Math 724, Fall 2013
Take-Home Test #1

Instructions: Write up your solutions using LaTeX. You may use books and notes, but you are not allowed to collaborate — you may not consult any human other than the instructor. Solutions are due at the start of class on **Friday, September 13.**

Problem #1 In a bridge deal, each of 4 players (North, South, West and East) is dealt a hand of 13 cards from a standard deck of 52 cards.

(#1a) [5 pts] How many bridge hands contain exactly four spades?

Answer: $\binom{13}{4}\binom{39}{9} = 151519319380$.

(#1b) [10 pts] How many bridge hands contain more spades than hearts?

Answer: The number of hands with the same number k of spades and hearts is $B = \sum_{k=0}^{6}\binom{13}{k}^2\binom{26}{13-2k} = 112142793596$. Therefore there are $\binom{52}{13} - B = 522870766004$ hands with different numbers of spades and hearts. Half of these, or 261435383002, have more spades than hearts.

You can also calculate this number as $\sum_{s=1}^{13}\sum_{h=0}^{s-1}\binom{13}{s}\binom{13}{h}\binom{26}{13-s-h}$, but I like the first way better.

(#1c) [5 pts] How many different possible deals are there?

Answer: $52!/13!^4V = V\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13} = 53644737765488792839237440000$.

Problem #2 [10 pts] Recall from Supplementary Problem 1 that a composition is an expression $n = a_1 + \cdots + a_k$, where the a_i are positive integers. A *weak composition* is the same thing, except that the a_i's are only required to be nonnegative feather than positive. Count the weak compositions of n into k parts.

Answer: Given a weak composition of n into k parts, adding 1 to each part produces a (non-weak) composition of $n+k$ into k parts. This is a bijection. By Problem 1, the number of these things is $\binom{n+k-1}{k-1}$.

Problem #3 [20 pts] Call a 3-digit number (in base 10) *purple* if its digits are in strictly increasing order. E.g., 314 and 288 are not purple, but 159 is purple. In order to solve the problem, give a bijection between the set of purple 3-digit numbers and something easily counted. You do not have to spend a lot of time proving that the bijection you construct is a bijection.

Answer: There is a bijection between purple numbers and the set of 3-element subsets of [9]. (Given such a subset, sort its elements in increasing order to get a purple number.) Therefore, the solution is $\binom{9}{3} = 84$.

Problem #4 [20 pts] Let n be an integer not divisible by 2 or 5. Prove that there is some multiple of n whose decimal expansion consists of all 9's.

Answer: Consider the sequence $9, 99, 999, 9999, \dots, 10^x - 1, \dots$. There are only finitely many elements of $\mathbb{Z}/n\mathbb{Z}$, so by pigeonhole, there exist two elements of this sequence, say $10^x - 1$ and $10^y - 1$ with $x < y$, that are congruent to each other modulo n. Then n divides $10^y - 10^x = 10^x(10^{y-x} - 1)$. By hypothesis, n is coprime to 10^x, so n divides $10^{y-x} - 1$, which is a number consisting of all 9's, as desired.

Problem #5 [10 pts] A *standard tableau of shape* $2 \times n$ is a $2 \times n$ grid filled with the numbers $1, \ldots, 2n$, using each number once, so that every row increases left to right and every column increases top to bottom. For example, there are five standard tableaux of shape 2×3:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}$$

Prove that for all $n > 0$, the number of standard tableaux of shape $2 \times n$ is the Catalan number C_n.

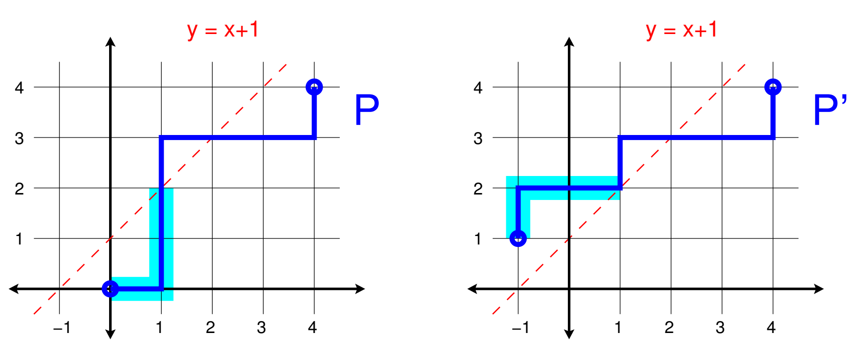
Answer: Given a standard tableau T, construct a Catalan path $S = \phi(T) = (s_1, s_2, \ldots, s_{2n})$ in which

$$s_i = \begin{cases} u & \text{if } i \text{ is in the top row of } T, \\ d & \text{if } i \text{ is in the bottom row of } T, \end{cases}$$

where u and d stand for up-steps and down-steps respectively (see problem #52). Then S is a lattice path from $(0,0)$ to $(2n, 0)$, and it never goes below $y = 0$ because for every k, the k^{th} up-step occurs before the k^{th} down-step (because the k^{th} column has a smaller entry in the top row). On the other hand, given a Catalan path S of length $2n$, we can construct a standard tableau of shape $2 \times n$ by recording the positions of the up- and down-steps in the top and bottom rows respectively; again, the Catalan condition ensures that each column is increasing. Therefore ϕ is a bijection and the desired conclusion follows.

Problem #6 [20 pts] Let $m \geq n \geq 0$ be integers. Let $C(m, n)$ be the number of lattice paths from $(0,0)$ to (m, n) that do not go above the line $y = x$. (So if $m = n$, then $C(m, n)$ is just the Catalan number C_n. Find a simple formula for $C(m, n)$ that generalizes the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$. (Hint: Generalize the method of problem 51 in the textbook.)

Answer: Given a lattice path P from $(0,0)$ to (m, n) that hits the line $y = x + 1$, find the first place where it does so; call that point $(a, a+1)$. Take the part of P from $(0,0)$ to $(a, a+1)$ and reflect it across the line $y = x + 1$, leaving the rest of P alone; this turns P into a lattice path P' from $(-1, 1)$ to (m, n).



This gives a bijection from (i) lattice paths from $(0,0)$ to (m, n) that do go above the line $y = x$, and (ii) lattice paths from $(-1, 1)$ to (m, n). Therefore: $C(m, n)$ is the number of paths from $(0,0)$ to (m, n) minus the number of paths from $(-1,1)$ to (m,n), i.e.,

$$\begin{aligned} C(m,n) &= \binom{m+n}{m} - \binom{m+n}{m+1} = \frac{(m+n)!}{m!\,n!} - \frac{(m+n)!}{(m+1)!\,(n-1)!} = \frac{(m+n)!\,(m+1) - (m+n)!\,n}{(m+1)!\,n!} \\ &= \frac{(m+n)!\,(m+1-n)}{(m+1)!\,n!} = \frac{m+1-n}{m+1}\frac{(m+n)!}{m!\,n!} = \boxed{\frac{m+1-n}{m+1}\binom{m+n}{m}.} \end{aligned}$$

Note that if $m=n$ then this formula specializes to $C_n = \frac{1}{n+1}\binom{2n}{n}$.