Extended 0/1 Generalized Permutahedra

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Polyhedra

Definition

A **polyhedron** is a (possibly unbounded) convex subset of Euclidean space $\mathbb{R}^n$ defined by linear equations and inequalities. A **polytope** is a bounded polyhedron.

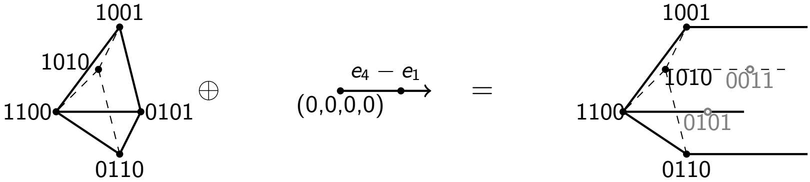
Definition

A **face** of a polyhedron P is the set of points in P maximized by some linear functional f. A face of dimension 0 is called a **vertex**. A face of dimension 1 is called an **edge**.

Polyhedra

Theorem

A polyhedron P has a decomposition $P = P' \oplus R(P)$ where $P' \subseteq P$ is a polytope and $R(P)$ is the recession cone *(cone of unbounded directions) of P.*



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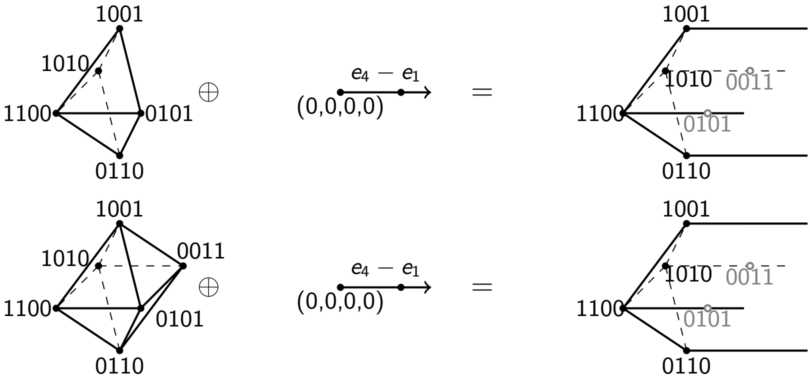
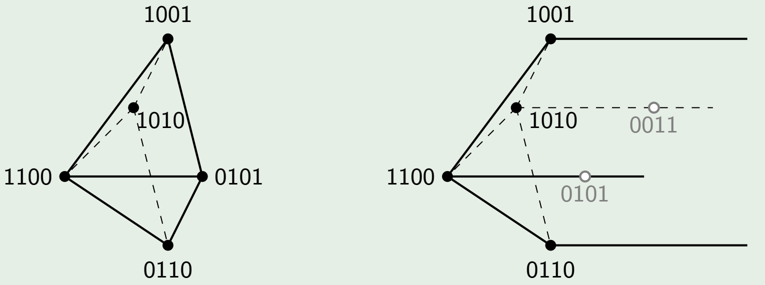


Polyhedra

Example (Two Polyhedra in $\mathbb{R}^4$)



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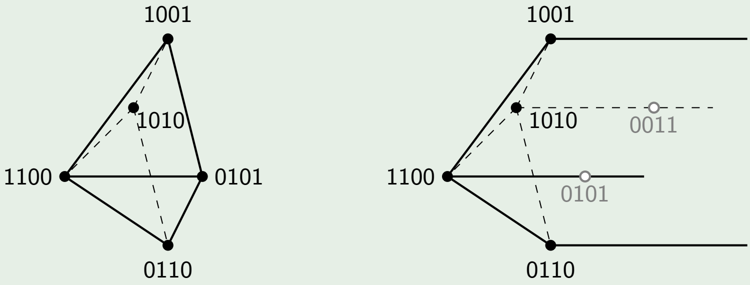

Polyhedra

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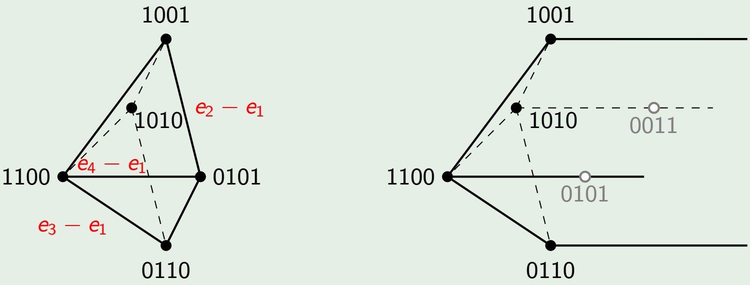
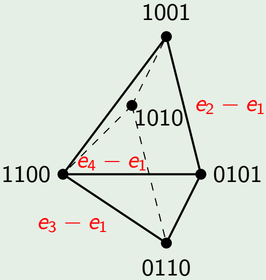


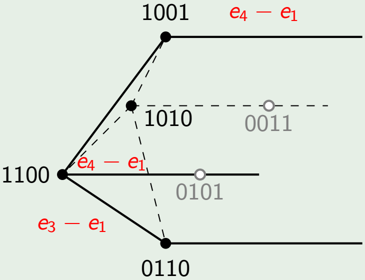
Definition

A **generalized permutahedron** is a polytope in $\mathbb{R}^n$ such that every edge is parallel to some difference of coordinate vectors $e_i - e_j$ and every vertex is in $\mathbb{R}^n_{\geq 0}$.

Polyhedra

Example (Two Polyhedra in $\mathbb{R}^4$)



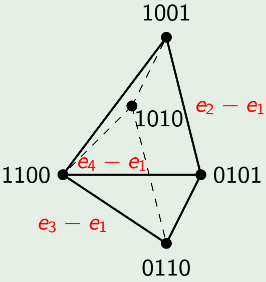


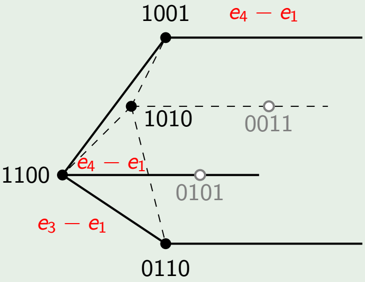
Definition

An **extended generalized permutahedron** is a polyhedron in $\mathbb{R}^n$ such that every edge or ray is parallel to some difference of coordinate vectors $e_i - e_j$ and every vertex is in $\mathbb{R}^n_{\geq 0}$.

Polyhedra

Example (Two Polyhedra in $\mathbb{R}^4$)

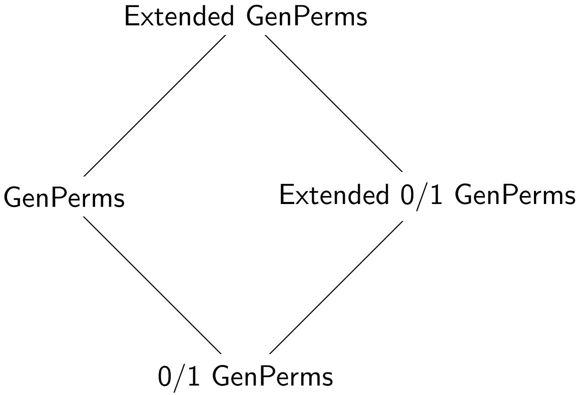




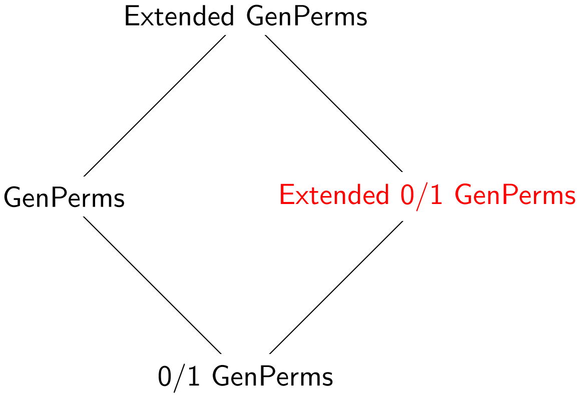
Definition

A polyhedron in $\mathbb{R}^n$ is $\mathbf{0/1}$ if all of its vertices are vectors in $\{0,1\}^n$.

Polyhedra


Extended GenPerms
GenPerms
Extended 0/1 GenPerms
0/1 GenPerms

Polyhedra


Extended GenPerms
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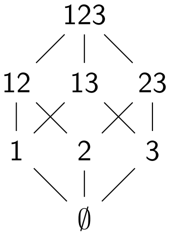
Distributive Lattices

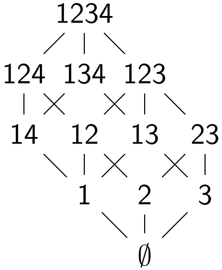
Definition

A **distributive lattice** on a set E is a subset $\mathcal{D}$ of 2^E such that, for $A, B \in E$:

1. $A \cup B \in \mathcal{D}$
2. $A \cap B \in \mathcal{D}$

All lattices $\mathcal{D}$ are assumed to be **accessible**, meaning that the rank of some $A \in \mathcal{D}$ is $|A|$.





Submodular Functions

Definition

A **matroid rank function** on ground set $[n]$ is a function $\rho : 2^{[n]} \to \mathbb{Z}$ which satisfies for $A, B \subseteq [n]$ and $e \in [n]$:

1. $\rho(\emptyset) = 0$
2. $\rho(A \cup e) \leq \rho(A) + 1$ **(unit increase)**
3. $A \subseteq B \implies \rho(A) \leq \rho(B)$ **(monotonicity)**
4. $\rho(A) + \rho(B) \geq \rho(A \cap B) + \rho(A \cup B)$ **(submodular inequality)**

Submodular Functions

Definition

A **polymatroid rank function** on ground set $[n]$ is a function $\rho : 2^{[n]} \to \mathbb{R}$ which satisfies for $A, B \subseteq [n]$ and $e \in [n]$:

1. $\rho(\emptyset) = 0$
2. $A \subseteq B \implies \rho(A) \leq \rho(B)$ **(monotonicity)**
3. $\rho(A) + \rho(B) \geq \rho(A \cap B) + \rho(A \cup B)$ **(submodular inequality)**

- We map to $\mathbb{R}$ and do not require unit increase.

Submodular Functions

Definition

A **submodular system** $S = (\mathcal{D}, \rho)$ on $[n]$ is a distributive lattice $\mathcal{D}$ on $2^{[n]}$ containing $\emptyset$ and $[n]$ with a function $\rho : \mathcal{D} \to \mathbb{R}$ satisfying for $A, B \subseteq [n]$:

1. $\rho(\emptyset) = 0$
2. $A \subseteq B \implies \rho(A) \leq \rho(B)$ **(monotonicity)**
3. $\rho(A) + \rho(B) \geq \rho(A \cap B) + \rho(A \cup B)$ **(submodular inequality)**

- We map to $\mathbb{R}$ and do not require unit increase.
- We work over a distributive lattice.

Submodular Functions

Submodular System

work over a distributive lattice

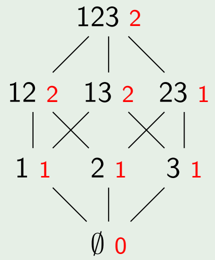
Polymatroid Rank Function

not integer-valued

not bounded by rank

Matroid Rank Function

Example



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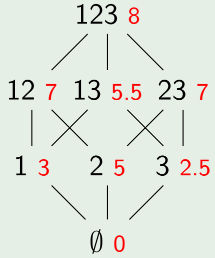

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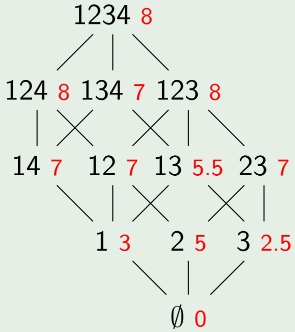


Base Polyhedra

Definition

The **base polyhedron** $B(S)$ of a submodular system $S = (\mathcal{D}, \rho)$ on $[n]$ is defined as

$$B(S) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}(A) \leq \rho(A)\ (\forall A \in \mathcal{D}) \text{ and } \mathbf{x}([n]) = \rho([n])\}$$

where, for a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we write $\mathbf{x}(A) = \sum_{i \in A} x_i$.

- The recession cone (set of unbounded directions) of $B(S)$ is determined by the lattice $\mathcal{D}$.
  - Smaller lattices give larger recession cones
- $\mathcal{D}$ is accessible $\iff$ $B(S)$ has at least one vertex.

Base Polyhedra

Taking base polyhedra gives the following correspondences:

Submodular Systems
work over a distributive lattice
$\longleftrightarrow$
Extended GenPerms
possibly unbounded

Polymatroid Rank Functions
not integer-valued
no unit-increase
$\longleftrightarrow$
GenPerms
vertices in $\mathbb{R}^n_{\geq 0}$

Matroid Rank Functions $\longleftrightarrow$ $0/1$ GenPerms

D-Matroids

To model extended 0/1 generalized permutahedra using rank functions, we work over a distributive lattice without getting rid of the unit increase property of matroids.

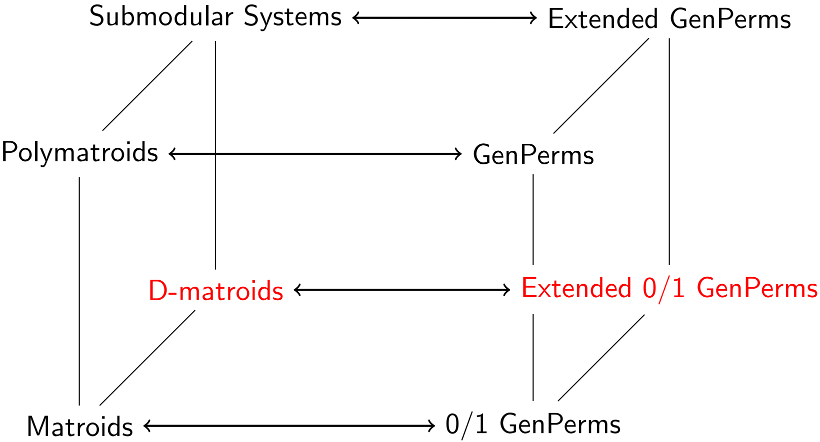
Definition

A **D-matroid rank function** $(\mathcal{D}, \rho)$ on $[n]$ is a submodular function satisfying $\rho(A \cup e) \leq \rho(A) + 1$ for any $A \in \mathcal{D}$ and $e \in E$ such that $A \cup e \in \mathcal{D}$.

Theorem

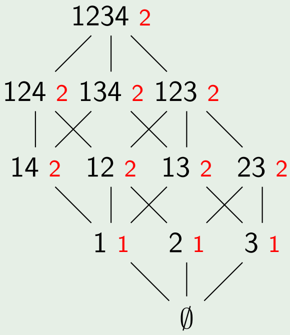
Taking base polyhedra gives a correspondence between D-matroid rank functions and extended 0/1 generalized permutahedra.

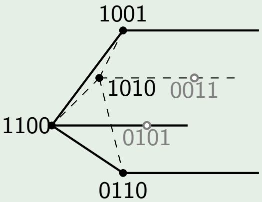
D-Matroids


Submodular Systems
Extended GenPerms
Polymatroids
GenPerms
D-matroids
Extended 0/1 GenPerms
Matroids
0/1 GenPerms

D-Matroids

Example





D-Matroids

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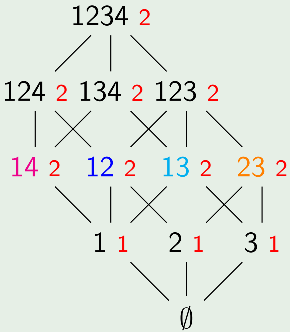

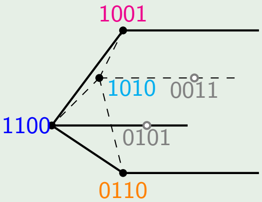

D-Matroids

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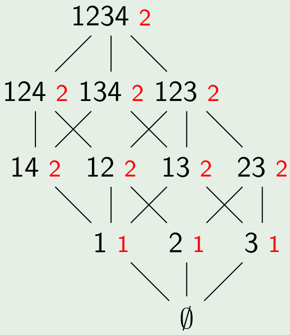
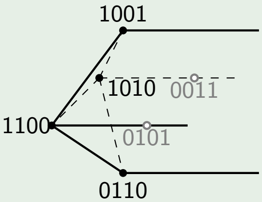
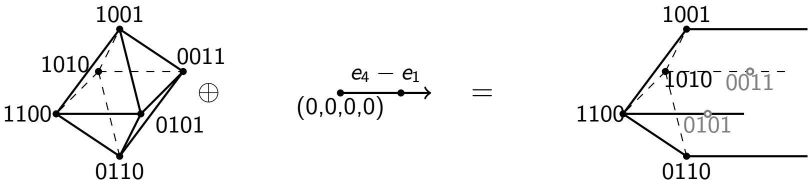




This D-matroid is the **restriction** of the uniform matroid. On the polyhedral side, this means that this polyhedron is the sum of the uniform matroid polytope with the ray $e_4 - e_1$ associated to $\mathcal{D}$.

D-Matroids


1001
1010
0011
1100
0101
0110
⊕
$e_4 - e_1$
(0,0,0,0)
=
1001
1010
0011
1100
0101
0110

New D-Matroids from Old

- A **restriction** of a D-matroid S corresponds to a sum of the base polytope $B(S)$ with the recession cone associated to $\mathcal{D}$.
- One D-matroid may have multiple matroid **extensions**.
  - These correspond to all ways of decomposing the base polyhedron as the Minkowski sum of a $0/1$ genperm with the recession cone.
- We will see that any D-matroid has a canonical largest matroid extension.

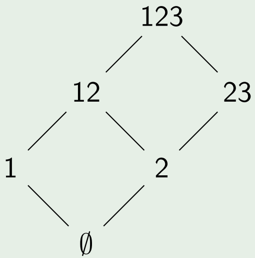
New D-Matroids from Old

Definition

Suppose that $e \in E \backslash \mathrm{Atom}(\mathcal{D})$. Let $\mathcal{D}[e]$ be the distributive sublattice of 2^E generated by $\mathcal{D} \cup \{\{e\}\}$.

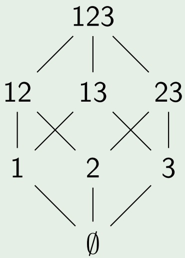
Example

$\mathcal{D} = \langle 1, 2, 23 \rangle$



$\mathrm{Atom}(\mathcal{D}) = \{1, 2\}$

$\mathcal{D}[3] = \langle 1, 2, 3 \rangle$



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Generous Extensions

Definition

The **generous atom extension** of ρ to $\mathcal{D}[e]$ is the function $\rho_a : \mathcal{D}[e] \to \mathbb{N}$ defined by

$$\rho_e(S) = \begin{cases} \rho(S) & \text{if } S \in \mathcal{D}, \\ \rho(S-e) & \text{if } S \notin D \text{ and} \\ & \exists S \subseteq S' \in \mathcal{D} \ : \ \rho(S') = \rho(S-e) \\ \rho(S-e)+1 & \text{otherwise} \end{cases}$$

We ignore the submodular inequality and rank every element as high as possible without violating unit increase or monotonicity.

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Theorem (BMS)

The generous extension ρ_e is submodular.

Generous Extensions

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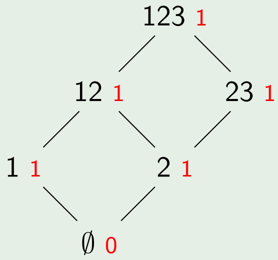

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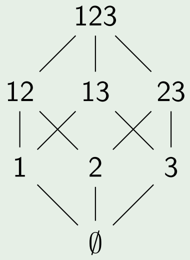

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Generous Extensions

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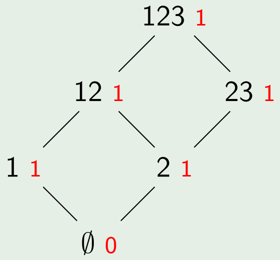

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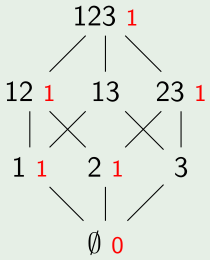

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Generous Extensions

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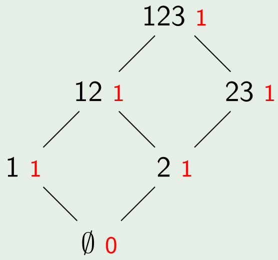

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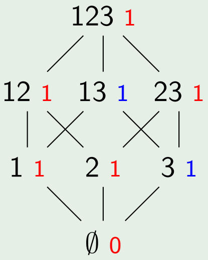

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Generous Extensions

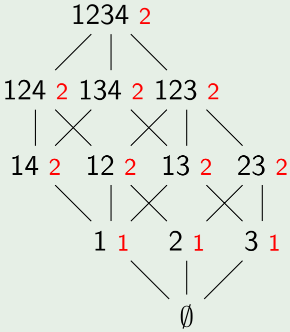
- We may repeatedly generously extend until we get a matroid rank function.
- The result is called the **generous matroid extension** of ρ.

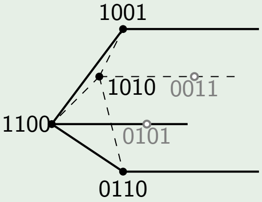
Theorem (BMS)

The generous extension $\widehat{\rho}$ of ρ is independent of the order of generous atom extensions. Moreover, it dominates *all extensions of ρ: for any other submodular extension ρ' of ρ, $\widehat{\rho}(A) \geq \rho'(A)$ for all $A \subseteq E$.*

Generous Extensions

Example





- The generous extension of this polyhedron is the convex hull of all 0/1 points inside of it. We will now see that this is always the case.

Base Polyhedra (again, for reference)

Definition

The **base polyhedron** $B(S)$ of a submodular system $S = (\mathcal{D}, \rho)$ on $[n]$ is defined as

$$B(S) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}(A) \leq \rho(A) \ (\forall A \in \mathcal{D}) \text{ and } \mathbf{x}([n]) = \rho([n])\}$$

where, for a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we write $\mathbf{x}(A) = \sum_{i \in A} x_i$.

- Each $A \in \mathcal{D}$ gives a supporting hyperplane defined by $\mathbf{x}(A) \leq \rho(A)$.

Generous Extensions

Theorem (BMS)

Let $(\mathcal{D}, \rho)$ a D-matroid and $(2^{[n]}, \hat{\rho})$ its generous matroid extension. Then the base polyhedron $B(\hat{\rho})$ is precisely the convex hull of the 0/1 vectors in $B(\rho)$.

Proof.

- Let $\mathbf{x}$ be a 0,1-vector in $B(\rho)$; $\mathbf{x}(E) = \rho(E) = \hat{\rho}(E)$.
- Choose an atom $d \in E \backslash \mathrm{Atom}(\mathcal{D})$.
- Show that $\mathbf{x}(A) \leq \rho_a(A)$ for all $A \in \mathcal{D}[e]$.

First, if $A \in \mathcal{D}$, then evidently $\mathbf{x}(A) \leq \rho(A) = \rho_e(A)$. Second, if $A \notin \mathcal{D}$, then $A - e \in \mathcal{D}$, so $\mathbf{x}(A - e) \leq \rho(A - e)$. Moreover,

$$\mathbf{x}(A) \in \{\mathbf{x}(A-e), \mathbf{x}(A-e)+1\}, \qquad \rho_e(A) \in \{\rho(A-e), \rho(A-e)+1\}.$$

Generous Extensions

Proof (continued).

In particular, if $\mathbf{x}(A) > \rho_e(A)$, then it must be the case that

$$\mathbf{x}(A) = \mathbf{x}(A - e) + 1 = \rho(A - e) + 1 = \rho_e(A) + 1.$$

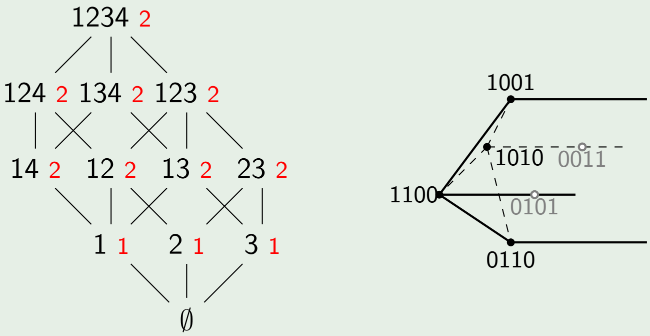
Then the definition of the generous extension gives $A \subseteq A' \in \mathcal{D}$ with $\rho(A - e) = \rho(A')$. We calculate

$$\begin{aligned}
\rho(A - e) &= \rho(A') \\
&\geq \mathbf{x}(A') && \text{(since } A' \in \mathcal{D} \text{ and } \mathbf{x} \in B(\rho)) \\
&\geq \mathbf{x}(A) && \text{(since } \mathbf{x} \in [0,1]^E \text{ and } A \subseteq A') \\
&= \rho(A - e) + 1 && \text{(by the previous equation),}
\end{aligned}$$

which is a contradiction. We conclude that $\mathbf{x}(A) \leq \rho_e(A)$ in all cases. $\square$

Generous Extensions

Example



- There are three matroid extensions of this D-matroid. The generous extension is the uniform matroid.

Ongoing Research

1. There are other **cryptomorphic** definitions of D-matroids.
   - We have a combinatorial characterization of D-matroid **closure operators** and D-matroid **lattices of flats**.

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   - We have a combinatorial characterization of D-matroid **closure operators** and D-matroid **lattices of flats**.
   - D-matroid **basis systems** ought to be the supports of vertex sets of D-matroid base polyhedra.
     - The bases of a D-matroid generate a shellable simplicial complex!
     - What is the right analogue of matroid basis exchange?

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     - The bases of a D-matroid generate a shellable simplicial complex!
     - What is the right analogue of matroid basis exchange?

2. Barnabei et al. connected **subspace arrangements** to a certain special class of D-matroids. Can we use D-matroids to understand subspace arrangements, e.g., the cohomology of their complements??

Bibliography

J. Berggren, J. Martin, J. Samper. Extended 0/1 Generalized Permutahedra

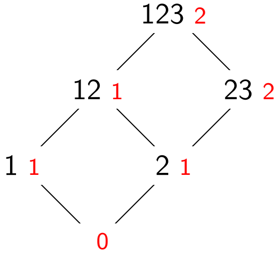
Fujishige, Satoru. Submodular functions and optimization

F. Castillo, J. Martin, J. Samper. Hopf monoids of ordered simplicial complexes

M. Barnabei, G. Nicoletti, L. Pezzoli. Matroids on partially ordered sets

I. M. Gel'fand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova. Combinatorial geometries, convex polyhedra, and Schubert cells.

A Counterexample to Several Plausible Conjectures



- The bases are 13 and 23. In particular, $13 \notin \mathcal{D}$. These are the only $0/1$ points interior to the base polyhedron.