Vizing's Conjecture and Techniques from Computer Algebra

Susan Margulies
Computational and Applied Math, Rice University

joint work in progress with I.V. Hicks[1]

April 17, 2010

[1] funded by VIGRE and NSF-CMMI-0926618 and NSF-DMS-0729251

Definition of Dominating Set Problem

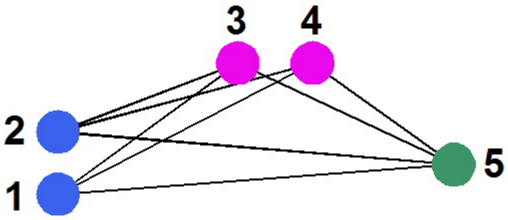
- **Dominating Set:** Given a graph G and an integer k, does there exist a subset of vertices D, with $|D| = k$, such that every vertex in the graph is in, or adjacent to, a vertex in D?

Definition of Dominating Set Problem

- **Dominating Set:** Given a graph G and an integer k, does there exist a subset of vertices D, with $|D| = k$, such that every vertex in the graph is in, or adjacent to, a vertex in D?
- **Definition:** The *domination number* of a graph G is the size of a minimum dominating set, and is denoted by $\gamma(G)$.

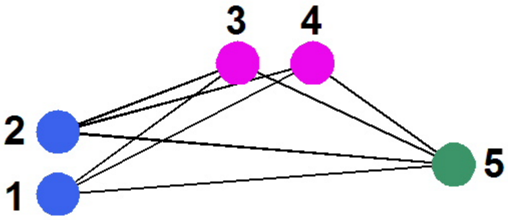
Definition of Dominating Set Problem

- **Dominating Set:** Given a graph G and an integer k, does there exist a subset of vertices D, with $|D| = k$, such that every vertex in the graph is in, or adjacent to, a vertex in D?
- **Definition:** The *domination number* of a graph G is the size of a minimum dominating set, and is denoted by $\gamma(G)$.
- **Turán Graph $T(5, 3)$:**



Definition of Dominating Set Problem

- **Dominating Set:** Given a graph G and an integer k, does there exist a subset of vertices D, with $|D| = k$, such that every vertex in the graph is in, or adjacent to, a vertex in D?
- **Definition:** The *domination number* of a graph G is the size of a minimum dominating set, and is denoted by $\gamma(G)$.
- **Turán Graph $T(5,3)$:** $\gamma(\ T(5,3)\ ) = 1$.



Cartesian Product Graph, $G \Box H$

- **Cartesian Product:** Given graphs G and H, the cartesian product graph, denoted $G \Box H$, has vertex set

$$V(G) \times V(H)$$

Cartesian Product Graph, $G \Box H$

- **Cartesian Product:** Given graphs G and H, the cartesian product graph, denoted $G \Box H$, has vertex set

$$V(G) \times V(H)$$

Given vertices $iu, jv \in V(G \Box H)$, there is an edge between iu and jv **if** $i = j$ and $(u, v) \in E[H]$, **or** $u = v$ and $(i, j) \in E[G]$.

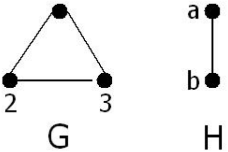
Cartesian Product Graph, $G \square H$

- **Cartesian Product:** Given graphs G and H, the cartesian product graph, denoted $G \square H$, has vertex set

$$V(G) \times V(H)$$

  Given vertices $iu, jv \in V(G \square H)$, there is an edge between iu and jv **if** $i = j$ and $(u, v) \in E[H]$, **or** $u = v$ and $(i, j) \in E[G]$.

- **Example:** Consider a triangle and an edge:



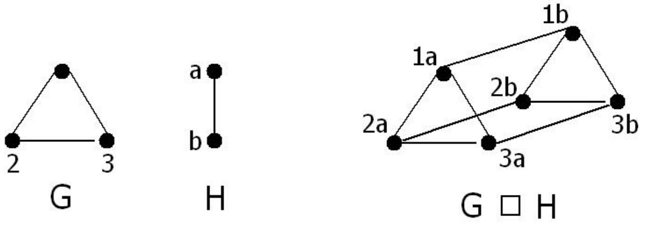
Cartesian Product Graph, $G \square H$

- **Cartesian Product:** Given graphs G and H, the cartesian product graph, denoted $G \square H$, has vertex set

$$V(G) \times V(H)$$

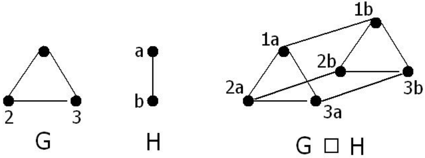
Given vertices $iu, jv \in V(G \square H)$, there is an edge between iu and jv **if** $i = j$ and $(u, v) \in E[H]$, **or** $u = v$ and $(i, j) \in E[G]$.

- **Example:** Consider a triangle and an edge:



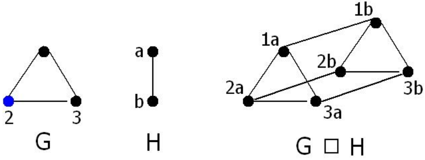
Cartesian Product Graphs and Dominating Sets

- **Example:** Consider a triangle and an edge:



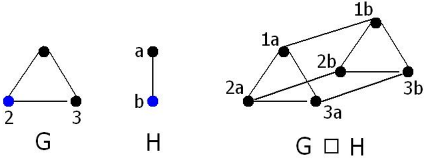
Cartesian Product Graphs and Dominating Sets

- **Example:** Consider a triangle and an edge:



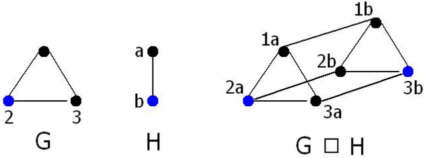
Cartesian Product Graphs and Dominating Sets

- **Example:** Consider a triangle and an edge:



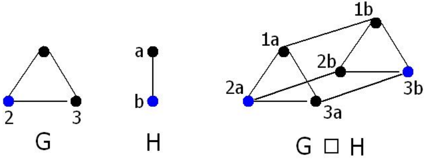
Cartesian Product Graphs and Dominating Sets

- **Example:** Consider a triangle and an edge:



Cartesian Product Graphs and Dominating Sets

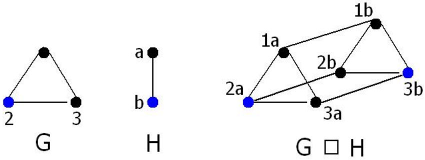
- **Example:** Consider a triangle and an edge:



$\gamma(G) = 1, \gamma(H) = 1$ and $\gamma(G \Box H) = 2$.

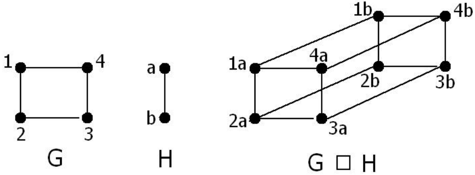
Cartesian Product Graphs and Dominating Sets

- **Example:** Consider a triangle and an edge:



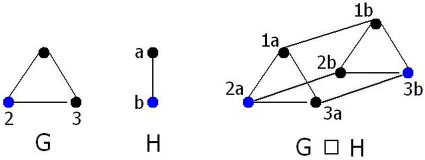
$\gamma(G) = 1, \gamma(H) = 1$ and $\gamma(G \square H) = 2$.

- **Example:** Consider a square and an edge:



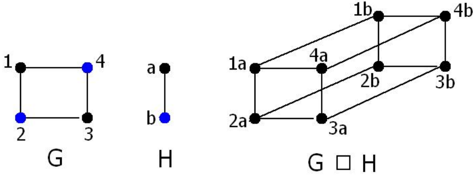
Cartesian Product Graphs and Dominating Sets

- **Example:** Consider a triangle and an edge:



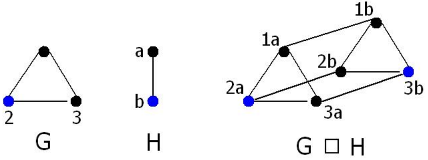
$\gamma(G) = 1, \gamma(H) = 1$ and $\gamma(G \Box H) = 2$.

- **Example:** Consider a square and an edge:



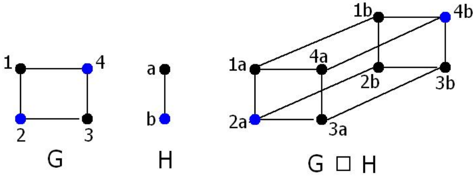
Cartesian Product Graphs and Dominating Sets

- **Example:** Consider a triangle and an edge:



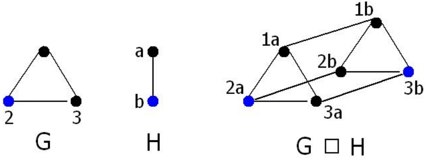
$\gamma(G) = 1, \gamma(H) = 1$ and $\gamma(G \square H) = 2$.

- **Example:** Consider a square and an edge:



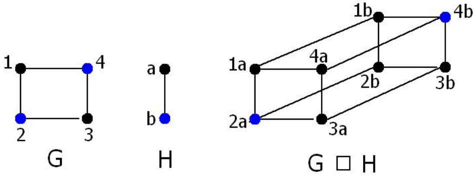
Cartesian Product Graphs and Dominating Sets

- **Example:** Consider a triangle and an edge:



$\gamma(G) = 1, \gamma(H) = 1$ and $\gamma(G \square H) = 2$.

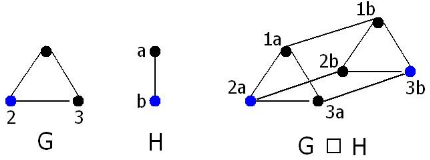
- **Example:** Consider a square and an edge:



$\gamma(G) = 2, \gamma(H) = 1$ and $\gamma(G \square H) = 2$.

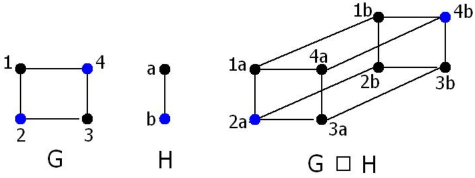
Cartesian Product Graphs and Dominating Sets

- **Example:** Consider a triangle and an edge:



$\gamma(G) = 1, \gamma(H) = 1$ and $\gamma(G \Box H) = 2$. $\gamma(G)\gamma(H) < \gamma(G \Box H)$.

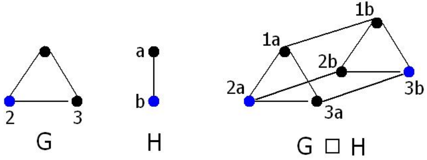
- **Example:** Consider a square and an edge:



$\gamma(G) = 2, \gamma(H) = 1$ and $\gamma(G \Box H) = 2$.

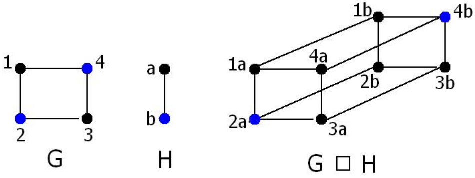
Cartesian Product Graphs and Dominating Sets

- **Example:** Consider a triangle and an edge:



$\gamma(G) = 1, \gamma(H) = 1$ and $\gamma(G \Box H) = 2$. $\gamma(G)\gamma(H) < \gamma(G \Box H)$.

- **Example:** Consider a square and an edge:



$\gamma(G) = 2, \gamma(H) = 1$ and $\gamma(G \Box H) = 2$. $\gamma(G)\gamma(H) = \gamma(G \Box H)$.

Vizing's Conjecture

Vizing's Conjecture (1963)

Given graphs G and H,

$$\gamma(G)\gamma(H) \leq \gamma(G \square H) \; .$$

Brief History of Progress

- Vizing proposes his conjecture in 1963.

Brief History of Progress

- Vizing proposes his conjecture in 1963.
- In 1979, Barcalkin and German prove that Vizing's conjecture holds for a large class of graphs ("A-class" graphs).

Brief History of Progress

- Vizing proposes his conjecture in 1963.
- In 1979, Barcalkin and German prove that Vizing's conjecture holds for a large class of graphs ("A-class" graphs).
- In 1990, Faudree, Schelp and Shreve prove that Vizing's conjecture holds for graphs that satisfy a special "coloring property".

Brief History of Progress

- Vizing proposes his conjecture in 1963.
- In 1979, Barcalkin and German prove that Vizing's conjecture holds for a large class of graphs ("A-class" graphs).
- In 1990, Faudree, Schelp and Shreve prove that Vizing's conjecture holds for graphs that satisfy a special "coloring property".
- In 1991, El-Zahar and Pareek show that Vizing's conjecture holds for cycles.

Brief History of Progress

- Vizing proposes his conjecture in 1963.
- In 1979, Barcalkin and German prove that Vizing's conjecture holds for a large class of graphs ("A-class" graphs).
- In 1990, Faudree, Schelp and Shreve prove that Vizing's conjecture holds for graphs that satisfy a special "coloring property".
- In 1991, El-Zahar and Pareek show that Vizing's conjecture holds for cycles.
- In 2000, Clark and Suen show that $\gamma(G)\gamma(H) \leq 2\gamma(G \square H)$.

Brief History of Progress

- Vizing proposes his conjecture in 1963.
- In 1979, Barcalkin and German prove that Vizing's conjecture holds for a large class of graphs ("A-class" graphs).
- In 1990, Faudree, Schelp and Shreve prove that Vizing's conjecture holds for graphs that satisfy a special "coloring property".
- In 1991, El-Zahar and Pareek show that Vizing's conjecture holds for cycles.
- In 2000, Clark and Suen show that $\gamma(G)\gamma(H) \leq 2\gamma(G \Box H)$.
- In 2003, Sun proves that Vizing's conjecture holds if $\gamma(G) \leq 3$.

An arbitrary graph G in n vertices and a dominating set of size k

Lemma

The following zero-dimensional system of polynomial equations has a solution if and only if there exists a graph G in n vertices that has a dominating set of size k.

$$
\begin{aligned}
x_i^2 - x_i &= 0\ , \quad \textit{for } i = 1, \ldots, n\ , \\
e_{ij}^2 - e_{ij} &= 0\ , \quad \textit{for } i, j = 1, \ldots, n \textit{ with } i < j\ , \\
(1 - x_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} (1 - e_{ij} x_j) &= 0\ , \quad \textit{for } i = 1, \ldots, n\ , \\
-k + \sum_{i=1}^{n} x_i &= 0\ .
\end{aligned}
$$

An arbitrary graph G in n vertices and an arbitrary dominating set of size k

Let S_n^k denote the set of k-subsets of $\{1, 2, \ldots, n\}$.

An arbitrary graph G in n vertices and an arbitrary dominating set of size k

Let S_n^k denote the set of k-subsets of $\{1, 2, \ldots, n\}$.

Lemma

The following zero-dimensional system has a solution if and only if there exists a graph G in n vertices that has a dominating set of size k.

$$e_{ij}^2 - e_{ij} = 0 \ , \quad \text{for } 1 \leq i < j \leq n,$$

$$\prod_{S \in S_n^k} \left(\sum_{i \notin S} \left(\prod_{j \in S} (1 - e_{ij}) \right) \right) = 0 \ .$$

Notation Definitions

Let $\mathscr{P}_G$ be the set of polynomials representing a graph G in n vertices with a dominating set of size k:

$$e_{ij}^2 - e_{ij} = 0 \ , \quad \text{for } 1 \leq i < j \leq n,$$

$$\prod_{S \in S_n^k} \left(\sum_{i \notin S} \left(\prod_{j \in S} (1 - e_{ij}) \right) \right) = 0 \ .$$

Notation Definitions

Let $\mathscr{P}_G$ be the set of polynomials representing a graph G in n vertices with a dominating set of size k:

$$e_{ij}^2 - e_{ij} = 0 \;, \quad \text{for } 1 \leq i < j \leq n,$$

$$\prod_{S \in S_n^k} \left(\sum_{i \notin S} \left(\prod_{j \in S} (1 - e_{ij}) \right) \right) = 0 \;.$$

Let $\mathscr{P}_H$ be the set of polynomials representing a graph H in n' vertices with a dominating set of size l:

$$e'^2_{ij} - e'_{ij} = 0 \;, \quad \text{for } 1 \leq i < j \leq n',$$

$$\prod_{S \in S_{n'}^l} \left(\sum_{i \notin S} \left(\prod_{j \in S} (1 - e'_{ij}) \right) \right) = 0 \;.$$

Notation Definitions (continued)

Let $\mathscr{P}_{G\square H}$ be the set of polynomials representing the cartesian product graph $G\square H$ with a dominating set of size r:

For $i = 1, \ldots, n$ and $j = 1, \ldots, n'$,

$$z_{ij}^2 - z_{ij} = 0 \ ,$$

$$(1 - z_{ij}) \prod_{k=1}^{n} (1 - e_{ik} z_{kj}) \prod_{k=1}^{n'} (1 - e'_{jk} z_{ik}) = 0 \ ,$$

and

$$-r + \sum_{i=1}^{n} \sum_{j=1}^{n'} z_{ij} = 0 \ ,$$

The ideal I_k^l and variety V_k^l

Lemma

The system of polynomial equations $\mathscr{P}_G, \mathscr{P}_H$ and $\mathscr{P}_{G\Box H}$ has a solution if and only if there exist graphs G, H in n, n' vertices respectively with dominating sets of size k, l respectively such that their cartesian product graph $G\Box H$ has a dominating set of size r.

The ideal I_k^l and variety V_k^l

Lemma

The system of polynomial equations $\mathscr{P}_G, \mathscr{P}_H$ and $\mathscr{P}_{G\square H}$ has a solution if and only if there exist graphs G, H in n, n' vertices respectively with dominating sets of size k, l respectively such that their cartesian product graph $G \square H$ has a dominating set of size r.

Let $I_k^l := I(n, k, n', l, r = kl - 1) := \langle \mathscr{P}_G, \mathscr{P}_H, \mathscr{P}_{G\square H} \rangle$.

The ideal I_k^l and variety V_k^l

Lemma

The system of polynomial equations $\mathscr{P}_G, \mathscr{P}_H$ and $\mathscr{P}_{G\Box H}$ has a solution if and only if there exist graphs G, H in n, n' vertices respectively with dominating sets of size k, l respectively such that their cartesian product graph $G\Box H$ has a dominating set of size r.

Let $I_k^l := I(n, k, n', l, r = kl - 1) := \langle \mathscr{P}_G, \mathscr{P}_H, \mathscr{P}_{G\Box H} \rangle$.
Let $V_k^l := V(I_k^l)$.

The ideal I_k^l and variety V_k^l

Lemma

The system of polynomial equations $\mathscr{P}_G, \mathscr{P}_H$ and $\mathscr{P}_{G\Box H}$ has a solution if and only if there exist graphs G, H in n, n' vertices respectively with dominating sets of size k, l respectively such that their cartesian product graph $G\Box H$ has a dominating set of size r.

Let $I_k^l := I(n, k, n', l, r = kl - 1) := \langle \mathscr{P}_G, \mathscr{P}_H, \mathscr{P}_{G\Box H} \rangle$.
Let $V_k^l := V(I_k^l)$.
Note that $I(V_k^l) = I_k^l$ since the ideal I_k^l is *radical*.

Unions and Vizing's Conjecture

Theorem

Vizing's conjecture is true $\iff V^{l}_{k-1} \cup V^{l-1}_{k} = V^{l}_{k}$.

Unions and Vizing's Conjecture

Theorem

Vizing's conjecture is true $\iff V_{k-1}^{l} \cup V_{k}^{l-1} = V_{k}^{l}$.

Proof.

Every point in the variety corresponds to a G, H pair.

□

Unions and Vizing's Conjecture

Theorem

Vizing's conjecture is true $\iff V_{k-1}^{l} \cup V_{k}^{l-1} = V_{k}^{l}$.

Proof.

Every point in the variety corresponds to a G, H pair. Since dominating sets can always be extended, $V_{k-1}^{l} \cup V_{k}^{l-1} \subseteq V_{k}^{l}$.

□

Unions and Vizing's Conjecture

Theorem

Vizing's conjecture is true $\Longleftrightarrow V_{k-1}^{l} \cup V_{k}^{l-1} = V_{k}^{l}$.

Proof.

Every point in the variety corresponds to a G, H pair. Since dominating sets can always be extended, $V_{k-1}^{l} \cup V_{k}^{l-1} \subseteq V_{k}^{l}$. If $V_{k}^{l} \subseteq V_{k-1}^{l} \cup V_{k}^{l-1}$, then for every G, H pair, either k or l is *strictly less* than $\gamma(G), \gamma(H)$ respectively.

□

Unions and Vizing's Conjecture

Theorem

Vizing's conjecture is true $\iff V_{k-1}^{l} \cup V_{k}^{l-1} = V_{k}^{l}$.

Proof.

Every point in the variety corresponds to a G, H pair. Since dominating sets can always be extended, $V_{k-1}^{l} \cup V_{k}^{l-1} \subseteq V_{k}^{l}$.
If $V_{k}^{l} \subseteq V_{k-1}^{l} \cup V_{k}^{l-1}$, then for every G, H pair, either k or l is *strictly less* than $\gamma(G), \gamma(H)$ respectively.
Thus, Vizing's conjecture is true $\iff V_{k-1}^{l} \cup V_{k}^{l-1} = V_{k}^{l}$. □

Intersections and Vizing's Conjecture

Corollary

Vizing's conjecture is true $\iff I^{l}_{k-1} \cap I^{l-1}_{k} = I^{l}_{k}$.

Computing the Intersection of Two Ideals

- **Definition:** Given $I = \langle f_1, \ldots, f_s \rangle$ and $J = \langle g_1, \ldots, g_t \rangle$, then the *product ideal* $I \cdot J := \langle f_i g_j \ : \ 1 \leq i \leq s, 1 \leq j \leq t \rangle$.

Computing the Intersection of Two Ideals

- **Definition:** Given $I = \langle f_1, \ldots, f_s \rangle$ and $J = \langle g_1, \ldots, g_t \rangle$, then the *product ideal* $I \cdot J := \langle f_i g_j \ : \ 1 \leq i \leq s, 1 \leq j \leq t \rangle$.
- **Fact:** Given radical ideals $I, J \in \mathbb{K}[x_1, \ldots, x_n]$, $\sqrt{I \cdot J} = I \cap J$.

Computing the Intersection of Two Ideals

- **Definition:** Given $I = \langle f_1, \dots, f_s \rangle$ and $J = \langle g_1, \dots, g_t \rangle$, then the *product ideal* $I \cdot J := \langle f_i g_j \ : \ 1 \le i \le s, 1 \le j \le t \rangle$.
- **Fact:** Given radical ideals $I, J \in \mathbb{K}[x_1, \dots, x_n]$, $\sqrt{I \cdot J} = I \cap J$.

Lemma

Let I, J be ideals in $\mathbb{K}[x_1, \dots, x_n]$ such that $I = \langle f_1, \dots, f_s \rangle$ and $J = \langle g_1, \dots, g_t \rangle$. Furthermore, let $\{f_1, \dots, f_n\} = \{g_1, \dots, g_n\}$ be square-free univariate polynomials such that $f_i = g_i$ is univariate in x_i.

Computing the Intersection of Two Ideals

- **Definition:** Given $I = \langle f_1, \ldots, f_s \rangle$ and $J = \langle g_1, \ldots, g_t \rangle$, then the *product ideal* $I \cdot J := \langle f_i g_j \; : \; 1 \leq i \leq s, 1 \leq j \leq t \rangle$.
- **Fact:** Given radical ideals $I, J \in \mathbb{K}[x_1, \ldots, x_n]$, $\sqrt{I \cdot J} = I \cap J$.

Lemma

Let I, J be ideals in $\mathbb{K}[x_1, \ldots, x_n]$ such that $I = \langle f_1, \ldots, f_s \rangle$ and $J = \langle g_1, \ldots, g_t \rangle$. Furthermore, let $\{f_1, \ldots, f_n\} = \{g_1, \ldots, g_n\}$ be square-free univariate polynomials such that $f_i = g_i$ is univariate in x_i. Then, $\sqrt{I \cdot J} = \langle f_i g_j : 1 \leq i \leq s, 1 \leq j \leq t \rangle + \langle f_i : 1 \leq i \leq n \rangle$.

Vizing's Conjecture and Linear Algebra

Corollary

Vizing's conjecture is true $\Longleftrightarrow$
$I^l_{k-1} \cdot I^{l-1}_k + \langle e_i^2 - e_i, {e'}_j^2 - {e'}_j, z_{ij}^2 - z_{ij} \rangle = I^l_k$.

Vizing's Conjecture and Linear Algebra

Corollary

Vizing's conjecture is true $\iff$
$I^l_{k-1} \cdot I^{l-1}_k + \langle e_i^2 - e_i, {e'}_j^2 - e'_j, z_{ij}^2 - z_{ij} \rangle = I^l_k$.

Let

$$P_{G_{k-1}} := \prod_{S \in S_n^{k-1}} \left(\sum_{i \notin S} \left(\prod_{j \in S} (1 - e_{ij}) \right) \right) ,$$

$$P_{H_{l-1}} := \prod_{S \in S_{n'}^{l-1}} \left(\sum_{i \notin S} \left(\prod_{j \in S} (1 - {e'}_{ij}) \right) \right) .$$

Vizing's Conjecture and Linear Algebra

Corollary

Vizing's conjecture is true $\Longleftrightarrow$
$I_{k-1}^{l} \cdot I_{k}^{l-1} + \langle e_i^2 - e_i, {e'}_j^2 - {e'}_j, z_{ij}^2 - z_{ij} \rangle = I_k^l$.

Let

$$P_{G_{k-1}} := \prod_{S \in S_n^{k-1}} \left(\sum_{i \notin S} \left(\prod_{j \in S} (1 - e_{ij}) \right) \right) ,$$

$$P_{H_{l-1}} := \prod_{S \in S_{n'}^{l-1}} \left(\sum_{i \notin S} \left(\prod_{j \in S} (1 - {e'}_{ij}) \right) \right).$$

Since $V_{k-1}^{l} \cup V_{k}^{l-1} \subseteq V_k^l$, this implies $I_k^l \subseteq I_{k-1}^{l} \cap I_k^{l-1}$.

Vizing's Conjecture and Linear Algebra

Corollary

Vizing's conjecture is true $\iff$
$I_{k-1}^{l} \cdot I_{k}^{l-1} + \langle e_i^2 - e_i, {e'}_j^2 - {e'}_j, z_{ij}^2 - z_{ij} \rangle = I_k^l$.

Let

$$P_{G_{k-1}} := \prod_{S \in S_n^{k-1}} \left(\sum_{i \notin S} \left(\prod_{j \in S} (1 - e_{ij}) \right) \right) ,$$

$$P_{H_{l-1}} := \prod_{S \in S_{n'}^{l-1}} \left(\sum_{i \notin S} \left(\prod_{j \in S} (1 - {e'}_{ij}) \right) \right).$$

Since $V_{k-1}^{l} \cup V_{k}^{l-1} \subseteq V_k^l$, this implies $I_k^l \subseteq I_{k-1}^{l} \cap I_{k}^{l-1}$.

Proving $I_k^l \subseteq I_{k-1}^{l} \cap I_{k}^{l-1}$ is equivalent to proving

Vizing's Conjecture and Linear Algebra

Corollary

Vizing's conjecture is true $\iff$
$I_{k-1}^{l} \cdot I_{k}^{l-1} + \langle e_i^2 - e_i, {e'}_j^2 - {e'}_j, z_{ij}^2 - z_{ij} \rangle = I_k^l$.

Let

$$P_{G_{k-1}} := \prod_{S \in S_n^{k-1}} \left(\sum_{i \notin S} \left(\prod_{j \in S} (1 - e_{ij}) \right) \right) ,$$

$$P_{H_{l-1}} := \prod_{S \in S_{n'}^{l-1}} \left(\sum_{i \notin S} \left(\prod_{j \in S} (1 - {e'}_{ij}) \right) \right).$$

Since $V_{k-1}^{l} \cup V_{k}^{l-1} \subseteq V_k^l$, this implies $I_k^l \subseteq I_{k-1}^{l} \cap I_k^{l-1}$.

Proving $I_k^l \subseteq I_{k-1}^{l} \cap I_k^{l-1}$ is equivalent to proving
$P_{G_{k-1}} \cdot P_{H_{l-1}} \in I_k^l$.

Vizing's Conjecture and Gröbner Bases

Let

$$\mathscr{P}'_{G\square H} := \mathscr{P}_{G\square H} \setminus \left\{ -(kl - l) + \sum_{i=1}^{n} \sum_{j=1}^{n'} z_{ij} \right\}$$

Vizing's Conjecture and Gröbner Bases

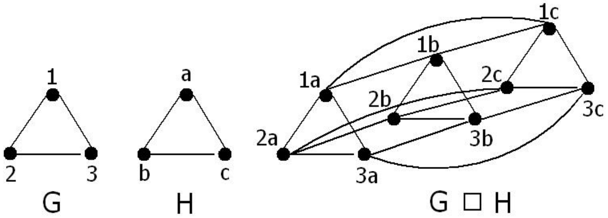
Let

$$\mathscr{P}'_{G\Box H} := \mathscr{P}_{G\Box H} \setminus \left\{ -(kl - l) + \sum_{i=1}^{n} \sum_{j=1}^{n'} z_{ij} \right\}$$

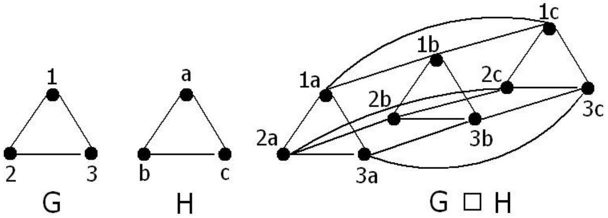
Conjecture via Experimental Observation

The following set of polynomials (described by cases 1 through 6) is a graph-theoretic interpretation of the unique, reduced Gröbner basis of $\mathscr{P}'_{G\Box H}$.

Vizing's Conjecture and Gröbner Bases: Degree



Vizing's Conjecture and Gröbner Bases: Degree

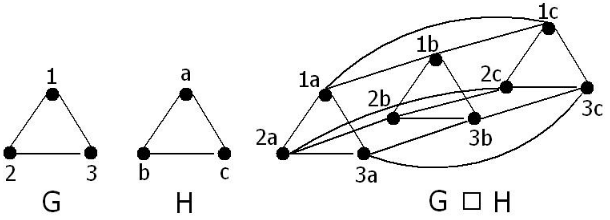


Every polynomial in the Gröbner basis has the following form:

$$(x_{i_1} - 1)(x_{i_d} - 1) \cdots (x_{i_D} - 1) \ ,$$

where $D := (n-1) + (n'-1) + 1 := n + n' - 1$.

Vizing's Conjecture and Gröbner Bases: Degree



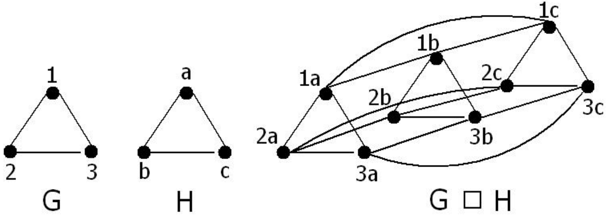
Every polynomial in the Gröbner basis has the following form:

$$(x_{i_1} - 1)(x_{i_d} - 1) \cdots (x_{i_D} - 1) \ ,$$

where $D := (n-1) + (n'-1) + 1 := n + n' - 1$.

In the $\mathscr{P}'_{\text{tri}\square\text{tri}}$ example, the degree equals five.

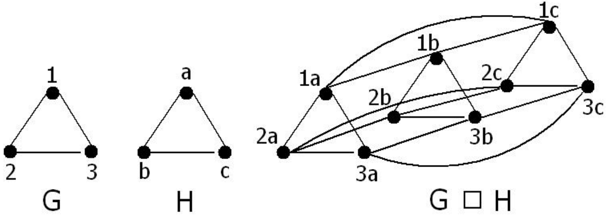
Vizing's Conjecture and Gröbner Bases: Case 1



Notation: Let $\mathscr{G}$ represent the set of G-levels in $G\Box H$. Given a level $l \in \mathscr{G}$, let

$$p(l) := \prod_{i \in V(l)} (x_i - 1) .$$

Vizing's Conjecture and Gröbner Bases: Case 1



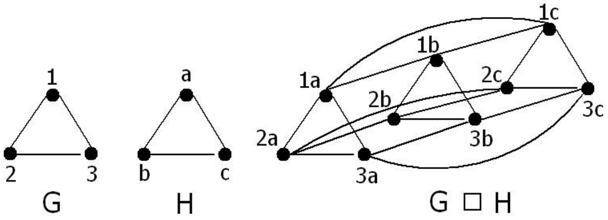
Notation: Let $\mathscr{G}$ represent the set of G-levels in $G \Box H$. Given a level $l \in \mathscr{G}$, let

$$p(l) := \prod_{i \in V(l)} (x_i - 1) \ .$$

Example: Consider the a-level in tri$\Box$tri. Then,

$$p(a) := (z_{1a} - 1)(z_{2a} - 1)(z_{3a} - 1) \ .$$

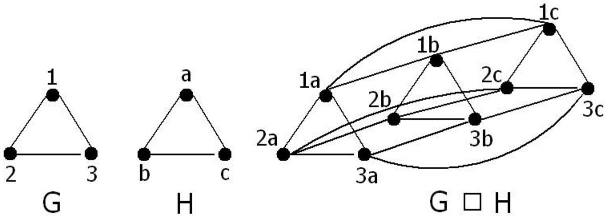
Vizing's Conjecture and Gröbner Bases: Case 1



Case 1: There are $|G| \cdot |H|$ polynomials of the form:

$$p(g) \cdot \prod_{\substack{l \in \mathscr{G}: \\ l \neq g}} (x[l_i] - 1) \ , \qquad \text{for each } i \in V(G) \text{ and each level } g \in \mathscr{G}.$$

Vizing's Conjecture and Gröbner Bases: Case 1



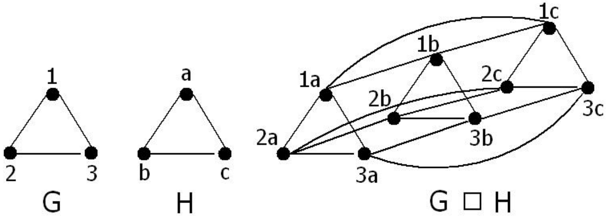
Case 1: There are $|G| \cdot |H|$ polynomials of the form:

$$p(g) \cdot \prod_{\substack{l \in \mathscr{G}: \\ l \neq g}} (x[l_i] - 1) \ , \qquad \text{for each } i \in V(G) \text{ and each level } g \in \mathscr{G}.$$

Example: For $g =$ a-level and $i = 1$, then

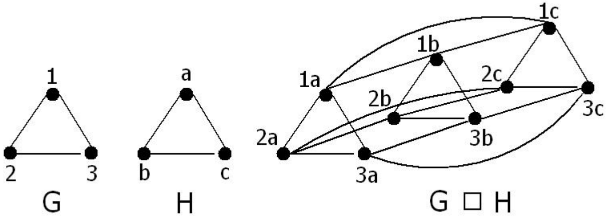
$$(z_{1a} - 1)(z_{2a} - 1)(z_{3a} - 1)(z_{1b} - 1)(z_{1c} - 1)$$

Vizing's Conjecture and Gröbner Bases: Case 2



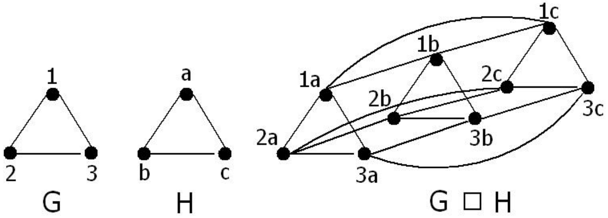
Notation: Let $e \in E[H]$. In $G \Box H$, the lexicographic order defined for the Gröbner basis also defines a direction on the edges in $G \Box H$.

Vizing's Conjecture and Gröbner Bases: Case 2



Notation: Let $e \in E[H]$. In $G \Box H$, the lexicographic order defined for the Gröbner basis also defines a direction on the edges in $G \Box H$. In particular, let $h(e)$ define the G-level that where the edge originates (according to the lexicographic order), and let $t(e)$ denote the G-level where the edge terminates.

Vizing's Conjecture and Gröbner Bases: Case 2

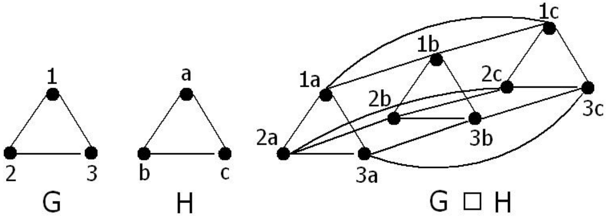


Notation: Let $e \in E[H]$. In $G\Box H$, the lexicographic order defined for the Gröbner basis also defines a direction on the edges in $G\Box H$. In particular, let $h(e)$ define the G-level that where the edge originates (according to the lexicographic order), and let $t(e)$ denote the G-level where the edge terminates.

Example: Consider the edge e'_{ac} and the c-level in tri$\Box$tri. Then,

$$p(h(e)) := (z_{1a} - 1)(z_{2a} - 1)(z_{3a} - 1) \ ,$$
$$p(t(e)) := (z_{1c} - 1)(z_{2c} - 1)(z_{3c} - 1) \ .$$

Vizing's Conjecture and Gröbner Bases: Case 2

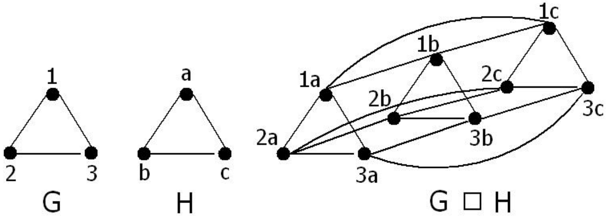


Case 2: There are $2||H|| \cdot |G| + 2||G|| \cdot |H|$ polynomials of the following form:

$$(x_e - 1)p(h(e)) \prod_{\substack{g \in \mathcal{G}:\ g \neq \mathcal{G}[t(e)] \\ \text{and } g \neq \mathcal{G}[h(e)]}} (g_i - 1)\ , \qquad \text{for each } e \in E(H) \text{ and each } i \in V(G)$$

$$(x_e - 1)p(t(e)) \prod_{\substack{g \in \mathcal{G}:\ g \neq \mathcal{G}[t(e)] \\ \text{and } g \neq \mathcal{G}[h(e)]}} (g_i - 1)\ , \qquad \text{for each } e \in E(H) \text{ and each } i \in V(G)$$

Vizing's Conjecture and Gröbner Bases: Case 2



Case 2: There are $2||H|| \cdot |G| + 2||G|| \cdot |H|$ polynomials of the following form:

$$(x_e - 1)p(h(e)) \prod_{\substack{g \in \mathscr{G}:\ g \neq \mathscr{G}[t(e)] \\ \text{and } g \neq \mathscr{G}[h(e)]}} (g_i - 1)\ , \qquad \text{for each } e \in E(H) \text{ and each } i \in V(G)$$

$$(x_e - 1)p(t(e)) \prod_{\substack{g \in \mathscr{G}:\ g \neq \mathscr{G}[t(e)] \\ \text{and } g \neq \mathscr{G}[h(e)]}} (g_i - 1)\ , \qquad \text{for each } e \in E(H) \text{ and each } i \in V(G)$$

Example: For $e = e'_{ac}$ and $i = 1$, then

$$(e'_{ac} - 1)(z_{1a} - 1)(z_{2a} - 1)(z_{3a} - 1)(z_{1b} - 1)\ ,$$
$$(e'_{ac} - 1)(z_{1c} - 1)(z_{2c} - 1)(z_{3c} - 1)(z_{1b} - 1)\ .$$

Summary and Future Work

- Represented Vizing's conjecture as a computational problem.

Summary and Future Work

- Represented Vizing's conjecture as a computational problem.
    - Can we use symmetry to speed up the computations?

Summary and Future Work

- Represented Vizing's conjecture as a computational problem.
  - Can we use symmetry to speed up the computations?
- Conjectured a graph-theoretic interpretation of the Gröbner basis of $\mathscr{P}'_{G\Box H}$ (presented only cases 1 and 2).

Summary and Future Work

- Represented Vizing's conjecture as a computational problem.
  - Can we use symmetry to speed up the computations?
- Conjectured a graph-theoretic interpretation of the Gröbner basis of $\mathscr{P}'_{G \square H}$ (presented only cases 1 and 2).
  - Code it up and check it!

Summary and Future Work

- Represented Vizing's conjecture as a computational problem.
  - Can we use symmetry to speed up the computations?
- Conjectured a graph-theoretic interpretation of the Gröbner basis of $\mathscr{P}'_{G\Box H}$ (presented only cases 1 and 2).
  - Code it up and check it!

Thank you for your kind attention!

Questions, comments, thoughts and suggestions are most welcome.