Math 724, Fall 2013
Homework #1

Problem #13: (a) The answer is 12^3 (which happens to be 1728). We have to make three independent choices, one for each element of the domain. Each choice consists of one of the twelve elements of the range.

(b) The answer is n^m. We have to make m independent choices, one for each element of the domain. Each choice consists of one of the n elements of the range. Another way to say this is that there is a bijection

$$\{\text{functions } f : [m] \to [n]\} \;\to\; \{(x_1, \dots, x_m) :\; x_i \in [n]\}$$

given by $f \mapsto (f(1), f(2), \dots, f(n))$, and the size of the right-hand set is n^m (by the Product Principle, for instance).

(c) Well, because the cardinalities match up. Also, because as in (b), you can think of these functions as ordered tuples of elements of N, indexed by elements of M. (E.g., we call the Euclidean plane $\mathbb{R}^2$ because its points correspond bijectively to ordered pairs of real numbers, or equivalently to functions $\{x, y\} \to \mathbb{R}$.)

Problem #19: Label the fruits and the children by $[k]$ and $[n]$ respectively. We are counting one-to-one functions $[k] \to [n]$. The number of ways is

$$n(n-1)(n-2)\dots(n-k+1) \tag{0.1}$$

because the first piece of fruit can go to any of n children; the second piece to any of $n-1$ children. $\dots$, the i^{th} piece can go to any of the $n-i+1$ children who has not yet received a piece of fruit; etc.

If $k > n$ then the number of ways to do this is 0, because if all the fruit is distributed then some child must receive more than one piece (by the Pigeonhole Principle, which we will soon see), and this is not allowed. This is not a trick question — 0 is a perfectly valid answer to a counting problem! The formula (0.1) actually works correctly in this case, because if $k > n$ then the product has at least $n+1$ terms, and the $(n+1)^{st}$ term is $n - n = 0$.

Problem #20 is essentially the same as #19, and for the same reasons.

Chapter 1 Supplementary Problem #1: (a) We need $n-1$ plus signs. (This observation seems trivial, but it's key in what follows.)

(b) The answer is $\binom{n-1}{k-1}$, and here's why. A composition of n with k parts requires $k-1$ plus signs. Start with the expression $n = 1 + 1 + \cdots + 1$, choose $k-1$ of the $n-1$ plus signs to keep, and erase the others (so you will erase $(n-1)-(k-1) = n-k$ plus signs). Replace each block of 1's by the number of 1's in it. Voila, a composition with k parts! For example, if $n = 5$ and $k = 3$, we pick out two of the plus signs in the expression $1+1+1+1+1$:

$$\begin{array}{lll}
1\underline{+}1\underline{+}1+1+1 & 1+1+111 & 1+1+3 \\
1\underline{+}1+1\underline{+}1+1 & 1+11+11 & 1+2+2 \\
1\underline{+}1+1+1\underline{+}1 & 1+111+1 & 1+3+1 \\
1+1\underline{+}1\underline{+}1+1 & 11+1+11 & 2+1+2 \\
1+1\underline{+}1+1\underline{+}1 & 11+11+1 & 2+2+1 \\
1+1+1\underline{+}1\underline{+}1 & 111+1+1 & 3+1+1
\end{array}$$

Chapter 1 Supplementary Problem #2: The answer is 2^{n-1}. There is a bijection between compositions of n and subsets of an $(n-1)$-element set (namely, the set of plus signs in the expression $n = 1 + \cdots + 1$). Also, the binomial theorem implies that $\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}$.

Chapter 1 Supplementary Problem #8: I can think of at least two proofs. Algebraically:

$$\begin{aligned}\binom{n}{k}\binom{n-k}{m} &= \frac{n!}{(k!)(n-k)!}\frac{(n-k)!}{(m!)(n-k-m)!}\\ &= \frac{n!}{(k!)(m!)(n-k-m)!}\\ &= \frac{n!(n-m)!}{(k!)(m!)(n-k-m)!(n-m)!}\\ &= \frac{n!}{(m!)(n-m)!}\frac{(n-m)!}{(k!)(n-k-m)!} \quad = \quad \binom{n}{m}\binom{n-m}{k}.\end{aligned}$$

Combinatorially: Suppose we have a set of n objects. The left-hand side can be interpreted as the number of ways of painting k of them red, and then painting m of the unpainted ones blue. The right-hand side can be interpreted as the number of ways of painting m of them blue, and then painting k of the unpainted ones red. In both cases we wind up painting k objects red and m objects blue, and we get each such choice of colors exactly once.

Extra Problem (#7): The equation I'm looking for is

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \tag{0.2}$$

These are the numbers in the central column of Pascal's triangle.

The right-hand side is evidently the number of ways of taking $2n$ objects and painting n of them blue.

To interpret the left-hand side, suppose we arbitrarily divide the set of $2n$ objects into two subsets of size n (call them "apples" and "oranges") before we paint them. If we want to paint k apples blue, then we will have to paint $n-k$ oranges blue as well (in order to have a total of n blue objects). The number of ways to choose this set of objects to paint is therefore $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$. Since the number of blue apples can be anything in the range $0, \ldots, k$, we can sum over all k to get the total number of ways to paint n things blue, and this gives the left-hand side of (0.2).

To be a bit more technically precise, suppose we have two disjoint sets A, B, each of cardinality n, and let $X = A \cup B$. Consider the map

$$\begin{aligned}\phi : \{Y \subseteq X : \ |Y| = n\} &\to \{(P,Q) : \ P \subseteq A, \ Q \subseteq B, \ |P| + |Q| = n\}\\ Y &\mapsto (Y \cap A, Y \cap B)\end{aligned}$$

This is a bijection because it is invertible: $\phi^{-1}(P,Q) = P \cup Q$. Therefore the two sets have the same cardinalities, which are given by the right- and left-hand sides of (0.2) (interpreting k as $|P|$).