Shape Optimization with Unfitted Finite Element Methods

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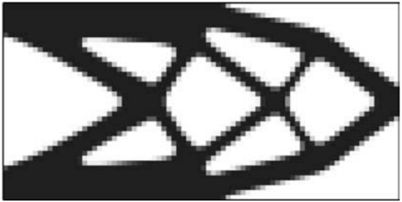
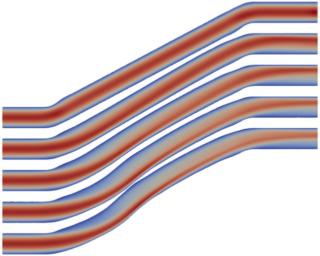
October 13, 2024

Introduction

- Allaire, Jouve, Toader, "Structural optimization using sensitivity analysis and a level-set method", Journal of Computational Physics, 2004





- Paganini, Wechsung, "Fireshape: a shape optimization toolbox for Firedrake" from Structural and Multidisciplinary Optimization Vol. 63, 2121

Model Problem

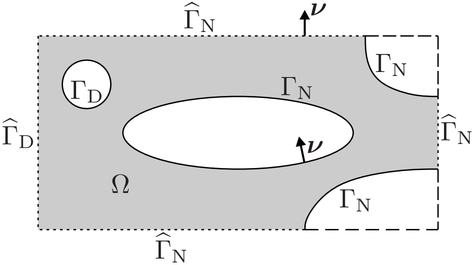
Consider the following linear elasticity problem:

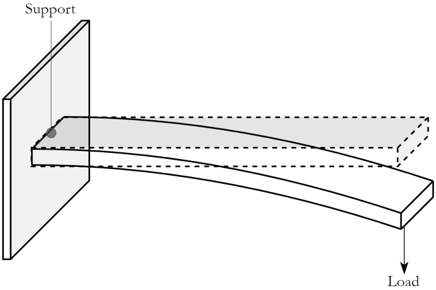
- $\Omega \subset \mathbb{R}^d$
- $\partial\Omega := \Gamma \cup \hat{\Gamma}$
- $\Gamma_D \cap \Gamma_N, \hat{\Gamma}_D \cap \hat{\Gamma}_N = \emptyset$
- $\boldsymbol{u}$: displacement field
- μ, λ are Lamé parameters
- $\epsilon(\nabla \boldsymbol{u}) := [\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T]/2$

$$\begin{cases} -Div(\boldsymbol{\sigma}) = \boldsymbol{f} & \text{in } \Omega \\ \boldsymbol{\sigma} = 2\mu\epsilon(\nabla \boldsymbol{u}) + \lambda tr(\epsilon(\nabla \boldsymbol{u}))I & \text{in } \Omega \\ \boldsymbol{u} = 0 & \text{on } \Gamma_D \cup \hat{\Gamma}_D \\ \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = \boldsymbol{g}_N & \text{on } \Gamma_N \cup \hat{\Gamma}_N \end{cases} \tag{1}$$

$\hat{\Gamma}$ is the inactive boundary and Γ is the active boundary.

Domain





- The design domain $\widehat{\mathfrak{D}}$ is the dashed rectangular region. The fixed background mesh.
- Γ is the active boundary and is described by a levelset function.
- $\hat{\Gamma}$ is the inactive boundary and $\hat{\Gamma} = \partial\widehat{\mathfrak{D}} \cap \partial\Omega$.

We view (1) as solving for the displacement of a cantilever beam.

Weak Formulation

We define the following linear and bilinear forms:

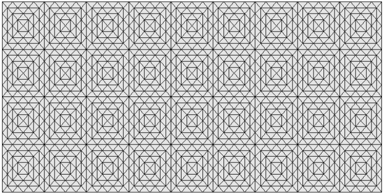
$$\chi(\Omega; \boldsymbol{v}) := (\boldsymbol{f}, \boldsymbol{v})_\Omega + (\boldsymbol{g}_N, \boldsymbol{v})_{\Gamma_N \cup \hat{\Gamma}_N},$$
$$a(\Omega; \boldsymbol{u}, \boldsymbol{v}) := 2\mu(\epsilon(\nabla \boldsymbol{u}), \epsilon(\nabla \boldsymbol{v}))_\Omega + \lambda(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})_\Omega.$$

And we state the weak formulation of (1):

Let $V_D := \{\boldsymbol{v} \in H^1(\Omega) : \boldsymbol{v}|_{\Gamma_D \cup \hat{\Gamma}_D} = 0\}$ and find $\boldsymbol{u} \in V_D$ such that

$$a(\Omega; \boldsymbol{u}, \boldsymbol{v}) = \chi(\Omega; \boldsymbol{v}) \quad \forall \boldsymbol{v} \in V_D(\Omega). \tag{2}$$

Discretization



Let $\widehat{\mathcal{T}}_h$ be a conforming shape regular mesh of $\widehat{\mathfrak{D}}$ and we define the following Lagrange finite element space

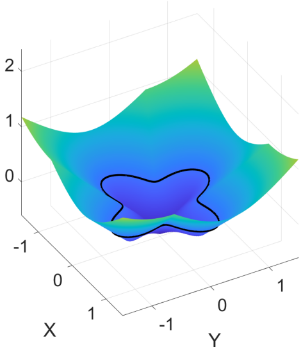
$$B_h = \{\boldsymbol{v}_h \in C^0(\widehat{\mathfrak{D}}) : \boldsymbol{v}_h|_T \in \mathcal{P}_k(T), \text{ for all elements } T\}, \quad k \geq 1,$$

where $\mathcal{P}_k(T)$ denotes the collection of polynomials of degree k on T.

Levelset Formulation

- $\phi_h \in B_h$ a Lipschitz levelset function

- $\Omega_h := \{\boldsymbol{x} \in \widehat{\mathfrak{D}} : \phi_h(\boldsymbol{x}) < 0\}$
- $\Gamma_h := \{\boldsymbol{x} \in \widehat{\mathfrak{D}} : \phi_h(\boldsymbol{x}) = 0\}$



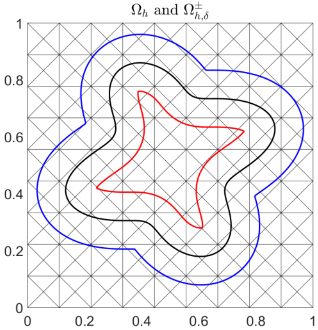
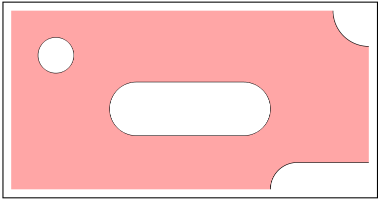
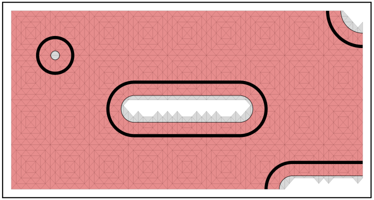


Figure: Interface (black) and shell region.

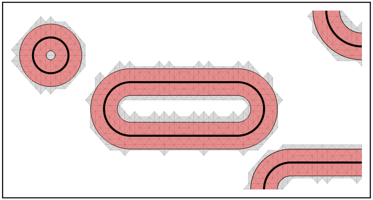
Subdomains Definitions



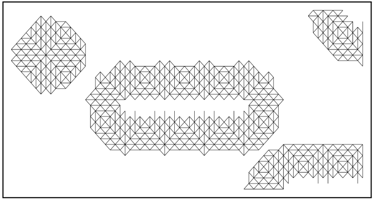
(a) The domain Ω_h is shown in red. The active boundary is shown in other figures as a thick black line.



(b) Active mesh $\mathcal{T}_{h,\delta}$, consists of elements that intersect the extended domain $\Omega_{h,\delta}$ (red).



(c) The selection of elements, $\mathcal{T}_{\Sigma^\pm}$, around the shell region $\Sigma^\pm_{h,\delta}$ (red).



(d) We have the facet selection $\mathcal{F}_{\Sigma^\pm}$ depicted by the above.

Stabilization Forms

We define the stabilized Nitsche Form:

$$a_h(\Omega_h; \boldsymbol{u}, \boldsymbol{v}) := a(\Omega_h; \boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu}, \boldsymbol{v})_{\Gamma_{h,D}} - (\boldsymbol{u}, \boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{\nu})_{\Gamma_{h,D}}$$
$$+ \gamma_D h^{-1} b(\Omega_h; \boldsymbol{u}, \boldsymbol{v}) + \gamma_N h(\boldsymbol{\sigma}(\boldsymbol{u})\nu, \boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{\nu})_{\Gamma_{h,N}},$$
$$b(\Omega_h; \boldsymbol{u}, \boldsymbol{v}) := 2\mu(\boldsymbol{u}, \boldsymbol{v})_{\Gamma_{h,D}} + \lambda(\boldsymbol{u} \cdot \boldsymbol{\nu}, \boldsymbol{v} \cdot \boldsymbol{\nu})_{\Gamma_{h,D}},$$
$$\chi_h(\Omega_h; \boldsymbol{u}, \boldsymbol{v}) := \chi(\Omega_h; \boldsymbol{v}) + \gamma_N h(\boldsymbol{g}_N, \boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{\nu})_{\Gamma_{h,N}},$$

- where $\gamma_D > 0$ and $\gamma_N \geq 0$ (we choose $\gamma_N = 0$).

Stabilization Forms

We define the stabilized Nitsche Form:

$$a_h(\Omega_h;\boldsymbol{u},\boldsymbol{v}) := a(\Omega_h;\boldsymbol{u},\boldsymbol{v}) - (\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu},\boldsymbol{v})_{\Gamma_{h,D}} - (\boldsymbol{u},\boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{\nu})_{\Gamma_{h,D}} + \gamma_D h^{-1} b(\Omega_h;\boldsymbol{u},\boldsymbol{v}) + \gamma_N h(\boldsymbol{\sigma}(\boldsymbol{u})\nu,\boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{\nu})_{\Gamma_{h,N}},$$

$$b(\Omega_h;\boldsymbol{u},\boldsymbol{v}) := 2\mu(\boldsymbol{u},\boldsymbol{v})_{\Gamma_{h,D}} + \lambda(\boldsymbol{u}\cdot\boldsymbol{\nu},\boldsymbol{v}\cdot\boldsymbol{\nu})_{\Gamma_{h,D}},$$

$$\chi_h(\Omega_h;\boldsymbol{u},\boldsymbol{v}) := \chi(\Omega_h;\boldsymbol{v}) + \gamma_N h(\boldsymbol{g}_N,\boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{\nu})_{\Gamma_{h,N}},$$

- where $\gamma_D > 0$ and $\gamma_N \geq 0$ (we choose $\gamma_N = 0$).

And we define the Facet stabilization form:

$$s_h(\mathcal{F};\boldsymbol{u},\boldsymbol{v}) := \sum_{\forall F\in\mathcal{F}} h^{-2} \int_{\omega_F} (\boldsymbol{u}_1 - \boldsymbol{u}_2)\cdot(\boldsymbol{v}_1 - \boldsymbol{v}_2)d\boldsymbol{x}.$$

- T_1 and T_2 neighboring elements around facet F
- $\omega_F := T_1 \cup T_2$
- $\boldsymbol{u}_i, \boldsymbol{v}_i$ are the canonical extensions of $\boldsymbol{u}_i|_{T_i}$ and $\boldsymbol{v}_i|_{T_i}$
- $F := T_1 \cap T_2$

Unfitted Finite Element Scheme

With this, we define the restricted finite element space on $\mathcal{T}_{h,\delta}$:

$$V_h \equiv V_h(\Omega_h) = \{\boldsymbol{v}_h \in C^0(\mathcal{T}_{h,\delta}) : \boldsymbol{v}_h = \hat{\boldsymbol{v}}_h|_{\mathcal{T}_{h,\delta}}, \text{ for some } \hat{\boldsymbol{v}}_h \in \mathring{B}_h\}.$$

We define a new bilinear form including the stabilization forms:

$$A_h(\Omega; \boldsymbol{u}, \boldsymbol{v}) := a_h(\Omega; \boldsymbol{u}, \boldsymbol{v}) + \gamma_s s_h\left(\mathcal{F}_{\Sigma^{\pm}_{\delta,D}}; \boldsymbol{u}, \boldsymbol{v}\right) + \gamma_s h^2 s_h\left(\mathcal{F}_{\Sigma^{\pm}_{\delta,N}}; \boldsymbol{u}, \boldsymbol{v}\right).$$

Full Scheme:
Find $\boldsymbol{u}_h \in V_h(\Omega_h)$ such that

$$A_h(\Omega_h; \boldsymbol{u}_h, \boldsymbol{v}_h) = \chi_h(\Omega_h; \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h(\Omega_h). \tag{3}$$

See also [Hansbo, Larson, Larsson] "Cut Finite Element Methods for Linear Elasticity Problems", 2017.

Minimization Problem

We focus on minimizing the compliance. Compliance measures the internal elastic energy and is defined as

$$J(\Omega; \boldsymbol{v}) := \chi(\Omega; \boldsymbol{v}) + a_0|\Omega|, \qquad a_0 > 0. \tag{4}$$

We state our minimization problem:

$$J(\Omega_{h,min}; \boldsymbol{u}_h(\Omega_{h,min})) = \min_{\forall \Omega_h \in \mathcal{A}, \text{ s.t. } \boldsymbol{v}_h \text{ solves (3)}} J(\Omega_h; \boldsymbol{v}_h). \tag{5}$$

- $\mathcal{A}$ is the set of admissible shapes
- Ω_h is the discrete domain

Lagrangian Formulation

In order to free the PDE constraint of our minimization problem (5), we introduce a Lagrangian

$$L(\Omega_h; \boldsymbol{v}_h, \boldsymbol{q}_h) := J(\Omega_h; \boldsymbol{v}_h) - A_h(\Omega_h; \boldsymbol{v}_h, \boldsymbol{q}_h) + \chi_h(\Omega_h; \boldsymbol{q}_h),$$
$$L(\overline{\Omega}_h; \overline{\boldsymbol{u}}_h, \overline{\boldsymbol{p}}_h) = \min_{\forall \Omega_h \in \mathcal{A}, \forall \boldsymbol{v}_h \in V_h(\Omega_h)} \max_{\forall \boldsymbol{q}_h \in V_h(\Omega_h)} L(\Omega_h; \boldsymbol{v}_h, \boldsymbol{q}_h),$$

which leads to the first order conditions:

$$\delta_{\boldsymbol{q}_h} L(\overline{\Omega}_h; \overline{\boldsymbol{u}}_h, \overline{\boldsymbol{p}}_h)(\boldsymbol{v}_h) = 0, \qquad \delta_{\boldsymbol{v}_h} L(\overline{\Omega}_h; \overline{\boldsymbol{u}}_h, \overline{\boldsymbol{p}}_h)(\boldsymbol{w}_h) = 0,$$

which implies that $\overline{\boldsymbol{u}}_h$ and $\overline{\boldsymbol{p}}_h$ solve the variational problems

$$A_h(\overline{\Omega}_h; \overline{\boldsymbol{u}}_h, \boldsymbol{v}_h) = \chi_h(\overline{\Omega}_h; \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in V_h(\overline{\Omega}_h),$$
$$A_h(\overline{\Omega}_h; \boldsymbol{w}_h, \overline{\boldsymbol{p}}_h) = \delta_{\boldsymbol{v}_h} J(\overline{\Omega}_h; \boldsymbol{v}_h)(\boldsymbol{w}_h) \qquad \forall \boldsymbol{w}_h \in V_h(\overline{\Omega}_h).$$

Fréchet Differentiability of Shape Functionals

We also note that we have the shape optimality condition:

$$L'(\overline{\Omega}_h; \overline{\boldsymbol{u}}_h, \overline{\boldsymbol{p}}_h)(\boldsymbol{Y}) = 0 \quad \forall \boldsymbol{Y} \in [W^{1,\infty}(\mathbb{R}^d)]^d.$$

Given some bulk shape functional $J(\Omega)$, we first define the perturbation of the identity as follows:

$$\boldsymbol{\Phi}_{\boldsymbol{U}}(\boldsymbol{x}) := I(\boldsymbol{x}) + \boldsymbol{U}(\boldsymbol{x}), \qquad \Omega_{\boldsymbol{U}} := \boldsymbol{\Phi}_{\boldsymbol{U}}(\Omega),$$

for some vector field $\boldsymbol{U}$. And now if there exists an operator, say $J'(\Omega)(\boldsymbol{U}) \in \mathcal{L}([W^{1,\infty}(\mathbb{R}^d)]^d, \mathbb{R})$, such that

$$\lim_{\|\boldsymbol{U}\|_{W^{1,\infty}} \to 0} \frac{|J(\Omega_{\boldsymbol{U}}) - J(\Omega) - J'(\Omega)(\boldsymbol{U})|}{\|\boldsymbol{U}\|_{W^{1,\infty}}} = 0, \tag{6}$$

then $J(\Omega)$ is Fréchet differentiable with Fréchet derivative $J'(\Omega)(\boldsymbol{U})$.

Fréchet Shape Derivative Lemmas

We use the following Lemmas from [Delfour, Zolesio] "Shapes and Geometries" and [Henrot, Pierre] "Shape Variation and Optimization".

Lemma (1)

Given $f \in L^1(\mathbb{R}^d)$ *and* $\boldsymbol{U} \in [W^{1,\infty}(\mathbb{R}^d)]^d$ *we have that*

$$\lim_{\|\boldsymbol{U}\|_{W^{1,\infty}} \to 0} \frac{\int_\Omega [f(\Phi_{\boldsymbol{U}}(\boldsymbol{a})) - f(\boldsymbol{a})] (\nabla_{\boldsymbol{a}} \boldsymbol{U}(\boldsymbol{a})) d\boldsymbol{a}}{\|\boldsymbol{U}\|_{W^{1,\infty}}} = 0$$

Lemma (2)

Given $f \in W^{1,1}(\mathbb{R}^d)$ *and* $\boldsymbol{U} \in [W^{1,\infty}(\mathbb{R}^d)]^d$ *we have that*

$$\lim_{\|\boldsymbol{U}\|_{W^{1,\infty}} \to 0} \frac{\int_\Omega f(\Phi_{\boldsymbol{U}}(\boldsymbol{a})) - f(\boldsymbol{a}) d\boldsymbol{a} - \int_\Omega \nabla f(\boldsymbol{a}) \cdot \boldsymbol{U}(\boldsymbol{a}) d\boldsymbol{a}}{\|\boldsymbol{U}\|_{W^{1,\infty}}} = 0.$$

Fréchet Shape Derivative

We have the following theorem due to [Delfour, Zolesio] "Shapes and Geometries" and [Henrot, Pierre] "Shape Variation and Optimization".

Theorem

Let $f(\boldsymbol{x})$ be defined on $\widehat{\mathfrak{D}}$ and independent of the shape Ω. Then for the shape functional $J(\Omega) := \int_\Omega f(\boldsymbol{x})d\boldsymbol{x}$ with $f \in W^{1,1}(\mathbb{R}^d)$ we have that $J(\Omega)$ is shape differentiable at Ω (in the sense of (6)) with Fréchet derivative

$$J'(\Omega)(\boldsymbol{U}) = \int_{\partial\Omega} f(\boldsymbol{a})\boldsymbol{U}(\boldsymbol{a}) \cdot \boldsymbol{\nu}(\boldsymbol{a})d\boldsymbol{a}$$

for all $\boldsymbol{U} \in [W^{1,\infty}(\mathbb{R}^d)]^d$.

The proof of which follows from the previous lemmas, a classical determinant expansion, and the divergence theorem.

Discretize-then-Optimize VS Optimize-then-Discretize

Optimize-then-Discretize: One first derives the optimality conditions for the continuous problem and then discretizes while still using the optimality conditions for the continuous problem.

Discretize-then-Optimize: One first discretizes the problem and then computes the optimality conditions for the discrete problem exactly.

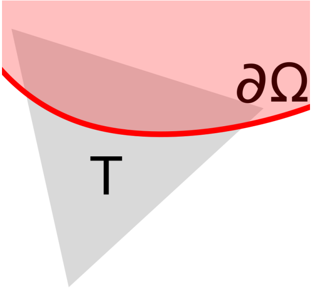
Our Contributions:

- Shape derivative of bulk functionals can be computed exactly for discretize unfitted finite element formulations.
- Discrete shape derivative formula agrees with the optimality conditions for the continuous problem.
- We gain benefits from both methods.

Shape Derivative on a Cut Element

Let $f(x)$ be defined on $\widehat{\mathfrak{D}}$ and independent of the shape Ω and define the shape functional restricted to an element T

$$J_T(\Omega) := \int_{\Omega \cap T} f(\boldsymbol{x}) d\boldsymbol{x}.$$



We show that we can obtain the exact shape derivative of $J_T(\Omega)$, so long as the following assumption holds:

Assumption (1)

Assume that $\partial\Omega \cap \partial T$ has vanishing $\mathbb{R}^{d-1}$ Lebesgue measure.

Shape Derivative on a Cut Element

Theorem

Given the shape functional $J_T(\Omega) := \int_{\Omega \cap T} f(\boldsymbol{x}) d\boldsymbol{x}$ *with* $f \in W^{1,1}(\mathbb{R}^d)$ *we have that* $J_T(\Omega)$ *is shape differentiable at* Ω *(in the sense of Definition* (6)*) with Fréchet derivative*

$$J'_T(\Omega)(\boldsymbol{U}) = \int_{\partial\Omega \cap T} f(\boldsymbol{a}) \boldsymbol{U}(\boldsymbol{a}) \cdot \boldsymbol{\nu}(\boldsymbol{a}) dS(\boldsymbol{a})$$

for all $\boldsymbol{U} \in [W^{1,\infty}(\mathbb{R}^d)]^d$*, provided assumption* (1) *holds.*

We sketch part of the proof.

Sketch of Proof

We prove a generalization of Lemma 2, where we show

$$\lim_{\|\boldsymbol{U}\|_{W^{1,\infty}}\to 0} \frac{\int_{\Omega\cap T} f(\Phi_{\boldsymbol{U}}(\boldsymbol{a})) - f(\boldsymbol{a})d\boldsymbol{a} - \int_{\Omega\cap T} \nabla f(\boldsymbol{a})\cdot \boldsymbol{U}(\boldsymbol{a})d\boldsymbol{a}}{\|\boldsymbol{U}\|_{W^{1,\infty}}} = 0.$$

To deal with this restriction to $\Omega \cap T$ we use a smooth approximation, $\rho_\epsilon(\boldsymbol{a})$, of the characteristic function $\chi_T(\boldsymbol{a})$ and consider

$$\lim_{\epsilon\to 0} \frac{\int_{\Omega} f(\Phi_{\boldsymbol{U}}(\boldsymbol{a}))\rho_\epsilon(\Phi_{\boldsymbol{U}}(\boldsymbol{a})) - f(\boldsymbol{a})\rho_\epsilon(\boldsymbol{a})d\boldsymbol{a} - \int_{\Omega} \nabla[f(\boldsymbol{a})\rho_\epsilon(\boldsymbol{a})]\cdot \boldsymbol{U}(\boldsymbol{a})d\boldsymbol{a}}{\|\boldsymbol{U}\|_{W^{1,\infty}}}.$$

Sketch of Proof

$$\int_\Omega f(\Phi_{\boldsymbol{U}}(\boldsymbol{a}))\rho_\epsilon(\Phi_{\boldsymbol{U}}(\boldsymbol{a})) - f(\boldsymbol{a})\rho_\epsilon(\boldsymbol{a})d\boldsymbol{a} - \int_\Omega \nabla[f(\boldsymbol{a})\rho_\epsilon(\boldsymbol{a})] \cdot \boldsymbol{U}(\boldsymbol{a})d\boldsymbol{a}$$

$$= \int_\Omega \int_0^1 [\nabla [f(\Phi_{s\boldsymbol{U}}(\boldsymbol{a}))\rho_\epsilon(\Phi_{s\boldsymbol{U}}(\boldsymbol{a}))] - \nabla[f(\boldsymbol{a})\rho_\epsilon(\boldsymbol{a})]] \cdot \boldsymbol{U}(\boldsymbol{a})dsd\boldsymbol{a} \qquad (MVT)$$

$$= \overbrace{\int_\Omega \int_0^1 [\nabla [f(\Phi_{s\boldsymbol{U}}(\boldsymbol{a}))\rho_\epsilon(\Phi_{s\boldsymbol{U}}(\boldsymbol{a}))] - \nabla[f(\Phi_{s\boldsymbol{U}}(\boldsymbol{a}))\rho_\epsilon(\boldsymbol{a})]] \cdot \boldsymbol{U}(\boldsymbol{a})dsd\boldsymbol{a}}^{I_\epsilon(\boldsymbol{U}):=}$$

$$+ \int_\Omega \int_0^1 [\nabla [f(\Phi_{s\boldsymbol{U}}(\boldsymbol{a}))\rho_\epsilon(\boldsymbol{a})] - \nabla[f(\boldsymbol{a})\rho_\epsilon(\boldsymbol{a})]] \cdot \boldsymbol{U}(\boldsymbol{a})dsd\boldsymbol{a}$$

Sketch of Proof

We expand $I_\epsilon(\boldsymbol{U})$ as follows:

$$I_\epsilon(\boldsymbol{U}) = \int_\Omega \int_0^1 \left[\nabla \left[f(\Phi_{s\boldsymbol{U}}(\boldsymbol{a}))\rho_\epsilon(\Phi_{s\boldsymbol{U}}(\boldsymbol{a}))\right] - \nabla[f(\Phi_{s\boldsymbol{U}}(\boldsymbol{a}))\rho_\epsilon(\boldsymbol{a})]\right] \cdot \boldsymbol{U}(\boldsymbol{a}) ds d\boldsymbol{a}$$

$$= \overbrace{\int_0^1 \int_\Omega \nabla \cdot \left[f(\Phi_{s\boldsymbol{U}}(\boldsymbol{a}))\left(\rho_\epsilon(\Phi_{s\boldsymbol{U}}(\boldsymbol{a})) - \rho_\epsilon(\boldsymbol{a})\right)\boldsymbol{U}(\boldsymbol{a})\right] d\boldsymbol{a} ds}^{A_\epsilon(\boldsymbol{U}) :=}$$

$$- \int_0^1 \int_\Omega f(\Phi_{s\boldsymbol{U}}(\boldsymbol{a}))\left(\rho_\epsilon(\Phi_{s\boldsymbol{U}}(\boldsymbol{a})) - \rho_\epsilon(\boldsymbol{a})\right) \nabla \cdot \boldsymbol{U}(\boldsymbol{a}) d\boldsymbol{a} ds$$

Other terms are omitted as they use similar techniques.

Sketch of Proof

$$
\begin{aligned}
\lim_{\epsilon\to 0} A_\epsilon &= \lim_{\epsilon\to 0} \int_0^1 \int_\Omega \nabla \cdot \left[f(\mathbf{\Phi}_{s\boldsymbol{U}}(\boldsymbol{a}))\left(\rho_\epsilon(\mathbf{\Phi}_{s\boldsymbol{U}}(\boldsymbol{a})) - \rho_\epsilon(\boldsymbol{a})\right)\boldsymbol{U}(\boldsymbol{a})\right] d\boldsymbol{a} ds \\
&= \lim_{\epsilon\to 0} \int_0^1 \int_{\partial\Omega} f(\mathbf{\Phi}_{s\boldsymbol{U}}(\boldsymbol{a}))\left[\rho_\epsilon(\mathbf{\Phi}_{s\boldsymbol{U}}(\boldsymbol{a})) - \rho_\epsilon(\boldsymbol{a})\right]\boldsymbol{U}(\boldsymbol{a}) \cdot \boldsymbol{\nu} d\boldsymbol{a} ds \\
&= \int_0^1 \int_{\partial\Omega} f(\mathbf{\Phi}_{s\boldsymbol{U}}(\boldsymbol{a}))\left[\chi_T(\mathbf{\Phi}_{s\boldsymbol{U}}(\boldsymbol{a})) - \chi_T(\boldsymbol{a})\right]\boldsymbol{U}(\boldsymbol{a}) \cdot \boldsymbol{\nu} d\boldsymbol{a} ds
\end{aligned}
$$

$$
\lim_{\epsilon\to 0} \frac{A_\epsilon}{\|\boldsymbol{U}\|_{W^{1,\infty}}} \leq \int_0^1 \int_{\partial\Omega} |f(\mathbf{\Phi}_{s\boldsymbol{U}}(\boldsymbol{a}))| \cdot |\chi_T(\mathbf{\Phi}_{s\boldsymbol{U}}(\boldsymbol{a})) - \chi_T(\boldsymbol{a})| \, d\boldsymbol{a} ds
$$

Sketch of Proof

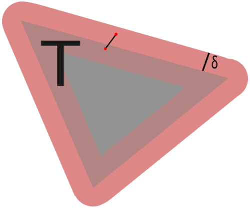
We add and subtract a smooth function f_k having the property $\lim_{k \to \infty} ||f - f_k||_{W^{1,1}} = 0$

$$\lim_{\epsilon \to 0} \frac{A_\epsilon}{\|\boldsymbol{U}\|_{W^{1,\infty}}} \leq \int_0^1 \int_{\partial\Omega} |(f - f_k) \circ (\boldsymbol{\Phi}_{s\boldsymbol{U}}(\boldsymbol{a}))| \cdot |\chi_T(\boldsymbol{\Phi}_{s\boldsymbol{U}}(\boldsymbol{a})) - \chi_T(\boldsymbol{a})| \, d\boldsymbol{a}ds$$
$$+ \int_0^1 \int_{\partial\Omega} |f_k(\boldsymbol{\Phi}_{s\boldsymbol{U}}(\boldsymbol{a}))| \cdot |\chi_T(\boldsymbol{\Phi}_{s\boldsymbol{U}}(\boldsymbol{a})) - \chi_T(\boldsymbol{a})| \, d\boldsymbol{a}ds$$
$$\leq C||f - f_k||_{W^{1,1}} + C_k \int_0^1 \int_{\partial\Omega} \left|\chi_{\tilde{T}}(\boldsymbol{a}) - \chi_T(\boldsymbol{a})\right| d\boldsymbol{a}ds,$$

where $\tilde{T} = \boldsymbol{\Phi}_{s\boldsymbol{U}}^{-1}(T)$ and note that $\left|\chi_{\tilde{T}}(\boldsymbol{a}) - \chi_T(\boldsymbol{a})\right| = \chi_{\tilde{T} \triangle T}$.

Sketch of Proof

Let $||\boldsymbol{U}||_{L^\infty} = \delta$. For $x \in \tilde{T} \triangle T := (\tilde{T} \setminus T) \cup (T \setminus \tilde{T})$ and notice that we have,



$$x \in (\tilde{T} \setminus T) \implies x + \boldsymbol{U}(x) \in T \text{ and } x \notin T$$
$$x \in (T \setminus \tilde{T}) \implies x + \boldsymbol{U}(x) \notin T \text{ and } x \in T$$

$$\implies \tilde{T} \triangle T \subset \mathcal{O}_\delta(\partial T).$$

Note: The line segment between $x + \boldsymbol{U}(x)$ and x must intersect ∂T and the length of the line is bounded by δ hence $dist(x, \partial T) \leq \delta$ for $x \in \tilde{T} \triangle T$.

Sketch of Proof

So let $\mathcal{O}_\delta(\partial T)$ be the δ neighborhood of ∂T, and so we have the following:

$$C||f - f_k||_{W^{1,1}} + C_k \int_0^1 \int_{\partial\Omega} \left|\chi_{\tilde{T}}(\boldsymbol{a}) - \chi_T(\boldsymbol{a})\right| d\boldsymbol{a}ds,$$

$$\leq C||f - f_k||_{W^{1,1}} + C_k \int_{\partial\Omega\cap\mathcal{O}_\delta(\partial T)} 1 d\boldsymbol{a}.$$

Hence as $\|\boldsymbol{U}\|_{W^{1,\infty}} \to 0$ we have that $\delta \to 0$ and we get

$$\lim_{\|\boldsymbol{U}\|_{W^{1,\infty}}\to 0} \lim_{\epsilon\to 0} \frac{A_\epsilon}{\|\boldsymbol{U}\|_{W^{1,\infty}}} = C||f - f_k||_{W^{1,1}} + C_k \int_{\partial\Omega\cap\partial T} 1 d\boldsymbol{a}.$$

And this tends towards 0 as $k \to \infty$ so long as assumption (1) holds.

Connection to Level Set Formulation

Theorem

Assume $\phi \in W^{2,\infty}(\mathbb{R}^d)$ such that $1 \sim |\nabla\phi| \geq c > 0$ in some δ-neighborhood of Γ. Let $\Omega(\phi + \eta)$ be the sub-zero level set of $\phi + \eta$. For the shape functional $J_T(\Omega) := \int_{\Omega \cap T} f(\boldsymbol{x}) d\boldsymbol{x}$ with $f \in W^{1,1}(\mathbb{R}^d)$ we have that $J_T(\Omega)$ is level set shape differentiable at Ω with Fréchet derivative

$$J'_T(\Omega)(\eta) = \int_{\partial\Omega \cap T} f(\boldsymbol{a}) \left(-\eta|\nabla\phi(\boldsymbol{a})|^{-1}\right) d\boldsymbol{a}$$

for all $\eta \in W^{2,\infty}(\mathbb{R}^d)$.

Given $\boldsymbol{U} = -\frac{\nabla\tilde{\phi}}{|\nabla\tilde{\phi}|^2}\eta$ we have $J'_T(\Omega)(\eta) = J'_T(\Omega)(\boldsymbol{U})$, where $\tilde{\phi}(\boldsymbol{x}, t) = \phi(\boldsymbol{x}) + t\eta(\boldsymbol{x})$.

Discrete Shape Derivatives

We make the following assumptions:

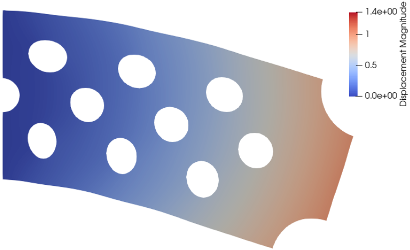
- $\hat{\Gamma} = \hat{\Gamma}_D \cup \hat{\Gamma}_N$ is fixed
- $\Gamma_D = \emptyset$
- $\boldsymbol{f} = 0$
- $\boldsymbol{g} \neq 0$ on a subset of $\hat{\Gamma}_N$

The exact shape derivative of our discrete functional is

$$\mathcal{J}'(\phi_h)(\eta_h) = \int_{\Gamma_h(\phi_h)} (2\mu|\epsilon(\nabla \boldsymbol{u}_h)|^2 + \lambda|\nabla \cdot \boldsymbol{u}_h|^2 + a_0)\frac{\eta_h}{|\nabla \phi_h|},$$

where $\boldsymbol{u}_h$ solves (3).

Algorithm



Given initial shape Ω, we define the stabilization region.

Then solve for $\overline{\boldsymbol{u}}_h$ and $\overline{\boldsymbol{p}}_h$ from

$$\begin{aligned} A_h(\Omega; \overline{\boldsymbol{u}}_h, \boldsymbol{v}_h) &= \chi_h(\Omega; \boldsymbol{v}_h) && \forall \boldsymbol{v}_h \in V_h(\Omega), \\ A_h(\Omega; \boldsymbol{w}_h, \overline{\boldsymbol{p}}_h) &= \delta_{\boldsymbol{v}_h} J(\Omega; \boldsymbol{v}_h)(\boldsymbol{w}_h) && \forall \boldsymbol{w}_h \in V_h(\Omega). \end{aligned}$$

A_h is self-adjoint due to our choice of $J(\Omega; \boldsymbol{v}_h)$, hence $\overline{\boldsymbol{u}}_h = \overline{\boldsymbol{p}}_h$.

Algorithm

Then compute the descent direction: Find $\delta\phi_h \in B_h$ such that

$$(\delta\phi_h, \eta_h)_{H^1(\widehat{\mathfrak{D}})} = -\mathcal{J}'(\phi_h)(\eta_h) \quad \forall \eta_h \in B_h.$$

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Update $\phi_h \leftarrow \phi_h + \alpha\delta\phi_h$ where $\alpha > 0$ is determined through a backtracking line search.

Note: the stabilization region remains **unchanged** until after α is determined.

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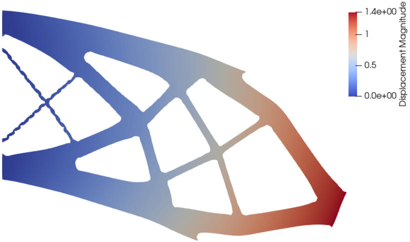
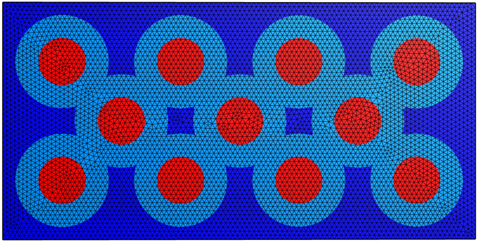
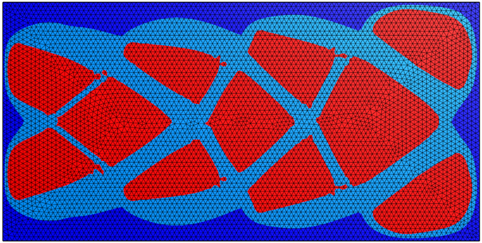
Repeat until stopping criteria is met.



Cantilever with 11 holes, $g_N = (0, -1)$ on far right

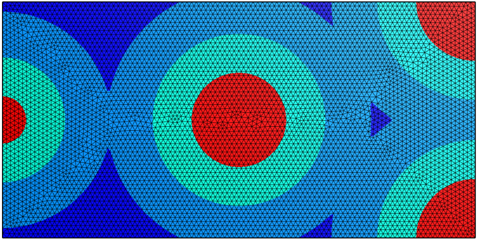


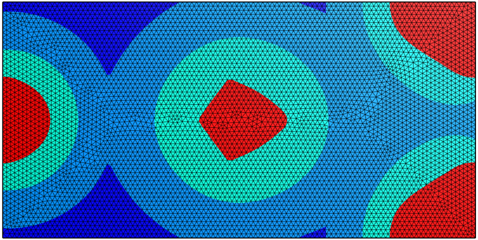


We start with an initial shape (top) having 11 holes spaced evenly and we do not update the levelset function along the boundary so that it retains its rectangular shape.

The resulting image (bottom) is shown.

Cantilever with 4 holes, $g_N = (0, -1)$ on far right

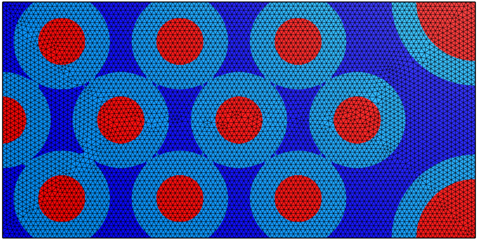


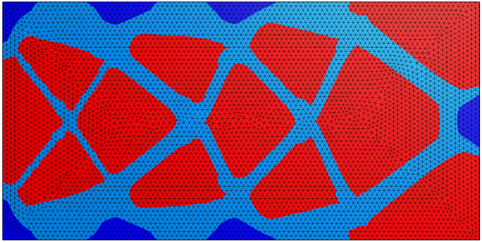


This time we allow the levelset function to be updated on a portion of the boundary, but still restrict the levelset on the boundary depicted in black here:

The initial guess (top) and resulting shape (bottom) are displayed.

Cantilever with 12 holes, $g_N = (0, -1)$ on far right





We again restrict the update of the levelset on the boundary depicted in black here:

We produce qualitative results similar to that of [Burman, et al] "Shape Optimization using the Cut Finite Element Method", 2018.

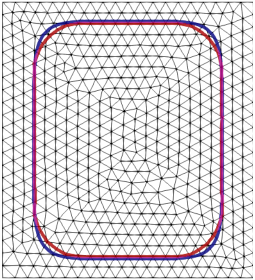
Shape Derivative Test

We test our shape derivative formula using some arbitrary perturbation with step size ϵ and then compare it with a finite difference approximation of the shape derivative.

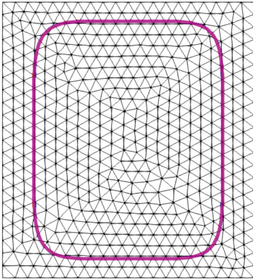
$$\delta J(FD) = \lim_{\epsilon \to 0} \frac{J - J_{prev}}{\epsilon}, \qquad \zeta = \frac{|\delta J(FD) - \delta J(exact)|}{\epsilon}.$$

eps	k	J prev	J	δJ (exact)	δJ (FD)	|difference|	ζ
1.00E-05	1	0.04199774	0.04198771	-1.00312053	-1.00300318	1.173e-04	11.74
	2	0.04192908	0.04191913	-0.99564238	-0.99552669	1.157e-04	11.57
	3	0.04192911	0.04191916	-0.99568833	-0.99557262	1.157e-04	11.57
1.00E-06	1	0.05471722	0.05471599	-1.23297870	-1.23296633	1.238e-05	12.38
	2	0.05461154	0.05461031	-1.23218803	-1.23217509	1.293e-05	12.93
	3	0.05461150	0.05461027	-1.23212173	-1.23210872	1.301e-05	13.01
1.00E-07	1	0.06983341	0.06983326	-1.50440084	-1.50439885	1.995e-06	19.95
	2	0.06977494	0.06977479	-1.49662564	-1.49662738	1.738e-06	17.38
	3	0.06977483	0.06977468	-1.49666321	-1.49666503	1.820e-06	18.20
1.00E-08	1	0.08731274	0.08731272	-1.78759980	-1.78759962	1.775e-07	17.75
	2	0.08720557	0.08720555	-1.78220514	-1.78220560	4.586e-07	45.86
	3	0.08720552	0.08720550	-1.78221505	-1.78221539	3.340e-07	33.40

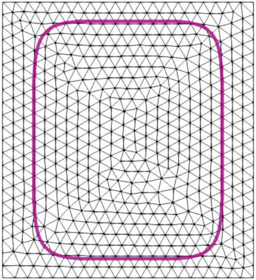
Superellipse Test



(a) FE Degree = 1



(b) FE Degree = 2



(c) FE Degree = 3

$$u(x, y) = \frac{1}{4}\left(\frac{1}{\alpha}x^4 + \frac{1}{\beta}y^4\right), \quad J(\Omega) = \int_\Omega |\nabla u|^2 \, dA - \lambda\left(|\Omega| - A_0\right),$$

$$\delta J(\Omega; \boldsymbol{V}) = \int_{\partial\Omega} \left(|\nabla u|^2 - \lambda\right) \boldsymbol{V} \cdot \boldsymbol{\nu} dS.$$

Superellipse Test

Error of Cost of Superellipse

maxh	FE Degree = 1	FE Degree = 2	FE Degree = 3
0.1	1.16996184e-03	3.60375786e-05	1.96245579e-07
0.05	7.32926122e-04	4.80178400e-06	9.54815926e-09
0.025	1.70979836e-04	2.09071780e-07	1.96049010e-10

- NGSolve uses an isoparametric mapping, which does not work directly with our shape derivative formula for higher order finite element methods.
- NGSolve can use a subdivision to compute integrals instead of the isopoarametric mapping.

Conclusion

Highlights:

- We obtain exact shape derivative of discrete bulk functionals in unfitted finite element methods with a level set framework.
- Easily works within level set framework.

Extensions:

- Boundary functionals

Special Thanks to

NSF