Math 290 Test #3 Solution Set
November 9, 2006

(#1a) We need to check that V is closed under addition and under scalar multiplication. Closure under addition means that $\underline{u} + \underline{v}$ belongs to V for all vectors $\underline{u}, \underline{v}$ in V. Closure under scalar multiplication means that $c\underline{u}$ belongs to V for all vectors $\underline{u}$ in V and all scalars c. Since V is a subset of the vector space R^4, these two properties are the only ones we need to check. ■

— It is not quite true that the other eight properties of a vector space follow from these two. Some of them do—for example, if V is closed under scalar multiplication, then it has an additive identity and every element has an additive inverse. However, properties like commutativity, associativity and distributivity do not follow from closure, but rather from the status of V as a subset of a larger vector space.

(#1b) *Closure under addition*: Let $\underline{x} = (x_1, x_2, x_3, x_4)$ and $\underline{y} = (y_1, y_2, y_3, y_4)$ be vectors in V. That is, we know that

$$x_1 = x_4, \qquad x_2 = -x_3, \qquad y_1 = y_4, \qquad y_2 = -y_3.$$

Then

$$\underline{x} + \underline{y} \;=\; (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$$

and the previous equation implies that $x_1 + y_1 = x_4 + y_4$ and that $x_2 + y_2 = -(x_3 + y_3)$. Therefore, $\underline{x} + \underline{y}$ belongs to V.

Closure under scalar multiplication: Let $\underline{x} = (x_1, x_2, x_3, x_4)$ be a vector in V, and let c be a scalar. So

$$c\underline{x} \;=\; c(x_1, x_2, x_3, x_4) \;=\; (cx_1, cx_2, cx_3, cx_4)$$

and $cx_1 = cx_4$ and $cx_2 = c(-x_3) = -cx_3$. So $c\underline{x}$ belongs to V. ■

(#1c) Every vector $\underline{x} = (x_1, x_2, x_3, x_4)$ in V can be rewritten as

$$\begin{aligned} \underline{x} &= (x_1, x_2, -x_2, x_1) \\ &= x_1(1, 0, 0, 1) + x_2(0, 1, -1, 0). \end{aligned}$$

So $\{(1, 0, 0, 1), (0, 1, -1, 0)\}$ is a basis for V. (There are of course many other possible answers.) Its dimension is 2. ■

— One popular approach was to think of some vectors in V, make them into the rows of a matrix M, and put the matrix in row-echelon form. I gave full credit for this if you ended up with a valid basis, but the theoretical problem with this approach is that the set of vectors you come up with might not be a spanning set for V. That is, the row space of M might be a subspace of V that is smaller than V itself.

— Also, you need to be sure to make these vectors into the *rows*, not the *columns* of M. Doing the latter will express V as the column space of M, which we know is changed by row operations. In fact, trying to do the problem this way will ultimately get you a collection of vectors that don't even belong to V, much less form a basis for it.

(#2a) Y is not a vector space because it's not closed under addition; the sum of two matrices of different sizes is undefined. (By the way, Y is in fact closed under scalar multiplication and additive inverses.) ■

(#2b) Z is not a vector space because it's not closed under scalar multiplication. For example, $\underline{v} = (3, 2)$ is a vector in R^2 because $3 \geq 2$, but multiplying $\underline{v}$ by the scalar -2 gives the vector $(-6, -4)$, which does not belong to Z. (By the way, Z *is* closed under addition, and it does contain the additive identity $\underline{0} = (0, 0)$.) ■

— Some people gave the following argument: "If (x, y) is in V and c is a negative scalar, then (cx, cy) is not in V." That's 90% correct (although I gave it full credit). The nitpicky point is that the argument fails if $x = y$. A completely precise reformulation would be as follows: "If $x > y$ and c is a negative scalar, then $cx < cy$, so (x, y) is in V but $c(x, y) = (cx, cy)$ is not in V."

(#3)

	Linearly independent?	Spanning set?	Basis?
$\{(-1,-1,2),\ (3,-1,3),\ (5,1,-1)\}$	No	No	No
$\{(-1,-1,2),\ (3,-1,3),\ (5,1,0)\}$	Yes	Yes	Yes
$\{(-1,-1,2),\ (3,-1,3)\}$	Yes	No	No

(#4) Let M be the matrix whose columns are the vectors of S. Since S consists of 3 vectors in R^3, it will be a basis exactly when $\det(M) \neq 0$. Calculate $\det(M)$, for instance by expanding on the first row:

$$\begin{aligned}
\begin{vmatrix} q & 5 & 2 \\ 2 & q & -2 \\ -2 & -5 & q \end{vmatrix} &= q\begin{vmatrix} q & -2 \\ -5 & q \end{vmatrix} - 5\begin{vmatrix} 2 & -2 \\ -2 & q \end{vmatrix} + 2\begin{vmatrix} 2 & q \\ -2 & -5 \end{vmatrix} \\
&= q(q^2 - 10) - 5(2q - 4) + 2(-10 + 2q) \\
&= q^3 - 10q - 10q + 20 - 20 + 4q \\
&= q^3 - 16q \\
&= q(q-4)(q+4).
\end{aligned}$$

We see that $\det(M) = 0$ if and only if q equals 0, 4, or -4. So these are the values for which S is not a basis. ■

— You can also evaluate the determinant by first simplifying it with row and column operations, but that method will produce a lot of messy rational functions in q, making it much easier to make a mistake. The calculation would look something like this:

$$\begin{vmatrix} q & 5 & 2 \\ 2 & q & -2 \\ -2 & -5 & q \end{vmatrix} = \begin{vmatrix} q & 5 & 2 \\ 2 & q & -2 \\ 0 & q-5 & q-2 \end{vmatrix} \qquad R_3 \leftarrow R_3 + R_2$$

$$= \begin{vmatrix} q & 5 & 2 \\ 0 & q-\frac{10}{q} & -2-\frac{4}{q} \\ 0 & q-5 & q-2 \end{vmatrix} \qquad R_2 \leftarrow R_2 - \frac{2}{q}R_1$$

$$= \begin{vmatrix} q & 5 & 2 \\ 0 & \frac{q^2-10}{q} & \frac{-2q-4}{q} \\ 0 & q-5 & q-2 \end{vmatrix} \qquad \text{(simplifying)}$$

$$= \begin{vmatrix} q & 5 & 2 \\ 0 & \frac{q^2-10}{q} & \frac{-2q-4}{q} \\ 0 & 0 & (q-2) - \frac{q(q-5)}{q^2-10}\frac{-2q-4}{q} \end{vmatrix} \qquad R_3 \leftarrow R_3 - \frac{q(q-5)}{q^2-10}R_2$$

$$= \begin{vmatrix} q & 5 & 2 \\ 0 & \frac{q^2-10}{q} & \frac{-2q-4}{q} \\ 0 & 0 & \frac{q(q^2-16)}{q^2-10} \end{vmatrix} \qquad \text{(simplifying)}$$

$$= q\frac{q^2-10}{q}\frac{q(q^2-16)}{q^2-10} = q(q^2-16).$$

(#5a) To find a basis for the row space of A, put A in row-echelon form:

$$A = \begin{bmatrix} -1 & 3 & 7 \\ 3 & -3 & -3 \\ 1 & 2 & 8 \\ 4 & -3 & -1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2/3} \begin{bmatrix} -1 & 3 & 7 \\ 1 & -1 & -1 \\ 1 & 2 & 8 \\ 4 & -3 & -1 \end{bmatrix}$$

$$\xrightarrow{\substack{R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 + R_1 \\ R_4 \leftarrow R_4 + 4R_1}} \begin{bmatrix} -1 & 3 & 7 \\ 0 & 2 & 6 \\ 0 & 5 & 15 \\ 0 & 9 & 27 \end{bmatrix}$$

$$\xrightarrow{\substack{R_2 \leftarrow R_2/2 \\ R_3 \leftarrow R_3/5 \\ R_4 \leftarrow R_4/9}} \begin{bmatrix} -1 & 3 & 7 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{\substack{R_3 \leftarrow R_3 - R_2 \\ R_4 \leftarrow R_4 - R_2}} \begin{bmatrix} -1 & 3 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We now have a row-echelon form of A, so we could stop here and write down a basis for the row space of A consisting of the nonzero rows:

$$\{(-1, 3, 7), \quad (0, 1, 3)\}.$$

If we wish, we could continue by finding the reduced row-echelon form of A:

$$\begin{bmatrix} -1 & 3 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftarrow -R_1 - R_2} \begin{bmatrix} 1 & -3 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 + 3R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So another basis for the rowspace of A is $\{(1, 0, 2), \quad (0, 1, 3)\}$. (This is not necessary now, but will be useful in #5c.) ■

(#5b) To find a basis for the column space of A, put

$$A^T = \begin{bmatrix} -1 & 3 & 1 & 4 \\ 3 & -3 & 2 & -3 \\ 7 & -3 & 8 & -1 \end{bmatrix}$$

into row-echelon form. I'll omit the details of the calculation, but two common answers were

$$\{(-1,3,1,4),\ (0,6,5,9)\} \qquad \text{and} \qquad \{(1,0,\tfrac{3}{2},\tfrac{1}{2}),\ (0,1,\tfrac{5}{6},\tfrac{3}{2})\}.$$

Notice that your answers to #5a and #5b should have the same size, because the row space and column space of a matrix always have the same dimension. If your answers don't have the same size, you should start looking for a mistake in your calculations somewhere! ■

(#5c) The nullspace of A is defined as the set of all vectors $\underline{x} = (x_1, x_2, x_3)$ such that $A\underline{x} = \underline{0}$. This matrix equation can be viewed as a system of linear equations in the variables x_1, x_2, x_3, which is equivalent to the system obtained by putting A in reduced row-echelon form. From #5a, that system is

$$\left\{ \begin{array}{ccccc} x_1 & & + & 2x_3 & = & 0 \\ & x_2 & + & 3x_3 & = & 0 \end{array} \right.$$

and the solution space can be parametrized by

$$x_1 = -2s, \quad x_2 = -3s, \quad x_3 = s,$$

i.e., every solution vector has the form $\underline{x} = (-2s, -3s, s) = s(-2,-3,1)$. Therefore, $\{(-2,-3,1)\}$ is a basis for the nullspace of A. ■

(#5d) The rank of A is 2 (the dimension of either the row space or column space). The nullity is 1 (the dimension of the nullspace). Notice that 2+1=3 is the number of columns of A. ■