Betti Numbers of Weighted Oriented Graphs

(Joint with Beata Casiday)

Algebraic, Geometric and
Topological Combinatorics
AMS Sectional at UTEP

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Outline

1 Motivation

2 Edge Ideals of Weighted Oriented Graphs
- History of Edge Ideals of WOG
- Difficulties

3 Algebraic Invariants of Weighted Oriented Graphs
- 1. Maximum Projective Dimension
- 2. Complete Graphs
- 3. Weight Reduction Process

Set-Up

Throughout the talk

- k is a field
- $R = k[x_1, \ldots, x_n]$ is standard graded polynomial ring
- I is a homogeneous ideal in R $\rightsquigarrow I = (x_1x_2 + x_3^2, x_2^3x_3)$

Main Question

How can we describe the structure of I?

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Example

Consider the ideal $I = (xy, xz)$ in the polynomial ring $k[x, y, z]$.

- generators of I: xy and xz
- first relations: these are the ones between xy and xz

  syzygy

$$z(xy) - y(xz) = 0$$

- second relations: relations between first relations (none)
- terminate

- Hilbert's approach: free resolutions!
- Hilbert Syzygy Theorem: The length of a free resolution of I is at most n.

Example

Let $I = (xy, xz)$ in $R = k[x, y, z]$. Then the minimal free resolution of I is of the following form:

$$0 \to R \xrightarrow{\begin{pmatrix} z \\ -y \end{pmatrix}} R \oplus R \xrightarrow{\begin{pmatrix} xy & xz \end{pmatrix}} I \to 0$$

The minimal **graded** free resolution of I is

$$0 \to R(-3) \xrightarrow{\begin{pmatrix} z \\ -y \end{pmatrix}} R(-2) \oplus R(-2) \xrightarrow{\begin{pmatrix} xy & xz \end{pmatrix}} I \to 0$$

Minimal Free Resolutions

The minimal graded free resolution of I is of the following form

$$0 \to \underbrace{\bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{p,j}(I)}}_{p^{\text{th}} \text{ step}} \xrightarrow{\partial_p} \cdots \to \underbrace{\bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{0,j}(I)}}_{0^{\text{th}} \text{ step}} \xrightarrow{\partial_0} I \to 0.$$

- $\beta_{i,j}(I)$ is called the $(i,j)^{\text{th}}$ Betti number of I
- $\beta_{i,j}(I)$: number of degree j syzygies at the i^{th} step

Betti Table and Algebraic Invariants

	0	1	2	$\cdots$	i	$\cdots$	pdim $\rightsquigarrow$ length
0	$\beta_{0,0}$	$\beta_{1,0}$	$\beta_{2,0}$	$\cdots$			
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\cdots$			
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\cdots$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$			
j					$\beta_{i,i+j}(I)$		
width $\leftsquigarrow$ reg							
Total Betti numbers	β_0	β_1	β_2	$\cdots$	β_i	$\cdots$	β_{pdim}

- $\text{pdim}(I) = \max \{i : \beta_{i,i+j}(I) \neq 0\}$
- $\text{reg}(I) = \max \{j : \beta_{i,i+j}(I) \neq 0\}$

Example continued

The minimal **graded** free resolution of $I = (xy, xz)$ is

$$0 \to \underbrace{R(-3)}_{\beta_{1,3}=1} \xrightarrow{\begin{pmatrix} z \\ -y \end{pmatrix}} \underbrace{R(-2) \oplus R(-2)}_{\beta_{0,2}=2} \xrightarrow{\begin{pmatrix} xy & xz \end{pmatrix}} I \to 0$$

Discussion

- It is hard to provide a general formula for Betti numbers for any homogeneous ideal.
- Even for the coarser invariants, classification is a difficult task.

Questions

- Establish bounds for large classes of ideals.
- Compute algebraic invariants for special classes.
- Find combinatorial descriptions.

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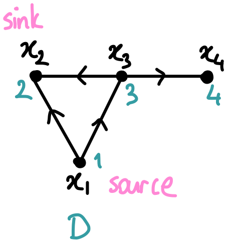
Weighted Oriented Graphs (WOG)

Definition

A **weighted oriented graph** is a triple $\mathcal{D} = (V(\mathcal{D}), E(\mathcal{D}), \omega)$.

- $V(\mathcal{D}) = \{x_1, \ldots, x_n\}$
- $E(\mathcal{D}) = \{(x_i, x_j) : (x_i, x_j)$ is a directed edge from x_i to $x_j\}$
- $\omega : V(\mathcal{D}) \to \mathbb{N}^+$, $\omega_i := \omega(x_i)$



$V(D) = \{x_1, x_2, x_3, x_4\}$

$E(D) = \{(x_1,x_2), (x_1,x_3), (x_3,x_2), (x_3,x_4)\}$

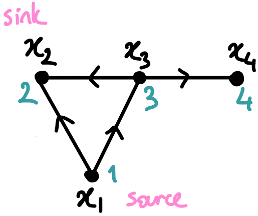
$w_1 = 1, \quad w_2 = 2, \quad w_3 = 3, \quad w_4 = 4$

Edge Ideals of WOG

Definition

The **edge ideal of** $\mathcal{D}$ is denoted by $I(\mathcal{D})$ and defined as

$$I(\mathcal{D}) = \left(x_i x_j^{\omega_j} \mid (x_i, x_j) \in E(\mathcal{D}) \right) \subseteq R = k[x_1, \ldots, x_n].$$



$\mathcal{D}$

$$I(\mathcal{D}) = (x_1 x_2^2, x_1 x_3^3, x_3 x_2^2, x_3 x_4^4)$$

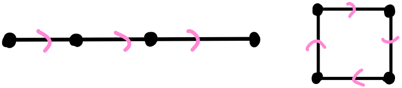
Why Edge Ideals of WOG?

- If $\omega_i = 1$ for all $i \in [n] \implies$ the usual edge ideal of a graph
- Edge ideals of WOGs can be considered as a generalization of edge ideals.
- Edge ideals of graphs are well-studied objects.
- Even though there is an extensive literature on edge ideals of graphs, we still don't have a complete picture.
- One of the known appearances of edge ideals of WOGs is in the field of algebraic coding theory.

$$I = (x_i x_j^{w_j} : 1 \leq i < j \leq n) \text{ where } 2 \leq w_1 \leq \cdots \leq w_n$$

What is known?

- Regularity and projective dimension of $I(\mathcal{D})$:
  - [Zhu, Xu, Wang, Tang (2019)] Cycles and paths: natural orientation, non-trivial weight at each vertex



  - [Biermann, -, Lin, O'Keefe (2020)] Cycles and paths: natural orientation, any weight distribution

- Not much is known about the Betti numbers of $I(\mathcal{D})$.

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Difficulties

- $I(\mathcal{D})$ is not necessarily squarefree.
- Generators of $I(\mathcal{D})$ depend on the orientation and positions of non-trivial weights.
- This makes it difficult to provide general formulas for the algebraic invariants.

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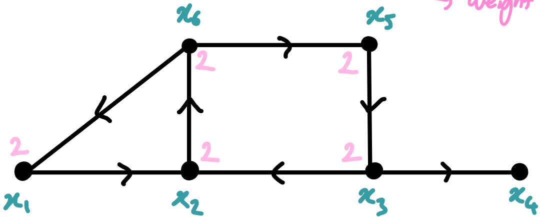
Theorem

Let $\mathcal{D}$ be a weighted oriented graph on the vertices $\{x_1, \ldots, x_n\}$. Then

$$\mathrm{pdim}(\mathcal{D}) = n$$

if and only if there is an edge $e = (x_j, x_i)$ oriented towards x_i for each $x_i \in V(\mathcal{D})$ such that x_j has a non-trivial weight.

weight is at least two.



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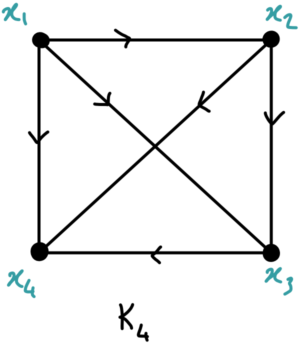
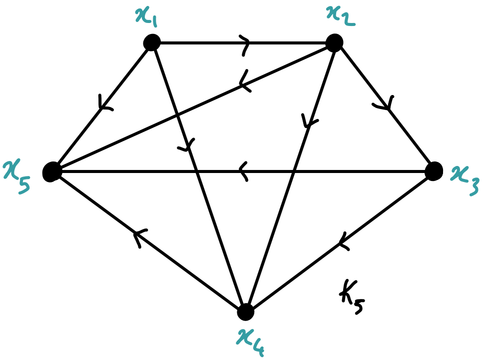
Complete Graphs

Naturally Oriented

A weighted oriented complete graph $\mathcal{K}_n$ is called **naturally oriented** if $E(\mathcal{K}_n) = \{(x_i, x_j) : 1 \le i < j \le n\}$.





Naturally oriented, (at least) one non-trivial weight

Theorem

Let $\mathcal{K}_n$ be a weighted naturally oriented complete graph. Suppose $w_p > 1$ for some $p \geq 2$. Then

(a) $\operatorname{pdim}(\mathcal{K}_n) = n - 1$ *and*

(b) $\operatorname{reg}(\mathcal{K}_n) = \sum_{i=1}^{n} w_i - n + 1.$

* This edge ideal $I(\mathcal{K}_n) = (x_i x_j^{w_j} : 1 \leq i < j \leq n)$ is a more general version of the ideal that appears in the algebraic coding theory.

$\rightsquigarrow$ relaxing the conditions

$\longrightarrow$ local conditions

One sink vertex

Theorem

Let $\mathcal{K}_n$ denote a weighted oriented complete graph on the vertices $\{x_1, \ldots, x_n\}$. If x_n is a sink in $\mathcal{K}_n$, we have

(a) $\operatorname{pdim}(\mathcal{K}_n) \in \{n-1, n\}$

(b) $\operatorname{reg}(\mathcal{K}_n) = \operatorname{reg}(\mathcal{K}_{n-1}) + (w_n - 1)$

* $\mathcal{K}_{n-1}$ is obtained from $\mathcal{K}_n$ by deleting the vertex x_n.

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Weight Reduction Process

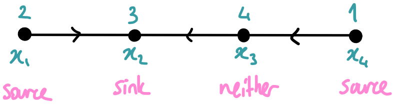
Definition

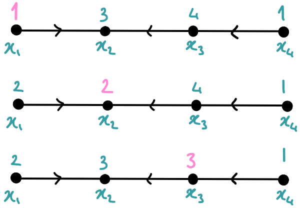
Let $\mathcal{D}$ be a weighted oriented graph.

Weight reduction process:

- preserves the vertices and oriented edges of $\mathcal{D}$,
- preserves weights of all vertices of $\mathcal{D}$ other than one vertex of non-trivial weight, say x, and reduces the weight of x by one.

$$\mathcal{D} \xrightarrow{\text{weight reduction}} \mathcal{D}' \text{: a weight reduction of } \mathcal{D}$$





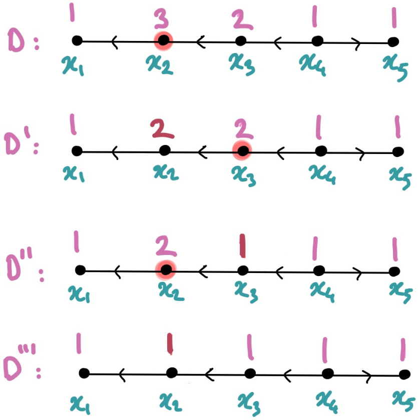
Weight reduction at a sink vertex

Theorem

Let $\mathcal{D}$ be a weighted oriented graph with a sink vertex x_p of weight $w > 1$ and let $\mathcal{D}'$ be the weight reduction of $\mathcal{D}$ on x_p. Then

$$\mathrm{pdim}(\mathcal{D}) = \mathrm{pdim}(\mathcal{D}').$$

Example



D

	0	1	2	3	4
0:	1	-	-	-	-
1:	-	2	-	-	-
2:	-	1	2	-	-
3:	-	1	2	1	-
4:	-	-	2	3	1
Tot:	1	4	6	4	1

D'

	0	1	2	3	4
0:	1	-	-	-	-
1:	-	2	-	-	-
2:	-	2	-	-	-
3:	-	-	3	4	1
Tot:	1	4	6	4	1

D''

	0	1	2	3
0:	1	-	-	-
1:	-	3	1	-
2:	-	1	4	2
Tot:	1	4	5	2

D''

	0	1	2	3
0:	1	-	-	-
1:	-	4	3	-
2:	-	-	1	1
Tot:	1	4	4	1

$\mathrm{pdim}(\mathcal{D}') = \mathrm{pdim}(\mathcal{D}'') + 1$ and $\mathrm{reg}(\mathcal{D}'') = \mathrm{reg}(\mathcal{D}''')$

Closing Remarks

- $\mathcal{D}$ be a weighted oriented graph
- Suppose $w_i \geq 2$ for some vertex x_i
- $\mathcal{D}'$ be a weight reduction of $\mathcal{D}$ on x_i.

Questions

(a) When is $\beta_i(\mathcal{D}) = \beta_i(\mathcal{D}')$ for all $i \geq 0$?

(b) Is there any relation between $\beta_{i,j}(\mathcal{D})$ and $\beta_{i,j}(\mathcal{D}')$?

(c) When is $\operatorname{pdim}(\mathcal{D}) = \operatorname{pdim}(\mathcal{D}')$?

(d) When is $\operatorname{reg}(\mathcal{D}) = \operatorname{reg}(\mathcal{D}') + 1$? or $\operatorname{reg}(D) = \operatorname{reg}(D')$?

The End

Thank You!

→ Share with your students! =)

* Check out meetamathematician.com!

** The Coalition Storytelling Event at JMM 2023!!