Vertex order shellings

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Simplicial complexes

Simplicial complex: Collection Δ such that

$$\text{if } \sigma \in \Delta \text{ and } \tau \subseteq \sigma, \text{ then } \tau \in \Delta.$$

closed under taking subst

Face: Element $\sigma \in \Delta$. **Facet**: Maximal element $F \in \Delta$.

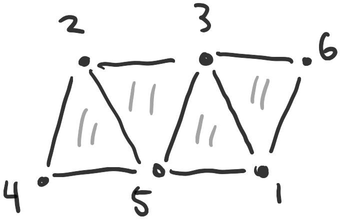
Dimension: $\dim \sigma := |\sigma| - 1$, $\dim \Delta := \max\{\dim \sigma \mid \sigma \in \Delta\}$.

Pure: All facets have the same dimension.

An example

$\Delta = \langle 135, 136, 235, 245 \rangle$

facets



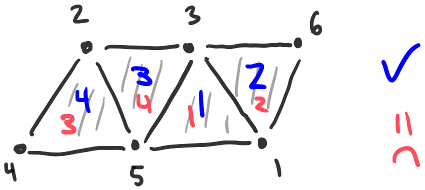
Throughout, all complexes will be **pure** with n **vertices**.

Shellability

A pure d-dimensional complex Δ is **shellable** if there exists an order on its facets $F_1, F_2, \ldots, F_k$ such that

$$\langle F_1, \ldots, F_i \rangle \cap \langle F_{i+1} \rangle$$

is pure and $(d-1)$-dimensional for each $1 \leq i \leq k-1$.



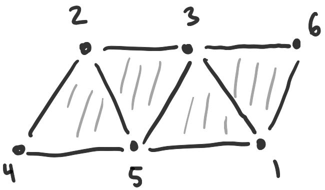
Shellability and Matroids

indepent sets of a matroid

Theorem (Björner)

A pure simplicial complex is a matroid independence complex if and only if every order on its vertices induces a shelling.

↳ lex



SC $\dfrac{1<2<3<4<5<6}{135,\ 136,\ 235,\ 245}$ ✓

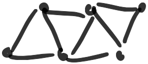
NOT SC $\dfrac{1<4<3<2<5<6}{135,\ 136,\ 245,\ 235}$

Not a matroid

The Lex-Shellability Statistic

An order on the vertices of Δ is **shelling compatible** (or **sc**) if it induces a shelling.

Lex-shelling statistic: $\mathcal{L}(\Delta) = \dfrac{\#\text{sc orders on vertices of } \Delta}{n!}$

Δ:  $\mathcal{L}(\Delta) = \dfrac{1}{2}$

<u>Björner:</u> matroid iff $\mathcal{L}(\Delta) = 1$

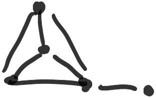
Only an epsilon away...

Let $\varepsilon > 0$

* Complexes w/ $0 < \mathcal{L}(\Delta) < \varepsilon$

$\mathcal{L}(P_n) \to 0$ as $n \to \infty$

* Complexes w/ $1 - \varepsilon < \mathcal{L}(\Delta) < 1$



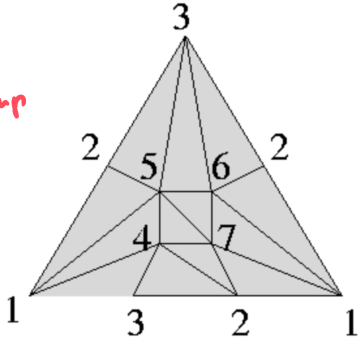
$\mathcal{L}(\overline{K_n}) \to 1$ as $n \to \infty$

plus edge

Vertex decomposability

Hachimori

Not vertex decomp



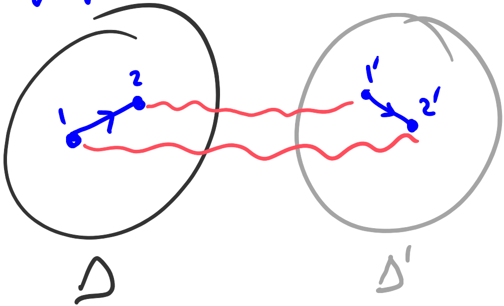
Out 7! Options, exactly 7 induce shelling

Does have "good" order : $6 < 7 < 5 < 3 < 2 < 1 < 4$

Vertex decomposability, Pt II

$\Delta = \langle 123, 135, 234, 346, 124, 136, 156, 345, 245, 256 \rangle$ ← shellable

Every "good" order $1 < 2$



Quasi-matroidal classes

Recently introduced by Samper:

Generalize matroids

$\Delta \in$ PURE $\Rightarrow$ $\mathcal{L}(\Delta) > 0$

$\Delta \in$ LEX $\Rightarrow$ $\mathcal{L}(\Delta) > 0$

$\not\Leftarrow$

The **other** lex shellability

EL-shelling:

↳ Label edges of poset ⇝ shelling of order complex of poset

(Tiansi Li)

²,

GDL: If $\mathcal{L}(\Delta) > 0$ then face poset is EL-shelling

The end

Thanks!