Math 821 Problem Set #3
Posted: Friday 2/25/11
Due date: Monday 3/7/11

Problem #1 Let p, q be distinct points on S^2, and let X be the space obtained by gluing them together. Determine the homotopy type of X.

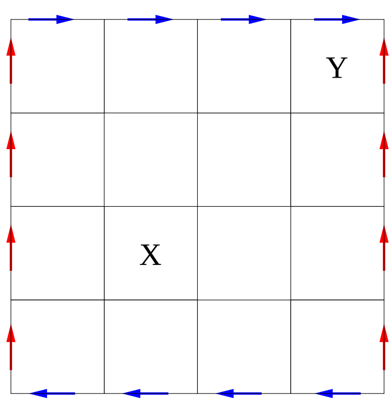
Solution: $X \simeq S^1 \vee S^2$. This is Example 0.11 in Hatcher (p.13), so I did not assign points for it. But it is helpful to be able to see the homotopy equivalence for yourself. Here is my description of it.

To see this, start with $S^1 \vee S^2$, which looks like a sphere glued to a circle at a point p. Let's put the circle inside the sphere (figure; left) Think of the circle as a closed loop from p to itself. Move the other one endpoint of the loop to another point $q \neq p$ (figure; center); this move is a homotopy equivalence by Proposition 0.18. Then collapse the resulting non-closed path to a point (figure; right); this is a homotopy equivalence by Prop. 0.17, and the resulting space is X.

Problem #2 For $k \geq 1$, let T_n denote the n-holed torus.

(#2a) Construct a cell complex structure on T_n.

Solution: Again, this is an example in Hatcher (p.5), so I did not assign points. There are lots of ways to impose a cell complex structure . Here's one, which stems from the observation that $T_n = T_1 \# T_{n-1}$, where $\#$ is the operation of *connected sum*: cut out a small disk from each of the operands, then glue their boundaries together.[1] This basically reduces the problem to finding a fine enough cell structure on the torus that is compatible with this operation. For example, start with the cell structure on the 1-hole torus shown below. Make n photocopies, delete the cells $X_1, Y_2, X_2, \ldots, X_{n-1}, Y_n$ (here, e.g., Y_2 means "the cell labeled Y in the 2nd photocopy") and identify the boundaries: $\partial X_1 = \partial Y_2$, $\ldots$, $\partial X_{n-1} = \partial Y_n$.



(#2b) Show that T_n is homotopy-equivalent to the cell complex $S^1 \times (\bigvee_{i=1}^n S^1)$ (that is, take n circles and wedge them together at a point to get a "bouquet", then take the product with another circle).

Solution: This is false. In fact, I think that for $n \geq 2$, the n-holed torus is not homotopy-equivalent to anything you can get using just spheres, direct products and wedges.

[1]This is not a purely topological operation, in the sense that you need to know the dimensions of two spaces in order to build their connected sum.

Problem #3 (Hatcher, p.20, #22) Let X be a finite graph lying in a half-plane $P \subset \mathbb{R}^3$ and intersecting the edge e of P in a subset of its vertices. Describe the homotopy type of the "surface of revolution" obtained by rotating X about e.

Solution: Let $R(X)$ be the surface of revolution thus obtained. First, we can assume that X is connected — if it has multiple connected components X_α, then $R(X)$ is the disjoint union of the $R(X_\alpha)$.

Second, observe that if a is an edge of X that has at least one endpoint not in e, then $R(X) \simeq R(X/a)$; this corresponds to deformation-retracting an annulus (specifically, the piece of $R(X)$ corresponding to a) onto a circle. (We just need to make sure that no other point hits the boundary during the contraction process. For example, if such a contraction produces an edge between two vertices on e, then that edge needs to be some kind of an arc rather than a line segment.)

By contracting enough edges, we will eventually end up with a graph Y all of whose vertices lie on e, and such that $R(Y) \simeq R(X)$. Let

$$\begin{aligned} n &= \text{total number of vertices of } X, \\ b &= \text{number of vertices of } X \text{ lying on } e, \\ m &= \text{number of edges of } X. \end{aligned}$$

Then Y has b vertices and $m - (n - b)$ edges.

Let T be a spanning tree of $R(Y)$. Then T has $b - 1$ edges, each of which gets rotated into a 2-sphere, and so $R(T) \simeq \bigvee_{b-1} S^2$. (This notation means "the wedge sum of $b - 1$ copies of S^2".) For each additional edge $a \notin T$, the revolution $R(a)$ is either a sphere attached to $R(T)$ at two points (if a is not a loop) or a sphere with two of its points identified, then attached to $R(T)$ (if a is a loop). In either case, the edge a contributes an additional $S^1 \vee S^2$ to the homotopy type. We conclude that

$$\begin{aligned} R(X) \simeq R(Y) &\simeq \bigvee_{b-1} S^2 \ \vee \ \bigvee_{m-(n-b)-(b-1)} (S^1 \vee S^2) \\ &= \bigvee_{b-1} S^2 \ \vee \ \bigvee_{m-n+1} (S^1 \vee S^2) \\ &= \bigvee_{m-n+b} S^2 \ \vee \ \bigvee_{m-n+1} S^1. \end{aligned}$$

(The graph pictured has $n = 9$, $b = 4$, $m = 15$, $m - n + b = 10$, $m - n + 1 = 7$.)

A comment: I should have required that each component of X have at least one vertex on e. (Otherwise, if for example X consists of one vertex off e and a loop attached to it, then $R(X)$ is a torus rather than a wedge of spheres.)

Problem #4 Recall that for a space X and base point $x \in X$, we have defined $\pi_1(X, x)$ to be the set of homotopy classes of x, x-paths on X — or equivalently of continuous functions $S^1 \to X$. Accordingly, define $\pi_0(X, x)$ to be the set of homotopy classes of continuous functions $S^0 \to X$.

Describe $\pi_0(X, x)$ intrinsically in terms of X.

Solution: Recall that S^0 consists of two points, with the discrete topology; call them a and b. The definition in the problem was a little imprecise; it should have said something like "continuous functions $f : S^0 \to X$ such that $f(a) = x$." (Since S^0 is discrete, the condition of continuity is vacuous.) The homotopy type of such a thing is determined by the path-connected component of X containing $f(b)$. Therefore, a reasonable interpretation for $\pi_0(X, x)$ is as the set of connected components. (There's no clear way to impose a topologically sensible group structure on this set.)

Problem #5 How generally does the lifting property (used in the proof of $\pi_1(S^1) \cong \mathbb{Z}$) hold? That is, suppose that we have continuous functions $f : X \to Y$ and $p : Z \twoheadrightarrow Y$, and we want to find $\tilde{f}$ such that the following diagram is commutative.

$$\begin{tikzcd}
 & Z \arrow[dd, "p"] \\
X \arrow[ur, "\tilde{f}"] \arrow[dr, "f"'] & \\
 & Y
\end{tikzcd}$$

What conditions on X, Y, Z, p guarantee that such a lift exists? (Whatever you come up with should include the case we needed for the calculation of $\pi_1(S^1)$, i.e., $X = I$, $Y = S^1$, $Z = \mathbb{R}$, $p(t) = e^{2\pi i t}$.)

Problem #6 (Hatcher, p.38, #2) Show that the change-of-basepoint homomorphism β_h (see p.28) depends only on the homotopy class of the path h.

Solution: Recall the setup: $x_0, x_1 \in X$; h is a x_0, x_1-path in X; and β_h is the map $\pi_1(X, x_1) \to \pi_1(X, x_0)$ given by $[f] \mapsto [h \cdot f \cdot \bar{h}]$.

Suppose that h_t is a homotopy of x_0, x_1-paths. Then $h_t \cdot f \cdot \overline{h_t}$ is a homotopy of x_0-loops. In particular, if $h \simeq h'$, then $\beta_h[f] \simeq \beta_{h'}[f]$.

Problem #7 (Hatcher, p.38, #7) Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on *one* of the boundary circles, but not by any homotopy f_t that is stationary on *both* boundary circles.

Solution: Visualize $S^1 \times I$ as a cylinder made of rubber, and f as a full twist of the cylinder. (Imagine opening a jar full of extremely old rubber cement.)

(i) Define $f_t : S^1 \times I \to S^1 \times I$ by
$$f_t(\theta, s) = (\theta + 2\pi t s, s).$$
This is evidently a homotopy (it is continuous in each of θ, s, t); f_0 is the identity map; and f_1 is the given map f. Moreover, f_t is stationary on the circle $S^1 \times \{0\}$, i.e., $f_t(\theta, 0) = (\theta_0)$.

(ii) Suppose that f_t is a homotopy that is stationary on both boundary circles. That is, $f_t : S^1 \times I \to S^1 \times I$ with
$$f_0(\theta, s) = (\theta, s), \qquad f_t(\theta, 0) = (\theta, 0),$$
$$f_1(\theta, s) = (\theta + 2\pi s, s), \qquad f_t(\theta, 1) = (\theta, 1).$$

We want to derive a contradiction. The idea is to draw a line down the side of the cylinder, so that twisting by f wraps the line around the outside in a spiral. Projecting these two paths from $S^1 \times I$ to S^1 will give two closed loops in S^1, one trivial and one that winds once around the circle — so they cannot be homotopic.

Here is a precise argument. Fix some basepoint $\theta_0 \in S^1$. Let g_t be the loop at θ_0 obtained from f_t by restricting its domain to $\{\theta_0\} \times I$, then projecting onto the S^1 factor. That is,
$$g_t(s) = p(f_t(\theta_0, s))$$

where p is the projection map $S^1 \times I \to S^1$. I claim that $\{g_t\}$ is a path homotopy. It certainly is a continuously varying family of functions $I \to S^1$, and

$$g_t(0) = p(f_t(\theta_0, 0)) = (\theta_0, 0) = \theta_0,$$
$$g_t(1) = p(f_t(\theta_0, 1)) = (\theta_0, 1) = \theta_0,$$

which says that each g_t defines a closed loop with basepoint θ_0.

We then have

$$\begin{aligned} g_0(s) &= p(f_0(\theta_0, s)) \\ &= p(\theta_0, s) \\ &= \theta_0, \\ g_1(s) &= p(f_1(\theta_0, s)) \\ &= p(\theta_0 + 2\pi s, s)) \\ &= \theta_0 + 2\pi s. \end{aligned}$$

But these paths have winding numbers 0 and 1 respectively (since they lift to $\widetilde{g_0}(s) = 0$ and $\widetilde{g_1}(s) = 2\pi s$ respectively; recall that the winding number of a loop can be computed from any lift). Therefore, by what we know about $\pi_1(S^1)$, they cannot be homotopic. This is a contradiction and says that no such homotopy f_t can exist.