f-Vectors of Triangulated Balls

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1. Background

2. Excluding Possible h-vectors: A Combinatorial Approach

3. Excluding Possible h-vectors: An Algebraic Approach

4. Constructions and a Conjecture for Dimension Five

f,h, and g vectors

Let Δ be a simplicial complex of dimension $(d-1)$. The *face numbers* of Δ, denoted by $f_i(\Delta)$, are the number of i-dimensional faces of Δ.

The f*-vector* of Δ is the list of the face numbers, $f(\Delta) := (f_{-1}, f_0, f_1, \dots, f_{d-1})$.

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The *h-vector* of Δ, $h(\Delta) = (h_0, h_1, \ldots, h_d)$ contains the same combinatorial information as the f-vector but is often easier to use. Its entries are defined from the face numbers by

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Define $g_i(\Delta) := h_i(\Delta) - h_{i-1}(\Delta)$ (and $g_0 = 1$).

Homology Manifolds

A pure simplicial complex Δ of dimension $(d-1)$ is a *homology $(d-1)$-manifold* if for every non-empty face $F \in \Delta$ the link of F has the same homology as the $(d-1-|F|)$-sphere or $(d-1-|F|)$-ball.

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A *homology* $(d-1)$*-sphere* is a homology $(d-1)$-manifold with empty boundary and the same homology as the $(d-1)$-sphere.

A *homology* $(d-1)$*-ball* is an homology $(d-1)$-manifold with the same homology as the $(d-1)$-ball and boundary a homology $(d-2)$-sphere.

The g-conjecture

$(h_0, h_1, \ldots)$ is an M-vector if and only if it is the degree sequence of an order ideal of monomials.

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Conjecture

(The g-Conjecture) An integer vector $(h_0, h_1, \ldots, h_d)$ with $h_0 = 1$ is the h-vector of a homology d-sphere if and only if

1. $h_i = h_{d-i}$ *for* $0 \leq i \leq \lfloor d/2 \rfloor$
2. $(1, g_1, g_2, \ldots, g_{\lfloor d/2 \rfloor})$ *is an M-vector, where* $g_i = h_i - h_{i-1}$.

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The g-conjecture is known to hold for boundaries of polytopes.

The initial part $(1, g_1, g_2)$ of the g-vector of a homology sphere is known to be a M-vector.

Generalized Dehn-Sommerville Equations

Theorem

(Generalized Dehn-Sommerville Equations, MacDonald '71)
Let Δ *be a* $(d-1)$*-dimensional homology manifold with boundary. Then*

$$h_{d-i}(\Delta) - h_i(\Delta) = \binom{d}{i}(-1)^{d-1-i}\tilde{\chi}(|\Delta|) - g_i(\partial\Delta)$$

for all $0 \leq i \leq d$.

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If Δ is a $(d-1)$ dimensional homology ball, this reduces to

$$h_i(\Delta) - h_{d-i}(\Delta) = g_i(\partial\Delta)$$

Conjectured Conditions on Homology Balls

Conjecture

If $(h_0, \ldots, h_d)$ is the h-vector of homology $(d-1)$-ball and we take $h_i = 0$ for $i > d$ then $(h_0 - h_{d+k}, h_1 - h_{d+k-1}, \ldots, h_m - h_{d+k-m})$ is an M-vector for $k = 0, \ldots, d+1$, $m = \lfloor (d+k-1)/2 \rfloor$.

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Billera and Lee ('81) conjectured that these conditions are sufficient for the *h*-vectors of homology $(d-1)$-balls.

The sufficiency part of this conjecture was verified in dimensions less than and equal to four by Lee and Schmidt ('09).

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A Combinatorial Approach

Given an h-vector $(1, h_1, \ldots, h_d)$ and corresponding f-vector $(1, f_0, f_1, \ldots, f_{d-1})$ look at all of the graphs with f-vector $(1, f_0, f_1)$.

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For each graph, compute the h-vector $(1, h_1, h_2, h_3')$ of $(d-1)$-skeleton of the flag complex induced by the graph (i.e. add in all possible triangles and compute the start of the h-vector as if your complex had dimension $(d-1)$).

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Since removing triangles from a complex Δ decreases $h_3(\Delta)$ and adding faces of dimension greater than two does not change $h_3(\Delta)$, in order for a graph to be the one skeleton of our complex we must have $h_3' \geq h_3$.

Allowed Vertex Degree

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Any vertex of degree less than $(d-1)$ would not be contained in any facet of our complex.

If there was a vertex of degree $(d-1)$ this vertex would be contained in exactly one facet.

Removing this facet from our homology ball would create a new homology ball with the same h-vector except with h_1 decreased by one.

In some cases, we know that this new h-vector can not be the h-vector of a homology ball

h-vectors Unobtainable with Homology Balls

Consider the vector $(1, 4, 5, 7, 3, 2, 0)$.

Following the above procedure, the only possible one skeletons have a vertex of degree five.

However, if $(1, 3, 5, 7, 3, 2, 0)$ was the h-vector of a homology ball the g-vector of the corresponding boundary sphere would be $(1, 1, 2)$, contradicting a known part of the g-conjecture.

A General Counterexample to the Billera-Lee Conjecture

More generally, if x, y are integers with $x > 4$ and $1 < y < x$ then

$$\left(1, x, \binom{x}{2}, \binom{x+1}{3} - 2, \binom{x+1}{3} - 2, \ldots, \right.$$
$$\left. \binom{x+1}{3} - 2, \binom{x}{2} - \left(\binom{y}{2} + 1\right), x - y, 0\right)$$

is not the h-vector of a homology ball.

This provides a counterexample to the Billera-Lee conjecture in dimensions five and higher.

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The Face Ring (Stanley-Reisner Ring)

Let k be an infinite field of arbitrary characteristic and $R = k[x_1, \ldots, x_n]$.

For a simplicial complex Δ on vertex set $[n]$ the *face ring* (or *Stanley-Reisner ring*) is

$$k[\Delta] := R/I_\Delta$$

where

$$I_\Delta = (x_{i_1}x_{i_2}\cdots x_{i_k} : i_1 < i_2 < \cdots < i_k, \{i_1, i_2, \cdots, i_k\} \notin \Delta).$$

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A *linear system of parameters* (l.s.o.p.) for $k[\Delta]$ is a collection of degree one elements $\theta_1, \ldots, \theta_d \in k[\Delta]$ such that $k[\Delta]/(\theta_1, \ldots, \theta_d)$ is finite dimensional over k.

Define $k(\Delta) := k[\Delta]/(\theta_1, \ldots, \theta_d)$ for any l.s.o.p. $\theta_1, \ldots, \theta_d$.

Minimal Free Resolutions

Let $S = k[\Delta]$ or $k(\Delta)$.
Thinking of S as an R-module the minimal free resolution of S has form

$$\begin{aligned} 0 \rightarrow \bigoplus_j S[-j]^{\beta_{l,j}} \rightarrow \bigoplus_j S[-j]^{\beta_{l-1,j}} \rightarrow \cdots \\ \rightarrow \bigoplus_j S[-j]^{\beta_{1,j}} \rightarrow \bigoplus_j S[-j]^{\beta_{0,j}} \rightarrow S \rightarrow 0 \end{aligned}$$

The $\beta_{i,j}$ are called the *Betti numbers* of S.

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We can relate the Betti numbers of $k[\Delta]$ to the toplogy of Δ using Hochster's Formula:

$$\beta_{i,j}(k[\Delta]) = \sum_{W \subset V, |W|=j} \dim_k(\tilde{H}_{j-i-1}(\Delta_W; k))$$

Showing that a Ball can be Split

Combining some results about linear systems of parameters with Hochster's formula and a homology result due to Hibi ('94) yields:

$\beta_{n,n+1}(k(\Delta))$ is non-zero if and only if there exists a $(d-2)$-face of Δ whose removal splits Δ into two homology $(d-1)$-balls.

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Using a cancellation result due to Peeva ('04), there are some vectors $\mathbf{v}$ for which we can show that any simplicial complex Δ such that $h(\Delta) = \mathbf{v}$ must have $\beta_{n,n+1}(k(\Delta)) > 0$.

For some h-vectors, we can check that no such division exists.

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Construction

Theorem

Let $(d-1)$ be even and let $(1, h_1, h_2, \ldots, h_{d-1}, 0)$ satisfy the following conditions:

- $(1, h_1 - 1, h_2 - h_1, \ldots, h_{(d-3)/2} - h_{(d-5)/2}, \max\{h_{(d-1)/2} - h_{(d-3)/2}, 0\})$ *is an M-vector.*
- $(1, h_1 - h_{d-1}, h_2 - h_{d-2}, \ldots, h_{(d-1)/2} - h_{(d+1)/2})$ *is an M-vector.*
- $h_{(d+1)/2} \geq h_{(d+3)/2} \geq \cdots \geq h_{d-1}$.

Then there exists a (shellable) triangulated $(d-1)$-ball with h-vector $(1, h_1, h_2, \ldots, h_{d-1}, 0)$.

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Then there exists a (shellable) triangulated $(d-1)$-ball with h-vector $(1, h_1, h_2, \ldots, h_{d-1}, 0)$.

With this theorem we obtain all possible h-vectors for balls up to dimension four.

A Dimension Five Conjecture

Conjecture

A vector $h = (1, h_1, h_2, h_3, h_4, h_5, 0)$ *is the h-vector of a five-ball if any only if there exists some integer* $m > 0$ *such that* $h = (1, h_1 - m, h_2, h_3, h_4, h_5, 0)$ *satisfies the conditions of the previous theorem.*

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Using our construction and stacking new facets on the boundary of a ball we can construct five-balls with all of the h-vectors in our conjecture.

Using Betti number calculations, many of the 'small' h-vectors that do not satisfy this conjecture cannot be the h-vectors of five-balls.

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