Lecture 7: Matchings

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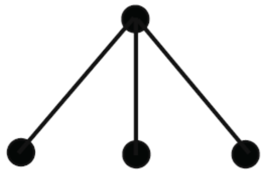
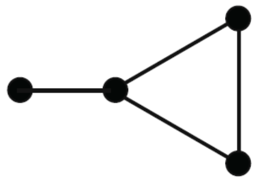
https://shuaili8.github.io

https://shuaili8.github.io/Teaching/CS445/index.html

Pattern-free & Hamiltonian



$K_{1,3}$



Z_1

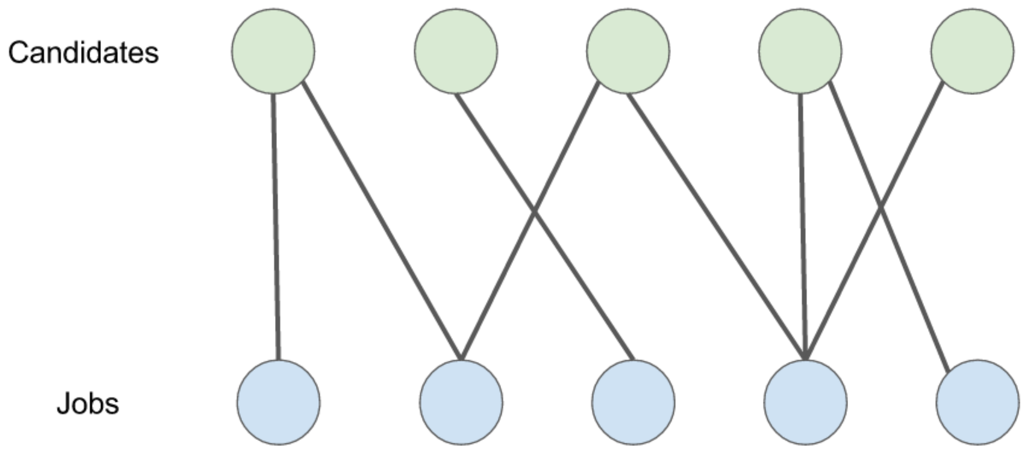
- G is H-free if G doesn't contain a copy of H as induced subgraph
- Theorem (1.25, H) If G is 2-connected and $\{K_{1,3}, Z_1\}$-free, then G is Hamiltonian

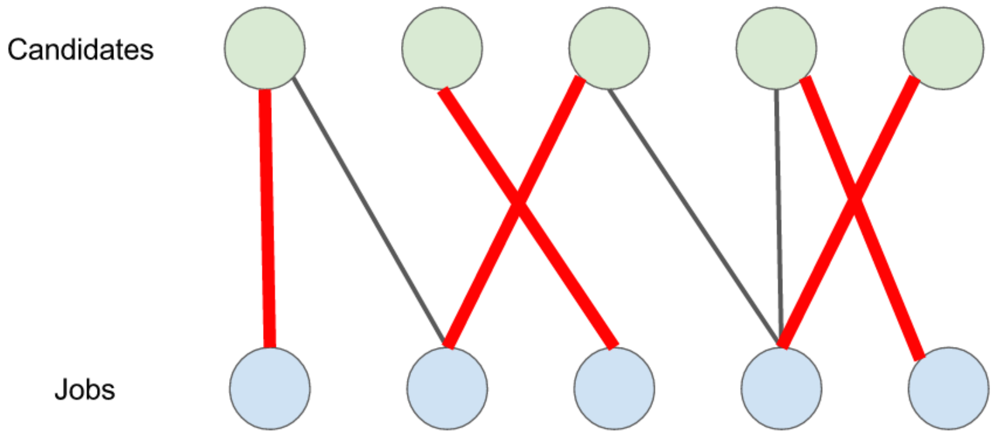
(Ex14, S1.1.2, H) $\kappa(G) \geq 2$ implies G has at least one cycle

- The condition 2-connectivity is necessary
- (Ex2, S1.4.3, H) If G is Hamiltonian, then G is 2-connected

Matchings

Motivating example





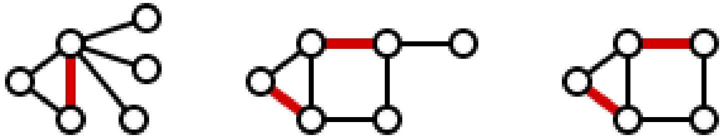
Definitions

- A matching is a set of independent edges, in which no pair shares a vertex
- The vertices incident to the edges of a matching M are M-saturated; the others are M-unsaturated
- A perfect matching in a graph is a matching that saturates every vertex
- Example (3.1.2, W) The number of perfect matchings in $K_{n,n}$ is $n!$
- Example (3.1.3, W) The number of perfect matchings in K_{2n} is

$$f_n = (2n-1)(2n-3)\cdots 1 = (2n-1)!!$$

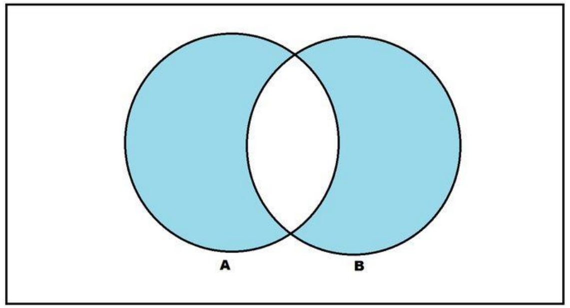
Maximal/maximum matchings 极大/最大

- A maximal matching in a graph is a matching that cannot be enlarged by adding an edge
- A maximum matching is a matching of maximum size among all matchings in the graph
- Example: P_3, P_5

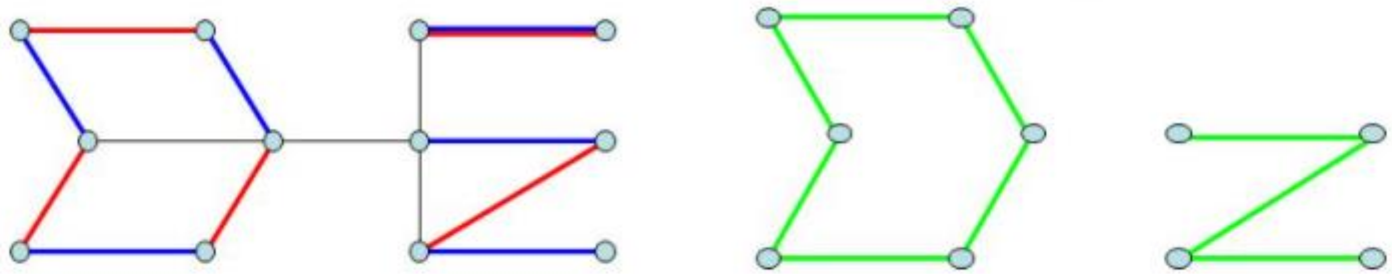


- Every maximum matching is maximal, but not every maximal matching is a maximum matching

Symmetric difference of matchings

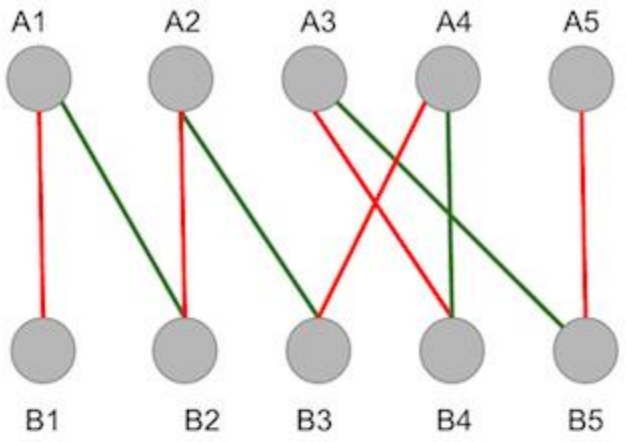


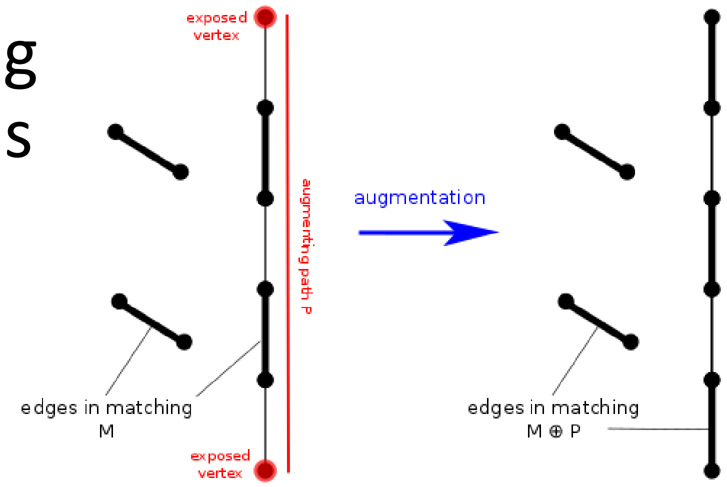
- The symmetric difference of M, M' is $M \Delta M' = (M - M') \cup (M' - M)$
- Lemma (3.1.9, W) Every component of the symmetric difference of two matchings is a path or an even cycle



Maximum matching and augmenting path

- Given a matching M, an M-alternating path is a path that alternates between edges in M and edges not in M
- An M-alternating path whose endpoints are M-unsaturated is an M-augmenting path
- Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching M in a graph G is a maximum matching in G $\Leftrightarrow$ G has no M-augmenting path





Hall’s theorem (TONCAS)

- Theorem (3.1.11, W; 1.51, H; 2.1.2, D; Hall 1935) Let G be a bipartite graph with partition X, Y.
  G contains a matching of $X \Leftrightarrow |N(S)| \geq |S|$ for all $S \subseteq X$

> Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching M in a graph G is a maximum matching in $G \Leftrightarrow G$ has no M-augmenting path

- Exercise. Read the other two proofs in Diestel.
- Corollary (3.1.13, W; 2.1.3, D) Every k-regular ($k > 0$) bipartite graph has a perfect matching

Application to SDR

- Given some family of sets X, a system of distinct representatives for the sets in X is a 'representative' collection of distinct elements from the sets of X

$$
\begin{aligned}
S_1 &= \{2,8\},\\
S_2 &= \{8\},\\
S_3 &= \{5,7\},\\
S_4 &= \{2,4,8\},\\
S_5 &= \{2,4\}.
\end{aligned}
$$

The family $X_1 = \{S_1, S_2, S_3, S_4\}$ does have an SDR, namely $\{2,8,7,4\}$. The family $X_2 = \{S_1, S_2, S_4, S_5\}$ does not have an SDR.

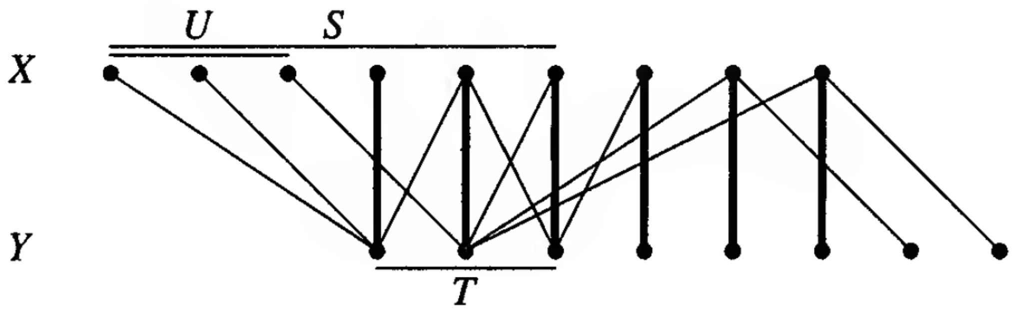
- Theorem(1.52, H) Let $S_1, S_2, \dots, S_k$ be a collection of finite, nonempty sets. This collection has SDR $\Leftrightarrow$ for every $t \in [k]$, the union of any t of these sets contains at least t elements

König-Egeváry Theorem (Min-max theorem)

- A set $U \subseteq V$ is a (vertex) cover of E if every edge in G is incident with a vertex in U
- Example:
  - Art museum is a graph with hallways are edges and corners are nodes
  - A security camera at the corner will guard the paintings on the hallways
  - The minimum set to place the cameras?
- Theorem (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931) Let G be a bipartite graph. The maximum size of a matching in G is equal to the minimum size of a vertex cover of its edges

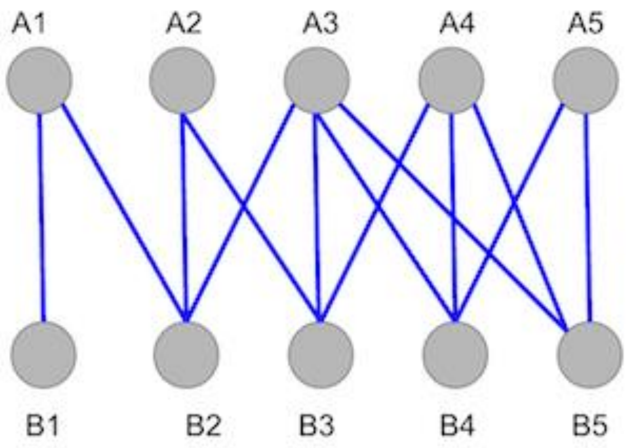
> Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching M in a graph G is a maximum matching in $G \Leftrightarrow G$ has no M-augmenting path

Find the maximum matching: Augmenting path algorithm

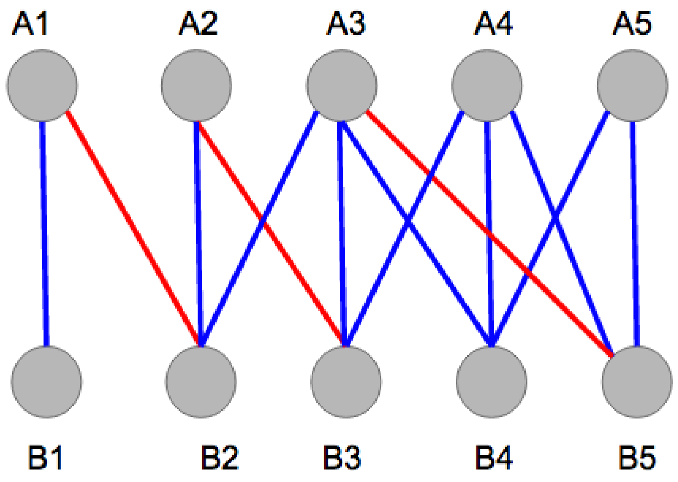


- **Input**: $G = B(X, Y)$, a matching M in G
  $U = \{M\text{-unsaturated vertices in } X\}$
- **Idea**: Explore M-alternating paths from U
  letting $S \subseteq X$ and $T \subseteq Y$ be the sets of vertices reached
- **Initialization**: $S = U, T = \emptyset$ and all vertices in S are unmarked
- **Iteration**:
  - If S has no unmarked vertex, stop and report $T \cup (X - S)$ as a minimum cover and M as a maximum matching
  - Otherwise, select an unmarked $x \in S$ to explore
    - Consider each $y \in N(x)$ such that $xy \notin M$
      - If y is unsaturated, terminate and report an M-augmenting path from U to y
      - Otherwise, $yw \in M$ for some w
        - include y in T (reached from x) and include w in S (reached from y)
    - After exploring all such edges incident to x, mark x and iterate.

Example



A random matching



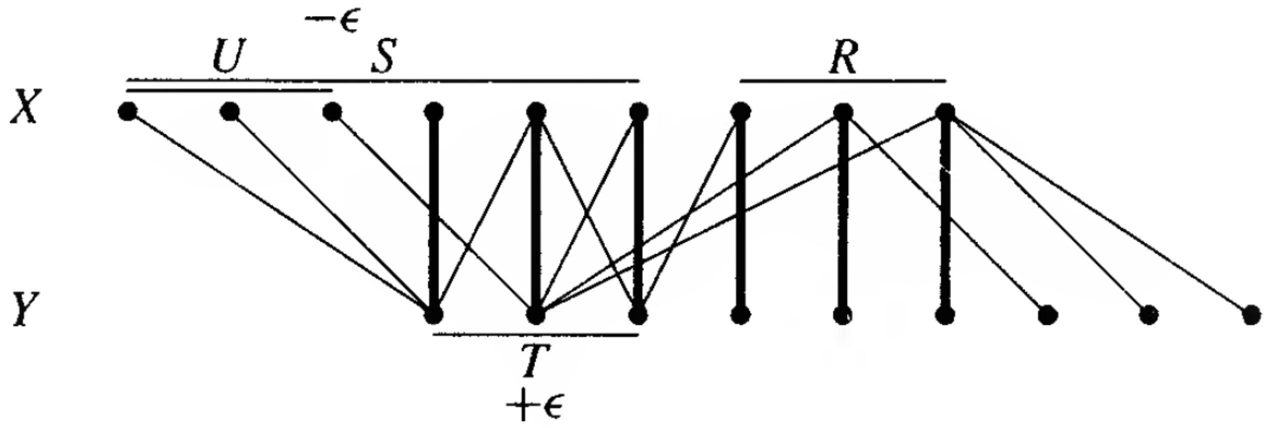
Theoretical guarantee for Augmenting path algorithm

- Theorem (3.2.2, W) Repeatedly applying the Augmenting Path Algorithm to a bipartite graph produces a matching and a vertex cover of equal size

Weighted bipartite matching

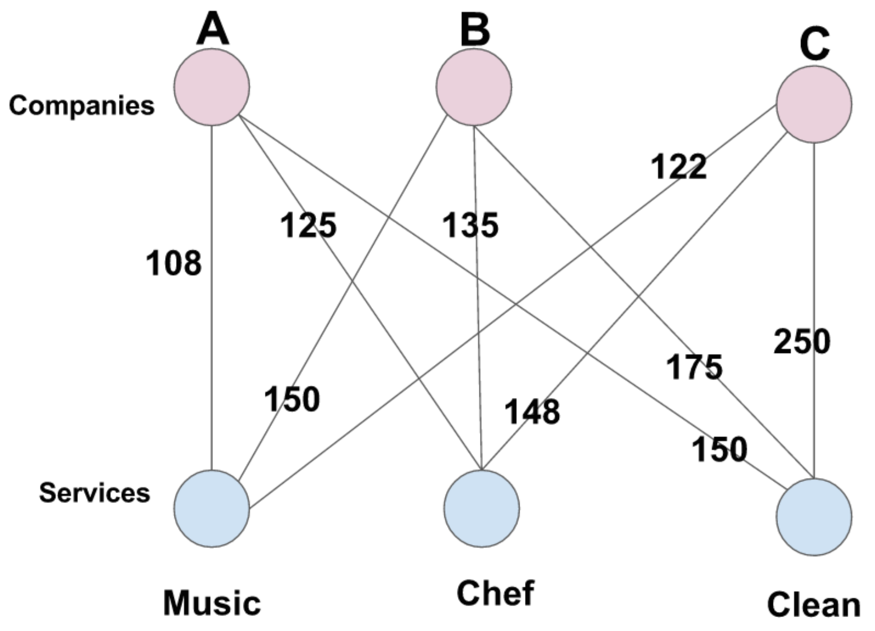
- The maximum weighted matching problem is to seek a perfect matching M to maximize the total weight $w(M)$
- A (weighted) cover is a choice of labels $u_1, \dots, u_n$ and $v_1, \dots, v_n$ such that $u_i + v_j \geq w_{i,j}$ for all i, j
  - The cost $c(u, v)$ of a cover (u, v) is $\sum_i u_i + \sum_j v_j$
  - The minimum weighted cover problem is that of finding a cover of minimum cost
- Lemma (3.2.7, W) For a perfect matching M and cover (u, v) in a weighted bipartite graph G, $c(u, v) \geq w(M)$
  $c(u, v) = w(M) \Leftrightarrow M$ consists of edges $x_i y_j$ such that $u_i + v_j = w_{i,j}$
  In this case, M and (u, v) are optimal.

Hungarian algorithm

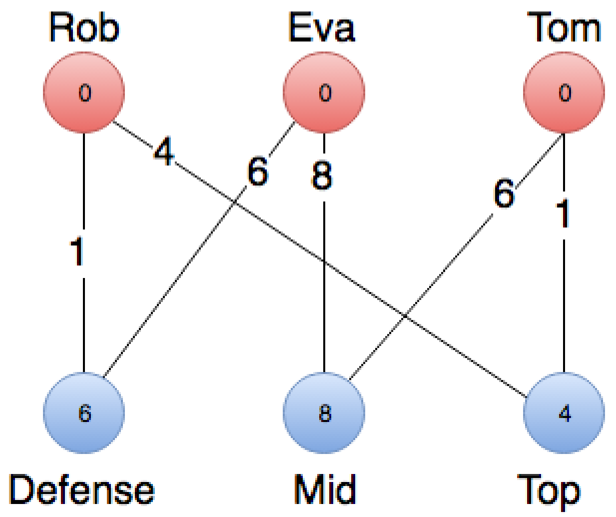


- The equality subgraph $G_{u,v}$ for a cover (u,v) is the spanning subgrapn of $K_{n,n}$ having the edges $x_i y_j$ such that $u_i + v_j = w_{i,j}$
- **Input**: Weighted $K_{n,n} = B(X,Y)$
- **Idea**: Iteratively adjusting the cover (u,v) until the equality subgraph $G_{u,v}$ has a perfect matching
- **Initialization**: Let (u,v) be a cover, such as $u_i = \max_j w_{i,j}$, $v_j = 0$
- **Iteration**: Find a maximum matching M in $G_{u,v}$
  - If M is a perfect matching, stop and report M as a maximum weight matching
  - Otherwise, let Q be a vertex cover of size $|M|$ in $G_{u,v}$
    - Let $R = X \cap Q, T = Y \cap Q$
    $$\epsilon = \min\{u_i + v_j - w_{i,j} : x_i \in X - R, y_j \in Y - T\}$$
    - Decrease u_i by ϵ for $x_i \in X - R$ and increase v_j by ϵ for $y_j \in T$
  - Form the new equality subgraph and repeat

Example



A
B
C
Companies
108
125
135
122
150
148
175
250
150
Services
Music
Chef
Clean



Rob
Eva
Tom
0
0
0
4
6
8
6
1
1
6
8
4
Defense
Mid
Top

Theoretical guarantee for Hungarian algorithm

- Theorem (3.2.11, W) The Hungarian Algorithm finds a maximum weight matching and a minimum cost cover

Matchings in general graphs

Perfect matchings

- K_{2n}, C_{2n}, P_{2n} have perfect matchings

Corollary (3.1.13, W; 2.1.3, D) Every k-regular ($k > 0$) bipartite graph has a perfect matching

- Theorem(1.58, H) If G is a graph of order $2n$ such that $\delta(G) \geq n$, then G has a perfect matching

Theorem (1.22, H, Dirac) Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian

Tutte’s Theorem (TONCAS)

- Let $q(G)$ be the number of connected components with odd order
- Theorem (1.59, H) Let G be a graph of order $n \geq 2$. G has a perfect matching $\Leftrightarrow q(G - S) \leq |S|$ for all $S \subseteq V$

Petersen’s Theorem

- Theorem (1.60, H) Every bridgeless, 3-regular graph contains a perfect matching