Lecture Notes on Algebraic Combinatorics

Jeremy L. Martin

jlmartin@ku.edu

February 3, 2015

Foreword

The starting point for these lecture notes was my notes from Vic Reiner's Algebraic Combinatorics course at the University of Minnesota in Fall 2003. I currently use them for graduate courses at the University of Kansas. They will always be a work in progress. Please use them and share them freely for any research purpose. I have added and subtracted some material from Vic's course to suit my tastes, but any mistakes are my own; if you find one, please contact me at `jlmartin@ku.edu` so I can fix it. Thanks to those who have suggested additions and pointed out errors, including but not limited to: Logan Godkin, Alex Lazar, Nick Packauskas, Billy Sanders, and Tony Se.

1. The Fundamentals: Posets, Simplicial Complexes, and Polytopes

1.1. Posets.

Definition 1.1. A **partially ordered set** or **poset** is a set P equipped with a relation $\leq$ that is reflexive, antisymmetric, and transitive. That is, for all $x, y, z \in P$:

(1) $x \leq x$ (reflexivity).
(2) If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry).
(3) If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

We say that x is **covered** by y, written $x \lessdot y$, if $x < y$ and there exists no z such that $x < z < y$. Two posets P, Q are **isomorphic** if there is a bijection $\phi : P \to Q$ that is order-preserving; that is, $x \leq y$ in P iff $\phi(x) \leq \phi(y)$ in Q.

We'll usually assume that P is finite.

Definition 1.2. A poset L is a **lattice** if every pair $x, y \in L$ has a unique **meet** $x \wedge y$ and **join** $x \vee y$. That is,

$$x \wedge y = \max\{z \in L \mid z \leq x, y\},$$
$$x \vee y = \min\{z \in L \mid z \geq x, y\}.$$

We'll have a lot more to say about lattices in Section **??**.

Example 1.3 (**Boolean algebras**)**.** Let $[n] = \{1, 2, \ldots, n\}$ (a standard piece of notation in combinatorics) and let $2^{[n]}$ be the power set of $[n]$. We can partially order $2^{[n]}$ by writing $S \leq T$ if $S \subseteq T$. A poset isomorphic to $2^{[n]}$ is called a **Boolean algebra of rank n.**

Note that $2^{[n]}$ is a lattice, with meet and join given by intersection and union respectively.