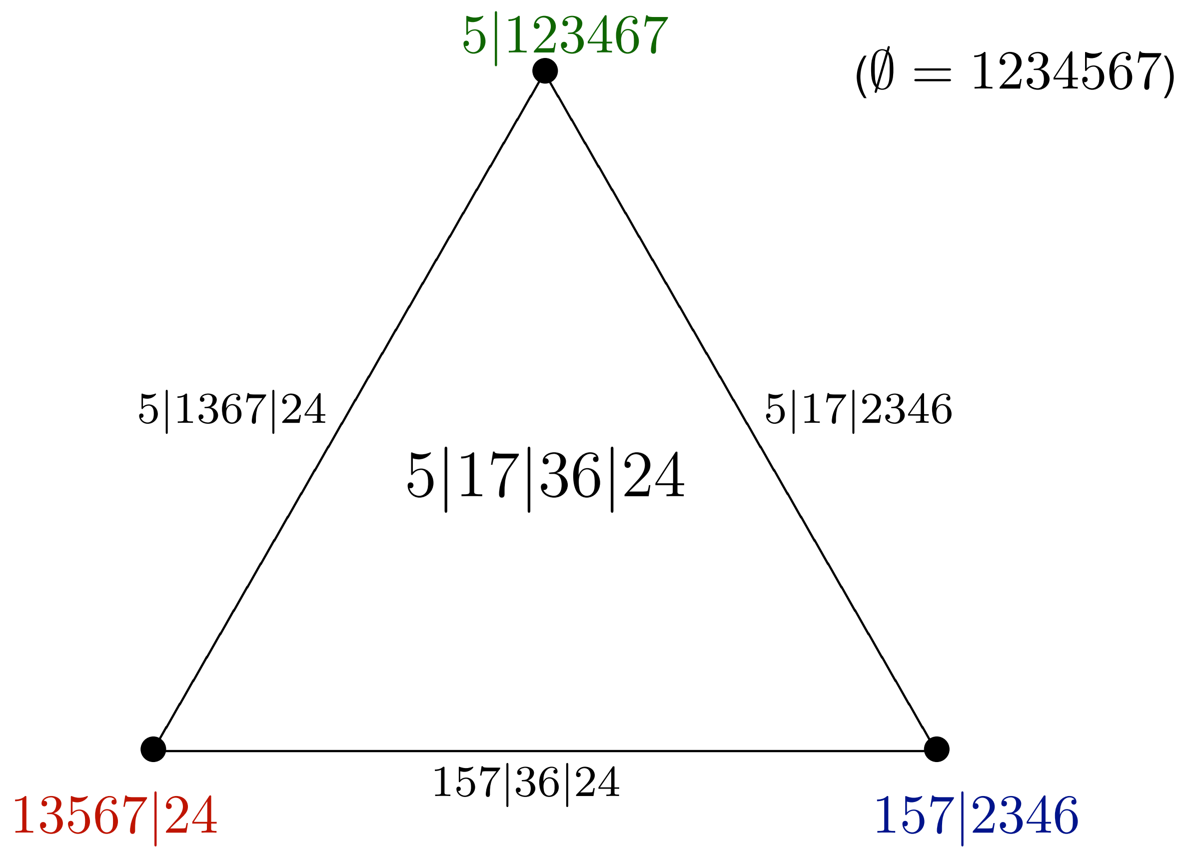
$$A_2(t) = 1 + 4t + t^2$$

$$\gamma^{(3,1)} = (1, 2)$$

Balanced complex for $\gamma^{(n+1,1)}$

We can identify elements of $\widehat{S}_{n+1}$ with faces a simplicial complex, denoted $\Gamma(A_n)$.

A facet of $\Gamma(A_6)$:



The color of a vertex with peak at position i is $\lceil \frac{i}{2} \rceil$.

$\gamma^{(n+1,j)}$ is GOOD for $\gamma^{(n+1,1)}$

Recall: $\gamma(\mathrm{sd}(\Delta)) = \sum_{i=0}^{\lfloor d/2 \rfloor} h_i(\Delta)\gamma^{(d+1,i+1)}$.

The idea:

- We know $\gamma^{(d+1,1)}$ is the f-vector of the balanced complex $\Gamma(A_d)$, hence also of its $\lfloor \frac{d}{2} \rfloor$-compression, denoted $Comp(d)$.
- Show for $i > 0$, $\gamma^{(d+1,i+1)} = (0, b_1) + ... + (0, b_m)$ where each b_j is the f-vector of a subcomplex of $Comp(d)$ s.t by coning with new vertices over these subcomplexes obtain a balanced complex with obvious coloring. Warning: d odd case!
- Key: analyzing the order preserving map $\phi_d : \mathrm{revlex}\binom{\mathbb{N}}{\mathrm{k}}_{\mathrm{d}-1} \longrightarrow \mathrm{revlex}\binom{\mathbb{N}}{\mathrm{k}}_{\mathrm{d}}$ $\phi_d((d-1)i + j)) = di + j$.

h- and g-vectors consequences

Proposition:(N.-Petersen-Tenner)

1. If $\gamma(\Delta)$ is the f-vector of a simplicial complex then so are $g(\Delta)$ and $h(\Delta)$.

2. If $\gamma(\Delta)$ is the f-vector of a balanced simplicial complex then so is $h(\Delta)$ (but $g(\Delta)$ may not be).

Example: $\Delta = \mathrm{susp}(\mathrm{C_{k+2}} * \mathrm{C_{k+2}})$ is a flag 4-sphere, if $k > 1$. $\gamma(\Delta) = (1, 2(k-2), (k-2)^2)$ and $g(\Delta) = (1, 2k, k^2+1)$.

The following consequence of N.-Petersen conjecture is open:

Conjecture: If Δ is a flag homology sphere then $g(\Delta)$ is the f-vector of a simplicial complex and $h(\Delta)$ is the f-vector of a balanced simplicial complex.

THANK YOU !