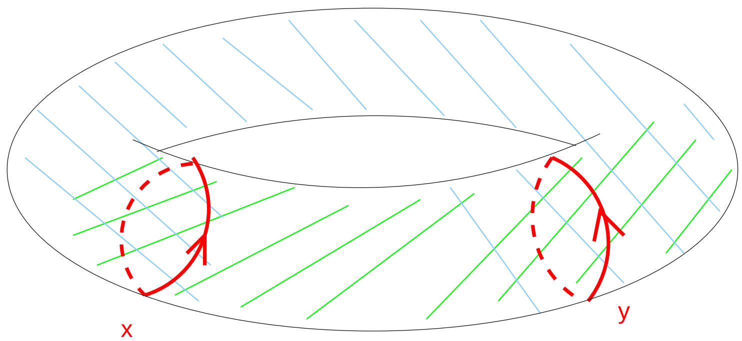
Decompose a torus $T = S^1 \times S^1$ into two cylinders, as shown.



Here T is the whole torus;
A is the green subspace, homotopy equivalent to S^1;
B is the blue subspace, homotopy equivalent to S^1;
$A \cap B$ is homotopy equivalent to the union of the two red triangles x and y, and in particular $H_1(A \cap B) \cong \mathbb{Z}^2$ is the free $\mathbb{Z}$-module on x and y, regarded as cycles with the orientations given.

Notice that x and y both map to the same generator of $H_1(A) \cong \mathbb{Z}$ and to the same generator of $H_1(B) \cong \mathbb{Z}$. If we call those generators a and b respectively, then the map

$$g : H_1(A \cap B) \to H_1(A) \oplus H_1(B)$$

in the Mayer-Vietoris sequence is given by $x \mapsto (a, b)$, $y \mapsto (a, b)$, i.e., represented by the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Reversing one of the red arrows, say y, would now make x and y map to opposite generators in both $H_1(A)$ and $H_1(B)$. But the point is that *the relative orientation of x with respect to y is the same whether we regard them as cycles in A or in B.* The matrix for g might be any of these things (in fact, the eight 2×2 matrices with an even number each of 1's and -1's):

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

but in all cases the kernel and image are both copies of $\mathbb{Z}$, spanned by vectors that extend to bases of their respective modules, so the Mayer-Vietoris calculation will yield the same result.

Now, what happens with the Klein bottle? If we attach the two cylinders near x as shown and then give the green cylinder B a twist before gluing near y, then what we have done is precisely to change the sign of the b-coordinate in $g(y)$. The map g is therefore

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and as we saw in class, we now have $\ker g = 0$ and $\operatorname{coker} g \cong \mathbb{Z}_2$. Changing bases for any of the homology groups might, for example, multiply one or more rows or columns of this matrix by -1, but it would not change the isomorphism type of the kernel or cokernel. For this decomposition of the Klein bottle, *the relative orientation of x with respect to y is the opposite in B of whatever it is in A.*