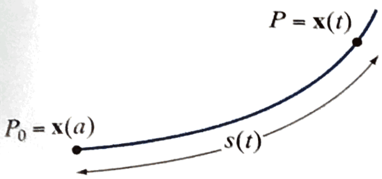

Figure 3.26 The arclength function.

Arclength function: (See Figure 3.26.)

$$s(t) = \int_a^t \|\mathbf{x}'(\tau)\|\, d\tau \quad \text{(basepoint is } P_0 = \mathbf{x}(a)\text{)}$$

Intrinsic quantities:

The moving frame:

$$\text{Unit tangent vector } \mathbf{T} = \frac{d\mathbf{x}}{ds} = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}.$$

$$\text{Principal normal vector } \mathbf{N} = \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\|} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}.$$

$$\text{Binormal vector } \mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

$$\text{Curvature } \kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{\|d\mathbf{T}/dt\|}{ds/dt}.$$

$$\text{Torsion } \tau \text{ is defined so that } \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

Additional formulas:

$$\mathbf{v}(t) = \dot{s}\,\mathbf{T} \quad (\dot{s} \text{ is speed}).$$

$$\mathbf{a}(t) = \ddot{s}\,\mathbf{T} + \kappa\dot{s}^2\,\mathbf{N} \quad (\ddot{s} \text{ is derivative of speed}).$$

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}.$$

Addendum: More About Torsion and the Frenet–Serret Formulas

We now derive formula (13), the basis for the definition of the torsion of a curve. That is, we show that the derivative of the binormal vector $\mathbf{B}$ (with respect to arclength) is always parallel to the principal normal $\mathbf{N}$ (i.e., that $d\mathbf{B}/ds$ is a scalar function times $\mathbf{N}$). The two main ingredients in our derivation are part 1 of Proposition 2.3 and the product rule.

We begin by noting that, since the ordered triple of vectors $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ forms a frame for $\mathbf{R}^3$, any moving vector, including $d\mathbf{B}/ds$, can be expressed as a **linear combination** of these vectors; that is, we must have

$$\frac{d\mathbf{B}}{ds} = a(s)\mathbf{T} + b(s)\mathbf{N} + c(s)\mathbf{B}, \tag{18}$$

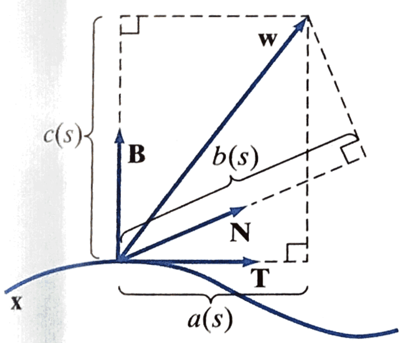

Figure 3.27 $\mathbf{w}(s) = a\mathbf{T} + b\mathbf{N} + c\mathbf{B}$.

where a, b, and c are appropriate scalar-valued functions. (Because $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are mutually perpendicular unit vectors, any (moving) vector $\mathbf{w}$ in $\mathbf{R}^3$ can be decomposed into its components with respect to $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ in much the same way that it can be decomposed into $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components—see Figure 3.27.) To find the particular values of the component functions a, b, and c, it turns out that we can solve for each function by applying appropriate dot products to equation (18). Specifically,

$$\begin{aligned}\frac{d\mathbf{B}}{ds}\cdot\mathbf{T} &= a(s)\mathbf{T}\cdot\mathbf{T}+b(s)\mathbf{N}\cdot\mathbf{T}+c(s)\mathbf{B}\cdot\mathbf{T}\\ &= a(s)\cdot 1 + b(s)\cdot 0 + c(s)\cdot 0\\ &= a(s),\end{aligned}$$

and, similarly,

$$\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}=b(s),\qquad \frac{d\mathbf{B}}{ds}\cdot\mathbf{B}=c(s).$$

From Proposition 1.7, $d\mathbf{B}/ds$ is perpendicular to $\mathbf{B}$ and, hence, c must be zero. To find a, we use an ingenious trick with the product rule: Because $\mathbf{T}\cdot\mathbf{B}=0$, it follows that $d/ds(\mathbf{T}\cdot\mathbf{B})=0$. Now, by the product rule,

$$\frac{d}{ds}(\mathbf{T}\cdot\mathbf{B})=\mathbf{T}\cdot\frac{d\mathbf{B}}{ds}+\frac{d\mathbf{T}}{ds}\cdot\mathbf{B}.$$

Consequently, $(d\mathbf{B}/ds)\cdot\mathbf{T}=-(d\mathbf{T}/ds)\cdot\mathbf{B}$. Thus,

$$\begin{aligned}a(s)=\frac{d\mathbf{B}}{ds}\cdot\mathbf{T} &= -\frac{d\mathbf{T}}{ds}\cdot\mathbf{B}\\ &= -\kappa\mathbf{N}\cdot\mathbf{B} \qquad \text{by formula (11),}\\ &= 0,\end{aligned}$$

and equation (18) reduces to

$$\frac{d\mathbf{B}}{ds}=b(s)\mathbf{N}.$$

No further reductions are possible, and we have proved that the derivative of $\mathbf{B}$ is parallel to $\mathbf{N}$. The torsion τ can, therefore, be defined by $\tau(s)=-b(s)$.

Formulas (11) and (13) gave us intrinsic expressions for $d\mathbf{T}/ds$ and $d\mathbf{B}/ds$, respectively. We can complete the set by finding an expression for $d\mathbf{N}/ds$. The method is the same as the one just used. Begin by writing

$$\frac{d\mathbf{N}}{ds}=a(s)\mathbf{T}+b(s)\mathbf{N}+c(s)\mathbf{B}, \tag{19}$$

where a, b, and c are suitable scalar functions. Taking the dot product of equation (19) with, in turn, $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$, yields the following:

$$a(s)=\frac{d\mathbf{N}}{ds}\cdot\mathbf{T},\qquad b(s)=\frac{d\mathbf{N}}{ds}\cdot\mathbf{N},\qquad c(s)=\frac{d\mathbf{N}}{ds}\cdot\mathbf{B}.$$

The "product rule trick" used here then reveals that

$$\begin{aligned}a(s)=\frac{d\mathbf{N}}{ds}\cdot\mathbf{T} &= -\mathbf{N}\cdot\frac{d\mathbf{T}}{ds}\\ &= -\mathbf{N}\cdot\kappa\mathbf{N} \qquad \text{by formula (11)}\\ &= -\kappa,\end{aligned}$$

and

$$\begin{aligned}c(s)=\frac{d\mathbf{N}}{ds}\cdot\mathbf{B} &= -\mathbf{N}\cdot\frac{d\mathbf{B}}{ds}\\ &= -\mathbf{N}\cdot(-\tau\mathbf{N}) \qquad \text{by formula (13)}\\ &= \tau.\end{aligned}$$

Moreover, we may differentiate the equation $\mathbf{N}\cdot\mathbf{N}=1$ to find

$$b(s)=\frac{d\mathbf{N}}{ds}\cdot\mathbf{N}=-\mathbf{N}\cdot\frac{d\mathbf{N}}{ds},$$

which implies that $b(s)$ is zero. Hence, equation (19) becomes

$$\frac{d\mathbf{N}}{ds}=-\kappa\mathbf{T}+\tau\mathbf{B}.$$

The formulas for $d\mathbf{T}/ds$, $d\mathbf{N}/ds$, and $d\mathbf{B}/ds$ are usually taken together as

$$\begin{cases}\mathbf{T}'(s)=\kappa\mathbf{N}\\ \mathbf{N}'(s)=-\kappa\mathbf{T}+\tau\mathbf{B}\\ \mathbf{B}'(s)=-\tau\mathbf{N}\end{cases}$$

and are known as the **Frenet–Serret formulas** for a curve in space. They are so named for Frédéric-Jean Frenet and Joseph Alfred Serret, who published them separately in 1852 and 1851, respectively. The Frenet–Serret formulas give a system of differential equations for a curve and are key to proving a result like Theorem 2.5. They are often written in matrix form, in which case, they have an especially appealing appearance, namely,

$$\begin{bmatrix}\mathbf{T}'\\ \mathbf{N}'\\ \mathbf{B}'\end{bmatrix}=\begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}.$$

3.2 Exercises

Calculate the length of each of the paths given in Exercises 1–6.

1. $\mathbf{x}(t)=(2t+1, 7-3t)$, $-1\le t\le 2$

2. $\mathbf{x}(t)=t^2\,\mathbf{i}+\frac{2}{3}(2t+1)^{3/2}\,\mathbf{j}$, $0\le t\le 4$

3. $\mathbf{x}(t)=(\cos 3t, \sin 3t, 2t^{3/2})$, $0\le t\le 2$

4. $\mathbf{x}(t)=7\mathbf{i}+t\,\mathbf{j}+t^2\mathbf{k}$, $1\le t\le 3$

5. $\mathbf{x}(t)=(t^3, 3t^2, 6t)$, $-1\le t\le 2$

6. $\mathbf{x}(t)=(\ln(\cos t), \cos t, \sin t)$, $\frac{\pi}{6}\le t\le\frac{\pi}{3}$

7. $\mathbf{x}(t)=(\ln t, t^2/2, \sqrt{2}t)$, $1\le t\le 4$

8. $\mathbf{x}(t)=(2t\cos t, 2t\sin t, 2\sqrt{2}t^2)$, $0\le t\le 3$

9. The path $\mathbf{x}(t)=(a\cos^3 t, a\sin^3 t)$, where a is a positive constant, traces a curve known as an **astroid** or a **hypocycloid of four cusps.** Sketch this curve and find its total length. (Be careful when you do this.)

10. If f is a continuously differentiable function, show how Definition 2.1 may be used to establish the formula

$$L=\int_a^b\sqrt{1+(f'(x))^2}\,dx$$

for the length of the curve $y=f(x)$ between $(a, f(a))$ and $(b, f(b))$.

11. Use Exercise 10 or Definition 2.1 (or both) to calculate the length of the line segment $y=mx+b$ between (x_0, y_0) and (x_1, y_1). Explain your result with an appropriate sketch.

12. (a) Calculate the length of the line segment determined by the path

$$\mathbf{x}(t)=(a_1t+b_1, a_2t+b_2)$$

as t varies from t_0 to t_1.

(b) Compare your result with that of Exercise 11.

(c) Now calculate the length of the line segment determined by the path $\mathbf{x}(t)=\mathbf{a}\,t+\mathbf{b}$ as t varies from t_0 to t_1.

13. This problem concerns the path $\mathbf{x}=|t-1|\,\mathbf{i}+|t|\,\mathbf{j}$, $-2\le t\le 2$.

(a) Sketch this path.

(b) The path fails to be of class C^1 but is piecewise C^1. Explain.

(c) Calculate the length of the path.

14. Consider the path $\mathbf{x}(t)=(e^{-t}\cos t, e^{-t}\sin t)$.