Math 724, Fall 2013
Take-Home Test #3 Solutions

Problem #1 Recall that in the game of bridge, each player is dealt a hand of 13 cards from a standard deck of 52 cards.

(#1a) [10 pts] Bridge players call a hand "balanced" if it contains at least 2 cards in every suit, and no more than 8 cards in any two suits. How many possible bridge hands are balanced?

Solution: $4\binom{13}{4}\binom{13}{3}^3 + 12\binom{13}{4}^2\binom{13}{3}\binom{13}{2} + 12\binom{13}{5}\binom{13}{3}^2\binom{13}{2} = 302292822832.$

(#1b) [10 pts] How many possible bridge hands contain at least one card of every suit?

Solution: Let $X = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$. For $S \subseteq X$, let V_S be the set of hands that contain voids in the suits of S. We are trying to calculate the cardinality of

$$\{\text{all hands}\} \setminus \bigcup_{x \in X} V_{\{x\}}.$$

By inclusion/exclusion, this number is

$$\begin{aligned}\sum_{S \subseteq X} (-1)^{|S|} |V_S| &= \sum_{s=0}^{4} (-1)^s \binom{4}{s}\binom{13s}{13} \\ &= \texttt{sum((-1)\^{}s * binomial(4,s) * binomial(13*s,13) for s in range(4))} \\ &= 602586261420.\end{aligned}$$

Problem #2 [10 pts] Give a combinatorial proof of the identity

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

In other words, describe a set that is counted in two different ways by the left- and right-hand sides of the desired equation.

Solution: I want to choose a committee of k people, including a committee chair, from a set of n people. One way to do this is to first choose the chair (n possibilities) and then choose any subset of the remaining $n-1$ people to form the rest of the committee (2^{n-1} possibilities.) Alternately, I could choose the entire committee first — if there are k people on the committee then there are $\binom{n}{k}$ possibilities — and then choose one of the k of them to serve as chair.

Alternate solution: The left-hand side equals $\sum_{S \subseteq [n]} |S|$, which equals $\sum_{x \in [n]} \#\{S \subseteq [n] :\ x \in S\}$. On the other hand, this second sum has n summands, each of which equals 2^{n-1}, because the subsets of $[n]$ containing x are precisely of the form $T \cup \{x\}$, where $T \subseteq [n] \setminus \{x\}$.

Problem #3 [20 pts] How many ways are there of making change for a three-dollar bill with pennies, nickels, dimes, and quarters that use at least one, but no more than ten, of each kind of coin?

Solution: The answer is the coefficient of x^{300} in $\left(\frac{x}{1-x}\right)\left(\frac{x^5}{1-x^5}\right)\left(\frac{x^{10}}{1-x^{10}}\right)\left(\frac{x^{25}}{1-x^{25}}\right)$.

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sage: Facto = lambda v: sum(x^(v*i) for i in range(1,11))
sage: Genf = expand(mul(Facto(v) for v in [1, 5, 10, 25]))
sage: Genf.coeff(x^300)
38
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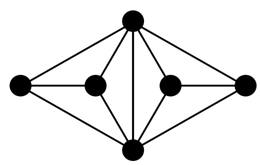
Problem #4 Let G be a graph in which vertices $v_1, \ldots, v_s$ form a clique. Let H be a graph constructed from G by creating a new vertex x and edges $xv_1, \ldots, xv_s$.

(#4a) [10 pts] Give a formula for the chromatic polynomial $\chi(H,k)$ of H in terms of $\chi(G,k)$.

(#4b) [10 pts] Calculate the chromatic polynomial of the graph shown below.



Solution: (a) In any proper k-coloring of G, the vertices $v_1, \ldots, v_s$ must receive different colors (precisely because they form a clique). So there are $k-s$ possible colors that can be assigned to x. Therefore $\boxed{\chi(H;k) = (k-s)\chi(G;k).}$

(b) First color the four vertices on the left — they form a K_4 so there are $k(k-1)(k-2)(k-3)$ possibilities. Then attach the vertices on the right, to a 2-clique and 3-clique respectively. It follows that the chromatic polynomial of the graph in the figure is $\boxed{k(k-1)(k-2)^2(k-3)^2.}$

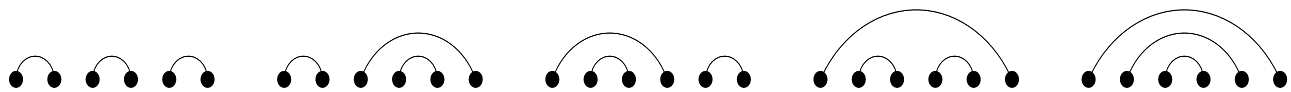
Problem #5 [10 pts] Give a combinatorial interpretation for the coefficient of $x^n q^k$ in the power series expansion of the infinite product

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-qx^2}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-qx^4}\right)\left(\frac{1}{1-x^5}\right)\left(\frac{1}{1-qx^6}\right)\cdots.$$

In other words, describe a set of combinatorial objects whose cardinality is the coefficient of $x^n y^k$.

Solution: The number of partitions of n containing k even parts.

Problem #6 [20 pts] A *noncrossing matching of order* n consists of $2n$ points arranged in a line, together with n arcs linking the points in pairs, such that no two arcs cross. For example, the noncrossing matchings of order 3 are shown below. Prove that the number of noncrossing matchings of order n is the Catalan number C_n.



Solution: Given a noncrossing matching M, construct a lattice path $P = f(M)$ from $(0,0)$ to $(2n,0)$ in which the i^{th} step is up or down according as i is the left or right end of an arc. Since every left end precedes the corresponding right end, P is a Catalan path (i.e., it never falls below the line $y = 0$).

Given a Catalan path P, label the steps $1, \dots, 2n$ and define a matching $M = g(P)$ by drawing an arc from each up-step U to the first subsequent down-step D at the same height as U. Note that

- there must be at least one such down-step after U;
- M really is a matching, i.e., different up-steps U, U' are paired with different down-steps. This is clear if U and U' are not at the same height. Suppose they are both at height h and (WLOG) that U' lies on the right of U. Then there must be at least one down-step at height h between U and U' (since the path had to descend to the left endpoint of U somehow), so U is certainly matched to something on the left of U' — in particular, something other than whatever U' is matched to.
- every up-step between U and D is matched to a down-step also between U and D, and every up-step after D is of course matched to a down-step after D. Therefore (U, D) does not cross any arc with $U' > U$. This is true of fall U; it follows that the entire matching is noncrossing.

The definition implies that $f \circ g$ is the identity, so these maps are bijections.

Alternate solution: Let M_n be the number of non-crossing n-matchings. Note that $M_0 = 1$ (by convention if you insist, although I think it is perfectly OK to say that there is exactly one way of matching zero points into pairs, and there certainly aren't any crossings!) and $M_1 = 1$.

Let X be a non-crossing n-matching with $n \geq 2$. Then X contains one arc a connecting $2n$ with some $i \in [2n-1]$. No other arc of X can have one endpoint in $[1, i-1]$ and the other in $[i+1, 2n-1]$, because if it did, it would cross q. Therefore, $X = a \cup X' \cup X''$, where X' and X'' are non-crossing matchings on $[1, i-1]$ and $[i+1, 2n-1]$ respectively. In particular, i must be odd, so $i = 2j-1$ for some $j \in [n]$ and

$$|[1, i-1]| = |[1, 2j-2] = 2(j-1) \qquad \text{and} \qquad [i+1, 2n-1] = [2j, 2n-1] = 2(n-j).$$

Thus X' and X'' are respectively a non-crossing $(j-1)$-matching and a non-crossing $(n-j)$-matching. Meanwhile, given data j, X', X'', we can construct a non-crossing matching X. Therefore

$$M_n = \sum_{j=1}^{n} M_{j-1} M_{n-j}$$

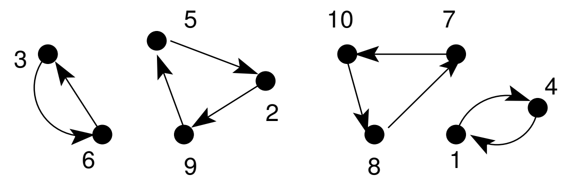
which is precisely (one form of) the Catalan recurrence. Together with the base case $M_0 = 1 = C_0$, it follows by induction that $M_n = C_n$.

Problem #7 [20 pts] The numbers $1, \dots, n$ are to be placed in the first quadrant in $\mathbb{R}^2$ according to the following rules. The squares must be left- and bottom-justified, and each row of tiles must be in increasing order left to right. Moreover, for every pair of rows of equal length, the leftmost number in the higher row must be greater than the leftmost number in the lower row. How many ways are there to arrange the numbers $1, \dots, 10$ in this way with exactly 4 nonempty rows?

Answer: These arrangements are in bijection with the set partitions of $[10]$ into exactly 4 blocks. Given a set partition, make a row labeled by the elements in each block, written in increasing order right to left.

Sort the rows top to bottom in increasing order of size, using the smallest element as a tiebreaker. So the answer is the Stirling number $S(10, 4) = 34105$.

Problem #8 For $n \in \mathbb{N}$, let $G(n)$ be the set of directed graphs on vertex set $[n]$ in which every component is either a pair of opposite edges, or a 3-cycle with edges oriented cyclically. (Equivalently, $G(n)$ is the set of permutations of $[n]$ in which every cycle has length 2 or 3.) For example, here is an element of $G(10)$.



Let $g(n) = |G(n)|$. By convention, we will set $g(0) = 1$, and $g(n) = 0$ for $n < 0$.

(#8a) [5 pts] What are the numbers $g(1), \ldots, g(5)$?

(#8b) [10 pts] Find a recurrence for $g(n)$. (Hint: Consider the cycle containing vertex n.)

(#8c) [10 pts] Let y be the exponential generating function $\sum_{n=0}^{\infty} g(x)x^n/n!$. Translate the recurrence you just found into a differential equation for y.

(#8d) [5 pts] Solve the differential equation to obtain a nice closed form for y.

(#8e) (Extra credit) Show that the e.g.f. for the set of directed graphs on vertex set $[n]$ in which every component is an oriented cycle of *odd* length is

$$\sqrt{\frac{1+x}{1-x}}.$$

Solution: (a) $g(1) = 0$; $g(2) = 1$; $g(3) = 2$ (two ways to orient a 3-cycle); $g(4) = 3$ (three ways to separate the four vertices into two 2-cycles; no choice about the orientation); $g(5) = 2\binom{5}{3} = 20$.

(b) Let C be the cycle containing n. If C has length 2 then there are $(n-1)g(n-2)$ possibilities. If C has length 3 then there are $(n-1)(n-2)g(n-3)$ possibilities. So

$$g_n = (n-1)g_{n-2} + (n-1)(n-2)g(n-3).$$

(c,d) Let $y = \sum_{n\geq 0} g_n \frac{x^n}{n!}$ be the e.g.f. Observe that $y' = dy/dx$ is given by

$$y' = \sum_{n\geq 1} g_n \frac{x^{n-1}}{(n-1)!}.$$

To obtain an equation for the e.g.f., multiply both sides of the recurrence by $nx^n/n!$ and sum for all $n \geq 3$:

$$\begin{aligned}
\sum_{n\geq 3} ng(n)\frac{x^n}{n!} &= \sum_{n\geq 3}\left(n(n-1)g(n-2) + n(n-1)(n-2)g(n-3)\right)\frac{x^n}{n!} \\
x\sum_{n\geq 3} g(n)\frac{x^{n-1}}{(n-1)!} &= x^2\sum_{n\geq 3} g(n-2)\frac{x^{n-2}}{(n-2)!} \;+\; x^3\sum_{n\geq 3} g(n-3)\frac{x^{n-3}}{(n-3)!} \\
x(y' - xg(2) - g(1)) &= x^2(y - g(0)) + x^3 y \\
y' - x &= x(y-1) + x^2 y \;=\; xy - x + x^2 y \\
y' &= xy + x^2 y \\
y'/y &= x + x^2 \\
\int y'/y\, dx &= \int (x + x^2)\, dx \\
\ln y &= \frac{x^2}{2} + \frac{x^3}{3} + C
\end{aligned}$$

and so $y = \exp\left(\frac{x^2}{2} + \frac{x^3}{3} + C\right)$. Setting $x = 0$ gives $g(0) = 1 = e^C$ and so $C = 0$, so the solution is

$$\boxed{y = \exp\left(\frac{x^2}{2} + \frac{x^3}{3}\right).}$$

Extra credit: Even more generally, let S be any set of positive integers and let $f(n, S)$ be the number of digraphs on $[n]$ such that every opponent is a cycle whose length is a number in S. The e.g.f. is

$$\phi(S, x) := \sum_{n\geq 0} f(n, S)\frac{x^n}{n!} = \exp\left(\sum_{s\in S}\frac{x^s}{s}\right).$$

For example, if $\mathcal{O}$ is the set of all odd numbers, then

$$\phi(\mathcal{O}, x) = \exp\underbrace{\left(x + x^3/3 + x^5/5 + \cdots\right)}_{z}$$

To obtain a closed form for z, observe that

$$z'' = 2x + 4x^3 + 6x^5 + \cdots = 3x\sum_{n\geq 0} n(x^2)^n = \frac{2x}{(1-x^2)^2}.$$

Integrating twice with respect to x, we obtain

$$z' = \frac{1}{1-x^2}, \qquad z = \log\sqrt{\frac{1+x}{1-x}}$$

so

$$\phi(\mathcal{O}, x) = \left(\frac{1+x}{1-x}\right)^{1/2}.$$

As another example, if $\mathcal{E}$ is the set of all odd numbers, then

$$\phi(\mathcal{E}, x) = \left(\frac{1}{1-x^2}\right)^{1/2}$$

(proof left to the reader).