Chapter 16: Counting, Probabilities, and Odds

Counting, Probabilities, and Odds (Chapter 16)

Big Questions For Chapter 16

- What does probability mean? How can we model it mathematically?
- How do we calculate probabilities of events?
- How can we count big sets without having to list all their members?
- How do we decide if a game of chance is fair?

Sample Problems: Rolling Dice

- Roll one (fair) die. What is the probability of rolling a 3?
- Roll two fair dice and add them up. What is the probability that the <u>total</u> is 4? What is the most likely total? What is the probability of rolling it?
- Roll 11 fair dice and add them up. What is the most likely total?
- Roll 11 fair dice and add them up. What is the probability that the total is a multiple of 6?

Sample Problems: Poker Hands

A poker hand consists of five cards from a 52-card deck.

- How many possible poker hands are there?
- What if the order the five cards are dealt matters?
- What is the probability of being dealt a flush (i.e., five cards of the same suit)?
- If you are dealt four hearts, what is the probability that the fifth card you are dealt will also be a heart?
- How does the answer change if you can see the ♠A face up on the table?

Sample Problems: Birthdays

- What is the probability that some pair of people in this room have the same birthday?

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- How many people would there have to be in the room to **guarantee** that two have the same birthday?
- How many people would there have to be to make it **even odds** that two have the same birthday? (The answer may surprise you.)

The Monty Hall "Paradox"

A game-show host offers you a choice of three doors. Behind one door there is a new sports car. Behind two of the other doors are goats.

You pick one of the doors (let's say Door #1).

Before revealing your prize, the host opens one of the doors you didn't choose (say Door #3) and opens it to reveal a goat.

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Before revealing your prize, the host opens one of the doors you didn't choose (say Door #3) and opens it to reveal a goat.

The host then offers you the chance to switch your choice to Door #2.

Should you switch? Does it matter? ★

Probability Terminology

Random experiment: an activity whose result cannot be predicted in advance (e.g., a die roll, a coin toss, a basketball free throw)

Sample space: the set of possible outcomes of a random experiment (e.g., “die comes up 2”, “full house”)
Typically denoted S.

Experiment	**Sample space**
Coin toss	$\{h, t\}$
Die roll	$\{1, 2, 3, 4, 5, 6\}$

Probability Terminology

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Probability Terminology

Experiment	**Sample space**
Coin toss	$\{h, t\}$
Die roll	$\{1, 2, 3, 4, 5, 6\}$
Free throw	$\{0, 1\}$
Two coin tosses	$\{hh, ht, th, tt\}$
Coin toss + die roll	$\{h1, h2, \ldots, h6, t1, t2, \ldots, t6\}$

Probability Terminology

Experiment	Sample space
Coin toss	$\{h, t\}$
Die roll	$\{1, 2, 3, 4, 5, 6\}$
Free throw	$\{0, 1\}$
Two coin tosses	$\{hh, ht, th, tt\}$
Coin toss + die roll	$\{h1, h2, \ldots, h6, t1, t2, \ldots, t6\}$
Two free throws	$\{00, 01, 10, 11\}$
One-and-one free throw	$\{0, 01, 11\}$

Same Experiment, Different Sample Spaces

The same experiment can have different sample spaces, depending on what we are trying to measure.

Same Experiment, Different Sample Spaces

Experiment: roll **two** dice, one red and one blue.
What is the sample space S?

Same Experiment, Different Sample Spaces

Experiment: roll **two** dice, one red and one blue.
What is the sample space S?

(i) If we care about the numbers on both dice:

$$S = \left\{ \begin{array}{cccccc} 11, & 12, & 13, & 14, & 15, & 16, \\ 21, & 22, & 23, & 24, & 25, & 26, \\ 31, & 32, & 33, & 34, & 35, & 36, \\ 41, & 42, & 43, & 44, & 45, & 46, \\ 51, & 52, & 53, & 54, & 55, & 56, \\ 61, & 62, & 63, & 64, & 65, & 66 \end{array} \right\}$$

Same Experiment, Different Sample Spaces

Experiment: roll **two** dice, one red and one blue.
What is the sample space S?

(ii) If we don't care which number is on which color die:

$$S = \left\{ \begin{array}{cccccc} 11, & 12, & 13, & 14, & 15, & 16, \\ & 22, & 23, & 24, & 25, & 26, \\ & & 33, & 34, & 35, & 36, \\ & & & 44, & 45, & 46, \\ & & & & 55, & 56, \\ & & & & & 66 \end{array} \right\}$$

Same Experiment, Different Sample Spaces

Experiment: roll **two** dice, one red and one blue.
What is the sample space S?

(iii) If we only care about the **sum** of the two numbers:

$$S = \{2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12\}.$$

Probability Terminology: Events

Suppose we toss a coin three times and record all the results in order. The sample space is

$$S = \{hhh,\ hht,\ hth,\ htt,\ thh,\ tht,\ tth,\ ttt\}.$$

Probability Terminology: Events

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$$S = \{hhh,\ hht,\ hth,\ htt,\ thh,\ tht,\ tth,\ ttt\}.$$

The number of tails is odd for the outcomes

$$hht,\ hth,\ thh,\ ttt$$

(and for only those outcomes).

Probability Terminology: Events

Suppose we toss a coin three times and record all the results in order. The sample space is

$$S = \{hhh,\ hht,\ hth,\ htt,\ thh,\ tht,\ tth,\ ttt\}.$$

The number of tails is odd for the outcomes

$$hht,\ hth,\ thh,\ ttt$$

(and for only those outcomes).

The **event** "the number of tails is odd" **is** the subset $\{hht, hth, tht, ttt\}$ of S.

Probability Terminology: Events

Experiment: Three coin flips; record all results in order

Sample space: $S = \{hhh,\ hht,\ hth,\ htt,\ thh,\ tht,\ tth,\ ttt\}$

Event	**Subset of S**
Exactly one head	$\{htt,\ tht,\ tth\}$
At least one tail	$\{hht,\ hth,\ htt,\ thh,\ tht,\ tth,\ ttt\}$
Two heads in a row	$\{hhh,\ hht,\ thh\}$
More heads than tails	$\{hhh,\ hht,\ thh,\ hth\}$

Impossible, Simple, and Certain Events

An event can include any number of outcomes, from 0 up to the size of the sample space S.

Event	**Subset of S**	**# of outcomes**	
Exactly three tails	$\{ttt\}$	1	(**simple** event)
At least six tails	$\{\}$	0	(**impossible** event)
At most six tails	S	All	(**certain** event)

Equiprobable Spaces

A space S is **equiprobable** if all outcomes are equally likely.

- rolling a fair die
- tossing a fair coin
- tossing a fair coin N times, recording all results in order

The **probability** of an event E in an equiprobable space S is the ratio

$$\frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}$$

Counting Sample Spaces

Suppose we toss a coin **eight** times and record all the results. The sample space is

$S = \{$hhhhhhhh, thhhhhhh, hthhhhhh, tthhhhhh, hhthhhhh, ththhhhh, htthhhhh, ttthhhhh, hhhthhhh, thhthhhh, hththhhh, tththhhh, hhtthhhh, thtthhhh, httthhhh, tttthhhh, hhhhthhh, thhhthhh, hthhthhh, tthhthhh, hhththhh, thththhh, httththhh, ttththhh, hhhtthhh, thhtthhh, hthtthhh, tthtthhh, hhttthhh, thttthhh, httttthhh, tttttthhh, hhhhhthh, thhhhthh, hthhhthh, tthhhthh, hhthhthh, ththhthh, htthhthh, ttthhthh, hhhththh, thhththh, hthththh, tthththh, hhttththh, thttththh, htttththh, ttttththh, hhhhtthh, thhhtthh, hthhtthh, tthhtthh, hhthtthh, ththtthh, htthtthh, ttthtthh, hhhttthh, thhttthh, hthttthh, tthttthh, hhtttthh, thtttthh, httttthh, tttttthh, hhhhhhth, thhhhhth, hthhhhth, tthhhhth, hhthhhth, ththhhth, htthhhth, ttthhhth, hhhthhth, thhthhth, hththhth, tththhth, hhtthhth, thtthhth, httthhth, tttthhth, hhhhthth, thhhthth, hthhthth, tthhthth, hhththth, thththth, httththth, ttththth, hhhttht h, thhtthth, hthtthth, tthtthth, hhttthth, thttthth, httttthth, tttttthth, hhhhhtth, thhhhtth, hthhhtth, tthhhtth, hhthhtth, ththhtth, htthhtth, ttthhtth, hhhthtth, thhthtth, hththtth, tththtth, hhtthtth, thtthtth, httthtth, tttthtth, hhhhttth, thhhttth, hthhttth, tthhttth, hhthttth, ththttth, htthttth, ttthttth, hhhtttth, thhtttth, hthtttth, tthtttth, hhttttth, thttttth, httttttth, tttttttth, hhhhhhht, thhhhhht, hthhhhht, tthhhhht, hhthhhht, ththhhht, htthhhht, ttthhhht, hhhthhht, thhthhht, hththhht, tththhht, hhtthhht, thtthhht, httthhht, tttthhht, hhhhthht, thhhthht, hthhthht, tthhthht, hhththht, thththht, httththht, ttththht, hhhtthht, thhtthht, hthtthht, tthtthht, hhttthht, thttthht, httttthht, tttttthht, hhhhhtht, thhhhtht, hthhhtht, tthhhtht, hhthhtht, ththhtht, htthhtht, ttthhtht, hhhththt, thhththt, hthththt, tthththt, hhtththt, thtththt, httththt, tttththt, hhhhttht, thhhttht, hthhttht, tthhttht, hhthttht, ththttht, htthttht, ttthttht, hhhtttht, thhtttht, hthtttht, tthtttht, hhttttht, thttttht, httttttht, tttttttht, hhhhhhtt, thhhhhtt, hthhhhtt, tthhhhtt, hhthhhtt, ththhhtt, htthhhtt, ttthhhtt, hhhthhtt, thhthhtt, hththhtt, tththhtt, hhtthhtt, thtthhtt, httthhtt, tttthhtt, hhhhthtt, thhhthtt, hthhthtt, tthhthtt, hhththtt, thththtt, httththtt, ttththtt, hhhtthtt, thhtthtt, hthtthtt, tthtthtt, hhttthtt, thttthtt, httttthtt, tttttthtt, hhhhhttt, thhhhttt, hthhhttt, tthhhttt, hhthhttt, ththhttt, htthhttt, ttthhttt, hhhthttt, thhthttt, hththttt, tththttt, hhtthttt, thtthttt, httthttt, tttthttt, hhhhtttt, thhhtttt, hthhtttt, tthhtttt, hhthtttt, ththtttt, htthtttt, ttthtttt, hhhttttt, thhttttt, hthttttt, tthttttt, hhtttttt, thtttttt, httttttt, tttttttt$\}$

How big is S? (Don't stare too hard.)

The Multiplication Rule

Each time you flip a coin, there are two possible outcomes.

$$S = \{h, t\}$$

Flip a coin twice: $2 \times 2 = 4$ possible outcomes.

$$S = \{hh, ht, th, tt\}$$

Flip a coin and then roll a die: $2 \times 6 = 12$ possible outcomes.

$$S = \{h1, h2, h3, h4, h5, h6, t1, t2, t3, t4, t5, t6\}$$

The Multiplication Rule

The Multiplication Rule: If there are m different ways to do X and n different ways to do Y, then there are mn different ways to do X followed by doing Y.

Warning: The number of ways to do X and Y must be independent of each other.

(In other words, the outcome of X must not affect the number of ways to do Y.)

Extensions of the Multiplication Rule

Multiplication Rule (2): If there are

- m different ways to do X,
- n different ways to do Y, and
- p different ways to do Z,

then there are mnp different ways to do X, Y, Z in that order (provided that these three things are independent).

Extensions of the Multiplication Rule

Multiplication Rule (3): If there are

- m_1 different ways to do X_1,
- m_2 different ways to do X_2,
- $\ldots, \ldots,$
- m_K different ways to do X_K,

then there are $m_1 m_2 \cdots m_K$ different ways to do $X_1, X_2, \ldots, X_K$ in that order (provided that these K things are independent).

Extensions of the Multiplication Rule

Multiplication Rule (4): If there are m different ways to do X, then the number of ways to repeat X a total of T times is

$$m^T.$$

For example, if we toss a coin 8 times and record the results in order, the sample space has size

$$2^8 = 256.$$

Independence

In order to apply the Multiplication Rule, the number of ways to do the different experiments must be independent of each other.

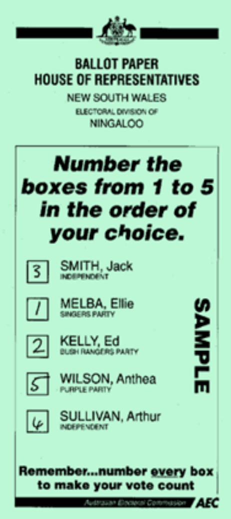
In other words, the outcome of X must not affect the number of ways to do Y.

Examples of independent events:
separate dice rolls, separate coin flips

Example of non-independent events:
one-and-one free throw

The Multiplication Rule

Problem: How many ways are there to fill out this ballot?



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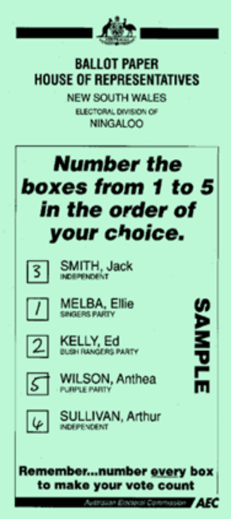

Favorite candidate: **5 choices.**

The Multiplication Rule

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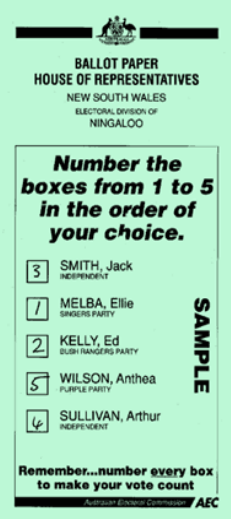

Favorite candidate:	**5 choices.**
Second favorite:	**4 choices.**

The Multiplication Rule

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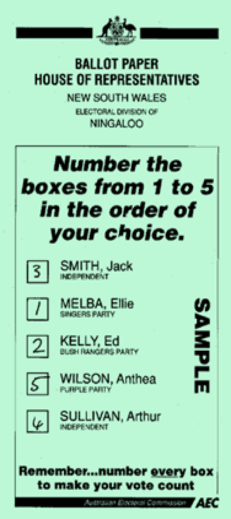
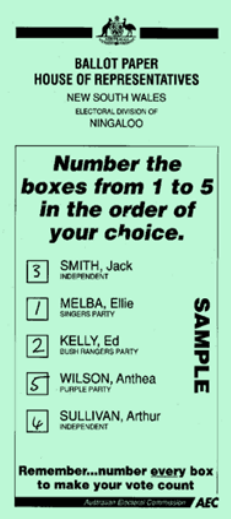


Favorite candidate: **5 choices.**
Second favorite: **4 choices.**
Third favorite: **3 choices.**

The Multiplication Rule

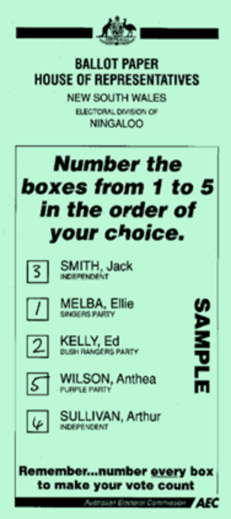
Problem: How many ways are there to fill out this ballot?



Favorite candidate:	**5 choices.**
Second favorite:	**4 choices.**
Third favorite:	**3 choices.**
Fourth favorite:	**2 choices.**

The Multiplication Rule

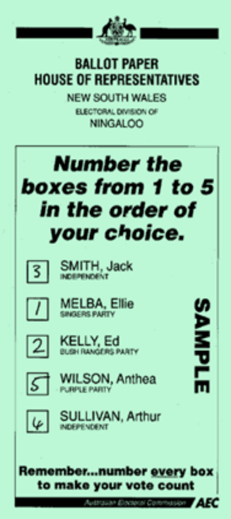
Problem: How many ways are there to fill out this ballot?



Favorite candidate:	**5 choices.**
Second favorite:	**4 choices.**
Third favorite:	**3 choices.**
Fourth favorite:	**2 choices.**
Least favorite:	**1 choice.**

The Multiplication Rule

Problem: How many ways are there to fill out this ballot?

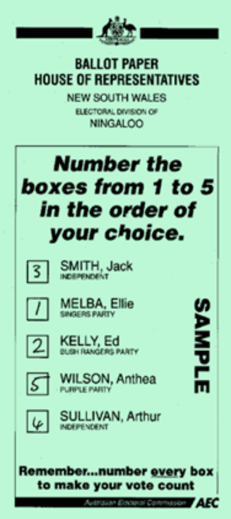


Favorite candidate:	**5 choices.**
Second favorite:	**4 choices.**
Third favorite:	**3 choices.**
Fourth favorite:	**2 choices.**
Least favorite:	**1 choice.**

Answer: $5 \times 4 \times 3 \times 2 \times 1 =$ **120**.

The Multiplication Rule

Problem: How many ways are there to fill out this ballot?


BALLOT PAPER
HOUSE OF REPRESENTATIVES
NEW SOUTH WALES
ELECTORAL DIVISION OF
NINGALOO

Number the boxes from 1 to 5 in the order of your choice.

3 SMITH, Jack
INDEPENDENT
1 MELBA, Ellie
SINGERS PARTY
2 KELLY, Ed
BUSH RANGERS PARTY
5 WILSON, Anthea
PURPLE PARTY
4 SULLIVAN, Arthur
INDEPENDENT

SAMPLE

Remember...number every box to make your vote count

Australian Electoral Commission AEC

Favorite candidate:	**5 choices.**
Second favorite:	**4 choices.**
Third favorite:	**3 choices.**
Fourth favorite:	**2 choices.**
Least favorite:	**1 choice.**

Answer: $5 \times 4 \times 3 \times 2 \times 1 =$**120**.

Note that the **number** of choices at each stage is independent of the previous choices.

Factorials

Shuffle a deck of 52 cards and lay them out on the table left to right. How many different layouts can there be?

- Top card: 52 possibilities
- Second card: 51 possibilities
- ...
- n^{th} card: $52 - (n-1) = 53 - n$ possibilities (number of cards not already face up)
- ...
- Last card: 1 possibility

Total:

$$52 \times 51 \times 50 \times \cdots \times (53-n) \times \cdots \times 3 \times 2 \times 1$$
$$= \mathbf{52}! \quad \text{(pronounced "52 factorial.")}$$

Factorials

$$
\begin{aligned}
n! &= n(n-1)(n-2)\cdots(3)(2)(1) \\
&= \text{number of arranging } n \text{ things in order}
\end{aligned}
$$

$$
\begin{array}{rclcl}
1! & = & 1 & & \\
2! & = & 2 \times 1 & = & 2 \\
3! & = & 3 \times 2 \times 1 & = & 6 \\
4! & = & 4 \times 3 \times 2 \times 1 & = & 24 \\
5! & = & 5 \times 4! & = & 120 \\
6! & = & 6 \times 5! & = & 720 \\
10! & = & 10 \times 9! & = & 3,628,800 \\
0! & = & 1 & &
\end{array}
$$

$0! = 1$ (Really!)

Why do we define $0! = 1$? If you have no objects to arrange, there's one way to arrange them.

(Don't confuse zero **objects** with zero **arrangements**!)

$0! = 1$ (Really!)

Why do we define $0! = 1$? If you have no objects to arrange, there's one way to arrange them.

(Don't confuse zero **objects** with zero **arrangements**!)

Also, many formulas and patterns work better with this definition.

$$\begin{array}{lcl cl}
5!/4! & = & 120/24 & = & 5 \\
4!/3! & = & 24/6 & = & 4 \\
3!/2! & = & 6/2 & = & 3 \\
2!/1! & = & 2/1 & = & 2 \\
1!/0! & = & 1/\mathbf{?} & = & \mathbf{?}
\end{array}$$

Factorials Get Really Big

Shuffle a deck of 52 cards and lay them out left to right. There are 52! possibilities. How big is that number?

[1] Source: http://education.jlab.org/qa/mathatom_05.html

Factorials Get Really Big

Shuffle a deck of 52 cards and lay them out left to right. There are 52! possibilities. How big is that number?

80,658,175,170,943,878,571,660,636,856,403,766,975, 289,505,440,883,277,824,000,000,000,000. 68 digits

Estimate of number of atoms in the Earth[1]:
150,000,000,000,000,000,000,000,000,000,000,000,000, 000,000,000,000,000 (51 digits)

The number 70! has **101 digits.**

[1] Source: http://education.jlab.org/qa/mathatom_05.html

Factorials Get Really, REALLY Big

How many ways can the 305 students enrolled in Math 105 be assigned seats in Budig 120 (which has a capacity of 976)?

- First student: 976 possible seats
- 2nd student: 975 possible seats
- ...
- 305th student: $976 - 304 = 672$ possible seats

By the Multiplication Rule, the answer is

$$\begin{aligned} &976 \times 975 \times \cdots \times 673 \times 672 \\ &= \frac{976!}{671!} = \frac{976!}{(976-305)!} \\ &= (\text{enormous number with } \textbf{890 digits}) \end{aligned}$$

Permutations and Combinations

Sylas and Maddy's Ice Cream[2] (1014 Massachusetts St.) has 36 flavors. You have three favorites:

- Dinosaur Egg (not sure what it is but it's blue)
- Queen Of Hearts (vanilla, raspberry chocolate chips)
- Rock Chocolate Jayhawk (chocolate chunks, brownies)

You want to get two scoops of different flavors. How many possibilities are there?

[2]Prof. Martin is not affiliated with Sylas and Maddy's, nor was he paid to include this example. He just likes their ice cream.

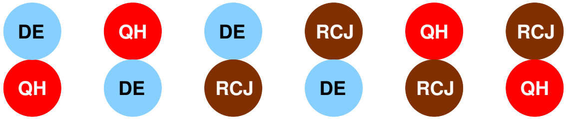
Permutations and Combinations

Answer: It depends.

Permutations and Combinations

Answer: It depends.

For some people, it is very important which scoop is on top. For them, there are 6 different possibilities.



Permutations and Combinations

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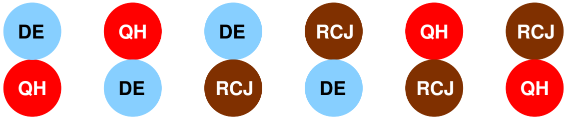
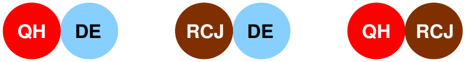


Some people don't care. For them, there are 3 possibilities.



Permutations and Combinations

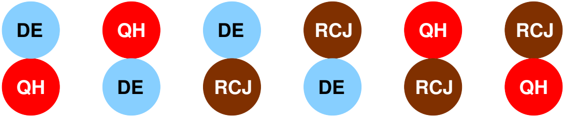
Let's say you want to choose r items from a set of n items. (E.g., you want to choose $r = 2$ scoops out of $n = 3$ available flavors.)
Permutation: An **ordered** selection of r items. (You care which scoop is on top.)
Combination: An **unordered** selection of r items. (You don't care about the order of the scoops.)

Notation for Permutations and Combinations

Notation: The number of **permutations** of r items chosen from a set of n items is written as

$$_nP_r$$

For example, $_3P_2 = 6$.



Notation for Permutations and Combinations

Notation: The number of **combinations** of r items chosen from a set of n items is written as

$${}_nC_r$$

For example, ${}_3C_2 = 3$.

Notation for Permutations and Combinations

If you want to choose r items out of a set of n items:

${}_nP_r =$ number of **permutations**
${}_nC_r =$ number of **combinations**

Memory aid:

- If you turn the letter P upside down, it changes.
- If you turn the letter C upside down, it stays the same!

Permutations and Combinations: More Examples

- Number of triple scoops at Sylas and Maddy's for someone who is **P**icky about the order of scoops: $_{36}P_3 = 42840$
- Number of triple scoops at Sylas and Maddy's for someone who is **C**areless about the order of scoops: $_{36}C_3 = 7140$

Permutations and Combinations: More Examples

- Number of triple scoops at Sylas and Maddy's for someone who is **P**icky about the order of scoops: $_{36}\mathbf{P}_{3} = \mathbf{42840}$

- Number of triple scoops at Sylas and Maddy's for someone who is **C**areless about the order of scoops: $_{36}\mathbf{C}_{3} = \mathbf{7140}$

- Number of ways of assigning seats to 297 students in a 976-seat lecture hall: $_{976}\mathbf{P}_{297}$ (866 digits)

- Number of ways of choosing which seats will be occupied (but not who sits where): $_{976}\mathbf{C}_{297}$ (259 digits)

Permutations and Combinations: More Examples

- Number of blackjack hands = number of ways to deal one card face down, followed by one card face up: $\mathbf{{}_{52}P_2 = 2652}$

Permutations and Combinations: More Examples

- Number of blackjack hands = number of ways to deal one card face down, followed by one card face up:
  $\mathbf{{}_{52}P_2 = 2652}$

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Permutations and Combinations: More Examples

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- Number of ways you can rank 5 candidates for office from favorite to least favorite: $\mathbf{{}_5P_5 = 5! = 120}$
- Number of games played in a round-robin tournament among 10 teams = number of ways of choosing two of the teams (in no particular order!) = $\mathbf{{}_{10}C_2 = 45}$

Permutations and Combinations: The Formulas

Here is one way to choose a quintuple scoop at Sylas and Maddy's (if you're one of the **P**icky people):

- Ask (nicely) to borrow the 36 wooden slats that the flavors are printed on.
- Shuffle them like a deck of cards.
- The top five "cards" tell you what to order. Ignore the other 31 cards.

Number of ways the deck could be shuffled $=$ **36!**

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Permutations and Combinations: The Formulas

Number of ways the deck could be shuffled = **36!**
Number of ways to arrange the cards you don't care about = (36-5)! = 31!
Therefore,

$$ {}_{36}P_5 = \frac{36!}{31!} = \frac{36!}{(36-5)!} = 45239040. $$

By the same logic, the general formula for ${}_nP_r$ is:

$$ \boxed{{}_{\mathbf{n}}\mathbf{P}_{\mathbf{r}} = \frac{\mathbf{n!}}{\mathbf{(n-r)!}}} $$

(Warning: Don't accidentally write $n! - r!$ instead of $(n-r)!$. If you do, you will probably get rubbish.)

Permutations and Combinations: The Formulas

$$\boxed{{}_n\mathbf{P}_\mathbf{r} = \frac{\mathbf{n!}}{\mathbf{(n-r)!}}}$$

Some important consequences of this formula:

- ${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0} = \frac{n!}{1} = n!$. (Makes sense — ${}_nP_n$ counts all $n!$ permutations of n objects.)
- ${}_nP_1 = \frac{n!}{(n-1)!} = n$. ("Permutation of one item" = "item".)
- ${}_nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$. (There is exactly one way to order zero scoops of ice cream.)

Permutations and Combinations: The Formulas

What's the formula for ${}_nC_r$?

Permutations and Combinations: The Formulas

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Key: How are ${}_nP_r$ and ${}_nC_r$ related?

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For example, to order a quintuple-scoop **bowl**...

- Wait until your picky friend is done picking 5 "cards".
- Forget what order those cards are in, since you don't care.
- You have $\frac{1}{5!} = \frac{1}{120}$ as many choices as your picky friend.

$${}_{36}C_5 = \frac{{}_{36}P_5}{5!} = \frac{\frac{36!}{31!}}{5!} = \frac{36!}{(31!)(5!)} = 376992$$

$$\boxed{\mathbf{{}_nC_r = \frac{n!}{r!\,(n-r)!}}}$$