The Augmented External Activity Complex of a Matroid

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Matroids

Definition

A matroid *independence system*, $\mathcal{I} \subset 2^{[n]}$, with ground set $[n]$ satisfies the following axioms:

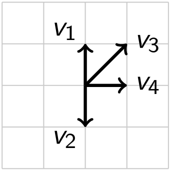
(I0) $\emptyset \in \mathcal{I}$

(I1) If $J \in \mathcal{I}$ and $I \subset J \implies I \in \mathcal{I}$

(I2) (Donation) If $I, J \in \mathcal{I}$ and $|I| < |J| \implies$ there exists $j \in J \setminus I$ so that $I \cup \{j\} \in \mathcal{I}$.

- A maximal independent set is called a *basis*.
- All bases have size, $r = \mathit{rank}$.

Example



$$\mathcal{I} = \left\{ \begin{array}{ccc} \emptyset, & 1, & 34, \\ & 2, & 24, \\ & 3, & 14, \\ & 4, & 23, \\ & & 13 \end{array} \right\}$$

Simplicial complexes

Definition

A *simplicial complex* $\Delta \subset 2^{[n]}$ with *vertex set* $[n]$ satisfies the following axioms:

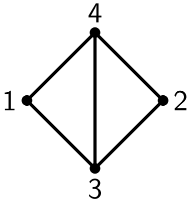
(I0) $\emptyset \in \Delta$

(I1) If $J \in \Delta$ and $I \subset J \implies I \in \Delta$.

- The elements (maximal) of Δ are called *faces* (*facets*).
- If all facets have the same size we call our complex *pure*.

Example

$\Delta = \langle 34, 24, 14, 23, 13 \rangle$



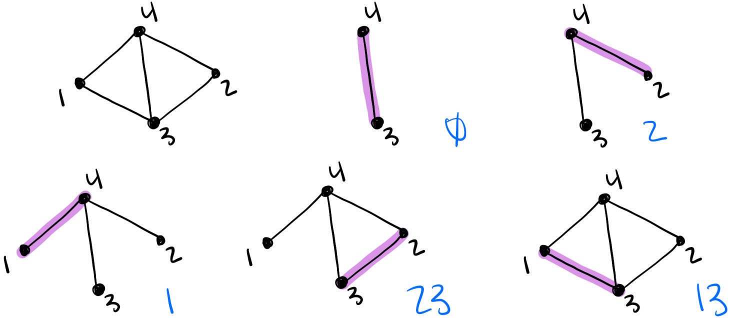
Shellability

Definition

A (pure) simplicial complex, $\Delta = \langle F_1, F_2, \ldots, F_s \rangle$, is *shellable* if we can order the facets in a *shelling order*.
For every $j \geq 2$, the facet F_j introduces a ***unique*** *minimal new face* to the complex generated by $F_1, F_2, \ldots, F_{j-1}$.

Example

$\Delta = \langle 34, 24, 14, 23, 13 \rangle$



Matroids, simplicial complexes & shellability

Theorem (Provan-Billera 1980)

For a matroid M, the independent set complex, $\mathcal{I}$, is shellable. Any lexicographic ordering of the bases gives a shelling.

Theorem (Björner 1980)

The order complex of the lattice of flats, $\Delta(\mathcal{F}(M))$, is shellable.

Theorem (B-K-R-R-S-S-S-T-Z-Z 2022)

The augmented Bergman complex is shellable.

Theorem (Ardila-Castillo-Samper 2016)

The external activity complex, $\underline{\Delta}_M$, is shellable. Any linear extension of Las Vergnas's internal/external order gives a shelling.

Matroid activities

Our ground set $1 < 2 < \cdots < n$ is equipped with the natural order.

Definition (Las Vergnas 2001)

Let B be a basis with $e \notin B$. We say that e is *externally active* iff

$$\text{for any } b \in B \text{ with } B \cup e \setminus b \text{ is a basis then } e > b.$$

Otherwise, e is *externally passive*. *Internal activity* is obtained by *matroid duality* and external activity.

Example

B	$EA(B)$	$EP(B)$	$IA(B)$	$IP(B)$
34	$\emptyset$	12	34	$\emptyset$
24	$\emptyset$	13	4	2
14	2	3	4	1
23	4	1	$\emptyset$	23
13	24	$\emptyset$	$\emptyset$	13

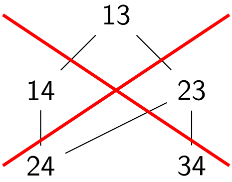
Las Vergnas's active orders (for matroid bases)

Definition (Las Vergnas 2001)

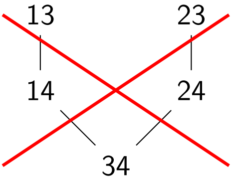
Suppose that A and B are bases.

$$A \leq_{int/ext} B :\Longleftrightarrow IP(A) \cup EA(A) \subset IP(B) \cup EA(B)$$

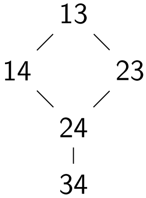
Example



$\mathcal{B}, \leq_{ext}$



$\mathcal{B}, \leq_{int}$



$\mathcal{B}, \leq_{int/ext}$

The external activity complex

Definition (Ardila-Boocher 2016)

The *external activity complex of a matroid*, $\underline{\Delta}_M$, is a simplicial complex generated by the facets:

$$B \mapsto F(B) := \{x_i : i \in B \cup EP(B)\} \cup \{z_j : j \in B \cup EA(B)\}$$

Example

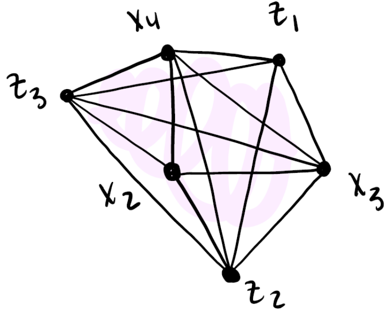
B	$EA(B)$	$EP(B)$
34	$\emptyset$	12
24	$\emptyset$	13
14	2	3
23	4	1
13	24	$\emptyset$

$$F(34) = x_1x_2x_3x_4z_3z_4$$
$$F(24) = x_1x_2x_3x_4z_2z_4$$
$$F(14) = x_1x_3x_4z_1z_2z_4$$
$$F(23) = x_1x_2x_3z_2z_3z_4$$
$$F(13) = x_1x_3z_1z_2z_3z_4$$

Notice: x_1 and z_4 belong to every facet.

The external activity complex (shelling orders)

Shelling orders of $\underline{\Delta}_M$ (Ardila-Castillo-Samper 2016): Take any linear extension of $\leq_{int/ext}$. The minimal new faces (restriction sets) record internal passivity.



B :	34	24	14	23	13
$F'(B)$:	$x_2x_3x_4z_3$	$x_2x_3x_4z_2$	$x_3x_4z_1z_2$	$x_2x_3z_2z_3$	$x_3z_1z_2z_3$
R(B):	$\emptyset$	z_2	z_1	z_2z_3	z_1z_3

Observation: For shelling orders of $\underline{\Delta}_M$, linear extensions of $\leq_{int/ext}$ cannot be relaxed.

Question: Is $\underline{\Delta}_M$ contained in a (shellable) simplicial complex that incorporates both external and internal activities?

Answer: Yes.

Las Vergnas's active orders (for independent sets)

Theorem (Crapo 1969)

Every independent set I can be written uniquely as $I = B \setminus Y$ where B is a basis and $Y \subset IA(B)$.

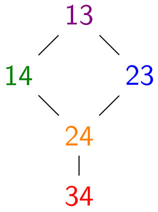
Example

34 has $IA(34) = 34$	24 has $IA(24) = 4$	14 has $IA(14) = 4$
$34 \setminus \emptyset = 34$	$24 \setminus \emptyset = 24$	$14 \setminus \emptyset = 14$
$34 \setminus 3 = 4$	$24 \setminus 4 = 2$	$34 \setminus 4 = 1$
$34 \setminus 4 = 3$		
$34 \setminus 34 = \emptyset$		

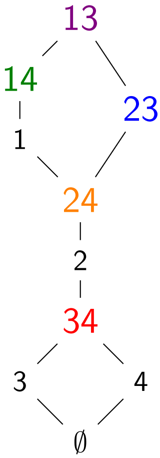
23 has $IA(23) = \emptyset$	13 has $IA(13) = \emptyset$
$23 \setminus \emptyset = 23$	$13 \setminus \emptyset = 13$

Las Vergnas's active orders (for independent sets)

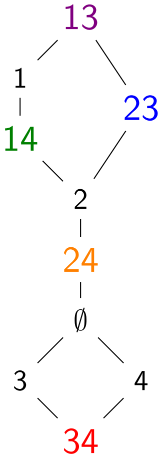
Example



$\mathcal{B}, \leq_{int/ext}$



$\mathcal{I}, \leq_{int/ext}$



$\mathcal{I}, \leq'_{int/ext}$

4. The augmented external activity complex

Definition (Berget-M. 2024)

The *augmented external activity complex* of a matroid is a simplicial complex generated by the facets,

$$I \mapsto F(I) := \{x_i : i \in I \cup EP(I)\} \cup \{y_i : i \in Y\} \cup \{z_j : j \in I \cup EA(I)\}$$

for every independent set $I = B \setminus Y$ and $Y \subset IA(B)$ is unique.

Example

Recall $IA(14) = 4$. We have $14 = 14 \setminus \emptyset$. Also, $1 = 14 \setminus 4$.

$$F(14) = x_1 x_3 x_4 z_1 z_2 z_4$$
$$F(1) = x_1 x_3 x_4 y_4 z_1 z_2$$

4. The augmented external activity complex

Theorem (Berget-M. 2024)

i. *The augmented external activity complex, Δ_M, contains $\underline{\Delta}_M$ as a subcomplex.*

ii. Δ_M *is shellable.*

   a. *Any linear extension of $\leq_{int/ext}$ on the independent sets gives a shelling.*
      - *The restriction set of $F(I)$ is $R(I) = z_I$.*
      - *The h-polynomial of Δ_M is the f-polynomial of $\mathcal{I}(M)$.*

   b. *Any linear extension of $\leq'_{int/ext}$ on the independent sets gives a shelling.*
      - *Let $I = B \setminus Y$. The restriction set of $F(I)$ is $R(I) = y_Y z_{IP(B)}$.*

Thank you!