Math 821 Final Exam
Monday 5/14/14

Problem #1 Let X be a topological space. Show that a continuous map $f : S^1 \to X$ is homotopic to a constant map if and only if it extends to a map $D^2 \to X$.

Solution: Coordinatize $D^2 = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1\}$ and $S^1 = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| = 1\}$ so that we can do vector arithmetic.

($\Longleftarrow$) Suppose $f : S^1 \to X$ extends to a map $\tilde{f} : D^2 \to X$. Then the homotopy $F : S^1 \times I \to X$ defined by $F(\mathbf{x}, t) = \tilde{f}(t\mathbf{x})$ has $F(\mathbf{x}, 0) = 0$ (a constant map) and $F(\mathbf{x}, 1) = f(\mathbf{x})$.

($\Longrightarrow$) Suppose that f is nullhomotopic, specifically that $F : S^1 \times I \to X$ is a homotopy between f and a constant map, say with $F(\mathbf{x}, 0) = c$ and $F(\mathbf{x}, 1) = f(\mathbf{x})$. We can then define $g : D^2 \to X$ by

$$g(\mathbf{x}) = \begin{cases} F(\mathbf{x}/\|\mathbf{x}\|, \|\mathbf{x}\|) & \text{if } \|\mathbf{x}\| > 0, \\ F(\text{anything}, 0) & \text{if } \mathbf{x} = 0. \end{cases}$$

Equivalently, if $\mathbf{x}$ has polar coordinates (r, θ), then

$$g(r, \theta) = F(\theta, r) = F_r(\theta).$$

The function g is well-defined and continuous on D^2, and $g(\mathbf{x}) = f(\mathbf{x})$ for $\mathbf{x} \in S^1$. $\square$

Alternate proof of ($\Longleftarrow$): The space D^2 is contractible, i.e., homotopy-equivalent to a one-point space. That is, there are functions $h : D^2 \to \{\bullet\}$ and $g : \{\bullet\} \to D^2$ such that $g \circ h \simeq \mathbb{1}_{D^2}$. Therefore

$$F \circ g \circ h \simeq F \circ \mathbb{1}_{D^2} = F$$

and $F \circ g \circ h$ is a constant map.

Alternate proof of ($\Longrightarrow$): Suppose that we have a homotopy $f_t : S^1 \to X$ with f_0 a constant map (say $f_0(s) = c$ for all $s \in S^1$) and $f_1 = f$. Certainly f_0 can be extended to the constant map $\overline{f_0}(d) = c$ for all $d \in D^2$, and since S^1 can be made into a CW-subcomplex of D^2, f_t extends to a homotopy $\overline{f_t} : D^2 \to X$ by the Homotopy Extension Property [HEP], and in particular $\overline{f_1}$ extends $f_1 = f$. (This is correct, but is using a sledgehammer to kill a fly, since it is nontrivial to prove the HEP for CW-pairs.)

Problem #2 Let X be a path-connected space. Show that $\pi_1(X)$ is abelian if and only if all basepoint-change homomorphisms β_h depend only on the endpoints of the path h.

Solution: Recall that if h is a path from x_0 to x_1, then there is an isomorphism $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ given by $\beta_h[f] = [h \cdot f \cdot \bar{h}]$.

($\Longrightarrow$): Suppose $\pi_1(X)$ is abelian. Let h, k be two paths from x_0 to x_1; we want to show that $\beta_h[f] = \beta_k[f]$, which is equivalent to showing that $\beta_h[f]\beta_k[\bar{f}] = 0$. Indeed,

$$\begin{aligned}\beta_h[f]\beta_k[\bar{f}] &= [h \cdot f \cdot (\bar{h} \cdot k) \cdot \bar{f} \cdot \bar{k}] \\ &= [h \cdot (\bar{h} \cdot k) \cdot f \cdot \bar{f} \cdot \bar{k}]\end{aligned}$$

(since f and $\bar{h} \cdot k$ are both loops at x_0, hence represent elements of the abelian group $\pi_1(X, x_0)$)

$$\begin{aligned}&= [(h \cdot \bar{h}) \cdot k \cdot (f \cdot \bar{f}) \cdot \bar{k}] \\ &= [k \cdot \bar{k}] \;=\; 1.\end{aligned}$$

($\Longleftarrow$): Let $[f], [g] \in \pi_1(X, x_0)$. By hypothesis, the basepoint-change homomorphisms $\beta_f, \beta_g : \pi_1(X, x_0) \to \pi_1(X, x_0)$ are equal. Therefore

$$\beta_f[f] = [f \cdot f \cdot \bar{f}] = \underline{[f]} = \beta_g[f] = [g \cdot f \cdot \bar{g}] = \underline{[g][f][g]^{-1}}$$

and multiplying the underlined terms on the right by $[g]$ gives $[f][g] = [g][f]$, so $\pi_1(X, x_0)$ is abelian.

Problem #3 (#3a) Explicitly describe all the connected covering spaces of $\mathbb{S}^1$ and their associated covering maps. Your description should include a criterion for when one covering space $\tilde{X}$ covers another covering space $\tilde{X}'$.

(#3b) What algebraic objects correspond bijectively to the connected covering spaces of $\mathbb{S}^1 \times \mathbb{S}^1$? (A single-sentence answer suffices — you don't need to explain.)

(#3c) What algebraic objects correspond bijectively to the *finite-sheeted* connected covering spaces of $\mathbb{S}^1 \times \mathbb{S}^1$? (Again, a single-sentence answer suffices.)

Solution: (a) For each integer n, there is a unique n-sheeted covering space $\tilde{X}_n \cong \mathbb{S}^1$, with covering map $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ given by $p(z) = z^n$ (where $\mathbb{S}^1$ is regarded as the unit circle in $\mathbb{C}$). For $n, m \in \mathbb{Z}$, the space X_n covers $\tilde{X}_m$ if and only if m divides n; in this case the covering map $\tilde{X}_n \to \tilde{X}_m$ is given by $z \mapsto z^{n/m}$. In addition, there is the universal covering space $\tilde{X}_\infty = \mathbb{R}$, with covering map $p_\infty : \mathbb{R} \to \mathbb{S}^1$ given by $p_\infty(t) = e^{2\pi i t}$. The covering maps $\mathbb{R} \to \tilde{X}_n$ is $t \mapsto e^{2\pi i t/n}$.

(b) We know that $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \pi_1(\mathbb{S}^1) \times \pi_1(\mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z}$. Therefore, the connected covering spaces of $\mathbb{S}^1 \times \mathbb{S}^1$ correspond to the subgroups of $\mathbb{Z} \times \mathbb{Z}$ (or equivalently the spans of 2×2 matrices over $\mathbb{Z}$).

(c) If $p : (\tilde{X}, \tilde{x}) \to (X, x)$ is a covering space, then the number of sheets of the cover is the index $[\pi_1(X, x) : p_*\pi_1(\tilde{X}, \tilde{x})]$. So the answer is finite-index subgroups of $\mathbb{Z}^2$ — i.e., the spans of *full-rank* 2×2 matrices over $\mathbb{Z}$.

Problem #4 Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

in which both rows are exact. Show that if α and γ are isomorphisms, then so is β. (In fact this is a special case of the Snake Lemma — but don't use the Snake Lemma in your proof; recreate the diagram chase.)

Solution: **First**, we show that β is injective. Suppose that $x \in \ker \beta$. Then:

$$\begin{array}{lll} & \gamma g x = g'\beta x = 0 & \text{(commutativity of second square)} \\ \therefore & gx \in \ker\gamma & \\ \therefore & gx = 0 & \text{(because } \gamma \text{ is an isomorphism)} \\ \therefore & x \in \ker g = \operatorname{im} f & \\ \therefore & x = fy \ \text{ for some } y \in A & \\ \therefore & f'\alpha y = \beta f y = \beta x = 0 & \text{(first square; remember } x \in \ker\beta) \\ \therefore & \alpha y \in \ker f' & \\ \therefore & \alpha y = 0 & \text{(because } f' \text{ is injective)} \\ \therefore & y = 0 & \text{(because } \alpha \text{ is an isomorphism)} \\ \therefore & x = fy = 0. & \end{array}$$

Second, we show that β is surjective. Let $y \in B'$, $z = g'y \in C'$, and $w \in C$ such that $\gamma w = z$ (since γ is an isomorphism). Let $x \in B$ such that $gx = w$ (since g is surjective). Then:

$$\begin{array}{lll} & g'\beta x = \gamma g x = \gamma w = z = g'y & \\ \therefore & \beta x - y \in \ker g' = \operatorname{im} f' & \\ \therefore & \beta x - y = f'v = f'\alpha u \ \text{ for some } v \in A',\ u \in A & \text{(because } \alpha \text{ is an isomorphism)} \\ \therefore & \beta f u = f'\alpha u = \beta x - y & \text{(commutativity of first square)} \\ \therefore & y = \beta x - \beta f u = \beta(x - fu) & \\ \therefore & y \in \operatorname{im}\beta. & \end{array}$$

Problem #5 Using only the definition of singular homology, prove that the (unreduced) singular homology groups of a one-point space X are $H_0(X) = \mathbb{Z}$ and $H_n(X) = 0$ for $n > 0$.

Solution: There is only one singular n-simplex for each n, namely the constant map $k_n : \Delta^n \to X$. Therefore $C_n(X) = \mathbb{Z}\langle k_n \rangle \cong \mathbb{Z}$ for all n. The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is given by

$$\partial_n(k_n) = \sum_{i=0}^{n} (-1)^i k_{n-1} = \begin{cases} k_{n-1} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

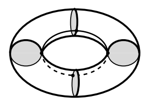
Therefore the singular chain complex of X is

$$\cdots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{\cong} C_1(X) = \mathbb{Z} \xrightarrow{0} C_0(X) = \mathbb{Z} \to 0$$

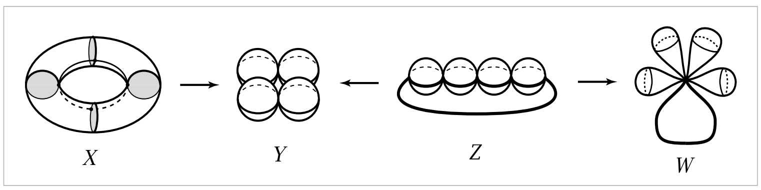
and the homology groups are

$$H_n(X) = \begin{cases} 0/0 = 0 & \text{if } n > 0 \text{ is even,} \\ \mathbb{Z}/\mathbb{Z} = 0 & \text{if } n \text{ is odd,} \\ \mathbb{Z}/0 = \mathbb{Z} & \text{if } n = 0. \end{cases}$$

Problem #6 What are the homology groups of the space X shown below?

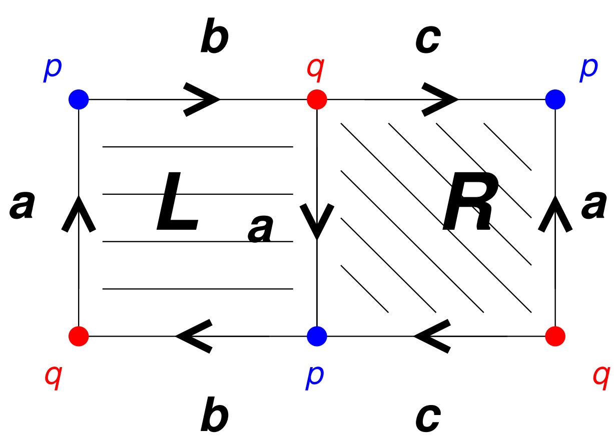


Solution: The space is homotopy-equivalent to $\mathbb{S}^1 \wedge (\mathbb{S}^2)^{\wedge 4}$ by the following sequence of deformation retractions and "unretractions":



Therefore, $\boxed{\tilde{H}_2(X) \cong \mathbb{Z}^4,\ \tilde{H}_1(X) \cong \mathbb{Z},\ \tilde{H}_0(X) = 0.}$

Problem #7 Use cellular homology to calculate the homology groups of the two-dimensional cell complex X obtained by gluing together two copies of $\mathbb{R}P^2$ as shown below.



Solution: Let E_n denote the set of n-cells in X. Note that there are two 2-cells L, R and two vertices (call them p, q as shown). The cellular chain complex of X is

$$0 \to \mathbb{Z}E_2 = \mathbb{Z}^2 \xrightarrow{\qquad \partial_2 \qquad} \mathbb{Z}E_1 = \mathbb{Z}^3 \xrightarrow{\qquad\qquad \partial_1 \qquad\qquad} \mathbb{Z}E_0 = \mathbb{Z}^2 \to 0.$$

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 2 & -2 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \end{array} \qquad\qquad \begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} p \\ q \end{array} & \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \end{array}$$

We therefore have

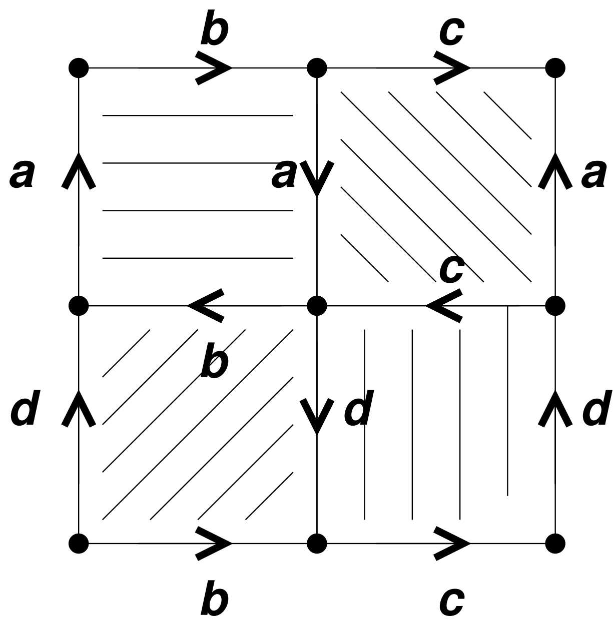
$$H_2(X) = \ker \partial_2 = 0$$

(since the matrix of ∂_2 has linearly independent columns),

$$\begin{aligned} H_1(X) &= \ker \partial_1 / \operatorname{im} \partial_2 \\ &= \mathbb{Z}\langle (1,1,0), (-1,0,1) \rangle \;/\; \mathbb{Z}\langle (2,2,0), (-2,0,2) \rangle \\ &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \end{aligned}$$

$$H_0(X) = \operatorname{coker} \partial_1 = \mathbb{Z}^2 / \mathbb{Z}\langle (1,-1) \rangle \cong \mathbb{Z}.$$

Problem #8 Use Problem #7 and a Mayer-Vietoris sequence to calculate $H_2(Y)$, where Y is the two-dimensional complex shown below (obtained from two copies of X by more gluing).



Solution: Let X and X' denote the two copies of X being glued together (so X consists of the two top squares and X' of the two bottom squares). Note that $X \cap X' \cong \mathbb{S}^1$ (the 2-cycle with edges b, c). The reduced Mayer-Vietoris sequence

$$0 \to H_2(X \cap X') \to H_2(X) \oplus H_2(X') \to H_2(Y) \to H_1(X \cap X') \to H_1(X) \oplus H_1(X') \to H_1(Y) \to 0$$

simplifies to

$$0 \to H_2(Y) \xrightarrow{f} \mathbb{Z} \to \mathbb{Z}_2^4 \to H_1(Y) \to 0.$$

The map f can't be zero, since there is no injection from $\mathbb{Z}$ into the finite group $\mathbb{Z}_2^4$. Therefore f identifies $H_2(Y)$ with a nonzero subgroup of $\mathbb{Z}$, which means that $H_2(Y) \cong \mathbb{Z}$.