A counterexample on contractible transformations on graphs

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Motivation

People, in Topological Data Analisys, are interested in finding the form of a data cloud through its homology. But they handle an enormous amount of data, and it is necessary to reduce them without altering the homology to study the form.

- Frías-Armenta, M.E. . A counterexample on contractible transformations on graphs. Discrete Math. Volume 343, Issue 8, August 2020, 111915.
- Espinoza, J.F., Frías-Armenta, M.E., Hernández-Hernández, H.A. Collapsibility and homological properties of I-contractible transformations. Bol. Soc. Mat. Mex. 28, 42 (2022).
- Dochtermann A., Espinoza, J.F., Frías-Armenta, ME. Hernández-Hernández H.A., Minimal graphs for contractible and dismantlable properties, submitted to Discrete Math.

A better title for this talk,

"A counterexample on contractible transformations on graphs and obstructions to find an algorithm to reduce a graph without change its homology"

1. Ivashchenko A.V., Contractible transformations do not change the homology groups of graphs, Discrete Math. 126 (1) (1994) 159–170,
2. Ivashchenko A.V., Some properties of contractible transformations on graphs, Discrete Math. 133 (1) (1994) 139–145.

Let $G = (V(G), E(G)) \in \mathfrak{G}$ be a graph, and let $v \in V(G)$ be a vertex. We denote $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ and $N_G(v, w) := N_G(v) \cap N_G(w)$. In addition, by abuse of notation we identify the graph with its set of vertices.

The *clique complex* $\Delta(G)$ of the graph G is the (finite) simplicial complex with all complete subgraphs of G as simplices. The 1-skeleton of $\Delta(G)$ can be identified with G itself. A simplicial complex Δ is called a flag complex if there exists a graph G such that $\Delta = \Delta(G)$.
Some barycentric division (not necessarily the first one) of any simplicial complex is a clique complex of some graph: for example the dunce hat needs the second barycentric division.

In the first paper of Ivashchenko, the next family of graphs was defined, and its elements are called contractible graphs.

Definition

Let $\mathfrak{I} \subset \mathfrak{G}$ be the family of graphs defined by

1. The trivial graph $K(1)$ is in $\mathfrak{I}$.
2. Any graph of $\mathfrak{I}$ can be obtained from $K(1)$ by the following transformations.
   - (I1) Deleting of a vertex v. A vertex v of a graph G can be deleted if $N_G(v) \in \mathfrak{I}$.
   - (I2) Gluing of a vertex v. If a subgraph G_1 of the graph G is in $\mathfrak{I}$, then the vertex v can be glued to the graph G in such way that $N_G(v) = G_1$.
   - (I3) Deleting of an edge $\{v_1, v_2\}$. The edge $\{v_1, v_2\}$ of a graph G can be deleted if $N_G(v_1, v_2) \in \mathfrak{I}$.
   - (I4) Gluing of an edge $\{v_1, v_2\}$. Let two vertices v_1 and v_2 of a graph G be nonadjacent. The edge $\{v_1, v_2\}$ can be glued if $N_G(v_1, v_2) \in \mathfrak{I}$.

The transformations (*I1*)-(*I4*) were named by Ivashchenko as contractible transformations. He proved that contractible transformations do not change the homology groups of a graph, for any commutative group of coefficients A, so the elements of $\mathfrak{I}$ have trivial groups of A-homology.

Given a simplicial complex Δ, a couple of simplices $\sigma, \tau \in \Delta$ are called a *free face* if the following conditions are satisfied:

1. $\sigma \subsetneq \tau$,
2. τ is a maximal face of Δ with respect to inclusion,
3. and no other maximal face of Δ contains σ.

A *simplicial collapse* of Δ in $\tilde{\Delta}$ is obtained by removal of all simplices $\gamma \in \Delta$ such that $\sigma \subsetneq \gamma \subseteq \tau$, provided that (σ, τ) is a free face, we will write $\Delta \searrow \tilde{\Delta}$. Additionally, if $\dim(\tau) = \dim(\sigma) + 1$, then $\Delta \searrow \tilde{\Delta}$ is called an *elementary simplicial collapse*. It is not hard to see that any simplicial collapse can be realized by elementary ones.

If there are $\Delta_1, \Delta_2, \ldots, \Delta_n$ simplicial complexes such that $\Delta_1 \searrow \Delta_2 \searrow \cdots \searrow \Delta_n$, we say that Δ_1 is collapsible to Δ_n. We denote this also by $\Delta_1 \searrow \Delta_n$. In particular, if Δ is collapsible to $\Delta^0 = \{point\}$ we just say that Δ is collapsible.

Definition

Let $\mathfrak{C}$ denote the family of graphs $G \in \mathfrak{G}$ such that $\Delta(G)$ is collapsible. A graph $G \in \mathfrak{C}$ is called a collapsible graph.

For example, any complete graph $K(n)$ is collapsible: $\Delta(K(n)) \searrow \Delta^0$.

Definition

Let $\mathfrak{SIC} \subset \mathfrak{G}$ be the family of graphs defined by

1. The trivial graph $K(1)$ is in $\mathfrak{SIC}$.
2. Any graph of $\mathfrak{SIC}$ can be obtained from $K(1)$ by the following transformations.
   - (I2) Gluing of a vertex v. If a subgraph G_1 of the graph G is in $\mathfrak{SIC}$, then the vertex v can be glued to the graph G in such way that $N_G(v) = G_1$.
   - (I4) Gluing of an edge $\{v_1, v_2\}$. Let two vertices v_1 and v_2 of a graph G be nonadjacent. The edge $\{v_1, v_2\}$ can be glued if $N_G(v_1, v_2) \in \mathfrak{SIC}$.

If G belongs to $\mathfrak{SIC}$, then G is called a strong I-contractible graph.

The next theorem proves that every strong I-contractible graph is also a collapsible graph.

Theorem

If G is a strong I-contractible graph, then $\Delta(G)$ is a collapsible simplicial complex.

The before Theorem has as collorary that contractible deleting of a vertex does not change the homology group of a graph i.e.

Theorem

If G is a graph and v is a vertex in G such that $N_G(v) \in \mathfrak{I}$, then

$$H_*(\Delta(G)) \cong H_*(\Delta(G - v))$$

That was the main result of the first paper of Ivashchenko.

Axiom 3.4. Suppose that G is a contractible graph, and a vertex $v, v \in G$, is not adjacent to some vertices of G. Then there exists a nonadjacent vertex $u, u \in G$, such that the subgraph $O(vu)$ is contractible.

Figure: Axiom 3.4 [Second paper of Ivashchenko]

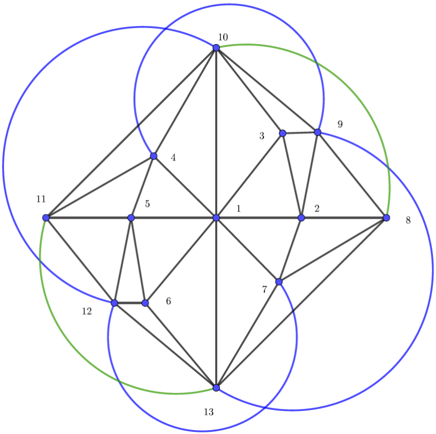

Figure: We can see in this figure, that the axiom of 3.4, of [Second paper of Ivashchenko], is false, we can not put any edge in vertex 1, as the axiom 3.4 in that paper states.

Remark

In the figure 2 we can see that axiom 3.4 of [Second paper of Ivashchenko] is false, So all the results of that paper are based in the axiom 3.4, and so the Theorems 3.5, 3.8, 3.9,3.10,3.13 and corollary 3.6 are clearly false, the figure 2 showed this directly. Ultimately, the proofs of Theorem 3.7, 3.11 and 3.12 were badly proven since theirs proofs were based on Axiom 3.4 or some of its consequences. We believe however, that a significant portion of these three results may be true.

In this talk we have mentioned three reductions, contractible transformations, strong contractible transformations, collapses. None of them change homology. There are more, for example strong vertices I-contractible (Only (I2) is permited) and k-dismantlings, for $k \in \mathbb{N}$.

k-dismantlings

For a graph G and vertex $v \in G$, we say that v is 0-*dismantlable* if its open neighborhood $N_G(v)$ is a cone. A graph G is 0-*dismantlable* (or simply *dismantlable*) if it can be reduced to a single vertex by successive deletions of 0-dismantlable vertices. Proceeding inductively, a vertex $v \in G$ is k-*dismantlable* if its open neighborhood $N_G(v)$ is $(k-1)$-dismantlable, and a graph G is k-*dismantlable* if it can be reduced to a single vertex by successive deletions of k-dismantlable vertices.

Which are the minimal graphs that make different that concepts?

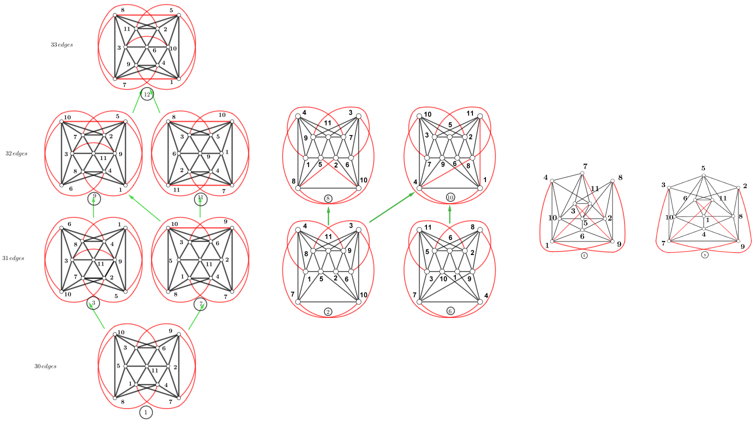

Figure: (Theorem) The minimal strong *I*-contractible graphs that they are not vertices strong *I*-Contractible.

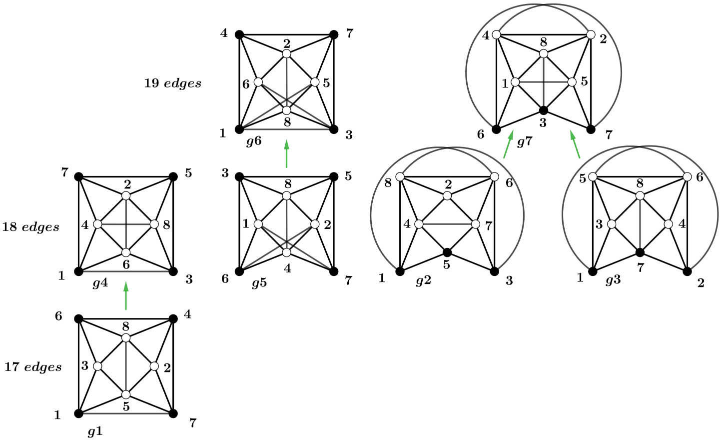

Figure: (Theorem) The minimal 1-dismantlable graphs that they are not 0-dismantlable

The question, ”what is the better way to reduce a graph for reduced the time of calculated homology?” Remains. The easier algorithm is only delete vertices or edges I-contractible, but not ever we can reduce to the minimal the graphs, for example:

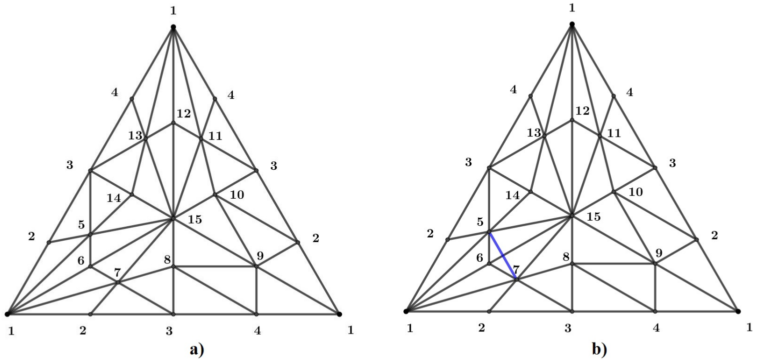

Figure: The left graph has 15 vertices (Dunce Hat), it is *I*-contractible and it is not strong *I*-contractible, Conjecture it is the minimal graph with this properties. (Conjecture the right graph is the minimal that the order of erase edges or vertices matter.)

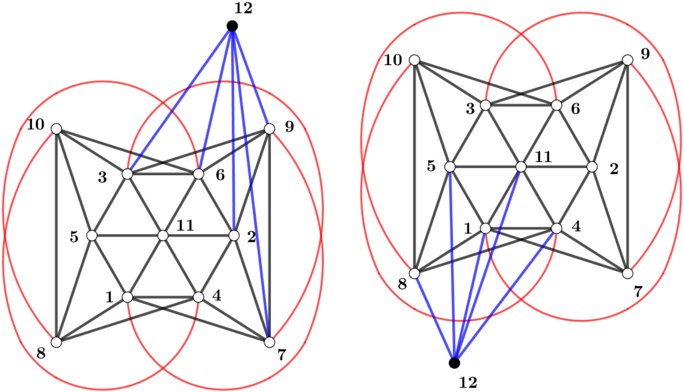

Figure: (Theorem) Minimal graphs where the order of erase vertices matter.

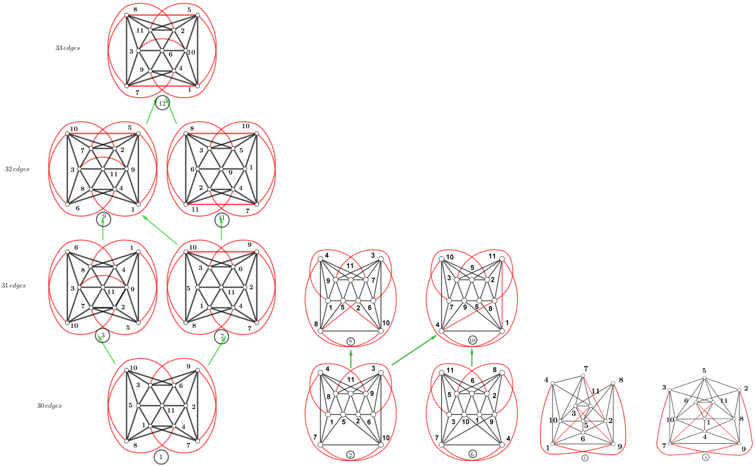

Figure: (Theorem) The minimal no-evasive graphs.

Thank you