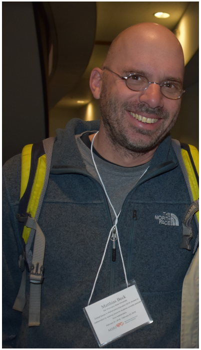
Decompositions of Ehrhart h^*-polynomials for rational polytopes

Andrés R. Vindas Meléndez

Department of Mathematics
University of Kentucky

AMS Sectional
12-September-2020

People



Matthias Beck
(SF State & FU Berlin)

Benjamin Braun
(Univ. of Kentucky)

This material is based upon work supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. 1247392.

Main Characters: (Rational) Polytopes!

Main Characters: (Rational) Polytopes!

- Let P to be a rational d-polytope in $\mathbb{R}^d$, i.e., convex hull of finitely many points in $\mathbb{Q}^d$.

Main Characters: (Rational) Polytopes!

- Let P to be a rational d-polytope in $\mathbb{R}^d$, i.e., convex hull of finitely many points in $\mathbb{Q}^d$.
- For a positive integer t, let $L_P(t)$ denote the number of integer lattice points in tP.

Main Characters: (Rational) Polytopes!

- Let P to be a rational d-polytope in $\mathbb{R}^d$, i.e., convex hull of finitely many points in $\mathbb{Q}^d$.
- For a positive integer t, let $L_P(t)$ denote the number of integer lattice points in tP.

Theorem: (Ehrhart 1962) Given a rational polytope P, the counting function $L_P(t) := |tP \cap \mathbb{Z}^d|$ is a quasipolynomial of the form

$$\mathrm{vol}(P)t^d + k_{d-1}(t)t^{d-1} + \cdots + k_1(t)t + k_0(t),$$

where $k_0(t), k_1(t), \ldots, k_{d-1}(t)$ are periodic functions in t.

Main Characters: (Rational) Polytopes!

- Let P to be a rational d-polytope in $\mathbb{R}^d$, i.e., convex hull of finitely many points in $\mathbb{Q}^d$.
- For a positive integer t, let $L_P(t)$ denote the number of integer lattice points in tP.

Theorem: (Ehrhart 1962) Given a rational polytope P, the counting function $L_P(t) := |tP \cap \mathbb{Z}^d|$ is a quasipolynomial of the form

$$\mathrm{vol}(P)t^d + k_{d-1}(t)t^{d-1} + \cdots + k_1(t)t + k_0(t),$$

where $k_0(t), k_1(t), \ldots, k_{d-1}(t)$ are periodic functions in t. We call $L_P(t)$ the *Ehrhart quasipolynomial* of P, and each period of $k_0(t), k_1(t), \ldots, k_{d-1}(t)$ divides the *denominator* q of P, which is the least common multiple of all its vertex coordinate denominators.

Ehrhart Quasipolynomials

Ehrhart Quasipolynomials

A quasipolynomial $L_P(t)$ is a function $\mathbb{Z} \to \mathbb{R}$ of the form

$$L_P(t) = k_d(t)t^d + \cdots + k_1(t)t + k_0(t),$$

where $k_0, \cdots, k_d$ are periodic functions in the integer variable t.

Ehrhart Quasipolynomials

A quasipolynomial $L_P(t)$ is a function $\mathbb{Z} \to \mathbb{R}$ of the form

$$L_P(t) = k_d(t)t^d + \cdots + k_1(t)t + k_0(t),$$

where $k_0, \cdots, k_d$ are periodic functions in the integer variable t.

Alternatively, for a quasipolynomial, there exist a positive integer q and polynomials $f_0, \ldots, f_{p-1}$, such that

$$L_P(t) = \begin{cases} f_0(t) & \text{if } t \equiv 0 \mod q \\ f_1(t) & \text{if } t \equiv 1 \mod q \\ \vdots & \\ f_{p-1}(t) & \text{if } t \equiv q-1 \mod q. \end{cases}$$

Ehrhart Series

Ehrhart Series

The *Ehrhart series* is the rational generating function

$$\mathrm{Ehr}(P;z) := \sum_{t\geq 0} L_P(t)z^t = \frac{h^*(P;z)}{(1-z^q)^{d+1}},$$

where $h^*(P;z)$ is a polynomial of degree less than $q(d+1)$ called the h^*-polynomial of P.

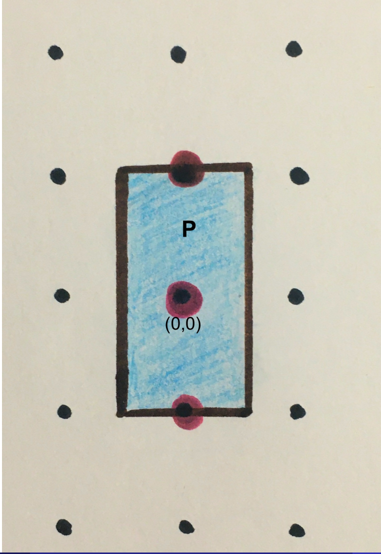
Ehrhart Theory of Rational Polytopes

Ehrhart Theory of Rational Polytopes

Let $P = \text{conv}\{\left(\frac{-1}{2}, 1\right), \left(\frac{-1}{2}, -1\right), \left(\frac{1}{2}, 1\right), \left(\frac{1}{2}, -1\right)\}$.

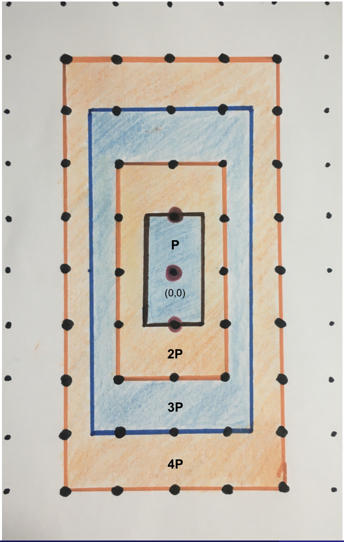
Ehrhart Theory of Rational Polytopes

Let $P = \text{conv}\{\left(\frac{-1}{2}, 1\right), \left(\frac{-1}{2}, -1\right), \left(\frac{1}{2}, 1\right), \left(\frac{1}{2}, -1\right)\}$.



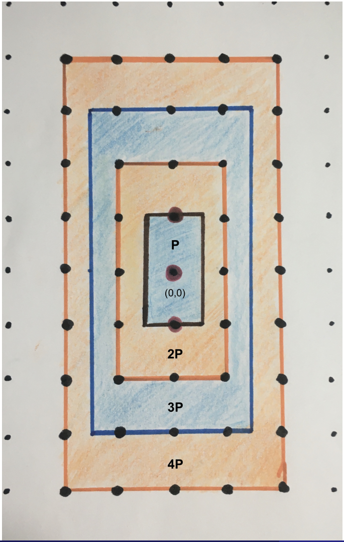
Ehrhart Theory of Rational Polytopes

Let $P = \text{conv}\{\left(\frac{-1}{2},1\right),\left(\frac{-1}{2},-1\right),\left(\frac{1}{2},1\right),\left(\frac{1}{2},-1\right)\}$.

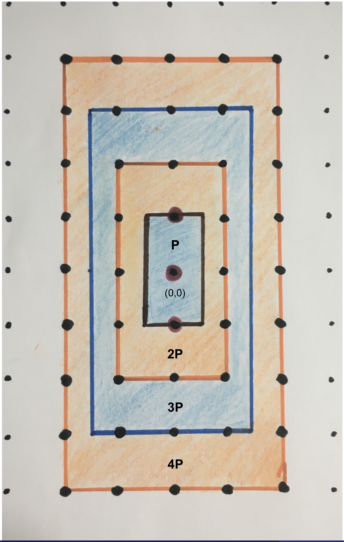


Ehrhart Theory of Rational Polytopes

Let $P = \operatorname{conv}\{\left(\frac{-1}{2}, 1\right), \left(\frac{-1}{2}, -1\right), \left(\frac{1}{2}, 1\right), \left(\frac{1}{2}, -1\right)\}$.



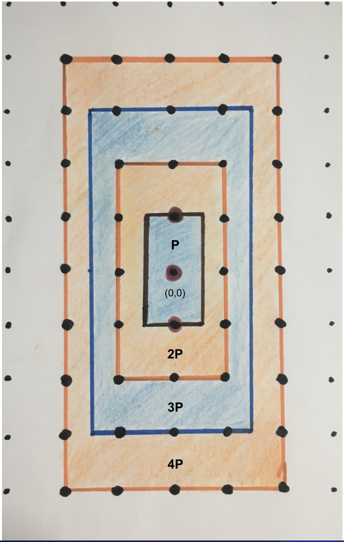
Let $P = \text{conv}\{\left(\frac{-1}{2}, 1\right), \left(\frac{-1}{2}, -1\right), \left(\frac{1}{2}, 1\right), \left(\frac{1}{2}, -1\right)\}$.



$$L_P(t) = \begin{cases} 2t^2 + 3t + 1 & \text{when } t \text{ is even,} \\ 2t^2 + t & \text{when } t \text{ is odd.} \end{cases}$$

Ehrhart Theory of Rational Polytopes

Let $P = \mathrm{conv}\{\left(\frac{-1}{2}, 1\right), \left(\frac{-1}{2}, -1\right), \left(\frac{1}{2}, 1\right), \left(\frac{1}{2}, -1\right)\}$.



$$L_P(t) = \begin{cases} 2t^2 + 3t + 1 & \text{when } t \text{ is even,} \\ 2t^2 + t & \text{when } t \text{ is odd.} \end{cases}$$

$$\begin{aligned} \mathrm{Ehr}(P; z) &= \sum_{t \geq 0} L_P(t) z^t \\ &= \sum_{\substack{t \geq 0 \\ t \text{ even}}} (2t^2 + 3t + 1) z^t + \sum_{\substack{t \geq 1 \\ t \text{ odd}}} (2t^2 + t) z^t \\ &= \frac{3z^4 + 12z^2 + 1}{(1 - z^2)^3} + \frac{z^5 + 12z^3 + 3z}{(1 - z^2)^3} \\ &= \frac{z^5 + 3z^4 + 12z^3 + 12z^2 + 3z + 1}{(1 - z^2)^3}, \end{aligned}$$

Ehrhart Theory of Rational Polytopes

Ehrhart Theory of Rational Polytopes

Theorem: (Ehrhart–Macdonald Reciprocity, 1971)
Let P be a rational polytope. Then $L_P(-t) = (-1)^d L_{P^\circ}(t)$.
Similarly, $\operatorname{Ehr}\left(P; \frac{1}{z}\right) = (-1)^{d+1} \operatorname{Ehr}(P^\circ; z)$.

Ehrhart Theory of Rational Polytopes

Theorem: (Ehrhart–Macdonald Reciprocity, 1971)
Let P be a rational polytope. Then $L_P(-t) = (-1)^d L_{P^\circ}(t)$.
Similarly, $\mathrm{Ehr}\left(P; \frac{1}{z}\right) = (-1)^{d+1}\,\mathrm{Ehr}(P^\circ; z)$.

Theorem:(Stanley's Non-negativity Result, 1980)
For a rational d-polytope with $\mathrm{Ehr}(P; z) = \frac{h^*(P;z)}{(1-z^q)^{d+1}}$, the coefficients of the h^*-polynomial are non-negative integers, i.e., $h_j^* \geq \mathbb{Z}_{\geq 0}$.

Ehrhart Theory of Rational Polytopes

Theorem: (Ehrhart–Macdonald Reciprocity, 1971)
Let P be a rational polytope. Then $L_P(-t) = (-1)^d L_{P^\circ}(t)$.
Similarly, $\mathrm{Ehr}\left(P; \frac{1}{z}\right) = (-1)^{d+1}\,\mathrm{Ehr}(P^\circ; z)$.

Theorem:(Stanley's Non-negativity Result, 1980)
For a rational d-polytope with $\mathrm{Ehr}(P; z) = \frac{h^*(P;z)}{(1-z^q)^{d+1}}$, the coefficients of the h^*-polynomial are non-negative integers, i.e., $h_j^* \geq \mathbb{Z}_{\geq 0}$.

Theorem: (Stanley's Monotonicity Result, 1993) For $P \subseteq Q$, where qP and qQ are integral for some $q \in \mathbb{Z}_{>0}$, $h^*(P) \leq h^*(Q)$.

Goals

1. Present a generalization of a decomposition of the h^*-polynomial for lattice polytopes due to Betke and McMullen (1985).
    (i) Use this decomposition to provide another proof of Stanley's Monotonicity Result.
2. Present a generalization of the h^*-polynomial for lattice polytopes due to Stapledon (2009).
    (i) Application of this decomposition.

Set-Up and Notation

- A *rational pointed simplicial cone* is a set of the form

$$K(\mathbf{W}) = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{w}_i : \lambda_i \geq 0 \right\},$$

where $\mathbf{W} := \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a set of linearly independent vectors in $\mathbb{Z}^d$.

Set-Up and Notation

- A *rational pointed simplicial cone* is a set of the form

$$K(\mathbf{W}) = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{w}_i : \lambda_i \geq 0 \right\},$$

  where $\mathbf{W} := \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a set of linearly independent vectors in $\mathbb{Z}^d$.

- Define the *open parallelepiped* associated with $K(\mathbf{W})$ as

$$\text{Box}(\mathbf{W}) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{w}_i : 0 < \lambda_i < 1 \right\}.$$

Set-Up and Notation

- A *rational pointed simplicial cone* is a set of the form

$$K(\mathbf{W}) = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{w}_i : \lambda_i \geq 0 \right\},$$

  where $\mathbf{W} := \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a set of linearly independent vectors in $\mathbb{Z}^d$.

- Define the *open parallelepiped* associated with $K(\mathbf{W})$ as

$$\mathrm{Box}(\mathbf{W}) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{w}_i : 0 < \lambda_i < 1 \right\}.$$

- Let $u : \mathbb{R}^d \to \mathbb{R}$ denote the projection onto the last coordinate. We then define the *box polynomial* as

$$B(\mathbf{W}; z) := \sum_{\mathbf{v} \in \mathrm{Box}(\mathbf{W}) \cap \mathbb{Z}^d} z^{u(\mathbf{v})}.$$

Set-Up and Notation

Set-Up and Notation

Example: Let $\mathbf{W} = \{(1,3),(2,3)\}$. Then

$$\mathrm{Box}(W) = \{\lambda_1(1,3) + \lambda_2(2,3) : 0 < \lambda_1, \lambda_2 < 1\}.$$

Set-Up and Notation

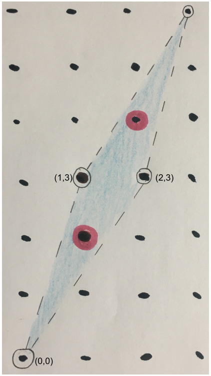
Example: Let $\mathbf{W} = \{(1,3),(2,3)\}$. Then

$$\text{Box}(W) = \{\lambda_1(1,3) + \lambda_2(2,3) : 0 < \lambda_1, \lambda_2 < 1\}.$$

Set-Up and Notation

Example: Let $\mathbf{W} = \{(1,3),(2,3)\}$. Then

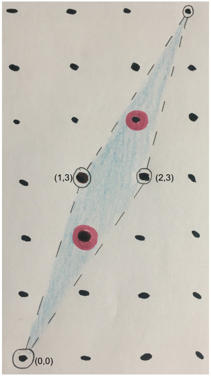
$$\mathrm{Box}(W) = \{\lambda_1(1,3) + \lambda_2(2,3) : 0 < \lambda_1, \lambda_2 < 1\}.$$



Set-Up and Notation

Example: Let $\mathbf{W} = \{(1,3),(2,3)\}$. Then

$$\mathrm{Box}(W) = \{\lambda_1(1,3) + \lambda_2(2,3) : 0 < \lambda_1, \lambda_2 < 1\}.$$



Thus,
$\mathrm{Box}(\mathbf{W}) \cap \mathbb{Z}^2 = \{(1,2),(2,4)\}$
and its associated box polynomial is

$$B(\mathbf{W}; z) = z^2 + z^4.$$

Set-Up and Notation

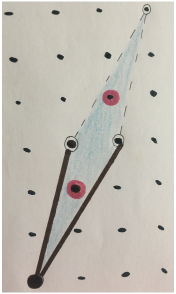
- Define the *fundamental parallelepiped* $\Pi(\mathbf{W})$ to be the half-open variant of $\text{Box}(\mathbf{W})$, namely

$$\Pi(\mathbf{W}) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{w}_i : 0 \leq \lambda_i < 1 \right\}.$$

Set-Up and Notation

- Define the *fundamental parallelepiped* $\Pi(\mathbf{W})$ to be the half-open variant of $\mathrm{Box}(\mathbf{W})$, namely

$$\Pi(\mathbf{W}) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{w}_i : 0 \leq \lambda_i < 1 \right\}.$$



Set-Up and Notation

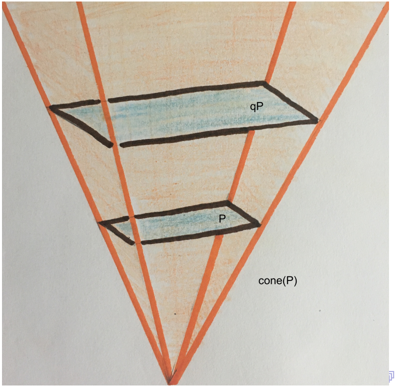
- For a rational polytope $P \subset \mathbb{R}^d$ with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{Q}^d$, we lift the vertices into $\mathbb{R}^{d+1}$ by appending a 1 as the last coordinate. Then the *cone* of P is

$$\operatorname{cone}(P) = \left\{ \sum_{i=1}^{n} \lambda_i (\mathbf{v}_i, 1) : \lambda_i \geq 0 \right\}.$$

Set-Up and Notation

- For a rational polytope $P \subset \mathbb{R}^d$ with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{Q}^d$, we lift the vertices into $\mathbb{R}^{d+1}$ by appending a 1 as the last coordinate. Then the *cone* of P is

$$\text{cone}(P) = \left\{ \sum_{i=1}^{n} \lambda_i(\mathbf{v}_i, 1) : \lambda_i \geq 0 \right\}.$$

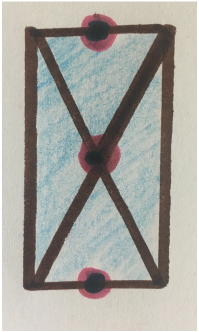


Set-Up and Notation

- A *triangulation* T of a d-polytope P is a subdivision of P into simplices (of all dimensions) that is closed under taking faces.

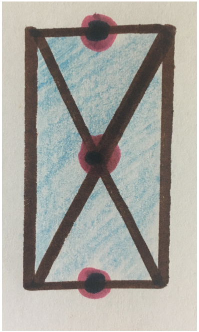
Set-Up and Notation

- A *triangulation* T of a d-polytope P is a subdivision of P into simplices (of all dimensions) that is closed under taking faces.



Set-Up and Notation

- A *triangulation* T of a d-polytope P is a subdivision of P into simplices (of all dimensions) that is closed under taking faces.



- If all vertices of T are rational points, define the *denominator* of T to be the least common multiple of all vertex coordinate denominators of the faces of T.

Set-Up and Notation

Set-Up and Notation

- For each $\Delta \in T$, we define the *h-polynomial* of Δ with respect to T as

$$h_T(\Delta; z) := (1 - z)^{d - \dim(\Delta)} \sum_{\Delta \subseteq \Phi \in T} \left(\frac{z}{1 - z} \right)^{\dim(\Phi) - \dim(\Delta)},$$

where the sum is over all simplices $\Phi \in T$ containing Δ.

Set-Up and Notation

- For each $\Delta \in T$, we define the *h-polynomial* of Δ with respect to T as

$$h_T(\Delta; z) := (1-z)^{d-\dim(\Delta)} \sum_{\Delta \subseteq \Phi \in T} \left(\frac{z}{1-z}\right)^{\dim(\Phi)-\dim(\Delta)},$$

where the sum is over all simplices $\Phi \in T$ containing Δ.

- For a simplex Δ with denominator p, let $\mathbf{W}$ be the set of integral ray generators of cone(Δ) at height p. We define the *h^*-polynomial* of Δ as the generating function of the last coordinate of integer points in $\Pi(\mathbf{W}) := \Pi(\Delta)$, that is,

$$h^*(\Delta; z) = \sum_{\mathbf{v} \in \Pi(\Delta) \cap \mathbb{Z}^{d+1}} z^{u(\mathbf{v})}.$$

Decomposition à la Betke–McMullen

Decomposition à la Betke–McMullen

- Let P be a rational d-polytope and T be a triangulation with denominator q.

Decomposition à la Betke–McMullen

- Let P be a rational d-polytope and T be a triangulation with denominator q.
- For an m-simplex $\Delta \in T$, let $\mathbf{W} = \{(\mathbf{r}_1, q), \ldots, (\mathbf{r}_{m+1}, q)\}$, where the $(\mathbf{r}_i, q)$ are the integral ray generators for $\text{cone}(\Delta)$ at height q.

Decomposition à la Betke–McMullen

- Let P be a rational d-polytope and T be a triangulation with denominator q.
- For an m-simplex $\Delta \in T$, let $\mathbf{W} = \{(\mathbf{r}_1, q), \ldots, (\mathbf{r}_{m+1}, q)\}$, where the $(\mathbf{r}_i, q)$ are the integral ray generators for cone(Δ) at height q.
- Set $B(\mathbf{W}; z) =: B(\Delta; z)$ and $\mathrm{Box}(\mathbf{W}) =: \mathrm{Box}(\Delta)$.

Decomposition à la Betke–McMullen

- Let P be a rational d-polytope and T be a triangulation with denominator q.
- For an m-simplex $\Delta \in T$, let $\mathbf{W} = \{(\mathbf{r}_1, q), \ldots, (\mathbf{r}_{m+1}, q)\}$, where the $(\mathbf{r}_i, q)$ are the integral ray generators for cone(Δ) at height q.
- Set $B(\mathbf{W}; z) =: B(\Delta; z)$ and $\mathrm{Box}(\mathbf{W}) =: \mathrm{Box}(\Delta)$.

Lemma: Fix a triangulation T with denominator q of a rational d-polytope P and let $\Delta \in T$. Then $h^*(\Delta; z) = \sum_{\Omega \subseteq \Delta} B(\Omega; z)$.

Decomposition à la Betke–McMullen

Decomposition à la Betke–McMullen

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Fix a triangulation T with denominator q of a rational d-polytope P,

$$\mathrm{Ehr}(P;z) = \frac{\sum_{\Omega \in T} B(\Omega;z)h(\Omega;z^q)}{(1-z^q)^{d+1}}.$$

Decomposition à la Betke–McMullen

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Fix a triangulation T with denominator q of a rational d-polytope P,

$$\mathrm{Ehr}(P; z) = \frac{\sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^q)}{(1 - z^q)^{d+1}}.$$

Proof Sketch:

Decomposition à la Betke–McMullen

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Fix a triangulation T with denominator q of a rational d-polytope P,

$$\mathrm{Ehr}(P; z) = \frac{\sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^q)}{(1 - z^q)^{d+1}}.$$

Proof Sketch:

- Write P as the disjoint union of all open nonempty simplices in T $(\mathrm{Ehr}(P; z) = 1 + \sum_{\Delta \in T \setminus \emptyset} \mathrm{Ehr}(\Delta^\circ; z))$.

Decomposition à la Betke–McMullen

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Fix a triangulation T with denominator q of a rational d-polytope P,

$$\mathrm{Ehr}(P; z) = \frac{\sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^q)}{(1 - z^q)^{d+1}}.$$

Proof Sketch:

- Write P as the disjoint union of all open nonempty simplices in T $(\mathrm{Ehr}(P; z) = 1 + \sum_{\Delta \in T \setminus \emptyset} \mathrm{Ehr}(\Delta^\circ; z))$.
- Use Ehrhart–Macdonald reciprocity.

Decomposition à la Betke–McMullen

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Fix a triangulation T with denominator q of a rational d-polytope P,

$$\mathrm{Ehr}(P; z) = \frac{\sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^q)}{(1 - z^q)^{d+1}}.$$

Proof Sketch:

- Write P as the disjoint union of all open nonempty simplices in T ($\mathrm{Ehr}(P; z) = 1 + \sum_{\Delta \in T \setminus \emptyset} \mathrm{Ehr}(\Delta^\circ; z)$).
- Use Ehrhart–Macdonald reciprocity.
- Apply previous lemma.

Decomposition à la Betke–McMullen

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Fix a triangulation T with denominator q of a rational d-polytope P,

$$\mathrm{Ehr}(P; z) = \frac{\sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^q)}{(1 - z^q)^{d+1}}.$$

Proof Sketch:

- Write P as the disjoint union of all open nonempty simplices in T $(\mathrm{Ehr}(P; z) = 1 + \sum_{\Delta \in T \setminus \emptyset} \mathrm{Ehr}(\Delta^\circ; z))$.
- Use Ehrhart–Macdonald reciprocity.
- Apply previous lemma.
- Use the symmetry of box polynomials.

Decomposition à la Betke–McMullen

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Fix a triangulation T with denominator q of a rational d-polytope P,

$$\operatorname{Ehr}(P;z) = \frac{\sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^q)}{(1-z^q)^{d+1}}.$$

Proof Sketch:

- Write P as the disjoint union of all open nonempty simplices in T ($\operatorname{Ehr}(P;z) = 1 + \sum_{\Delta \in T \setminus \emptyset} \operatorname{Ehr}(\Delta^\circ; z)$).
- Use Ehrhart–Macdonald reciprocity.
- Apply previous lemma.
- Use the symmetry of box polynomials.
- Use the definition of the h-polynomial.

Rational h^*-Monotonicity

Theorem: (Stanley 1993) Suppose $P \subseteq Q$ are rational polytopes with qP and qQ integral (for minimal possible $q \in \mathbb{Z}_{>0}$). Define the h^*-polynomials via

$$\mathrm{Ehr}(P;z) = \frac{h^*(P;z)}{(1-z^q)^{\dim(P)+1}} \qquad \text{and} \qquad \mathrm{Ehr}(Q;z) = \frac{h^*(Q;z)}{(1-z^q)^{\dim(Q)+1}}.$$

Then $h_i^*(P;z) \leq h_i^*(Q;z)$ coefficient-wise.

Rational h^*-Monotonicity

Rational h^*-Monotonicity

Lemma: (Beck–Braun–Vindas-Meléndez 2020+) Suppose P is a polytope and T a triangulation of P.

Rational h^*-Monotonicity

Lemma: (Beck–Braun–Vindas-Meléndez 2020+) Suppose P is a polytope and T a triangulation of P. Let $P \subseteq Q$ be a polytope and T' be a triangulation of Q such that T' restricted to P is T.

Rational h^*-Monotonicity

Lemma: (Beck–Braun–Vindas-Meléndez 2020+) Suppose P is a polytope and T a triangulation of P. Let $P \subseteq Q$ be a polytope and T' be a triangulation of Q such that T' restricted to P is T. Further, if $\dim(P) < \dim(Q)$, assume that there exists a set of affinely independent vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of Q outside the affine span of P such that

Rational h^*-Monotonicity

Lemma: (Beck–Braun–Vindas-Meléndez 2020+) Suppose P is a polytope and T a triangulation of P. Let $P \subseteq Q$ be a polytope and T' be a triangulation of Q such that T' restricted to P is T. Further, if $\dim(P) < \dim(Q)$, assume that there exists a set of affinely independent vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of Q outside the affine span of P such that

1. the join $T * \mathrm{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subcomplex of T' and

Rational h^*-Monotonicity

Lemma: (Beck–Braun–Vindas-Meléndez 2020+) Suppose P is a polytope and T a triangulation of P. Let $P \subseteq Q$ be a polytope and T' be a triangulation of Q such that T' restricted to P is T. Further, if $\dim(P) < \dim(Q)$, assume that there exists a set of affinely independent vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of Q outside the affine span of P such that

1. the join $T * \mathrm{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subcomplex of T' and
2. $\dim(P * \mathrm{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}) = \dim(Q)$.

Rational h^*-Monotonicity

Lemma: (Beck–Braun–Vindas-Meléndez 2020+) Suppose P is a polytope and T a triangulation of P. Let $P \subseteq Q$ be a polytope and T' be a triangulation of Q such that T' restricted to P is T. Further, if $\dim(P) < \dim(Q)$, assume that there exists a set of affinely independent vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of Q outside the affine span of P such that

1. the join $T * \mathrm{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subcomplex of T' and
2. $\dim(P * \mathrm{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}) = \dim(Q)$.

For every face $\Omega \in T$, the coefficient-wise inequality $h_T(\Omega; z) \leq h_{T'}(\Omega, z)$ holds.

Rational h^*-Monotonicity

Rational h^*-Monotonicity

Theorem: (Stanley 1993) Suppose $P \subseteq Q$ are rational polytopes with qP and qQ integral. Then $h_i^*(P;z) \leq h_i^*(Q;z)$ coefficient-wise.

Rational h^*-Monotonicity

Theorem: (Stanley 1993) Suppose $P \subseteq Q$ are rational polytopes with qP and qQ integral. Then $h_i^*(P; z) \leq h_i^*(Q; z)$ coefficient-wise.

Proof Sketch:

Rational h^*-Monotonicity

Theorem: (Stanley 1993) Suppose $P \subseteq Q$ are rational polytopes with qP and qQ integral. Then $h_i^*(P; z) \leq h_i^*(Q; z)$ coefficient-wise.

Proof Sketch:

- Let P contained in Q and let T be a triangulation of P and T' a triangulation of Q such that $T'|_P$ is T, where if $\dim(P) < \dim(Q)$ the triangulation T' satisfies the conditions from the previous lemma.

Rational h^*-Monotonicity

Theorem: (Stanley 1993) Suppose $P \subseteq Q$ are rational polytopes with qP and qQ integral. Then $h_i^*(P;z) \leq h_i^*(Q;z)$ coefficient-wise.

Proof Sketch:

- Let P contained in Q and let T be a triangulation of P and T' a triangulation of Q such that $T'|_P$ is T, where if $\dim(P) < \dim(Q)$ the triangulation T' satisfies the conditions from the previous lemma.
- By rational Betke–McMullen, $h^*(P;z) = \sum_{\Omega \in T} B(\Omega;z)h(\Omega;z^q)$.

Rational h^*-Monotonicity

Theorem: (Stanley 1993) Suppose $P \subseteq Q$ are rational polytopes with qP and qQ integral. Then $h_i^*(P;z) \leq h_i^*(Q;z)$ coefficient-wise.

Proof Sketch:

- Let P contained in Q and let T be a triangulation of P and T' a triangulation of Q such that $T'|_P$ is T, where if $\dim(P) < \dim(Q)$ the triangulation T' satisfies the conditions from the previous lemma.
- By rational Betke–McMullen, $h^*(P;z) = \sum_{\Omega \in T} B(\Omega;z)h(\Omega;z^q)$.
- Since $P \subseteq Q$, $h^*(Q;z) = \sum_{\Omega \in T} B(\Omega;z)h_{T'|_P}(\Omega;z^q) + \sum_{\Omega \in T' \setminus T} B(\Omega;z)h_{T'}(\Omega;z^q)$.

Rational h^*-Monotonicity

Theorem: (Stanley 1993) Suppose $P \subseteq Q$ are rational polytopes with qP and qQ integral. Then $h_i^*(P; z) \leq h_i^*(Q; z)$ coefficient-wise.

Proof Sketch:

- Let P contained in Q and let T be a triangulation of P and T' a triangulation of Q such that $T'|_P$ is T, where if $\dim(P) < \dim(Q)$ the triangulation T' satisfies the conditions from the previous lemma.
- By rational Betke–McMullen, $h^*(P; z) = \sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^q)$.
- Since $P \subseteq Q$,
  $h^*(Q; z) = \sum_{\Omega \in T} B(\Omega; z) h_{T'|_P}(\Omega; z^q) + \sum_{\Omega \in T' \setminus T} B(\Omega; z) h_{T'}(\Omega; z^q)$.
- By the lemma, the coefficients of $\sum_{\Omega \in T} B(\Omega; z) h_{T'|_P}(\Omega; z^q)$ dominate the coefficients of $\sum_{\Omega \in T} B(\Omega; z) h_{T'}(\Omega; z^q)$.

Rational h^*-Monotonicity

Theorem: (Stanley 1993) Suppose $P \subseteq Q$ are rational polytopes with qP and qQ integral. Then $h_i^*(P;z) \leq h_i^*(Q;z)$ coefficient-wise.

Proof Sketch:

- Let P contained in Q and let T be a triangulation of P and T' a triangulation of Q such that $T'|_P$ is T, where if $\dim(P) < \dim(Q)$ the triangulation T' satisfies the conditions from the previous lemma.
- By rational Betke–McMullen, $h^*(P;z) = \sum_{\Omega \in T} B(\Omega;z)h(\Omega;z^q)$.
- Since $P \subseteq Q$, $h^*(Q;z) = \sum_{\Omega \in T} B(\Omega;z)h_{T'|_P}(\Omega;z^q) + \sum_{\Omega \in T' \setminus T} B(\Omega;z)h_{T'}(\Omega;z^q)$.
- By the lemma, the coefficients of $\sum_{\Omega \in T} B(\Omega;z)h_{T'|_P}(\Omega;z^q)$ dominate the coefficients of $\sum_{\Omega \in T} B(\Omega;z)h_{T'}(\Omega;z^q)$.
- $\sum_{\Omega \in T} B(\Omega;z)h(\Omega;z^q) \leq \sum_{\Omega \in T} B(\Omega;z)h_{T'|_P}(\Omega;z^q) \leq \sum_{\Omega \in T} B(\Omega;z)h_{T'|_P}(\Omega;z^q) + \sum_{\Omega \in T' \setminus T} B(\Omega;z)h_{T'}(\Omega;z^q)$.

Decomposition from Boundary Triangulation

Decomposition from Boundary Triangulation

Set-up:

- Fix a boundary triangulation T with denominator q of a rational d-polytope P.

Decomposition from Boundary Triangulation

Set-up:

- Fix a boundary triangulation T with denominator q of a rational d-polytope P.
- Take $\ell \in \mathbb{Z}_{>0}$, such that ℓP contains a lattice point $\mathbf{a}$ in its interior. Thus $(\mathbf{a}, \ell) \in \mathrm{cone}(P)^\circ \cap Z^{d+1}$ is a lattice point in the interior of the cone of P at height ℓ and $\mathrm{cone}((\mathbf{a}, \ell))$ is the ray through the point $(\mathbf{a}, \ell)$.

Decomposition from Boundary Triangulation

Set-up:

- Fix a boundary triangulation T with denominator q of a rational d-polytope P.
- Take $\ell \in \mathbb{Z}_{>0}$, such that ℓP contains a lattice point $\mathbf{a}$ in its interior. Thus $(\mathbf{a}, \ell) \in \text{cone}(P)^\circ \cap Z^{d+1}$ is a lattice point in the interior of the cone of P at height ℓ and $\text{cone}((\mathbf{a}, \ell))$ is the ray through the point $(\mathbf{a}, \ell)$.
- We cone over each $\Delta \in T$ and define $\mathbf{W} = \{(\mathbf{r}_1, q), \ldots, (\mathbf{r}_{m+1}, q)\}$ where the $(\mathbf{r}_i, q)$ are integral ray generators of $\text{cone}(\Delta)$ at height q.

Decomposition from Boundary Triangulation

Set-up:

- Fix a boundary triangulation T with denominator q of a rational d-polytope P.
- Take $\ell \in \mathbb{Z}_{>0}$, such that ℓP contains a lattice point $\mathbf{a}$ in its interior. Thus $(\mathbf{a}, \ell) \in \text{cone}(P)^\circ \cap Z^{d+1}$ is a lattice point in the interior of the cone of P at height ℓ and $\text{cone}((\mathbf{a}, \ell))$ is the ray through the point $(\mathbf{a}, \ell)$.
- We cone over each $\Delta \in T$ and define $\mathbf{W} = \{(\mathbf{r}_1, q), \ldots, (\mathbf{r}_{m+1}, q)\}$ where the $(\mathbf{r}_i, q)$ are integral ray generators of $\text{cone}(\Delta)$ at height q.
- Let $B(\mathbf{W}; z) =: B(\Delta; z)$ and $\mathbf{W}' = \mathbf{W} \cup \{(\mathbf{a}, \ell)\}$ be the set of generators from $\mathbf{W}$ together with $(\mathbf{a}, \ell)$ and set $\text{cone}(\Delta')$ to be the cone generated by $\mathbf{W}'$, wich associated box polynomial $B(\mathbf{W}'; z) =: B(\Delta'; z)$.

Decomposition from Boundary Triangulation

Decomposition from Boundary Triangulation

Set-up (continued):

Decomposition from Boundary Triangulation

Set-up (continued):

- A point $\mathbf{v} \in \text{cone}(\Delta)$ can be uniquely expressed as $\mathbf{v} = \sum_{i=1}^{m+1} \lambda_i(\mathbf{r}_i, q)$ for $\lambda_i \geq 0$.

Decomposition from Boundary Triangulation

Set-up (continued):

- A point $\mathbf{v} \in \text{cone}(\Delta)$ can be uniquely expressed as $\mathbf{v} = \sum_{i=1}^{m+1} \lambda_i(\mathbf{r}_i, q)$ for $\lambda_i \geq 0$.
- Define $I(\mathbf{v}) := \{i \in [m+1] : \lambda_i \in \mathbb{Z}\}$ and $\overline{I(\mathbf{v})} := [m+1] \setminus I(\mathbf{v})$.

Decomposition from Boundary Triangulation

Set-up (continued):

- A point $\mathbf{v} \in \text{cone}(\Delta)$ can be uniquely expressed as $\mathbf{v} = \sum_{i=1}^{m+1} \lambda_i(\mathbf{r}_i, q)$ for $\lambda_i \geq 0$.
- Define $I(\mathbf{v}) := \{i \in [m+1] : \lambda_i \in \mathbb{Z}\}$ and $\overline{I(\mathbf{v})} := [m+1] \setminus I(\mathbf{v})$.
- For each $\mathbf{v} \in \text{cone}(P)$ we associate two faces $\Delta(\mathbf{v})$ and $\Omega(\mathbf{v})$ of T, where $\Delta(\mathbf{v})$ is chosen to be the minimal face of T such that $\mathbf{v} \in \text{cone}(\Delta'(\mathbf{v}))$ and we define $\Omega(\mathbf{v}) := \text{conv}\left\{\frac{\mathbf{r}_i}{q} : i \in \overline{I(\mathbf{v}))}\right\} \subseteq \Delta(\mathbf{v})$.

Decomposition from Boundary Triangulation

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Consider a rational d-polytope P that contains an interior point $\frac{\mathbf{a}}{\ell}$, where $\mathbf{a} \in \mathbb{Z}^d$ and $\ell \in \mathbb{Z}_{>0}$.

Decomposition from Boundary Triangulation

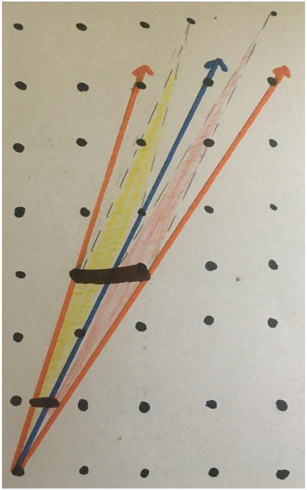
Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Consider a rational d-polytope P that contains an interior point $\frac{\mathbf{a}}{\ell}$, where $\mathbf{a} \in \mathbb{Z}^d$ and $\ell \in \mathbb{Z}_{>0}$. Fix a boundary triangulation T of P with denominator q.

Decomposition from Boundary Triangulation

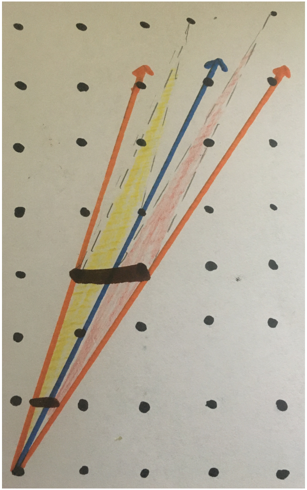
Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Consider a rational d-polytope P that contains an interior point $\frac{\mathbf{a}}{\ell}$, where $\mathbf{a} \in \mathbb{Z}^d$ and $\ell \in \mathbb{Z}_{>0}$. Fix a boundary triangulation T of P with denominator q. Then

$$
\begin{aligned}
h^*(P; z) &= \frac{1-z^q}{1-z^\ell} \sum_{\Omega \in T} \left(B(\Omega; z) + B(\Omega'; z) \right) h(\Omega; z^q) \\
&= \frac{1+z+\cdots+z^{q-1}}{1+z+\cdots+z^{\ell-1}} \sum_{\Omega \in T} \left(B(\Omega; z) + B(\Omega'; z) \right) h(\Omega; z^q).
\end{aligned}
$$

Decomposition from Boundary Triangulation

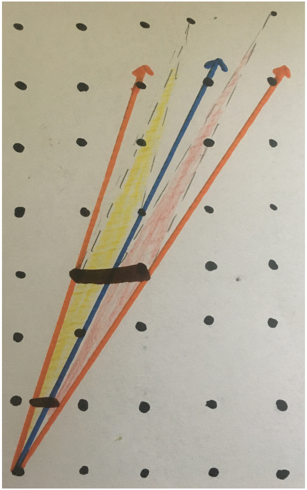


Decomposition from Boundary Triangulation



Let $P = \left[\frac{1}{3}, \frac{2}{3}\right]$.

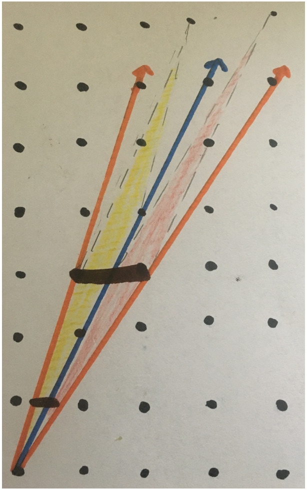
Decomposition from Boundary Triangulation



Let $P = \left[\frac{1}{3}, \frac{2}{3}\right]$.

- Boundary triangulation with denominator 3

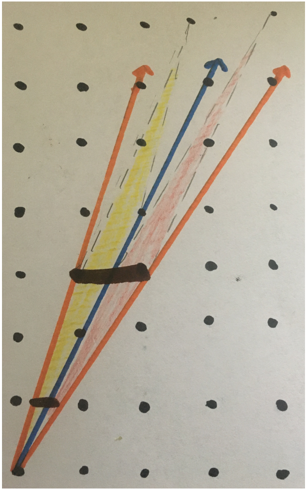
Decomposition from Boundary Triangulation



Let $P = \left[\frac{1}{3}, \frac{2}{3}\right]$.

- Boundary triangulation with denominator 3
- $(\mathbf{a}, \ell) = (2, 4)$

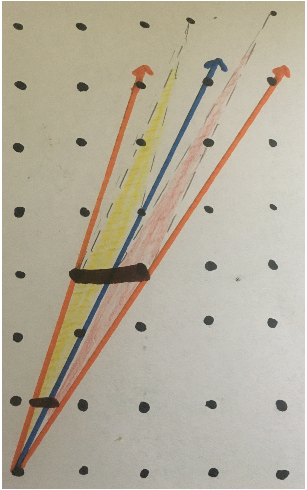
Decomposition from Boundary Triangulation



Let $P = \left[\frac{1}{3}, \frac{2}{3}\right]$.

- Boundary triangulation with denominator 3
- $(\mathbf{a}, \ell) = (2, 4)$
- simplices in T: empty face $\emptyset$ and vertices $\Delta_1 = \frac{1}{3}$ and $\Delta_2 = \frac{2}{3}$

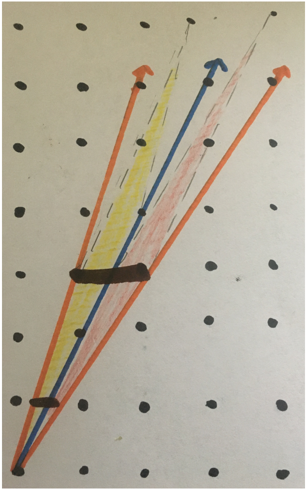
Decomposition from Boundary Triangulation



Let $P = \left[\frac{1}{3}, \frac{2}{3}\right]$.

- Boundary triangulation with denominator 3
- $(\mathbf{a}, \ell) = (2, 4)$
- simplices in T: empty face $\emptyset$ and vertices $\Delta_1 = \frac{1}{3}$ and $\Delta_2 = \frac{2}{3}$
- $\mathbf{W}_1 = \{(1,3)\}$ and $\mathbf{W}_2 = \{(2,3)\}$

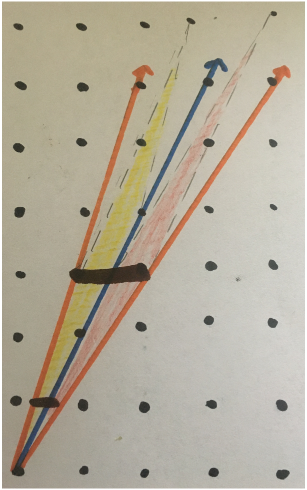
Decomposition from Boundary Triangulation



Let $P = \left[\frac{1}{3}, \frac{2}{3}\right]$.

- Boundary triangulation with denominator 3
- $(\mathbf{a}, \ell) = (2, 4)$
- simplices in T: empty face $\emptyset$ and vertices $\Delta_1 = \frac{1}{3}$ and $\Delta_2 = \frac{2}{3}$
- $\mathbf{W}_1 = \{(1, 3)\}$ and $\mathbf{W}_2 = \{(2, 3)\}$
- For $\mathbf{v} \in \text{cone}(P)$ then the only options for $\Delta(\mathbf{v})$ to be chosen as a minimal face of T such that $\mathbf{v} \in \text{cone}\,\Delta'(\mathbf{v})$ are again to consider $\emptyset$, Δ_1, and Δ_2. In this example, $\Omega(\mathbf{v}) = \Delta(\mathbf{v})$.

Decomposition from Boundary Triangulation

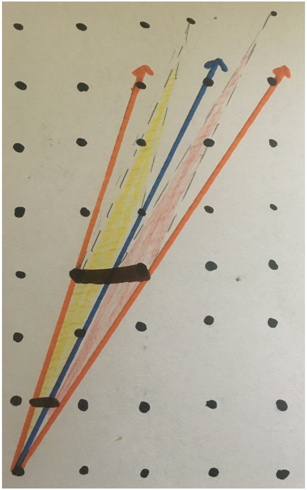


Let $P = \left[\frac{1}{3}, \frac{2}{3}\right]$.

- Boundary triangulation with denominator 3
- $(\mathbf{a}, \ell) = (2, 4)$
- simplices in T: empty face $\emptyset$ and vertices $\Delta_1 = \frac{1}{3}$ and $\Delta_2 = \frac{2}{3}$
- $\mathbf{W}_1 = \{(1,3)\}$ and $\mathbf{W}_2 = \{(2,3)\}$
- For $\mathbf{v} \in \text{cone}(P)$ then the only options for $\Delta(\mathbf{v})$ to be chosen as a minimal face of T such that $\mathbf{v} \in \text{cone}\,\Delta'(\mathbf{v})$ are again to consider $\emptyset$, Δ_1, and Δ_2. In this example, $\Omega(\mathbf{v}) = \Delta(\mathbf{v})$.

$\Omega \in T$	$\dim(\Omega)$	$B(\Omega; z)$	$B(\Omega'; z)$	$h(\Omega, z^3)$
Δ_1	0	0	0	1
Δ_2	0	0	0	1
$\emptyset$	-1	1	z^2	$1 + z^3$

Decomposition from Boundary Triangulation



Let $P = \left[\frac{1}{3}, \frac{2}{3}\right]$.

- Boundary triangulation with denominator 3
- $(\mathbf{a}, \ell) = (2, 4)$
- simplices in T: empty face $\emptyset$ and vertices $\Delta_1 = \frac{1}{3}$ and $\Delta_2 = \frac{2}{3}$
- $\mathbf{W}_1 = \{(1,3)\}$ and $\mathbf{W}_2 = \{(2,3)\}$
- For $\mathbf{v} \in \text{cone}(P)$ then the only options for $\Delta(\mathbf{v})$ to be chosen as a minimal face of T such that $\mathbf{v} \in \text{cone}\,\Delta'(\mathbf{v})$ are again to consider $\emptyset$, Δ_1, and Δ_2. In this example, $\Omega(\mathbf{v}) = \Delta(\mathbf{v})$.

$\Omega \in T$	$\dim(\Omega)$	$B(\Omega; z)$	$B(\Omega'; z)$	$h(\Omega, z^3)$
Δ_1	0	0	0	1
Δ_2	0	0	0	1
$\emptyset$	-1	1	z^2	$1 + z^3$

$$h^*(P; z) = \frac{1 - z^3}{1 - z^4}\left(1 + z^3 + z^2 + z^5\right)$$
$$= 1 + z^2 + z^4.$$

Rational Stapledon Decomposition and Inequalities

Proposition: (Beck–Braun–Vindas-Meléndez 2020+) Let P be a rational d-polytope with denominator q and Ehrhart series

$$\mathrm{Ehr}(P;z) = \frac{h^*(P;z)}{(1-z^q)^{d+1}}\,.$$

Then $\deg h^*(P;z) = s$ if and only if $(q(d+1)-s)P$ is the smallest integer dilate of P that contains an interior lattice point.

Rational Stapledon Decomposition and Inequalities

Rational Stapledon Decomposition and Inequalities

Next, we turn our attention to the polynomial

$$\overline{h^*}(P;z) := \left(1 + z + \cdots + z^{\ell-1}\right) h^*(P;z)\,.$$

Rational Stapledon Decomposition and Inequalities

Next, we turn our attention to the polynomial

$$\overline{h^*}(P;z) := \left(1 + z + \cdots + z^{\ell-1}\right) h^*(P;z)\,.$$

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Let P be a rational d-polytope with denominator q, and let $s := \deg h^*(P;z)$.

Rational Stapledon Decomposition and Inequalities

Next, we turn our attention to the polynomial

$$\overline{h^*}(P;z) := \left(1 + z + \cdots + z^{\ell-1}\right) h^*(P;z)\,.$$

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Let P be a rational d-polytope with denominator q, and let $s := \deg h^*(P;z)$. Then $\overline{h^*}(P;z)$ has a unique decomposition

$$\overline{h^*}(P;z) = a(z) + z^{\ell} b(z)\,,$$

where $\ell = q(d+1) - s$ and $a(z)$ and $b(z)$ are polynomials with integer coefficients satisfying $a(z) = z^{q(d+1)-1} a\left(\frac{1}{z}\right)$ and $b(z) = z^{q(d+1)-1-\ell} b\left(\frac{1}{z}\right)$.

Next, we turn our attention to the polynomial

$$\overline{h^*}(P;z) := \left(1 + z + \cdots + z^{\ell-1}\right) h^*(P;z)\,.$$

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Let P be a rational d-polytope with denominator q, and let $s := \deg h^*(P;z)$. Then $\overline{h^*}(P;z)$ has a unique decomposition

$$\overline{h^*}(P;z) = a(z) + z^{\ell} b(z)\,,$$

where $\ell = q(d+1) - s$ and $a(z)$ and $b(z)$ are polynomials with integer coefficients satisfying $a(z) = z^{q(d+1)-1} a\left(\frac{1}{z}\right)$ and $b(z) = z^{q(d+1)-1-\ell} b\left(\frac{1}{z}\right)$. Moreover, the coefficients of $a(z)$ and $b(z)$ are nonnegative.

Rational Stapledon Decomposition and Inequalities

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Let P be a rational d-polytope with denominator q, let $s := \deg h^*(P; z)$ and $\ell := q(d+1) - s$.

Rational Stapledon Decomposition and Inequalities

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Let P be a rational d-polytope with denominator q, let $s := \deg h^*(P;z)$ and $\ell := q(d+1) - s$. The h^*-vector $(h_0^*, \ldots, h_{q(d+1)-1}^*)$ of P satisfies the following inequalities:

$$h_0^* + \cdots + h_{i+1}^* \geq h_{q(d+1)-1}^* + \cdots + h_{q(d+1)-1-i}^* \,, \qquad i = 0, \ldots, \left\lfloor \frac{q(d+1)-1}{2} \right\rfloor - 1\,, \tag{1}$$

$$h_s^* + \cdots + h_{s-i}^* \geq h_0^* + \cdots + h_i^* \,, \qquad i = 0, \ldots, q(d+1)-1\,. \tag{2}$$

Rational Reflexive Polytopes

Rational Reflexive Polytopes

- A lattice polytope is *reflexive* if its dual is also a lattice polytope.

Rational Reflexive Polytopes

- A lattice polytope is *reflexive* if its dual is also a lattice polytope.
- Hibi (1992): A lattice polytope P is the translate of a reflexive polytope if and only if $\mathrm{Ehr}\left(P;\frac{1}{z}\right) = (-1)^{d+1} z\,\mathrm{Ehr}(P;z)$ as rational functions, that is, $h^*(z)$ is palindromic.

Rational Reflexive Polytopes

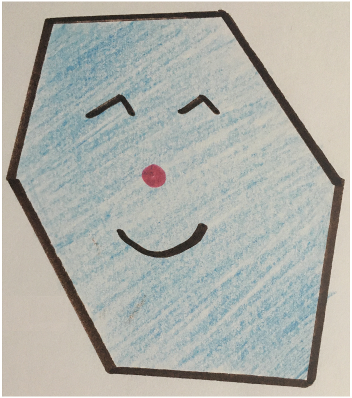
- A lattice polytope is *reflexive* if its dual is also a lattice polytope.
- Hibi (1992): A lattice polytope P is the translate of a reflexive polytope if and only if $\mathrm{Ehr}\left(P; \frac{1}{z}\right) = (-1)^{d+1} z \, \mathrm{Ehr}(P; z)$ as rational functions, that is, $h^*(z)$ is palindromic.
- Fiset–Kaspryzk (2008): A rational polytope P whose dual is a lattice polytope has a palindromic h^*-polynomial.

Rational Reflexive Polytopes

- A lattice polytope is *reflexive* if its dual is also a lattice polytope.
- Hibi (1992): A lattice polytope P is the translate of a reflexive polytope if and only if $\mathrm{Ehr}\left(P; \frac{1}{z}\right) = (-1)^{d+1} z \, \mathrm{Ehr}(P; z)$ as rational functions, that is, $h^*(z)$ is palindromic.
- Fiset–Kaspryzk (2008): A rational polytope P whose dual is a lattice polytope has a palindromic h^*-polynomial.

Theorem: (Beck–Braun–Vindas-Meléndez 2020+) Let P be a rational polytope containing the origin. The dual of P is a lattice polytope if and only if $\overline{h^*}(P; z) = h^*(z) = a(z)$, that is, $b(z) = 0$ in the a/b-decomposition of $\overline{h^*}(P; z)$.

The End



¡Gracias!