Map Coloring

Problem #1: Color a map of the USA so that adjoining states are always colored differently.

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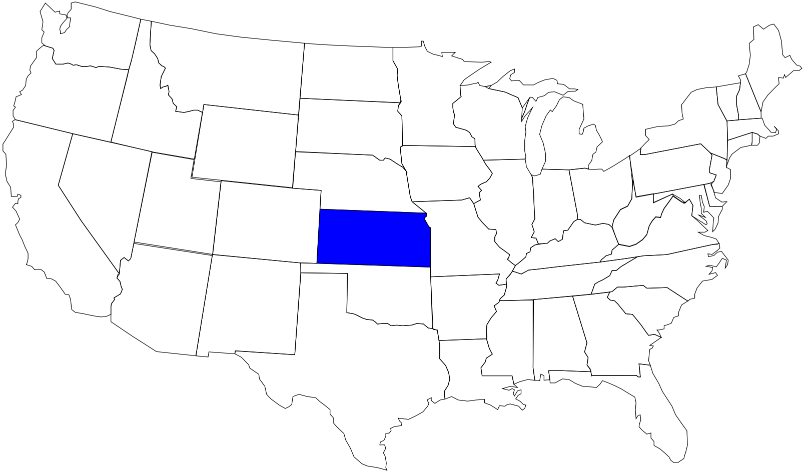

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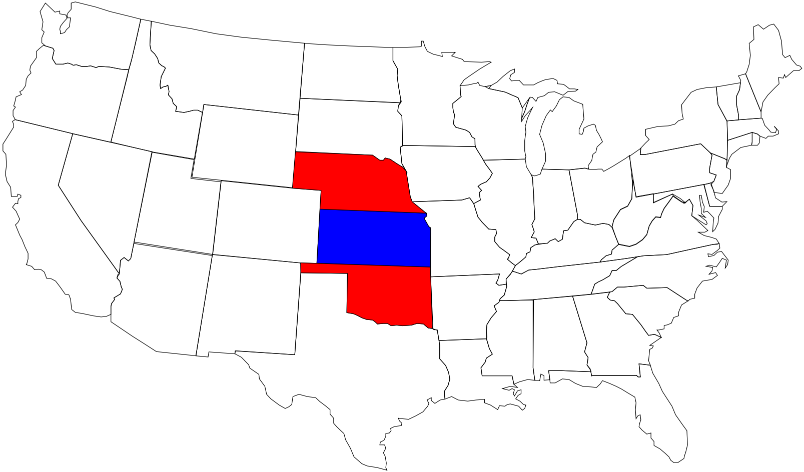

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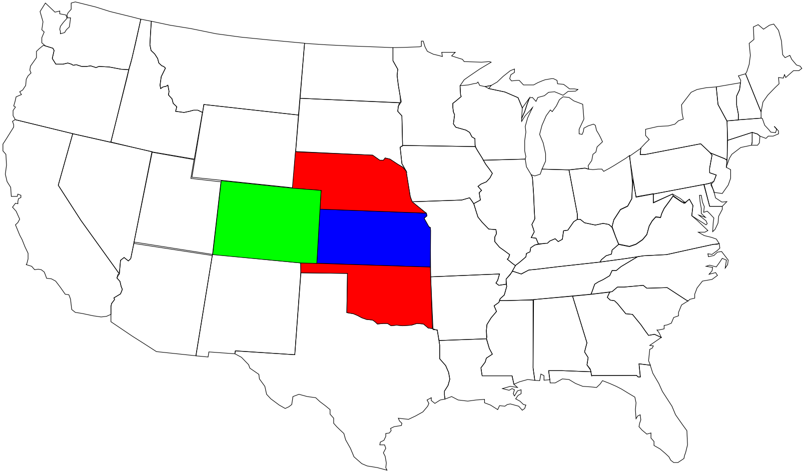

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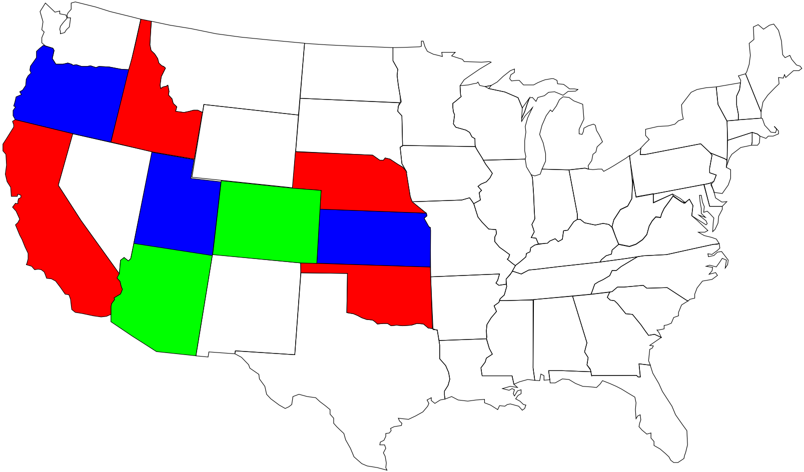

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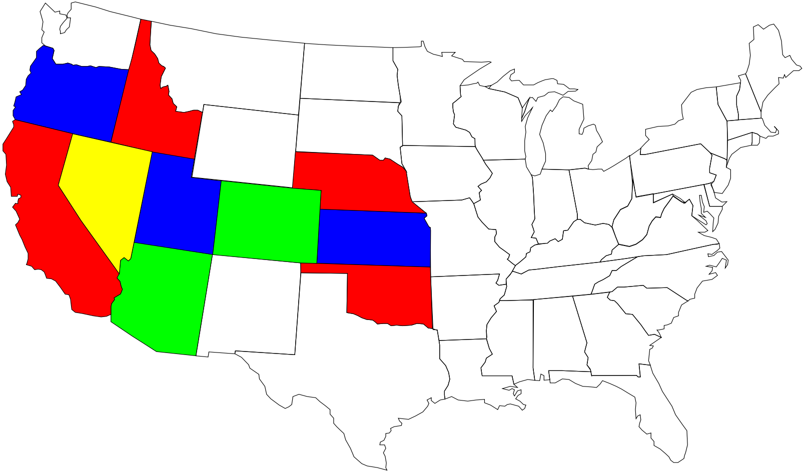

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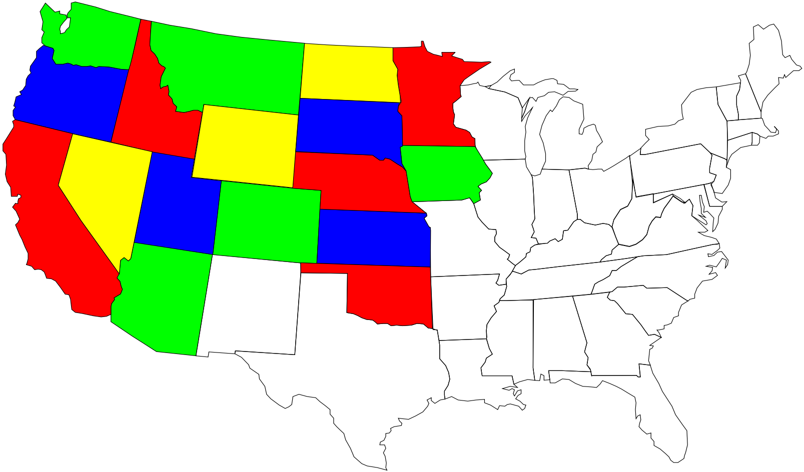

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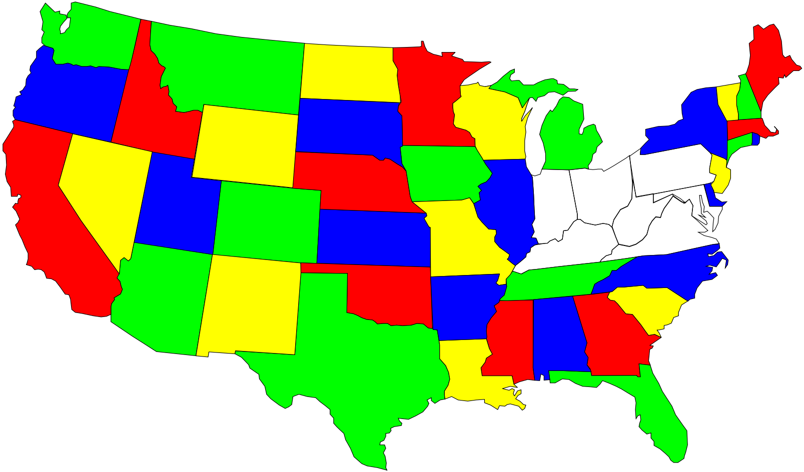

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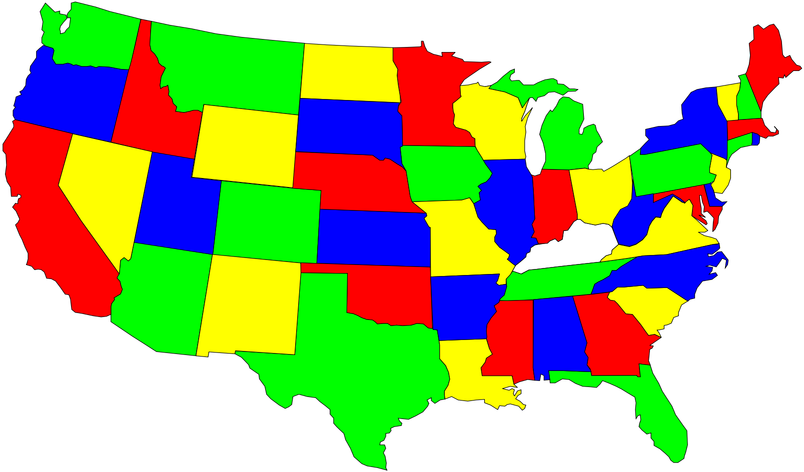

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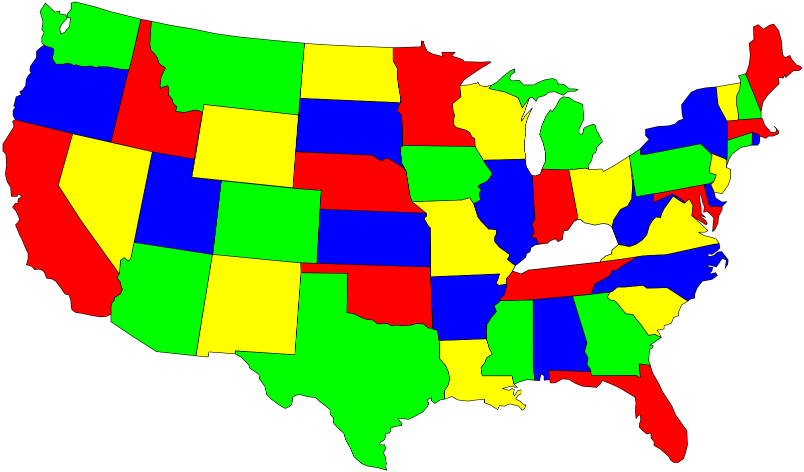

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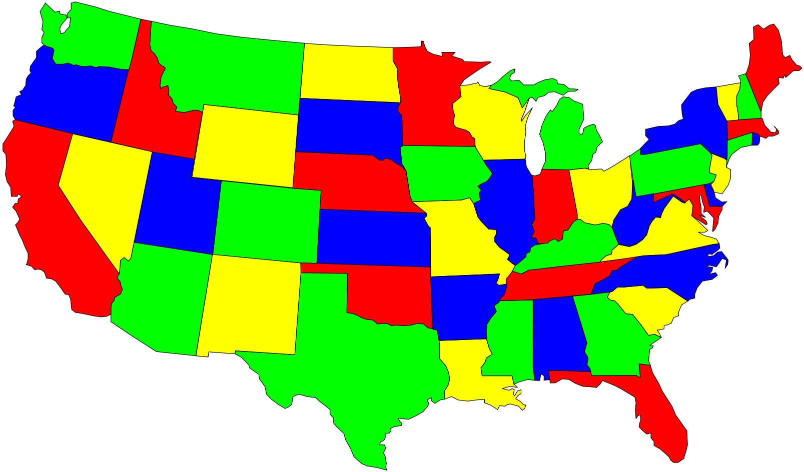

A Seating Problem

Problem #2: Eight cousins are at a family reunion. Some of them don't like each other.

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Cousin	Doesn't like
Alice	Bob, Chloe, Fred, Horace
Bob	Alice, Dave, Gina, Horace
Chloe	Alice
Dave	Bob, Fred, Horace
Edith	Horace
Fred	Alice, Dave
Gina	Bob, Horace
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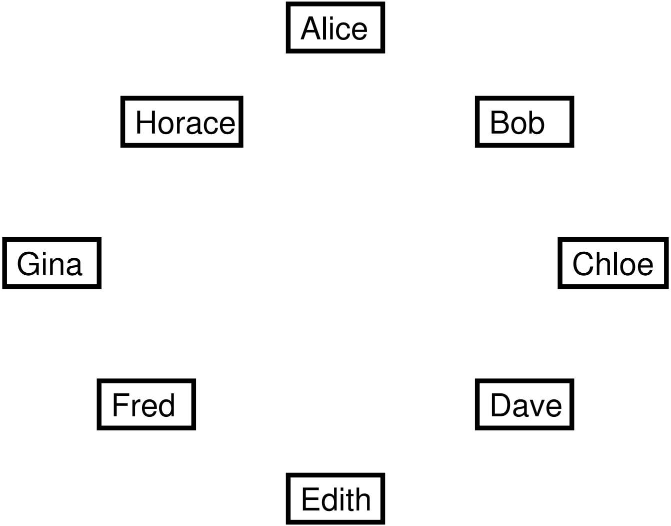
How many tables are necessary so that no one has to sit at a table with someone s/he doesn't like?

A Seating Problem



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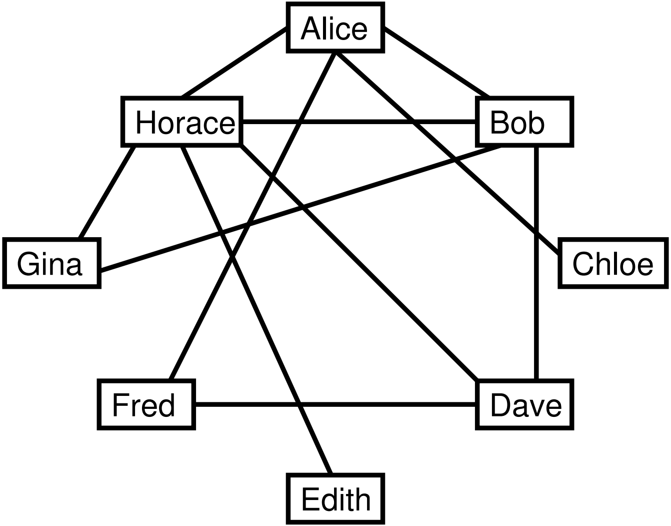

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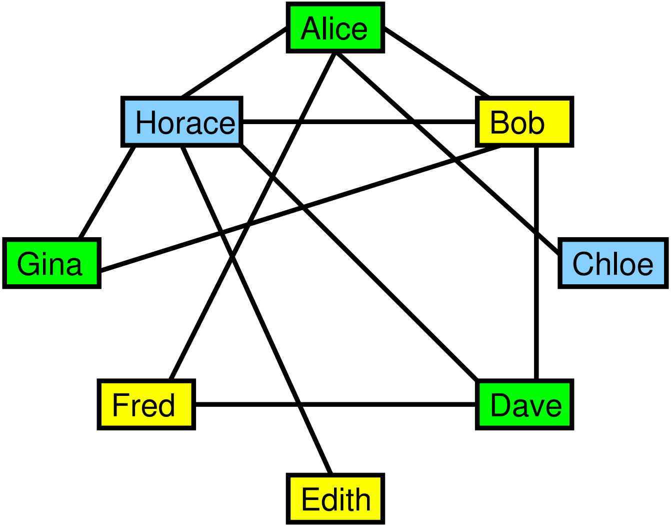
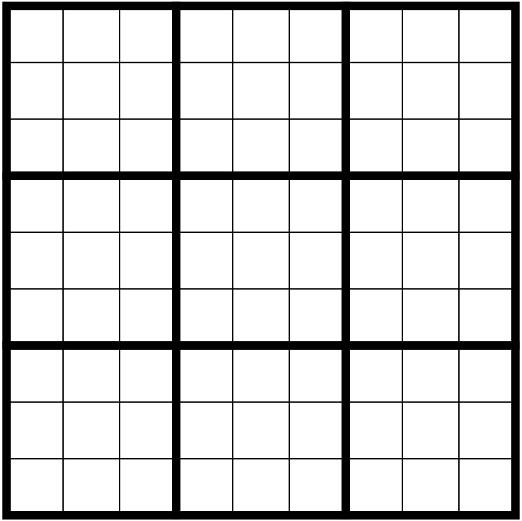


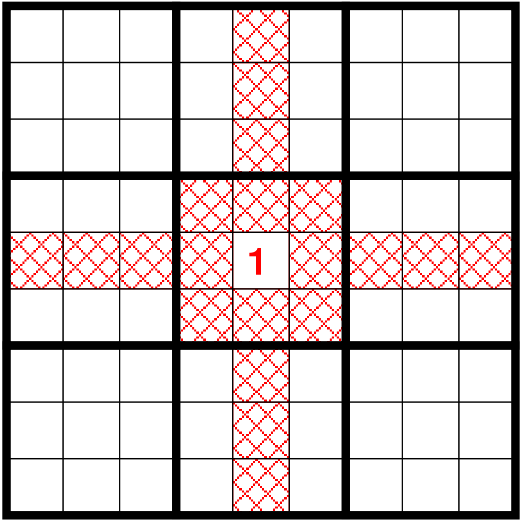
Sudoku



Sudoku

8	9	7	1	2	3	4	6	5
5	3	2	7	6	4	1	9	8
4	6	1	8	5	9	2	3	7
9	8	3	4	7	6	5	2	1
6	2	4	5	1	8	9	7	3
1	7	5	9	3	2	8	4	6
3	1	6	2	4	5	7	8	9
7	4	9	6	8	1	3	5	2
2	5	8	3	9	7	6	1	4

Sudoku



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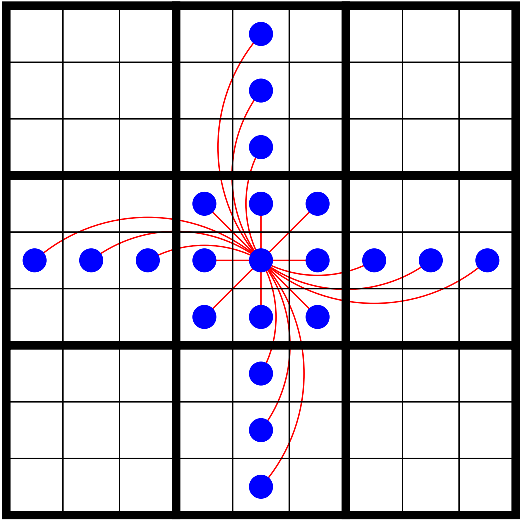

Sudoku

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5	3	2	7	6	4	1	9	8
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9	8	3	4	7	6	5	2	1
6	2	4	5	1	8	9	7	3
1	7	5	9	3	2	8	4	6
3	1	6	2	4	5	7	8	9
7	4	9	6	8	1	3	5	2
2	5	8	3	9	7	6	1	4

Graph Coloring

Suppose that G is a graph and k is a positive integer.

Definition: A k-**coloring** of G is a coloring of the vertices of G using k colors, such that adjacent vertices are colored with different colors.

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Definition: The **chromatic number** of G is the smallest number k for which a k-coloring of G is possible. (Notation: $\chi(G)$.)

Definition: A coloring of G is **optimal** if it uses exactly $\chi(G)$ colors (and no more).

Graph Coloring

- What are the chromatic numbers of special graphs that we know (e.g., complete graphs, circuits, trees, ...)
- How can we calculate the chromatic number of an arbitrary graph?
- How can we construct an optimal coloring?

Graph Coloring

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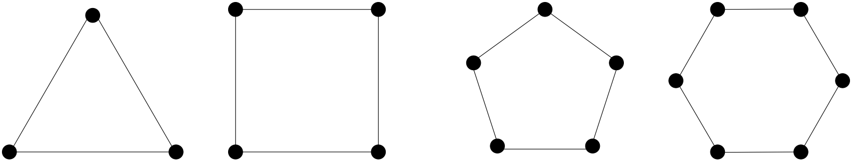
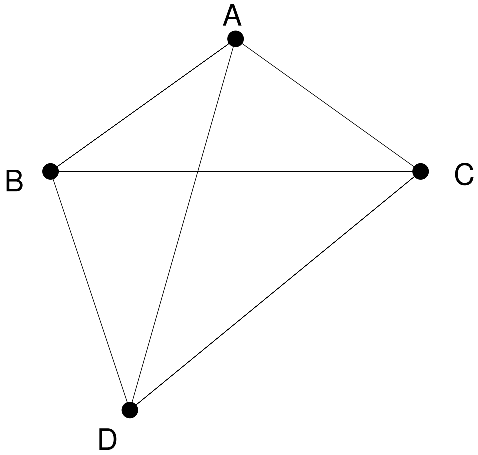
- In a circuit C_N...



Graph Coloring Examples



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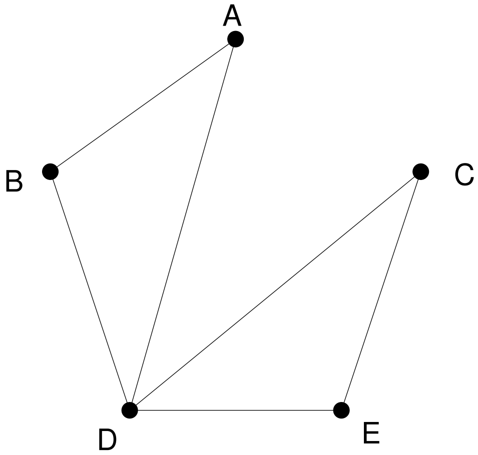

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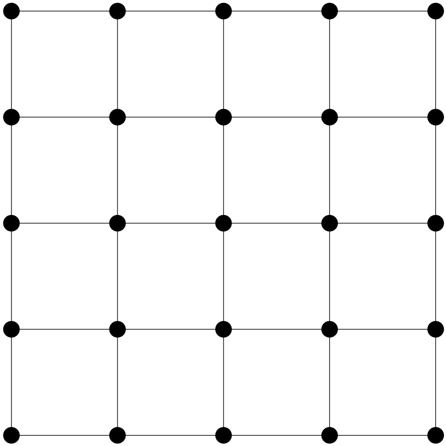

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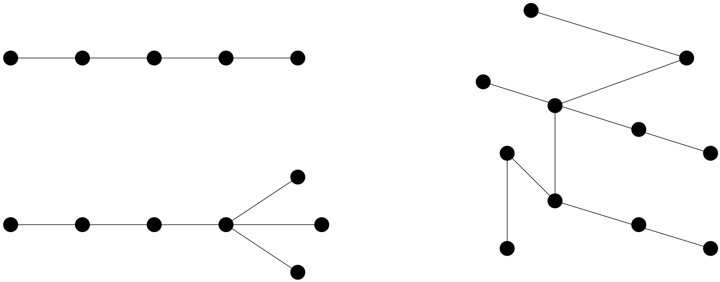


A Graph Coloring Algorithm

What are we doing when we color a graph?

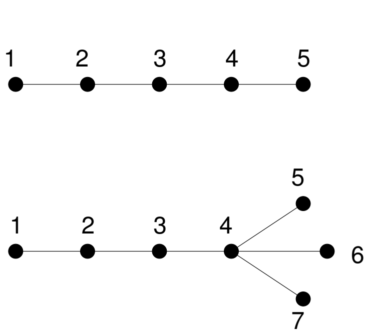
1. List the vertices in order: v_1, v_2, ..., v_n.
2. Number the colors 1, 2, ..., k.
3. Color the vertices in that order. Assign each vertex the smallest possible color.

(E.g., if vertex v_6 has neighbors that have already been colored 1 and 2, but has no neighbors that have been colored 3, then assign color 3 to v_6.)

A Graph Coloring Algorithm

Question: Does this algorithm always produce an optimal coloring?

That is, does coloring a graph G in this way always use **only** $\chi(G)$ colors, never more than that?


1
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4
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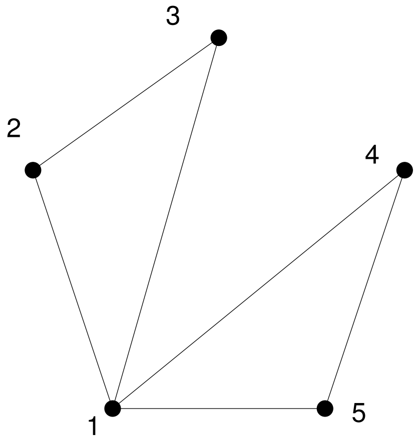
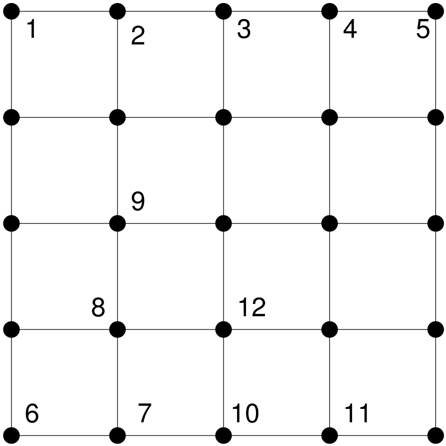
10
11


3
2
4
1
5


1
2
3
4
5
9
8
12
6
7
10
11

Oops.

The Bad News

- This algorithm is **efficient** but **not optimal** — the number of colors required might be depends on the order in which the vertices are chosen.

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- This algorithm is **efficient** but **not optimal** — the number of colors required might be depends on the order in which the vertices are chosen.
- There is always some order that produces an optimal coloring; however, it is very hard to know in advance what order to use.
- Finding an optimal coloring is theoretically **just as hard as the Traveling Salesman Problem** — there is no efficient algorithm known.

The Good News

1. List the vertices in order: v_1, v_2, ..., v_n.
2. Number the colors 1, 2, ..., k.
3. Color the vertices in that order. Assign each vertex the smallest possible color.

Observation: If vertex v_i has degree d, then at least one of the colors 1, 2, ..., $d + 1$ is available when it is v_i's turn to get colored.

That means we can say at least something about $\chi(G)$.

If every vertex in a graph G has degree Δ or less, then $\chi(G) \leq \Delta + 1$.

Actually, we can do even better than that!

Brooks' Theorem: If every vertex in a graph G has degree Δ or less, then $\chi(G) \leq \Delta$, unless G is a complete graph or a cycle of odd length.