The **F**undamental **T**heorem of **A**lgebra

and. . .

combinatorial basketballs??

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(joint with David Savitt and Theodore Singer)
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Menu

1. Matchings, noncrossing matchings, and the Catalan numbers

2. Gauss's proof of the FTA

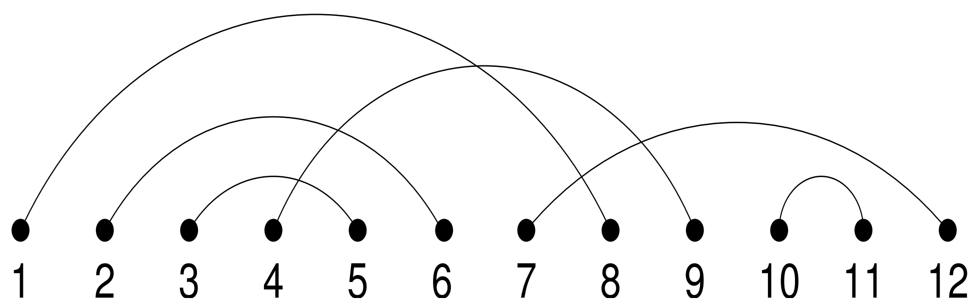
3. Basketballs

4. What we know (not much)

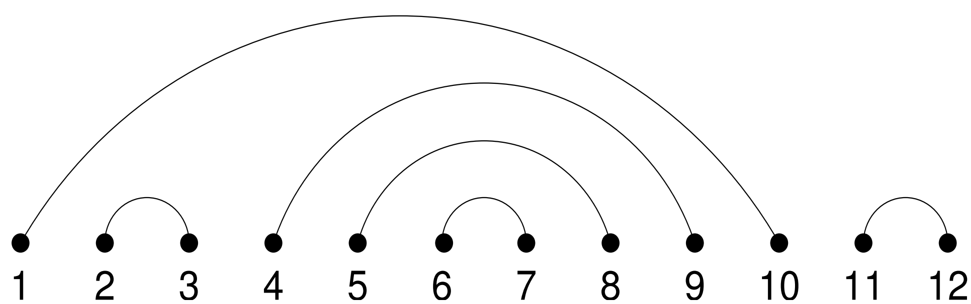
5. What we'd like to know (plenty)

1. (Noncrossing) Matchings

A **matching of order n** is a partition of $[2n] = \{1, 2, \ldots, 2n\}$ into n pairs.



A matching is **noncrossing** if no two pairs cross.



Equivalently, if $\{i < j\}$ and $\{k < \ell\}$ are pairs in a noncrossing matching and $i < k$, then either

$$\text{either} \quad i < j < k < \ell \quad \text{or} \quad i < k < \ell < j.$$

- Even vertices are paired with odd vertices.
- How many noncrossing matchings of order n are there?

Theorem: The number of noncrossing matchings of order n is

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n!)}{(2n+1)n!^2},$$

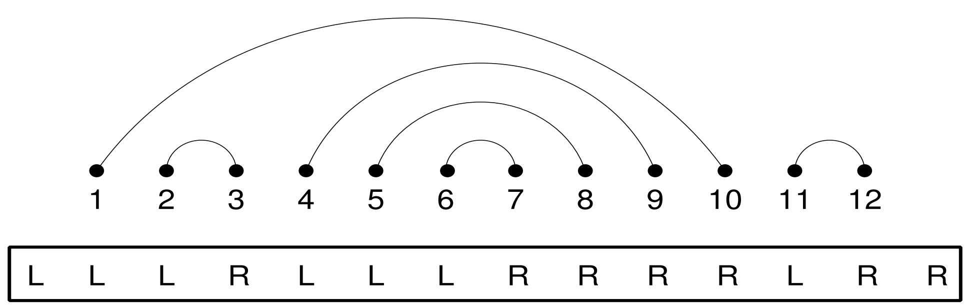
the n^{th} *Catalan number.*

Proof. Encode each NCM of order n by a $(2n+2)$-tuple

$$(\mathsf{L},\, a_1,\, a_2,\, \ldots,\, a_{2n},\, \mathsf{R})$$

where

$$a_i = \begin{cases} \mathsf{L} & \text{if } i < \operatorname{sp}(i), \\ \mathsf{R} & \text{if } i > \operatorname{sp}(i). \end{cases}$$



Each $(2n+2)$-tuple that arises in this way

(1) begins with L and ends with R
(2) contains n other L's and n other R's
(3) has the property

$$\{i \le k :\ a_i = \mathsf{L}\} > \{i \le k :\ a_i = \mathsf{R}\}$$

for $1 \le k \le 2n-1$.

There are $\binom{2n}{n}$ sequences that satisfy (1) and (2).
They occur in "rotation sets" of cardinality $n+1$:

$$
\begin{array}{l}
\boxed{\mathsf{L},}\ \mathsf{L, L, R, L, L, L, R, R, R, R, L, R, R} \\
\mathsf{L, L, R, L, L, L, R, R, R, R, L, R, R,}\ \boxed{\mathsf{L}} \\
\mathsf{L, R, L, L, L, R, R, R, R, L, R, R,}\ \boxed{\mathsf{L},}\ \mathsf{L} \\
\mathsf{L, L, L, R, R, R, R, L, R, R,}\ \boxed{\mathsf{L},}\ \mathsf{L, L, R} \\
\mathsf{L, L, R, R, R, R, L, R, R,}\ \boxed{\mathsf{L},}\ \mathsf{L, L, R, L} \\
\mathsf{L, R, R, R, R, L, R, R,}\ \boxed{\mathsf{L},}\ \mathsf{L, L, R, L, L} \\
\mathsf{L, R, R,}\ \boxed{\mathsf{L},}\ \mathsf{L, L, R, L, L, L, R, R, R, R}
\end{array}
$$

- The subsequence before $\boxed{\mathsf{L}}$ contains more R's than L's.

— So every rotation set contains at most one member corresponding to an NCM.

- For any sequence (a_i) satisfying (1) and (2), find k such that $a_k = \mathsf{L}$ maximizing

$$\{i \leq k : \ a_i = \mathsf{R}\} - \{i \leq k : \ a_i = \mathsf{L}\}.$$

— Rotating the sequence so that it starts at a_k produces a sequence satisfying (3).

Therefore the number of NCM's is

$$\frac{\text{number of sequences}}{\text{size of a rotation set}} = \frac{1}{n+1}\binom{2n}{n}. \quad \blacksquare$$

2. Gauss's Proof of the FTA

The Fundamental Theorem of Algebra: Let $f(z)$ be a complex polynomial of degree n:

$$f(z) \;=\; z^n \,+\, a_{n-1}z^{n-1} \,+\, \ldots \,+\, a_1 z \,+\, a_0 \qquad (*)$$

where $a_i \in \mathbb{C}$ for all i. Then f has exactly n complex roots (counting multiplicities).

Proof. (Gauss 1799; Gersten–Stallings 1988)

First, if $f(\alpha) = 0$ for $\alpha \in \mathbb{C}$, then $z - \alpha$ divides $f(z)$. So f can have **at most** n roots.

Consider the plane algebraic curves

$$R = \{z : \ \mathrm{Re}\, f(z) = 0\},$$
$$I = \{z : \ \mathrm{Im}\, f(z) = 0\},$$

$\mathrm{Re}\, f(z)$ and $\mathrm{Im}\, f(z)$ are polynomials in x and y, and

$$R \cap I = \{z : \ f(z) = 0\}.$$

In polar coordinates,

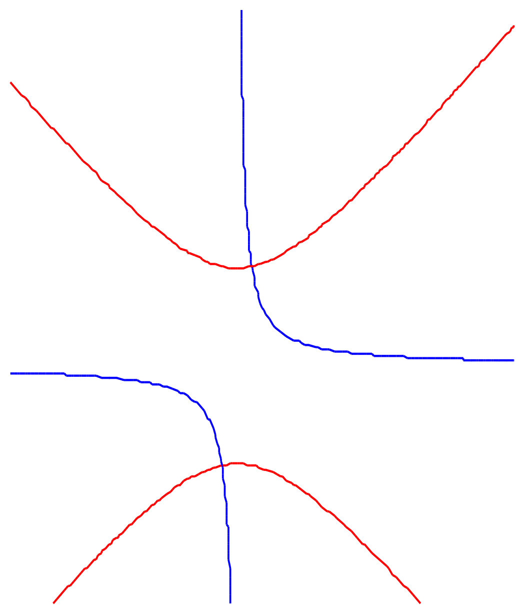
$$\begin{aligned} f(z) &= r^n e^{in\theta} + \ldots \text{(lower-order terms)}, \\ &= r^n(\cos n\theta + \sin n\theta) + \ldots . \end{aligned}$$

That is,

$$\mathrm{Re}\, f(z) = r^n \cos n\theta + \ldots,$$
$$\mathrm{Im}\, f(z) = r^n \sin n\theta + \ldots .$$

Example: $f(z) = z^2 + (1+2i)z + 2$

$$\operatorname{Re} f(z) = x^2 - y^2 + x - 2y + 2$$
$$\operatorname{Im} f(z) = 2xy + 2x + y$$

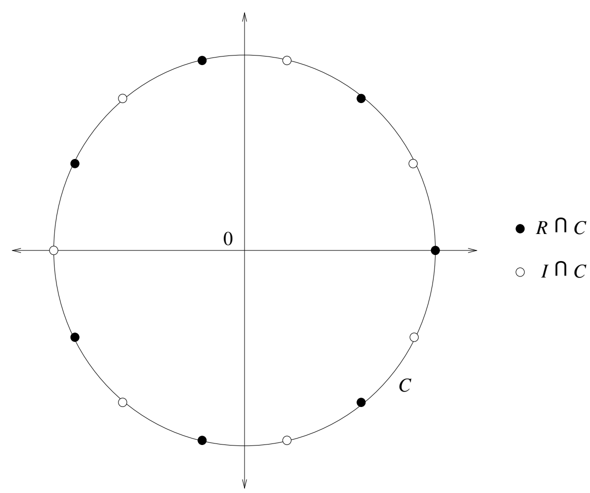


Let C be a circle of radius $r \gg 0$. Then $R \cap C$ consists of $2n$ points, one at each angle θ for which $\cos n\theta = 0$—that is, near the angles

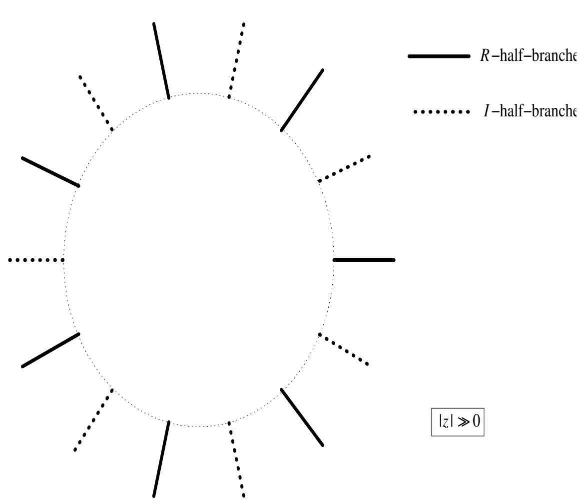
$$0,\ \frac{\pi}{n},\ \frac{2\pi}{n},\ \ldots,\ \frac{(2n-1)\pi}{n}.$$

Likewise, $I \cap C$ consists of $2n$ points at the angles

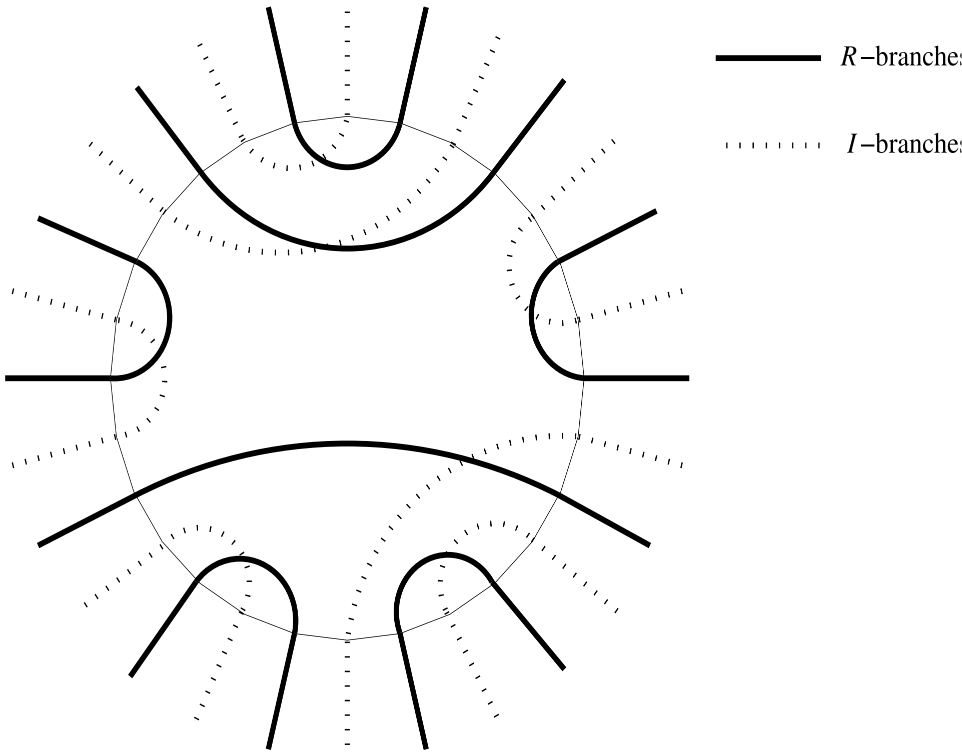
$$\frac{\pi}{2n},\ \frac{3\pi}{2n},\ \ldots,\ \frac{(4n-1)\pi}{2n}.$$



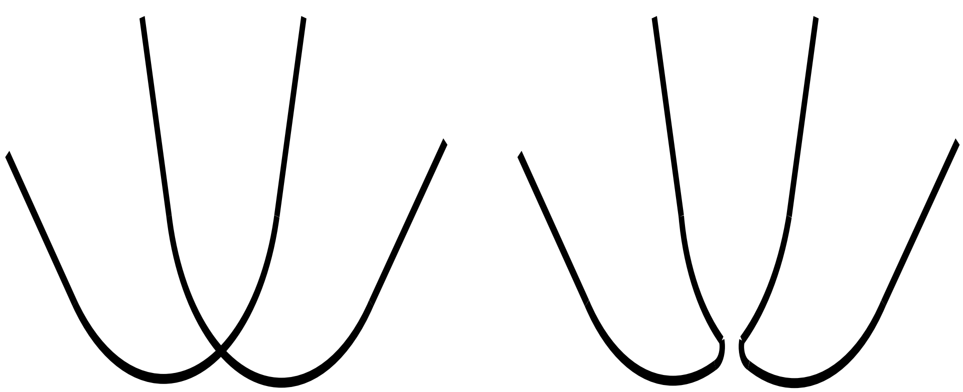
Toward infinity, each of R and I will consist of $2n$ disjoint "half-branches" asymptotic to the lines $\theta = k\pi/2n$ with n even (for R) or odd (for I).



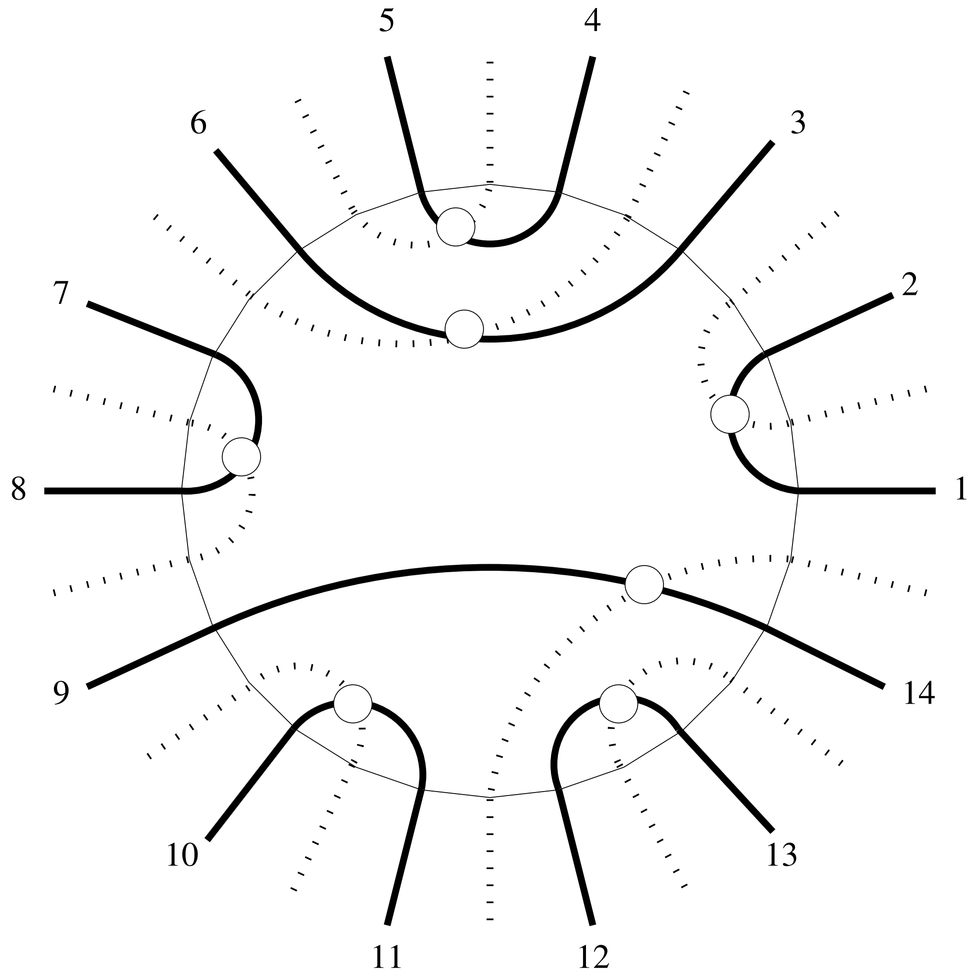
When we move further in, each R-half-branch must connect with another one to form a full R-branch (and likewise for I).



We obtain two matchings of order n, which we may assume are noncrossing.

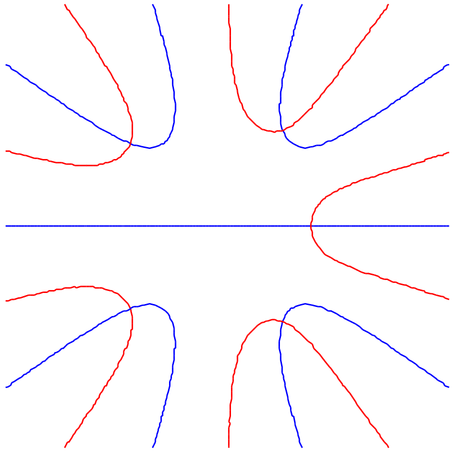


But then each branch of R has an odd number of I-half-branches on each side of it. So it must cross at least one I-branch (in fact, an odd number).

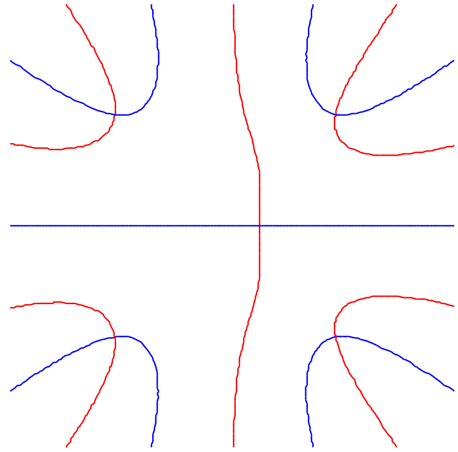


Therefore $|R \cap I| = n$ (counting multiplicities), and f has at least n zeroes. ■

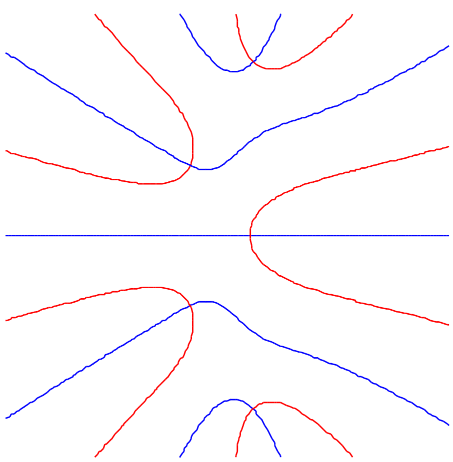
(Unfortunately, the proof requires the Jordan Curve Theorem, so it isn't really that elementary...)



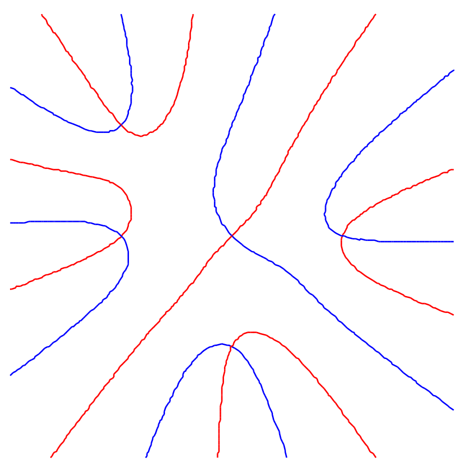
$z^5 + z + 1$



$z^5 + z^2 + 1$



$z^5 + 6z^3 + 3z^2 + 5z - 2$



$z(z-1)(z+1)(z+i)(z+1-i)$

3. Basketballs

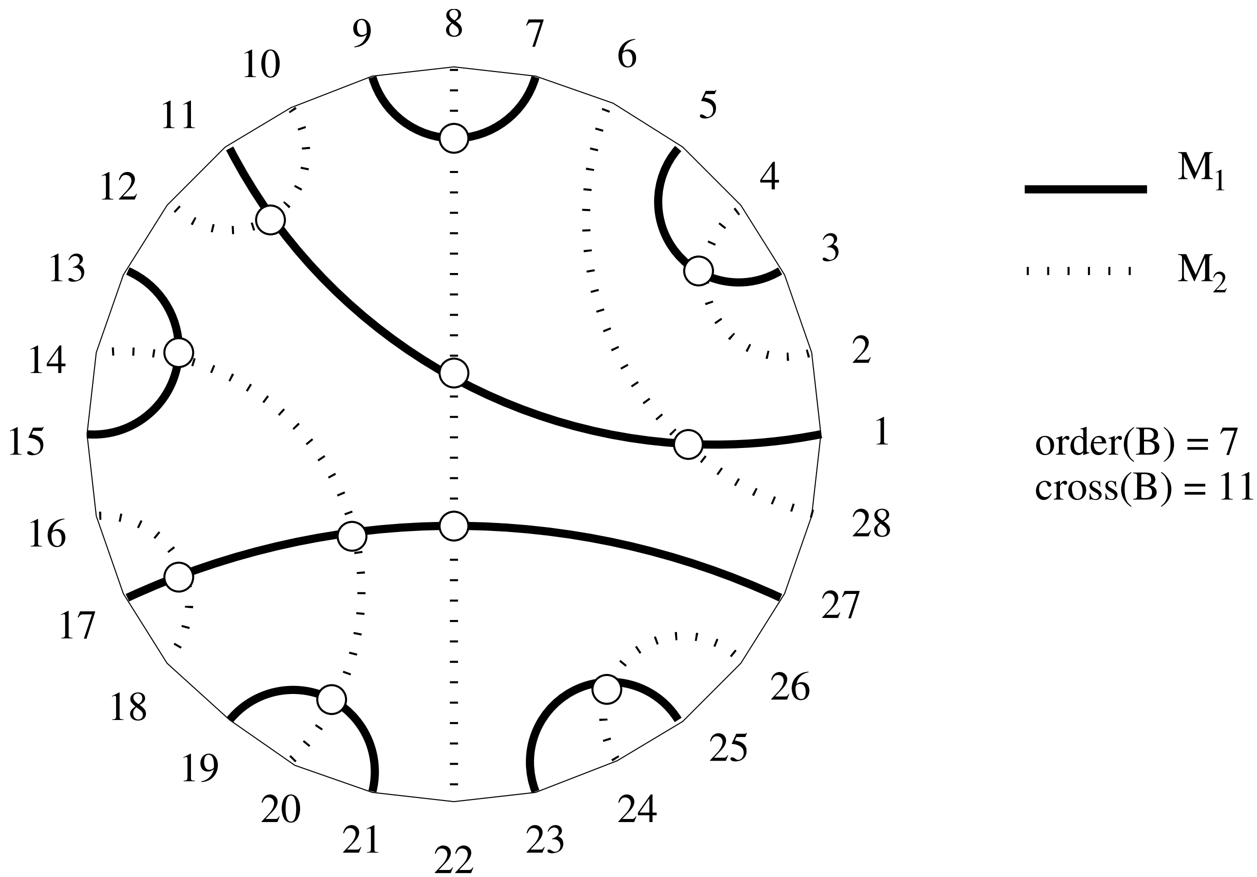
Definition: An ***n*-basketball** is a pair $B = (M_1, M_2)$, where

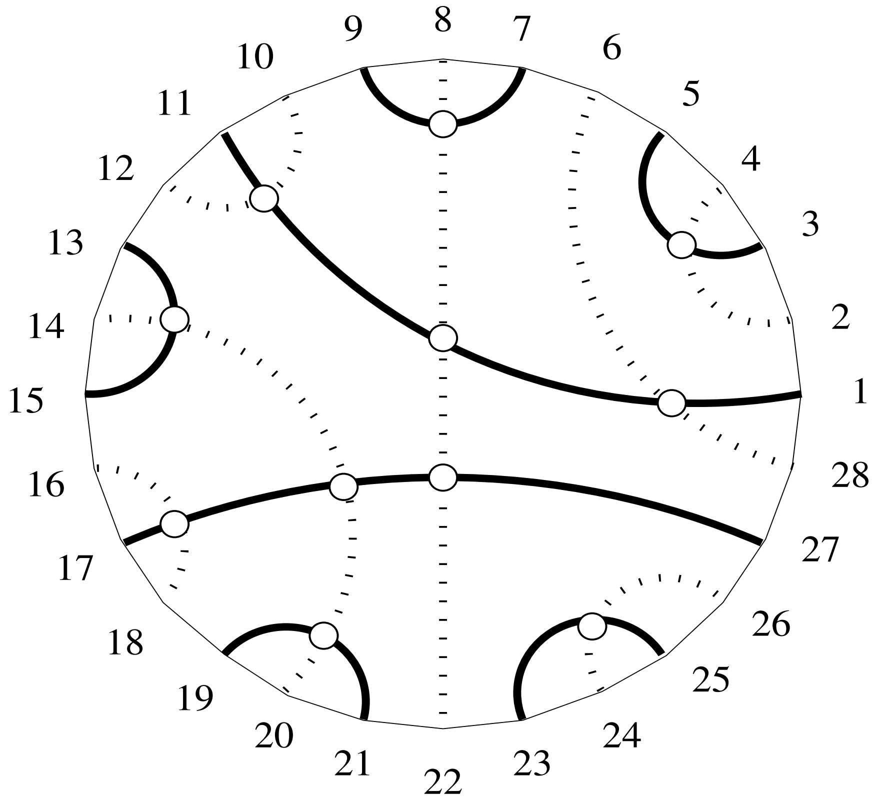
$M_1 =$ "black" noncrossing matching on $\{1, 3, \ldots, 4n-1\}$,

$M_2 =$ "red" noncrossing matching on $\{2, 4, \ldots, 4n\}$.

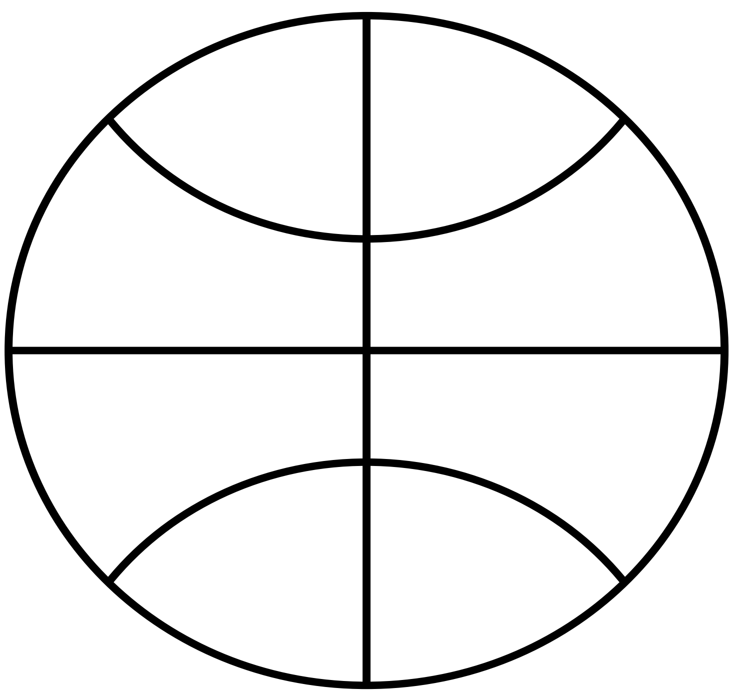
The **order** of B is n.

The **crossing number** of B is the number of times a red pair crosses a black pair.





A combinatorial basketball

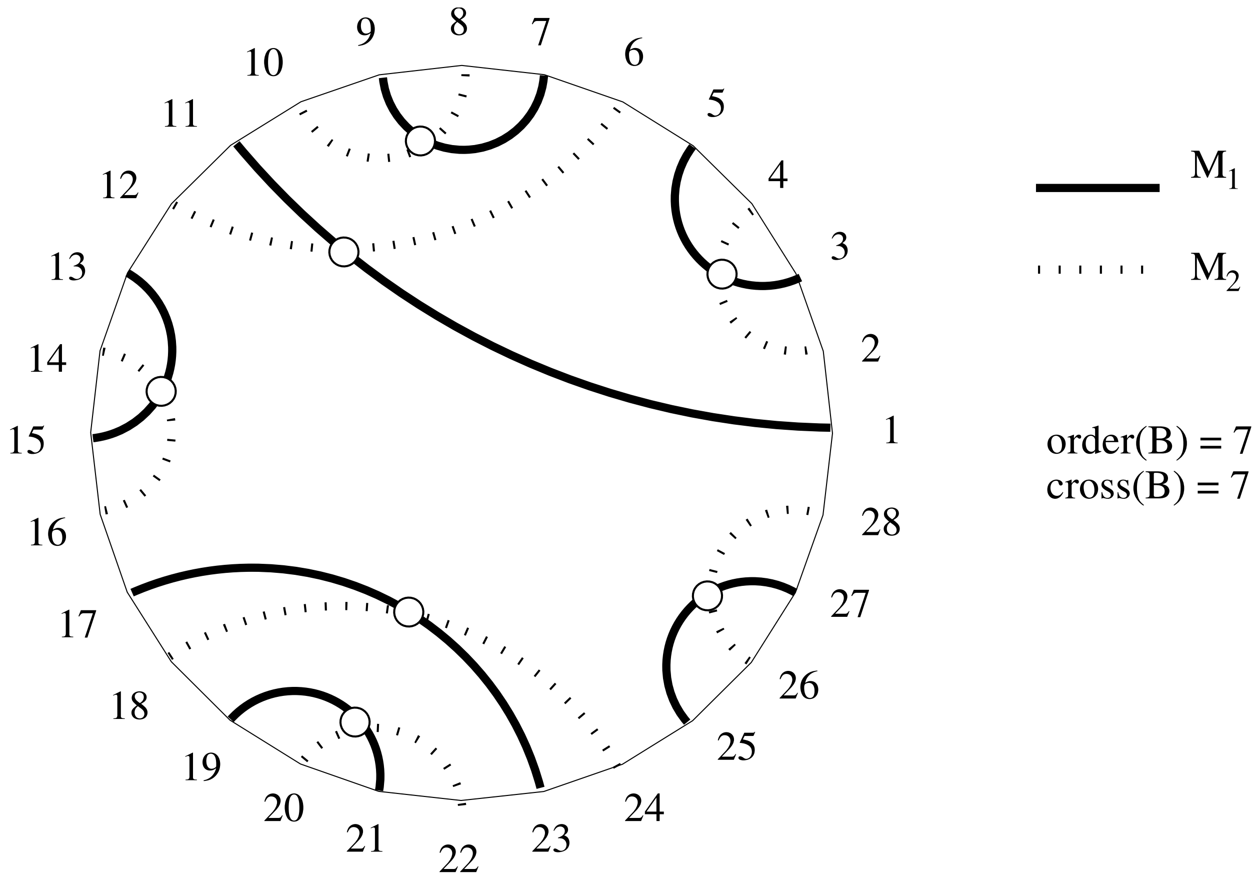


An NBA−approved basketball

- Each red pair crosses an odd number of black pairs. Therefore,

$$\text{cross}(B) \geq n \qquad \text{and} \qquad \text{cross}(B) \equiv n \pmod 2.$$

Call B **simple** if $\text{cross}(B) = n$. (These are the basketballs that arise in Gauss's proof of FTA.)



- How many simple n-basketballs are there?
- How many n-basketballs are there with k crossings?
- Do all simple basketballs arise from some complex polynomial?

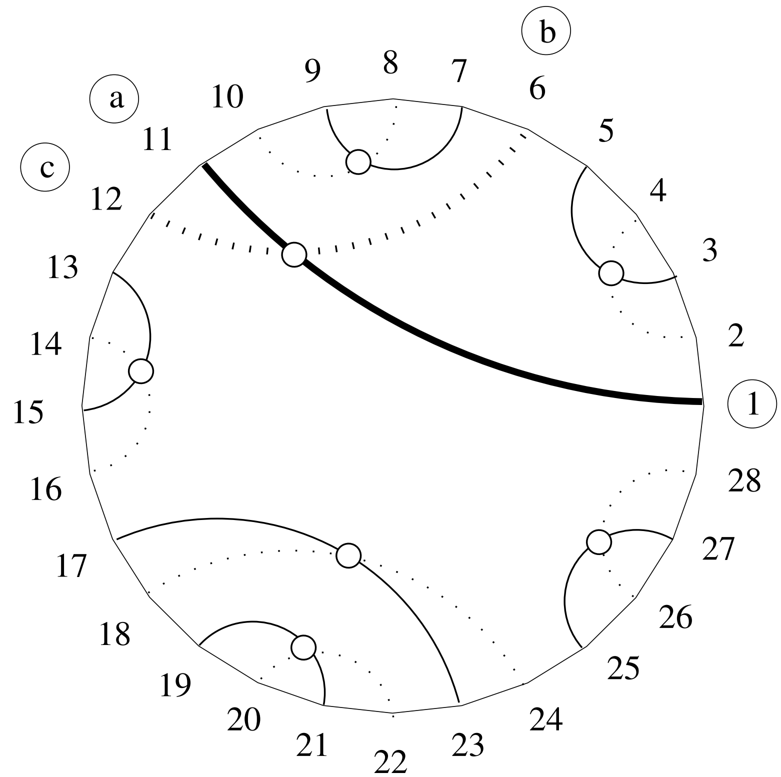
4. Enumerating Basketballs

$$\begin{aligned}
\mathcal{B}(n) &= \{\text{basketballs of order } n\},\\
\mathcal{B}(n,k) &= \{B \in \mathcal{B}(n) : \ \text{cross}(B) = k\},\\
b(n) &= \#\mathcal{B}(n,n) \ = \ \text{number of simple } n\text{-basketballs.}
\end{aligned}$$

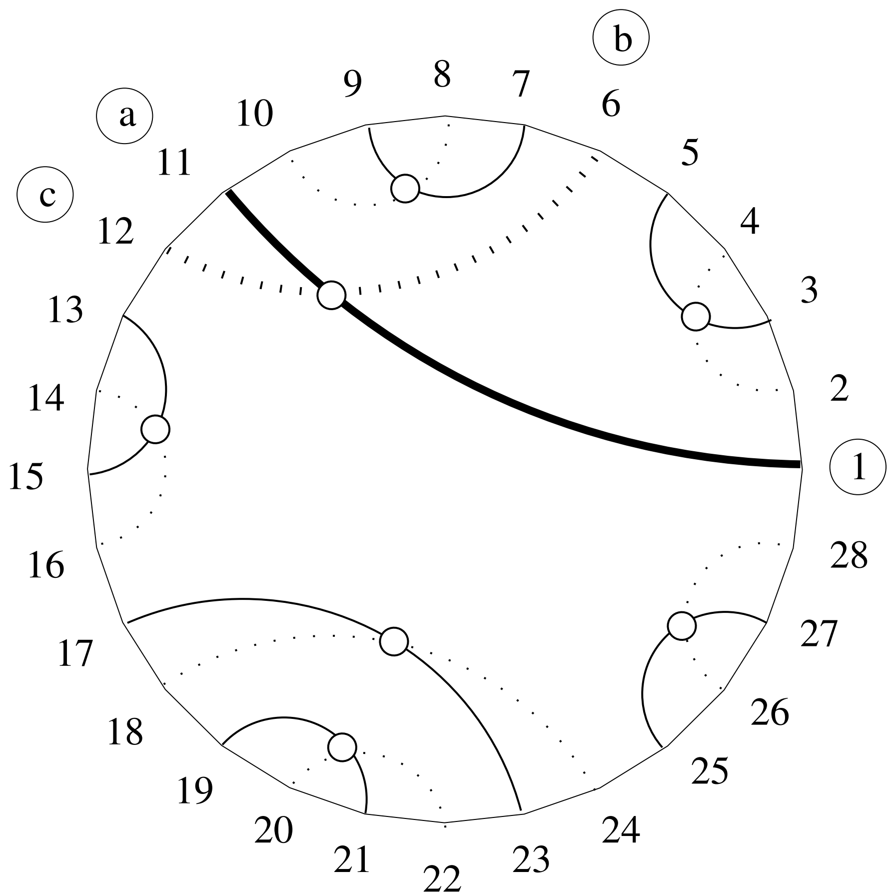
Note that $\#\mathcal{B}(n) = C_n^2 = \left(\frac{1}{n+1}\binom{2n}{n}\right)^2$.

Theorem (Savitt-Singer): $\quad b(n) = \dfrac{1}{3n+1}\dbinom{4n}{n}$.

Proof. Suppose that $\{1 < a\}$ is a black pair. Let $\{b < c\}$ be the unique red pair that it crosses (so $1 < b < a < c$.)



We have divided the circle into four subregions.



Each subregion contains a mini-basketball (after relabeling, and possibly empty). The orders p, q, r, s of the four mini-basketballs add up to $n-1$. Therefore

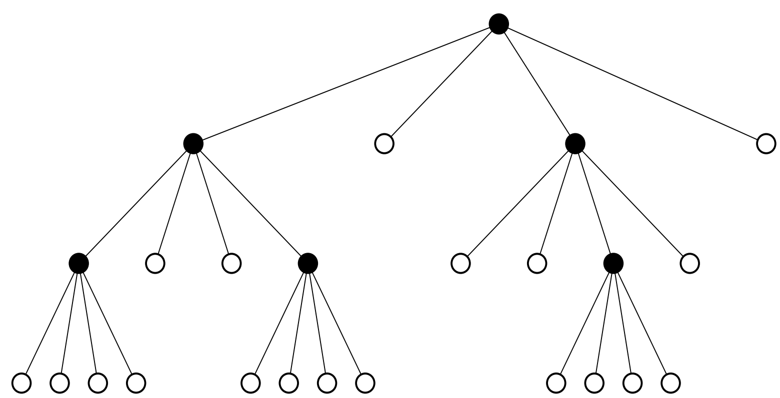
$$b(n) \;=\; \sum_{p+q+r+s=n-1} b(p)b(q)b(r)b(s).$$

The recurrence

$$b(n) \;=\; \sum_{p+q+r+s=n-1} b(p)b(q)b(r)b(s)$$

holds also for $t_4(n)$, where

$$t_k(n) \;=\; \text{number of } k\text{-ary trees with } n \text{ internal nodes.}$$



Fact: The number of k-ary trees with n internal nodes is

$$t_k(n) \;=\; \frac{1}{(k-1)n+1}\binom{kn}{n}. \qquad (\heartsuit)$$

Putting $k = 4$ in $(\heartsuit)$, we get

$$b(n) \;=\; \gamma_4(n) \;=\; \frac{1}{3n+1}\binom{4n}{n}.$$

■

5. Further Questions

Define $b(n,k) \;=\; \#\mathcal{B}(n,k) \;=\; \#$ of n-basketballs with k crossings.

Question 1: What is $\boldsymbol{b(n,k)}$?

Define $c(n,k) = b(n, n+2k)$, so that $c(n,0) = b(n,n) = b(n)$.

	$c(n,k)$						
	$k=0$	1	2	3	4	5	6
$n=1$	1						
2	4						
3	22	3					
4	140	48	8				
5	969	570	190	30	5		
6	7084	6072	3036	920	252	48	12

First column: $b(n) = \dfrac{1}{3n+1}\dbinom{4n}{n}$.

We'd like to find the bivariate generating function

$$\sum_{\text{basketballs } B} x^{\text{order(B)}} y^{\text{cross}(B)}.$$

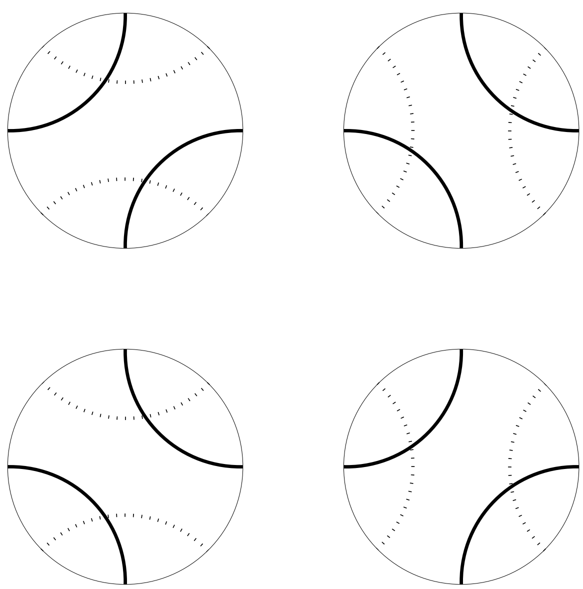
Got any ideas?

Question 2a: Does every simple n-basketball arise from some honest-to-goodness complex polynomial of degree n?

- The answer is yes for $n \leq 3$.

Question 2b: What does the basketball associated with a particular polynomial tell us about that polynomial?

- For $n = 2$, the four basketballs are



…and the basketball corresponding to $f(z) = z^2 + bz + c$ is determined by the *discriminant*

$$\Delta = b^2 - 4c.$$

Question 3: How many (simple) basketballs are there...
...up to rotation?
...up to rotation and reflection?
...up to rotation, reflection and swapping colors?

— These operations correspond to multiplying $f(z)$ by a scalar, or replacing z with $\bar{z}$.

Question 4: What kind of basketball is produced by a "generic" polynomial (whatever that means)?

Question 5: What about basketballs with k colors?

— Connection with k-ary trees??