A Taste of Hyperplane Arrangements

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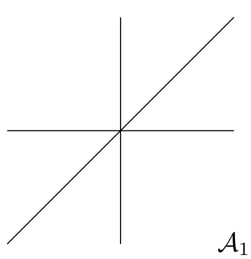
These notes are intended to accompany two lectures at RECO 2023 (Recorrido por la Combinatoria) at Pontificia Universidad Católica de Chile on December 4–5, 2023. They are adapted for an undergraduate audience from my graduate-level Lecture Notes on Algebraic Combinatorics.

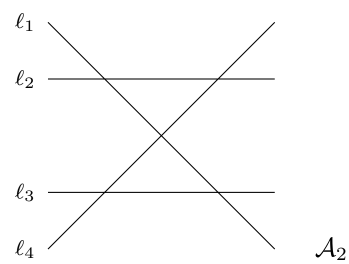
An excellent source for the combinatorial theory of hyperplane arrangements is Stanley's book section [Sta07], which is accessible to newcomers, and includes a self-contained treatment of topics such as the Möbius function and characteristic polynomial. Another canonical (but harder) source is the monograph by Orlik and Terao [OT92]. Of course, the standard reference for all things combinatorial is Stanley's *Enumerative Combinatorics* [Sta97, Sta99].

Lecture 1: Basics of Hyperplane Arrangements

Definition 1.1. A **linear hyperplane** in $\mathbb{R}^n$ is a vector subspace of codimension 1. An **affine hyperplane** is a translate of a linear hyperplane. A **hyperplane arrangement** $\mathcal{A} \subseteq \mathbb{R}^n$ is a finite set of (distinct) hyperplanes $H_1, \dots, H_k \subseteq \mathbb{R}^n$. The number n is called the **dimension** of $\mathcal{A}$, and the space $\Bbbk^n$ is its **ambient space**.

Example 1.2. Here are two line arrangements in $\mathbb{R}^2$. The arrangement $\mathcal{A}_1$ consists of the lines $x = 0$, $y = 0$, and $x = y$. The arrangement $\mathcal{A}_2$ consists of the four lines $\ell_1, \ell_2, \ell_3, \ell_4$ given by the equations $y = 1$, $x = y$, $x = -y$, $y = -1$ respectively.





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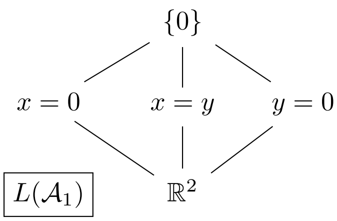
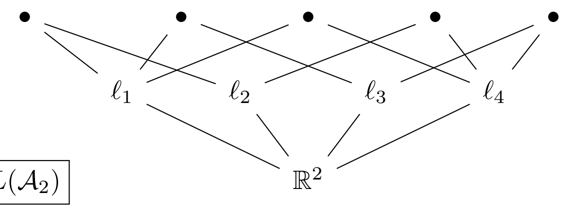
To describe the structure of a hyperplane arrangement, we draw a schematic diagram including the hyperplanes themselves and all the subspaces where they intersect, like this:



$\boxed{L(\mathcal{A}_1)}$



$\boxed{L(\mathcal{A}_2)}$

The lines in the diagrams indicate which subspaces contain each other. As you move up the diagram, the subspaces get smaller. Specifically, if you can walk up from a subspace P to a subspace Q along the lines, then $P \supseteq Q$. We'd say the subspaces are "ordered by reverse inclusion."

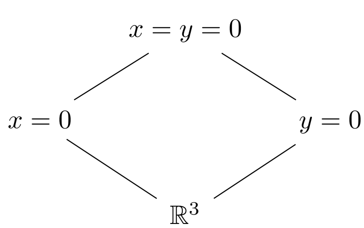
By the way, we are going to regard the space in which the hyperplanes live as a perfectly good subspace: it's the intersection of no hyperplanes! (If this seems strange, think about it this way: the more sets you intersect, the smaller the intersection is. So if you want as big an intersection as possible, make the family of sets you are intersecting as small as possible. In other words, make it the empty set.)

These structures are the **intersection posets** of the arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$. The word "poset" is shorthand for "partially ordered set". The containment relation is an ordering because it is *antisymmetric* (P and Q can't contain each other unless they are equal) and *transitive* (if $P \supseteq Q \supseteq R$, then $P \supseteq R$). But it is a *partial* ordering because there are pairs of subspaces such that neither one contains the other (for example, any two hyperplanes in the original arrangement).

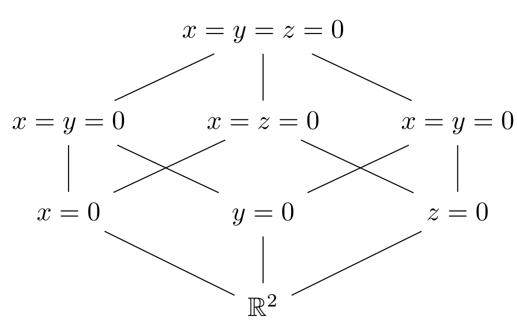
Here is a precise definition of $L(\mathcal{A})$, now that you have some examples and hopefully a bit of intuition. For every subarrangement $\mathcal{B} \subseteq \mathcal{A}$, let $\cap\mathcal{B}$ denote the intersection of all the hyperplanes in $\mathcal{B}$. The intersection poset $L(\mathcal{A})$ then consists of all nonempty spaces $\cap\mathcal{B}$, ordered by reverse inclusion. (Note that there is *not* a one-to-one correspondence between subarrangements of $\mathcal{A}$ and elements of $L(\mathcal{A})$. In general there are more of the former, because different subarrangements can have the same intersection, and we discard subarrangements with $\cap\mathcal{B} = \emptyset$.)

The intersection poset is the most important combinatorial tool we have to study arrangements.

Example 1.3. The **Boolean arrangement** Bool_n (or **coordinate arrangement**) consists of the n coordinate hyperplanes in n-space: $x_1 = 0, \dots, x_n = 0$. Its intersection poset is called the **Boolean lattice**, also denoted by Bool_n, and looks like an n-dimensional cube **(draw examples)**. This is the one case where there *is* a one-to-one correspondence between subarrangements of $\mathcal{A}$ and elements of $L(*\mathcal{A})$.



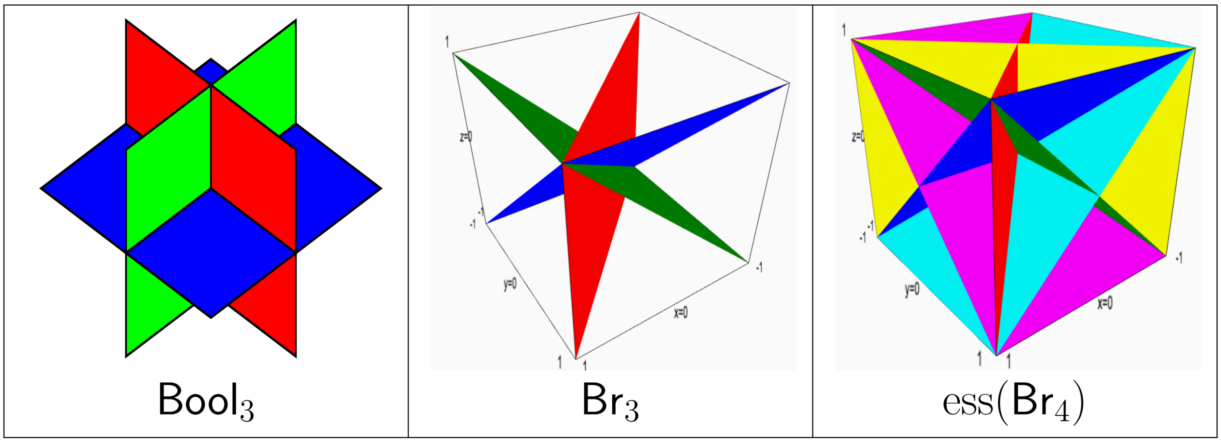
$L(\mathsf{Bool}_2)$



$L(\mathsf{Bool}_3)$

There are several useful ways to think about the Boolean lattice. For example, it can be regarded as the subsets of $[n]$,[1] ordered by inclusion. Alternatively, you can think of its elements as strings of 0's and 1's, all of length n: going up in the poset means changing one or more 0's to 1's. ◀

Example 1.4. The **braid arrangement** Br_n consists of the $\binom{n}{2}$ hyperplanes $x_i = x_j$ in n-space.



For example, Br_2 (not shown) consists of the single line $x = y$ in $\mathbb{R}^2$; Br_3 consists of the three planes $x = y, x = z, y = z$ in $\mathbb{R}^3$; and Br_4 consists of six 3-dimensional spaces in $\mathbb{R}^4$. Fortunately, since every hyperplane in Br_n contains the line $x_1 = x_2 = \cdots = x_n$, we can project $\mathbb{R}^4$ to $\mathbb{R}^3$ along that line (a process called **essentialization**) in order to picture Br_4 as an arrangement in $\mathbb{R}^3$.

(More precisely: Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$. Imagine translating every hyperplane in $\mathcal{A}$ so that it contains the origin, and then intersecting the resulting hyperplanes to get a space $Z = Z(\mathcal{A})$. The simplest case is that $Z = 0$, when we say that $\mathcal{A}$ is **essential**. If $Z \neq 0$, then we can pass from $\mathbb{R}^n$ to the quotient space $\mathbb{R}^n/Z$ to get an essential arrangement that has the exact same combinatorics as $\mathcal{A}$; this new arrangement is defined as the essentialization of $\mathcal{A}$, denoted $\mathrm{ess}(\mathcal{A})$.)

What does a typical element of Br_n look like? For example, if we intersect the hyperplanes in Br_9 given by

$$x_1 = x_2, \quad x_2 = x_5, \quad x_3 = x_6, x_4 = x_8, \quad x_4 = x_9, \quad x_8 = x_9,$$

we get the subspace

$$\{(x_1, x_2, \dots, x_9) \in \mathbb{R}^9 \colon x_1 = x_2 = x_5,\ x_3 = x_6,\ x_4 = x_8 = x_9\}$$

(which has dimension 4, by the way — 9 minus the number of equals signs). Instead of writing all that, we could keep track of this information by a **set partition** of the set $[9]$:

$$\big\{\{1,2,5\},\ \{3,6\},\ \{4,8,9\},\ \{7\}\big\}$$

or, even more simply,

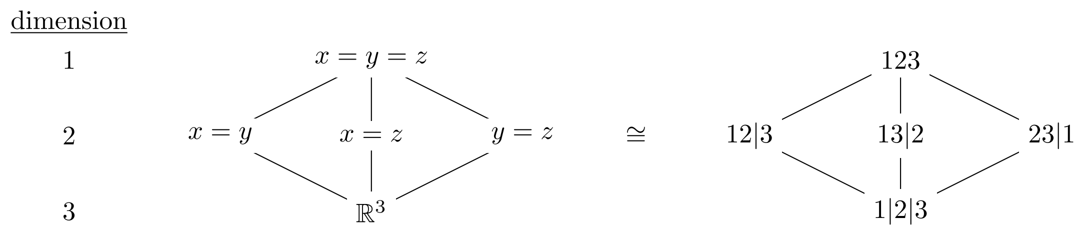
$$125|36|489|7.$$

A set partition is the same thing as an equivalence relation: a way of sorting the elements into blocks. There is no ordering specified on the set of blocks, nor internally in any block, so for example

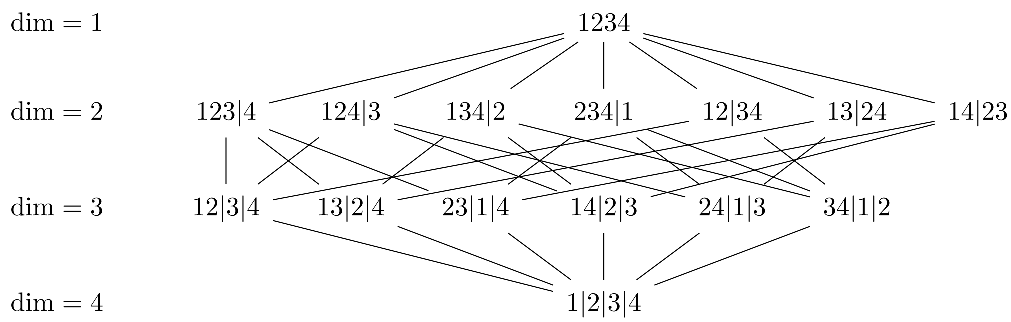
$$125|36|489|7 = 948|215|7|63.$$

[1] This is a standard abbreviation for the set $\{1, \dots, n\}$, since it comes up so often.

The set of all partitions is called the **partition lattice** and written Π_n. Every set of equalities among $x_1, \ldots, x_n$ defines an equivalence relation on $[n]$, and every equivalence relation can be obtained in this way, so the intersection poset of Br_n is isomorphic to the partition lattice Π_n. The dimension of the space $X \in L(\mathsf{Br}_n)$ corresponding to a partition π is just the number of blocks in π (since this is the number of coordinates you need to specify a point in X). For $n = 3$, the isomorphism is shown below.



And here is Π_4:



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Here are some elementary but important observations about the partition poset.

1. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation, then $L(T(\mathcal{A})) \cong L(\mathcal{A})$, where $T(\mathcal{A}) = \{T(H) \colon H \in \mathcal{A}\}$. In fact, the intersection poset is invariant under any *affine* transformation. (The group of affine transformations is generated by the invertible linear transformations together with translations.)

2. For every $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, the space $\cap(\mathcal{B} \cap \mathcal{C})$ contains both $\cap\mathcal{B}$ and $\cap\mathcal{C}$; in fact it is the smallest element of $L(\mathcal{A})$ containing both of them, so it is their greatest common lower bound (called the *meet* in poset terminology, and notated $\cap\mathcal{B} \wedge \cap\mathcal{C}$). There is also always a unique bottom element, namely $\mathbb{R}^n$. In poset theory, the bottom element of a poset (if it has one) is referred to by the special symbol $\hat{\mathbf{0}}$.

3. $L(\mathcal{A})$ has a top element if and only if the **center** $\cap\mathcal{A}$ is nonempty. Such an arrangement is called **central**. In this case it turns out every two elements $X, Y \in L(\mathcal{A})$ have a *join* (least common upper bound) $X \vee Y$: if $X = \cap\mathcal{B}$ and $Y = \cap\mathcal{C}$ then $X \vee Y = \cap(\mathcal{B} \cup \mathcal{C})$. Such a poset is called a **lattice**. Since translation does not affect whether an arrangement is central (or indeed any of its combinatorial structure), we will typically assume that $\cap\mathcal{A}$ contains the zero vector, which is to say that every hyperplane in $\mathcal{A}$ is a *linear* hyperplane in $\Bbbk^n$.

4. The poset $L(\mathcal{A})$ is **ranked**. You can see in the examples that its elements naturally fall into horizontal well-defined "levels". To make this precise, pick an element $x \in L(\mathcal{A})$ and observe that every chain from $\hat{\mathbf{0}}$ to x uses the same number of steps. (Think about why this is the case!) We call this number the **rank** of x, denoted $\rho(x)$; notice that $\rho(x) = n - \dim x$ (where n is the dimension of the space containing $\mathcal{A}$).

5. The rank function satisfies the **submodular inequality**:

$$\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y) \qquad \forall x, y \in L(\mathcal{A}).$$

We'll prove this under the assumption that x, y are vector spaces (convince yourself that this assumption is harmless; see #3 above). Every two vector spaces satisfy the equality

$$\rho(x) + \rho(y) = \rho(x \cap y) + \rho(x + y).$$

Now $x \vee y = x \cap y$, but $x \wedge y \supseteq x + y$. So replacing $x + y$ with $x \wedge y$ in the equality can only make the right-hand side smaller.

We now introduce two important algebraic gadgets we need to work with posets.

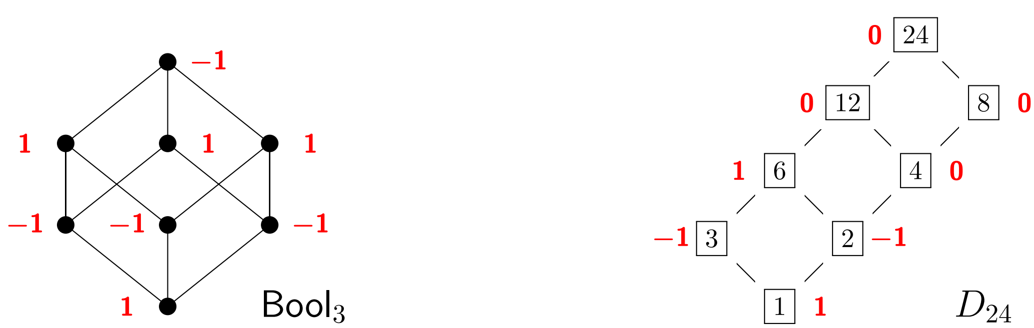
Definition 1.5. Let P be a poset with a bottom element $\hat{\mathbf{0}}$. The **Möbius function** μ_P is defined recursively as follows:

1. $\mu_P(\hat{\mathbf{0}}) = 1$.
2. For all $x \in P$, we define $\mu_P(x) = -\sum_{y<x} \mu_P(y)$.

This looks weird at first, so let's do some examples.

Example 1.6. Here are the Möbius functions $\mu_P(x) = \mu_P(\hat{\mathbf{0}}, x)$ for two posets called N_5 and M_5. (The former cannot be the intersection poset of any hyperplane arrangement — why not? — but it is still a good example.)

And here are the Boolean lattice Bool_3 and the divisor lattice D_{24} (which is the poset of all divisors of the number 24, ordered by divisibility).



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Problem 1.1. Prove that the Möbius function of the Boolean lattice Bool_n (regarded as the family of subsets of $[n]$) is $\mu(S) = (-1)|S|$. (There are many ways to do this: if you use the definition directly, then the binomial theorem will be helpful.)

Problem 1.2. Work out the Möbius functions of the partition lattices Π_3 and Π_4. Make a conjecture about the Möbius function of the top element of Π_n.

The most basic reason the Möbius function is useful is that it gives us a more general form of inclusion/exclusion, called the **Möbius inversion formula.** The usual inclusion-exclusion formula says that for sets $A_1, \dots, A_n$, the cardinality of their union is given by

$$\begin{aligned}|A_1 \cup \cdots \cup A_n| &= |A_1| + |A_2| + \cdots + |A_n| \\ &\quad - |A_1 \cap A_2| - \cdots - |A_{n-1} \cap A_n| \\ &\quad + \cdots \\ &\quad + (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| \\ &\quad + \cdots \\ &\quad + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|\end{aligned}$$

which can be rewritten more efficiently in terms of the Möbius function:

$$|A_1 \cup \cdots \cup A_n| = \sum_{K \subseteq [n]} (-1)^{|K|-1} \left| \bigcap_{i \in K} A_i \right| = -\sum_{K \subseteq [n]} \mu(K) \left| \bigcap_{i \in K} A_i \right|$$

Definition 1.7. Let $\mathcal{A}$ be a hyperplane arrangement and $L(\mathcal{A})$ its intersection poset. The **characteristic polynomial** of $\mathcal{A}$ is defined as

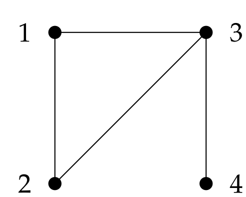
$$\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(x) t^{\dim x}. \tag{1}$$

Remarkably, this artificial-looking polynomial holds a wealth of information about $\mathcal{A}$!

Example 1.8. If $G = (V, E)$ is a simple graph on vertex set $V = [n]$, then the corresponding **graphic arrangement** $\mathcal{A}_G$ is the subarrangement of Br_n consisting of those hyperplanes $x_i = x_j$ for which $ij \in E$.

For instance, if $G = K_n$ is the complete graph on n vertices (i.e., every pair of vertices shares one edge), then $\mathcal{A}_G$ is just the braid arrangement Br_n.

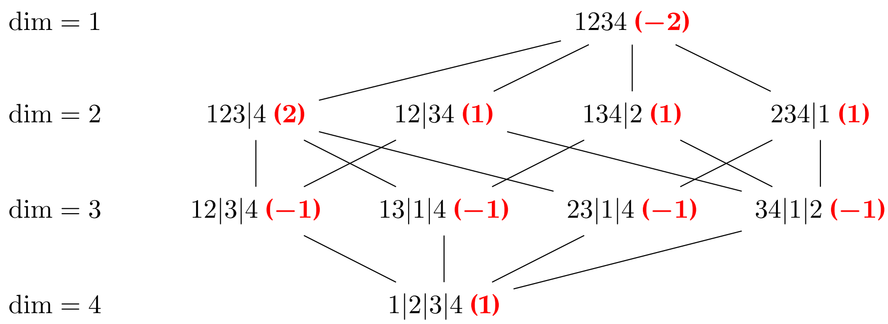
As another example, let G be the graph on four vertices shown below.



Then

$$\mathcal{A}_G = \{x_1 = x_2,\ \ x_1 = x_3,\ \ x_2 = x_3,\ \ x_3 = x_4\}$$

and the intersection poset is the following subposet of Π_4:



The partitions that show up are those such that each block induces a connected subgraph of G (that is, for two vertices v, w in the same block, it is possible to walk from v to w along edges of G). The characteristic polynomial is also something interesting. The numbers in red above indicate the values of the Möbius function, so

$$\chi_{\mathcal{A}_G}(t) = t^4 - 4t^3 + 5t^2 - 2t = t(t-1)^2(t-2).$$

There is a remarkable connection between the characteristic polynomial of $\mathcal{A}_G$ and the theory of colorings of graphs. A **coloring** of G is a function $\kappa : V(G) \to \{1, 2, \dots, \}$, where $V(G)$ means the set of vertices of G (the natural numbers are regarded as colors). A coloring is called **proper** if every two adjacent vertices receive different colors. It turns out the function

$$p_G(t) = \text{ number of colorings } \kappa : V(G) \to [t]$$

is a polynomial in t, called the **chromatic polynomial**. For the graph in the example above, we can calculate it as follows. To count the number of colorings with t colors, color the vertices in the order 1,2,3,4. Then:

- There are t choices for $\kappa(1)$;
- There are $t-1$ choices for $\kappa(2)$ (it can't equal $\kappa(1)$);
- There are $t-2$ choices for $\kappa(3)$ (it can't equal $\kappa(1)$ or $\kappa(2)$);
- There are $t-1$ choices for $\kappa(4)$ (it can't equal $\kappa(3)$).

So $p_G(t) = t(t-1)^2(t-2) = \chi_{A_G}$. This is not a coincidence!

Here is the connection between the graphic arrangement and the chromatic polynomial. In this example, the ambient space for $\mathcal{A}_G$ is $\mathbb{R}^4$. What are the integer points in the hypercube $[1,t] \times [1,t] \times [1,t] \times [1,t]$ that do *not* lie on any hyperplane of $\mathcal{A}_G$? They are

$$\begin{aligned}[1,t]^4 \cap (\mathbb{R}^4 \setminus \mathcal{A}_G) &= \{(x_1, x_2, x_3, x_4) \colon 1 \leq x_i \leq t,\ x_1 \neq x_2,\ x_1 \neq x_2,\ x_2 \neq x_3,\ x_3 \neq x_4\} \\ &= \{\text{proper } t\text{-colorings of } G\}.\end{aligned}$$

So proper colorings have something to do with the complement of the hyperplane arrangement $\mathcal{A}_G$. This doesn't entirely explain the equality $p_G(t) = \chi_{A_G}$, but it does tell you that the two topics are related. ◀

Problem 1.3. As a warmup for the next lecture, what can you say about the number of regions between the hyperplanes in an arrangement? Fo the arrangements pictured in Example 1.4, you can easily see that Bool_3 and Br_3 have 8 and 6 regions respectively, and with a little more visualization you should be able to convince yourself that Br_4 has 24 regions. Can you extrapolate from what you know to guess a formula for the number of regions of Bool_n and Br_n in terms of n? Or for $\mathcal{A}_G$ in terms of some structure on the graph G? Is there a systematic way to solve this problem?

Lecture 2: Counting Regions

Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a real hyperplane arrangement. The **regions** of $\mathcal{A}$ are the connected components of $\mathbb{R}^n \setminus \mathcal{A}$. Each component is the interior of a (bounded or unbounded) polyhedron; in particular, it is homeomorphic to $\mathbb{R}^n$. We call a region **relatively bounded** if the corresponding region in $\operatorname{ess}(\mathcal{A})$ is bounded.[2] Let

$$\begin{aligned} r(\mathcal{A}) &= \text{ number of regions of } \mathcal{A}, \\ b(\mathcal{A}) &= \text{ number of relatively bounded regions of } \mathcal{A}. \end{aligned}$$

Example 2.1. For the arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ shown in Example 1.2,

$$\begin{aligned} r(\mathcal{A}_1) &= 6, & r(\mathcal{A}_2) &= 10, \\ b(\mathcal{A}_1) &= 0, & b(\mathcal{A}_2) &= 2. \end{aligned}$$ ◀

Example 2.2. The Boolean arrangement Bool_n consists of the n coordinate hyperplanes in $\mathbb{R}^n$. The complement $\mathbb{R}^n \setminus \mathsf{Bool}_n$ is $\{(x_1, \ldots, x_n) \colon x_i \neq 0 \text{ for all } i\}$, and the connected components are the open orthants, specified by the signs of the n coordinates. Therefore

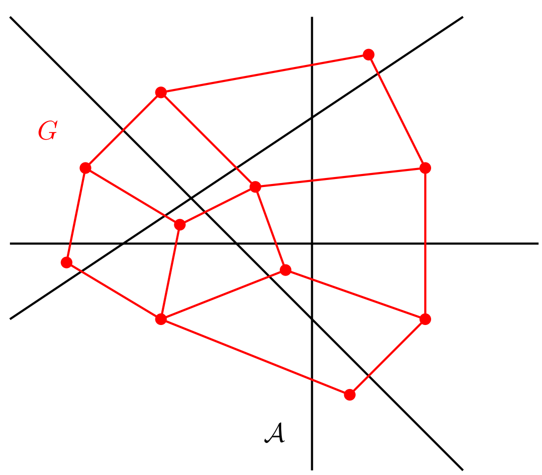
$$r(\mathsf{Bool}_n) = 2^n, \quad b(\mathsf{Bool}_n) = 0.$$

The Boolean arrangement is central and essential, and its intersection lattice is the Boolean lattice of rank n, so

$$\chi_{\mathsf{Bool}_n}(t) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} t^k = (t-1)^n.$$

How might you obtain the numbers $r(\mathsf{Bool}_n)$ and $b(\mathsf{Bool}_n)$ from $\chi_{\mathsf{Bool}_n}(t)$? ◀

Example 2.3. Let $\mathcal{A}$ consist of m lines in $\mathbb{R}^2$ in *general position*: that is, no two lines are parallel and no three are coincident. Draw the **dual graph** G, whose vertices are the regions of $\mathcal{A}$, with an edge between every two regions that share a common border.



Let $r = r(\mathcal{A})$ and $b = b(\mathcal{A})$, and let v, e, f denote the numbers of vertices, edges and faces of G, respectively. (In the example above, $(v, e, f) = (11, 16, 7)$.) Each bounded face of G is a quadrilateral that contains exactly

[2]If $\mathcal{A}$ is not essential then every region is unbounded, because it contains a translate of $Z(\mathcal{A})^{\perp}$. Thus, we have to essentialize for the problem of counting bounded regions to make sense.

one point where two lines of $\mathcal{A}$ meet, and the unbounded face is a cycle of length $r-b$. Therefore,

$$v \;=\; r, \tag{2a}$$

$$f \;=\; 1+\binom{m}{2} = \frac{m^2-m+2}{2} \tag{2b}$$

$$4(f-1)+(r-b) \;=\; 2e. \tag{2c}$$

Moreover, the number $r-b$ of unbounded regions of $\mathcal{A}$ is just $2m$. (Take a walk around a very large circle. You will enter each unbounded region once, and will cross each line twice.) Therefore, from (2c) and (2b) we obtain

$$e \;=\; m+2(f-1) \;=\; m^2. \tag{2d}$$

Euler's formula for planar graphs says that $v-e+f=2$. Substituting in (2a), (2b) and (2d) and solving for r gives

$$r \;=\; \frac{m^2+m+2}{2}$$

and therefore

$$b \;=\; r-2m \;=\; \frac{m^2-3m+2}{2} \;=\; \binom{m-1}{2}.$$

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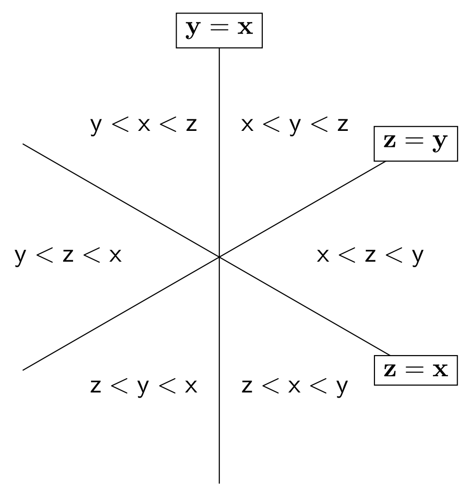
Example 2.4. The braid arrangement Br_n consists of the $\binom{n}{2}$ hyperplanes $H_{ij} = \{\mathbf{x}\colon x_i = x_j\}$ in $\mathbb{R}^n$. The complement $\mathbb{R}^n \setminus \mathsf{Br}_n$ consists of all vectors in $\mathbb{R}^n$ with no two coordinates equal, and the connected components of this set are specified by the ordering of the set of coordinates as real numbers. Here is the picture for $\operatorname{ess}(\mathsf{Br}_3)$.



Therefore,

$$r(\mathsf{Br}_n) = n!.$$

(Stanley: "Rarely is it so easy to compute the number of regions!") Furthermore,

$$\chi_{\mathsf{Br}_n}(t) = t(t-1)(t-2)\cdots(t-n+1).$$

which by the way is the chromatic polynomial of the complete graph (recall that $\mathsf{Br}_n = \mathcal{A}_{K_n}$). **How might you obtain the numbers $r(\mathsf{Br}_n)$ and $b(\mathsf{Br}_n)$ from $\chi_{\mathsf{Br}_n}(t)$?** ◀

Example 2.5. Let $G = (V, E)$ be a simple graph with $V = [n]$, and let $\mathcal{A}_G$ be its graphic arrangement (see Example 1.8). The regions of $\mathbb{R}^n \setminus \mathcal{A}_G$ are the open polyhedra whose defining inequalities include either $x_i < x_j$ or $x_i > x_j$ for each edge $ij \in E$. Suppose we draw an arrow on each edge of G to indicate which the inequality goes; for instance, have the arrow point toward the vertex corresponding to the larger coordinate. This collection of arrows is called an **orientation** of G. In fact it is an **acyclic orientation**: there are no consistently oriented cycles. (Why?) To put it another way, no walk following the arrows can ever reach the same vertex more than once. In fact, what we have described is a *bijection* between regions and acyclic orientations (prove this!), so

$$r(\mathcal{A}_G) \;=\; \text{number of acyclic orientations of } G. \qquad \blacktriangleleft$$

Problem 2.1. Calculate the characteristic polynomials and numbers of regions for (i) the four-vertex graph at the end of Example 1.8; (i) the graph obtained by deleting an edge (any edge, it doesn't matter) from K_4. **How might you obtain the numbers $r(\mathsf{Br}_n)$ and $b(\mathsf{Br}_n)$ from $\chi_{\mathsf{Br}_n}(t)$?**

I will end the suspense. Here is the punchline, historically the first major theorem about hyperplane arrangements.

Theorem 2.6 (Zaslavsky's Theorem [Zas75])**.** *Let $\mathcal{A}$ be a real hyperplane arrangement, and let $\chi_{\mathcal{A}}$ be the characteristic polynomial of its intersection poset. Then*

$$r(\mathcal{A}) = (-1)^{\dim \mathcal{A}} \chi_{\mathcal{A}}(-1) \text{ and} \tag{3}$$

$$b(\mathcal{A}) = (-1)^{\operatorname{rank} \mathcal{A}} \chi_{\mathcal{A}}(1). \tag{4}$$

Here $\dim \mathcal{A}$ means the dimension of the space containing $\mathcal{A}$, and $\operatorname{rank} \mathcal{A} = \dim \operatorname{ess}(\mathcal{A})$.

The proof has the following parts:

1. Show that r and b satisfy restriction/contraction recurrences in terms of associated hyperplane arrangements $\mathcal{A}'$ and $\mathcal{A}''$ (Prop. 2.8).
2. Show that $\chi_{\mathcal{A}}(-1)$ and $\chi_{\mathcal{A}}(1)$ satisfy analogous recurrences (being careful about signs).

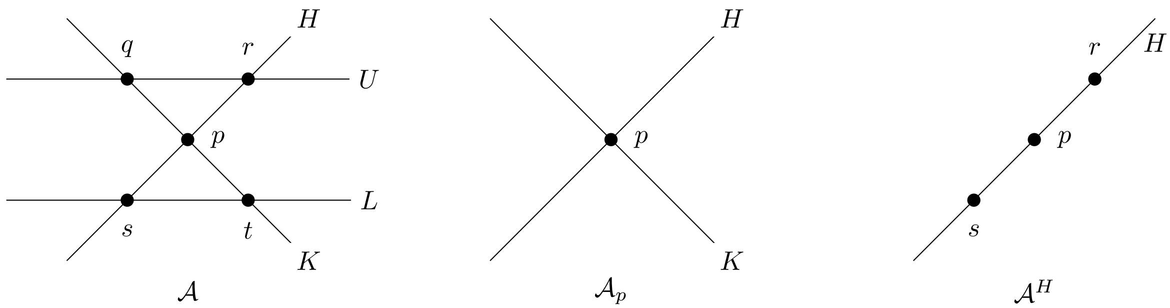
I will show you the first part in detail because it is geometric and intuitive; I will summarize the second part because it is more algebraic and relies on some machinery we don't have time to develop fully.

Let $x \in L(\mathcal{A})$, i.e., x is a nonempty affine space formed by intersecting some of the hyperplanes in $\mathcal{A}$. Define

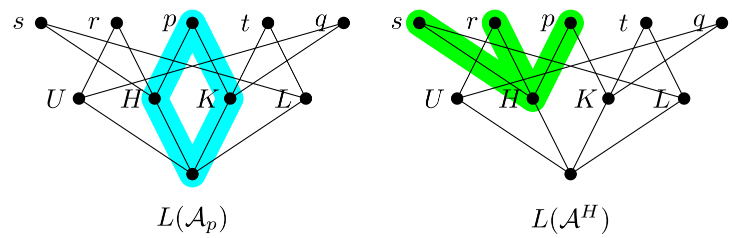
$$\begin{aligned} \mathcal{A}_x &= \{H \in \mathcal{A} \colon H \supseteq x\}, \\ \mathcal{A}^x &= \{H \cap x \colon H \in \mathcal{A} \setminus \mathcal{A}_x\}. \end{aligned} \tag{5}$$

In other words, $\mathcal{A}_x$ is obtained by deleting the hyperplanes not containing x, while $\mathcal{A}^x$ is obtained by restricting $\mathcal{A}$ to x so as to get an arrangement whose ambient space is x itself.

Example 2.7. Let $\mathcal{A}$ be the 2-dimensional arrangement shown on the left, with the line H and point p as shown. Then $\mathcal{A}_p$ and $\mathcal{A}^H$ are shown on the right.



The lattice $L(\mathcal{A})$ and its subposets $L(\mathcal{A}_p)$ and $L(\mathcal{A}^H)$ are shown below. This picture explains the notation $\mathcal{A}_x$ and $\mathcal{A}^x$: $L(\mathcal{A}_x)$ consists of the spaces below x in $L(\mathcal{A})$, while $L(\mathcal{A}^x)$ consists of the spaces above x.



Proposition 2.8. *Fix $H \in \mathcal{A}$ and let $\mathcal{A}' = \mathcal{A} \setminus H$ and $\mathcal{A}'' = \mathcal{A}^H$. The invariants r and b satisfy the following recurrences:*

1. $r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'')$.
2. $b(\mathcal{A}) = \begin{cases} 0 & \textit{if } \operatorname{rank} \mathcal{A} = \operatorname{rank} \mathcal{A}' + 1, \\ b(\mathcal{A}') + b(\mathcal{A}'') & \textit{if } \operatorname{rank} \mathcal{A} = \operatorname{rank} \mathcal{A}'. \end{cases}$

Proof. (1) Consider what happens when we add H to $\mathcal{A}'$ to obtain $\mathcal{A}$. Some regions of $\mathcal{A}'$ will remain the same, while others will be split into two regions.

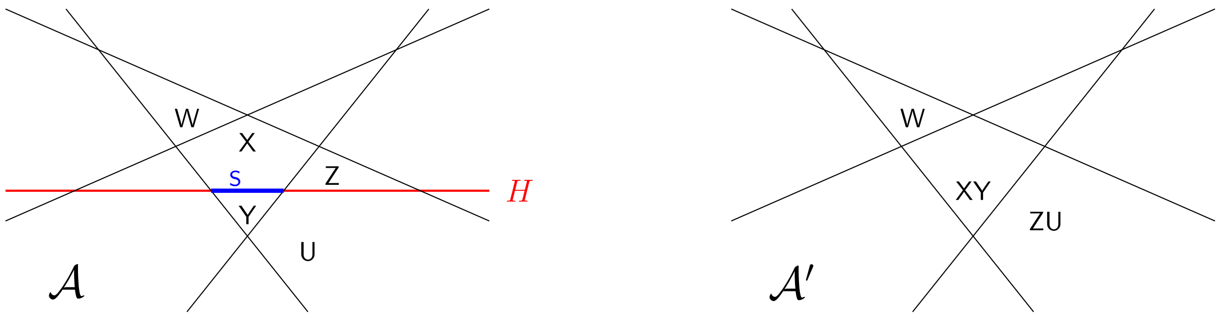
Let s and u be the numbers of split and unsplit regions of $\mathcal{A}'$ (so in the figure above, $s = 2$ and $u = 4$). The unsplit regions each contribute 1 to $r(\mathcal{A})$. The split regions each contribute 2 to $r(\mathcal{A})$, but they also correspond bijectively to the regions of $\mathcal{A}''$. (See, e.g., Example 2.7.) So

$$r(\mathcal{A}') = s + u, \qquad r(\mathcal{A}) = 2s + u, \qquad r(\mathcal{A}'') = s$$

and so $r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'')$, proving the first assertion of Proposition 2.8.

(2) Now we count bounded regions.

- If $\operatorname{rank} \mathcal{A} = \operatorname{rank} \mathcal{A}' + 1$, then $\mathcal{A}'$ is not essential, and, as we have already observed, $b(\mathcal{A}) = 0$.
- If $\operatorname{rank} \mathcal{A} = \operatorname{rank} \mathcal{A}'$, then the relatively bounded regions of $\mathcal{A}$ come in a few different flavors.
  - If W is a bounded region not bordered by H, then it is also a bounded region of $\mathcal{A}'$.
  - If X, Y are two bounded regions separated by part of H, then they merge into one bounded region XY in $\mathcal{A}'$. Also, the part S of H that separates them is a bounded region of $\mathcal{A}''$.
  - If Z is a bounded region of $\mathcal{A}$ whose neighbor across H is an unbounded region U, then the merged region ZU is unbounded in $\mathcal{A}'$, but $\partial Z \cap H$ is nevertheless bounded in $\mathcal{A}''$.



In short, here are the possibilities:

		Contributions to...		
	Description	$b(\mathcal{A})$	$b(\mathcal{A}')$	$b(\mathcal{A}'')$
(W)	bounded regions that don't touch H	1	1	0
(X, Y)	pairs of bounded regions separated by H	2	1	1
(Z)	bounded, neighbor across H is unbounded	1	0	1

In all cases the contribution to $b(\mathcal{A})$ equals the sum of those to $b(\mathcal{A}')$ and $b(\mathcal{A}'')$, establishing the second desired recurrence. $\square$

The second part of the proof of Zaslavsky's theorems is to show that the numbers $\chi_{\mathcal{A}}(-1)$ and $\chi_{\mathcal{A}}(1)$ satisfy analogous recurrences. As promised, I will omit a lot of details and just sketch the argument.

The first step is to prove that every hyperplane arrangement $\mathcal{A}$ satisfies

$$\chi_{\mathcal{A}}(t) \;=\; \sum_{\text{central } \mathcal{B}\subseteq\mathcal{A}} (-1)^{|\mathcal{B}|} t^{\dim \mathcal{A} - \operatorname{rank}(\mathcal{B})} \tag{6}$$

which is known as the **Whitney formula** (since it resembles a similar formula for the chromatic polynomial). Here $\operatorname{rank}(\mathcal{B}) = \dim \operatorname{ess}(\mathcal{B})$. This looks like an awkward way to calculate the characteristic polynomial, but it is actually well-suited for the next step, which is to show that $\chi_{\mathcal{A}}$ satisfies the recurrence

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t) \tag{7}$$

where $\mathcal{A}'$ and $\mathcal{A}''$ are defined as before. To give you the idea, the first step in the proof is to split up the Whitney formula for $\mathcal{A}$ as

$$\chi_{\mathcal{A}}(t) \;=\; \sum_{\text{central } \mathcal{B}\subseteq\mathcal{A}:\ H\notin\mathcal{B}} (-1)^{|\mathcal{B}|} t^{n-\operatorname{rank}\mathcal{B}} \quad+\quad \sum_{\text{central } \mathcal{B}\subseteq\mathcal{A}:\ H\in\mathcal{B}} (-1)^{|\mathcal{B}|} t^{n-\operatorname{rank}\mathcal{B}}$$

and observe that the first sum is just the Whitney formula for $\mathcal{A}'$, meaning that we just need to show that the second sum is $\chi_{\mathcal{A}''}(t)$. This is a little trickier, because different hyperplanes in $\mathcal{A}$ can have the same intersection with H, which means that multiple subarrangements of $\mathcal{A}$ can give rise to the same subarrangement of $\mathcal{A}''$, so some technique is necessary to handle the cancellations, but this is all doable. All that remains is to show that the numbers $(-1)^{\dim \mathcal{A}}\chi_{\mathcal{A}}(-1)$ and $(-1)^{\operatorname{rank} \mathcal{A}}\chi_{\mathcal{A}}(1)$ satisfy the same recurrences as r and b respectively (Prop. 2.8), and the proof is complete.

Remark 2.9. The recurrence (7) is strongly reminiscent of the *chromatic recurrence*: if $p_G(k)$ is the chromatic polynomial of a simple graph G, and e is an edge, then

$$p_G(k) = p_{G-e}(k) - p_{G/e}(k).$$

Indeed, if $\mathcal{A} = \mathcal{A}_G$ is a graphic arrangement in $\mathbb{R}^n$, e is an edge of G, and H_e is the corresponding hyperplane in $\mathcal{A}_G$, then $\mathcal{A}_{G\setminus e} = \mathcal{A}_G \setminus \{H_e\}$. In addition, two hyperplanes $H_f, H_{f'}$ will have the same intersection with H_e if and only if f, f' become parallel upon contracting e, so $\mathcal{A}_{G/e}$ can be identified with $(\mathcal{A}_G)^{H_e}$ (where the coordinates on $H_e \cong \mathbb{R}^{n-1}$ are given by equating the coordinates for the two endpoints of e).

Example 2.10. Let $s \geq n$, and let $\mathcal{A}$ be an arrangement of s linear hyperplanes in general position in $\mathbb{R}^n$; that is, every k hyperplanes intersect in a space of dimension $n-k$ (or 0 if $k > n$). Therefore,

$$\begin{aligned} r(\mathcal{A}) &= \sum_{k=0}^{s} \binom{s}{k} (-1)^{k-\min(n,k)} \\ &= \sum_{k=0}^{n} \binom{s}{k} + \sum_{k=n+1}^{s} \binom{s}{k} (-1)^{k-n} \\ &= \sum_{k=0}^{n} \binom{s}{k} (1-(-1)^{k-n}) + \underbrace{\sum_{k=0}^{s} \binom{s}{k} (-1)^{k-n}}_{=0} \\ &= 2\left(\binom{s}{n-1} + \binom{s}{n-3} + \binom{s}{n-5} + \cdots\right). \end{aligned}$$

For instance, if $n = 3$ then

$$r(\mathcal{A}) = 2\binom{s}{2} + 2\binom{s}{0} = s^2 - s + 2.$$

◀

Notice that this is not the same as the number of regions formed by s *affine* lines in general position in $\mathbb{R}^2$. The calculation of $r(\mathcal{A})$ and $b(\mathcal{A})$ for that arrangement is left to the reader (Problem 3.1).

Lecture 3: The finite field method

The following very important result is implicit in the work of Crapo and Rota [CR70] and was stated explicitly by Athanasiadis [Ath96]:

Theorem 3.1. *Let $\mathbb{F}_q$ be the finite field of order q, and let $\mathcal{A} \subseteq \mathbb{F}_q^n$ be a hyperplane arrangement. Then*

$$|\mathbb{F}_q^n \setminus \mathcal{A}| = \chi_{\mathcal{A}}(q).$$

This result gives a combinatorial interpretation of the values of the characteristic polynomial. In practice, it is often used to calculate the characteristic polynomial of a hyperplane arrangement by counting points in its complement over $\mathbb{F}_q$ (which can be regarded as regions of the complement, if you endow $\mathbb{F}_q^n$ with the discrete topology).

Proof. By inclusion-exclusion,

$$|\mathbb{F}_q^n \setminus \mathcal{A}| \;=\; \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} \left| \bigcap \mathcal{B} \right|.$$

If $\mathcal{B}$ is not central, then by definition $|\bigcap \mathcal{B}| = 0$. Otherwise, $|\bigcap \mathcal{B}| = q^{n - \operatorname{rank} \mathcal{B}}$. So the sum becomes

$$|\mathbb{F}_q^n \setminus \mathcal{A}| \;=\; \sum_{\text{central } \mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} q^{n - \operatorname{rank} \mathcal{B}}$$

which is just the Whitney formula for $\chi_{\mathcal{A}}(q)$, eqn. (6). ☐

(It is also possible to prove the result directly from the definition of the characteristic polynomial.)

This fact has a much more general application, which was systematically mined by Athanasiadis, e.g., [Ath96].

Definition 3.2. Let $\mathcal{A} \subseteq \mathbb{R}^n$ be an integral hyperplane arrangement (i.e., whose hyperplanes are defined by equations with integer coefficients). For a prime p, let $\mathcal{A}_p = \mathcal{A} \otimes \mathbb{F}_p$ be the arrangement in $\mathbb{F}_p^n$ defined by regarding the equations in $\mathcal{A}$ as equations over $\mathbb{F}_p$. We say that $\mathcal{A}$ **reduces correctly modulo p** if $L(\mathcal{A}_p) \cong L(\mathcal{A})$. (We need only consider the prime case, since if q is a power of p, then $L(\mathcal{A}_q) = L(\mathcal{A}_p)$.)

A sufficient condition for correct reduction is that no minor of the matrix of normal vectors is a nonzero multiple of p (so that rank calculations are the same over $\mathbb{F}_p$ as over $\mathbb{Z}$). In particular, if we choose p larger than the absolute value of any minor of M, then each set of columns of M is linearly independent over $\mathbb{F}_p$ iff it is independent over $\mathbb{Q}$. There are infinitely many such primes, implying the following highly useful result:

Theorem 3.3 (The finite field method). *Let $\mathcal{A} \subseteq \mathbb{R}^n$ be an integral hyperplane arrangement and q a power of a large enough prime. Then $\chi_{\mathcal{A}}(q)$ is the polynomial that counts points in the complement of $\mathcal{A}_q$.*

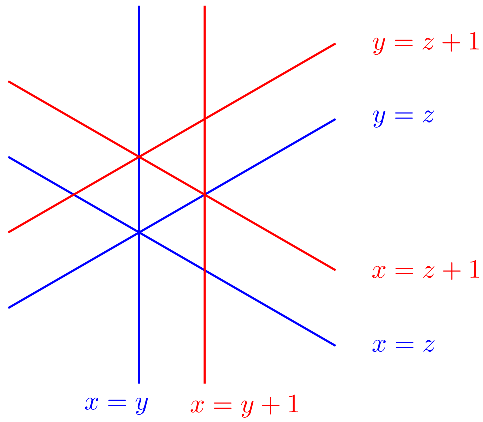
Example 3.4. Let $G = ([n], E)$ be a simple graph and let $\mathcal{A}_G$ be the corresponding graphic arrangement in $\mathbb{R}^n$. Note that $\mathcal{A}_G$ reduces correctly over every finite field $\mathbb{F}_q$ (because graphic matroids are regular). A point $(x_1, \ldots, x_n) \in \mathbb{F}_q^n$ can be regarded as the q-coloring of G that assigns color x_i to vertex i. The proper q-colorings are precisely the points of $\mathbb{F}_q^n \setminus \mathcal{A}_G$. The number of such colorings is $p_G(q)$ (the chromatic polynomial of G evaluated at q). On the other hand, by Theorem 3.1, it is also the characteristic polynomial $\chi_{\mathcal{A}_G}(q)$. Since $p_G(q) = \chi_{\mathcal{A}_G}(q)$ for infinitely many q (namely, all integer prime powers), the polynomials must be equal. In particular, by Zaslavsky's theorems, the number of regions is $p_G(-1)$, and on the other hand we know that regions of $\mathcal{A}_G$ are in bijection with acyclic orientations of G (see Example 2.5), so we now have a geometric proof that $|p_G(-1)| = a(G)$. ◀

Example 3.5. The **Shi arrangement** is the arrangement of $n(n-1)$ hyperplanes in $\mathbb{R}^n$ defined by

$$\mathsf{Shi}_n = \{x_i = x_j,\ x_i = x_j + 1 \mid 1 \le i < j \le n\}.$$

In other words, take the braid arrangement, clone it, and nudge each of the cloned hyperplanes a little bit in the direction of the bigger coordinate. The Shi arrangement has rank $n-1$ (every hyperplane in it contains a line parallel to the all-ones vector), so we may project along that line to obtain the essentialization in $\mathbb{R}^{n-1}$. Thus $\operatorname{ess}(\mathsf{Shi}_2)$ consists of two points on a line, while $\operatorname{ess}(\mathsf{Shi}_3)$ is shown below.

ess(Shi$_3$)



We will prove that the characteristic polynomial of the Shi arrangement is

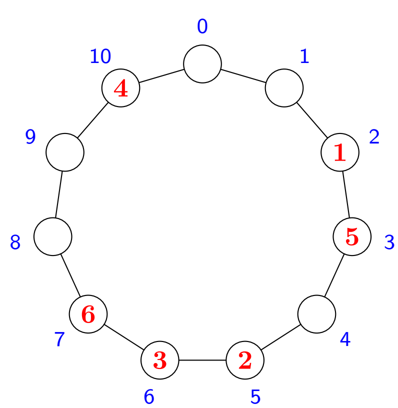
$$\chi_{\mathsf{Shi}_n}(q) = q(q-n)^{n-1}. \tag{8}$$

In particular, the numbers of regions and bounded regions are respectively

$$r(\mathsf{Shi}_n) = |\chi(-1)| = (n+1)^{n-1}, \qquad b(\mathsf{Shi}_n) = |\chi(1)| = (n-1)^{n-1}. \tag{9}$$

(The number $(n+1)^{n-1}$ may look familiar; by Cayley's formula, it is the number of spanning trees of the complete graph K_{n+1}. It also counts many other things of combinatorial interest, including parking functions.)

The following proof is from [Sta07, §5.2]. By Theorem 3.3, it suffices to count the points in $\mathbb{F}_q^n \setminus \mathsf{Shi}_n$ for a large enough prime q. Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{F}_q^n \setminus \mathsf{Shi}_n$. Draw a necklace with q beads labeled by the elements $0, 1, \dots, q-1 \in \mathbb{F}_q$, and for each $k \in [n]$, put a big red $\boldsymbol{k}$ on the x_k-th bead. For example, let $n = 6$ and $q = 11$. Then the necklace for $\mathbf{x} = (2, 5, 6, 10, 3, 7)$ is as follows:



The requirement that $\mathbf{x}$ avoids the hyperplanes $x_i = x_j$ implies that the red numbers are all on different beads. If we read the red numbers clockwise, starting at **1** and putting in a divider sign $|$ for each bead without a red number, we get

$$15 \mid 236 \mid \mid 4 \mid$$

which can be regarded as the *ordered weak partition* (or OWP)

$$\Pi(x) = 15, 236, \emptyset, 4, \emptyset$$

that is, a $(q-n)$-tuple $B_1, \dots, B_{q-n}$, where the B_i are pairwise disjoint sets (possibly empty; that's what the "weak" means) whose union is $[n]$, and $1 \in B_1$. (We've omitted the divider corresponding to the bead just counterclockwise of **1**; stay tuned.)

Note that each block of $\Pi(\mathbf{x})$ corresponds to a contiguous set of values among the coordinates of $\mathbf{x}$. For example, the block 236 occurs because the values 5,6,7 occur in coordinates x_2, x_3, x_6. In order to avoid the hyperplanes $x_i = x_j + 1$ for $i < j$, each contiguous block of beads must have its red numbers in strictly increasing order counterclockwise. (In particular the bead just counterclockwise of **1** must be unlabeled, which is why we could omit that divider.)

Thus we have bijections

$$\mathbb{F}_q^n \setminus \mathsf{Shi}_n \quad \longleftrightarrow \quad \{\text{necklaces}\} \quad \longleftrightarrow \quad \{\text{OWPs of } [n] \text{ with } q-n \text{ blocks}\}.$$

To get a necklace from an OWP, write out each block in increasing order, with bars between successive blocks.

Meanwhile, an OWP is given by a function $f : [n] \to [q-n]$, where $f(i)$ is the index of the block containing i (so $f(1) = 1$). There are $(q-n)^{n-1}$ such things. Since there are q choices for the bead containing the red **1**, we obtain

$$\left|\mathbb{F}_q^n \setminus \mathsf{Shi}_n\right| = q(q-n)^{n-1} = \chi_{\mathsf{Shi}_n}(q).$$

This proves (8), and (9) follows from Zaslavsky's theorems. ◀

Exercises

Problem 3.1. Let $m > n$, and let $\mathcal{A}$ be the arrangement of m affine hyperplanes in general position in $\mathbb{R}^n$. Here "general position" means that every k of the hyperplanes intersect in an affine linear space of dimension $n-k$; if $k > n$ then the intersection is empty. (Compare Example 2.10, where the hyperplanes are linear.) Calculate $\chi_{\mathcal{A}}(k)$, $r(\mathcal{A})$, and $b(\mathcal{A})$.

Problem 3.2. (Stanley, HA, 2.5) Let G be a graph on n vertices, let $\mathcal{A}_G$ be its graphic arrangement in $\mathbb{R}^n$, and let $\mathcal{B}_G = \mathsf{Bool}_n \cup \mathcal{A}_G$. (That is, $\mathcal{B}$ consists of the coordinate hyperplanes $x_i = 0$ in $\mathbb{R}^n$ together with the hyperplanes $x_i = x_j$ for all edges ij of G.) Calculate $\chi_{\mathcal{B}_G}(q)$ in terms of $\chi_{\mathcal{A}_G}(q)$.

Problem 3.3. (Stanley, EC2, 3.115) Determine the characteristic polynomial and the number of regions of the *type B braid arrangement* and the *type D braid arrangement* $\mathcal{B}_n, \mathcal{D}_n \subset \mathbb{R}^n$, which are defined by

$$\begin{aligned}
\mathcal{B}_n &= \{x_i = x_j :\ 1 \le i < j \le n\} \cup \{x_i = -x_j :\ 1 \le i < j \le n\} \cup \{x_i = 0 :\ 1 \le i \le n\}, \\
\mathcal{D}_n &= \{x_i = x_j :\ 1 \le i < j \le n\} \cup \{x_i = -x_j :\ 1 \le i < j \le n\}.
\end{aligned}$$

(Hint: Work out $\mathcal{B}_n$ first and use the result to understand $\mathcal{D}_n$.)

Problem 3.4 (Stanley [Sta07], Exercise 5.9(a))**.** Find the characteristic polynomial and number of regions of the arrangement $\mathcal{A}_n \subseteq \mathbb{R}^n$ with hyperplanes $x_i = 0$, $x_i = x_j$, and $x_i = 2x_j$, for all $1 \le i \ne j \le n$.

Problem 3.5. Recall that each permutation $w = (w_1, \ldots, w_n) \in \mathfrak{S}_n$ corresponds to a region of the braid arrangement Br_n, namely the open cone $C_w = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_{w_1} < x_{w_2} < \cdots < x_{w_n}\}$. Denote its closure by $\overline{C_w}$. For any set $W \subseteq \mathfrak{S}_n$, consider the closed fan

$$F(W) = \bigcup_{w \in W} \overline{C_w} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_{w_1} \leq \cdots \leq x_{w_n} \text{ for some } w \in W\}.$$

Prove that $F(W)$ is a convex set if and only if W is the set of linear extensions of some poset P on $[n]$. (A *linear extension* of P is a total ordering $\prec$ consistent with the ordering of P, i.e., if $x <_P y$ then $x \prec y$.)

Problem 3.6. The runners in a sprint are seeded $1, \ldots, n$ (stronger runners are assigned higher numbers). To even the playing field, the rules specify that you earn one point for each higher-ranked opponent you beat, and one point for each lower-ranked opponent you beat *by at least one second*. (If a higher-ranked runner beats a lower-ranked runner by less than 1 second, no one gets the point for that matchup.) Let s_i be the number of points scored by the ith player and let $s = (s_1, \ldots, s_n)$ be the *score vector*.

(a) Show that the possible score vectors are in bijection with the regions of the Shi arrangement.
(b) Work out all possible score vectors in the cases of 2 and 3 players. Conjecture a necessary and sufficient condition for $(s_1, \ldots, s_n)$ to be a possible score vector for n players. Prove it if you can.

Notational Index

◀	End of an example
2^S	power set of a set S
$[n]$	$\{1, \ldots, n\}$
$\hat{\mathbf{0}}, \hat{\mathbf{1}}$	unique minimum and maximum elements of a poset (if they exist)
Π_n	lattice of all set partitions of $[n]$
$\wedge, \vee$	meet, join
Bool_n	Boolean lattice of rank n
D_n	lattice of divisors of an integer n
$\mathbb{F}_q$	finite field of order q
$\mathcal{A}, \mathcal{B}, \ldots$	hyperplane arrangements
$L(\mathcal{A})$	intersection poset of arrangement $\mathcal{A}$
$\mathrm{ess}(\mathcal{A})$	essentialization of $\mathcal{A}$
$\mu_{\mathcal{A}}$	Möbius function of $\mathcal{A}$
$\chi_{\mathcal{A}}(x)$	characteristic polynomial of $\mathcal{A}$
Bool_n	Boolean arrangement
Br_n	braid arrangement
$\mathcal{A}_G$	arrangement associated with a graph G
$p_G(k)$	chromatic polynomial of a graph G
$r(\mathcal{A})$	number of regions of a real arrangement $\mathcal{A}$
$b(\mathcal{A})$	number of relatively bounded regions of $\mathcal{A}$
Shi_n	Shi arrangement

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