Math 724, Fall 2013
Homework #2

Chapter 1 Supplementary Problem #5 The answer is the Catalan number

$$\boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$$

where n is the number of pairs of parentheses. There is a bijection between balanced lists of parentheses and Dyck paths: left parentheses correspond to northeast steps, right parentheses correspond to southeast steps. The balancing condition corresponds to the condition that a Dyck path always stays on or above the x-axis.

Chapter 1 Supplementary Problem #11 This is the number of weak set partitions of an n-element set into four parts (see Test #1). Here's a way to do it from scratch: add three sticks to the n balls and arrange the resulting $n+3$ objects in a line, e.g.:

$$******|***********||***$$

Think of the sticks as separators between the balls to be colored red, white, blue and green, respectively. The figure above would correspond to painting 6 balls red, 11 white, 0 blue, and 3 green. So the answer is

$$\boxed{\binom{n+3}{3}.}$$

(This technique is often called "stars and bars.")

The problem does not require that each color be used at least once, but if it did, then the answer would be $\binom{n-1}{3}$. (Start by painting one ball each color, as mandated. Now there are $n-4$ balls left to be colored arbitrarily, which reduces to the original problem.

Problem #37 Since each child can get at most one ping-pong ball and the balls are all identical, the problem reduces to choosing which k children receive a ball. So the answer is

$$\boxed{\binom{n}{k}.}$$

(This was not supposed to be hard.)

Problem #49 (a) The set of points reachable from $(0,0)$ is

$$\boxed{\{(x,y)\in\mathbb{Z}^2 :\ x+y \text{ is even and } x \geq |y|\}.}$$

(b) The length of a diagonal lattice path from $(0,0)$ to (m,n) is just $\boxed{m.}$ (Of course, by part (a), we had better have $m \geq |n|$ if any such paths are to exist.

(c) Say there are a upsteps and b downsteps. Then $a+b=m$ (total length of the path) and $a-b=n$ (the eventual y-coordinate of the last point). Solving these equations for a,b gives $a=(m+n)/2$ and $b=(m-n)/2$. So the number of paths is

$$\boxed{\binom{m}{(m+n)/2} = \binom{m}{(m-n)/2}}$$

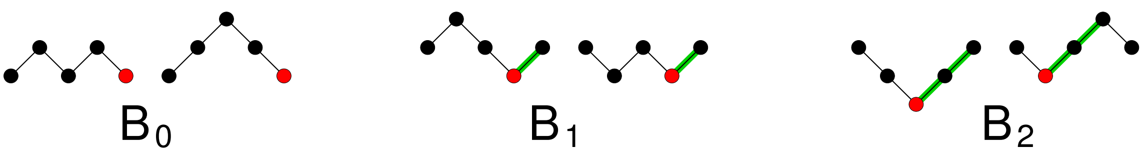
Problem #52 Terminological note: In this problem "Dyck path" means a diagonal lattice path that starts at $(0,0)$ and never goes below the x-axis, **but need not end on the x-axis** — if it does then it is called a "Catalan path".

(a) If a path never goes below the x-axis, then in particular it does not go below the x-axis during its first k steps.

(b) $\boxed{0}$ for sure. The definition of Dyck path implies that $(0,0)$ is an absolute minimum. Therefore, the last point is an absolute minimum (and thus the rightmost among all absolute minima) $\iff$ its y-coordinate is $0 \iff$ the Dyck path is in fact a Catalan path.

(c) The number i could be as small as 0 (if the path in question is a Catalan path) or as large as n (if the path consists of n up-steps followed by n down-steps. So the partition has $n+1$ blocks.

For example, if $n = 2$, the blocks are as follows:



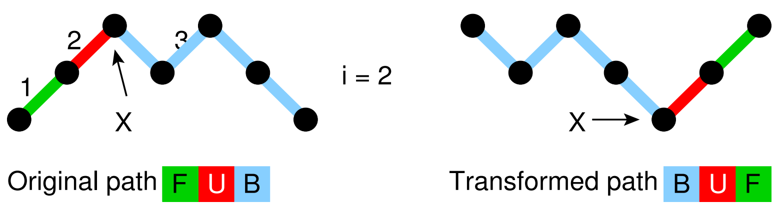
Note that B_0 is *precisely* the set of Catalan paths. To see this, let $P \in D$ and let (a,b) be the rightmost absolute minimum of P.

If $P \in B_0$, i.e., P has no up-steps after q, then it certainly has no down-steps after (a,b) either — otherwise, (a,b) wouldn't be an absolute minimum. So there are no steps at all after (a,b)! But that means that $(a,b) = (2n,0)$, and so P is a Catalan path by part (b).

On the other hand, if $P \in B_i$ for some $i > 0$, then $q = (a,b) \neq (2n,0)$ (because $a < 2n$). Therefore $(2n,0)$ is not the rightmost absolute minimum, hence not an absolute minimum, which implies that the path P must drop below the x-axis at some point.

(d) $\boxed{n.}$ (Again, nothing hard about this question; the idea is to get you to focus on the appropriate fact

(e) First, here is an example of the transformation:



Observe that in the transformed path BUF, the rightmost absolute minimum X is the starting point of U. Since F is a Dyck path, it never goes below its starting point; therefore, every point to the right of X in BUF has strictly greater y-coordinate than X. Meanwhile, the last point in B is an absolute minimum; this point is identified with X in BUF, so no point in B lies strictly above X (there might be points with the same y-coordinate, but they occur to the left of X). Therefore, the path BUF has exactly i up-steps after its rightmost absolute minimum — namely U itself and the $i-1$ up-steps in F. Therefore, the map ϕ sending FUB to BUF is a function from Dyck paths to B_i, and it is invertible because the subpaths B, U, F can be recovered from $\phi(P)$ — U must be the up-step emanating from the rightmost absolute minimum; B is everything before U; and F is everything after U. We conclude that ϕ is a bijection.

(f) The existence of a bijection implies that $|B_i| = |B_0|$ for every i. (Recall that B_0 is the set of Catalan paths by (c).) On the other hand, $D = B_0 \uplus B_1 \uplus \cdots \uplus B_n$ (where the symbol $\uplus$ means disjoint union) and $|D| = \binom{2n}{n}$ by Problem 49(c). Therefore, the number of Catalan paths is

$$\boxed{|B_0| = \frac{|D|}{n+1} = \frac{1}{n+1}\binom{2n}{n}.}$$

Problem #56 The answer is $\boxed{0.}$ Proof #1:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = (1-1)^n = 0$$

where the second equality comes from the Binomial Theorem.

Proof #2 (found by several of you): Using the Pascal recurrence we get

$$\begin{aligned}
&\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} \\
=&\binom{n}{0} - \left[\binom{n-1}{0} + \binom{n-1}{1}\right] + \left[\binom{n-1}{1} + \binom{n-1}{2}\right] - \cdots + (-1)^{n-1}\left[\binom{n-1}{n-2} + \binom{n-1}{n-1}\right] + (-1)^n\binom{n}{n} \\
=&\left[\binom{n}{0} - \binom{n-1}{0}\right] - \left[\binom{n-1}{1} - \binom{n-1}{1}\right] + \cdots + (-1)^{n-2}\left[\binom{n-1}{n-2} - \binom{n-1}{n-2}\right] + (-1)^{n-1}\left[\binom{n-1}{n-1} - \binom{n}{n}\right] \\
=&\ 0.
\end{aligned}$$

Problem #64 Let p be a prime other than 2 or 5. Then every power of p has either 1, 3, 7, or 9 as its last digit. In particular, there are only 4 possibilities for the last digit of p^k, hence at most 40 possibilities for the last two digits. By the pigeonhole principle, some pair of distinct elements of the list

$$1,\ p,\ p^2,\ \ldots,\ p^{39},\ p^{40}$$

must have the same last two digits. Call these p^a and p^b, with $0 \leq a < b \leq 40$. Then $p^b - p^a = p^a(p^{b-a} - 1)$ is a multiple of 100, and since $\gcd(p^a, 100) = 1$, it follows that $p^{b-a} - 1$ is a multiple of 100 — but that is equivalent to saying that p^{b-a} has 01 as its last two digits.

Problem #69 Let P be a set of people of cardinality n with n odd. For $p \in P$, let $d(p)$ be the number of people that are friends with p. Then $\sum_{p\in P} d(p) = 2F$, where F is the total number of friendships — note that each friendship contributes 2 to the sum. In particular, $\sum_{p\in P} d(p)$ is an even number, so the number of odd summands must be even and in particular cannot equal n. The same argument works equally well if "friend" is changed to "nonfriend".

Extra credit: Let $P < Q$ be positive integers. The students in a class are assigned to P groups for in-class group work. (Each group has at least one student in it.) One day a substitute teacher comes in and rearranges the students into Q groups. Prove that at least $Q - P + 1$ students end up in smaller groups.

For each student x, let $f(x)$ and $g(x)$ be the size of x's original group and her reassigned group. Note that $f(x)$ and $g(x)$ are positive integers. Then

$$\begin{aligned}
\sum_x \left(\frac{1}{g(x)} - \frac{1}{f(x)} \right) &= \sum_{\text{new groups } N} \sum_{x \in N} \frac{1}{|N|} - \sum_{\text{old groups } O} \sum_{x \in O} \frac{1}{|O|} \\
&= \sum_{\text{new groups } N} \frac{|N|}{|N|} - \sum_{\text{old groups } O} \frac{|O|}{|O|} \\
&= Q - P.
\end{aligned}$$

On the other hand, every summand lies in the open interval $(-1, 1)$. Therefore, at least $Q - P + 1$ summands must be positive (otherwise the sum would be strictly less than $Q - P$). That is, there are at least $Q - P + 1$ students x for which $1/g(x) - 1/f(x) > 0$, i.e., $g(x) < f(x)$,