1. Introduction to Bayesian Statistics

→ In Bayesian statistics probabilities represent „states of knowledge" or „degrees of belief"

→ Different than frequentist interpretation: Prob.'s express expected frequencies of events

→ Bayesian statistics tells us how to formally update our state of knowledge when new information (a measurement) becomes available

→ Bayesian statistics works with Conditional Probabilities

a) Working with conditional probabilities

X, Y: continuous random variables or discrete events

Conditional: $$P(X|Y) = \frac{P(X,Y)}{P(Y)} \quad (1)$$ ← joint

Marginal: $$P(Y) = \int dX \, P(X,Y) \quad (2)$$

OR $$\sum_{X \in \{X_i\}} P(X_i|Y)$$

b) <u>Bayes Theorem</u>

→ Tells us how to update our state of knowledge as new data becomes available

- <u>prior</u> state of belief $p(x)$
  $\underline{\underline{x}}$ could be parameters in a physics model

- additional knowledge gained through measurement $\underline{\underline{d}}$

  <u>Likelihood</u> $P(d|x)$

- updated knowledge

  <u>posterior</u> $p(x|d)$

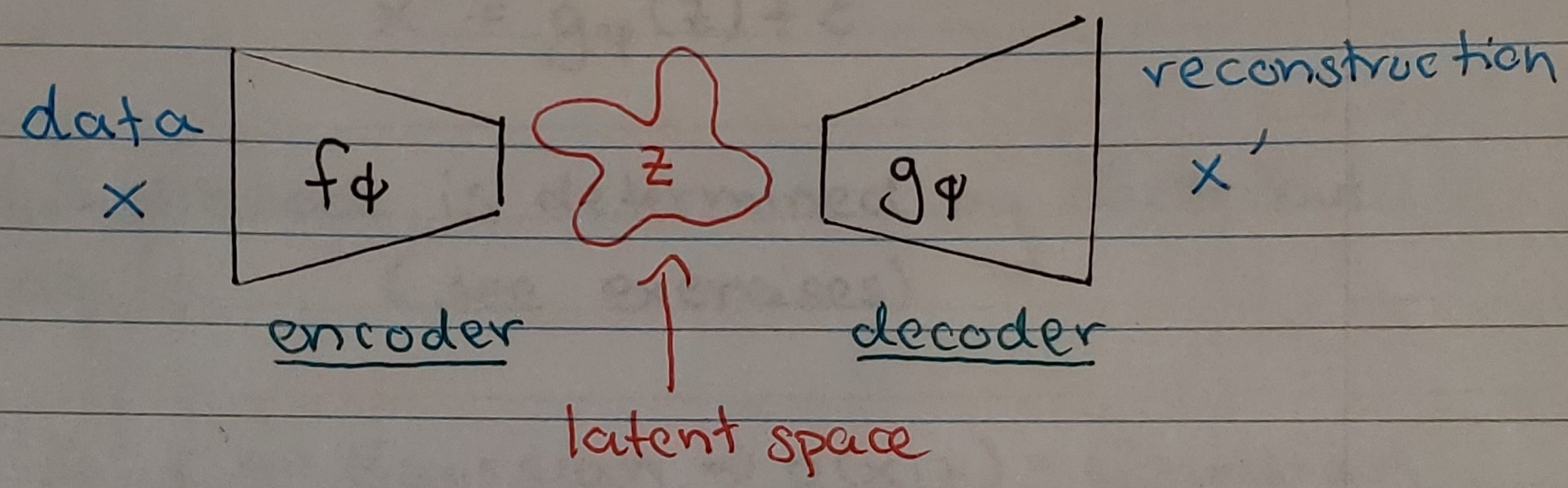
<u>Bayes Theorem</u>

$$P(x|d) = \frac{P(d|x)\,P(x)}{P(d)}$$

$P(d)$: normalization constant, <u>evidence</u>"

2. Why do I care?
OR
Introduction to Variational AEs

a) recap: Autoencoder



$$x' = g_\varphi(f_\phi(x)) \qquad \underbrace{\mathcal{L}_{AE} = \| x' - g_\varphi(f_\phi(x)) \|_2^2}_{\text{loss function}}$$

b) Probabilistic interpretation

-> Information loss in the compression
=> can't recover x precisely
=> implies probabilistic structure

-> data x follows probability distr. $p(x)$
-> all available data points are drawn from this distribution

$$p(x) = \int dz \, P(x,z) = \int dz \, P(x|z) P(z)$$

Likelihood $p(x|z)$

→ Information loss in compression implies probabilistic structure

$$x' = g_\psi(z) + \varepsilon$$

→ Likelihood is determined by distribution of ε (see exercises)

→ if $\varepsilon \leftarrow$ Gaussian $\Rightarrow p(x|z) =$ Gaussian

$$p_\psi(x|z) = \mathcal{G}(x \mid \mu = g_\psi(z), \Sigma = \varepsilon)$$

→ Likelihood depends on / is parameterized by network parameters

Prior $p(z)$

→ prior knowledge (before looking at data sample x) is average distribution of z

$$p(z) = \int dx \, p(z|x) \, p(x)$$

c) <u>Derivation of Evidence Lower Bound</u>

$$p(x) = \int dz \; p(x|z)\, p(z)$$

$$\log p(x) = \log \int dz \; p(x|z)\, p(z)$$

$$\log p(x) = \log \underbrace{\int dz \; q(z|x)\, p(x|z) \frac{p(z)}{q(z|x)}}_{E_{q(z|x)}\left[p(x|z) \frac{p(z)}{q(z|x)}\right]}$$

$$\log p(x) \geq \int dz \; q(z|x) \left[\log p(x|z) - \log\left[\frac{q(z|x)}{p(z)}\right]\right]$$

likelihood term = measure of reconstruction quality

$$\geq \underline{\int dz \; q(z|x) \log p(x|z)}$$

Gaussian likelihood $\log p(x|z) \hat{=} \frac{(x - g(z))^2}{\sigma^2}$

$$- \underline{\int dz \; q(z|x) \log \left[\frac{q(z|x)}{p(z)}\right]}$$

measures similarity between posterior & prior

Kullback-Leibler Divergence

$$D_{KL}(q \| p) \geq 0$$

- ELBO

d) <u>In practice</u>

-) $\underline{q_\phi(z|x)}$ is usually chosen to be a Gaussian with diagonal covariance *
Mean and covariance are parameterized by the encoder network

$$q_\phi(z|x) = \mathcal{G}(z|(\mu, \sigma) = f_\phi(x))$$

* note that this decorrelates latent variables

-) $\underline{p(z)}$ can be chosen. Typically a standard Normal distribution

$$p(z) = \mathcal{G}(z \mid \mu = 0, \sigma = 1)$$

We want to be able to sample from $p(z)$

-) <u>The ELBO</u> is evaluated stochastically

$$\text{ELBO} = \mathop{E}_{z \sim q_\phi(z|x)} \log p_\psi(x|z)$$

$$+ \mathop{E}_{z \sim q_\phi} \log \left[\frac{q_\phi(z|x)}{p(z)} \right]$$

Training requires derivative $\dfrac{\partial \text{ELBO}}{\partial(\psi, \phi)}$

-) How do we take this derivative?

Reparameterization Trick

sample: $z' \sim G(z' \mid 0, 1)$

compute: $z = \sigma_\phi z' + \mu_\phi$

e) What is the VAE good for?

① The VAE is a generative model
-> it allows us to generate artificial data
-> it allows us to approximately sample from $p(x)$

How? $z \leftarrow p(z)$

$x = g_\psi(z)$

② Different to the AE, we obtain a continuous latent space.
This allows us to interpolate between data points.

How? $z_1 = \mu_\phi(x_1)$ $z_2 = \mu_\phi(x_2)$

$z = z_1 + t \cdot (z_2 - z_1)$, $t \in [0, 1]$
$x = g_\psi(z)$

③ VAE allows us to explore the space of likely reconstructions

$$z \leftarrow q_\phi(z|x) \qquad x' = g_\psi(z)$$

„posterior analysis"

④ Take into account physical data properties (mask, noise)

⇒ tomorrow's exercise!