Math 724, Fall 2013
Homework #7

Instructions: Write up your solutions in LaTeX and hand in a hard copy in class on **Monday, December 9.** Collaboration is allowed (and in fact encouraged), but each student must write up his or her solutions independently and acknowledge all collaborators.

Problem #1 Bogart, Chapter 5, Supplementary Problem #2: There are m students attending a seminar in a room with n seats. The seminar is a long one, and in the middle the group takes a break. In how many ways may the students return to the room and sit down so that nobody is in the same seat as before?

For a set S of students, let $i = |S|$ and let A_S be the set of reseatings so that everyone in S (and possibly others) gets his/her own seat. An element of A_S is given by a way to seat the $m-i$ students not in S in the remaining $n-i$ other seats, so $|A_S| = (n-i)!/(n-m)!$. By inclusion, the answer is

$$\boxed{\sum_{i=0}^{m}(-1)^i\binom{m}{i}\frac{(n-i)!}{(n-m)!} \quad = \quad \sum_{i=0}^{m}(-1)^i\binom{m}{i}(n-i)^{\underline{m-i}}.}$$

Problem #2 Bogart, Chapter 5, Supplementary Problem #5. Suppose that n children join hands in a circle for a game at nursery school. The game involves everyone falling down (and letting go). In how many ways may they join hands in a circle again so that nobody has the same person immediately to the right both times the group joins hands?

Label the children $1, \ldots, n$. For a set C of children, let S_C be the set of rearrangements in which each child in C has the same right-hand neighbor (as though they had never let go). Then $|S_C|$ is the number of circular rearrangements of $n-|C|$ objects (where the "objects" are connected chains of children), so $|S_C| = (n-i-1)!$ where $i = |C|$ (unless $C = [n]$, in which case $|S_C| = 1$ rather than "$(-1)!$"). By inclusion/exclusion, the answer is

$$\boxed{(-1)^n + \sum_{i=0}^{n-1}(-1)^i\binom{n}{i}(n-i-1)!.}$$

Problem #3 Bogart, Chapter 5, Supplementary Problem #8. Suppose we take two graphs G_1 and G_2 with disjoint vertex sets, we choose one vertex on each graph, and connect these two vertices by an edge e to get a graph G_{12}. How does the chromatic polynomial of G_{12} relate to those of G_1 and G_2?

Call the vertices in question v_1 and v_2. Given a proper k-coloring of G_1 and a proper k-coloring of G_2, they together comprise a proper k-coloring of G_{12} iff v_1 and v_2 are assigned different colors. This will happen exactly $(k-1)/k$ of the time (since any particular color is equally likely to be assigned to any vertex). Therefore

$$\boxed{\chi(G_{12}, k) = \frac{k-1}{k}\chi(G_1, k)\chi(G_2, k).}$$

Problem #4 Let C_n be the cycle graph of length n — that is, the undirected graph with vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$. Find a formula for the chromatic polynomial of C_n.

Recall the inclusion/exclusion formula for the chromatic polynomial of a graph $G = (V, E)$:

$$\chi(G, k) = \sum_{A \subseteq E}(-1)^{|A|}k^{c(A)}$$

where $c(A)$ means the number of components of the graph (V, A). If $G = C_n$, then $c(A) = n - |A|$ for all $A \subsetneq E$ (for then A is acyclic) and $c(E) = 1$. So the formula gives

$$\begin{aligned} \chi(C_n, k) &= \left(\sum_{a=0}^{n-1} \binom{n}{a} (-1)^a k^{n-a} \right) + (-1)^n k \\ &= \left(\sum_{a=0}^{n} \binom{n}{a} (-1)^a k^{n-a} \right) + (-1)^n (k-1) \\ &= \boxed{(k-1)^n + (-1)^n (k-1)} \qquad \text{by the binomial theorem.} \end{aligned}$$

Problem #5 Let Y_n be the graph K_n with one edge removed. Find a formula for the chromatic polynomial of Y_n.

Let the missing edge be e. Deletion-contraction says that $\chi(K_n, e) = \chi(Y, e) - \chi(K_{n-1}, e)$. (The graph K_n/e by contracting an edge is not simple, but when calculating the chromatic polynomial we can ignore multiple edges, and the underlying simple graph of K_n/e is K_{n-1}. Therefore

$$\begin{aligned} \chi(Y, e) &= \chi(K_n, e) + \chi(K_{n-1}, e) = k(k-1) \cdots (k-n+1) + k(k-1) \cdots (k-n+2) \\ &= \boxed{\big(k(k-1) \cdots (k-n+3)\big)(k-n+2)^2.} \end{aligned}$$

Problem #6 . . . Let $A(G)$ denote the set of acyclic orientations of G, and let $\alpha(G) = |A(G)|$. (a) What is $\alpha(G)$ if G. . .

- **is a forest?**
- **is K_n (the complete graph with n vertices)?**
- **is C_n (the cycle graph)?**

If G is a forest with e edges then $\alpha(G) = 2^e$, because every orientation is automatically acyclic. There are exactly two orientations of C_n that are non-acyclic, so $\alpha(C_n) = 2^n - 2$

Let $\mathcal{O}$ be an acyclic orientation of K_n. I claim that $\mathcal{O}$ has a unique **sink** (a vertex with no out-edges). To see this, construct a walk by starting at any vertex and following edges in the direction of the arrows. The condition that there is no sink means that the walk can be extended indefinitely — but that means that eventually some vertex will have to be repeated, which means we have traversed a cycle, which is impossible. On the other hand, there cannot be more than one sink (if there were, how could the edge between them be oriented?) Therefore, the orientations on K_n are in bijection with the total orderings of the vertices (fid the unique sink, delete it, find the unique sink of the K_{n-1} remaining, delete it, . . .) It follows that $\alpha(K_n) = n!$.

(b) Show that $\alpha(G) = \alpha(G - e) + \alpha(G/e)$ for any edge e.

Let $e = vw$. Consider the map $\pi : A(G) \to A(G - e)$ given by forgetting the orientation of e. Given any $\mathcal{O} \in A(G - e)$, we can add an edge $v \to w$ (resp., $w \to v$) and obtain an acyclic orientation of G provided that $\mathcal{O}$ contains no path from w to v (resp., v to w). But $\mathcal{O}$ cannot contain both such paths — otherwise it would have a cycle! Therefore it is always possible to extend $\mathcal{O}$ to an orientation of G, i.e., π is surjective.

Call an orientation $\mathcal{O} \in A(G - e)$ *one-way* or *two-way* depending on the cardinality of $\pi^{-1}(\mathcal{O})$. The two-way orientations correspond bijectively to the acyclic orientations of G/e (since if there is no path in either direction between v and w then identifying them will not produce a cycle). Therefore,

$$\begin{aligned} \alpha(G) &= \#\{\text{one-way } \mathcal{O} \in A(G - e)\} + 2\#\{\text{two-way } \mathcal{O} \in A(G - e)\} \\ &= |A(G - e)| + \#\{\text{two-way } \mathcal{O} \in A(G - e)\} \\ &= \alpha(G - e) + \alpha(G/e). \end{aligned}$$

(c) Find a formula for $\alpha(G)$ in terms of the chromatic polynomial $\chi_G(k)$.

The formula is $\alpha(G) = (-1)^n \chi_G(-1)$, where $n = n(G)$ means the number of vertices. We'll prove this by induction on the number of *edges*. If G has no edges, then $\alpha(G) = 1$ and $\chi(G,k) = k^n$, so

$$(-1)^n \chi_G(-1) = (-1^n)(-1^n) = 1.$$

Meanwhile, if G has a loop then $\chi(G,k)$ and $\alpha(G)$ are both zero. Finally, if e is a nonloop edge, then by induction we have

$$\begin{aligned}
\alpha(G) &= \alpha(G-e) + \alpha(G/e) && \text{(previous problem)}\\
&= (-1)^n \chi(G-e,-1) + (-1)^{n-1}\chi(G/e,-1) && \text{(induction; note that } n(G) = n(G-e) = n(G/e)-1)\\
&= (-1)^n \big(\chi(G-e,-1) - \chi(G/e,-1)\big)\\
&= (-1)^n \big(\chi(G,-1)\big) && \text{(by the chromatic recurrence).}
\end{aligned}$$

as desired.

Problem #7 Let $V = [n]$ and let G be a graph with vertex set V. The *chromatic symmetric function* $X(G)$ of G is the formal power series defined by

$$X(G) = \sum_f \prod_{i=1}^{n} x_{f(i)}$$

where the sum ranges over all proper colorings f.

(a,b) What is $X(G)$ if G has no edges or if $G = K_n$?

If G has n vertices and no edges, then every coloring is proper, so

$$X(G) = \left(\sum_{i=1}^{\infty} x_i\right)^n.$$

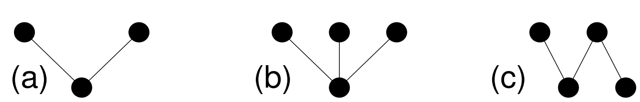
Meanwhile, if $G = K_n$ then the only colorings possible are those in which every vertex gets a distinct color. Given a set of n distinct colors, there are $n!$ ways to use them to properly color K_n, so

$$X(K_n) = n! \sum_{i_1 < \cdots < i_n} \prod_{j=1}^{n} x_{i_j}.$$

(c) Let k be a positive integer. Explain how to derive the number of proper k-colorings from $X(G)$.

Plug in $x_1 = x_2 = \cdots = x_k = 1$ and $x_m = 0$ for all $m > k$. The result is the number of colorings that use only the colors $1, \dots, k$ — that is, the number of k-colorings.

(d) Show that the two trees shown in (b) and (c) below do *not* have the same chromatic symmetric function.



It is possible to properly color tree (b) by coloring the top three vertices blue and the bottom one red. OTOH, every red/blue coloring of tree (c) uses each color twice. Therefore the coefficients of $x_1^2 x_2^2$ in the CSFs of the two trees are unequal. More specifically,

$$X(\text{Tree (b)}) = 24m_{1111} + 3m_{211} + m_{31}, \qquad X(\text{Tree (b)}) = 24m_{1111} + 3m_{211} + 2m_{22}$$

where m_λ is the monomial symmetric function corresponding to $\lambda \vdash n$ (that is, the sum of all monomials whose exponents are the parts of λ).

Extra credit. Let n be a positive integer. For $1 \leq i < j \leq n$, define a hyperplane $H_{ij} \subset \mathbb{R}^n$ by

$$H_{ij} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \colon x_i = x_j\}.$$

Now let G be a simple graph with vertex set $[n]$ and edge set E, and let

$$\mathcal{A}_G = \bigcup_{ij \in E} H_{ij}.$$

Thus $\mathcal{A}_G$ is a subset of $\mathbb{R}^n$ (it is called a *graphical hyperplane arrangement*). In terms of G, how many connected components does $\mathbb{R}^n \setminus \mathcal{A}_G$ have?

Each component of $\mathbb{R}^n \setminus \mathcal{A}_G$ is defined by a collection of inequalities: for each edge $ij \in E(G)$, either $x_i < x_j$ or $x_j < x_i$. Think of these inequalities as corresponding to orienting the edge ij as either $i \to j$ or $j \to i$ respectively. This orientation is acyclic, because a cycle would correspond to an infeasible list of inequalities. Furthermore, every acyclic orientation gives rise to at least one point in $\mathbb{R}^n \setminus \mathcal{A}_G$. (For example, assign the value 0 to every sink. Delete all the sinks and assign the value -1 to all sinks of the resulting graph. Delete all the sinks and assign the value $-2\ldots$) So there is a bijection between connected components of $\mathbb{R}^n \setminus \mathcal{A}_G$ and acyclic orientations!