Math 724, Fall 2013
Homework #6

Problem #1 Consider the sequence $T_0, T_1, T_2, T_3, \ldots$ defined recursively by

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \qquad \text{for } n \geq 3.$$

Find a closed-form expression (in terms of x, T_0, T_1, T_2) for the generating function

$$\Omega = \sum_{n\geq 0} T_n x^n.$$

Observe that

$$\begin{array}{rcccccccccccc} \Omega &=& T_0 &+& T_1x &+& T_2x^2 &+& T_3x^3 &+& T_4x^4 &+& \cdots \\ x\Omega &=& && T_0x &+& T_1x^2 &+& T_2x^3 &+& T_3x^4 &+& \cdots \\ x^2\Omega &=& && && T_0x^2 &+& T_1x^3 &+& T_2x^4 &+& \cdots \end{array}$$

Adding these up, we obtain

$$\begin{aligned}(1+x+x^2)\Omega &= T_0 + (T_0+T_1)x + \sum_{n=3}^{\infty}(T_{n-1}+T_{n-2}+T_{n-3})x^{n-1} \\ &= T_0 + (T_0+T_1)x + \sum_{n=3}^{\infty} T_n x^{n-1} \\ &= T_0 + (T_0+T_1)x + x^{-1}\sum_{n=3}^{\infty} T_n x^n\end{aligned}$$

and solving for Ω, we get

$$\boxed{\Omega \;=\; \frac{T_0 + (T_1-T_0)x + (T_2-T_1-T_0)x^2}{1-x-x^2-x^3} \;=\; \frac{T_0(1-x-x^2) + T_1(x-x^2) + T_2x^2}{1-x-x^2-x^3}.}$$

Problem #2 Give a combinatorial interpretation for the coefficient of $q^k x^\ell$ in the power series

$$\prod_{n=1}^{\infty}(1 + qx^n + qx^{2n} + qx^{3n} + qx^{4n} + \cdots).$$

The monomial qx^{jn} corresponds to choosing to use j parts of size n in a partition. So the coefficient of $q^k x^\ell$ is the number of partitions of ℓ with exactly k distinct part sizes.

Problem #3 Bogart #224.

(a) We did this in class; here is a variant of the argument. Call a Catalan path (i.e., a diagonal lattice path from $(0,0)$ to $(2n,0)$ consisting of n each northeast and southeast steps) *primitive* if it does not touch the x-axis anywhere other than its two endpoints. Given a Catalan path P, let $(i,0)$ be the leftmost point at which P touches the x-axis. (So $1 \leq i \leq n$.) Let P' and P'' be the parts of the path from $(0,0)$ to $(i,0)$ and from $(i,0)$ to $(n,0)$ respectively; by definition of i, P' is primitive. Trimming off the first and last steps of

P' gives a primitive path of length $2(i-1)$; this is a bijection. Meanwhile, P'' can be any Catalan path of length $2(n-i)$. Therefore

$$C_n = \sum_{i=1}^{n} C_{i-1} C_{n-i} \tag{1}$$

since the right-hand side counts the possibilities for the pair (P', P'').

(b) Let $y = \sum_{i=0}^{\infty} C_i x^i$. Then

$$y^2 \;=\; \left(\sum_{i=0}^{\infty} C_i x^i\right)\left(\sum_{j=0}^{\infty} C_j x^j\right) \;=\; \sum_{n=0}^{\infty} \sum_{\substack{i,j:\\ i+j=n}} C_i C_j x^n \;=\; \sum_{n=0}^{\infty} x^n \sum_{i=0}^{n} C_i C_{n-i}$$

This looks promising; the next step is to change variables to get something of the form (1). Looking at the subscripts there, the right change of variables is $i = j-1$ and $n = m-1$, which gives

$$y^2 \;=\; \sum_{m=1}^{\infty} x^{m-1} \sum_{j=1}^{n} C_{j-1} C_{m-j} \;=\; x^{-1}\left(-1 + \sum_{m=0}^{\infty} x^m \sum_{j=1}^{n} C_{j-1} C_{m-j}\right) \;=\; \frac{y-1}{x}.$$

Clearing denominators gives $xy^2 = y - 1$, or $xy^2 - y + 1 = 0$. Solving this quadratic and comparing the power series expansions of the two solutions, the one we are looking for is

$$\boxed{y = \sum_{i=0}^{\infty} C_i x^i = \frac{1-\sqrt{1-4x}}{2x}.}$$

(c) Expanding this power series using the extended binomial theorem gives yields

$$\begin{aligned}
\sum_{i=0}^{\infty} C_i x^i &= \frac{1-(1-4x)^{1/2}}{2x} \\
&= \frac{1}{2x}\left(1 - \sum_{i=0}^{\infty} \binom{1/2}{i} (-4x)^i\right) \\
&= \frac{1}{2x}\left(-\sum_{i=1}^{\infty} \binom{1/2}{i} (-1)^i 2^{2i} x^i\right) \\
&= \sum_{i=1}^{\infty} \binom{1/2}{i} 2^{2i-1} (-1)^{i-1} x^{i-1} && \text{Substitute } j = i-1,\ i = j+1. \\
&= \sum_{j=0}^{\infty} \binom{1/2}{j+1} 2^{2j+1} (-1)^j x^j
\end{aligned}$$

which says that

$$\begin{aligned}
C_i = (-1)^i 2^{2i+1} \binom{1/2}{i+1} &= (-1)^i 2^{2i+1} \frac{(1/2)(-1/2)(-3/2)\cdots((1-2i)/2)}{(i+1)!} \\
&= (-1)^i 2^{2i+1} \frac{(-1)^i (2i-1)(2i-3)\cdots(3)(1)}{2^{i+1}(i+1)!} \\
&= 2^i \frac{(2i-1)(2i-3)\cdots(3)(1)}{(i+1)!} \qquad (2)
\end{aligned}$$

Recall (or verify) that $(2i-1)(2i-3)\cdots(3)(1) = \frac{(2i)!}{2^i i!}$. Multiplying through by 2^i and plugging this into (2) gives our old friend

$$C_i \;=\; \frac{(2i!)}{i!(i+1)!} \;=\; \frac{1}{i+1} \cdot \frac{(2i!)}{(i!)^2} \;=\; \frac{1}{i+1}\binom{2i}{i}.$$

Problem #4 Bogart, Chapter 4, Supplementary Problem #4.

Let T_n denote the number of ways to triangulate a convex n-gon (for $n \geq 3$). For example, $T_3 = 1$ and $T_4 = 2$ (because triangulating a quadrilateral amounts to choosing one of the two diagonals). Let us also pass a law that $T_2 = 1$ (because whatever a convex 2-gon is, there is only one, presumably trivial, way to triangulate it).

Let's say we have a convex n-gon. Label the vertices $v_1, \ldots, v_n$. Let X be a triangulation. Say that a vertex v_i is *lonely* in X if it is not the endpoint of any diagonal — i.e., if X contains the triangle $v_{i-1}v_iv_{i+1}$ (taking all subscripts modulo n).

If v_1 is lonely, then X consists of the triangle $v_nv_1v_2$ together with a triangulation of the $(n-1)$-gon with vertices $v_2, v_3, \ldots, v_n$. The number of such triangulations is T_{n-1}.

If v_1 is not lonely, then let i be the smallest number such that X contains the diagonal v_1v_i. Note that $3 \leq i \leq n-1$. In this case, specifying X is equivalent to specifying triangulations T', T'' of the two polygons with vertex sets $v_1, v_2, \ldots, v_i$ and $v_i, v_{i+1}, \ldots, v_n, v_1$, whose lengths are i and $n-i+2$ respectively. The triangulation T'' can be anything, but in T', we have the additional condition that v_1 must be lonely (by definition of i). Therefore, T' really consists of the triangle $v_iv_1v_2$ together with a triangulation of the $(i-1)$-gon with vertices $v_2, v_3, \ldots, v_i$.

All in all, we get the recurrence

$$T_n = T_{n-1} + \sum_{i=3}^{n-1} T_{i-1}T_{n-i+2} = \sum_{i=3}^{n} T_{i-1}T_{n-i+2}$$

This looks awfully similar to the Catalan recurrence. Indeed, if we set $j = i-2$ and $T_n = U_{n-2}$, the recurrence becomes

$$U_{n-2} = \sum_{j=1}^{n-2} U_{j-1}U_{(n-2)-j}$$

which is precisely the Catalan recurrence (with the same initial terms: $U_0 = T_2 = 1$ and $U_1 = T_3 = 1$). So $T_n = U_{n-2} = C_{n-2}$.

Problem #5 Bogart, Chapter 4, Supplementary Problem #9.

Let $P_m(k)$ be the number of partitions of k in which each part is used at most m times. Then

$$\sum_{k=0}^{\infty} P_m(k)x^k \;=\; \prod_{n=1}^{\infty}(1 + x^n + x^{2n} + \cdots + x^{mn}) \;=\; \prod_{n=1}^{\infty}\sum_{j=0}^{m} x^{jn} \;=\; \prod_{n=1}^{\infty}\frac{1-x^{(m+1)n}}{1-x^n}.$$

Transposing the Ferrers diagram of such a partition produces the Ferrers diagram of a partition in which consecutive parts differ by at most m and the smallest part is also at most m. Since transposition is a bijection, the same formula is the generating function for such partitions.

Problem #6 Bogart #234.

$$\begin{aligned}\left|\bigcup_{i=1}^n A_i\right| &= \left(\sum_{1\le i\le n}|A_i|\right) - \left(\sum_{1\le i<j\le n}|A_i\cap A_j|\right) + \left(\sum_{1\le i<j<k\le n}|A_i\cap A_j\cap A_k|\right) \pm\cdots\\ &= \sum_{I\subset[n]:\ I\neq\emptyset}(-1)^{|I|-1}\left|\bigcap_{i\in I}A_i\right|\end{aligned}$$

so

$$\boxed{\left|\overline{\bigcup_{i=1}^n A_i}\right| = |A| - \sum_{I\subset[n]:\ I\neq\emptyset}(-1)^{|I|-1}\left|\bigcap_{i\in I}A_i\right| = \sum_{I\subset[n]}(-1)^{|I|}\left|\bigcap_{i\in I}A_i\right|.}$$

Problem #7 Bogart #238 and #239. (Once you do #238, problem #239 should be easy.)

For each $A\subseteq[n]$, let S_A denote the functions $[k]\to[n]$ whose range is a subset of A. Therefore $|S_A|=|A|^k$. By inclusion/exclusion, the number of onto functions $f:[k]\to[n]$ is

$$\sum_{A\subseteq[n]}(-1)^{n-|A|}|S_A| = \sum_{A\subseteq[n]}(-1)^{n-|A|}|A|^k = \boxed{\sum_{a=0}^n(-1)^{n-a}\binom{n}{a}a^k = \sum_{i=0}^n(-1)^i\binom{n}{i}(n-i)^k.}$$

(The summations could be $\sum_{a=1}^n$ and $\sum_{i=0}^{n-1}$; the omitted terms vanish.)

The Stirling number $S(k,n)$ counts set partitions of $[k]$ into n blocks. We can produce such a thing from an onto function $f:[k]\to[n]$: let the blocks be $B_1,\dots,B_n$, where $B_i=\{x\in[k]:\ f(x)=i\}$. Each set partition comes from $n!$ such functions (by permuting the elements of $[n]$), so

$$\boxed{S(k,n)=\frac{1}{n!}\sum_{a=1}^n(-1)^{n-a}\binom{n}{a}a^k.}$$

Problem #8 The game of *egdirb* uses a deck of 30 cards. There are three suits: artichokes, ferrets, and pumpkins. Each suit contains ten cards. In one deal of egdirb, each of three players (Larry, Curly and Moe) receives a hand of 10 cards. Use inclusion/exclusion to determine the probability that at least one player is dealt a void (i.e., zero cards) in at least one suit.

We might as well solve a more general problem: suppose that there are N cards in each suit (rather than 10). There are still 3 suits and 3 players.

Denote the players by L,C,M and the suits by A,F,P, and let $V=\{L,C,M\}\times\{A,F,P\}$. For $X\subset V$, let $D(X)$ denote the set of deals containing at least the voids specified by X. For example, $D(\emptyset)$ is the set of all deals; $D(MA,CP)$ is the set of deals in which Moe has an artichoke void and Curly has a pumpkin void (and there may be other voids as well); $D(LA, LF, LP)$ is the set of deals in which Larry has no cards at all, so $D(LA, LF, LP) = \emptyset$. By inclusion-exclusion, the number of deals with at least one void is

$$\sum_{X \subseteq V:\ X \neq \emptyset} (-1)^{|X|-1} d(X)$$

where $d(X) = |D(X)|$. Note that $d(X)$ does *not* depend only on the cardinality of X. However, if we represent X by a 3×3 grid with a row for each player and a column for each suit…

$$\emptyset = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \qquad \{MA, CP\} = \begin{array}{|c|c|c|}\hline \bullet & & \\ \hline & \bullet & \\ \hline & & \\ \hline \end{array} \qquad \{MA, MP\} = \begin{array}{|c|c|c|}\hline & & \\ \hline \bullet & \bullet & \\ \hline & & \\ \hline \end{array}$$

…then $d(X)$ is invariant under permuting rows and columns and under transposition:

$$d\left(\begin{array}{|c|c|c|}\hline \bullet & & \\ \hline \bullet & & \\ \hline & & \\ \hline \end{array}\right) = d\left(\begin{array}{|c|c|c|}\hline & & \\ \hline & \bullet & \\ \hline & \bullet & \\ \hline \end{array}\right) = d\left(\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline \bullet & & \bullet \\ \hline \end{array}\right).$$

Therefore, we can group the summands in the inclusion/exclusion by equivalence classes. Note that there are no deals with three dots in a row or column, or with a 2×2 square of dots, so we can omit these cases.

X	$\lvert X \rvert$	Number of equivalent sets	$d(X)$
$\begin{array}{\|c\|c\|c\|}\hline \bullet & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$	1	9	$\binom{2N}{N}^2$
$\begin{array}{\|c\|c\|c\|}\hline \bullet & \bullet & \\ \hline & & \\ \hline & & \\ \hline \end{array}$	2	18	$\binom{2N}{N}$
$\begin{array}{\|c\|c\|c\|}\hline \bullet & & \\ \hline & \bullet & \\ \hline & & \\ \hline \end{array}$	2	18	$\sum_{k=0}^{N} \binom{N}{k}\binom{N}{N-k}\binom{N+k}{N}$ (Note 1)
$\begin{array}{\|c\|c\|c\|}\hline \bullet & \bullet & \\ \hline \bullet & & \\ \hline & & \\ \hline \end{array}$	3	36	1
$\begin{array}{\|c\|c\|c\|}\hline \bullet & \bullet & \\ \hline & & \bullet \\ \hline & & \\ \hline \end{array}$	3	36	$\binom{2N}{N}$
$\begin{array}{\|c\|c\|c\|}\hline \bullet & & \\ \hline & \bullet & \\ \hline & & \bullet \\ \hline \end{array}$	3	6	$\sum_{k=0}^{N} \binom{N}{k}^3$ (Note 2)
$\begin{array}{\|c\|c\|c\|}\hline \bullet & \bullet & \\ \hline \bullet & & \bullet \\ \hline & & \\ \hline \end{array}$	4	36	1
$\begin{array}{\|c\|c\|c\|}\hline \bullet & \bullet & \\ \hline & & \bullet \\ \hline & & \bullet \\ \hline \end{array}$	4	9	$\binom{2N}{N}$
$\begin{array}{\|c\|c\|c\|}\hline \bullet & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array}$	4	36	1
$\begin{array}{\|c\|c\|c\|}\hline \bullet & \bullet & \\ \hline \bullet & & \bullet \\ \hline & \bullet & \\ \hline \end{array}$	5	36	1
$\begin{array}{\|c\|c\|c\|}\hline \bullet & \bullet & \\ \hline \bullet & & \bullet \\ \hline & \bullet & \bullet \\ \hline \end{array}$	6	6	1

Note 1: Say $X = \{MA, CP\}$. Deal Moe 0 artichokes, k pumpkins, and $N-k$ ferrets. Of the remaining cards, there are N artichokes and k ferrets available for Curly's hand; deal him N of them. Larry gets whatever's left.

Note 2: Say $X = \{MA, CP, LF\}$. Deal Moe k ferrets and $N-k$ pumpkins. The remaining $N-k$ ferrets must go to Curly and the remaining k pumpkins go to Larry. We now have to allocate k artichokes to Curly.

Totaling (and cleaning) all this up, the number of deals with at least one void is therefore

$$9\binom{2N}{N}^2 + 9\binom{2N}{N} - 18\sum_{k=0}^{N}\binom{N}{k}^2\binom{N+k}{N} + 6\sum_{k=0}^{N}\binom{N}{k}^3 - 6.$$

For $N = 10$, this comes out to 289977617340. Dividing by the total number of deals, namely $\binom{30}{10}\binom{20}{10} = \binom{30}{10,10,10} = 5550996791340$, gives the final answer:

$$\boxed{\frac{289977617340}{5550996791340} \approx 0.05224.}$$

Here is the Sage code and output for some particular values of N.

```
sage: b = binomial
sage: Deal = lambda N: b(3*N,N)*b(2*N,N)
sage: Void = lambda N: 9*b(2*N,N)^2 + 9*b(2*N,N) \
....: - 18*sum(b(N,k)^2*b(N+k,N) for k in range(N+1)) \
....: + 6*sum(b(N,k)^3 for k in range(N+1)) - 6
sage: Prob = lambda N: Void(N) / Deal(N)
sage: sage: [float(Prob(N)) for N in [5,10,15,20,100]]
[0.5084280798566513,
 0.052238837138653786,
 0.004011340260452221,
 0.00029552905608717776,
 1.959806036516882e-22]
```