**Math 290 Test #2 Solution Set**
**October 10, 2006**

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**(#1a)** Start by adjoining an identity matrix to $A$:

$$\left[\begin{array}{ccc|ccc} 8 & -10 & 7 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 3 & -4 & 3 & 0 & 0 & 1 \end{array}\right] \qquad R_1 \leftarrow R_1 - 7R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & -7 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 3 & -4 & 3 & 0 & 0 & 1 \end{array}\right] \qquad R_3 \leftarrow R_3 - 3R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & -7 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -3 & 1 \end{array}\right] \qquad R_3 \longleftrightarrow R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & -7 & 0 \\ 0 & -1 & 0 & 0 & -3 & 1 \\ 1 & -1 & 1 & 0 & 1 & 0 \end{array}\right] \qquad R_2 \leftarrow -R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & -7 & 0 \\ 0 & 1 & 0 & 0 & 3 & -1 \\ 1 & -1 & 1 & 0 & 1 & 0 \end{array}\right] \qquad R_1 \leftarrow R_1 + 3R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 1 & 0 & 0 & 3 & -1 \\ 1 & -1 & 1 & 0 & 1 & 0 \end{array}\right] \qquad R_3 \leftarrow R_3 - R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 1 & 0 & 0 & 3 & -1 \\ 0 & -1 & 1 & -1 & -1 & 3 \end{array}\right] \qquad R_3 \leftarrow R_3 - R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 1 & 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & 2 \end{array}\right]$$

Therefore $A^{-1} = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & -1 \\ -1 & 2 & 2 \end{bmatrix}$.

*The most common mistakes here were arithmetic errors somewhere along the way. It is a good idea to test your final answer, either by finding $A^{-1}$ on your calculator and making sure it matches your answer, or else by multiplying out $AA^{-1}$ and making sure you wind up with the identity matrix.*

*Another suggestion: try to avoid fractions if at all possible. For example, instead of the second step above ($R_3 \leftarrow R_3 - 3R_2$), we could have used the row operation $R_2 \leftarrow R_3 - \frac{1}{3}R_2$ to put a zero in the second row and third column. This wouldn't be wrong, but it would introduce fractions and thereby greatly increase the chances for arithmetic error when you are doing calculations by hand.*

**(#1b)** The given system of equations can be rewritten in matrix form as

$$\begin{bmatrix} 8 & -10 & 7 \\ 1 & -1 & 1 \\ 3 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}.$$

Multiplying both sides on the left by $A^{-1}$ gives

$$\begin{bmatrix} 1 & -2 & -3 \\ 0 & 3 & -1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 8 & -10 & 7 \\ 1 & -1 & 1 \\ 3 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 3 & -1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

and simplifying both sides of this equation yields

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ -2 \end{bmatrix}$$

which is the unique solution.

*Again, the most mistakes here involved arithmetic. As always, once you have found a solution to a system of equations, it doesn't hurt to confirm that it really is a solution by plugging it back in. Here, this amounts to confirming that*

$$\begin{aligned} 8(-4) \;-\; 10(-5) \;+\; 7(-2) \;&=\; 4, \\ (-4) \;-\; (-5) \;+\; (-2) \;&=\; -1, \\ 3(-4) \;-\; 4(-5) \;+\; 3(-2) \;&=\; 2. \end{aligned}$$

**(#2)** Yes, it is true. If $AB = BA$, then $(AB)^{-1} = (BA)^{-1}$, and the formula for the inverse of a product gives $B^{-1}A^{-1} = A^{-1}B^{-1}$.

*A common mistake on this problem was to write down two matrices $A$ and $B$ at random, calculate $A^{-1}B^{-1}$ and $B^{-1}A^{-1}$, and observe that these matrices are unequal. This approach has some merit; however, to make this work, you would also need to show that $AB = BA$, which will not generally be the case if $A$ and $B$ are chosen arbitrarily. (Notice that the problem isn't asking you to prove that $B^{-1}A^{-1} = A^{-1}B^{-1}$ for* all *invertible matrices $A, B$; just those for which $AB = BA$.)*

*Another common mistake was to assert that if $AB = BA$, then $A = B^{-1}$. For example, every matrix commutes with itself, but most matrices are not their own inverses. This mistake might have arisen from confusion with the correct statement "if $AB = I$, then $A = B^{-1}$."*

**(#3)** The abundance of zero entries makes it convenient to expand by cofactors. It is easiest to begin by expanding on the fourth row, which has only one nonzero entry:

$$\begin{vmatrix} 3 & 4 & -2 & 0 \\ 1 & -2 & 8 & 3 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 3 & 0 \end{vmatrix} = -3 \begin{vmatrix} 3 & 4 & 0 \\ 1 & -2 & 3 \\ 0 & 0 & 5 \end{vmatrix}$$

The minus sign shows up because of the $+/-$ sign pattern in the cofactor matrix. Next, expanding the resulting $3\times 3$ determinant on its third row gives a $2\times 2$ determinant that can be evaluated directly:

$$-3\begin{vmatrix}3 & 4 & 0\\ 1 & -2 & 3\\ 0 & 0 & 5\end{vmatrix} \;=\; (-3)(5)\begin{vmatrix}3 & 4\\ 1 & -2\end{vmatrix} \;=\; (-3)(5)\Big((3)(-2)-(4)(1)\Big) \;=\; (-3)(5)(-10) \;=\; 150.$$

Alternately, we can use row and column operations. Here the goal is to end up with a triangular matrix.

$$\begin{aligned}
&\begin{vmatrix}3 & 4 & -2 & 0\\ 1 & -2 & 8 & 3\\ 0 & 0 & -2 & 5\\ 0 & 0 & 3 & 0\end{vmatrix} && C_3 \longleftrightarrow C_4\\
= -&\begin{vmatrix}3 & 4 & 0 & -2\\ 1 & -2 & 3 & 8\\ 0 & 0 & 5 & -2\\ 0 & 0 & 0 & 3\end{vmatrix} && C_1 \leftarrow \tfrac{1}{2}C_2\\
= -&\begin{vmatrix}5 & 4 & 0 & -2\\ 0 & -2 & 3 & 8\\ 0 & 0 & 5 & -2\\ 0 & 0 & 0 & 3\end{vmatrix} \;=\; -(5)(-2)(5)(3) \;=\; 150.
\end{aligned}$$

**(#4a)** This problem can be modeled by the stochastic matrix

$$M = \begin{bmatrix}9/10 & 3/10\\ 1/10 & 7/10\end{bmatrix}$$

where the first (second) row represents working (not working) tomorrow, and the first (second) column represents working (not working) today. To find the probability that it is working in two days given that it is working today, calculate

$$M^2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}21/25\\ 4/25\end{bmatrix}$$

which says that the probability is $21/25 = 84\%$.

*A common and wrong (but not stupid!) answer was as follows: "The probability that it keeps working day to day is $90\% = 0.9$, so the probability that it is working in two days is $(0.9)^2 = 0.81 = 81\%$." However, it could also not work tomorrow but decide to work the next day. The chance of this happening is $10\% \times 30\% = (0.1)(0.3) = 0.03 = 3\%$, and adding that to 81% produces the right answer.*

**(#4b)** Oops! I messed this one up. The answer was supposed to be given by

$$M^{-1}\begin{bmatrix}1\\0\end{bmatrix} \;=\; \begin{bmatrix}7/6\\ -1/6\end{bmatrix}$$

but it makes no sense to say that the computer was $7/6 \approx 117\%$ likely to be working yesterday! So this problem is really unanswerable.

**(#5)** Take the determinant of both sides of the given equation:

$$\det(M^{-1}) \;=\; \det\left(\frac{1}{\det(M)}\operatorname{adj}(M)\right).$$

We know that the left side is $1/\det(M)$. To evaluate the right side, recall that if $A$ is any $n \times n$ matrix and $c$ is a scalar, then $\det(c \cdot A) = c^n \det(A)$. Therefore, the last equation becomes

$$\frac{1}{\det(M)} \;=\; \left(\frac{1}{\det(M)}\right)^n \det(\operatorname{adj}(M))$$

and solving for $\det(\operatorname{adj}(M))$ gives

$$\det(M)^{n-1} \;=\; \det(\operatorname{adj}(M)).$$

**(#6)** The number $\det(A^2) = \det(A)^2$ is a square, so it cannot be negative. Therefore no such $A$ can exist.

*Clarity of explanation was important here. For example, just writing "The square of a number cannot be negative" is not sufficient, because this neglects to specify what it is that you are squaring. The key property of determinants used here is that* $\det(AB) = \det(A)\det(B)$, *and this has to be mentioned explicitly.*

**(#7)** The volume is the absolute value of

$$\frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 4 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{vmatrix}.$$

We saw in class that the determinant of a "backwards-diagonal" matrix like this is, up to sign, the product of the elements on the "southwest-to-northeast" diagonal. So the volume is

$$\left|\tfrac{1}{6}(1)(-1)(4)(2)\right| \;=\; \frac{4}{3}.$$

You really need to mention that absolute value (or "$\pm$"). Otherwise you may get stuck saying something false like "$-4/3 = 4/3$" at the end.

**(Bonus)** Start with the equation $SS^{-1} \;=\; I$.

Transposing both sides gives $(S^{-1})^T S^T \;=\; I^T$.

Now $S$ is symmetric (by hypothesis) and $I$ is symmetric as well, so this simplifies to $(S^{-1})^T S \;=\; I$.

Multiplying both sides on the right by $S^{-1}$, we see that $(S^{-1})^T = S^{-1}$.

This last equation says that $S^{-1}$ is its own transpose. That is, it is symmetric, which is exactly what we were trying to prove.

*When a problem is phrased like this, it does not mean "Write down a specific example of a matrix $S$ that is symmetric and invertible, calculate its inverse, and verify that it is symmetric", because that doesn't imply that it is true for* every *symmetric invertible matrix. I'm looking for an argument like the one above, which* is *valid for every possible $S$.*