Graph Theory and Geometry

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February 24, 2011

Graphs

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- V is a finite set of *vertices*;
- E is a finite set of *edges*;
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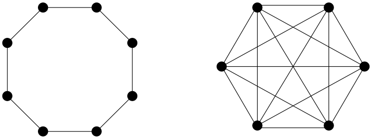
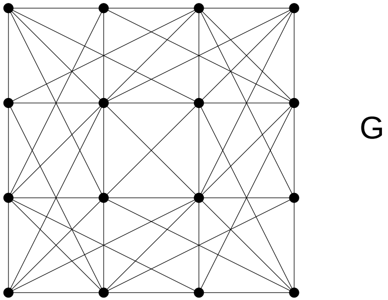


C_8 K_6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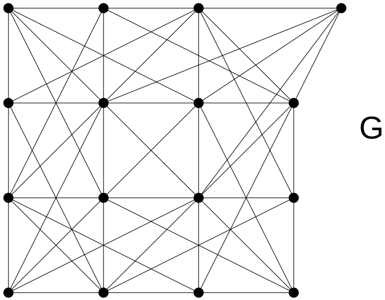
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Why study graphs?

- Real-world applications
  - Computer science (data structures, sorting, searching, networks, scheduling, discrete optimization, networks...)
  - Biology (evolutionary descent...)
  - Chemistry (molecular structure...)
  - Engineering (roads, rigidity...)
- Pure mathematics
  - Combinatorics (ubiquitous!)
  - Algebraic topology (1-dimensional cell complexes)
  - Discrete dynamical systems (chip-firing game...)
  - Algebra (quivers, Cayley graphs...)
  - Discrete geometry (polytopes, sphere packing...)

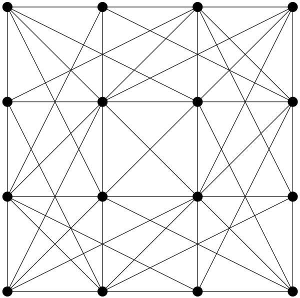
Spanning Trees

Definition A **spanning tree of G** is a set of edges T (or a subgraph (V, T)) such that:

1. (V, T) is **connected**: every pair of vertices is joined by a path
2. (V, T) is **acyclic**: there are no cycles
3. $|T| = |V| - 1$.

Any two of these conditions together imply the third.

Spanning Trees



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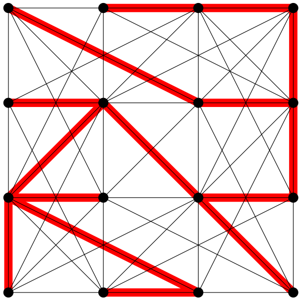

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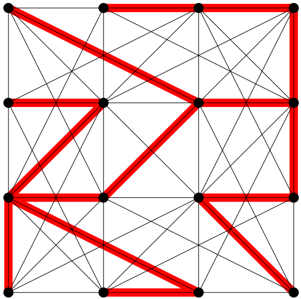

G
T

Counting Spanning Trees

$\tau(G)$ = number of spanning trees of G

- $\tau(\text{tree}) = 1$ (trivial)
- $\tau(C_n) = n$ (almost trivial)

- $\tau(K_n) = n^{n-2}$ (Cayley's formula; highly nontrivial!)
- Generalizations: complete bipartite graphs. . . threshold graphs. . . weighted enumeration. . .
- Many other enumeration formulas for "nice" graphs (e.g., hypercubes)

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Let $e \in E(G)$.

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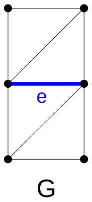

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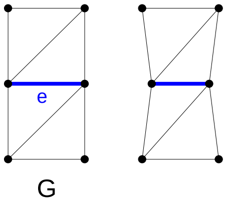

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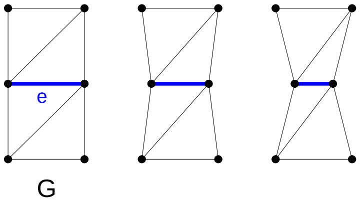

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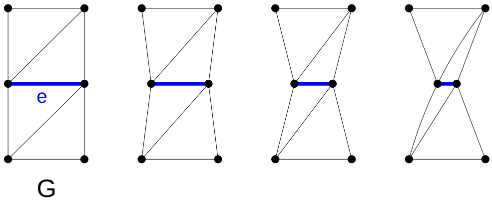

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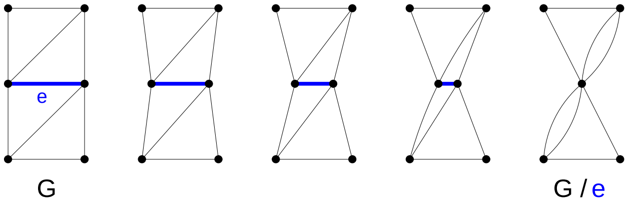

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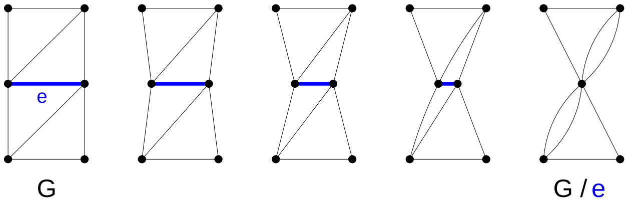

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- Therefore, we can calculate $\tau(G)$ recursively...
- ...but this is computationally inefficient (since $2^{|E|}$ steps must be considered)...
- ...and cannot be used to prove nice enumerative results (like Cayley's formula)

The Matrix-Tree Theorem

$G = (V, E)$: graph with no loops (parallel edges OK)
$V = \{1, 2, \ldots, n\}$

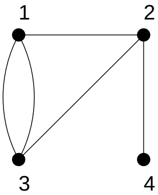
Definition The **Laplacian of G** is the $n \times n$ matrix $L = [\ell_{ij}]$:

$$\ell_{ij} = \begin{cases} \deg_G(i) & \text{if } i = j \\ -(\# \text{ of edges joining i,j}) & \text{otherwise.} \end{cases}$$

- rank $L = n - 1$.

The Matrix-Tree Theorem

Example

$G =$ 

$$L = \begin{bmatrix} 3 & -1 & -2 & 0 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The Matrix-Tree Theorem

The Matrix-Tree Theorem (Kirchhoff, 1847)

(1) Let $0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the eigenvalues of L. Then the number of spanning trees of G is

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(2) Let $1 \leq i \leq n$. Form the *reduced Laplacian* $\tilde{L}$ by deleting the i^{th} row and i^{th} column of L. Then

$$\tau(G) = \det \tilde{L} \,.$$

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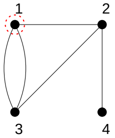
$G =$ 

$$L = \begin{bmatrix} 3 & -1 & -2 & 0 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 3 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad \tilde{L} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Eigenvalues: 0, 1, 4, 5

$(1 \cdot 4 \cdot 5)/4 = 5$

$\det \tilde{L} = 5$

Chip-Firing and the Critical Group

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- $K(G)$ is finite abelian of order $\tau(G)$.

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- Riemann-Roch theorem: relates behavior of divisors to topology of X

The Riemann-Roch Theorem for Graphs

- Graph-theoretic analogue of Riemann-Roch Theorem: M. Baker and S. Norine (2007)

Algebraic geometry	**Graph theory**
Riemann surface	Graph
Divisor	Chip configuration
Linear equivalence (i.e., equivalence mod principal divisors)	Sequence of chip-firing moves
Picard group	Critical group

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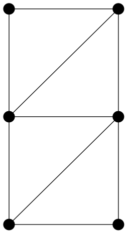

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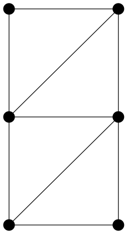

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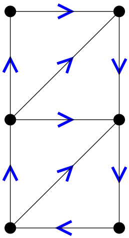

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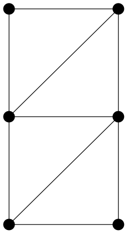

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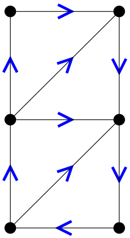

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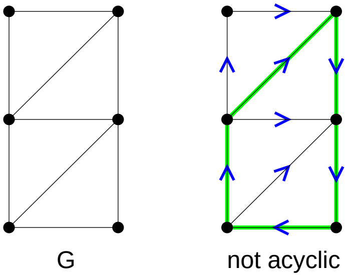

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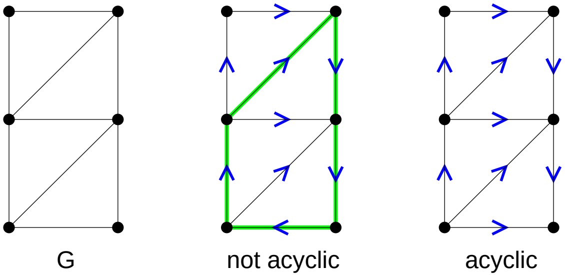

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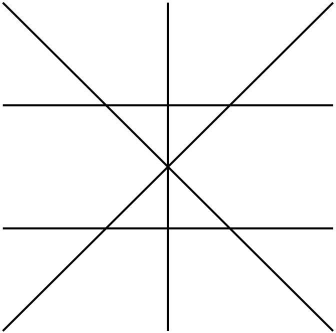
(Fact: Both $\alpha(G)$ and $\tau(G)$, as well as any other invariant satisfying a deletion-contraction recurrence, can be obtained from the *Tutte polynomial* $T_G(x,y)$.)

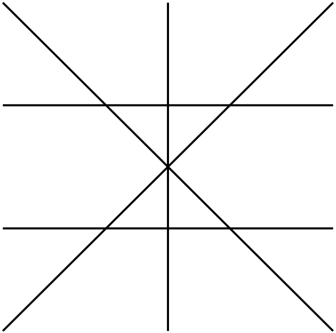
Hyperplane Arrangements

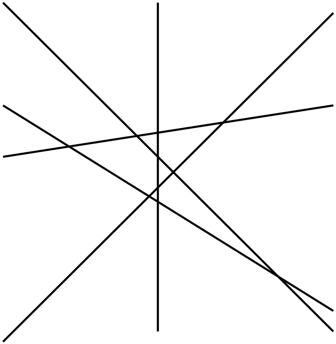
Definition A **hyperplane** H in $\mathbb{R}^n$ is an $(n-1)$-dimensional affine linear subspace.

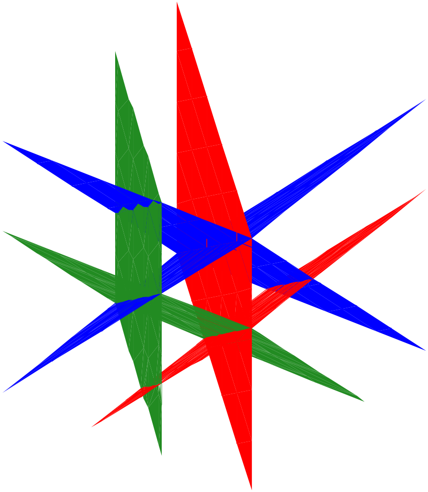
Definition A **hyperplane arrangement** $\mathcal{A} \subset \mathbb{R}^n$ is a finite collection of hyperplanes.

- $n = 1$: points on a line
- $n = 2$: lines on a plane
- $n = 3$: planes in 3-space









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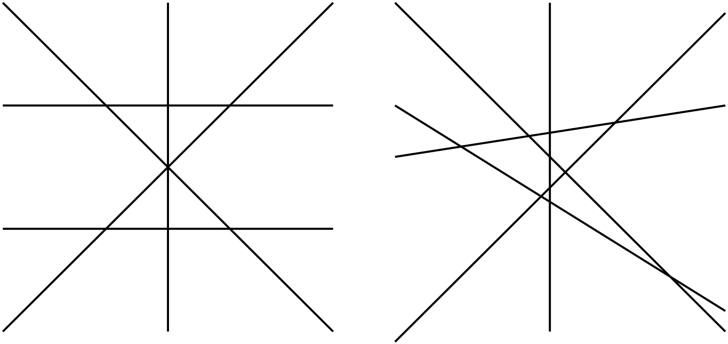

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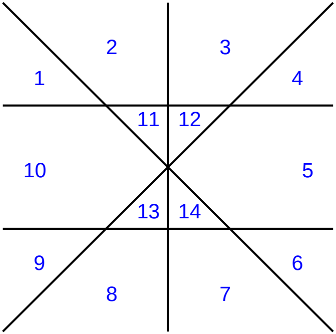
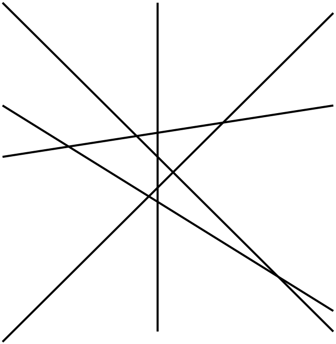


14 regions



16 regions

Counting Regions

Example $\mathcal{A} = n$ distinct lines in $\mathbb{R}^2$

- $n+1 \;\leq\; r(\mathcal{A}) \;\leq\; 1 + \binom{n+1}{2}$

Example $\mathcal{A} =$ the n coordinate hyperplanes in $\mathbb{R}^n$

- Regions of $\mathcal{A}$ = orthants
- $r(\mathcal{A}) = 2^n$

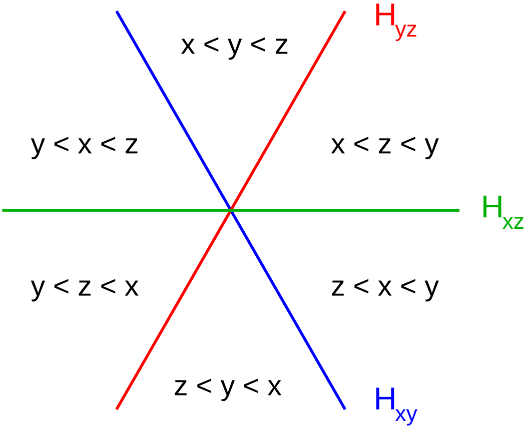
The Braid Arrangement

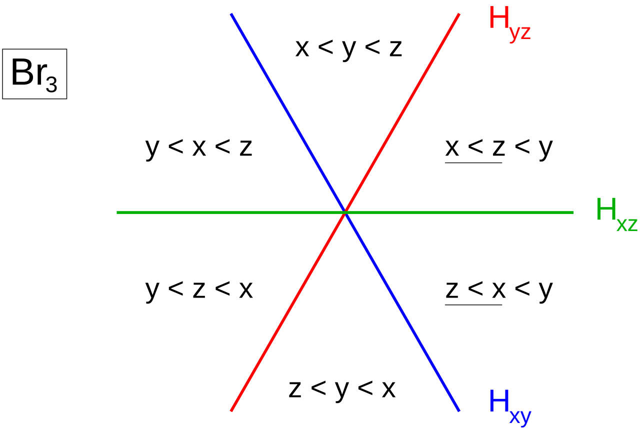
The *braid arrangement* $Br_n \subset \mathbb{R}^n$ consists of the $\binom{n}{2}$ hyperplanes

$$\begin{aligned} H_{12} &= \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_2\}, \\ H_{13} &= \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_3\}, \\ &\dots \\ H_{n-1,n} &= \{\mathbf{x} \in \mathbb{R}^n \mid x_{n-1} = x_n\}. \end{aligned}$$

- $\mathbb{R}^n \setminus Br_n = \{\mathbf{x} \in \mathbb{R}^n \mid \text{all } x_i \text{ are distinct}\}$.
- **How many regions does Br_n have?**

$\boxed{Br_3}$




Br_3
H_{yz}
$x < y < z$
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Graphic Arrangements

Let $G = (V, E)$ be a simple graph with $V = [n] = \{1, \dots, n\}$. The *graphic arrangement* $\mathcal{A}_G \subset \mathbb{R}^n$ consists of the hyperplanes

$$H_{ij} = \{\mathbf{x} \in \mathbb{R}^n \ \mid \ x_j = x_j\}$$

for all edges $ij \in E$. (So $\mathcal{A}_{K_n} = Br_n$.)

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Theorem There is a bijection between regions of $\mathcal{A}_G$ and acyclic orientations of G. In particular,

$$r(\mathcal{A}_G) = \alpha(G).$$

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In particular, $a_i \neq a_j$ for every edge ij. Orient that edge as

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The resulting orientation is acyclic.

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Sketch of proof: Suppose that $\mathbf{a} \in \mathbb{R}^n \setminus \mathcal{A}_G$.

In particular, $a_i \neq a_j$ for every edge ij. Orient that edge as

$$\begin{cases} i \to j & \text{if } a_i < a_j, \\ j \to i & \text{if } a_i > a_j. \end{cases}$$

The resulting orientation is acyclic.

Corollary $r(Br_n) = \alpha(K_n) = n!$.

Parking Functions

There are n parking spaces on a one-way street.

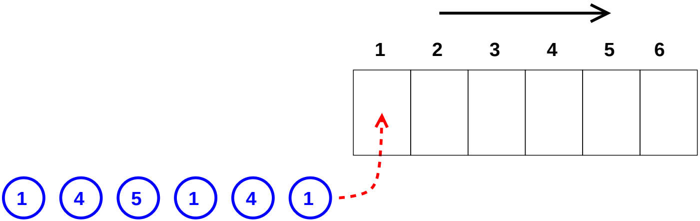
Cars $1, \ldots, n$ want to park in the spaces.

Each car has a preferred spot p_i.

Can all the cars park?

Parking Functions

Example #1: $n = 6$; $(p_1, \ldots, p_6) = (1, 4, 1, 5, 4, 1)$



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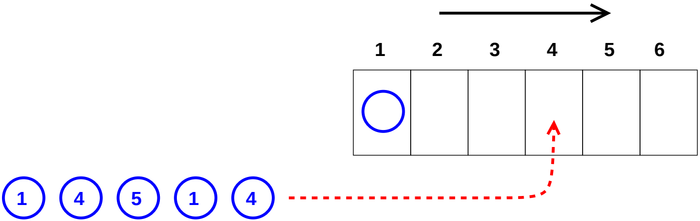

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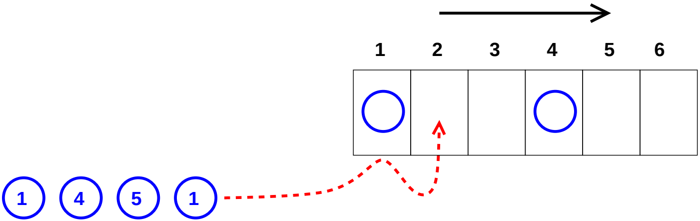

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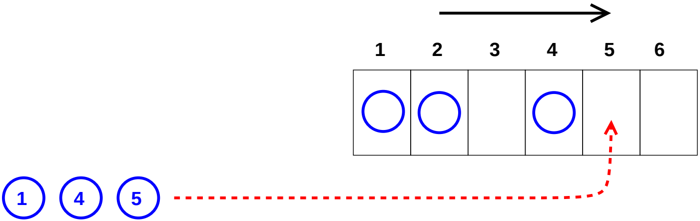

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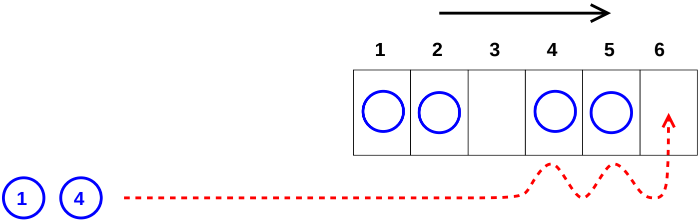

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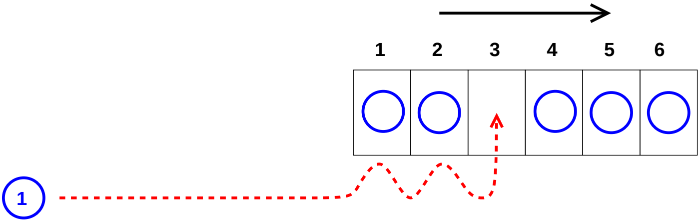

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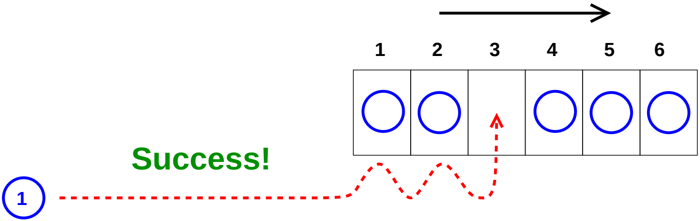

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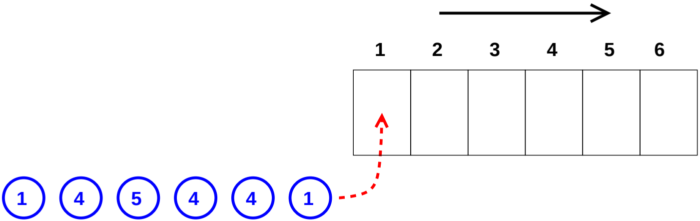

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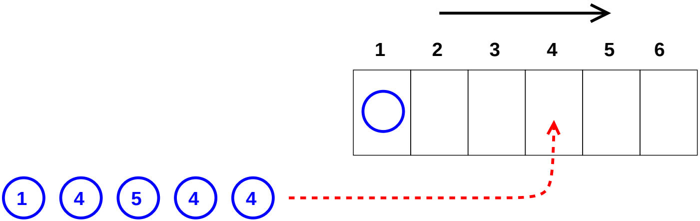

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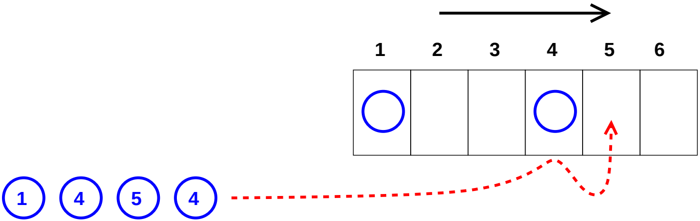

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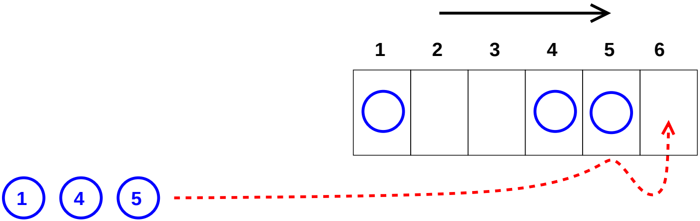

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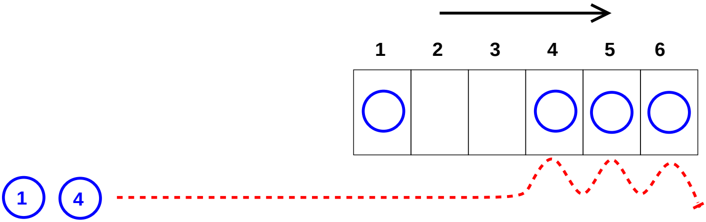

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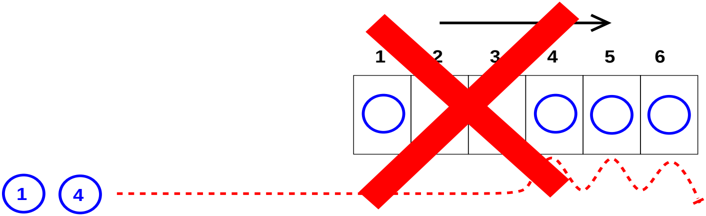

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- In particular, parking functions are invariant up to permutation.
- The number of parking functions of length n is $(n+1)^{n-1}$.

The Shi Arrangement

The *Shi arrangement* $Shi_n \subset \mathbb{R}^n$ consists of the $2\binom{n}{2}$ hyperplanes

$$\begin{array}{ll}
\{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_2\}, & \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_2 + 1\}, \\
\{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_3\}, & \{\mathbf{x} \in \mathbb{R}^n \mid x_1 = x_3 + 1\}, \\
\dots & \\
\{\mathbf{x} \in \mathbb{R}^n \mid x_{n-1} = x_n\}, & \{\mathbf{x} \in \mathbb{R}^n \mid x_{n-1} = x_n + 1\}.
\end{array}$$

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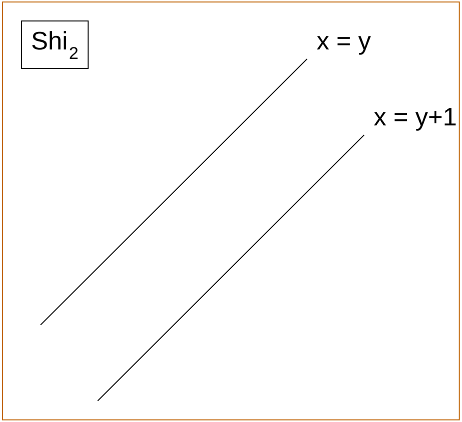
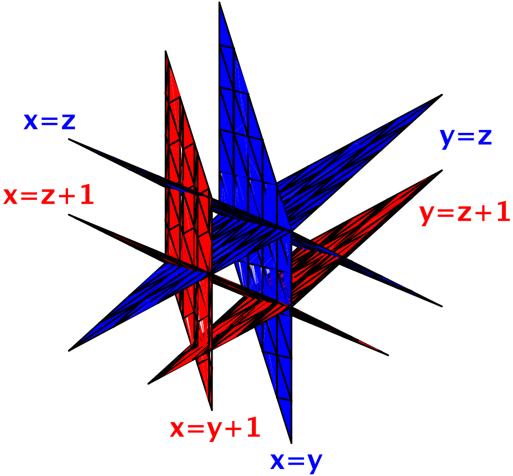
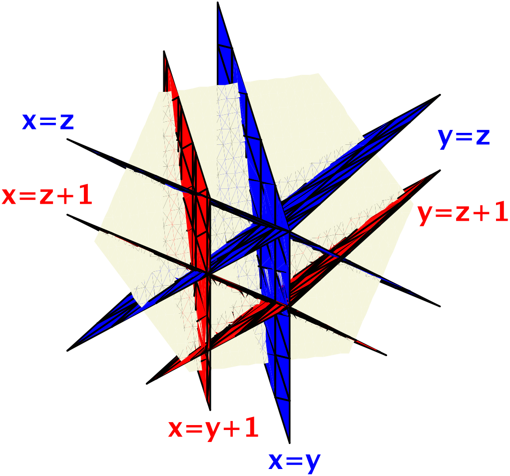



x=z
y=z
x=z+1
y=z+1
x=y+1
x=y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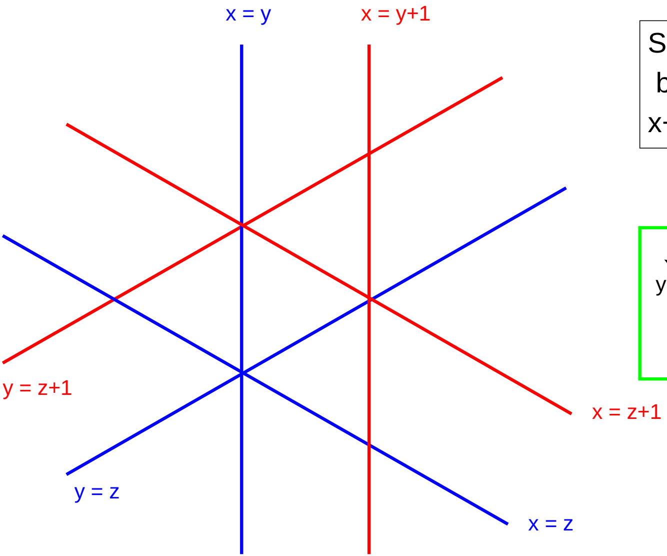
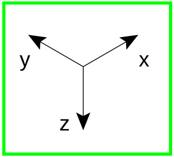


Slice of Shi_3 by plane $x+y+z = 0$



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Theorem The number of regions in Shi_n is $(n+1)^{n-1}$.

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- $(n+1)^{n-1}$ is the number of spanning trees of K_{n+1} (by Cayley's formula).
- It is also the number of parking functions of length n.
- This is all highly suspicious.

Score Vectors

Let $\mathbf{x} \in \mathbb{R}^n \setminus Shi_n$. For every $1 \leq i < j \leq n$:

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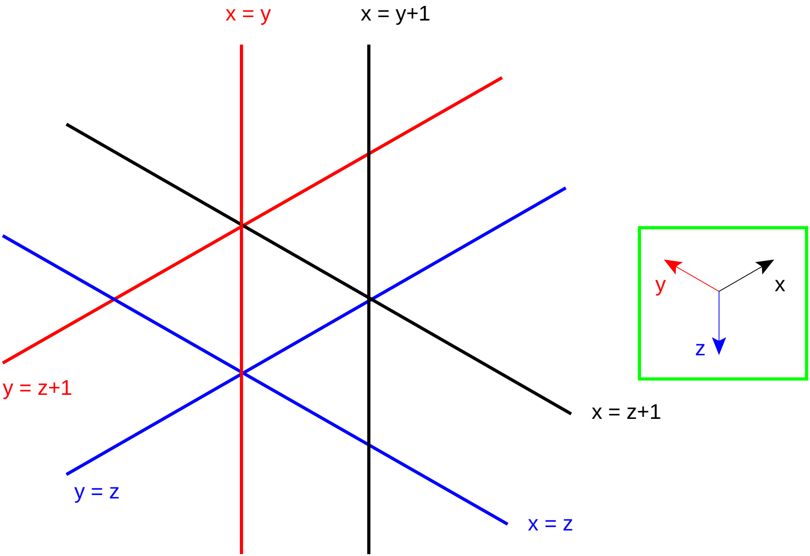
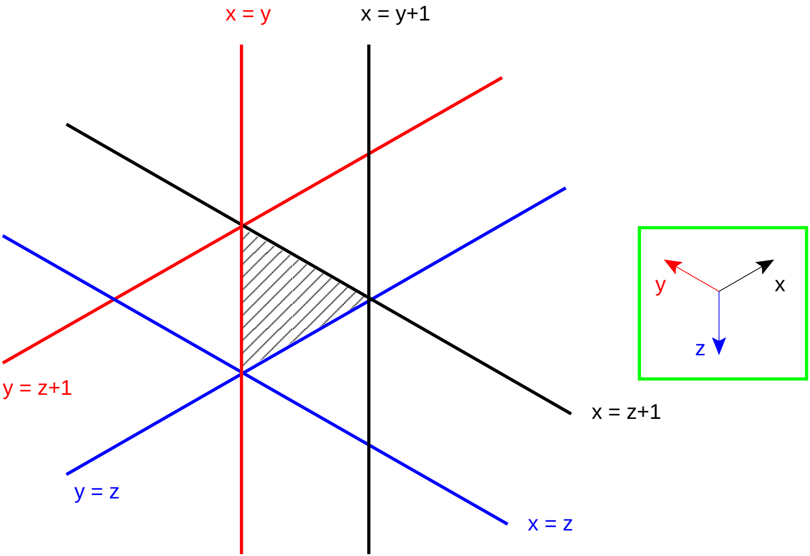
Example The score vector of $\mathbf{x} = (3.142,\ 2.010,\ 2.718)$ is $\mathbf{s} = (1, 0, 1)$.


x = y
x = y+1
y = z+1
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
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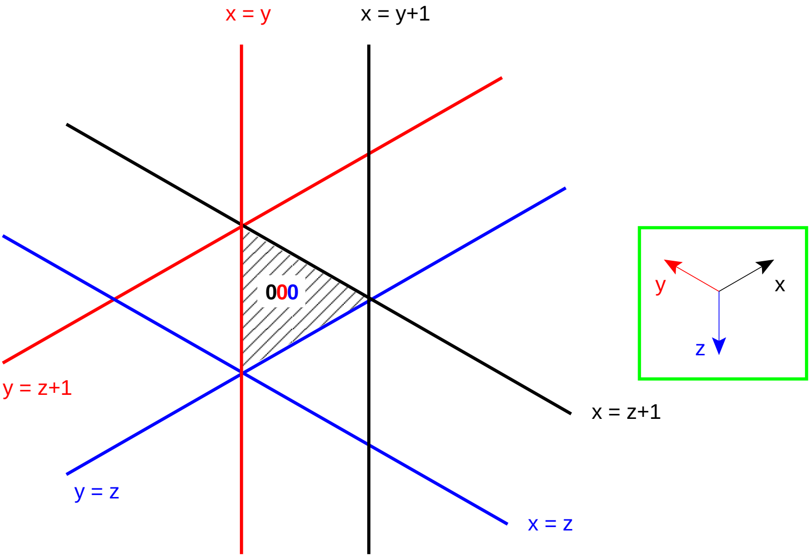
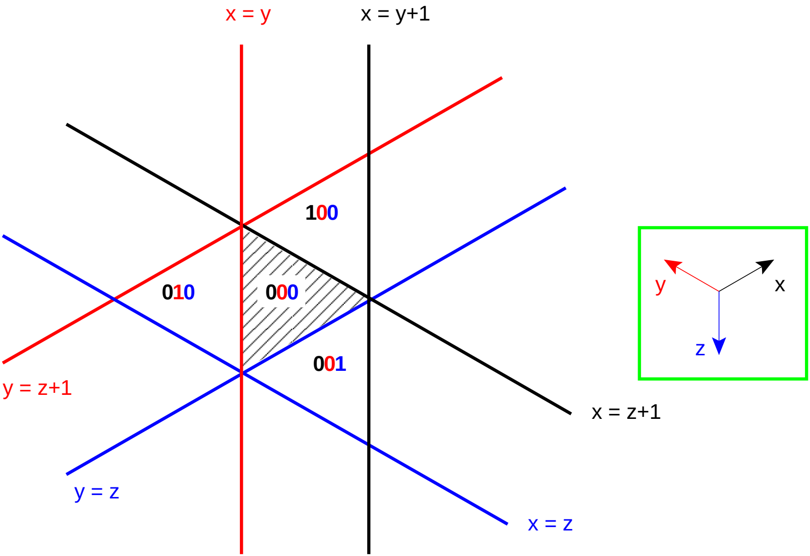
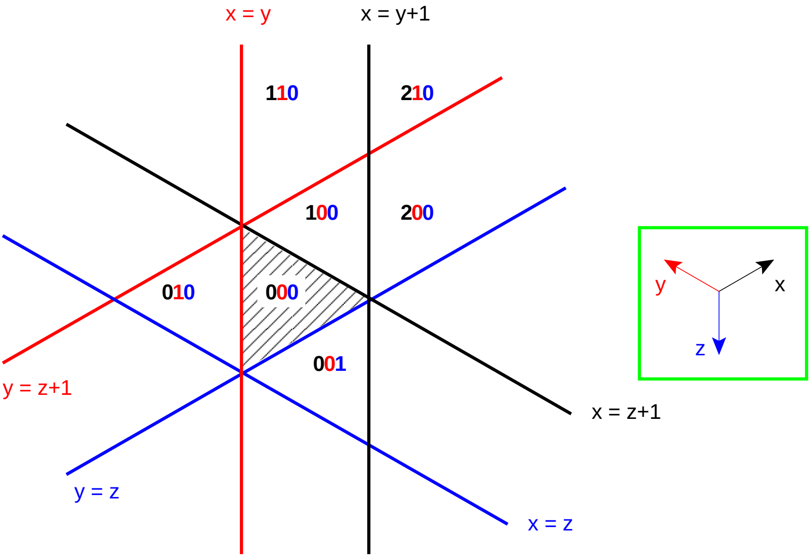
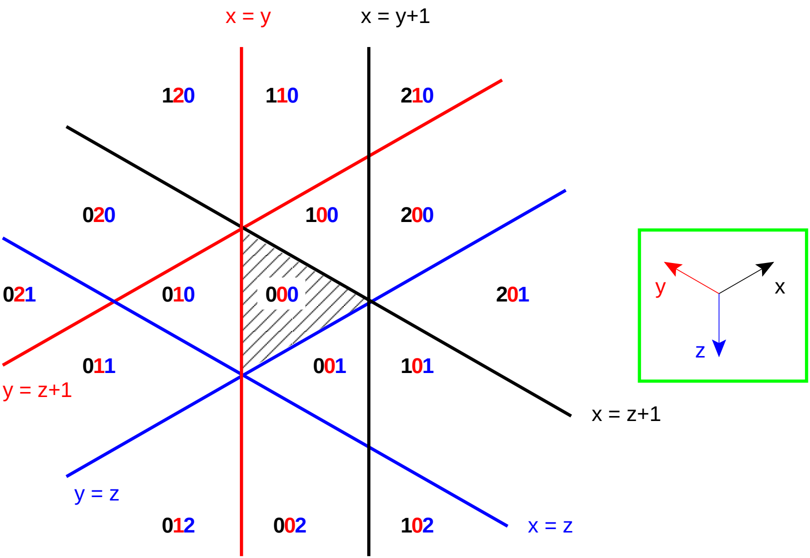
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
000
y = z+1
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
100
010
000
001
y = z+1
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
110
210
100
200
010
000
001
y = z+1
x = z+1
y = z
x = z
y
x
z


x = y
x = y+1
120
110
210
020
100
200
021
010
000
201
011
001
101
y = z+1
x = z+1
y = z
012
002
102
x = z
y
x
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$$\sum_{\text{regions } R \text{ of } Shi_n} y^{d(R_0,R)} = \sum_{\substack{\text{parking fns} \\ (p_1,\ldots,p_n)}} y^{p_1+\cdots+p_n} = T_{K_{n+1}(1,y)}$$

where d = distance, R_0 = base region.

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Example For $n = 3$: $T_{K_4}(1, y) = 1 + 3y + 6y^2 + 6y^3$.

Simplicial Complexes

Simplicial complex: a set family $\Delta \subseteq 2^{\{1,2,\dots,n\}}$ such that

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Simplicial complexes are topological spaces, with well-defined homology groups, Euler characteristic, ...

Simplicial Spanning Trees

Δ: pure simplicial complex of dimension d

Simplicial spanning tree (SST): a subcomplex $\Upsilon \subset \Delta$ such that

1. Υ contains all simplices of Δ of dimension $< d$;
2. $\tilde{H}_d(\Upsilon; \mathbb{Q}) = \tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = 0$.

(Think of condition 2 as generalizing the requirements that a tree be acyclic and connected.)

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  - Contractible complexes $\approx$ acyclic graphs
  - Some noncontractible complexes also qualify, notably $\mathbb{RP}^2$
- If Δ is a simplicial sphere: SSTs are $\Delta \setminus \{\sigma\}$, where $\sigma \in \Delta$ is any maximal face
  - Simplicial spheres $\approx$ cycle graphs

Kalai's Theorem

Let Δ be the d-skeleton of the n-vertex simplex, i.e.,

$$\Delta = \Big\{ F \subseteq \{1, 2, \ldots, n\} \;\Big|\; \dim F \leq d \Big\}$$

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Theorem [Kalai 1983]

$$\sum_{\Upsilon \in \mathcal{T}(\Delta)} |\tilde{H}_{d-1}(\Upsilon;\mathbb{Z})|^2 = n^{\binom{n-2}{d}}.$$

Kalai's Theorem

- Kalai's theorem reduces to Cayley's formula when $d = 1$ (i.e., when $\Delta = K_n$)
- Anticipated by Bolker (1976), who observed that $n^{\binom{n-2}{d}}$ gave an exact count of trees for small n, d, but failed for $n = 6$, $d = 2$ (the problem is $\mathbb{RP}^2$!)
- Adin (1992): Analogous formula for *complete colorful complexes* (generalizing known formula for complete bipartite graphs)
- Duval–Klivans–JLM (2007): More general *simplicial matrix-tree theorem* enumerating simplicial spanning trees of many simplicial complexes in terms of their combinatorial Laplacians

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- *Parking functions*: also doubtful
- *Combinatorial Riemann-Roch*: just maybe (but there's a lot of work to do!)

Thank you!