Rees products of posets, lexicographical shellability and multiset derangements

Michelle Wachs
University of Miami

Based on joint work with John Shareshian

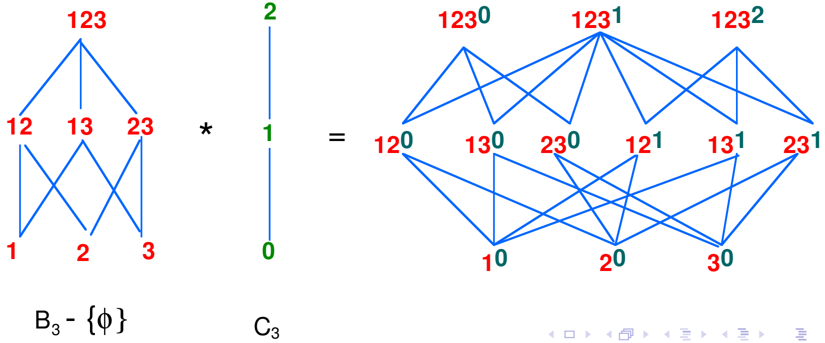
Rees Product-Björner & Welker, 2003

The Rees product of pure (ranked) posets P and Q is defined by

$$P * Q := \{(p,q) \in P \times Q : r(p) \geq r(q)\}$$

$(p_1, q_1) \leq (p_2, q_2)$ if the following holds

- $p_1 \leq_P p_2$
- $q_1 \leq_Q q_2$
- $r(p_2) - r(p_1) \geq r(q_2) - r(q_1)$



Rees Product-Björner & Welker, 2003

Theorem (Björner & Welker)

The Rees product of any Cohen-Macaulay poset with any acyclic Cohen-Macaualy poset is Cohen-Macaulay (CM means that homology of each interval vanishes below its top dimension.)

Conjecture (Björner & Welker)

$\dim \tilde{H}_{n-1}((B_n \setminus \{\emptyset\}) * C_n) = \#$ *derangements in* $\mathfrak{S}_n$.

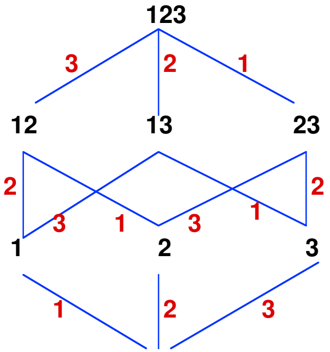
Proved by Jonsson using discrete Morse theory.

Lexicographic Shellability

Theorem (Björner 1980)

Suppose P is pure and admits an EL-labeling with m ascent-free maximal chains. Then $P - \{\hat{0}, \hat{1}\}$ has the homotopy type of a wedge of m spheres of dimension $\ell(P) - 2$.

Example: Number of ascent-free chains is 1.



EL-labeling of $R_n := ((B_n \setminus \{\emptyset\}) * C_n) \cup \{\hat{0}, \hat{1}\}$

Theorem (Shareshian & MW)

Let $d_n = \#$ derangements in $\mathfrak{S}_n$. Then R_n admits an EL-labeling with d_n ascent-free chains.

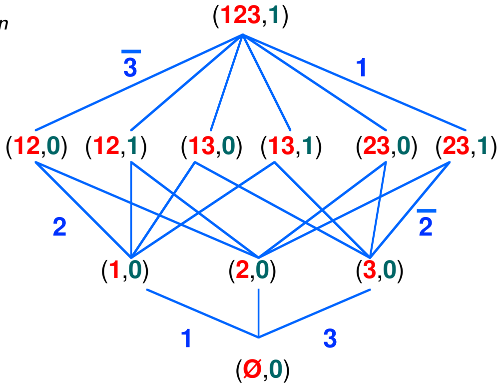
Poset of labels: product order on $\{1 < 2 < \cdots < n < \infty\} \times \{0, 1\}$

Edge	Label	Code
$(S, i) \longrightarrow (S \cup \{a\}, i)$	$(a, 0)$	a
$(S, i) \longrightarrow (S \cup \{a\}, i + 1)$	$(a, 1)$	$\bar{a}$
$([n], i) \longrightarrow \hat{1}$	$(\infty, 0)$	∞

View bottom element $\hat{0}$ as $(\emptyset, 0)$.

EL-labeling of $R_n := ((B_n \setminus \{\emptyset\}) * C_n) \cup \{\hat{0}, \hat{1}\}$

interval of R_n



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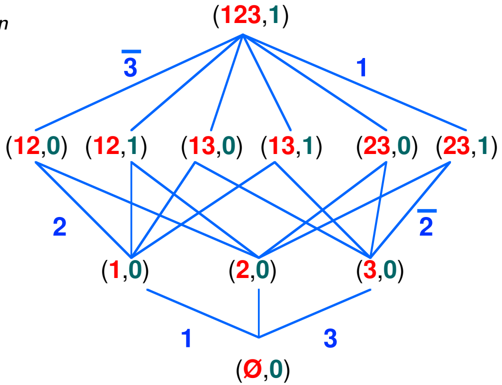

Maximal chains of R_n correspond bijectively to barred permutations in which the first letter is unbarred.

EL-labeling of $R_n := ((B_n \setminus \{\emptyset\}) * C_n) \cup \{\hat{0}, \hat{1}\}$

Ascent-free maximal chains correspond bijectively to the set $\mathcal{W}_n$ of barred permutations ω satisfying

- $\omega(1)$ is unbarred
- $\omega(n)$ is barred
- for each ascent $|\omega(i)| < |\omega(i+1)|$, first entry $\omega(i)$ is barred and second entry $\omega(i+1)$ is unbarred

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Examples: $3\bar{1}5\bar{2}76\bar{4}$ and $3\bar{1}5\bar{2}7\bar{6}\bar{4} \in \mathcal{W}_7$

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$\mathcal{W}_3 = \{32\bar{1}, 3\bar{2}\bar{1}\}$

$\mathcal{W}_4 = \{2\bar{1}4\bar{3}, 3\bar{1}4\bar{2}, 3\bar{2}4\bar{1}, 4\bar{1}3\bar{2}, 4\bar{2}3\bar{1}, 432\bar{1}, 4\bar{3}2\bar{1}, 43\bar{2}\bar{1}, 4\bar{3}\bar{2}\bar{1}\}$

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Claim: $|\mathcal{W}_n| = d_n$

q-analog

$B_n(q) :=$ lattice of subspaces of $\mathbb{F}_q^n$.
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Theorem (Shareshian & MW (2006))

$(B_n(q) \setminus \{(0)\}) * C_n$ *has the homotopy type of a wedge of*

$$\sum_{\sigma \in \mathcal{D}_n} q^{\binom{n}{2} - \mathrm{maj}(\sigma) + \mathrm{exc}(\sigma)}$$

$(n-1)$*-spheres.*

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Proof relies on (q,r)-analog of a formula of Euler for Eulerian polynomials due to Shareshian & MW (2006):

$$\sum_{n \geq 0} A_n(q,t,r) \frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(rz)}{\exp_q(ztq) - tq\exp_q(z)},$$

where

$$A_n(q,t,r) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)} r^{\mathrm{fix}(\sigma)}.$$

q-analog

There is an EL-labeling of $B_n(q)$ such that

- the label sequence of each maximal chain is a permutation in $\mathfrak{S}_n$
- the number of maximal chains with label sequence σ is $q^{inv(\sigma)}$.

We use this to show that

$$\dim \tilde{H}_{n-1}((B_n(q) \setminus \{(0)\}) * C_n) = \sum_{\omega \in \mathcal{W}_n} q^{\mathrm{inv}(|\omega|)}$$

Problem: Find a bijection $\phi : \mathcal{W}_n \to \mathcal{D}_n$ for which

$$\mathrm{inv}(|\omega|) = \binom{n}{2} - \mathrm{maj}(\phi(\omega)) + \mathrm{exc}(\phi(\omega))$$

Multiset analog

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a weak composition of n.
Set

$$B_\lambda := C_{\lambda_1+1} \times \cdots \times C_{\lambda_k+1}$$

If $\lambda_i = 1$ for all i then $B_\lambda = B_n$.

A derangement of the multiset $M_\lambda := \{1^{\lambda_1} 2^{\lambda_2} \ldots k^{\lambda_k}\}$ is a permutation σ of M_λ such that there are distinct entries in each column of the $2 \times n$ array whose top row is the increasing arrangement of M_λ and whose bottom row is σ

Let $\mathcal{D}_\lambda$ denote the set of derangements of M_λ.

$$33121 \in \mathcal{D}_\lambda$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 3 & 3 & 1 & 2 & 1 \end{bmatrix}$$

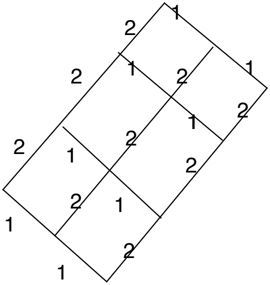
Multiset analog

Theorem

$(B_\lambda \setminus \{\hat{0}\}) * C_n$ has the homotopy type of a wedge of $|\mathcal{D}_\lambda|$ $(n-1)$-spheres.

Proof Idea: There is an natural EL-labeling of B_λ for which maximal chains of B_λ correspond bijectively to the permutations of M_λ.

$B_{(2,3)} = C_3 \times C_4 =$



Multiset analog

We use this EL-labeling of B_λ to obtain an EL-labeling of

$$((B_\lambda \setminus \{\hat{0}\}) * C_n) \cup \{\hat{0}, \hat{1}\}$$

Ascent-free maximal chains correspond bijectively to the set $\mathcal{W}_\lambda$ of barred multiset permutations ω of M_λ satisfying

- $\omega(1)$ is unbarred
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Then we show that $W_\lambda = D_\lambda$.

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Excedance in a multiset permutation:

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$$\mathrm{exc}(\textcolor{red}{33}121) = 2$$

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Theorem (Shareshian and MW (2009))

$$\sum_{\omega \in \mathcal{W}_\lambda} t^{\#\mathsf{bars}(\omega)} = \sum_{\sigma \in \mathcal{D}_\lambda} t^{\mathrm{exc}(\sigma)}$$

Proof uses a symmetric function result of Gessel on words with no double ascents and a symmetric function result of Askey and Ismail on multiset derangements.

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Let $B_\lambda(p)$ be the lattice of subgroups of the abelian p-group

$$\mathbb{Z}_{p^{\lambda_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\lambda_k}}$$

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Theorem (Butler (1986?))

There is an EL-labeling of $B_\lambda(p)$ such that the label sequence of each maximal chain is a permutation of multiset M_λ and the number of maximal chains with label sequence $w \in M_\lambda$ is $p^{\mathrm{cocharge}(w)}$.

p-analog of multiset analog

Lift this to an EL-labeling of the Rees product and conclude that

$$\dim \tilde{H}_{n-1}((B_\lambda(p) \setminus \{(0)\}) * C_n) = \sum_{\omega \in \mathcal{W}_\lambda} p^{\mathrm{cocharge}(|\omega|)}$$

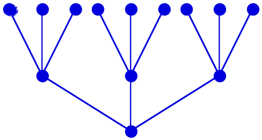
Use a bijection $\phi : \mathcal{D}_\lambda \to \mathcal{W}_\lambda$ to get

$$\dim \tilde{H}_{n-1}((B_\lambda(p) \setminus \{(0)\}) * C_n) = \sum_{\sigma \in \mathcal{D}_\lambda} p^{\mathrm{cocharge}(|\phi(\sigma)|)}$$

Problem: Find a ϕ for which $\mathrm{cocharge}(|\phi(\sigma)|)$ is a nice statistic on multiset derangements which generalizes $\binom{n}{2} - \mathrm{maj}(\sigma) + \mathrm{exc}(\sigma)$.

t-analog of a chain

Let $T_{t,n}$ be the poset whose Hasse diagram is the complete t-ary tree of height $n-1$ with the root at the bottom.

$T_{3,3} =$ 

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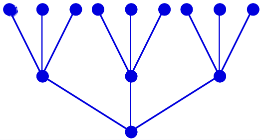
$T_{3,3} =$ 

Theorem (Shareshian & MW (2009))

$(B_\lambda(p) \setminus \{(0)\}) * T_{t,n}$ *has the homotopy type of a wedge of*

$$\sum_{w \in \mathcal{D}_\lambda} p^{\mathrm{stat}(\sigma)} t^{\mathrm{exc}(w)}$$

$(n-1)$*-spheres.*

When $\lambda = 1^n$, $\mathrm{stat}(\sigma) = \binom{n}{2} - \mathrm{maj}(\sigma) + \mathrm{exc}(\sigma)$

We use $\sum_{\omega \in \mathcal{W}_\lambda} t^{\#bars(\omega)} = \sum_{\sigma \in \mathcal{D}_\lambda} t^{\mathrm{exc}(\sigma)}$ in the proof.

Related work

- nontruncated version (Eulerian numbers) - Shareshian & MW (2005)
- equivariant version (representation of symmetric group) - Shareshian & MW (2005)
- type B analog - Shareshian & MW (2006), Muldoon and Readdy (2008)
- noncrossing partition lattice - Linusson and MW (2008)