Hopf Monoids

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Outline

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What is a Hopf monoid?

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    - Hopf algebras are built from unlabelled objects
    - Hopf monoids are built from labelled objects.

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  - Hopf algebras are built from unlabelled objects
  - Hopf monoids are built from labelled objects. we will be dealing with monoids in this talk.

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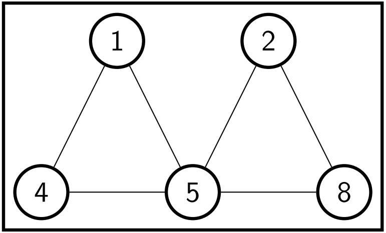
Breaking and Combining Graphs

Breaking



Combining

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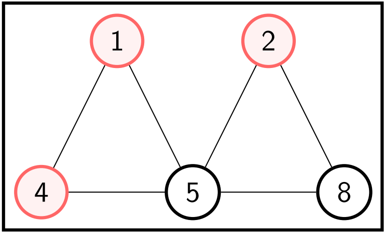

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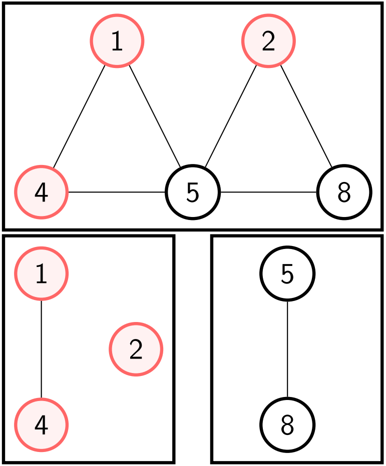

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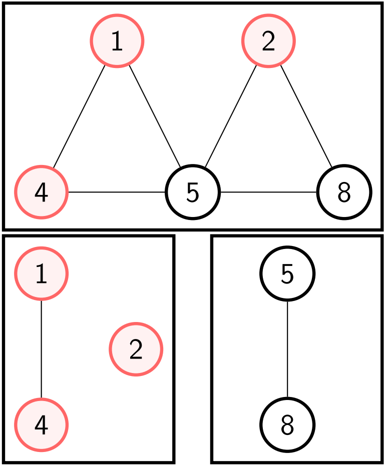

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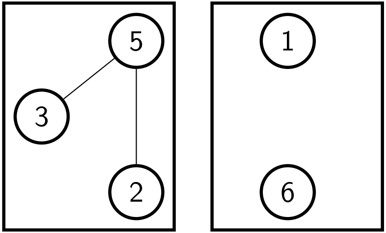

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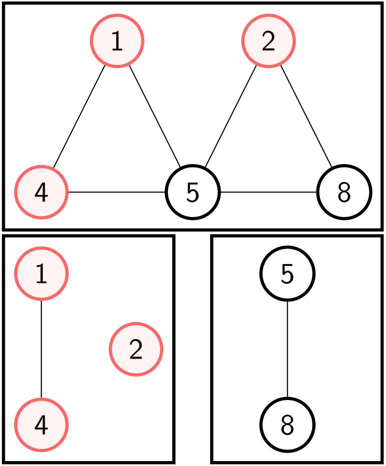

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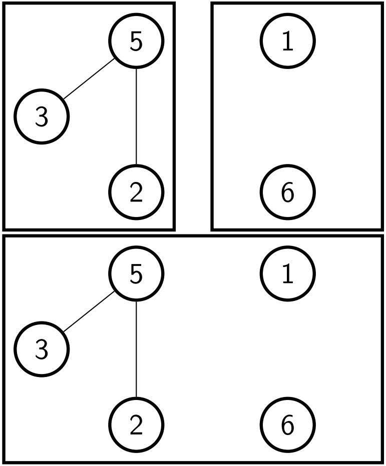


Breaking and Combining Linear Orders

Definition

A linear order on a set I is a particular arrangement of the elements of I. $\mathcal{L}[I]$ is the set of all linear orders of I.

Breaking

Given a linear order 1325468, we want to restrict to $\mathcal{L}[\{2,3,4,6\}]$

13259468

13259468

3246 – 1598

Combining

Given linear orders 6718 and 23, then there are two different ways to combine them:

$6718 * 23 = 671823$

$23 * 6718 = 236718$

Breaking and Combining Matroids

Let M be a matroid independence system on I, and $S \subseteq I$.

Definition

Then the restriction of M to S is $M|_S = \{\sigma \cap S | \sigma \in M\}$.

Definition

The contraction of M by S is
$M/_S = \{\sigma \in M | \sigma \sqcup \tau \in M, \tau$ is a basis of $M|_S\}$.

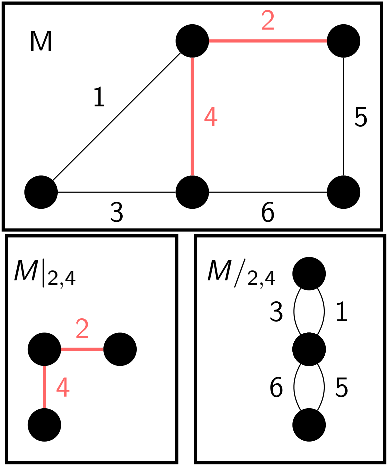
Definition

The direct sum of matroids M_1 and M_2 is $M_1 \oplus M_2 = \{\sigma \cup \tau | \sigma \in I, \tau \in J\}$.

Breaking and Combining Graphic Matroids

We are going to use graphic matroids for our example. The underlying independence complex in each picture consists of the acyclic sets of edges.

Breaking



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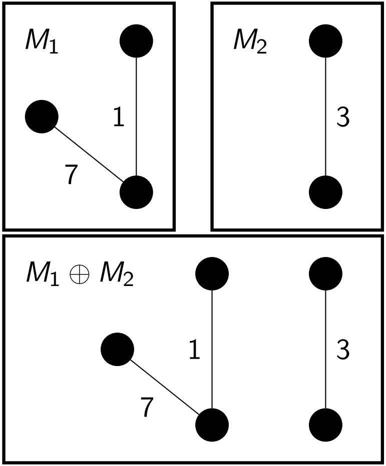


An Alternative Way to Break and Combine Matroids

In many instances, there are multiple compatible ways of breaking and combining objects. Kevin Marshall has been working on the Hopf monoid of set families, of which matroids form a submonoid with a different version of breaking.

Definition

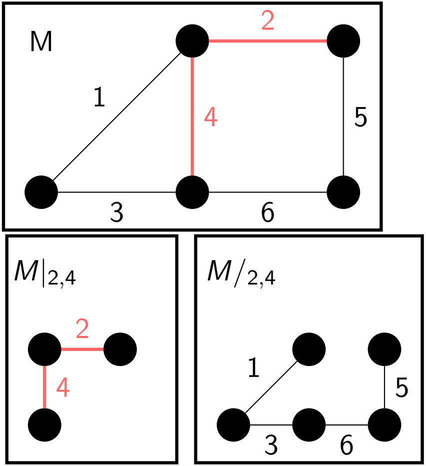
The contraction of a set family M to S is defined $M/_S = \{f \in M | f \cap S = \emptyset\}$.

When the set family M is a matroid independence complex (or any simplicial complex), this simplifies to $M/_S = M|_{S^c}$.

Breaking and Combining Matroids as Set Families

Here we use the same graphic matroid restricted to the same set

Breaking



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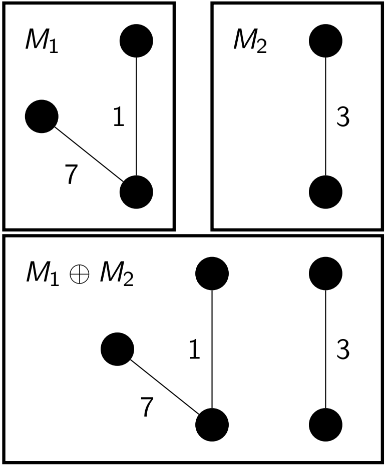

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Vector Species

Defining multiplication in a Hopf monoid is tricky because you can't combine any two objects together. If I had linear orders 124 and 216, and I tried to combine them into 124216 I would not be left with a linear order. We can only combine objects with distinct label sets. In order to keep track of these label sets, we introduce vector species.

Definition

A vector species H consists of two things:

- For each finite set I, a vector space $H[I]$
- For each bijection $\sigma : I \to J$, an isomorphism $H[\sigma] : H[I] \to H[J]$.

These bijections should be natural in that $H[\sigma \circ \tau] = H[\sigma] \circ H[\tau]$ and $H[id] = id$.

For combinatorial Hopf monoids, we should think of $H[I]$ as the set of all structures labelled by the set I.

The Definition of a Hopf monoid

Definition

A connected Hopf monoid in vector species is a vector species H with $H[\emptyset] = \Bbbk$, and for all decompositions $I = S \sqcup T$, linear maps

- $\mu_{S,T} : H[S] \otimes H[T] \to H[I]$
- $\Delta_{S,T} : H[I] \to H[S] \otimes H[T]$

that satisfy the following axioms:

1. naturality
2. unitality
3. associativity (and coassociativty)
4. compatibility

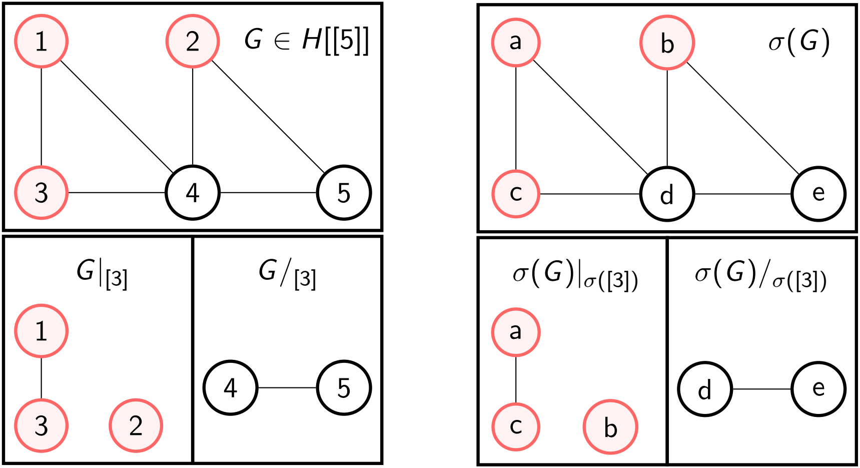
We should think of μ as how we combine objects, and Δ as how we break objects apart.

Notation

- $x * y := \mu_{S,T}(x \otimes y)$
- $\sum z|_S \otimes z/_S := \Delta_{S,T}(z)$
  - This is called Sweedler notation.
  - For most Hopf monoids we are concerned with, the coproduct of a single object will be a simple tensor, which we will refer to as $z|_S \otimes z/_S$.
- $x_1 * \ldots * x_n = \mu_{S_1,\ldots,S_n}(x_1, \ldots, x_n)$
- $\sum z_1 \otimes \ldots \otimes z_n = \Delta_{S_1,\ldots,S_n}(z)$

Naturality

The naturality axiom just says that your product and coproduct respect relabelling. In this example H is the Hopf monoid on graphs and σ relabels numbers as letters.



Unitality

The unitality axiom says that if you have a trivial decomposition $I = I \sqcup \emptyset$, then the product and coproduct preserve your structure.

Definition

The product map μ is unital if $\forall I$ and $\forall x \in H[I]$

$$\mu_{I,\emptyset}(x, 1) = \mu_{\emptyset,I}(1, x) = x$$

The coproduct map Δ is unital if $\forall I$ and $\forall x \in H[I]$

$$\Delta_{I,\emptyset}(x) = x \otimes 1, \quad \Delta_{\emptyset,I}(x) = 1 \otimes x$$

The 1 here is the identity element in the field $H[\emptyset] = \mathbb{k}$

Associativity

For the product map, associativity is exactly what we would expect. If you want to multiply 3 elements together, you can start with the first and second, or the second and third.

Definition

The product map μ is associative if $\forall I = R \sqcup S \sqcup T$ and $\forall x \in H[R], y \in H[S], z \in H[T]$

$$(x * y) * z = x * (y * z)$$

With associativity, we can unambiguously define $x * y * z = (x * y) * z = x * (y * z)$

Coassociativity

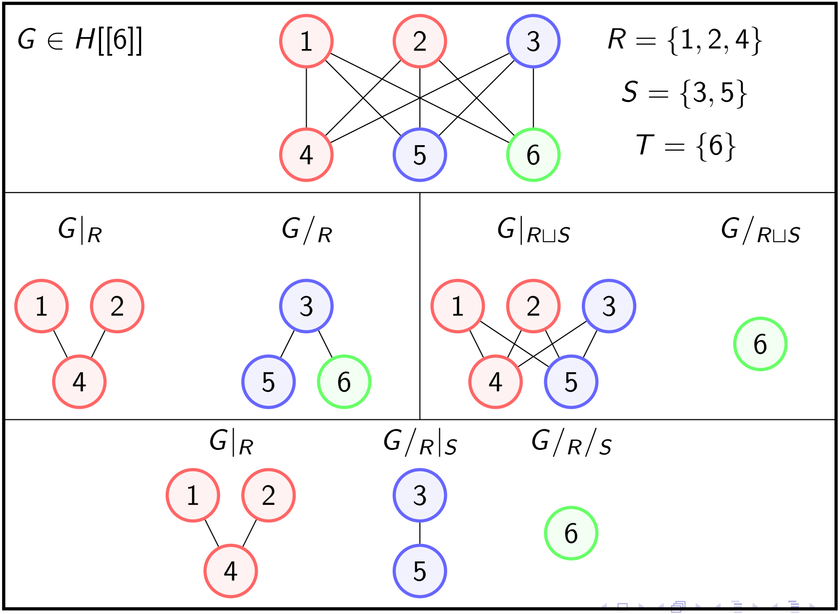
For the coproduct map, coassociativity tells us that if you want to break something into 3 pieces, you can start by breaking off the first piece or the third piece.

Definition

The coproduct map Δ is coassociative if $\forall I = R \sqcup S \sqcup T$ and $\forall w \in H[I]$

$$(w|_{R \sqcup S})|_R = w|_R, \quad (w|_{R \sqcup S})/_R = (w/_R)|_S, \quad (w/_R)/_S = w/_{R \sqcup S}$$

Example of Coassociativity



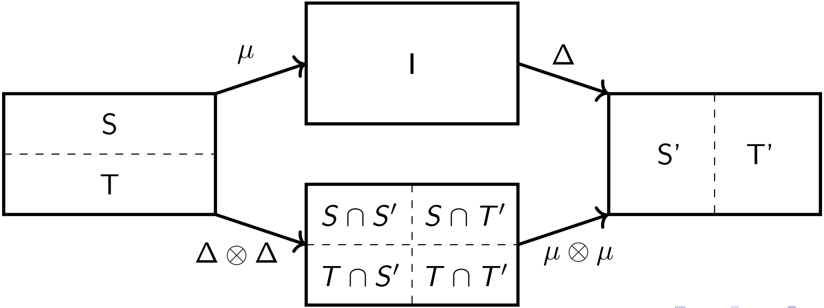
Compatibility

Definition

The product μ and coproduct Δ maps are compatible if for all decomposition pairs $I = S \sqcup T = S' \sqcup T'$ and $\forall x \in S, y \in T$

$$(x * y)|_{S'} = x|_{S' \cap S} * y|_{S' \cap T}$$

$$(x * y)/_{S'} = x/_{S' \cap S} * y/_{S' \cap T}$$



Example of Compatibility

Let $I = [9], S = \{1,2,3,4\}, T = \{5,6,7,8,9\}, S' = \{3,4,5,6,7\}$, and $T' = \{1,2,8,9\}$, and H be the Hopf monoid on linear orders. We start with $2413 \otimes 65798 \in H[S] \otimes H[T]$.

	S'	T'
S	$A = \{3,4\}$	$B = \{1,2\}$
T	$C = \{5,6,7\}$	$D = \{8,9\}$

$$\begin{array}{ccc} 2413 \otimes 65798 & \xrightarrow{\mu_{S,T}} & 241365978 \\ \Big\downarrow{\scriptstyle \Delta_{A,B} \otimes \Delta_{C,D}} & & \Big\downarrow{\scriptstyle \Delta_{S',T'}} \\ 43 \otimes 21 \otimes 657 \otimes 98 & \xrightarrow{\mu_{A,C} \otimes \mu_{B,D}} & 43657 \otimes 2198 \end{array}$$

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The Unit and Counit

Recall that $\Bbbk$ is the base field of the vector space. In order to set up the antipode, I need to introduce two additional linear maps: the unit, ι, and the counit, ε.

- The unit $\iota : \Bbbk \to H$ is defined $\iota(k) = k * 1$. It is the natural isomorphism of the base field with $H[\emptyset]$.
- The counit $\varepsilon : H \to \Bbbk$ is defined by $\varepsilon(1) = 1$, and $\forall x \in H[I], I \neq \emptyset, \varepsilon(x) = 0$. So the counit picks out the coefficient of 1 and ignores everything else.
- In particular, the composition $\iota \circ \varepsilon : H \to H$ takes a general element and removes everything but the constant term.

The Diagram of the Antipode

Every connected Hopf monoid in vector species has a unique map $S : H \to H$ called the antipode that ensures the following diagram commutes:

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S \otimes id"] & & H \otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & K \arrow[rr, "\iota"] & & H \\
 & H \otimes H \arrow[rr, "id \otimes S"] & & H \otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Takeuchi's Formula

Theorem (Takeuchi's Formula)

The formula for the antipode map $S_I : H[I] \to H[I]$ is given by:

$$S_I(x) = \sum_{k \geq 1, (S_1, \ldots, S_k) \models I} (-1)^k \mu_{S_1, \ldots, S_k} \circ \Delta_{S_1, \ldots, S_k}(x)$$

where $(S_1, \ldots, S_k) \models I$ are the set compositions of I.

Takeuchi's formula gives an explicit formula for the antipode. However, it tends to have severe cancellation, and is computationally expensive. Finding a cancellation free formula for the antipode of a particular Hopf monoid is often an active area of research.

Properties of the Antipode

The antipode map reverses products and coproducts.

- $S(x * y) = S(y) * S(x)$
- $S(\Delta_{S,T}(z)) = S(\sum z|_S \otimes z/_S) = \sum S(z/_T) \otimes S(z|_T)$

If the Hopf monoid is either commutative or cocommutative, then $S^2(x) = x$

Computed Example of the Antipode Map

We are going to use Takeuchi's formula to compute the antipode of the linear order $132 \in H[[3]]$.

$S_1 \sqcup \ldots \sqcup S_k$	$\Delta(132)$	$(-1)^k \mu \circ \Delta(132)$	$S(132)$
123	132	-132	
$1, 23$	$1 \otimes 32$	132	
$2, 13$	$2 \otimes 13$	213	
$3, 12$	$3 \otimes 12$	312	
$12, 3$	$12 \otimes 3$	123	
$13, 2$	$13 \otimes 2$	132	
$23, 1$	$32 \otimes 1$	321	
$1, 2, 3$	$1 \otimes 2 \otimes 3$	-123	
$1, 3, 2$	$1 \otimes 3 \otimes 2$	-132	
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Thank You

How did Jeremy keep track of his beer brewing experiments?

Thank You

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A Hops Monoid