Counting Facets of Symmetric Edge Polytopes

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AMS Fall Central Sectional

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Definition

For a simple graph $G = (V, E)$, the **symmetric edge polytope** *is defined*

$$P_G := conv\{\pm(e_v - e_w) : vw \in E\} \subset \mathbb{R}^V,$$

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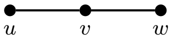

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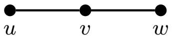

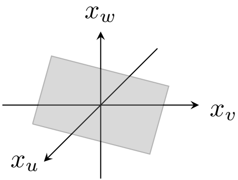

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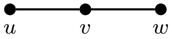

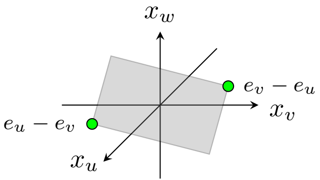

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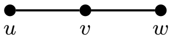

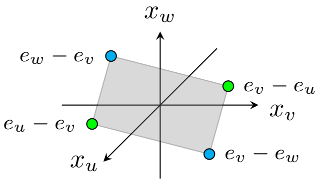

A Hyperplane Description

Every hyperplane that intersects P_G in a facet is of the form

$$H = \left\{ x \in R^V \ : \ \sum_{v \in V} f(v) x_v = 1 \right\}$$

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Theorem (Higashitani, Jochemko, Michałek 2019)

Let $G = (V, E)$ be a finite simple connected graph. Then $f : V \to \mathbb{Z}$ is facet-defining if and only if both of the following hold.

(i) *For any edge $e = uv$ we have $|f(u) - f(v)| \leq 1$.*

(ii) *The subset of edges $E_f = \{e = uv \in E \ : \ |f(u) - f(v)| = 1\}$ forms a spanning connected subgraph of G.*

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Functions $f, g : V \to \mathbb{Z}$ are identified if they differ by a common constant.

Describing Facet Subgraphs

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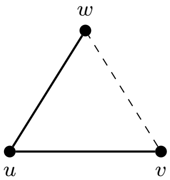

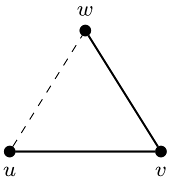

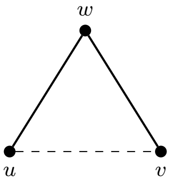

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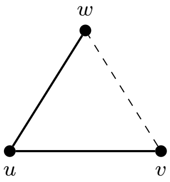

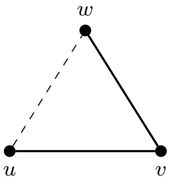

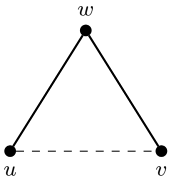

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Formulas and Bounds for Some Sparse Graphs

Let $N(G)$ denote the number of facets of P_G.

Proposition

Let $G \vee H$ denote the graph obtained by identifying the graphs G and H at a single vertex. Then

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- If T is a tree with n vertices, $N(T) = 2^{n-1}$.
- For any connected graph G with n vertices and n edges, $N(G)$ is completely determined by the length of the unique cycle in G.

Graphs with n Vertices and n Edges

Corollary

Let $C(n,m)$ be a connected graph with n vertices and n edges containing a cycle of length m. Then

$$N(C(n,m)) = \begin{cases} \binom{m}{m/2} \cdot 2^{n-m} & m \text{ even,} \\ \\ m\binom{m-1}{(m-1)/2} \cdot 2^{n-m} & m \text{ odd.} \end{cases}$$

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Theorem (Braun, B. 2022)

$N(G)$ is maximized in this family by

$$G = \begin{cases} C(n,n) & n \text{ odd,} \\ C(n,n-1) & n \text{ even.} \end{cases}$$

“Coffee Bean” Graphs

Definition

For a vector $\mathbf{m} \in \mathbb{N}^t$, let $CB(\mathbf{m})$ denote the graph made of t internally disjoint paths of lengths $m_1, m_2, \ldots, m_t$ connecting two endpoints.

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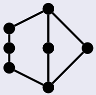
The graphs $CB(\mathbf{m})$ for which all entries of $\mathbf{m}$ are the same $m \in \mathbb{N}$ are sometimes called *theta graphs*, denoted by $\theta_{m,t}$.



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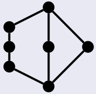
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Theorem (Braun, B. 2022)

For $\mathbf{m} \in \mathbb{N}^t$, we may permute the entries so that without loss of generality we have $m_1 \geq m_2 \geq \cdots \geq m_t$. If all the m_i's have the same parity, $N(CB(\mathbf{m}))$ is given by

$$F(\mathbf{m}) = \sum_{j=0}^{t} \binom{m_t}{j} \left[\prod_{k=1}^{t-1} \binom{m_k}{\frac{1}{2}(m_k - m_t) + j} \right].$$

Theorem (Braun, B. 2022)

For $\mathbf{m} \in \mathbb{N}^t$, *permute the entries so that* $\mathbf{m} = (e_1, \ldots, e_k, o_1, \ldots, o_\ell)$ *with* $e_1 \geq e_2 \geq \cdots \geq e_k$ *even and* $o_1 \geq o_2 \geq \cdots \geq o_\ell$ *odd and* $k, \ell \geq 1$, $k + \ell = t$. *Also, let* $\mathbf{m}_e$ *be the vector obtained by subtracting* 1 *from every even entry of* $\mathbf{m}$, *and* $\mathbf{m}_o$ *the vector obtained by subtracting* 1 *from every odd entry of* $\mathbf{m}$.

(i) *If all entries of* $\mathbf{m}$ *are at least 2,*

$$N(CB(\mathbf{m})) = \left(\prod_{j=1}^{k} e_j\right) F(\mathbf{m}_e) + \left(\prod_{j=1}^{\ell} o_j\right) F(\mathbf{m}_o).$$

(ii) *If* $o_{p+1} = \cdots = o_\ell = 1$ *(and* $o_p > 1$*),*

$$N(CB(\mathbf{m})) = \left(\prod_{j=1}^{k} e_j\right) F(\mathbf{m}_e) + \left(\prod_{j=1}^{\ell} o_j\right) N\left(\left(\vee_{j=1}^{k} C_{e_j}\right) \vee \left(\vee_{j=1}^{p} C_{o_j - 1}\right)\right).$$

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For $n \geq 3$, let $M(n)$ be the number of facets of P_G where

$$G := \begin{cases} C_{k+1} \vee C_{k-1} & n = 2k-1,\ k \text{ even} \\ C_k \vee C_k & n = 2k-1,\ k \text{ odd} \\ C_{k+1} \vee C_{k-1} \vee e & n = 2k,\ k \text{ even} \\ C_k \vee C_k \vee e & n = 2k,\ k \text{ odd} \end{cases}$$

Conjecture

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Randomly Generated Graphs

Definition

For each vertex v of a graph G, define the *local clustering coefficient* $C_{WS}(v)$ to be the number of edges connecting two neighbors of v divided by the number of possible edges between neighbors of v. The *average local clustering coefficient* (or *Watts-Strogatz clustering coefficient* is then defined as

$$C_{WS} = \frac{1}{|V(G)|} \sum_{v \in V(G)} C_{WS}(v)\,.$$

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- the space of graphs with a fixed degree sequence.

$G(n, p)$

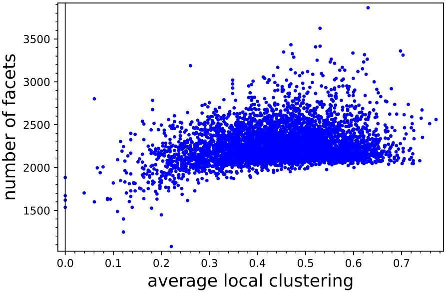


Figure 1: Data from a sample of 4874 connected graphs sampled from $G(11, 0.45)$.

Fixed Number of Edges

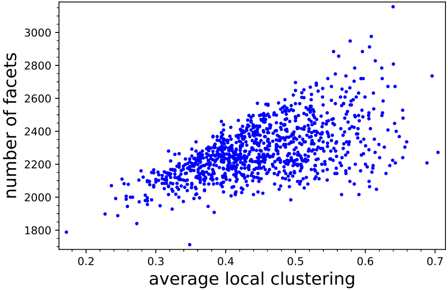


Figure 2: Data from an ensemble of 1001 graphs with 11 vertices and 25 edges.

Fixed Degree Sequence

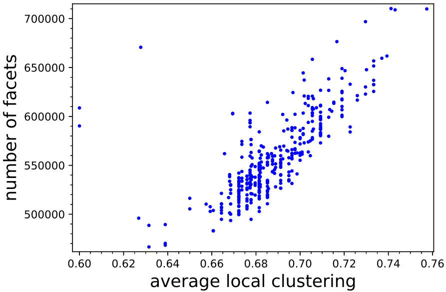
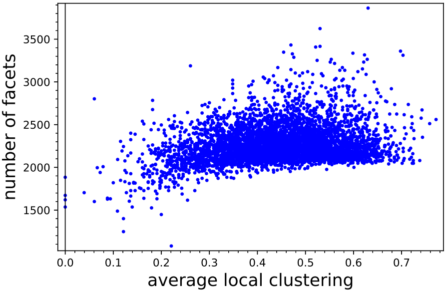


Figure 3: Data from 370 graphs having 18 vertices and degree sequence $[3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 16, 16]$.

Another Observation about $G(n, p)$



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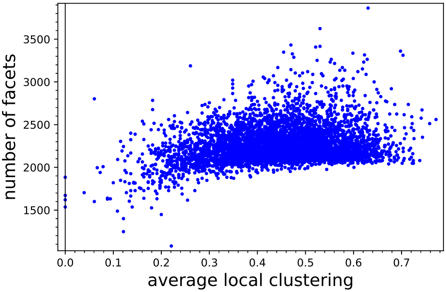

For graphs sampled from $G(n, p)$, there appears to be an approximate lower bound on number of facets given by $N(K_n) = 2^n - 2$.

Facet Subgraphs for $G(n, p)$

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For any bipartition $(A, V \setminus A)$ of the vertex set V of a graph G, we denote by $B(A, G)$ the induced bipartite subgraph for the bipartition $(A, V \setminus A)$.

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Theorem (Braun, B., Kahle 2022)

Let $G = (V, E) \sim G(n, p)$.

- *If $p < 1/2$ is fixed, then w.h.p. there exists an $\lfloor n/2 \rfloor$-subset A of V such that $B(A, G)$ is not connected.*
- *If $p > 1/2$ is fixed, then w.h.p. for every subset $A \subset V$, $B(A, G)$ consists of a single connected component unioned with isolated vertices.*
- *Further, if $p = 1/2 + \epsilon$ is fixed, then w.h.p. for every subset $A \subset V$ with $||A| - n/2| < \epsilon(1/2 - \epsilon)n$ we have that $B(A, G)$ is connected and spans V.*

A Colorful and Surprising Picture

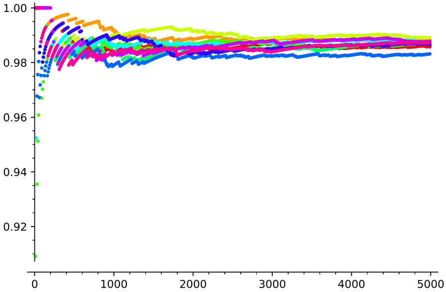

Figure 4: A collection of sequence plots for a sample of ten 11-regular connected graphs G on 5000 vertices showing how the fraction of sampled subsets A_i inducing a connected $B(A_i, G)$ changes over time and appears to stabilize near a value between 0.98 and 1.

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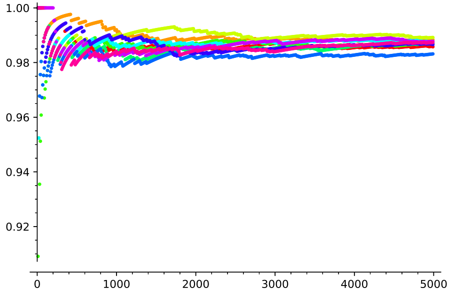

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Thank you!