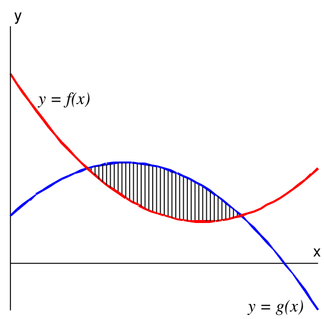
Mathematics 141 — Fall 2005 **Quiz #12 (12/2/05)**

For problems #1–3, let R be the region between the graphs of the functions $f(x) = x^2 - 5x + 8$ and $g(x) = -x^2 + 3x + 2$.

#1 [5 pts] Find the exact area of the region R.

The graphs of f and g are parabolas, opening up and down respectively, so the region R is as pictured below:



We need to know where the two graphs intersect. To find the intersection points, set $f(x) = g(x)$ and solve:

$$\begin{aligned} x^2 - 5x + 8 &= -x^2 + 3x + 2 \\ 2x^2 - 8x + 6 &= 0 \\ x^2 - 4x + 3 &= 0 \\ (x-3)(x-1) &= 0 \\ x &= 1, 3. \end{aligned}$$

Notice that $g(x) \geq f(x)$ on the interval $[1, 3]$. Therefore, the area of R is

$$\int_1^3 (g(x) - f(x))dx = \int_1^3 (-2x^2 + 8x - 6)dx = \frac{-2x^3}{3} + 4x^2 - 6x\bigg]_1^3 = (-18 + 36 - 18) - \left(\frac{-2}{3} + 4 - 6\right) = \boxed{\frac{8}{3}.}$$

#2 [5 pts] Set up, *but do not evaluate*, an integral that represents the volume of the solid obtained by rotating R about the x-axis.

Slice the solid perpendicular to the x-axis. The cross-section at x will be an annulus with outer radius $g(x)$ and inner radius $f(x)$. Therefore, the volume of the solid is

$$\int_1^3 \left[\pi g(x)^2 - \pi g(x)^2\right] \, dx \quad = \quad \boxed{\pi \int_1^3 \left[(-x^2 + 3x + 2)^2 - (x^2 - 5x + 8)^2\right] \, dx.}$$

#3 [5 pts] Set up, *but do not evaluate*, an integral that represents the volume of the solid obtained by rotating R about the y-axis.

It is most convenient to use the method of cylndrical shells. For $1 \leq x \leq 3$, the shell of radius x will have height $g(x) - f(x)$. Therefore, the volume of the solid is

$$\int_1^3 2\pi x(g(x) - f(x))\,dx \quad = \quad \boxed{\int_1^3 2\pi x(-2x^2 + 8x - 6)\,dx.}$$

It is possible to slice into annuli perpendicular to the y-axis, but the resulting expression is far more complicated. First, we must solve the equations $y = f(x)$ and $y = g(x)$ for x, so as to be able to express the outer and inner radii of each annulus in terms of y. Of course, the quadratic functions f and g are not invertible; we will have to make a choice of sign of a square root at some point.

Omitting the details of the algebra, the solutions of $y = f(x) = x^2 - 5x + 8$ are

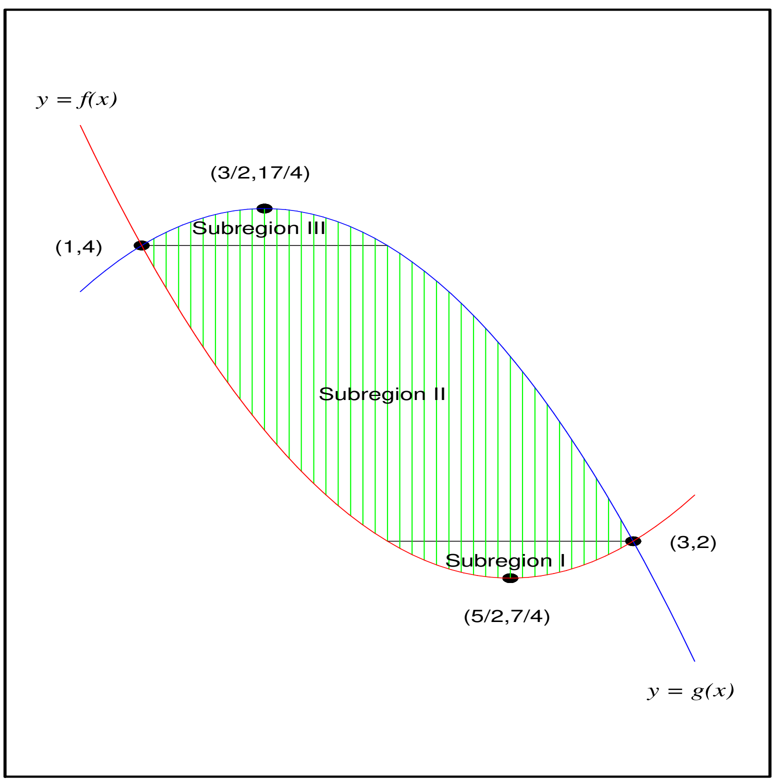
$$x = \frac{5 + \sqrt{4y - 7}}{2}, \qquad x = \frac{5 - \sqrt{4y - 7}}{2}$$

(the two solutions correspond respectively to the left and right halves of the parabola $y = f(x)$), and the solutions of $y = g(x) = -x^2 + 3x + 2$ are

$$x = \frac{3 - \sqrt{-4y + 17}}{2}, \qquad x = \frac{3 + \sqrt{-4y + 17}}{2}$$

(likewise).

Second, what are the limits of integration? Even though the points where the graphs of f and g meet are $(1, 4)$ and $(3, 2)$, there are relevant y-values that are *not* in the interval $[2, 4]$. The lowest point of R is the vertex of the parabola $y = f(x)$, namely $(5/2, 7/4)$, while the highest point of R is the vertex of the parabola $g(x)$, namely $(3/2, 17/4)$. So we have to be concerned about y-values in the interval $[7/4, 17/4]$. In fact, we need to divide R into three subregions, as shown.



Subregion	Range of y-values	Inner radius of annulus	Outer radius of annulus
I	$[\frac{7}{4}, 2]$	$\dfrac{5-\sqrt{4y-7}}{2}$	$\dfrac{5+\sqrt{4y-7}}{2}$
II	$[2, 4]$	$\dfrac{5-\sqrt{4y-7}}{2}$	$\dfrac{3+\sqrt{-4y+17}}{2}$
III	$[4, \frac{17}{4}]$	$\dfrac{3-\sqrt{-4y+17}}{2}$	$\dfrac{3+\sqrt{-4y+17}}{2}$

Therefore, the correct expression for the volume of the solid by slicing into annuli is as follows:

$$\int_{7/4}^{2} \pi \left(\left(\frac{5+\sqrt{4y-7}}{2} \right)^2 - \pi \left(\frac{5-\sqrt{4y-7}}{2} \right)^2 \right) dy$$
$$+ \int_{2}^{4} \pi \left(\left(\frac{3+\sqrt{-4y+17}}{2} \right)^2 - \pi \left(\frac{5-\sqrt{4y-7}}{2} \right)^2 \right) dy$$
$$+ \int_{4}^{17/4} \pi \left(\left(\frac{3+\sqrt{-4y+17}}{2} \right)^2 - \pi \left(\frac{3+\sqrt{-4y+17}}{2} \right)^2 \right) dy.$$

Wouldn't you rather use cylindrical shells?

#4 [5 pts] Find the length of the parametric curve defined by $x = e^t \cos t$, $y = e^t \sin t$, for $\pi \le t \le 2\pi$.

First, use the Product Rule to calculate

$$\frac{dx}{dt} = e^t(\cos t - \sin t), \qquad \frac{dy}{dt} = e^t(\sin t + \cos t).$$

Now use the arc length formula:

$$\begin{aligned}
\int_{\pi}^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt &= \int_{\pi}^{2\pi} \sqrt{(e^t(\cos t - \sin t))^2 + (e^t(\sin t + \cos t))^2}\, dt \\
&= \int_{\pi}^{2\pi} \sqrt{e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t) + e^{2t}(\sin^2 t + 2\sin t \cos t + \cos^2 t)}\, dt \\
&= \int_{\pi}^{2\pi} \sqrt{e^{2t}(2\cos^2 t + 2\sin^2 t)}\, dt \\
&= \int_{\pi}^{2\pi} \sqrt{2} e^t\, dt \\
&= \sqrt{2} e^t \Big]_{\pi}^{2\pi} = \boxed{\sqrt{2}\left(e^{2\pi} - e^{\pi}\right).}
\end{aligned}$$

[Bonus Problem] Let a and b be constants, and let S be the solid defined as follows: its base is the ellipse $x^2/a^2 + y^2/b^2 = 1$, and every cross-section perpendicular to the x-axis is a square. Find the volume of S.

The top half of the ellipse has the equation $y = b\sqrt{1 - x^2/a^2}$, and the "west pole" and "east pole" are respectively the points $(-a, 0)$ and $(a, 0)$. Therefore, for $-a \le x \le a$, the cross-section at x is a square with side length $2b\sqrt{1 - x^2/a^2}$, hence with area $A(x) = 4b^2(1 - x^2/a^2)$. Consequently, the volume of S is

$$\begin{aligned}
\int_{-a}^{a} A(x)\,dx &= \int_{-a}^{a} 4b^2(1 - x^2/a^2)\,dx \\
&= 4b^2 \left(x - \frac{x^3}{3a^2} \Big]_{-a}^{a} \right) \\
&= 4b^2 \left(\left(a - \frac{a}{3}\right) - \left(-a + \frac{a}{3}\right) \right) = \boxed{\frac{16ab^2}{3}}.
\end{aligned}$$