UPDOWN NUMBERS AND THE INITIAL MONOMIALS OF THE SLOPE VARIETY

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Abstract. Let I_n be the ideal of all algebraic relations on the slopes of the $\binom{n}{2}$ lines formed by placing n points in a plane and connecting each pair of points with a line. Under each of two natural term orders, the initial ideal $\mathrm{in}(I_n)$ is generated by monomials corresponding to permutations satisfying a certain pattern-avoidance condition. We show bijectively that these permutations are enumerated by the updown (or Euler) numbers, thereby obtaining a formula for the number of generators of $\mathrm{in}(I_n)$ in every degree.

The symbol $\mathbb{N}$ will denote the set of positive integers. For integers $m \leq n$, we put $[n] = \{1, 2, \ldots, n\}$ and $[m, n] = \{m, m+1, \ldots, n\}$. The set of all permutations of an integer set P will be denoted $\mathfrak{S}_P$, and the n^{th} symmetric group is $\mathfrak{S}_n$ $(= \mathfrak{S}_{[n]})$. We will write each permutation $w \in \mathfrak{S}_P$ as a word with $n = |P|$ digits, $w = w_1 \ldots w_n$, where $\{w_1, \ldots, w_n\} = P$. The symbol w_i^{-1} denotes the position of digit i in w; that is, $w_i^{-1} = j$ if and only if $w_j = i$. If necessary for clarity, we will separate the digits with commas. Concatenation will also be denoted with commas; for instance, if $w = 12$ and $w' = 34$, then $(w, w', 5) = 12345$. The *reversal* w^* of $w_1 w_2 \ldots w_{n-1} w_n$ is the word $w_n w_{n-1} \ldots w_2 w_1$. A *subword* of a permutation $w \in \mathfrak{S}_P$ is a word $w[i, j] = w_i w_{i+1} \cdots w_j$, where $[i, j] \subseteq [n]$. The subword is *proper* if $w[i, j] \neq w$. We write $w \approx w'$ if the digits of w are in the same relative order as those of w'; for instance, $58462 \approx 35241$.

Definition 1. Let $P \subset \mathbb{N}$ with $n = |P| \geq 2$. A permutation $w \in \mathfrak{S}_P$ is a *G-word* if it satisfies the two conditions

- **(G1)** $w_1 = \max(P)$ and $w_n = \max(P \setminus \{w_1\})$; and
- **(G2)** If $n \geq 4$, then $w_2 > w_{n-1}$.

It is an *R-word* if it satisfies the two conditions

- **(R1)** $w_1 = \max(P)$ and $w_n = \max(P \setminus \{w_1\})$; and
- **(R2)** If $n \geq 4$, then $w_2 < w_{n-1}$.

A G-word (resp., an R-word) is *primitive* if for every proper subword x of length ≥ 4, neither x nor x^* is a G-word (resp., an R-word). The set of all primitive G-words (resp., on $P \subset \mathbb{N}$, or on $[n]$) is denoted $\mathcal{G}$ (resp., $\mathcal{G}_P$, or $\mathcal{G}_n$). The sets $\mathcal{R}$, $\mathcal{R}_P$, $\mathcal{R}_n$ are defined similarly.

For example, the word 53124 is a G-word, but not a primitive one, because it contains the G-word $3124 = (4213)^*$ as a subword. The primitive G- and R-words of lengths up to 6 are as follows:

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$$
\begin{aligned}
\mathcal{G}_2 &= \{21\},\\
\mathcal{G}_3 &= \{312\},\\
\mathcal{G}_4 &= \{4213\},\\
\mathcal{G}_5 &= \{52314,\ 53214\},\\
\mathcal{G}_6 &= \{623415,\ 624315,\ 642315,\ 634215,\ 643215\},\\
\mathcal{R}_2 &= \{21\},\\
\mathcal{R}_3 &= \{312\},\\
\mathcal{R}_4 &= \{4123\},\\
\mathcal{R}_5 &= \{51324,\ 52134\},\\
\mathcal{R}_6 &= \{614235,\ 624135,\ 623145,\ 621435,\ 631245\}.
\end{aligned} \tag{1}
$$

Clearly, if $w \approx w'$, then either both w and w' are (primitive) G- (R-)words, or neither are; therefore, for all $P \subset \mathbb{N}$, the set $\mathcal{G}_P$ is determined by (and in bijection with) $\mathcal{G}_{|P|}$.

These families of permutations arose in [3] in the following way. Let $p_1 = (x_1, y_1), \ldots, p_n = (x_n, y_n)$ be points in $\mathbb{C}^2$ with distinct x-coordinates, let ℓ_{ij} be the unique line through p_i and p_j, and let $m_{ij} = (y_j - y_i)/(x_j - x_i) \in \mathbb{C}$ be the slope of ℓ_{ij}. Let $A = \mathbb{C}[m_{ij}]$, and let $I_n \subset A$ be the ideal of algebraic relations on the slopes m_{ij} that hold for all choices of the points p_i. Order the variables of A lexicographically by their subscripts: $m_{12} < m_{13} < \cdots < m_{1n} < m_{23} < \cdots$. Then [3, Theorem 4.3], with respect to graded lexicographic order on the monomials of A, the initial ideal of I_n is generated by the squarefree monomials $m_{w_1,w_2} m_{w_2 w_3} \cdots m_{w_{r-1} w_r}$, where $\{w_1, \ldots, w_r\} \subseteq [n]$, $r \geq 4$, and $w = w_1 w_2 \cdots w_r$ is a primitive G-word. Consequently, the number of degree-d generators of the initial ideal of I_n is

$$\binom{n}{d+1} |\mathcal{G}_{d+1}|. \tag{2}$$

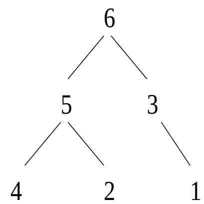
Similarly, under reverse lex order (rather than graded lex order) on A, the initial ideal of I_n is generated by the squarefree monomials corresponding to primitive R-words.

It was noted in [3, p. 134] that the first several values of the sequence $|\mathcal{G}_3|, |\mathcal{G}_4|, \ldots$ coincide with the *updown numbers* (or *Euler numbers*):

$$1, 1, 2, 5, 16, 61, 272, \ldots.$$

This is sequence A000111 in the Online Encyclopedia of Integer Sequences [4]. The updown numbers enumerate (among other things) the *decreasing 012-trees* [1, 2], which we now define.

Definition 2. A *decreasing 012-tree* is a rooted tree, with vertices labeled by distinct positive integers, such that (i) every vertex has either 0, 1, or 2 children; and (ii) $x < y$ whenever x is a descendant of y. The set of all decreasing 012-trees with vertex set P will be denoted $\mathcal{D}_P$. We will represent rooted trees by the recursive notation $T = [v, T_1, \ldots, T_n]$, where the T_i are the subtrees rooted at the children of v. Note that reordering the T_i in this notation does not change the tree T. For instance, $[6, [5, [4], [2]], [3, [1]]]$ represents the decreasing 012-tree shown below.



The purpose of this note is to verify that the updown numbers do indeed enumerate both primitive G-words and primitive R-words. Specifically:

Theorem 1. *Let $n \geq 2$. Then:*

(1) *The primitive G-words on $[n]$ are equinumerous with the decreasing 012-trees on vertex set $[n-2]$.*
(2) *The primitive R-words on $[n]$ are equinumerous with the decreasing 012-trees on vertex set $[n-2]$.*

To prove this theorem, we construct explicit bijections between G-words and decreasing 012-trees (Theorem 6) and between R-words and decreasing 012-trees (Theorem 7). Our constructions are of the same ilk as Donaghey's bijection [2] between decreasing 012-trees on $[n]$ and *updown permutations*, i.e., permutations $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ such that $w_1 < w_2 > w_3 < \cdots$.

Together with (2), Theorem 1 enumerates the generators of the graded-lex and reverse-lex initial ideals of I_n in each degree; for instance, I_6 is generated by $\binom{6}{4} \cdot 1 = 15$ cubic monomials, $\binom{6}{5} \cdot 2 = 12$ quartics, and $\binom{6}{6} \cdot 5 = 5$ quintics.

We start with three lemmas describing the recursive structure of G- and R-words.

Lemma 2. *Let $n \geq 3$, let $w \in \mathfrak{S}_n$, and let $k = w_{n-2}^{-1}$. Define words w_L, w_R by*

$$w_L = w_1 w_{k-1} w_{k-2} \cdots w_3 w_2 w_k, \qquad w_R = w_n w_{k+1} w_{k+2} \cdots w_{n-2} w_{n-1} w_k.$$

Then:

(1) *If w is a primitive G-word, then so are w_L and w_R.*
(2) *If w is a primitive R-word, then so are w_L and w_R.*

Proof. We will show that if w is a primitive G-word, then so is w_L; the other cases are all analogous. If $n = 3$, then the conclusion is trivial. Otherwise, we have $2 \leq k \leq n-2$ by definition of a G-word. If $k = 2$, then $w_L = w_1 w_2$, while if $k = 3$, then $w_L = w_1 w_3 w_2$; in both cases the conclusion follows by inspection. Now suppose that $k \geq 4$. Then the definition of k implies that w_L satisfies **(G1)**, and if $w_{k-1} < w_2$ then $w[1, k]$ is a G-word, contradicting the assumption that w is a primitive G-word. Therefore w_L is a G-word. Moreover, $w_L[i, j] \approx w[k+1-j, k+1-i]^*$ for every $[i, j] \subsetneq [k]$. No such subword of w is a G-word, so w_L is a primitive G-word as desired. $\square$

Lemma 3. *Let $n \geq 3$ and $x \in \mathfrak{S}_{n-1}$.*

(1) *If x is a primitive G-word, then so is*
$$w = (n,\ n-2,\ x_2,\ x_3,\ \ldots,\ x_{n-2},\ n-1).$$
(2) *If x is a primitive R-word, then so is*
$$w = (n,\ x_{n-2},\ x_{n-3},\ \ldots,\ x_2,\ n-2,\ n-1).$$

Proof. Suppose that x is a primitive G-word. By construction, w is a G-word in $\mathfrak{S}_n$. Let $w[i, j]$ be any proper subword of w. Then:

- If $i \geq 3$, or if $i = 2$ and $j < n$, then $w[i,j] = x[i-1, j-1]$ is not a G-word.
- If $i = 2$ and $j = n$, then $w_i < w_j$ but $w_{i+1} = x_2 > w_{j-1} = x_{n-2}$ (because x is a G-word), so $w[i,j]$ is not a G-word.
- If $i = 1$, then $j < n$, but then $w_{i+1} \geq w_j$, so $w[i,j]$ is not a G-word.

Therefore w is a primitive G-word. The proof of assertion (2) is similar. □

Lemma 4. *Let $n \geq 4$, and let P, Q be subsets of $[n]$ such that*

$$p = |P| \geq 3, \quad q = |Q| \geq 3, \quad P \cup Q = [n], \quad \text{and} \quad P \cap Q = \{n-2\}.$$

Let $x \in \mathfrak{S}_P$ and $y \in \mathfrak{S}_Q$ such that $x_p = n-2 = y_q$ and $x_{p-1} > y_{q-1}$. Then:

(1) *If x and y are primitive G-words, then so is*

$$w = (n,\ x_{p-1},\ \ldots,\ x_2,\ n-2,\ y_2,\ \ldots,\ y_{q-1},\ n-1).$$

(2) *If x and y are primitive R-words, then so is*

$$w = (n,\ y_{q-1},\ \ldots,\ y_2,\ n-2,\ x_2,\ \ldots,\ x_{p-1},\ n-1).$$

Proof. Suppose that x and y are primitive G-words. By construction, w is a G-word. We will show that no proper subword $w[i,j]$ of w is a G-word. Indeed:

- If $i < p < j$, then $w[i,j]$ cannot satisfy **(G1)**.
- If $i \geq p$, then either $[i,j] = [p,n]$, when $w_i = n-2 < w_j = n-1$ and $w_{i+1} = y_2 \geq w_{j-1} = y_{q-1}$ (because y is a G-word), or else $[i,j] \subsetneq [p,n]$, when $w[i,j] \approx y[i-p+1, j-p+1]$. In either case, $w[i,j]$ is not a G-word.
- Similarly, if $j \leq p$, then either $[i,j] = [1,p]$, when $w_i > w_j$ and $w_{i+1} = x_{p-1} \leq w_{j-1} = x_2$ (because x is a G-word), or else $[i,j] \subsetneq [1,p]$, when $w[i,j]^* \approx x[p-j+1, p-i+1]$. In either case, $w[i,j]$ is not a G-word.

Therefore, w is a primitive G-word. The proof of assertion (2) is similar. □

We pause to point out an elementary property about primitive G-words, which is not necessary for the sequel, but is easy to observe from (1) and can be proved by an argument similar to the preceding lemmas.

Proposition 5. *Let $n \geq 2$ and let $w \in \mathcal{G}_n$. Then $w_{n-1} = 1$.*

Proof. For $n \leq 6$, the result follows by inspection from (1). Otherwise, let $i = w_1^{-1}$. Note that $i \notin \{1, 2, n\}$ by the definition of G-word. Suppose that $i \neq n-1$ as well. By replacing w with w^* if necessary, we may assume that $w_{i-1} < w_{i+1}$. Let $A = \{j \in [1, i-2] \mid w_j > w_{i+1}\}$. In particular $\{1\} \subseteq A \subseteq [1, i-2]$. Let $k = \max(A)$. Then

$$\begin{aligned} w_k &= \max\{w_k, w_{k+1}, \ldots, w_{i+1}\}, \\ w_{i+1} &= \max\{w_{k+1}, \ldots, w_{i+1}\}, \\ w_{k+1} &> w_i = 1. \end{aligned}$$

So $w[k, i+1]$ is a G-word. It is a proper subword of w because $i+1 \leq n-1$, and its length is $i+2-k \geq i+2-(i-2) = 4$. Therefore $w \notin \mathcal{G}_n$. □

For the rest of the paper, let P be a finite subset of $\mathbb{N}$, let $n = |P|$, and let $m = \max(P)$. Define

$$\begin{aligned} \mathcal{G}'_P &= \{w \in \mathfrak{S}_P \mid (m+2, w, m+1) \in \mathcal{G}\}, \\ \mathcal{R}'_P &= \{w \in \mathfrak{S}_P \mid (m+2, w, m+1) \in \mathcal{R}\}. \end{aligned}$$

The elements of $\mathcal{G}'_P$ (resp., $\mathcal{R}'_P$) should be regarded as primitive G-words (resp., primitive R-words) on $P \cup \{m+1, m+2\}$, from which the first and last digits have been removed.

We now construct a bijection between $\mathcal{G}'_P$ and the decreasing 012-trees $\mathcal{D}_n$ on vertex set $[n]$. If $P = \emptyset$, then both these sets trivially have cardinality 1, so we assume henceforth that $P \neq \emptyset$. Since the cardinalities of $\mathcal{G}'_P$ and $\mathcal{D}_P$ depend only on $|P|$, this theorem is equivalent to the statement that the primitive G-words on $[n]$ are equinumerous with the decreasing 012-trees on vertex set $[n-2]$, which is the first assertion of Theorem 1.

Let $w \in \mathcal{G}'_P$ and $k = w_m^{-1}$. Note that if $n > 1$, then $w_n < w_1 \leq m$, so $k \neq n$. Define a decreasing 012-tree $\phi_G(w)$ recursively (using the notation of Definition 2) by

$$\phi_G(w) = \begin{cases} [m] & \text{if } n = 1; \\ [m, \phi_G(w[2,n])] & \text{if } n > 1 \text{ and } k = 1; \\ [m, \phi_G(w[1,k-1]^*), \phi_G(w[k+1,n])] & \text{if } n > 1 \text{ and } 2 \leq k \leq n-1. \end{cases}$$

Now, given $T \in \mathcal{D}_P$, recursively define a word $\psi_G(T) \in \mathfrak{S}_P$ as follows.

- If T consists of a single vertex v, then $\psi_G(T) = m$.
- If $T = [m, T']$, then $\psi_G(T) = (m, \psi_G(T'))$.
- If $T = [m, T', T'']$ with $\min(P) \in T''$, then $\psi_G(T) = (\psi_G(T')^*, m, \psi_G(T''))$.

For example, let T be the decreasing 012-tree shown in Definition 2. Then

$$\begin{aligned} \psi_G(T) &= \psi_G\left([6,[5,[4],[2]],[3,[1]]]\right) \\ &= (\psi_G([5,[4],[2]])^*,\ 6,\ \psi_G([3,[1]])) \\ &= ((452)^*,\ 6,\ 31) \\ &= 254631 \end{aligned}$$

which is an element of $\mathcal{G}_6$ because, as one may verify, 82546317 is a primitive G-word. Meanwhile, $\phi_G(254631) = T$.

Theorem 6. *The functions ϕ_G and ψ_G are bijections $\mathcal{G}'_n \to \mathcal{D}_n$ and $\mathcal{D}_n \to \mathcal{G}'_n$ respectively.*

Proof. First, we show by induction on $n = |P|$ that $\psi_G(T) \in \mathcal{G}'_P$. This is clear if $n = 1$; assume that it is true for all decreasing 012-trees on fewer than n vertices. If $T = [m, T']$, then $\psi_G(T) \in \mathcal{G}'_P$ by Lemma 3, and if $T = [m, T', T'']$, then $\psi_G(T) \in \mathcal{G}'_P$ by Lemma 4.

Next, we show that ϕ_G and ψ_G are mutual inverses. Again, we proceed by induction on n. The base case $n = 1$ is clear, so we assume henceforth $n > 1$.

Suppose inductively that $\psi_G(\phi_G(x)) = x$ for all x with $|x| < |w|$. Suppose $w_k = m = \max(P)$. If $k = 1$, then by induction

$$\psi_G(\phi_G(w)) = \psi_G([m, \phi_G(w[2,n])]) = (m, \psi_G\phi_G(w[2,n])) = (m, w[2,n]) = w$$

while if $2 \leq k \leq n-1$, then

$$\begin{aligned} \psi_G(\phi_G(w)) &= \psi_G([m, \phi_G(w[1,k-1]^*), \phi_G(w[k+1,n])]) \\ &= (\psi_G(\phi_G(w[1,k-1]^*))^*, m, \psi_G(\phi_G(w[k+1,n]))) \\ &= ((w[1,k-1]^*)^*, m, w[k+1,n]) = w. \end{aligned}$$

On the other hand, suppose inductively that $\phi_G(\psi_G(U)) = U$ for every tree $U \in \mathcal{D}_P$ with $|U| < n$. If $T = [m, T']$, then

$$\phi_G(\psi_G(T)) = \phi_G(m, (\psi_G(T'))) = [m, \phi_G(\psi_G(T'))] = [m, T'] = T$$

while if $T = [m, T', T'']$ with $\min(P) \in T''$, then

$$\begin{aligned}\phi_G(\psi_G(T)) &= \phi_G((\psi_G(T')^*, m, \psi_G(T''))) = [m, \phi_G(\psi_G(T')), \phi_G(\psi_G(T''))] \\ &= [m, T', T''] = T\end{aligned}$$

as desired. $\square$

Next, we construct the analogous bijections for primitive R-words. Let $w \in \mathcal{R}'_P$ with $k = w_m^{-1}$. Note that if $n > 1$, then $w_1 < w_n \leq m$, so $k \neq 1$. Define a decreasing 012-tree $\phi_R(w)$ recursively by

$$\phi_R(w) = \begin{cases} [m] & \text{if } n = 1; \\ [m, \phi_R(w[1, n-1]^*)] & \text{if } n > 1 \text{ and } k = n; \\ [m, \phi_R(w[1, k-1]^*), \phi_R(w[k+1, n])] & \text{if } n > 1 \text{ and } 2 \leq k \leq n-1. \end{cases}$$

Now, given $T \in \mathcal{D}_P$, we recursively define a word $\psi_R(T) \in \mathfrak{S}_P$ as follows.

- If T consists of a single vertex v, then $\psi_R(T) = v$.
- If $T = [v, T']$, then $\psi_R(T) = (\psi_R(T')^*, v)$.
- If $T = [v, T', T'']$, and the last digit of $\psi_R(T')$ is less than the last digit of $\psi_R(T'')$, then $\psi_R(T) = (\psi_R(T')^*, v, \psi_R(T''))$.

Again, if T is the decreasing 012-tree shown in Definition 2, then

$$\begin{aligned}\psi_R(T) &= \psi_R\left([6, [3, [1]], [5, [4], [2]]]\right) \\ &= (\psi_R([3, [1]])^*,\ 6,\ \psi_R([5, [2], [4]])) \\ &= ((13)^*,\ 6,\ 254) \\ &= 316254\end{aligned}$$

which is an element of $\mathcal{R}_6$ because, as one may verify, 83162547 is a primitive R-word. Meanwhile, $\phi_R(316254) = T$.

Theorem 7. *The functions ϕ_R and ψ_R are bijections $\mathcal{R}'_n \to \mathcal{D}_n$ and $\mathcal{D}_n \to \mathcal{R}'_n$ respectively.*

Proof. First, we show by induction on $n = |P|$ that $\psi_R(T) \in \mathcal{R}'_P$. This is clear if $n = 1$, so assume that it is true for all decreasing 012-trees on fewer than n vertices. If $T = [v, T']$, then $\psi_R(T) \in \mathcal{R}'_P$ by Lemma 3, and if $T = [v, T', T'']$, then $\psi_R(T) \in \mathcal{R}'_P$ by Lemma 4.

We have now constructed functions

$$\phi_R :\ \mathcal{R}'_n \to \mathcal{D}_n, \qquad \psi_R :\ \mathcal{D}_n \to \mathcal{R}'_n.$$

It remains to show that they are mutual inverses, which we do by induction on n. The base case $n = 1$ is clear, so we assume henceforth $n > 1$.

Suppose inductively that $\psi_R(\phi_R(x)) = x$ for all x with $|x| < |w|$. Suppose $w_k = m = \max(P)$. If $k = n$, then by induction

$$\begin{aligned}\psi_R(\phi_R(w)) &= \psi_R([m, \phi_R(w[1, n-1]^*)]) = (\psi_R(\phi_R(w[1, n-1])), m) \\ &= (w[1, n-1], m) = w\end{aligned}$$

while if $2 \leq k \leq n-1$, then

$$\begin{aligned}\psi_R(\phi_R(w)) &= \psi_R([m, \phi_R(w[1,k-1]^*), \phi_R(w[k+1,n])]) \\ &= (\psi_R(\phi_R(w[1,k-1]^*))^*, m, \psi_R(\phi_R(w[k+1,n]))) \\ &= ((w[1,k-1]^*)^*, m, w[k+1,n]) = w.\end{aligned}$$

On the other hand, suppose inductively that $\phi_R(\psi_R(U)) = U$ for all $U \in \mathcal{D}_P$ with $|P| < n$. If $T = [m, T']$, then

$$\begin{aligned}\phi_R(\psi_R(T)) &= \phi_R((\psi_R(T')^*, m)) = [m, \phi_R((\psi_R(T')^*)^*)] = [m, \phi_R(\psi_R(T'))] \\ &= [m, T'] = T\end{aligned}$$

while if $T = [m, T', T'']$ with $\psi_R(T')_1 < \psi_R(T'')_1$, then

$$\begin{aligned}\phi_R(\psi_R(T)) &= \phi_R((\psi_R(T')^*, m, \psi_R(T''))) = [m, \phi_R(\psi_R(T')), \phi_R(\psi_R(T''))] \\ &= [m, T', T''] = T\end{aligned}$$

as desired. □

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