Chromatic Symmetric Functions and Polynomial Invariants of Trees

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Chromatic Symmetric Functions of Graphs

Let $G = (V, E)$ be a simple graph with $V = [n] = \{1, \ldots, n\}$.

proper coloring: a function $f : V \to \mathbb{N}_{>0}$ such that $f(i) \neq f(j)$ whenever $ij \in E$.

chromatic symmetric function (CSF): the power series

$$\mathbf{X}_G = \mathbf{X}_G(x_1, x_2, \ldots) = \sum_{\substack{f : V \to \mathbb{N}_{>0} \\ \text{proper}}} x_{f(1)} \cdots x_{f(n)}.$$

- Symmetric and homogeneous of degree n
- Generalizes the chromatic polynomial:

$$\mathbf{X}_G(1^k, 0^\infty) = \text{ number of proper } k\text{-colorings}$$

Chromatic Symmetric Functions of Graphs

- Starting point: work of Chmutov–Duzhin–Lando on Vassiliev knot invariants in early '90s; most cited source is Stanley 1995

- **Related invariants:** Tutte symmetric function / U-polynomial (Noble–Welsh 1999), matroid quasisymmetric function (Billera–Jia–Reiner 2009)
- **Analogues:** noncommutative CSFs (Gebhard–Sagan 2001), quasisymmetric CSFs (Shareshian–Wachs 2016), ...
- **Applications:** combinatorial Hopf algebras (Aguiar–Bergeron–Sottile 2006), cohomology of Hessenberg subvarieties of flag manifolds (Shareshian–Wachs 2012)

Stanley's Uniqueness Problem

Question (Stanley)
Is a tree uniquely determined up to isomorphism by its CSF?

I.e., if T, T' are non-isomorphic trees, must $X(T) \neq X(T')$?

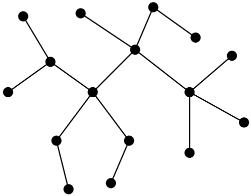
No one really has any idea (although some experts have opinions).

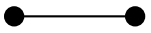
- The answer is yes for $n \leq 29$ [Heil–Ji 2019].
- Also yes for various very special classes of trees

Expanding CSFs in the Power Sum Basis

Let $G = (V, E)$ be a graph, $n = |V|$, $A \subseteq E$

type of A = partition of n whose parts are component sizes of $G|_A$





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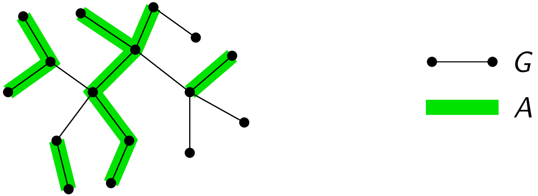

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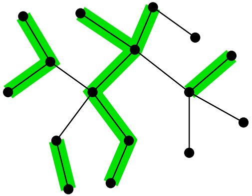

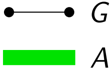

$\text{type}(A) = (6, 3, 2, 2, 1, 1, 1)$

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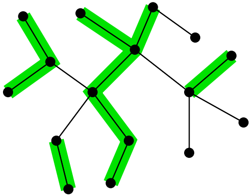

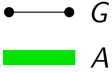

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Theorem (Stanley 1995)

$$\mathbf{X}_G = \sum_{A \subseteq E} (-1)^{n-|A|} p_{\text{type}(A)}.$$

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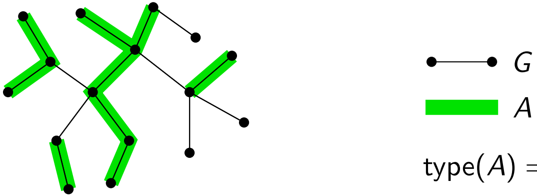

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Theorem (Stanley 1995)

$$\mathbf{X}_G = \sum_{A \subseteq E} (-1)^{n-|A|} p_{\text{type}(A)}.$$

Corollary If $T = (V, E)$ is a tree, then $\ell(\text{type}(A)) = n - |A|$, so there is no cancellation:

$$\boxed{[p_\lambda]\mathbf{X}_T = c_\lambda(T) = (-1)^{\ell(\lambda)} \#\{A \subseteq E \mid \text{type}(A) = \lambda\}.}$$

The Subtree Polynomial

Let $T = (V, E)$ be a tree. For a subtree $S \subseteq T$, define

$$\begin{aligned} e(S) &= \text{ number of edges of } S \\ \ell(S) &= \text{ number of leaf edges of } S \end{aligned}$$

(Henceforth "subtree" means "subtree with at least one edge.)

The **subtree polynomial** (STP) of T is

$$\mathbf{S}_T = \sum_{\text{subtrees } S \subseteq T} q^{e(S)} r^{\ell(S)}$$

The Subtree Polynomial

Theorem [Martin–Morin–Wagner 2008]
The STP can be obtained linearly from the CSF:

$$[q^i r^j]\mathbf{S}_T = \sum_{\lambda \vdash n} \phi(\lambda, i, j) c_\lambda(T).$$

where $\phi(\lambda, i, j)$ is independent of T.

Corollary
The CSF of a tree T determines its degree and distance sequences, i.e., the numbers

$$\#\{v \in V\colon \operatorname{deg}(v) = k\}, \qquad \#\{(v, w)\colon \operatorname{dist}(v, w) = k\}.$$

Trees with the Same STP

The STP is a strictly weaker isomorphism invariant than the CSF. Here's the smallest pair of trees with same STP and different CSF:

The Generalized Degree Polynomial of a Tree

The **generalized degree polynomial (GDP)** of T is

$$\mathbf{G}_T = \mathbf{G}_T(x, y, z) = \sum_{A \subseteq V} x^{|A|} y^{d(A)} z^{e(A)}.$$

where

$$\begin{aligned} e(A) &= \text{ number of internal edges (both endpoints in } A) \\ d(A) &= \text{ number of external edges (one endpoint in } A) \end{aligned}$$

- In particular, $d(\{v\}) = \deg(v)$.

Conjecture [Crew 2022]
The CSF of a tree determines its GDP.

The Half-Generalized Degree Polynomial

The **half-generalized degree polynomial (HDP)** of T is

$$\mathbf{H}_T = \mathbf{H}_T(x,y,z) = \sum_{\substack{A \subseteq V \\ T[A] \text{ connected}}} y^{d(A)} z^{e(A)}$$

$$= \sum_{\text{subtrees } S \subseteq T} y^{d(S)} z^{e(S)}$$

$$= \left(\text{sum of terms of } \mathbf{G}_T \text{ of the form } x^{c+1} y^b z^c\right)\Big|_{x=1}$$

Theorem [Wang-Yu-Zhang 2023]
The CSF of a tree determines its HDP.

(Key tool: use Stanley's formula for $\dfrac{\partial \mathbf{X}_T}{\partial p_k}$.)

Our Results

Theorem [Aliste-Prieto, Martin, Wagner, Zamora 2024]

1. *The CSF of a tree determines its GDP linearly*.

(This is Crew's conjecture.)

2. *The HDP and the STP of a tree determine each other.*

(Together, these two results imply the 2008 theorem of Martin–Morin–Wagner.)

3. *There exist arbitrarily large sets of trees with the same STP.*

(This implies a 2006 conjecture of Eisenstat and Gordon.)

The Hierarchy of Invariants

$\boxed{\begin{array}{c}\text{chromatic symmetric function} \\ \mathbf{X}_T \text{ (Stanley '95)}\end{array}}$

The Hierarchy of Invariants

chromatic symmetric function
$\mathbf{X}_T$ (Stanley '95)

Martin–Morin–Wagner '08

strict

subtree polynomial $\mathbf{S}_T$
(Eisenstat–Gordon '06)

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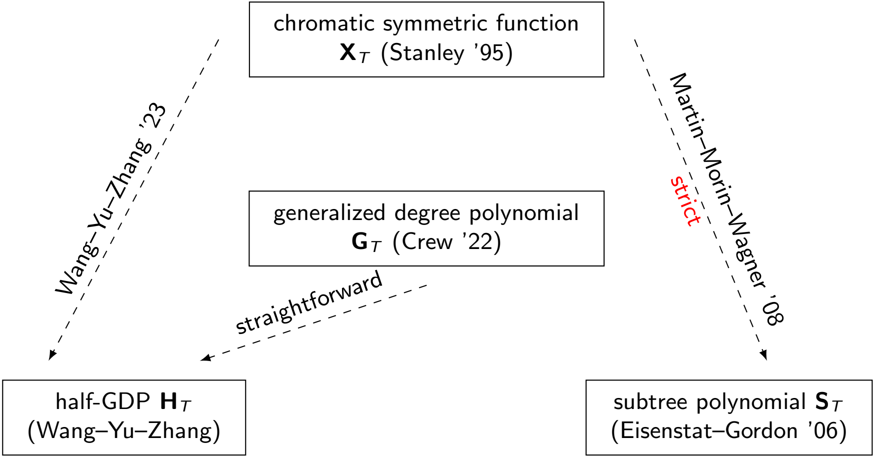

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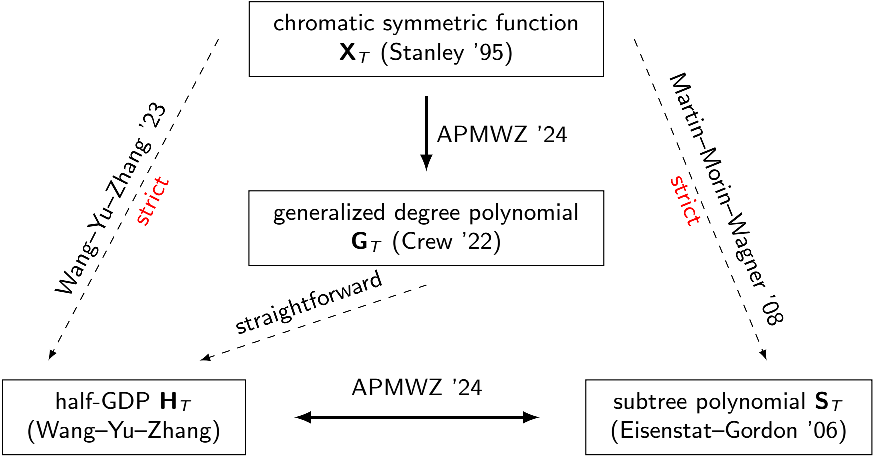

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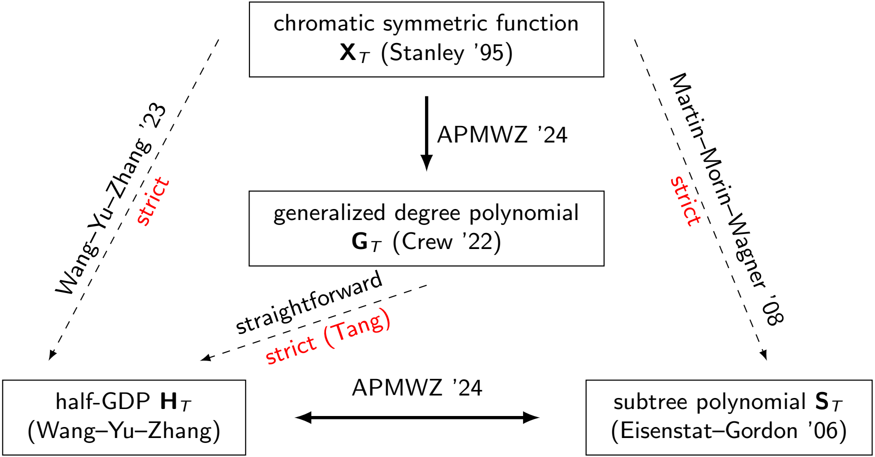


Crew's Conjecture: Obtaining the GDP from the CSF

Theorem [APMWZ 2024]
The coefficients

$$g_T(a, b, c) = \#\{A \subseteq V(T) \colon \ |A| = a, \ d(A) = b, \ e(A) = c\}$$

of $\mathbf{G}_T$ are given by

$$g_T(a, b, c) = \sum_{\lambda \vdash n} c_\lambda(T)\omega(\lambda, a, b, c)$$

where $c_\lambda(T) = [p_\lambda]\mathbf{X}_T$ and

$$\omega(\lambda, a, b, c) = (-1)^{n-b-1} \sum_{\mu \vdash a} \binom{a - \ell(\mu)}{c} \binom{\lambda}{\mu} \binom{n - \ell(\lambda) + \ell(\mu) - a}{n - b - c - 1}$$

and

$$\binom{\lambda}{\mu} := \prod_{i=1}^{n} \binom{\#\text{ of parts of } \lambda \text{ equal to } i}{\#\text{ of parts of } \mu \text{ equal to } i}$$

How to Prove Crew's Conjecture

1. Hope that the conjecture is true.
2. Compute the matrices of coefficients

   $$X = [c_\lambda(T)]_{T \in \mathcal{T}_n,\ \lambda \vdash n} \qquad G = [g_T(a,b,c)]_{T \in \mathcal{T}_n,\ \lambda \vdash n}$$

   Do this until the computer gets tired.
3. Solve the matrix equation $X\Omega = G$ for Ω (there will be a large solution space).
4. Find a matrix Ω in the solution space whose entries have a nice combinatorial form.
5. Write the proof (which involves manipulations of sums and a bijection or two).

Liu and Tang have since proved Crew's conjecture using Hopf algebraic techniques.

Equivalence of the HDP and STP

The HDP and the STP can be written as

$$\mathbf{H}_T = \sum_{b,c} h_T(b,c)\, y^b\, z^c, \qquad \mathbf{S}_T = \sum_{i,j} s_T(i,j)\, q^i\, r^j$$

where

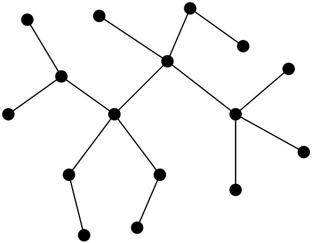
$$\begin{aligned} h_T(b,c) &= |\{U \subseteq T\colon\ d(U) = b,\ e(U) = c\}|, \\ s_T(i,j) &= |\{S \subseteq T\colon\ e(S) = i,\ \ell(S) = j\}|. \end{aligned}$$

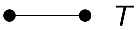
Computational evidence ($n \leq 18$) suggested to us that

$$\mathbf{H}_T = \mathbf{H}_{T'} \iff \mathbf{S}_T = \mathbf{S}_{T'}.$$

However, the method we used to approach Crew's Conjecture leads to very ugly matrices (with non-integer entries).

Equivalence of the HDP and STP





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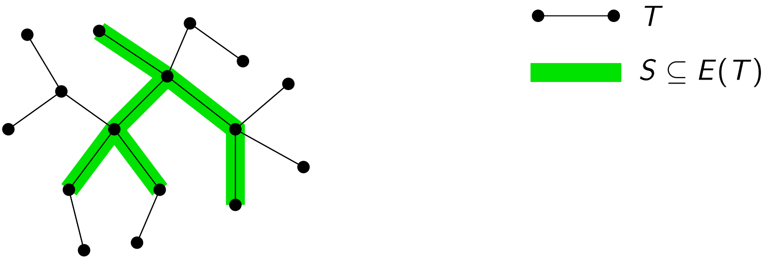

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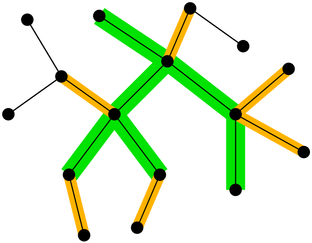

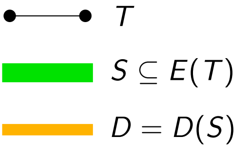

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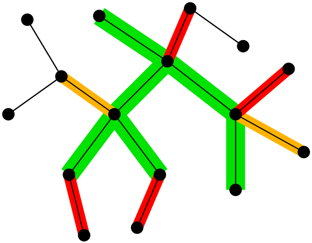

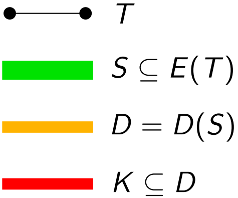

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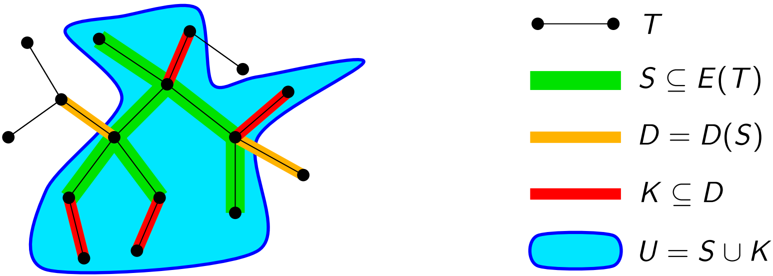

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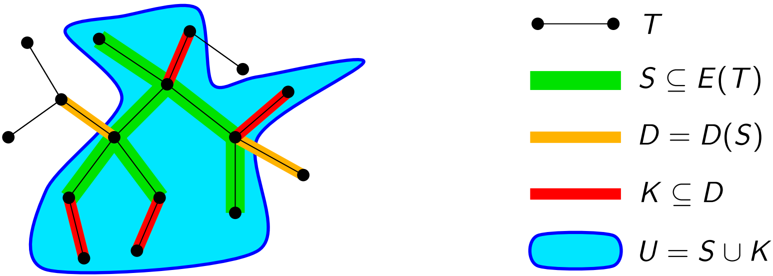

We have a bijection

$$\big\{(S,K)\colon S \subseteq T,\ K \subseteq D(S)\big\} \xrightarrow{\xi} \big\{(U,K)\colon U \subseteq T,\ K \subseteq L(U)\big\}$$

$$(S,K) \longmapsto (S \cup K, K)$$

$$(U \setminus K, K) \longleftarrow\!\!\shortmid\ (U,K)$$

Equivalence of the HDP and STP

The bijection implies the equalities

$$\sum_{b=k}^{n-1-a} \binom{b}{k} h(a,b) = \sum_{j=k}^{n-1} \binom{j}{k} s(a+k,j)$$

for all a, k. In matrix form, $MH = NS$, where

$$H = [h(a,b)]_{a,b=1}^{n}, \qquad S = [s(i,j)]_{i,j=1}^{n}.$$

- Entries of M and N are binomial coefficients
- M is unitriangular, hence invertible over $\mathbb{Z}$
- $\det N = n!$ by Gessel–Viennot lattice path theory

In particular, H and S, hence $\mathbf{H}_T$ and $\mathbf{S}_T$, determine each other. (But $M^{-1}N$ and $N^{-1}M$ are not combinatorially nice!)

Thank you!

Oh, and please read our paper or preprint!

J. Aliste-Prieto, J.L. Martin, J.D. Wagner, and J. Zamora, *Chromatic symmetric functions and polynomial invariants of trees*, Bull. Lond. Math. Soc., 56(11):3452-3476, 2024 or arXiv:2402.10333.

References

M. Aguiar, N. Bergeron, and F. Sottile, *Combinatorial Hopf algebras and generalized Dehn-Sommerville relations*, Compos. Math. **142** (2006), 1–30.

L.J. Billera, N. Jia, and V. Reiner, *A quasisymmetric function for matroids*, Eur. J. Combin. **30** (2009), 1727–1757.

L.J. Billera, H. Thomas, and S. van Willigenburg, *Decomposable compositions, symmetric quasisymmetric functions and equality of ribbon Schur functions*, Adv. Math. **204** (2006), 204–240.

L. Crew, *A note on distinguishing trees with the chromatic symmetric function*, Discrete Math. **345** (2022), 112682.

D. Eisenstat and G. Gordon, *Non-isomorphic caterpillars with identical subtree data*, Discrete Math. **306** (2006), 827–830.

D.D. Gebhard and B.E. Sagan, *A chromatic symmetric function in noncommuting variables*, J. Algebraic Combin. **13** (2001), no. 3, 227–255.

I. Gessel and G. Viennot, *Binomial determinants, paths, and hook length formulae*, Adv. Math. **58** (1985), 300–321.

S. Heil and C. Ji, *On an algorithm for comparing the chromatic symmetric functions of trees*, Australas. J. Combin. **75** (2019), 210–222.

J.L. Martin, M. Morin, and J.D. Wagner, *On distinguishing trees by their chromatic symmetric functions*, J. Combin. Theory Ser. A **115** (2008), 237–253.

S.D. Noble and D.J.A. Welsh, *A weighted graph polynomial from chromatic invariants of knots*, Ann. Inst. Fourier (Grenoble) **49** (1999), 1057–1087.

R. Orellana and G. Scott, *Graphs with equal chromatic symmetric functions*, Discrete Math. **320** (2014), 1–14.

R.P. Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*, Adv. Math. **111** (1995), 166–194.

J. Shareshian and M.L. Wachs, *Chromatic quasisymmetric functions*, Adv. Math. **295** (2016), 497–551.

Y. Wang, X. Yu, and X.-D. Zhang, *A class of trees determined by their chromatic symmetric functions*, Discrete Math. **347** (2024), 114096.