Math 821 Problem Set #6
Posted: Friday 4/15/11
Due date: Monday 4/25/11

Problem #1 [Hatcher p.131 #4] Compute the simplicial homology groups of the "triangular parachute" obtained from the standard 2-simplex Δ^2 by identifying its three vertices to a single point.

Solution: This Δ-complex has simplicial chain complex

$$C_2 = \mathbb{Z} \xrightarrow[\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}]{\partial_2} C_1 = \mathbb{Z}^3 \xrightarrow[\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}]{\partial_1} C_0 = \mathbb{Z}$$

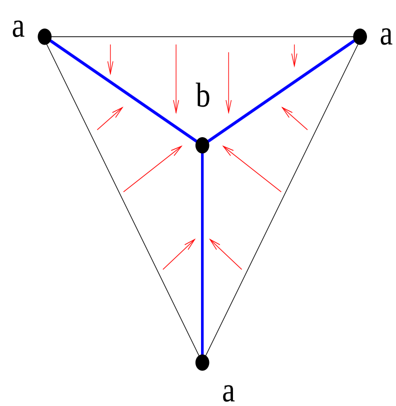
so we have

$$\begin{aligned} H_2 &= \ker \partial_2 = 0, \\ H_1 &= \ker \partial_1 / \operatorname{im} \partial_2 = \mathbb{Z}^3 / \operatorname{im} \partial_2 = \mathbb{Z}^2, \\ H_0 &= C_0/0 = \mathbb{Z}. \end{aligned}$$

Note that H_1 has no torsion because $\operatorname{im} \partial_1$ is not just an abstract copy of $\mathbb{Z}$, but in fact a direct summand of $\mathbb{Z}^3$, e.g., because

$$\mathbb{Z}^3 = \mathbb{Z} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \oplus \mathbb{Z} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \oplus \mathbb{Z} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Another argument: the parachute is homotopy equivalent to a graph with two vertices a, b and three parallel edges by the deformation retraction shown below – subdivide the triangle into three smaller triangles and push each smaller triangle onto the union of the two edges ab. This graph is in turn homotopy equivalent to $S^1 \vee S^1$, hence has the same homology.



Problem #2 [Hatcher p.131 #8] Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \ldots, T_n$ by the following two steps.

First, arrange the tetrahedra in a cyclic pattern as in the figure (see p. 131) so that each T_i shares a common vertical face with its two neighbors. For consistent notation, call the top and bottom vertices x and y respectively, and call the side vertices $v_1, \ldots, v_n$, so the tetrahedra are

$$T_1 = [x, v_1, v_2, y],\ T_2 = [x, v_2, v_3, y],\ \ldots, T_{n-1} = [x, v_{n-1}, v_n, y],\ T_n = [x, v_n, v_1, y].$$

Then, identify the bottom face of T_i with the top face of T_{i+1} for all i, that is, $[v_i, v_{i+1}, y] = [v_{i+1}, v_{i+2}, y]$.

Show that the simplicial homology groups of X in dimensions 0,1,2,3 are [isomorphic to] $\mathbb{Z}$, $\mathbb{Z}_n$, 0, $\mathbb{Z}$ respectively. (Here $\mathbb{Z}_n$ means $\mathbb{Z}/n\mathbb{Z}$.)

Solution: Start by taking a census of the simplices of the resulting Δ-complex. We already have a list of the 3-simplices (tetrahedra).

The 2-simplices (triangles) of X are as follows:

$$\begin{aligned}
U_1 &= [x, y, v_1], & Y_1 &= [x, v_1, v_2] = [y, v_n, v_1],\\
U_2 &= [x, y, v_2], & Y_2 &= [x, v_2, v_3] = [y, v_1, v_2],\\
&\cdots & &\cdots\\
U_{n-1} &= [x, y, v_{n-1}], & Y_{n-1} &= [x, v_{n-1}, v_n] = [y, v_{n-2}, v_{n-1}],\\
U_n &= [x, y, v_n], & Y_n &= [x, v_n, v_1] = [y, v_{n-1}, v_n].
\end{aligned}$$

The last column of equalities in this table come from the identifications in the second step of the construction of X. Those identifications force a bunch of the original edges to be identified, so that the 1-simplices (edges) of X are as follows:

$$\begin{aligned}
a &= [x, y],\\
b &= [v_1, v_2] = [v_2, v_3] = \cdots = [v_{n-1}, v_n] = [v_n, v_1],\\
e_1 &= [x, v_1] = [y, v_n],\\
e_2 &= [x, v_2] = [y, v_1],\\
&\cdots\\
e_{n-1} &= [x, v_{n-1}] = [y, v_{n-2}],\\
e_n &= [x, v_n] = [y, v_{n-1}].
\end{aligned}$$

Finally, X has just two vertices, because the identifications in step 2 force $x = y$ and $v_1 = \cdots = v_n$ (let's call them x and v respectively).

So the simplicial chain complex is going to be of the form

$$C_3 \cong \mathbb{Z}^n \xrightarrow{\partial_3} C_2 \cong \mathbb{Z}^{2n} \xrightarrow{\partial_2} C_1 \cong \mathbb{Z}^{n+2} \xrightarrow{\partial_1} C_0 \cong \mathbb{Z}^2.$$

The simplicial boundary maps are defined by

$$\begin{aligned}
\partial_3 T_i &= [v_i, v_{i+1}, y] - [x, v_{i+1}, y] + [x, v_i, y] - [x, v_i, v_{i+1}] = Y_{i+1} + U_{i+1} - U_i - Y_i, & \partial_1 a &= 0,\\
\partial_2 U_i &= [y, v_i] - [x, v_i] + [x, y] = e_{i+1} - e_i + a, & \partial_1 b &= 0,\\
\partial_2 Y_i &= [v_i, v_{i+1}] - [x, v_{i+1}] + [x, v_i] = b - e_{i+1} + e_i, & \partial_1 e_i &= x - v.
\end{aligned}$$

In matrix form, we have

$$\partial_3 = \left[\begin{array}{c} J \\ \hline J \end{array}\right] \in \mathbb{Z}^{2n \times n}$$

where J is the matrix with -1's on the main diagonal and 1's on the subdiagonal. The columns of ∂_3 represent the T_i's and the two row blocks the U_i's and Y_i's. Next,

$$\partial_2 = \left[\begin{array}{cccccc} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ \hline & & & & & \\ & J & & & -J & \end{array}\right] \in \mathbb{Z}^{n+2\times 2n}$$

and finally

$$\partial_1 = \begin{bmatrix} 0 & 0 & 1 & 1 & \cdots 1 \\ 0 & 0 & -1 & -1 & \cdots -1 \end{bmatrix} \in \mathbb{Z}^{2\times n+2}.$$

The columns of J add up to 0; that is the only linear dependendence among them. Therefore

$$\boxed{H_3(X) = \ker \partial_3 = \mathbb{Z} \begin{bmatrix} 1 \\ \ldots \\ 1 \end{bmatrix} \cong \mathbb{Z}.}$$

The rank of ∂_2 is $n+1$ (the e-rows have rank $n-1$, and the a- (resp. b-)row is the only one in which the sum of the entries in the first (resp. second) column block is nonzero). Therefore $\ker \partial_2$ has rank $2n-(n+1) = n-1$, and the following linearly independent vectors span it as a vector space:

$$-U_1 + U_2 - Y_1 + Y_2,\ -U_2 + U_3 - Y_2 + Y_3,\ \ldots,\ -U_{n-1} - U_n - Y_{n-1} + Y_n.$$

These correspond to the first $n-1$ columns of ∂_3. Since these columns are already in column-echelon form with 1's for the pivots, they in fact span $\ker \partial_2$ as a $\mathbb{Z}$-module. In particular, $\operatorname{im} \partial_3 = \ker \partial_2$ and so

$$\boxed{H_2(X) = 0.}$$

Next,

$$\begin{aligned} \operatorname{im} \partial_2 &= \mathbb{Z}\{a - e_1 + e_2,\ a - e_2 + e_3,\ \ldots,\ a - e_n + e_1,\ b + e_1 - e_2,\ b + e_2 - e_3,\ \ldots,\ b + e_n - e_1\} \\ &= \mathbb{Z}\{a - e_1 + e_2,\ a - e_2 + e_3,\ \ldots,\ a - e_n + e_1,\ a + b\} \\ &= \mathbb{Z}\{a - e_1 + e_2,\ a - e_2 + e_3,\ \ldots,\ na,\ a + b\} \end{aligned}$$

(This amounts to performing a bunch of $\mathbb{Z}$-column operations on the matrix.) On the other hand, we can take for a $\mathbb{Z}$-basis of $\ker \partial_1$

$$\{a,\ a + b,\ a - e_1 + e_2,\ a - e_2 + e_3,\ \ldots,\ a - e_{n-1} + e_n\}$$

and from these two descriptions it follows that

$$\boxed{H_1(X) \cong \mathbb{Z}_n.}$$

Problem #3 [Hatcher p.131 #11] Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Solution: Let i be that inclusion, and let $r : X \to A$ be a retraction, i.e., a map onto A that fixes A pointwise. So we have a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d, "i_\#"] & C_n(A) \arrow[r] \arrow[d, "i_\#"] & C_{n-1}(A) \arrow[r] \arrow[d, "i_\#"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[d, "r_\#"] & C_n(X) \arrow[r] \arrow[d, "r_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "r_\#"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(A) \arrow[r] & C_n(A) \arrow[r] & C_{n-1}(A) \arrow[r] & \cdots \end{tikzcd}$$

We have $r_\# i_\# = (r \circ i)_\# = \mathbb{1}_\# = \mathbb{1}$. Accordingly the induced map on homology $(r \circ i)_* = r_* i_*$ is the identity, in particular an isomorphism. Therefore i_* is injective, as desired (and r_* is surjective). $\square$

Problem #4 A [finite] *partially ordered set* or *poset* is a finite set P with an order relation $\leq$ such that for all $x, y, z \in P$, $x \leq x$; if $x \leq y$ and $y \leq x$, then $x = y$; if $x \leq y$ and $y \leq z$, then $x \leq z$. We write $x < z$ if $z \leq z$ and $x \neq z$. If $x \leq z$ or $z \leq x$, we say that x, z are *comparable*.

A *chain* in P (not to be confused with a simplicial or singular chain!) is a subset in which any two elements are comparable.

(#4a) Prove that the set $\Delta(P)$ of chains in P is a simplicial complex. (This is called the *order complex* of P.)

(#4b) Suppose that P has a unique maximal element. Prove that $\Delta(P)$ is contractible.

(#4c) For each $n \geq 1$, construct a poset for which $\Delta(P)$ is homeomorphic to an n-sphere.

(#4d) The *Möbius function* $\mu = \mu_P$ of P is defined as follows.

(1) Adjoin two new elements $\hat{0}, \hat{1}$ to P to obtain a poset $\hat{P}$, in which $\hat{0} < x < \hat{1}$ for every $x \in P$.
(2) Define μ recursively as follows: $\mu(\hat{0}) = 1$, and if $\mu(y)$ is defined for all $y < x$, then define

$$\mu(x) = -\sum_{y<x} \mu(y)$$

What is the relationship between $\mu_P(\hat{1})$ and $\chi(\Delta(P))$? (You may want to work out some small examples.)

Solution: (I didn't assign points for this one because there was some confusion over the definition of "poset".)

(a) Every singleton element is (trivially) a chain, and every subset of a chain is a chain.

(b) Call the unique maximal element x. Then $\Delta(P)$ is in fact a cone with base $\Delta(P \setminus \{x\})$ and x as the cone point, and every cone is contractible since it can be deformation-retracted to the cone point.

Note that the assumption that P has a unique maximal element is *not* the assumption that $\Delta(P)$ has a unique maximal element! (With the latter assumption, the problem becomes trivial, since a simplicial complex with a unique maximal face is a simplex, which is homeomorphic to a ball, hence contractible.)

(c) One possibility: let P be the poset with $2(n+1)$ elements $\{p_{\pm i} \mid 0 \leq i \leq n\}$ and relations $p_{\pm i} \leq p_{\pm j}$ iff $i \leq j$. Think of p_i as the i^{th} basis vector in $\mathbb{R}^{n+1}$ and p_{-i} as its negative. The 2^{n+1} maximal faces of the order complex are copies of the standard simplex, one in each orthant of $\mathbb{R}^{n+1}$, and the faces together form a simplicial n-sphere (called a *cross-polytope*). For example, the $n = 1$ case is a diamond, and $n = 2$ gives an octahedron.

(d) In fact $\chi(\Delta(P)) = 1 + \mu_P(\hat{1})$.