# Lecture 2: Girth, Connectivity and Bipartite Graphs

Shuai Li

John Hopcroft Center, Shanghai Jiao Tong University

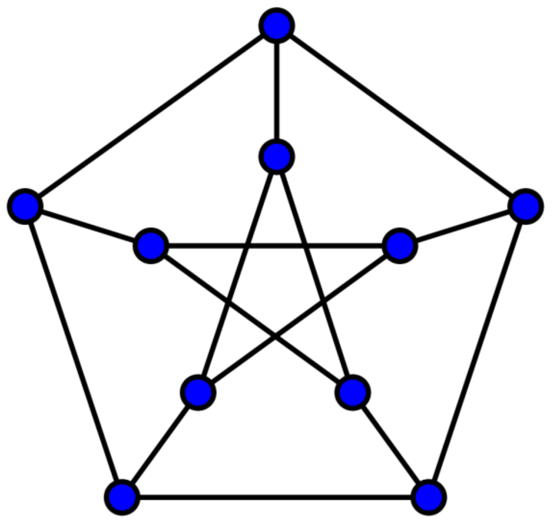
https://shuaili8.github.io

https://shuaili8.github.io/Teaching/CS445/index.html

# Girth

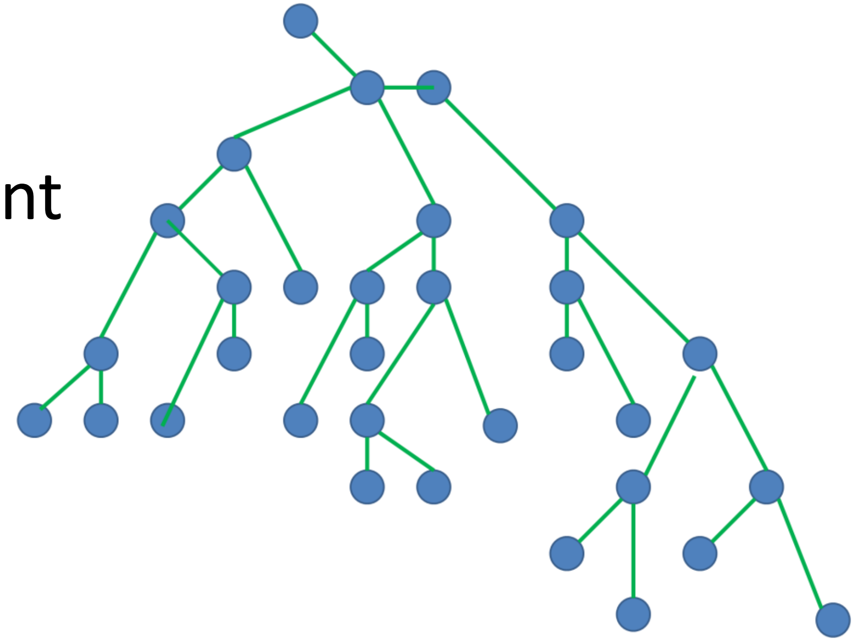
- The minimum length of a cycle in a graph $G$ is the girth $g(G)$ of $G$

- Example: The Peterson graph is the unique 5-cage
  - cubic graph (every vertex has degree 3)
  - girth = 5
  - smallest graph satisfies the above properties



# Girth (cont.)

- A tree has girth $\infty$
- Note that a tree can be colored with two different colors
- $\Longrightarrow$ A graph with large girth has small chromatic number?
- Unfortunately NO!
- Theorem (Erdős, 1959) For all $k, l$, there exists a graph $G$ with $g(G) > l$ and $\chi(G) > k$



# Girth and diameter

- Proposition (1.3.2, D) Every graph $G$ containing a cycle satisfies $g(G) \leq 2\,\mathrm{diam}(G) + 1$

- When the equality holds?

# Girth and minimal degree lower bounds graph size

- $n_0(\delta, g) := \begin{cases} 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^i, & \text{if } g = 2r + 1 \text{ is odd} \\ 2 \sum_{i=0}^{r-1} (\delta - 1)^i, & \text{if } g = 2r \text{ is even} \end{cases}$
- Exercise (Ex7, ch1, D) Let $G$ be a graph. If $\delta(G) \geq \delta \geq 2$ and $g(G) \geq g$, then $|G| \geq n_0(\delta, g)$
- Corollary (1.3.5, D) If $\delta(G) \geq 3$, then $g(G) < 2 \log_2 |G|$

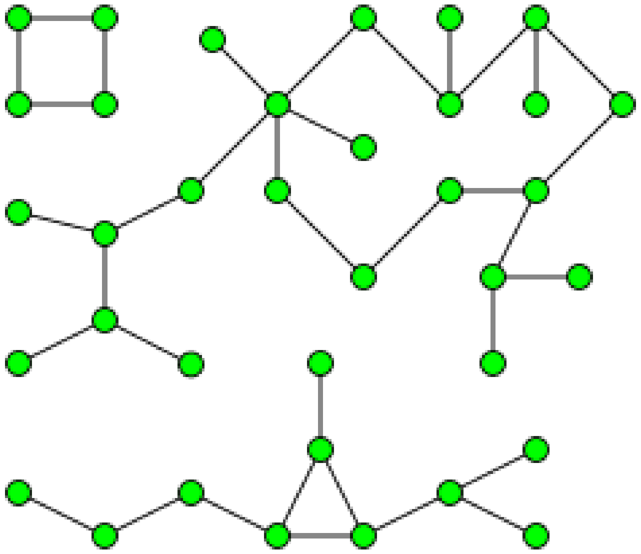
# Triangle-free upper bounds # of edges

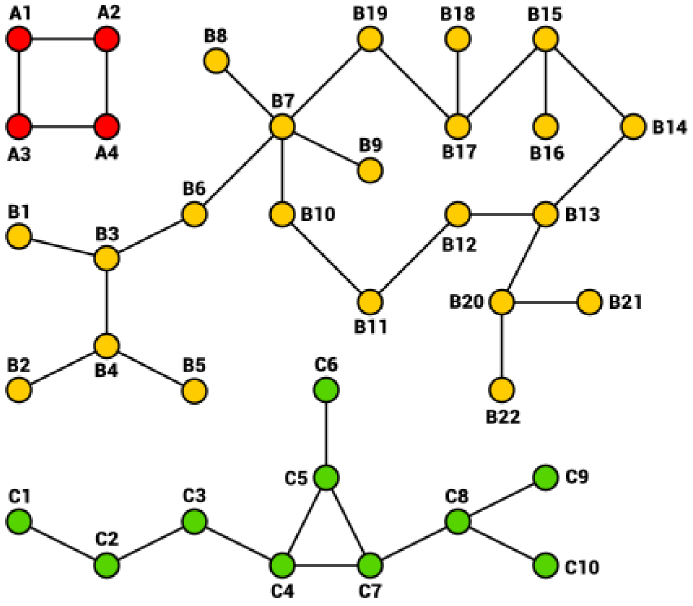
- Theorem (1.3.23, W, Mantel 1907) The maximum number of edges in an $n$-vertex triangle-free simple graph is $\lfloor n^2/4 \rfloor$

- The bound is best possible
- There is a triangle-free graph with $\lfloor n^2/4 \rfloor$ edges: $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$
- Extremal problems

# Connected, connected component

- A graph $G$ is connected if $G \neq \emptyset$ and any two of its vertices are linked by a path
- A maximal connected subgraph of $G$ is a (connected) component





# Quiz

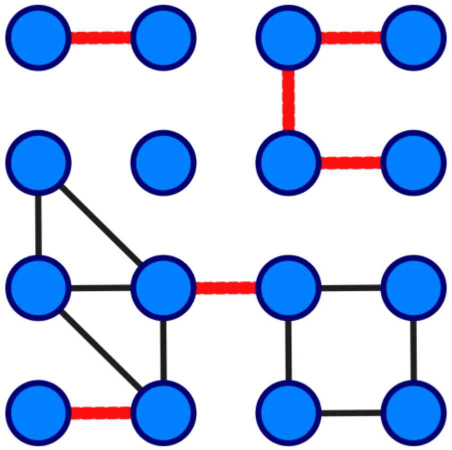
- Problem (1B, L) Suppose $G$ is a graph on $10$ vertices that is not connected. Prove that $G$ has at most $36$ edges. Can equality occur?
- More general (Ex9, S1.1.2, H) Let $G$ be a graph of order $n$ that is not connected. What is the maximum size of $G$?

# Connected vs. minimal degree

- Proposition (1.3.15, W) If $\delta(G) \geq \frac{n-1}{2}$, then $G$ is connected
- (Ex16, S1.1.2, H; 1.3.16, W)
  If $\delta(G) \geq \frac{n-2}{2}$, then $G$ need not be connected

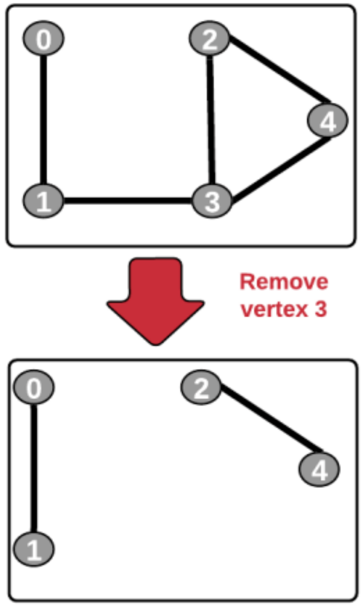
- Extremal problems
- "best possible" "sharp"

# Add/delete an edge



- Components are pairwise disjoint; no two share a vertex
- Adding an edge decreases the number of components by 0 or 1
  - ⇒ deleting an edge increases the number of components by 0 or 1
- Proposition (1.2.11, W)
  Every graph with $n$ vertices and $k$ edges has at least $n - k$ components
- An edge $e$ is called a bridge if the graph $G - e$ has more components
- Proposition (1.2.14, W)
  An edge $e$ is a bridge $\Leftrightarrow$ $e$ lies on no cycle of $G$
  - Or equivalently, an edge $e$ is not a bridge $\Leftrightarrow$ $e$ lies on a cycle of $G$

# Cut vertex and connectivity



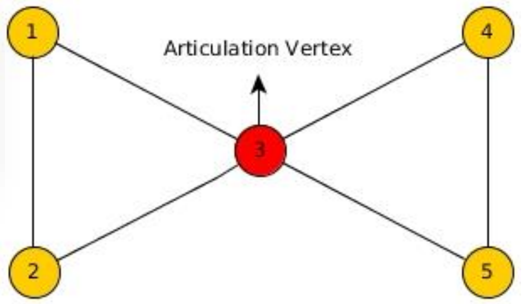

- A node $v$ is a cut vertex if the graph $G - v$ has more components
- A proper subset S of vertices is a vertex cut set if the graph $G - S$ is disconnected, or trivial (with only one vertex)
- The connectivity, $\kappa(G)$, is the minimum size of a cut set of $G$
  - The graph is $k$-connected for any $k \leq \kappa(G)$

# Connectivity properties

- $\kappa(K^n) := n - 1$
- If $G$ is disconnected, $\kappa(G) = 0$
  - $\Rightarrow$ A graph is connected $\Leftrightarrow \kappa(G) \geq 1$
- If $G$ is connected, non-complete graph of order $n$, then
$$1 \leq \kappa(G) \leq n - 2$$

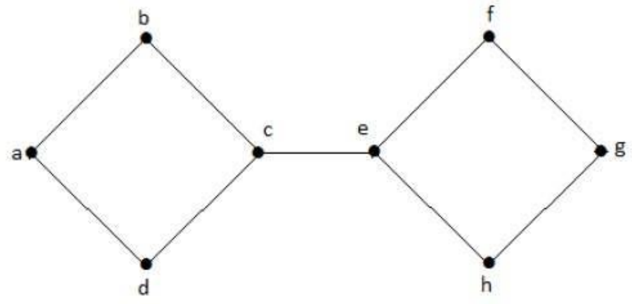
# Connectivity properties (cont.)

Proposition (1.2.14, W)
An edge $e$ is a bridge $\Leftrightarrow$ $e$ lies on no cycle of $G$
- Or equivalently, an edge $e$ is not a bridge $\Leftrightarrow$ $e$ lies on a cycle of $G$



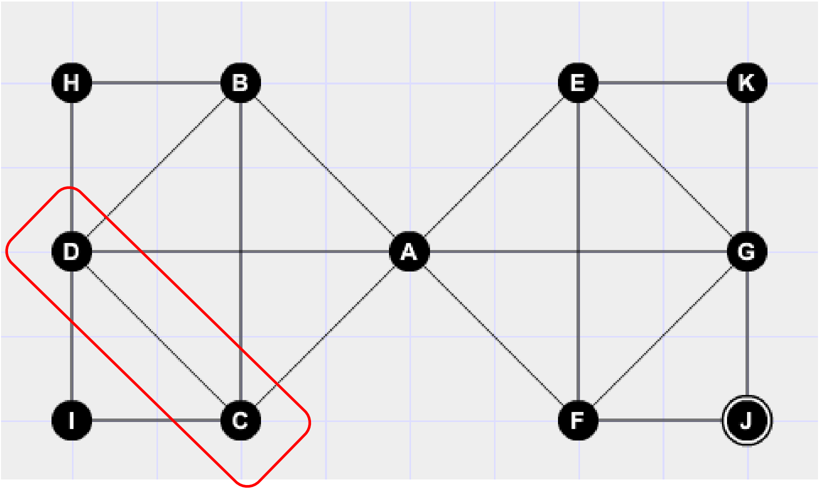

- $\kappa(G) \geq 2 \Leftrightarrow G$ is connected and has no cut vertices
- A vertex lies on a cycle $\not\Rightarrow$ it is not a cut vertex
  - $\Rightarrow$ (Ex13, S1.1.2, H) Every vertex of a connected graph $G$ lies on at least one cycle $\not\Rightarrow \kappa(G) \geq 2$
  - (Ex14, S1.1.2, H) $\kappa(G) \geq 2$ implies $G$ has at least one cycle

- (Ex12, S1.1.2, H) $G$ has a cut vertex vs. $G$ has a bridge



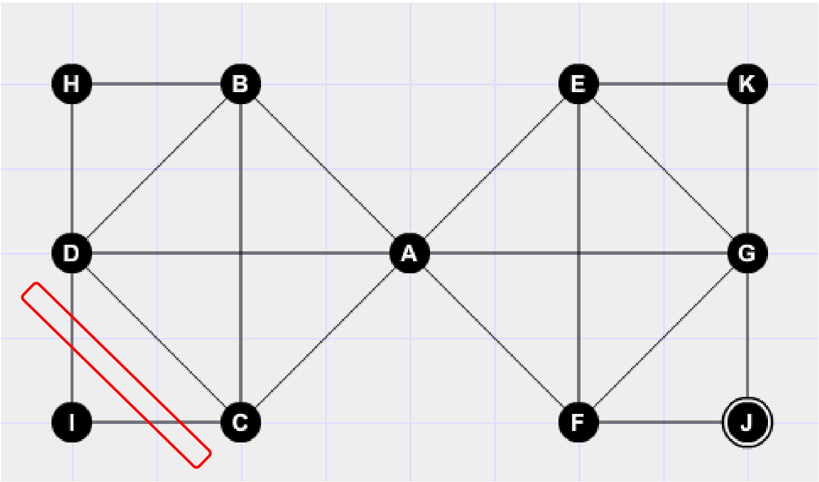
# Connectivity and minimal degree

- (Ex15, S1.1.2, H)
- $\kappa(G) \le \delta(G)$
- If $\delta(G) \ge n - 2$, then $\kappa(G) = \delta(G)$



# Edge-connectivity

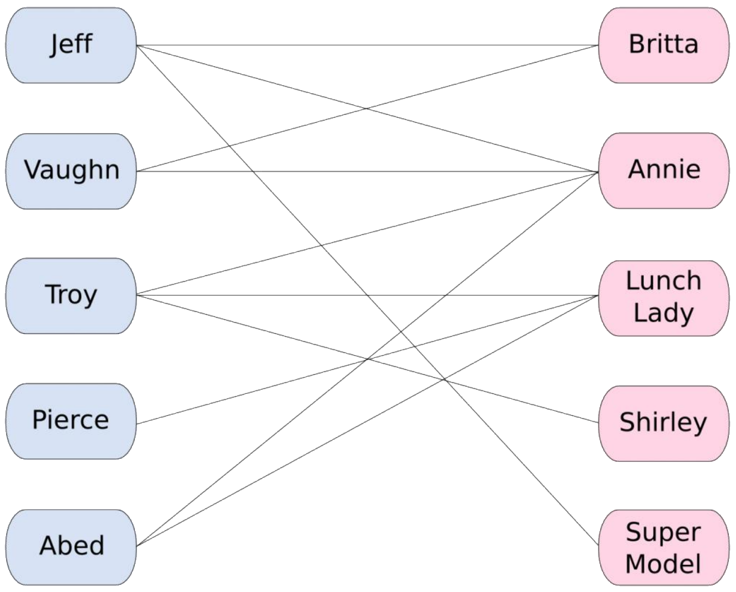
- A proper subset $F \subset E$ is edge cut set if the graph $G - F$ is disconnected
- The edge-connectivity $\lambda(G)$ is the minimal size of edge cut set
- $\lambda(G) = 0$ if $G$ is disconnected
- Proposition (1.4.2, D) If $G$ is non-trivial, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$



# Average (minimal) degree implies connectivity

- Theorem (1.4.3, D, Mader 1972) Every graph $G$ with $d(G) \geq 4k$ has a $(k+1)$-connected subgraph $H$ such that $d(H) > d(G) - 2k$.

# Bipartite graphs



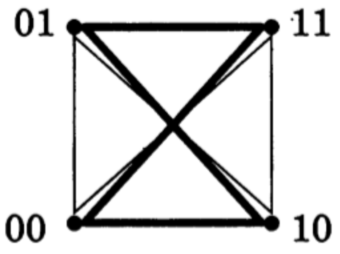

- Theorem (1.2.18, W, Kőnig 1936)
  A graph is bipartite $\Longleftrightarrow$ it contains no odd cycle

Proposition (1.2.15, W) Every closed odd walk contains an odd cycle

# Complete graph is a union of bipartite graphs

- The union of graphs $G_1, \dots, G_k$, written $G_1 \cup \cdots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$

- Consider an air traffic system with $k$ airlines
  - Each pair of cities has direct service from at least one airline
  - No airline can schedule a cycle through an odd number of cities
  - Then, what is the maximum number of cities in the system?




- Theorem (1.2.23, W) The complete graph $K_n$ can be expressed as the union of $k$ bipartite graphs $\Leftrightarrow n \leq 2^k$

# Bipartite subgraph is large

- Theorem (1.3.19, W) Every loopless graph $G$ has a bipartite subgraph with at least $|E|/2$ edges