Total Cut Complexes of Graphs

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joint work with

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Simplicial Complex

Δ: collection of subsets of $[n]$ closed under inclusion.

- Face: $\sigma \in \Delta$
- Facet: maximal face
- Pure: all facets have the same size
- Dimension of faces: $\dim(\sigma) = |\sigma| - 1$.

  of Δ: $\dim(\Delta) = \max\{\dim(\sigma) \mid \sigma \in \Delta\}$.

Shellable simplicial complexes

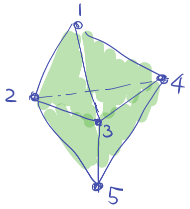
Idea of shelling:

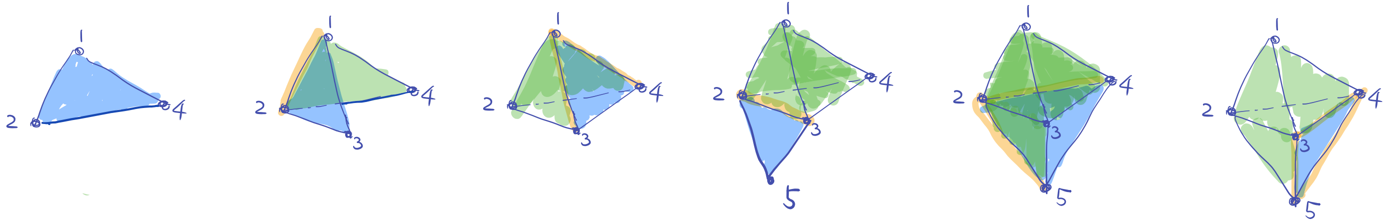
Build a complex by adding its facets one-by-one such that every facet fits "nicely" with previously added ones

Def.: An ordering $F_1, F_2, \cdots, F_m$ such that

$\left(\bigcup_{i=1}^{j-1} F_i \right) \cap F_j$ is a nonempty union of $(\dim(F_j) - 1)$- dim. faces.

A shellable complex

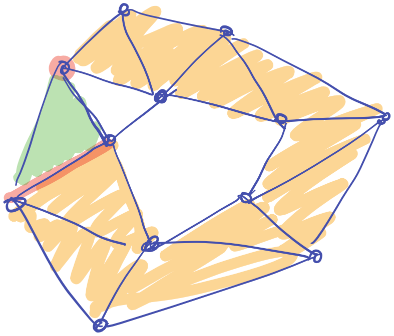




Theorem (Bruggesser, Mani, '70)

The boundary complex of a polytope is shellable.

A nonshellable complex



$\simeq S^1$

(NOT S^2)

$\Longrightarrow$ NONSHELLABLE

Theorem (Bjorner)

A pure shellable complex of dimension d either has the homology type of a wedge of spheres, all of dimension d, or it is contractible.

Graphs $\rightsquigarrow$ Simplicial complexes

- independence complex
- matching complex

...

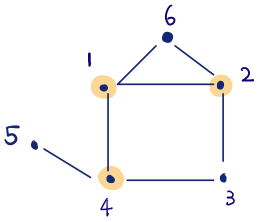
- cut complexes
- total cut complexes

Total cut complexes

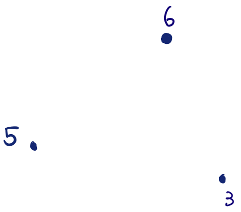
Def. $G=(V,E)$, $S\subseteq V$ is an independent k-set if $|S|=k$, and $G[S]$ has no edge.

$$I_k(G)=\{S\subseteq V \mid S \text{ is an independent } k\text{-set}\}.$$

If $S\in I_k(G)$, $|V|=n$, then $V\setminus S$ is a total cut $(n-k)$-set of G.



$F=\{1,2,4\}$ is a total cut 3-set



$G[V\setminus F]$

Def. k-**total cut complex** of the graph G is the simplicial complex $\Delta_k^t(G)$ whose facets are the total cut sets of G of size $n-k$.

$$\Delta_k^t(G) = \langle F \subseteq V \mid V \setminus F \in I_k(G) \rangle.$$

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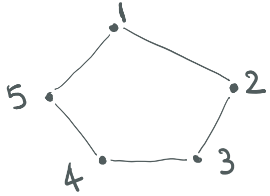
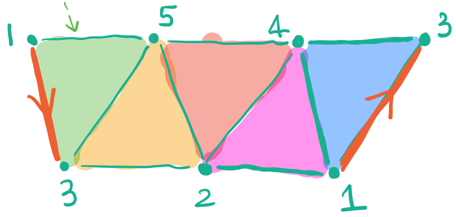
Example: $G = C_5$.



$$\Delta_2^t(C_5) = \langle 135, 235, 245, 124, 134 \rangle = \Delta_2(C_5)$$



(the ordinary 2-cut complex)

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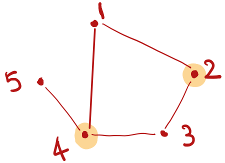
Example(s):

$G =$



$\Delta_3^t(G) = \langle 24 \rangle$

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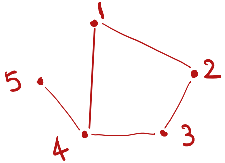

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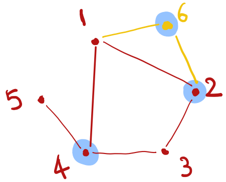

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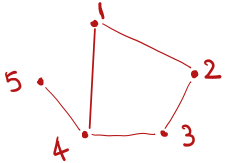

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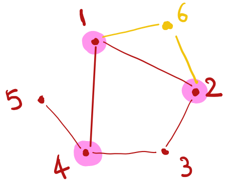

$$\Delta_3^t(G) = \langle 246, 124 \rangle$$

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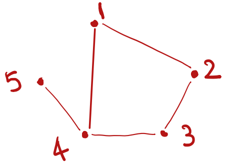
$G =$ 

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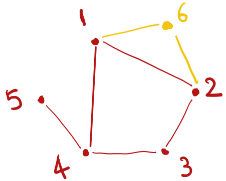
$G =$ 

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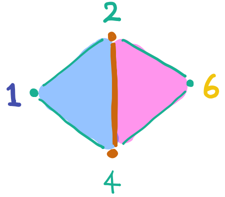

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$$\Delta_k^t(G) = \langle F \subseteq V \mid V \setminus F \in I_k(G) \rangle.$$

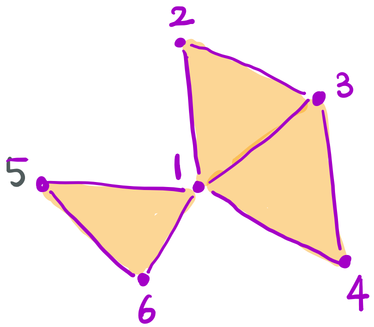
MAX size of independent set in G

Note:

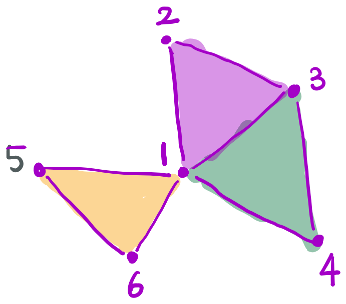
- If $k > \alpha(G)$, then $\Delta_k^t(G) = \emptyset$ (void **complex**). Otherwise, $\Delta_k^t(G)$ is a pure $(n-k-1)$-dim. simplicial complex.
- $\Delta_2^t(G) = \Delta_2(G)$
- $\Delta_k^t(G) \subsetneq \Delta_k(G)$

Which simplicial complexes are $\Delta_k^t(G)$?

- Any simplicial complex is a cut complex $\Delta_k(G)$ for some G and k.
- Not true for $\Delta_k^t(G)$.



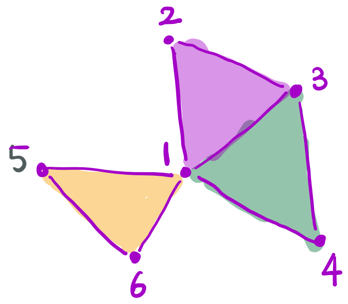
Which simplicial complexes are $\Delta_k^t(G)$?



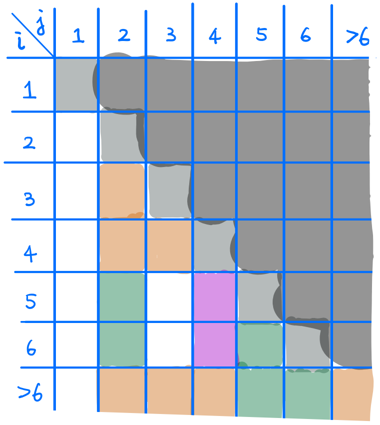
$G = ?$

156 is a facet $\Rightarrow$

Which simplicial complexes are $\Delta_k^t(G)$?



$\not\leftarrow$ G (DOES NOT EXIST!)



156 is a facet $\Rightarrow$ ij for $i, j \in \{2, 3, 4\}$ cannot be an edge.
ij for $i \in \{2, 3, 4\}$, $j > 6$ cannot be an edge.

134 is a facet $\Rightarrow$ 25, 26, 56 cannot be edges
5i for $i > 6$ cannot be an edge.

123 is a facet $\Rightarrow$ 4i for $i \geq 5$ cannot be edges

135, 136 must be facets!

Motivation: Fröberg's Theorem

Def. A graph is chordal if it contains NO induced subgraph that is a cycle C_m with $m \geq 4$.

(Fröberg, 1990) (Eagon, Reiner, 1996)

Theorem. G is chordal $\iff$ edge ideal of G^c has linear resolution. $\iff$ some simplicial complex is shellable

Motivation: Fröberg's Theorem

Def. A graph is chordal if it contains NO induced subgraph that is a cycle C_m with $m \geq 4$.

(Fröberg, 1990) (Eagon, Reiner, 1996)

Theorem. G is chordal $\iff$ edge ideal of G^c has linear resolution. $\iff$ $\Delta_2^t(G)$ is shellable

$\iff$ $\Delta_2^t(G)$ is vertex decomposable

Goal: Extend Fröberg's Theorem!

Star, link, and deletion

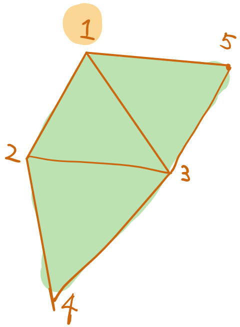
Δ: simplicial complex, $\sigma \in \Delta$.

Star: $\mathrm{St}_\Delta \sigma = \{\tau \in \Delta \mid \sigma \cup \tau \in \Delta\}$.

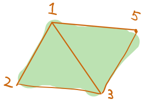
Link: $\mathrm{lk}_\Delta \sigma = \{\tau \in \Delta \mid \sigma \cap \tau = \phi \text{ and } \sigma \cup \tau \in \Delta\}$.

Deletion $\mathrm{del}_\Delta \sigma = \{\tau \in \Delta \mid \sigma \not\subseteq \tau\}$.

Ex:



$\mathrm{st}_\Delta\{1\} =$



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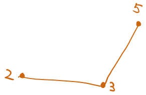

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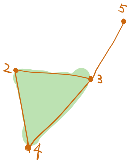
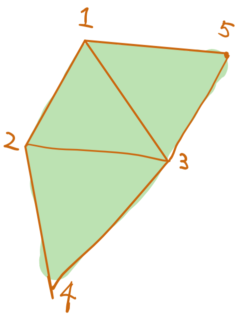


Def. simplicial complex Δ is **vertex decomposable** if either Δ is a simplex, or there is a vertex $v \in \Delta$ such that

(1). both $\mathrm{lk}_\Delta v$ and $\mathrm{del}_\Delta v$ are vertex decomposable

and (2). $\mathrm{del}_\Delta v$ is pure and d-dimensional.

$\Big\}$ v is called a **shedding vertex**



is vertex decomposable with shedding order $\{5, 1\}$.

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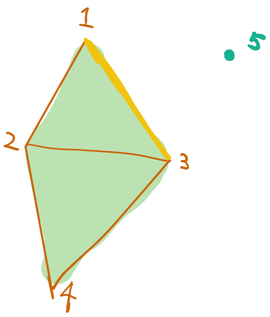

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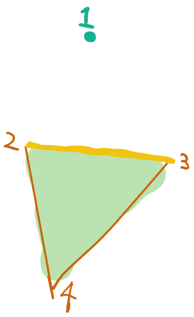

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Vertex decomposable $\Longrightarrow$ Shellable

Goal: Extend Fröberg's Theorem!

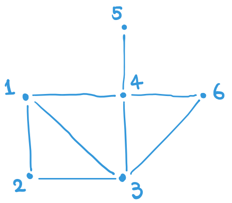
$\Delta_k^t(G)$ is shellable / vertex decomposable $\iff$ G is $\boxed{?}$

A property of chordal graphs.

Def. A vertex is simplicial in G if the subgraph generated by its neighbor(s) is a complete graph.

Property: If G is chordal, there exists an eliminating order of its vertices $v_1, v_2, \cdots, v_n$ such that v_i is simplicial in the graph $G[V \setminus \{v_1, \cdots, v_{i-1}\}]$.

Ex.

$G =$ 

Eliminating order: $\{6, 5, 4, 3, 2, 1\}$.

Chordal graphs:

Fröberg, Eagon-Reiner

$$\boxed{G \text{ is chordal} \iff \Delta_2^t(G) \text{ vertex decomposable}}$$

Chordal graphs:

BDMRSX

$\Delta_k^t(G)$ vertex decomposable $(\forall k \geq 2)$ $\Longleftrightarrow$ $\boxed{G \text{ is chordal} \Longleftrightarrow \Delta_2^t(G) \text{ vertex decomposable}}$

Fröberg, Eagon-Reiner

Chordal graphs :

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Theorem (BDMRSX) $\Delta_k^t(G)$ vertex decomposable $(\forall\, k \geq 2)$ $\Longleftarrow$ G is chordal

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Proof

- Link Lemma: $\mathrm{lk}_{\Delta_k^t(G)} v = \Delta_k^t(G \setminus v)$. (for any vertex v)

- Deletion Lemma: $\mathrm{del}_{\Delta_k^t(G)} v = \mathrm{st}_{\Delta_{k-1}^t(G \setminus v)} N(v)$. (for v simplicial in G)

- v is simplicial vertex in a graph G, then

$$\Delta_k^t(G) = \left(\Delta_k^t(G \setminus v) * v\right) \cup \mathrm{st}_{\Delta_{k-1}^t(G \setminus v)} N(v)$$

- Induction (Base case done by Eagon-Reiner) ∎

Theorem (BDMRSX) $\Delta_k^t(G)$ vertex decomposable $(\forall\, k \geq 2)$ $\Longleftarrow$ G is chordal

Moreover,

Theorem (BDMRSX) $\Delta_k^t(G)$ vertex decomposable $(\forall k \geq 2)$ $\Longleftarrow$ G is chordal

Moreover,

Theorem G is chordal with n vertices, c components, $k \geq 1$, then $\Delta_k^t(G)$ is homotopy equivalent to the wedge of $\binom{c-1}{k-1}$ spheres of dimension $n-k-1$ for $c \geq k$.
If $c < k$, $\Delta_k^t(G)$ is either contractible or void.

Corollary $G =$ tree, then $\Delta_k^t(G)$ is contractible (if $k \leq \alpha(G)$)

Total cut complexes are easy to work with

when we have simplicial vertices

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Prop. If v is an isolated vertex in G, then

$$\Delta_k^t(G) = \Delta_{k-1}^t(G \setminus v) \cup (\Delta_k^t(G \setminus v) * v).$$

Theorem If v is a simplicial vertex in graph G and $N(v) \neq \phi$, then

$$\Delta_k^t(G) \simeq_{\text{htpy}} \operatorname{susp}(\Delta_k^t(G \setminus v)) \quad (\text{if } \Delta_k^t(G \setminus v) \neq \phi).$$

Nonshellable total cut complexes

(homotomic equivalent to sphere of wrong dimension. Proof by discrete Morse matching.)

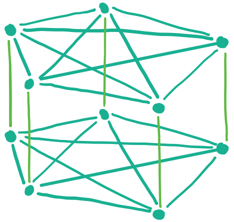
- $\Delta_k^t(C_n)$ is nonshellable for $n \geq 2k \geq 4$

  Proof: $\Delta_k^t(C_n) \simeq S^{n-2k}$.

- G_n = prism over complete graph $K_n = K_n \times K_2$

  $\Delta_2^t(G_n)$ is nonshellable.

  Proof: $\Delta_2^t(G_n) \simeq S^{2n-4}$ (but $\dim \Delta_2^t(G_n) = 2n-3$)

$G_5 =$ 

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Thank you!