**Math 821 Solution Set #1 (1/28/11)**

**Problem #1** Let $n$ and $m$ be nonnegative integers. Let $X = \mathbb{R}^n$ and $X' = \mathbb{R}^m$, both equipped with the standard topology. Prove that the product topology on $X \times X'$ is exactly the standard topology on $\mathbb{R}^{n+m}$.

**Solution:** Let $\mathscr{B}_n$ be the collection of nonempty open balls $B(x, r)$ in $\mathbb{R}^n$, i.e., $B(x, r) = \{y \in \mathbb{R}^n : \|x-y\| < r\}$ for $x \in \mathbb{R}^n$, $r \in \mathbb{R}_{>0}$. Then, in general, we have inclusions of nonempty open sets in $\mathbb{R}^{n+m}$:

$$\begin{aligned} B((x_1,\dots,x_n,x'_1,\dots,x'_m),\ \min(r,r')) &\subseteq B((x_1,\dots,x_n),r) \times B((x'_1,\dots,x'_m),r') \\ &\subseteq B((x_1,\dots,x_n,x'_1,\dots,x'_m),\ r+r') \end{aligned}$$

(details omitted), so the topology generated by $\mathscr{B}_n \times \mathscr{B}_m$ — i.e., the product topology — is the same as that generated by $\mathscr{B}_{n+m}$ — i.e., the standard topology.

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**Problem #2** Let $(X, \mathscr{T})$ and $(X', \mathscr{T}')$ be topological spaces, let $\mathscr{B}'$ be a basis for the topology $\mathscr{T}'$, and let $f : X \to X'$ be a function. Prove that $f$ is continuous if and only if $f^{-1}(U)$ is open for all $U \in \mathscr{B}'$.

**Solution:** The $\Longrightarrow$ direction is trivial, because by definition every element of a basis is open. For the other direction If $A$ is any open set, then it is the union of some family $\{B_j\}_{j\in J}$ of basis elements, and

$$f^{-1}(A) \;=\; \{x \in X \mid f(x) \in A\} \;=\; \{x \in X \mid f(x) \in B_i \text{ for some } i\} \;=\; \bigcup_i \{x \in X \mid f(x) \in B_i\} \;=\; \bigcup_i f^{-1}(B_i).$$

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**Problem #3** Let $X$ be a path-connected topological space. Prove that $X$ is connected. (Recall that the converse is not true — the topologists' sine curve is a counterexample.)

**Solution:** Let $U$ be a clopen subset of $X$, so that its complement $V = X \setminus U$ is also clopen. Suppose that $p \in U$ and that $q \in X \setminus U$. Let $P$ be a $p,q$-path in $X$ parametrized by $f : I = [0,1] \to X$. Then $P = (P \cap U) \cup\!\!\!\cdot\, (P \cap V)$. Let $A = f^{-1}(P \cap U)$ and $B = f^{-1}(P \cap V)$. Then

- $A \cup B = I$ (because $f(I) = P$);
- $A \cap B = \emptyset$ (because $f(A) \cap f(B) = \emptyset$);
- $A, B$ are open in $I$ (by continuity);
- $\emptyset \subsetneq A, B \subsetneq I$ (because $f^{-1}(p) = 0 \in A \setminus B$ and $f^{-1}(q) = 0 \in B \setminus A$).

But this is a contradiction because $I$ is connected. Therefore, no such pair $p, q$ can exist, which says that one of $U, X \setminus U$ is empty. It follows that $X$ is connected. (In principle, this argument uses the continuity of $f$ to reduce the "path-connected implies connected" statement about an arbitrary topological space to the same statement about the familiar topological space $I$.)

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**Problem #4** Let $\Gamma$ be a finite graph. Prove that if $\Gamma$ is connected, then it is path-connected. (In fact, this is true not merely for graphs, but for all finite cell complexes—so the distinction between "connected" and "path-connected" is insignificant in practice.)

**Solution:** I'll prove something slightly more general: every space $X$ that is both connected and locally path-connected is path-connected. ("Locally path-connected" means that every point $x$ has a connected neighborhood $K_x$.) Note that a graph is locally path-connected because every vertex has a neighborhood that looks like the vertex itself plus $d$ rays sticking out (where $d$ is the degree of the vertex — the number of edges attached to it, counting a loop as two edges) and every point on the interior of an edge has a neighborhood that looks like a line segment.

Let $x \in X$ and let $Y$ be the set of all points that are joined to $x$ by a path. Then $Y$ is open in $X$, because $Y = \bigcup_{y \in Y} K_y$. (Since $y \in K_y$, we have $\subseteq$; for $\supseteq$, note that path-connectedness is an equivalence relation, so if $y \in Y$ then $K_y \subseteq Y$.) On the other hand, the same argument works if we replace $Y$ with $X \setminus Y$. Therefore $X \setminus Y$ is open, and $Y$ is closed, hence clopen. Now, connectedness implies that $Y = X$ (not $Y = \emptyset$, because $x \in Y$), which implies that $X$ is path-connected.

In fact, we don't even need the assumption of finiteness — any cell complex will work. What this tells us, among other things, is that the topologists' sine curve cannot be realized as a cell complex!

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**Problem #5** Let $X$ and $Y$ be topological spaces and let $f : X \twoheadrightarrow Y$ be a continuous function that is onto. Prove that if $X$ is connected, then so is $Y$.

**Solution:** Suppose $Y = U \cup\!\!\!\cdot\, V$ is a nontrivial decomposition into open sets. Then $f^{-1}(Y)$ and $f^{-1}(V)$ are open by continuity, and $X = f^{-1}(U) \cup\!\!\!\cdot\, f^{-1}(V)$ (because every point in $X$ is in exactly one of these two sets). By connectedness, WLOG, $f^{-1}(U) = X$ and $f^{-1}(V) = \emptyset$, which says that $U = Y$ and $V = \emptyset$.

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**Problem #6** Let $X$ and $Y$ be topological spaces and let $f : X \twoheadrightarrow Y$ be a continuous function that is onto. Prove that if $X$ is compact, then so is $Y$.

**Solution:** Let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of $Y$. Then $\{f^{-1}(U_\alpha) \mid \alpha \in A\}$ is an open cover of $X$ (because every point in $X$ gets mapped to a point in at least one $U_\alpha$, hence belongs to $f^{-1}(U_\alpha)$). By compactness, it has a finite subcover: $\{f^{-1}(U_\alpha) \mid \alpha \in A'\}$, where $A' \subseteq A$ is finite. I.e., $X = \bigcup_{\alpha \in A'} f^{-1}(U_\alpha)$. This implies set-theoretically that $Y = \bigcup_{\alpha \in A'} U_\alpha$, so there we have the desired finite subcover.

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**Problem #7** Let $n \geq 0$ be an integer, and let $X_n$ be the space you get by taking a strip of paper, twisting it $n$ times, and gluing the ends together. (So $X_0$ is a cylinder and $X_1$ is the Möbius strip.) For which pairs $n, m$ are $X_n$ and $X_m$ homeomorphic?

**Solution:** $X_n \cong X_m$ iff $n \cong m \mod 2$. For the "if" direction, note that as topological spaces, $X_n$ and $X_m$ are defined identically as quotient spaces. (It doesn't matter that they don't behave identically as subspaces of $\mathbb{R}^3$ — their *inherent* topologies are isomorphic.)

For the "only if" direction, here is what I was looking for. Let $C$ be the central circle of $X_1$; then $C \cong S^1$ and $X_1 \setminus C$ is connected (in fact it is homeomorphic to $X_2$), as you can prove by constructing a M"obius strip and cutting it down the middle. Therefore, if $X_0 \cong X_1$, then there must be some subspace of $X_0$ homeomorphic to a circle whose complement is connected. But there isn't — this is pretty much by the Jordan curve theorem. (Yes, this last part is not a purely topological argument, but the idea you should take away from this is that we can understand topological space not just by looking at its own properties, but by looking at properties of cleverly chosen subspaces of it as well.)