The chromatic symmetric class function of a graph

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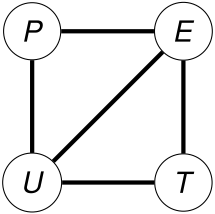
AMS Special Session: AGT

Chromatic Symmetric Function

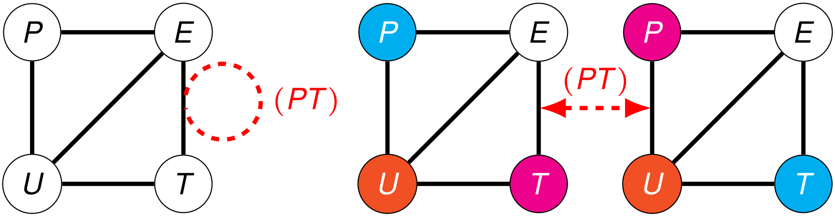


$\chi(G, \mathbf{x}) = \sum_{f} \prod_{v \in V} x_{f(v)}$ where the sum is over proper colorings of G.

$$\begin{aligned}\chi(\mathrm{UTEP}, \mathbf{x}) &= \sum_{i,j,k} x_i^2 x_j x_k + \sum_{i,j,k,l} x_i x_j x_k x_l \\ &= 2m_{2,1,1} + 24m_{1,1,1,1} \\ &= 2(F_{1,1,2} + F_{1,2,1} + F_{2,1,1}) + 18F_{1,1,1,1}\end{aligned}$$

Orbital Chromatic Symmetric Function

An **automorphism** of G is a bijection $\sigma : G \to G$ that preserves edges.
Let $\mathfrak{H}$ be a group of automorphisms of G.



$\mathfrak{H}$ acts on the set of proper colorings of G.
Given $f : V \to \mathbb{N}$, $\mathfrak{g}f = f'$, we have

$$\prod_{v \in V} x_{f(v)} = \prod_{v \in V} x_{f'(v)}.$$

Counting Fixed Points

Let $\chi^O(G, \mathfrak{H}, k) = \sum_O \prod_{x \in V} x_{f(v)}$ where we sum over orbits, and $f \in O$ is arbitrarily chosen. This is the **orbital chromatic symmetric function.** We have

$$\begin{aligned}\chi^O(\mathrm{UTEP}, \mathbb{Z}/2\mathbb{Z}, \mathbf{x}) &= 2m_{2,1,1} + 12m_{1,1,1,1} \\ &= 2(F_{2,1,1} + F_{1,2,1} + F_{1,1,2}) + 6F_{1,1,1,1}\end{aligned}$$

Conjecture

We claim that $[F_\alpha]\chi^O(G, \mathfrak{H}, \mathbf{x}) \geq 0$.

Fixed Points

For $\mathfrak{g} \in \mathfrak{H}$,

$$\chi(G, \mathfrak{H}, \mathbf{x}; \mathfrak{g}) = \sum_{f : \mathfrak{g} f = f} \prod_{v \in V} x_{f(v)}.$$

We have

$\chi(\mathrm{UTEP}, \mathbb{Z}/2\mathbb{Z}, \mathbf{x}; (PT)) = m_{2,1,1} = 2F_{2,1,1} + 2F_{1,2,1} + 2F_{1,1,2} - 6F_{1,1,1,1}.$

Character Table

Let's consider the characters of $\mathbb{Z}/2\mathbb{Z}$.

	$[F_{2,1,1}]$	$[F_{1,1,1,1}]$	χ_1	χ_2
e	2	18	1	1
(PT)	2	-6	1	-1

We see that $[F_{2,1,1}]\chi(\text{UTEP}, \mathbb{Z}/2\mathbb{Z}, \mathbf{x}) = 2\chi_1$ and $[F_{1,1,1,1}]\chi(\text{UTEP}, \mathbb{Z}/2\mathbb{Z}, \mathbf{x}) = 6\chi_1 + 12\chi_2$.

Chromatic Symmetric class function and Conjecture

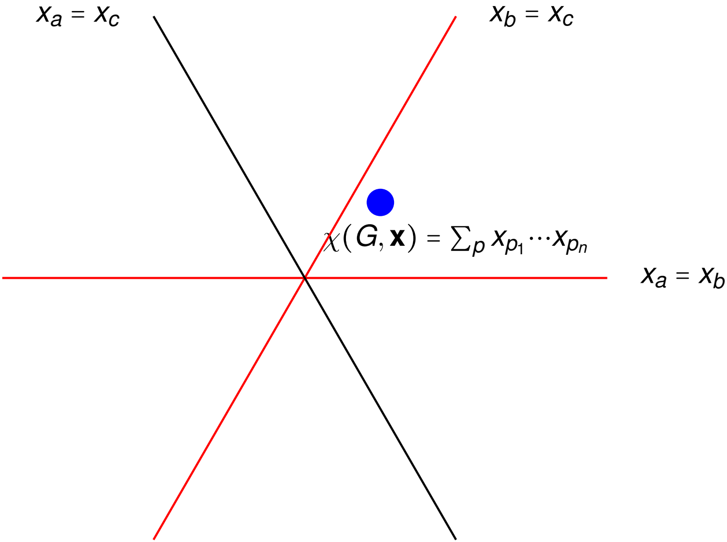
We can view $\chi(G, \mathfrak{H}, \mathbf{x}, -) : \mathfrak{H} \to Sym$ that is constant on conjugacy classes (hence is a symmetric class function).
$\chi(G, \mathfrak{H}, \mathbf{x})$ is the chromatic symmetric class function of $(G, \mathfrak{H})$.
We can write $\chi(G, \mathfrak{H}, \mathbf{x}) = \sum_{\alpha \vDash n} c_\alpha F_\alpha$ where the c_α are virtual characters.

$$\chi(UTEP, \mathbb{Z}/2\mathbb{Z}, \mathbf{x}) = 2\chi_1(F_{2,1,1} + F_{1,2,1} + F_{1,1,2}) + (6\chi_1 + 12\chi_2)F_{1,1,1,1}$$

Theorem

We have c_α is a character for all α.
Thus $[F_\alpha]\chi^O(G, \mathfrak{H}, \mathbf{x}) \geq 0$.

Coloring Complex



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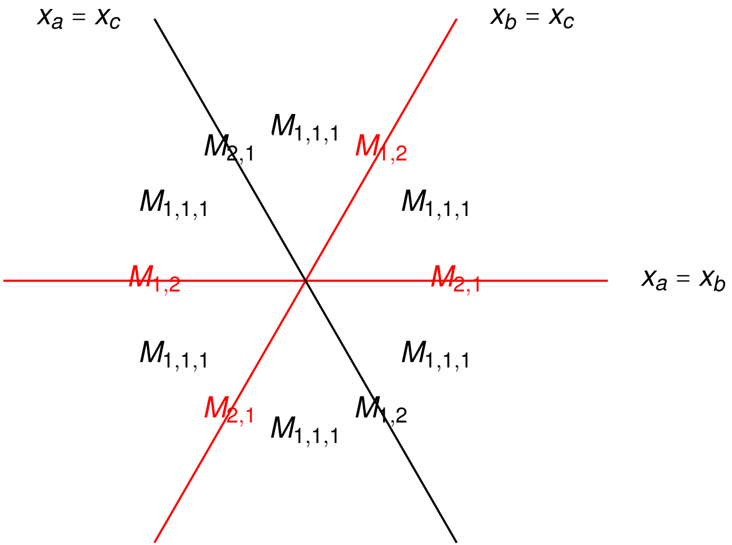

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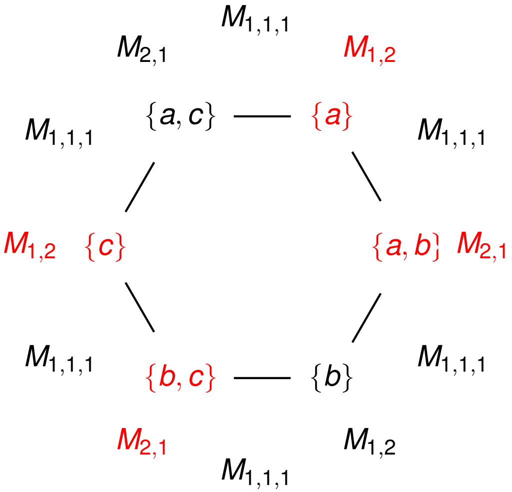

Balanced Relative Simplicial Complex

We define $f : 2^{V(G)} \setminus \{\emptyset, V\}$ by $f(S) = |S|$. Thus we have $\Phi(G) = (\Delta, \Gamma)$, which is a balanced relative simplicial complex:

1. For every $\rho \subseteq \sigma \subseteq \tau$, if $\rho, \tau \in \Phi(G)$, then $\sigma \in \Phi(G)$.
2. $f : 2^{V(G)} \setminus \{\emptyset, V\} \to [|V| - 1]$, such that $f(\sigma) = [|V| - 1]$ for every facet of $\Phi(G)$.

Given $S = \{s_1, \ldots, s_k\}$ with $s_1 < \cdots < s_k$, we let $\alpha(S) = (s_1, s_2 - s_1, \ldots, s_k - s_{k-1}, |V| - s_k)$.
We let $f_S(\Phi) = \#\{\sigma \in \Phi : f(\sigma) = S\}$.
Then $[M_{\alpha(S)}]\chi(G, \mathbf{x}) = f_S(\Phi(G))$.

Equivariant flag f-vector

We have $\mathfrak{H}$ acts on $\Phi(G)$.
For $\mathfrak{g} \in \mathfrak{H}$, we let $f_S(\Phi, \mathfrak{H}, \mathfrak{g}) = \#\{\sigma \in \Phi : f(\sigma) = S, \mathfrak{g}\sigma = \sigma\}$.
We define $h_S(\Phi, \mathfrak{H}, \mathfrak{g}) = \sum_{T \supset S} (-1)^{|T \setminus S|} f_T(G, \mathfrak{H}, \mathfrak{g})$.

Theorem

Let $\mathfrak{H}$ act on G. Then $[M_{\alpha(S)}]\chi(G, \mathfrak{H}, \mathbf{x}) = f_S(\Phi(G), \mathfrak{H}, \mathfrak{g})$.
We have $[F_\alpha]\chi(G, \mathfrak{H}, \mathbf{x}) = h_S(\Phi(G), \mathfrak{H}, \mathbf{x})$.

Relatively Cohen Macaulay

We say Φ is relatively Cohen-Macaulay if

$$\widetilde{H}_i(\mathrm{link}_\Delta(\sigma), \mathrm{link}_\Gamma(\sigma)) = 0$$

for $i < \dim \mathrm{link}_\Delta(\sigma)$ and $\sigma \in \Delta$.
For $S \subseteq [n-1]$, we let $\Phi|_S = \{\sigma \in \Phi : f(\sigma) \subseteq S\}$.

Theorem

If $\mathfrak{H}$ acts on a balanced, relatively Cohen-Macaulay complex Φ, then $\widetilde{H}_{|S|-1}(\Phi|_S)$ is an $\mathfrak{H}$-module with character $h_S(\Phi, \mathfrak{H})$.

Main Theorem

$\Phi(G)$ is relatively Cohen-Macaulay.

Theorem

If $\mathfrak{H}$ acts on a graph G, then $\widetilde{H}_{|S|-1}(\Phi(G)|_S)$ is an $\mathfrak{H}$-module with character $[F_{\alpha(S)}]\chi(G,\mathfrak{H},\mathbf{x})$ is a character.
Moreover, $[F_{\alpha(S)}]\chi^O(G,\mathfrak{H},\mathbf{x}) \geq 0$.

Coming soon

Eventually will be posted:
Equivariant flag f-vectors of balanced simplicial complexes
Other results: Study mixed graphs G and mixed graph coloring. Main change: $\Phi(G)$ is no longer relatively Cohen-Macaulay, but satisfies Serre's condition $(S_{\ell(G)})$ for some $\ell(G)$. This implies $[F_\alpha]\chi(G, \mathfrak{H}, \mathbf{x})$ is a character for $|S| \leq \ell(G)$.

Thank You