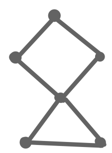
1. For each of the following, draw an Eulerian graph that satisfies the conditions, or prove that no such graph exists.

   (a) An even number of vertices, an even number of edges.
   (b) An even number of vertices, an odd number of edges.
   (c) An odd number of vertices, an even number of edges.
   (d) An odd number of vertices, an odd number of edges.

   Solution:

   (a) C_4
   (b) The following graph.



   (c) K_5
   (d) C_3

2. What follows is another algorithm for finding Eulerian cir-cuits. The method used here is to build the circuit, one edge at a time, mak-ing sure to make good choices along the way.
   Fleury's Algorithm for Identifying Eulerian Circuits
   Given: An Eulerian graph G, with all of its edges *unmarked*.

   i. Choose a vertexv, and call it the "lead vertex."
   ii. If all edges of G have been marked, then stop. Otherwise continue to step iii.
   iii. Among all edges incident with the lead vertex, choose, if possible, one that is not a bridge of the subgraph formed by the unmarked edges. Ifthis is not possible, choose any edge incident with the lead. Mark thisedge and let its other end vertex be the new lead vertex.
   iv. Go to step ii.

   Use both Fleury's algorithm and Hierholzer's algorithm to find an Eulerian circuit for the graph in Figure 1. Let a be your initial vertex.

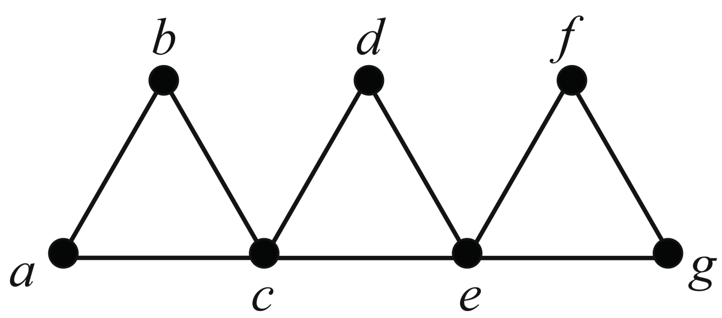


Figure 1:

Solution: Just strictly follow the steps of those 2 algorithms.

3. Prove Corollary 1.21 (Harris).

   Solution: ($\Rightarrow$) If the trail is a circuit, then there are zero vertices of odd degree. If the trail is not a circuit, then it has end vertices, say u and v. If $w \neq u$, v is on the trail, then each time the trail comes to w through one edge, it must leave through a different edge. Therefore deg(w) is divisible by two. However, the trail passes through the end vertex and never leaves, and it begins at the start vertex without arriving there first. Therefore, these two vertices have odd degree.

   ($\Leftarrow$)If there are zero vertices of odd degree, then G is Eulerian and the Eulerian circuit is also an Eulerian trail. If there are two vertices of odd degree, say u and v, then there are two cases.

4. Prove that if every edge of a graph G lies on an odd number of cycles, then G is Eulerian.

   Solution: We only need to prove that each vertex is of even degree. Consider $f(e)$ denoting the number of cycles containing edge e. For each vertex v, if a cycle contains v, there will be exactly two of the edges in the cycle adjacent to v. Thus, the number of cycles containing v equals $\frac{1}{2}\sum_{e\ is\ anedge\ adjacent\ to\ v} f(e)$. Since $f(e)$ are all odd, we can draw the conclusion that the number of edges adjacent to v is even.

5. Let $G = K_{n_1,n_2}$.

   (a) Find conditions on n_1 and n_2 that characterize when G will have an Eulerian trail.

   (b) Find conditions that characterize when G will be Eulerian.

   Solution:

   (a) Following the theorem ("An undirected graph has an Eulerian trail if and only if exactly zero or two vertices have odd degree, and all of its vertices with nonzero degree belong to a single connected component."), there will be 3 conditions for (a):

      i. n_1 and n_2 are both even.

      ii. $n_1 = 1$ and $n_2 = 1$.

      iii. $n_1 = 2, n_2 \in \mathbb{N}_+$ or $n_2 = 2, n_1 \in \mathbb{N}_+$.

   (b) n_1 and n_2 are both even.

6. Let $G = K_{n_1,\cdots,n_k}$, where $k \geq 3$

   (a) Find conditions on $n_1, \cdots, n_k$ that characterize when G will have an Eulerian trail.

   (b) Find conditions that characterize when G will be Eulerian.

   Solution:

   (a) Following the theorem ("An undirected graph has an Eulerian trail if and only if exactly zero or two vertices have odd degree, and all of its vertices with nonzero degree belong to a single connected component."), there will be 4 conditions for (a):

      i. n_i are all even.

      ii. n_i are all odd and k is odd.

      iii. There contains exactly two 1s in $\{n_i\}$ and the others are all even.

      iv. There is exactly one $n_i = 2$, the others are all odd and k is even.

(b) i. n_i are all even.
   ii. n_i are all odd and k is odd.

7. Prove Theorem 1.23 (Harris).

   Solution: Use the similar proof in class.

8. Prove that if G is of order n and is regular, then $\alpha(G) \leq n/2$.

   Solution: Assume the conclusion is not correct. Let S be a independent set with $|S|(> n/2)$ vertices. There won't be less than $k|S| > nk/2$ (where k is the degree) edges in G. However, there are only $kn/2$ edges in G, which makes contradicion.

9. Let $G = K_{n_1,n_2}$.

   (a) Find conditions on n_1 and n_2 that characterize the traceability of G.
   (b) Find conditions that characterize the Hamiltonicity of G.

   Solution:

10. Let n be a positive integer.

    (a) Prove that $K_{n,2n,3n}$ is Hamiltonian.
    (b) Prove that $K_{n,2n,3n+1}$ is not Hamiltonian.

    Solution:

    (a) The answer is $|n_1 - n_2| \leq 1$. We can easily find a Hamilton path under the condition. When $|n_1 - n_2| \geq 2$, since the path cannot contains 2 points from the same independent set adjacent to each other, there will be one more point out of the Hamilton path.
    (b) $n_1 = n_2$

    (a) We can find a Hamiltonian circuit in $K_{n,2n,3n}$. We denote the 3 independent sets as A, B and C, which contains separately $n, 2n, 3n$ vertices. We following the path $C \to A \to C \to B \to C \to B \to C \cdots$. Starting from a point in C, choose a not chosen point in A and back to C and then choose a not chosen point in A for two times with every time back to C. We repeat the processes, and we will get the circuits.
    (b) We can know if there is a Hamilton circuit, there will always be 2 points out of C adjacent to a point in C and this is not possible since $|C| = 3n + 1 > |A| + |B|$.