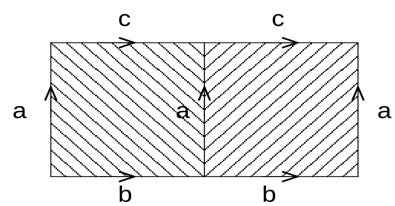
Math 821 Problem Set #7
Posted: Friday 4/29/11
Due date: Wednesday 5/11/11

Note: In all cases, "compute the homology groups" means "compute $H_n(X)$ for $n > 0$" − you don't have to incessantly repeat that $H_0(X) = \mathbb{Z}$ for path-connected spaces X.

Problem #1 [Hatcher p.156 #9b] Compute the homology groups of $S^1 \times (S^1 \vee S^1)$. (Note that this space is *not* homeomorphic to $(S^1 \times S^1) \vee (S^1 \times S^1)$.)

Solution: This space can be gotten by gluing two tori along meridional circles. It is the following cell complex, with two 2-cells, three 1-cells, and one 1-cell:



The cellular chain complex is $\mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z}$, with both boundary maps zero. So $\tilde{H}_2(X) = \mathbb{Z}^2$, $\tilde{H}_1(X) = \mathbb{Z}^3$, $\tilde{H}_0(X) = \mathbb{Z}$.

Alternate Solution: The reduced Mayer-Vietoris sequence is

$$0 \to \tilde{H}_2(T) \oplus \tilde{H}_2(T) \to \tilde{H}_2(X) \xrightarrow{f} \tilde{H}_1(S^1) \xrightarrow{g} \tilde{H}_1(T) \oplus \tilde{H}_1(T') \to \tilde{H}_1(X) \to 0$$

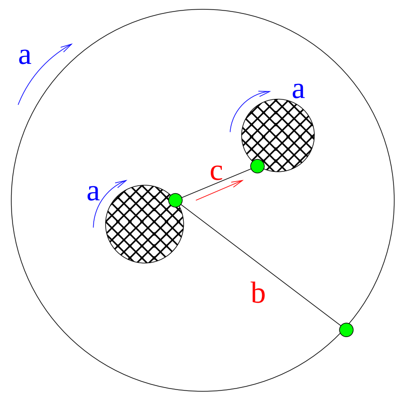
The map g maps S^1 to (a, a'), which can be extended to a basis of the image. in particular g is one-to-one, so f is the zero map and we can split the sequence into two pieces

$$0 \to \tilde{H}_2(T) \oplus \tilde{H}_2(T) \to \tilde{H}_2(X) \to 0, \qquad 0 \to \tilde{H}_1(S^1) \xrightarrow{h} \tilde{H}_1(T) \oplus \tilde{H}_1(T') \to \tilde{H}_1(X) \to 0$$

which gives $\tilde{H}_2(X) = \mathbb{Z}^2$, $\tilde{H}_1(X) = \mathbb{Z}^3$.

Problem #2 [Hatcher p.156 #9c] Compute the homology groups of the space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.

Solution: It is tempting to say that this space has a cell complex structure with one 1-cell a (the three boundary circles that have been identified with each other) and one 2-cell P (the interior of D^2 minus the two deleted disks), with $\partial P = 3a$. The problem is that P is not a 2-cell — it is not homeomorphic to $\mathbb{R}^2$. One way of dealing with this difficulty is to draw a couple of additional line segments b, c as shown.



Now the interior is a 2-cell, and we have a cell complex consisting of a vertex v (marked in green), three loops a, b, c, and a 2-cell whose boundary is identified with the loop $ac^{-1}ab^{-1}a^{-1}bc$ (in multiplicative notation), which abelianizes to simply a. So the cellular chain complex is

$$C_2 = \mathbb{Z} \xrightarrow{\begin{bmatrix}1\\0\\0\end{bmatrix}} C_1 = \mathbb{Z}^3 \xrightarrow{0} C_0 = \mathbb{Z}$$

giving $H_2 = 0$, $H_1 = \mathbb{Z}^2$, $H_0 = \mathbb{Z}$. Note that the Euler characteristic of this space is -1.

Problem #3 [Hatcher p.156 #9d] Compute the homology groups of the quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by $2\pi/m$ rotation and identifying points in the circle $\{x_0\} \times S^1$ that differ by $2\pi/n$ rotation.

Solution: Curiously enough, the n-fold and m-fold identifications don't make a difference in homology. The cell complex still has one vertex, two edges e, f and one 2-cell, with the 2-cell attached along the word $f^m e^n f^{-m} e^{-n}$. Abelianizing this thing kills it. So the cellular chain complex is the same as it is for the torus: $\mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z}$ with all maps zero, and we have $H_2 = C_2 = \mathbb{Z}$, $H_1 = C_1 = \mathbb{Z}^2$, $H_0 = C_0 = \mathbb{Z}$.

Problem #4 [Hatcher p.155 #2, modified] Given a map $f : S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. (Hint: Use the fact that S^{2n} is a covering space of $\mathbb{R}P^{2n}$.)

Solution: First, suppose that there exists a function f without the given property. Then we can construct a nonzero tangent vector field $T(x)$ on S^{2n} as follows: let $g(x)$ be the projection of $f(x)$ onto the tangent hyperplane to S^{2n} at x, and let $T(x)$ be the vector from x to $g(x)$. Note that $f(x) \notin \{x, -x\}$ implies that $g(x) \neq x$, so $T(x) \neq 0$. We have constructed an everywhere-nonzero tangent vector field on S^{2n}, which is impossible by the Hairy Ball Theorem.

(An equally good solution is to incorporate the argument of the Hairy Ball Theorem explicitly. Consider

$$F(x,t) = \frac{x \cos t + f(x) \sin t}{\|x \cos t + f(x) \sin t\|}$$

which is well-defined because x and $f(x)$ are linearly independent. Then $F(x,0) = x/\|x\| = x$ and $F(x,\pi) = -x/\|x\| = -x$, so F is a homotopy between the identity and antipodal maps, a contradiction because their degrees are 1 and -1 respectively. This is essentially the argument for property (g) of degree (Hatcher, p.134); you could even just cite that property.)

Second, let $p : S^{2n} \to \mathbb{R}P^{2n}$ be the usual two-to-one covering space map that identifies antipodal points of S^{2n}, so that $p(x) = p(-x)$. For any $g : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$, we have a commutative diagram

$$\begin{tikzcd} S^{2n} \arrow[r, "f"] \arrow[d, "p"] \arrow[dr, "g\circ p"] & S^{2n} \arrow[d, "p"] \\ \mathbb{R}P^{2n} \arrow[r, "g"] & \mathbb{R}P^{2n} \end{tikzcd}$$

where f is obtained by lifting $g \circ p$ (using the Lifting Criterion, Prop. 1.33, p.61 of Hatcher). (Note that we do not have to construct f explicitly!) By the first part of the problem, there is a point $x \in S^{2n}$ with $f(x) = x$ or $-x$, so $p(x)$ is a fixed point of g.

Problem #5 [Hatcher p.156 #15] Show that if X is a CW complex then $H_n(X^n)$ is free, by identifying it with the kernel of the cellular boundary map $H^n(X^n, X^{n-1}) \to H^{n-1}(X^{n-1}, X^{n-2})$. (Hint: If you understand how the diagram on p.139 is constructed, the proof is quite short.)

Solution: First of all, we know that $H^n(X^n, X^{n-1})$ is a free $\mathbb{Z}$-module and that any submodule of a free $\mathbb{Z}$-module is free. Second, from the diagram on p.139 we have

$$\ker d_n = \ker(j_{n-1} \circ \partial_n) = \partial_n^{-1}(\ker j_{n-1}) = \partial_n^{-1}(0) = \ker \partial_n$$

simply because j_{n-1} is injective.

Problem #6 [Hatcher p.158 #29] The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a closed 3-manifold X. Compute the homology groups of X via the Mayer-Vietoris sequence for this decomposition of X into two copies of R.

Solution: From Example 2.36, we have $\tilde{H}_1(M_g) = \mathbb{Z}^{2g}$, $\tilde{H}_2(M_g) = \mathbb{Z}$, and $\tilde{H}_n(M_g) = 0$ for $n \geq 3$.

$$\tilde{H}_2(M_g) = \mathbb{Z}\{M\}, \qquad \tilde{H}_1(M_g) = \mathbb{Z}\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$$

where the a_i are the longitudinal circles and the b_i are the meridional circles.

Meanwhile, R is homotopy equivalent to the wedge of the g longitudinal circles, so $\tilde{H}_1(R) = \mathbb{Z}^g$ and $\tilde{H}_n(R) = 0$ for $n \geq 2$.

For $n \geq 3$, the Mayer-Vietoris sequence includes the piece

$$\tilde{H}_n(R)^2 = 0 \to \tilde{H}_n(X) \to \tilde{H}_{n-1}(M_g) \to 0 = \tilde{H}_{n-1}(R)^2$$

whence $\tilde{H}_n(X) \cong \tilde{H}_{n-1}(M_g)$; this is zero for $n \geq 4$ and $\mathbb{Z}$ for $n = 3$.

That leaves the end of the sequence, which is

$$\tilde{H}_2(R)^2 = 0 \to \tilde{H}_2(X) \to \underbrace{\tilde{H}_1(M_g)}_{\mathbb{Z}^{2g}} \xrightarrow{\partial} \underbrace{\tilde{H}_1(R)^2}_{\mathbb{Z}^{2g}} \to \tilde{H}_1(X) \to 0$$

The map ∂ kills all the meridional circles and maps the longitudinal circles a_i to $(a_i, a_i) \in \tilde{H}_1(R)^2$; in particular $\tilde{H}_2(X) = \ker \partial \cong \mathbb{Z}^g$ and $\tilde{H}_1(X) = \operatorname{coker} \partial \cong \mathbb{Z}^g$.

In summary:

$$\tilde{H}_3(X) = \mathbb{Z}, \qquad \tilde{H}_2(X) = \mathbb{Z}^g, \qquad \tilde{H}_1(X) = \mathbb{Z}^g, \qquad \tilde{H}_0(X) = \mathbb{Z}$$

and $\tilde{H}_n(X) = 0$ for $n > 3$.