CRITICAL GROUPS OF SIMPLICIAL COMPLEXES

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Abstract. We generalize the theory of critical groups from graphs to simplicial complexes. Specifically, given a simplicial complex, we define a family of abelian groups in terms of combinatorial Laplacian operators, generalizing the construction of the critical group of a graph. We show how to realize these critical groups explicitly as cokernels of reduced Laplacians, and prove that they are finite, with orders given by weighted enumerators of simplicial spanning trees. We describe how the critical groups of a complex represent flow along its faces, and sketch another potential interpretation as analogues of Chow groups.

1. Introduction

Let G be a finite, simple, undirected, connected graph. The *critical group* of G is a finite abelian group $K(G)$ whose cardinality is the number of spanning trees of G. The critical group is an interesting graph invariant in its own right, and it also arises naturally in the theory of a discrete dynamical system with many essentially equivalent formulations — the *chip-firing game*, *dollar game*, *abelian sandpile model*, etc.— that has been discovered independently in contexts including statistical physics, arithmetic geometry, and combinatorics. There is an extensive literature on these models and their behavior: see, e.g., [3, 4, 7, 16, 25]. In all guises, the model describes a certain type of discrete flow along the edges of G. The elements of the critical group correspond to states in the flow model that are stable, but for which a small perturbation causes an instability.

The purpose of this paper is to extend the theory of the critical group from graphs to simplicial complexes. For a finite simplicial complex Δ of dimension d, we define its higher critical groups as

$$K_i(\Delta) := \ker \partial_i \,/\, \operatorname{im}(\partial_{i+1}\partial_{i+1}^*)$$

for $0 \leq i \leq d-1$; here ∂_j means the simplicial boundary map mapping j-chains to $(j-1)$-chains. The map $\partial_{i+1}\partial_{i+1}^*$ is called an *(updown) combinatorial Laplacian operator*. For $i = 0$, our definition coincides with the standard definition of the critical group of the 1-skeleton of Δ. Our main result (Theorem 3.4) states that, under certain mild assumptions on the complex Δ, the group $K_i(\Delta)$ is in fact isomorphic to the cokernel of a reduced version of the Laplacian. It follows from a simplicial analogue of the matrix-tree theorem [10, 11] that the orders $|K_i(\Delta)|$ of

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the higher critical groups are given by a torsion-weighted enumeration of higher-dimensional spanning trees (Corollary 4.2) and in terms of the eigenvalues of the Laplacian operators (Corollary 4.5). In the case of a simplicial sphere, we prove (Theorem 4.7) that the top-dimensional critical group is cyclic, with order equal to the number of facets, generalizing the corresponding statement [26, 27, 29] for cycle graphs. In the case that Δ is a skeleton of an n-vertex simplex, the critical groups are direct sums of copies of $\mathbb{Z}/n\mathbb{Z}$ (Theorem 4.9), which follows from a theorem of Maxwell [28] together with our main result. We also give a model of discrete flow (Section 5) on the codimension-one faces along facets of the complex whose behavior is captured by the group structure. Finally, we outline (Section 6) an alternative interpretation of the higher critical groups as discrete analogues of the Chow groups of an algebraic variety.

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2. Critical Groups of Graphs

2.1. **The chip-firing game.** We summarize the chip-firing game on a graph, omitting the proofs. For more details, see, e.g., Biggs [3].

Let $G = (V, E)$ be a finite, simple[1], connected, undirected graph, with $V = [n] \cup q = \{1, 2, \ldots, n, q\}$ and $E = \{e_1, \ldots, e_m\}$. The special vertex q is called the *bank* (or "root" or "government"). Let d_i be the degree of vertex i, i.e. the number of adjacent vertices. The chip-firing game is a discrete dynamical system whose state is described by a *configuration* vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{N}^n$. Each c_i is a nonnegative integer that we think of as the number of "chips" belonging to vertex i. (Note that the number c_q of chips belonging to the bank q is not part of the data of a configuration.)

Each non-root vertex is generous (it likes to donate chips to its neighbors), egalitarian (it likes all its neighbors equally), and prudent (it does not want to go into debt). Specifically, a vertex v_i is called *ready* in a configuration $\mathbf{c}$ if $c_i \geq d_i$. If a vertex is ready, it can *fire* by giving one chip to each of its neighbors. Unlike the other vertices, the bank is a miser. As long as other vertices are firing, the bank does not fire, but just collects chips.

As more and more chips accumulate at the bank, the game eventually reaches a configuration in which no non-bank vertex can fire. Such a configuration is called *stable*. At this point, the bank finally fires, giving one chip to each of its neighbors. Unlike the other vertices, the bank is allowed to go into debt: that is, we do not require that $c_q \geq d_q$ for the bank to be able to fire.

Denote by $\mathbf{c}(x_1, \ldots, x_r)$ the configuration obtained from $\mathbf{c}$ by firing the vertices $x_1, \ldots, x_r$ in order. This sequence (which may contain repetitions) is called a *firing sequence* for $\mathbf{c}$ if every firing is permissible: that is, for each i, either $x_i \neq q$ is ready to fire in the configuration $\mathbf{c}(x_1, \ldots, x_{i-1})$, or else $x_i = q$ and $\mathbf{c}(x_1, \ldots, x_{i-1})$ is stable. A configuration $\mathbf{c}$ is called *recurrent* if there is a nontrivial firing sequence X such that $\mathbf{c}(X) = \mathbf{c}$.

A configuration is called *critical* if it is both stable and recurrent. For every starting configuration $\mathbf{c}$, there is a uniquely determined critical configuration $[\mathbf{c}]$

[1]The chip-firing game and our ensuing results can easily be extended to allow parallel edges; we assume that G is simple for the sake of ease of exposition.

that can be reached from $\mathbf{c}$ by some firing sequence [3, Thm. 3.8]. The *critical group* $K(G)$ is defined as the set of these critical configurations, with group law given by $[\mathbf{c}]+[\mathbf{c}'] = [\mathbf{c}+\mathbf{c}']$, where the right-hand addition is componentwise addition of vectors.

The *abelian sandpile model* was first introduced in [7] as an illustration of "self-organized criticality"; an excellent recent exposition is [25]. Here, grains of sand (analogous to chips) are piled at each vertex, and an additional grain of sand is added to a (typically randomly chosen) pile. If the pile reaches some predetermined size (for instance, the degree of that vertex), then it *topples* by giving one grain of sand to each of its neighbors, which can then topple in turn, and so on. This sequence of topplings is called an *avalanche* and the associated operator on states of the system is called an *avalanche operator*. (One can show that the avalanche operator does not depend on the order in which vertices topple; this is the reason for the use of the term "abelian".) The sandpile model itself is the random walk on the stable configurations, and the critical group is the group generated by the avalanche operators.

The critical group can also be viewed as a discrete analogue of the Picard group of an algebraic curve. This point of view goes back at least as far as the work of Lorenzini [26, 27] and was developed, using the language of divisors, by Bacher, de la Harpe, and Nagnibeda [1] (who noted that their "setting has a straightforward generalization to higher dimensional objects"). It appears in diverse combinatorial contexts including elliptic curves over finite fields (Musiker [30]), linear systems on tropical curves (Haase, Musker and Yu [18]), and Riemann-Roch theory for graphs (Baker and Norine [2]).

2.2. **The algebraic viewpoint.** The critical group can be defined algebraically in terms of the Laplacian matrix.

Definition 2.1. Let G be a finite, simple, connected, undirected graph with vertices $\{1, \dots, n, q\}$. The Laplacian matrix of G is the symmetric matrix L (or, equivalently, linear self-adjoint operator) whose rows and columns are indexed by the vertices of G, with entries

$$\ell_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } ij \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Firing vertex i in the chip-firing game is equivalent to subtracting the i^{th} column of the Laplacian (ignoring the entry indexed by q) from the configuration vector $\mathbf{c}$. Equivalently, if $\mathbf{c}' = \mathbf{c}(x_1, \dots, x_r)$, then the configurations $\mathbf{c}$ and $\mathbf{c}'$ represent the same element of the cokernel of the Laplacian (that is, the quotient of $\mathbb{Z}^{n+1}$ by the column space of L).

It is immediate from the definition of L that $L(\mathbf{1}) = \mathbf{0}$, where $\mathbf{1}$ and $\mathbf{0}$ denote the all-ones and all-zeros vectors in $\mathbb{N}^{n+1}$. Moreover, it is not difficult to show that $\operatorname{rank} L = |V| - 1 = n$. In terms of homological algebra, we have a chain complex

$$\mathbb{Z}^{n+1} \xrightarrow{L} \mathbb{Z}^{n+1} \xrightarrow{S} \mathbb{Z} \to 0 \tag{1}$$

where $S(\mathbf{c}) = \mathbf{c} \cdot \mathbf{1} = c_q + c_1 + \cdots + c_n$. The equation $L(\mathbf{1}) = \mathbf{0}$ says that $\ker(S) \supseteq \operatorname{im}(L)$. Moreover, $\operatorname{rank} L = n = \operatorname{rank} \ker S$, so the abelian group $\ker(S)/\operatorname{im}(L)$ is finite.

Definition 2.2. The *critical group* of a graph G is $K(G) = \ker(S)/\operatorname{im}(L)$.

This definition of the critical group is equivalent to that in terms of the chip-firing game [3, Thm. 4.2]. The order of the critical group is the determinant of the *reduced Laplacian* formed by removing the row and column indexed by q [3, Thm. 6.2]. By the matrix-tree theorem, this is the number of spanning trees. As we will see, the algebraic description provides a natural framework for generalizing the critical group.

3. The Critical Groups of a Simplicial Complex

We first fix some notation concerning simplicial complexes and algebraic topology; see, e.g., Hatcher [20, Sects. 2.1 and 3.1] for more details. A *simplicial complex* Δ on vertex set V is a subset of the power set of V such that if $\sigma \subseteq \tau$ and $\tau \in \Delta$, then $\sigma \in \Delta$. The elements of Δ are the *faces* of Δ; we explicitly allow the empty set as a face of Δ. The *dimension* of a face σ is $\dim(\sigma) = |\sigma| - 1$, and the *dimension* of a simplicial complex is the maximum dimension of any of its faces. The symbol Δ_i will denote the set of faces of Δ of dimension i. The *f-vector* is $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \dots)$, where $f_i(\Delta)$ is the number of faces of dimension i. The *i-dimensional skeleton* or *i-skeleton* $\Delta_{(i)}$ of a simplicial complex Δ is the subcomplex consisting of all faces of dimension $\leq i$. A *facet* is a face of maximal dimension; a complex is *pure* if all facets have the same dimension.

Let Δ be a d-dimensional simplicial complex. For $-1 \leq i \leq d$, let $C_i(\Delta;\mathbb{Z})$ be the i^{th} simplicial chain group of Δ. We denote the simplicial boundary and coboundary maps respectively by

$$\begin{aligned} \partial_{\Delta,i} &: C_i(\Delta;\mathbb{Z}) \to C_{i-1}(\Delta;\mathbb{Z}), \\ \partial^*_{\Delta,i} &: C_{i-1}(\Delta;\mathbb{Z}) \to C_i(\Delta;\mathbb{Z}), \end{aligned}$$

where we have identified cochains with chains via the natural inner product. We will abbreviate the subscripts in the notation for boundaries and coboundaries whenever no ambiguity can arise. The i^{th} reduced homology group of Δ with coefficients in a ring R is denoted $\tilde{H}_i(\Delta;R)$. The *Betti numbers* of Δ are $\beta_i(\Delta) = \dim_{\mathbb{Q}} \tilde{H}_i(\Delta;\mathbb{Q})$.

Let $-1 \leq i \leq d$. The *i-dimensional combinatorial Laplacian*[2] of Δ is the operator

$$L_{\Delta,i} = \partial_{i+1}\partial^*_{i+1} \colon C_i(\Delta;\mathbb{Z}) \to C_i(\Delta;\mathbb{Z}).$$

Combinatorial Laplacian operators seem to have first appeared in the work of Eckmann [13] on finite dimensional Hodge theory. As the name suggests, they are discrete versions of the Laplacian operators on differential forms on a Riemannian manifold. In fact, Dodziuk and Patodi [8] showed that for suitably nice triangulations of a manifold, the eigenvalues of the discrete Laplacian converge in an appropriate sense to those of the usual continuous Laplacian. The eigenvalues of combinatorial Laplacians have been investigated for a number of classes of simplicial complexes. In particular, it is known that chessboard [14], matching [9], matroid [23], and shifted [12] complexes are Laplacian integral, i.e., all their Laplacian

[2]In other settings, our Laplacian might be referred to as the "up-down" Laplacian, L^{ud}. The i^{th} *down-up Laplacian* is $L_i^{\mathrm{du}} = \partial_i^*\partial_i$, and the i^{th} *total Laplacian* is $L_i^{\mathrm{tot}} = L_i + L_i^{\mathrm{du}}$. We adopt the notation we do since, except for one application (Theorem 4.9 below), we only need the up-down Laplacian.

eigenvalues are integers. For one-dimensional complexes, i.e., graphs, the combinatorial Laplacian is just the usual Laplacian matrix $L = D - A$, where D is the diagonal matrix of vertex degrees and A is the (symmetric) adjacency matrix.

In analogy to the chain complex of (1), we have the chain complex

$$C_i(\Delta;\mathbb{Z}) \xrightarrow{L} C_i(\Delta;\mathbb{Z}) \xrightarrow{\partial_i} C_{i-1}(\Delta;\mathbb{Z}),$$

where $L = L_{\Delta,i}$. (This is a chain complex because $\partial_i L = \partial_i\,\partial_{i+1}\partial^*_{i+1} = 0$.) We are now ready to make our main definition.

Definition 3.1. The i-dimensional critical group of Δ is

$$K_i(\Delta) := \ker\partial_i\,/\operatorname{im} L = \ker\partial_i\,/\operatorname{im}(\partial_{i+1}\partial^*_{i+1}).$$

Note that $K_0(\Delta)$ is precisely the critical group of the 1-skeleton of Δ.

3.1. **Simplicial spanning trees.** Our results about critical groups rely on the theory of simplicial and cellular spanning trees developed in [10], based on earlier work of Bolker [5] and Kalai [22]. Here we briefly review the definitions and basic properties, including the higher-dimensional analogues of Kirchhoff's matrix-tree theorem. For simplicity, we present the theory for simplicial complexes, the case of primary interest in combinatorics. Nevertheless, the definitions of spanning trees, their enumeration using a generalized matrix-tree theorem, and the definition and main result about critical groups are all valid in the more general setting of regular CW-complexes [11].

Definition 3.2. Let Δ be a pure d-dimensional simplicial complex, and let $\Upsilon \subseteq \Delta$ be a subcomplex such that $\Upsilon_{(d-1)} = \Delta_{(d-1)}$. We say that Υ is a *(simplicial) spanning tree* of Δ if the following three conditions hold:

(1) $\tilde{H}_d(\Upsilon;\mathbb{Z}) = 0$;
(2) $\tilde{H}_{d-1}(\Upsilon;\mathbb{Q}) = 0$ (equivalently, $|\tilde{H}_{d-1}(\Upsilon;\mathbb{Z})| < \infty$);
(3) $f_d(\Upsilon) = f_d(\Delta) - \beta_d(\Delta) + \beta_{d-1}(\Delta)$.

More generally, an *i-dimensional spanning tree* of Δ is a spanning tree of the i-dimensional skeleton of Δ.

In the case $d = 1$ (that is, Δ is a graph), we recover the usual definition of a spanning tree: the three conditions above say respectively that Υ is acyclic, connected, and has one more vertex than edge. Meanwhile, the 0-dimensional spanning trees of Δ are its vertices (more precisely, the subcomplexes of Δ with a single vertex), which are precisely the connected, acyclic subcomplexes of $\Delta_{(0)}$.

Just as in the graphical case, any two of the conditions of Definition 3.2 imply the third [10, Prop. 3.5]. In order for Δ to have a d-dimensional spanning tree, it is necessary and sufficient that $\tilde{H}_i(\Delta;\mathbb{Q}) = 0$ for all $i < d$; such a complex is called *acyclic in positive codimension*, or APC. Note that a graph is APC if and only if it is connected.

Example 3.3. Consider the *equatorial bipyramid*: the two-dimensional simplicial complex B with vertices $[5]$ and facets $123, 124, 125, 134, 135, 234, 235$. A geometric realization of B is shown in Figure 1. A two-dimensional spanning tree of B can be constructed by removing two facets F, F', provided that $F \cap F'$ contains neither of the vertices $4, 5$. A simple count shows that there are 15 such pairs F, F', so B has 15 two-dimensional spanning trees.

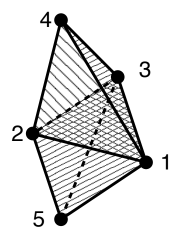

FIGURE 1. The equatorial bipyramid B.

A phenomenon arising only in dimension $d > 1$ is that spanning trees may have torsion: that is, $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ can be finite but nontrivial. For example, the 2-dimensional skeleton of a 6-vertex simplex has (several) spanning trees Υ that are homeomorphic to the real projective plane, and in particular have $\tilde{H}_1(\Upsilon; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. This cannot happen in dimension 1 (i.e., for graphs), in which every spanning tree is a contractible topological space. This torsion directly affects tree enumeration in higher dimension; see Section 4.

3.2. **The main theorem.** Our main result gives an explicit form for the critical group $K_i(\Delta)$ in terms of a reduced Laplacian matrix. This reduced form is both more convenient for computing examples, and gives a direct connection with the simplicial and cellular generalizations of the matrix-tree theorem [10, 11]. For a general reference on the homological algebra we will need, see, e.g., Lang [24].

Theorem 3.4. *Let Δ be a pure, d-dimensional, APC simplicial complex, and fix $i < d$. Let Υ be an i-dimensional spanning tree of $\Delta_{(i)}$, and let Θ be the set of i-dimensional faces of Δ* not *in Υ. Suppose that $\tilde{H}_{i-1}(\Upsilon; \mathbb{Z}) = 0$ (that is, Υ is a "torsion-free spanning tree"). Then*

$$K_i(\Delta) \cong \mathbb{Z}^{\Theta} / \operatorname{im} \tilde{L}$$

where $\tilde{L}$ denotes the reduced Laplacian obtained from L by removing the rows and columns corresponding to Υ (equivalently, by restricting L to the rows and columns corresponding to Θ).

Proof. We will construct a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im} L \arrow[r] \arrow[d, "f"'] & \ker \partial_{\Delta,i} \arrow[r] \arrow[d, "g"'] & K_i(\Delta) \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & \operatorname{im} \tilde{L} \arrow[r] & \mathbb{Z}^{\Theta} \arrow[r] & \mathbb{Z}^{\Theta} / \operatorname{im} \tilde{L} \arrow[r] & 0
\end{tikzcd} \tag{2}$$

where the rows are short exact sequences with the natural inclusions and quotient maps. The map f is defined by $f(L\theta) = \tilde{L}\theta$ for all $\theta \in \Theta$; we will show in Claim 4 that it is an isomorphism. The map g is projection onto $C_i(\Theta; \mathbb{Z})$; we will show in Claim 5 that it induces an isomorphism in the center column of (2). We will then show that $f(\gamma) = g(\gamma)$ for all $\gamma \in \operatorname{im} L$, so the left-hand square commutes. Having proven these facts, the map h is well-defined by a diagram-chase, and it is an isomorphism by the snake lemma [24, pp. 158–159]. We organize the proof into a series of claims.

Claim 1. $\operatorname{im}\partial_{\Upsilon,i} = \operatorname{im}\partial_{\Delta,i}$ as $\mathbb{Z}$-modules.

Indeed, we have $\operatorname{im}\partial_{\Upsilon,i} \subseteq \operatorname{im}\partial_{\Delta,i} \subseteq \ker\partial_{\Delta,i-1} = \ker\partial_{\Upsilon,i-1}$ (the last equality because $\Upsilon_{(i-1)} = \Delta_{(i-1)}$), so there is a short exact sequence of $\mathbb{Z}$-modules

$$0 \to \frac{\operatorname{im}\partial_{\Delta,i}}{\operatorname{im}\partial_{\Upsilon,i}} \to \frac{\ker\partial_{\Upsilon,i-1}}{\operatorname{im}\partial_{\Upsilon,i}} \to \frac{\ker\partial_{\Delta,i-1}}{\operatorname{im}\partial_{\Delta,i}} \to 0.$$

Since Υ is a torsion-free spanning tree, the middle term $\tilde{H}_{i-1}(\Upsilon;\mathbb{Z})$ is zero. Therefore, the first term is zero as well, proving Claim 1.

Claim 2. $\operatorname{coker}\partial_{\Upsilon,i}$ is a free $\mathbb{Z}$-module.

We will use some of the basic theory of projective modules [24, pp. 137–139]. The image of $\partial_{\Upsilon,i}$ is a submodule of $C_{i-1}(\Upsilon;\mathbb{Z})$, so it is free, hence projective. Therefore, the short exact sequence

$$0 \to \ker\partial_{\Upsilon,i} \to C_i(\Upsilon;\mathbb{Z}) \xrightarrow{\partial_{\Upsilon,i}} \operatorname{im}\partial_{\Upsilon,i} \to 0$$

is split: that is, $C_i(\Upsilon;\mathbb{Z}) = \ker\partial_{\Upsilon,i} \oplus F$, where F is a free $\mathbb{Z}$-module. On the other hand, $\operatorname{im}\partial_{\Upsilon,i} \subseteq \ker\partial_{\Upsilon,i-1}$, so

$$\operatorname{coker}\partial_{\Upsilon,i} = \frac{C_{i-1}(\Upsilon;\mathbb{Z})}{\operatorname{im}\partial_{\Upsilon,i}} = \frac{\ker\partial_{\Upsilon,i-1} \oplus F}{\operatorname{im}\partial_{\Upsilon,i}} = \frac{\ker\partial_{\Upsilon,i-1}}{\operatorname{im}\partial_{\Upsilon,i}} \oplus F = \tilde{H}_{i-1}(\Upsilon;\mathbb{Z}) \oplus F$$

and $\tilde{H}_{i-1}(\Upsilon;\mathbb{Z}) = 0$ by hypothesis, proving Claim 2.

Claim 3. The coboundary map $\partial^*_{\Upsilon,i}\colon C_{i-1}(\Upsilon;\mathbb{Z}) \to C_i(\Upsilon;\mathbb{Z})$ is surjective.

By the basic theory of finitely generated abelian groups [24, pp. 42–46], we may write

$$\partial_{\Upsilon,i} = P \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right] Q \tag{3}$$

where $P \in GL_{f_{i-1}}(\mathbb{Z})$, $Q \in GL_{f_i}(\mathbb{Z})$, and D is a diagonal matrix whose entries are the cyclic summands of the torsion submodule of $\operatorname{coker}\partial_{\Upsilon,i}$. In fact, these entries are all 1 by Claim 2. Moreover, the columns of $\partial_{\Upsilon,i}$ are linearly independent over $\mathbb{Q}$ and over $\mathbb{Z}$ because Υ is a simplicial tree, so in fact there are no zero columns in (3). Therefore

$$\partial_{\Upsilon,i} = P \left[\begin{array}{c} I \\ \hline 0 \end{array}\right] Q = P \left[\begin{array}{c} Q \\ \hline 0 \end{array}\right]$$

and transposing yields

$$\partial^*_{\Upsilon,i} = \left[\begin{array}{c|c} Q^T & 0 \end{array}\right] P^T$$

and so

$$\begin{aligned} \operatorname{im}\partial^*_{\Upsilon,i} &= \operatorname{im}\left[\begin{array}{c|c} Q^T & 0 \end{array}\right] && \text{(because } P\text{, hence } P^T\text{, is invertible over } \mathbb{Z}) \\ &= C_i(\Upsilon;\mathbb{Z}) && \text{(because } Q\text{, hence } Q^T\text{, is invertible over } \mathbb{Z}). \end{aligned}$$

We have proved Claim 3.

Claim 4. $L(C_i(\Delta;\mathbb{Z})) = L(C_i(\Theta;\mathbb{Z}))$.

Choose an arbitrary chain $\gamma \in C_i(\Upsilon;\mathbb{Z})$. By Claim 3, there is a chain $\eta \in C_{i-1}(\Upsilon;\mathbb{Z})$ such that $\partial^*_{\Upsilon,i}(\eta) = \gamma$. On the other hand, $\partial^*_{\Delta,i}(\eta) = \partial^*_{\Upsilon,i}(\eta) - \theta$ for some chain $\theta \in C_i(\Theta;\mathbb{Z})$. Hence

$$L(\gamma) - L(\theta) \;=\; L(\gamma - \theta) \;=\; L(\partial^*_{\Upsilon,i}(\eta) - \theta) \;=\; L\partial^*_{\Delta,i}(\eta) \;=\; \partial_{i+1}\partial^*_{i+1}\partial^*_i(\eta) \;=\; 0$$

and so $L(\gamma) = L(\theta)$. In particular, $L(\gamma) \in L(C_i(\Theta;\mathbb{Z}))$, which proves Claim 4.

Observe that

$$L \;=\; \left[\; L(C_i(\Upsilon;\mathbb{Z})) \;\middle|\; L(C_i(\Theta;\mathbb{Z})) \;\right] \;=\; \left[\begin{array}{c|c} L(C_i(\Upsilon;\mathbb{Z})) & \begin{array}{c} * \\ \hline \tilde{L} \end{array} \end{array}\right].$$

Thus Claim 4 says that the Θ-columns span the full column space of L. Since L is a symmetric matrix, this statement remains true if we replace "column" with "row". In particular, there is an isomorphism $f\colon L(C_i(\Theta;\mathbb{Z})) \to \tilde{L}(C_i(\Theta;\mathbb{Z}))$ given by deleting the Υ-rows; that is, $f(L\theta) = \tilde{L}\theta$.

By Claim 1, for each chain $\theta \in C_i(\Theta;\mathbb{Z})$, we can write

$$\partial_{\Delta,i}(\theta) = \sum_{\sigma\in\Upsilon_i} c_{\sigma\theta}\partial_{\Delta,i}(\sigma)$$

with $c_{\sigma\theta} \in \mathbb{Z}$. Therefore, the chain

$$\hat{\theta} := \theta - \sum_{\sigma\in\Upsilon_i} c_{\sigma\theta}\sigma$$

lies in $X := \ker \partial_{\Delta,i}$.

Claim 5. The set $\{\hat{\theta}\colon \theta \in \Theta\}$ is a $\mathbb{Z}$-module basis for X.

First, this set is linearly independent, since the support of each $\hat{\theta}$ in Θ is simply θ. Now we show that it spans X. Indeed, for any $\gamma = \sum_{\sigma\in\Delta_i} a_\sigma\sigma \in X$, let

$$\gamma' \;=\; \sum_{\sigma\in\Delta_i} a_\sigma\sigma \;-\; \sum_{\sigma\in\Theta} a_\sigma\hat{\sigma} \;=\; \sum_{\sigma\in\Upsilon_i} a_\sigma\sigma \;+\; \sum_{\sigma\in\Theta} a_\sigma(\sigma - \hat{\sigma}).$$

By the previous observation, we have $\gamma' \in X \cap C_i(\Upsilon;\mathbb{Z}) = \tilde{H}_i(\Upsilon;\mathbb{Z})$. On the other hand, $\tilde{H}_i(\Upsilon;\mathbb{Z}) = 0$ (because Υ is an i-dimensional simplicial tree), so in fact $\gamma' = 0$. Therefore, $\gamma = \sum_{\sigma\in\Theta} a_\sigma\hat{\sigma}$, proving Claim 5.

It follows from Claim 5 that the $\mathbb{Z}$-module projection $g : X \to \mathbb{Z}^\Theta$ defined by $g(\hat{\theta}) = \theta$, and extended $\mathbb{Z}$-linearly to all of X, is an isomorphism.

Claim 6. Suppose that the reduced Laplacian $\tilde{L}$ acts on $C_i(\Theta;\mathbb{Z})$ by

$$\tilde{L}\theta = \sum_{\sigma\in\Theta} \ell_{\theta\sigma}\sigma$$

for $\theta \in \Theta$. Then $L\theta = \sum_{\sigma\in\Theta} \ell_{\sigma\theta}\hat{\sigma}$.

Indeed, the chain $L\theta - \sum_{\sigma\in\Theta} \ell_{\sigma\theta}\hat{\sigma}$ belongs both to X and to $C_i(\Upsilon;\mathbb{Z})$, so it must be zero (as in the proof of Claim 5, because Υ is an i-dimensional simplicial tree), establishing Claim 6.

Therefore, for $\gamma = L\theta \in \operatorname{im} L$,

$$f(\gamma) = f(L\theta) = \tilde{L}\theta = \sum_{\sigma\in\Theta} \ell_{\theta\sigma}\sigma = g\left(\sum_{\sigma\in\Theta} \ell_{\theta\sigma}\hat{\sigma}\right) = g(L\theta) = g(\gamma).$$

It follows that the diagram (2) is commutative, completing the proof. □

Remark 3.5. The condition that Υ is torsion-free cannot be dropped. For example, let Δ be the 2-skeleton of the 6-vertex simplex; then $K_2(\Delta) \cong (\mathbb{Z}/6\mathbb{Z})^4$ (see Theorem 4.9 below). If we take Υ to be a triangulation of $\mathbb{R}P^2$ (a non-torsion-free simplicial spanning tree, since $\tilde{H}_1(\mathbb{R}P^2;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$), then direct calculation (e.g., using the computer algebra system Macaulay2 [17]) shows that $\operatorname{coker} \tilde{L} \cong (\mathbb{Z}/12\mathbb{Z}) \oplus (\mathbb{Z}/6\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z})$.

The first place we need the hypothesis that Υ is torsion-free is the end of Claim 2. But Claim 1 holds for any subcomplex $\Upsilon \subseteq \Delta$ *containing* a torsion-free spanning tree or, more interestingly, if Υ is "torsion-minimal", i.e., if $\tilde{H}_{i-1}(\Upsilon;\mathbb{Z}) = \tilde{H}_{i-1}(\Delta;\mathbb{Z})$. Not every APC complex need have a torsion-minimal spanning tree. For instance, for $k \in \mathbb{N}$, let $M_k = M(\mathbb{Z}/k\mathbb{Z}, 2)$ be the Moore space [20, p. 143] obtained by attaching a 2-cell to the circle S^1 by a map of degree k; let X be the complex obtained by identifying the 1-skeletons of M_2 and M_3 (thus, X is a CW-complex with two 2-cells, one 1-cell, and one 0-cell); and let Δ be a simplicial triangulation of X (in particular, note that Δ is a regular CW-complex with the same homology as X). Then $\tilde{H}_2(\Delta;\mathbb{Z}) \cong \mathbb{Z}$ and $\tilde{H}_i(\Delta;\mathbb{Z}) = 0$ for $i < 2$. On the other hand, the simplicial spanning trees of Δ are precisely the subcomplexes Υ obtained by deleting a single facet σ (just as though Υ were a simplicial sphere, which it certainly is not—see Remark 4.6 and subsequently), and each such Υ has $\tilde{H}_1(\Upsilon;\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$, according as σ is a face of M_2 or M_3. (We thank Vic Reiner for providing this example.)

Example 3.6. We return to the bipyramid B from Example 3.3 to illustrate Theorem 3.4. We must first pick a 1-dimensional spanning tree Υ; we take Υ to be the spanning tree with edges $12, 13, 14, 15$. (In general, we must also make sure Υ is torsion-free, but this is always true for 1-dimensional trees.) Let $L = L_{B,1}\colon C_1(B;\mathbb{Z}) \to C_1(B;\mathbb{Z})$ be the full Laplacian; note that L is a 9×9 matrix whose rows and columns are indexed by the edges of B. The reduced Laplacian $\tilde{L}$ is formed by removing the rows and columns indexed by the edges of Υ:

$$\tilde{L} \;=\; \begin{array}{c} \\ 23 \\ 24 \\ 25 \\ 34 \\ 35 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} 23 & 24 & 25 & 34 & 35 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 1 & 1 \\ -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 1 & -1 & 0 & 2 & 0 \\ 1 & 0 & -1 & 0 & 2 \end{pmatrix} \end{array}.$$

The critical group $K_1(B)$ is the cokernel of this matrix, i.e., $K_1(B) \cong \mathbb{Z}^5/\operatorname{im}\tilde{L}$. Since $\tilde{L}$ has full rank, it follows that $K_1(B)$ is finite; its order is $\det(\tilde{L}) = 15$. Thus

$$K_1(B) \cong \mathbb{Z}/15\mathbb{Z}.$$

Meanwhile, the graph Laplacian $L_{B,0} : C_0(B,\mathbb{Z}) \to C_0(B;\mathbb{Z})$ is the matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{pmatrix} \end{array}$$

with rows and columns labeled by vertices. Deleting any row and column (i.e., any vertex, or 0-dimensional spanning tree) and computing the cokernel of the resulting 4×4 matrix gives

$$K_0(B) \cong \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}.$$

4. The Structure of the Critical Group

4.1. The order of the critical group. The matrix-tree theorem implies that the order of the critical group of a graph equals the number of spanning trees. In this section, we explain how this equality carries over to the higher-dimensional setting.

Let Δ be a pure APC simplicial complex of dimension d. For $0 \leq i \leq d$, let $\mathcal{T}_i(\Delta)$ denote the set of all i-dimensional spanning trees of Δ (that is, spanning trees of the i-dimensional skeleton $\Delta_{(i)}$). As noted in Section 3.1, the APC condition is equivalent to the statement that $\mathcal{T}_i(\Delta) \neq \emptyset$ for every i. Accordingly, define

$$\tau_i = \sum_{\Upsilon \in \mathcal{T}_i(\Delta)} |\tilde{H}_{i-1}(\Upsilon; \mathbb{Z})|^2,$$

$$\pi_i = \text{ product of all nonzero eigenvalues of } L_{\Delta, i-1}.$$

Note that τ_0 equals the number of vertices of Δ and $\tau_{-1} = 1$. The following formulas relate the tree enumerators τ_i to the linear-algebraic invariants π_i.

Theorem 4.1 (The simplicial matrix-tree theorem)**.** [10, Thm. 1.3] *Let Δ be a pure APC simplicial complex of dimension d. For all $0 \leq i \leq d$, we have*

$$\pi_i = \frac{\tau_i \tau_{i-1}}{|\tilde{H}_{i-2}(\Delta; \mathbb{Z})|^2}.$$

Moreover, if Υ is any spanning tree of $\Delta_{(i-1)}$, then

$$\tau_i = \frac{|\tilde{H}_{i-2}(\Delta; \mathbb{Z})|^2}{|\tilde{H}_{i-2}(\Upsilon; \mathbb{Z})|^2} \det \tilde{L},$$

where $\tilde{L}$ is the reduced Laplacian formed by removing the rows and columns corresponding to Υ.

Recall that when $d = 1$, the number $\tau_1(\Delta)$ is simply the number of spanning trees of the graph Δ, and $\tau_0(\Delta)$ is the number of vertices (i.e., 0-dimensional spanning trees). Therefore, the formulas above specialize to the classical matrix-tree theorem.

Corollary 4.2. *Let $i < d$. Suppose that $\tilde{H}_{i-1}(\Delta; \mathbb{Z}) = 0$ and that Δ has an i-dimensional spanning tree Υ such that $\tilde{H}_{i-1}(\Upsilon; \mathbb{Z}) = 0$. Then the order of the i-dimensional critical group is the torsion-weighted number of $(i+1)$-dimensional spanning trees, i.e.,*

$$|K_i(\Delta)| = \tau_{i+1}.$$

Example 4.3. Suppose Δ is a d-dimensional APC complex such that $\tilde{H}_d(\Delta; \mathbb{Q}) = 0$ (equivalently, $\tilde{H}_d(\Delta; \mathbb{Z}) = 0$, since the top-dimensional homology is a submodule of $C_d(\Delta; \mathbb{Z})$, hence torsion-free). Then Δ is the only d-spanning tree of itself, so $\tau_d = 1$ and the group $K_{d-1}(\Delta)$ is trivial. In the case $d = 1$, the condition $\tilde{H}_d(\Delta; \mathbb{Q}) = 0$ is simply that Δ is an acyclic graph, and the hypothesis that Δ is APC means that $\tilde{H}_0(\Delta; \mathbb{Q}) = 0$, which implies Δ is a connected graph.

Example 4.4. Returning again to the bipyramid B, recall that 15 is both the number of its spanning trees (Example 3.3) and the order of its 1-dimensional critical group (Example 3.6), in each case because $\det \tilde{L} = 15$.

Another formula for the orders of the critical groups of Δ is as follows.

Corollary 4.5. *Under the conditions of Corollary 4.2, for every $i < d$, we have*

$$|K_i(\Delta)| = \prod_{j=0}^{i} \pi_j^{(-1)^{i-j}}.$$

Proof. Corollary 2.10 of [11] states that

$$\tau_i = \prod_{j=0}^{i} \pi_j^{(-1)^{i-j}},$$

which together with Corollary 4.2 implies the desired result. □

The condition that Δ and Υ be torsion-free is not too restrictive, in the sense that many simplicial complexes of interest in combinatorics are torsion-free and have torsion-free spanning trees. For instance, if Δ is (pure) shellable (see e.g., [32, Section III.2] for definitions and basic properties), then it is torsion-free, and has a torsion-free spanning tree Υ, constructed as follows: Let Θ be the facets of Δ equal to their own restriction faces (i.e., in the shelling order, facets added to the complex after their entire boundaries have been added), and let Υ be the subcomplex of Δ obtained by removing Θ. By definition, Υ contains the whole codimension 1 skeleton of Δ; it is also easy to check that Υ is contractible, hence $\mathbb{Q}$-acyclic, and hence a torsion-free spanning tree. Furthermore, Δ is homotopic to a wedge of spheres, and hence torsion-free itself.

Remark 4.6. When every spanning tree of Δ is torsion-free, the order of the critical group is exactly the number of spanning trees. This is a strong condition on Δ, but it does hold for some complexes — notably for simplicial spheres (that is, simplicial complexes homeomorphic to spheres), whose spanning trees are exactly the (contractible, hence acyclic) subcomplexes obtained by deleting a single facet. (In dimension 1, a graph is a simplicial sphere if and only if it is a cycle, when its spanning trees are the paths obtained by deleting a single edge.) Thus a given explicit bijection between spanning trees and elements of the critical group amounts to an abelian group structure on the set of facets of a simplicial sphere.

4.2. **The group structure of the critical group.** Determining the structure of the critical group seems to be a difficult problem. Even for graphs, the answer is only known for a handful of very special families; see, e.g., [6, 21]. One such answer was found by Lorenzini [26, 27] and Merris [29, Example 1(1.4)], who independently noted that the critical group of the cycle graph on n vertices is $\mathbb{Z}/n\mathbb{Z}$, the cyclic group on n elements. Simplicial spheres are the natural generalizations of cycle graphs from a tree-enumeration point of view. In fact, the theorem of Lorenzini and Merris carries over to simplicial spheres, as we now show.

Theorem 4.7. *Let Σ be a d-dimensional simplicial sphere with n facets. Then $K_{d-1}(\Sigma) \cong \mathbb{Z}/n\mathbb{Z}$.*

Proof. Let $K = K_{d-1}(\Sigma)$. Remark 4.6 implies that $|K| = n$, so it is sufficient to show that it is cyclic. In what follows, we use the standard terms "facets" and "ridges" for faces of Σ of dimensions d and $d-1$, respectively.

By definition, K is generated by $(d-1)$-dimensional cycles, that is, elements of $\ker \partial_{d-1}$. Since $\tilde{H}_{d-1}(\Sigma;\mathbb{Z}) = 0$, all such cycles are in fact $(d-1)$-dimensional boundaries of d-dimensional chains. Therefore, K is generated by the boundaries of facets, modulo the image of $L = \partial_d \partial_d^*$. We now show that for any two facets $\sigma, \sigma' \in \Sigma$, we have $\partial_d \sigma \equiv \pm \partial_d \sigma$ modulo $\operatorname{im} L$. This will imply that K can be generated by a single element as a $\mathbb{Z}$-module.

Since Σ is a sphere, it is in particular a pseudomanifold, so every ridge is in the boundary of at most two facets [32, p. 24]. Consequently, if two facets σ, σ' share a ridge ρ, then no other facet contains ρ, and we have

$$0 \equiv \partial\partial^*(\rho) = \partial(\pm\sigma \pm \sigma') = \pm(\partial\sigma \pm \partial\sigma')$$

(where $\equiv$ means "equal modulo $\operatorname{im} L$"). Hence $\partial(\sigma)$ and $\partial(\sigma')$ represent the same or opposite elements of K. Furthermore, the definition of pseudomanifold guarantees that for any two facets σ, σ', there is a sequence of facets $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ such that each σ_j and σ_{j+1} share a common ridge. Therefore, by transitivity, the boundary of any single facet generates K, as desired. □

Example 4.8. Theorem 4.7 says that simplicial spheres with the same dimension and f-vector cannot be distinguished by their top critical groups; on the other hand, their lower critical groups may be different. For instance, if O is the octahedron and S is the 2-sphere obtained from O by a bistellar flip (see Figure 2), then $K_1(O) \cong K_1(S) \cong \mathbb{Z}/8\mathbb{Z}$, since both complexes are simplicial 2-spheres with 8 facets, but their 1-skeletons do not have the same numbers of spanning trees, hence cannot have isomorphic critical groups in dimension 0. (In fact, computation reveals that $K_0(O) \cong \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $K_0(S) \cong \mathbb{Z}/336\mathbb{Z}$.)

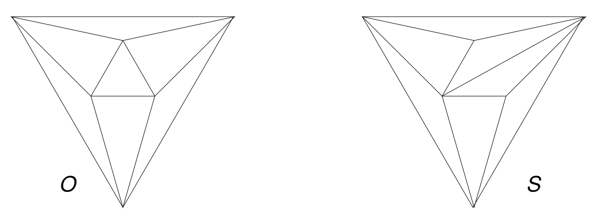

Figure 2. Two 2-dimensional spheres.

The condition in Theorem 4.7 that Σ be a simplicial sphere can be relaxed: in fact, the proof goes through for any d-dimensional pseudomanifold Σ such that $\tilde{H}_{d-1}(\Sigma;\mathbb{Z}) = 0$. On the other hand, if Σ is APC in addition to being a pseudomanifold (for example, certain lens spaces—see [20, p. 144]), then it has the rational homology type of either a sphere or a ball (because $\tilde{H}_d(\Sigma;\mathbb{Q})$ is either $\mathbb{Q}$ or 0; see [32, p. 24]).

For the special case of the skeleton of a simplex, we can determine the exact structure of the critical group. Let $\Delta = \Delta_n$ be the simplex on vertex set $[n]$, and let $i \leq n$. Kalai [22, Thm. 1] proved that $\tau_i(\Delta) = n^{\binom{n-2}{i}}$ for every n and i, generalizing Cayley's formula n^{n-2} for the number of labeled trees on n vertices.

By Corollary 4.2, it follows that $K_i(\Delta_n)$ is a finite abelian group of order $\tau_{i+1}(\Delta) = n^{\binom{n-2}{i+1}}$. The natural guess for its structure turns out to be correct:

Theorem 4.9. *For all $n \geq 1$ and $0 \leq i \leq n$, the group $K_i(\Delta_n)$ is a direct sum of $\binom{n-2}{i+1}$ copies of $\mathbb{Z}/n\mathbb{Z}$.*

The key step in the proof of Theorem 4.9 is due to Molly Maxwell [28], whose work we review before giving the proof. Maxwell studied the skew-symmetric matrix

$$A = \left[\begin{array}{c} \tilde{\partial}_{\Delta,i} \\ \hline -\tilde{\partial}^*_{\Delta,i+1} \end{array}\right]$$

where $\tilde{\partial}_{\Delta,i}$ denotes the $\binom{n-1}{i} \times \binom{n}{i+1}$ reduced boundary matrix obtained from the usual simplicial boundary $\partial_{\Delta,i}$ by deleting the rows corresponding to $(i-1)$-faces containing vertex 1, and $\tilde{\partial}^*_{\Delta,i+1}$ is the $\binom{n-1}{i+1} \times \binom{n}{i+1}$ matrix obtained from $\partial^*_{\Delta,i+1}$ by deleting the rows corresponding to $(i+1)$-faces *not* containing vertex 1. (Note that we index dimension by i where Maxwell uses k, and our $\partial_{\Delta,i}$ corresponds to Maxwell's $I^k_r(X)$.) In particular, Maxwell [28, Prop. 5.4] proved that

$$\operatorname{coker} A \cong (\mathbb{Z}/n\mathbb{Z})^{\binom{n-2}{i}}.$$

In light of Maxwell's results, the proof of Theorem 4.9, boils down to identifying the precise relationship between the matrix A and the combinatorial Laplacian.

Proof of Theorem 4.9. We have

$$AA^T = -A^2 = \left[\begin{array}{c} \tilde{\partial}_{\Delta,i} \\ \hline -\tilde{\partial}^*_{\Delta,i+1} \end{array}\right] \left[\begin{array}{c|c} \tilde{\partial}^*_{\Delta,i} & -\tilde{\partial}_{\Delta,i+1} \end{array}\right] = \left[\begin{array}{c|c} \tilde{L}^{\mathrm{ud}}_{i-1} & 0 \\ \hline 0 & \tilde{L}^{\mathrm{du}}_{i+1} \end{array}\right]$$

where "ud" and "du" stand for "up-down" and "down-up" respectively (see footnote 2). Therefore

$$\begin{aligned} \operatorname{coker}(AA^T) &\cong \operatorname{coker}(\tilde{L}^{\mathrm{ud}}_{i-1}) \oplus \operatorname{coker}(\tilde{L}^{\mathrm{du}}_{i+1}) \\ &\cong \operatorname{coker}(\tilde{L}^{\mathrm{ud}}_{i-1}) \oplus \operatorname{coker}(\tilde{L}^{\mathrm{ud}}_{i}) \\ &\cong K_{i-1}(\Delta) \oplus K_i(\Delta), \end{aligned}$$

where the second step follows from the general fact that MM^T and M^TM have the same multisets of nonzero eigenvalues for any matrix M, and the third step follows from Theorem 3.4. On the other hand, we have $\operatorname{coker}(AA^T) = \operatorname{coker}(-A^2) = \operatorname{coker}(A^2) \cong (\operatorname{coker} A) \oplus (\operatorname{coker} A)$. It follows from Maxwell's result that $K_i(\Delta_n)$ is a direct sum of copies of $\mathbb{Z}/n\mathbb{Z}$, and Corollary 4.2 implies that $|K_i(\Delta_n)| = n^{\binom{n-2}{i+1}}$, so the theorem follows. $\square$

5. The Critical Group as a Model of Discrete Flow

In this section, we describe an interpretation of the critical group in terms of flow, analogous to the chip-firing game. By definition, the elements of $K_i(\Delta)$ are integer vectors $\mathbf{c} = (c_F)_{F \in \Delta_i}$, modulo an equivalence relation given by the Laplacian L. These vectors are the analogues of chip configurations in the graph case ($i = 0$). When $i = 1$, it is natural to interpret c_F as a flow along the edge F, in the direction given by some predetermined orientation; a negative value on an edge corresponds to flow in the opposite direction. More generally, if F is an i-dimensional face, then we can interpret c_F as a generalized *i-flow*, again with the understanding that a negative value on a face means a i-flow in the opposite orientation. For instance, if F is a triangle then c_F measures circulation around F, either clockwise and counterclockwise depending on the sign of c_F.

When $i = 1$, the condition $\mathbf{c} \in \ker \partial_i$ means that flow neither accumulates nor depletes at any vertex; intuitively, matter is conserved. In general, we call an i-flow *conservative* if it lies in $\ker \partial_i$. For instance, ∂_2 converts circulation around a triangle into 1-flow along the three edges of its boundary in the natural way; for a 2-flow on Δ to be conservative, the sum of the resulting 1-flows on each edge must cancel out, leaving no net flow along any edge. In general, the sum of (the boundaries of) all the i-dimensional flows surrounding an $(i-1)$-dimensional face must cancel out along that face.

That the group $K_i(\Delta)$ is a quotient by the image of the Laplacian means that two configurations are equivalent if they differ by an integer linear combination of Laplacians applied to i-dimensional faces. This is analogous to the chip-firing game, where configurations are equivalent when it is possible to get from one to the other by a series of chip-firings, each of which corresponds to adding a column vector of the Laplacian. When $i = 1$, it is easy to see that firing an edge e (adding the image of its Laplacian to a configuration) corresponds to diverting one unit of flow around each triangle containing e (see Example 5.1). More generally, to fire an i-face F means to divert one unit of i-flow from F around each $(i+1)$-face containing F.

By Theorem 3.4, we may compute the critical group as the cokernel of the reduced Laplacian $\tilde{L}$. Passing from L to $\tilde{L}$ means ignoring the i-flow along each facet of an i-dimensional spanning tree Υ. In the graph case ($i = 0$), this spanning tree is simply the bank vertex. The higher-dimensional generalization is that the equivalence class of a configuration $\mathbf{c}$ is determined by the subvector $(c_F)_{F \in \Delta \setminus \Upsilon}$.

A remaining open problem is to identify the higher-dimensional "critical configurations", i.e., a set of stable and recurrent configurations that form a set of coset representatives for the critical group. Recall that in the chip-firing game, when vertex i fires, every vertex other than i either gains a chip or stays unchanged. It is tempting to define stability simply by the condition $\mathbf{c}_i < \deg(i)$ for every non-bank vertex i. On the other hand, when a higher-dimensional face fires, the flow along nearby faces can actually decrease. Therefore, it is not as easy to define stability. Alternatively, one could define a configuration to be stable if no face can fire without forcing some face (either itself or one of its neighbors) into debt. However, with this definition, there are examples (such as the 2-skeleton of the tetrahedron) for which some cosets of the Laplacian admit more than one critical configuration. Therefore, it is not clear how to choose a canonical set of coset representatives analogous to the critical configurations of the graphic chip-firing game.

Example 5.1. Again, let B be the equatorial bipyramid, let $\Upsilon = \{12, 13, 14, 15\}$ (a 1-dimensional spanning tree), and let $\Theta = B_1 \setminus \Upsilon = \{23, 24, 25, 34, 35\}$ (the remaining edges). If a conservative flow has the values on Θ in Figure 3(a), then the flows on Υ must be as in Figure 3(b). By Theorem 3.4, any conservative 1-flow is determined by its values on Θ, so we only show flows on Θ in subsequent diagrams.

Firing edge 23 yields the configuration shown in Figure 3(c). One unit of flow on edge 23 has been diverted across face 234 to edges 24 and 34, and another unit of flow has been diverted across 235 to 25 and 35. Note that the absolute value of flow on edge 25 has actually decreased, because of its orientation relative to 23. If we subsequently fire 24, we get the configuration shown in Figure 3(d). One unit of flow on 24 has been diverted across 234 to 23 and 34, and another unit of flow has been diverted across 124 to 12 and 14 (hence out of Θ).

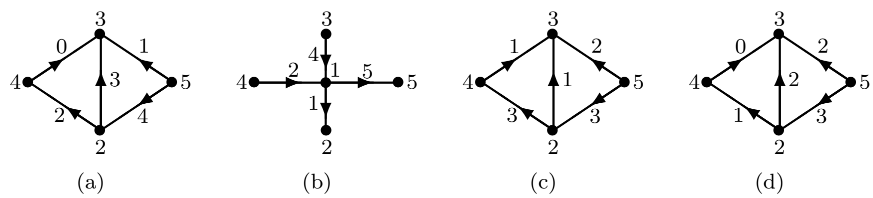

FIGURE 3. Conservative 1-flows and firings

6. CRITICAL GROUPS AS CHOW GROUPS

We close with a question for further research: can the higher-dimensional critical groups of a simplicial complex Δ be interpreted as simplicial analogues of the Chow groups of an algebraic variety? (For the algebraic geometry background, see, e.g., [19, Appendix A] or [15].) We regard Δ as the discrete analogue of a d-dimensional variety, so that divisors correspond to formal sums of codimension-1 faces. Even more generally, algebraic cycles of dimension i correspond to simplicial i-chains. The critical group $K_i(\Delta)$ consists of closed i-chains modulo conservative flows (in the language of Section 5), which is analogous to the Chow group of algebraic cycles modulo rational equivalence. This point of view has proved fruitful in the case of graphs [1, 2, 18, 26, 27]. In order to develop this analogy fully, the next step is to define a graded ring structure on $\bigoplus_{i\geq 0} K_i(\Delta)$ with a ring structure analogous to that of the Chow ring. The goal is to define a "critical ring" whose multiplication encodes a simplicial version of intersection theory on Δ. A starting point could be the "monodromy pairing" on the critical group of a graph, as developed by F. Shokrieh [31]. It may be necessary to dualize the theory of critical groups so that the grading is by codimension rather than dimension.

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