

Lecture 7: Coloring

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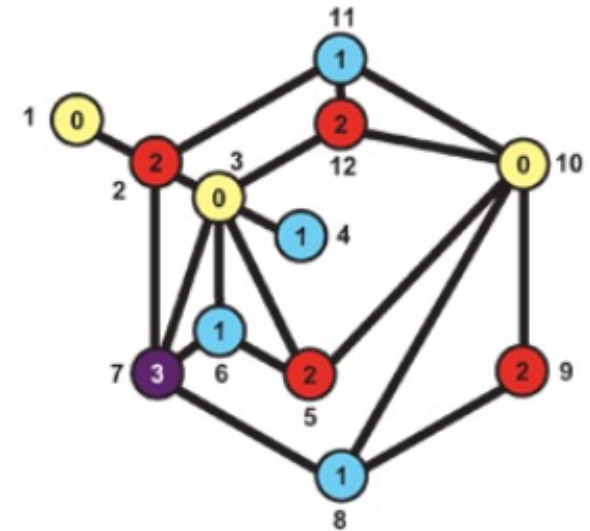
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<https://shuaili8.github.io>

<https://shuaili8.github.io/Teaching/CS445/index.html>

Motivation: Scheduling and coloring

- University examination timetabling
 - Two courses linked by an edge if they have the same students
- Meeting scheduling
 - Two meetings are linked if they have same member



Definitions

- Given a graph G and a positive integer k , a **k -coloring** is a function $K: V(G) \rightarrow \{1, \dots, k\}$ from the vertex set into the set of positive integers less than or equal to k . If we think of the latter set as a set of k “colors,” then K is an assignment of one color to each vertex.
- We say that K is a **proper k -coloring** of G if for every pair u, v of adjacent vertices, $K(u) \neq K(v)$ — that is, if adjacent vertices are colored differently. If such a coloring exists for a graph G , we say that G is **k -colorable**
- In a proper coloring, each color class is an independent set. Then G is k -colorable $\iff V(G)$ is the union of k independent sets

Chromatic number

- Given a graph G , the **chromatic number** of G , denoted by $\chi(G)$, is the smallest integer k such that G is k -colorable. G is said to be **k -chromatic**

- Examples

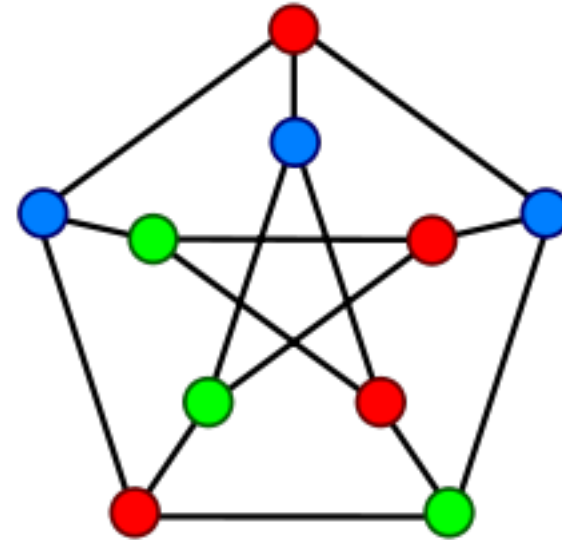
$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd,} \end{cases}$$

$$\chi(P_n) = \begin{cases} 2 & \text{if } n \geq 2, \\ 1 & \text{if } n = 1, \end{cases}$$

$$\chi(K_n) = n,$$

$$\chi(E_n) = 1, \leftarrow \text{Empty graph}$$

$$\chi(K_{m,n}) = 2.$$



- (Ex5, S1.6.1, H) A graph G of order at least two is bipartite \Leftrightarrow it is 2-colorable

Theorem (1.2.18, W, König 1936)

A graph is bipartite \Leftrightarrow it contains no odd cycle

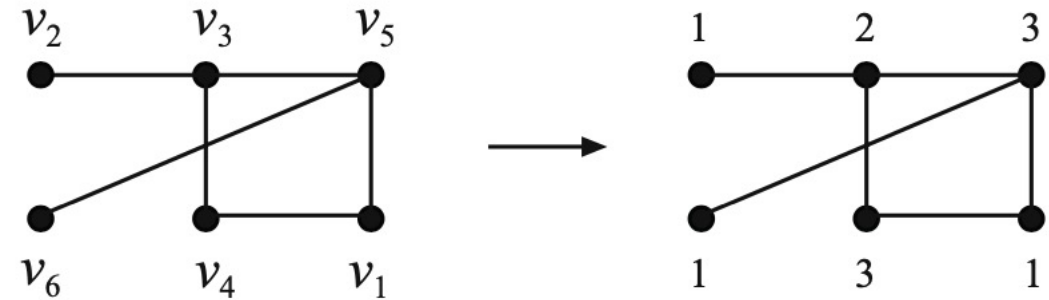
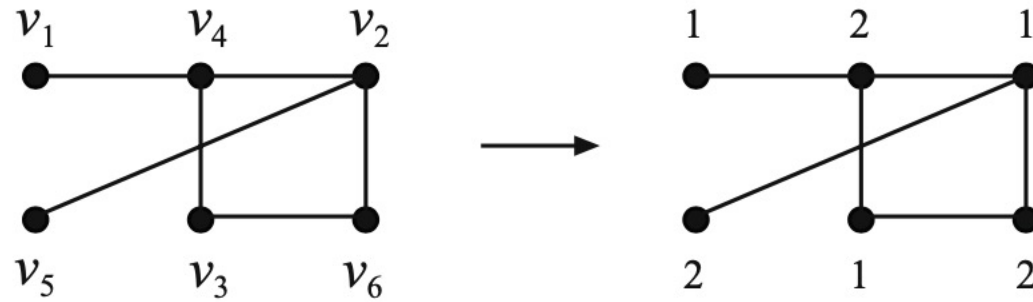
Bounds on Chromatic number

- **Theorem** (1.41, H) For any graph G of order n , $\chi(G) \leq n$
- It is tight since $\chi(K_n) = n$
- $\chi(G) = n \iff G = K_n$

Greedy algorithm

- First label the vertices in some order—call them v_1, v_2, \dots, v_n
- Next, order the available colors $(1, 2, \dots, n)$ in some way
 - Start coloring by assigning color 1 to vertex v_1
 - If v_1 and v_2 are adjacent, assign color 2 to vertex v_2 ; otherwise, use color 1
 - To color vertex v_i , use the first available color that has not been used for any of v_i 's previously colored neighbors

Examples: Different orders result in different number of colors

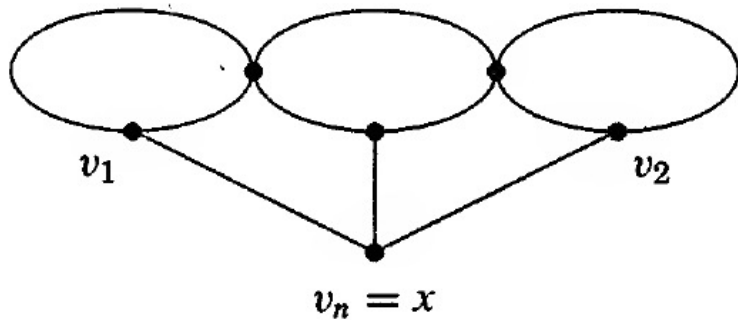


Bound using the greedy algorithm

- **Theorem** (1.42, H) For any graph G , $\chi(G) \leq \Delta(G) + 1$
The equality is obtained for complete graphs and odd cycles

Brooks's theorem

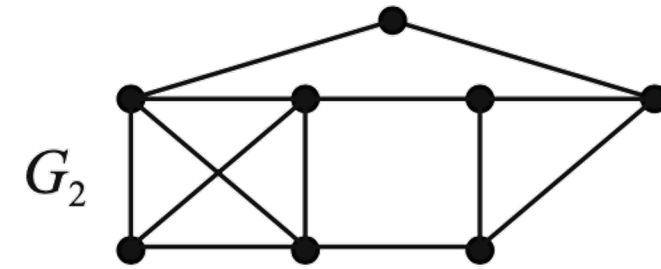
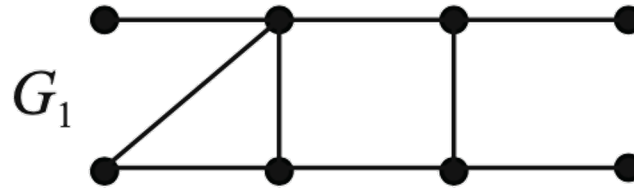
- **Theorem** (1.43, H; 5.1.22, W; 5.2.4, D; Brooks 1941)
If G is a connected graph that is neither an odd cycle or a complete graph, then $\chi(G) \leq \Delta(G)$



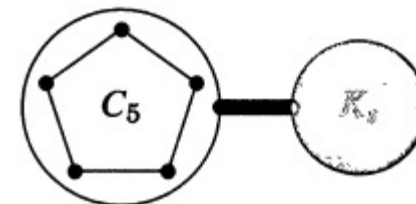
- \Rightarrow The Petersen graph is 3-colorable

Chromatic number and clique number

- The **clique number** $\omega(G)$ of a graph is defined as the order of the largest complete graph that is a subgraph of G
- Example: $\omega(G_1) = 3, \omega(G_2) = 4$



- **Theorem** (1.44, H; 5.1.7, W) For any graph G , $\chi(G) \geq \omega(G)$
- Example (5.1.8, W) For $G = C_{2r+1} \vee K_s$, $\chi(G) > \omega(G)$



Chromatic number and independence number

- **Theorem** (1.45, H; 5.1.7, W; Ex6, S1.6.2, H) For any graph G of order n ,

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G)$$

The **independence number** of a graph G , denoted as $\alpha(G)$, is the largest size of an independent set

In a proper coloring, each color class is an independent set. Then G is k -colorable $\Leftrightarrow V(G)$ is the union of k independent sets

Extremal properties for k -chromatic graphs

- **Proposition** (5.2.5, W) Every k -chromatic graph with n vertices has **at least** $\binom{k}{2}$ edges
 - Equality holds for a complete graph plus isolated vertices.

In a proper coloring, each color class is an independent set. Then G is k -colorable $\Leftrightarrow V(G)$ is the union of k independent sets

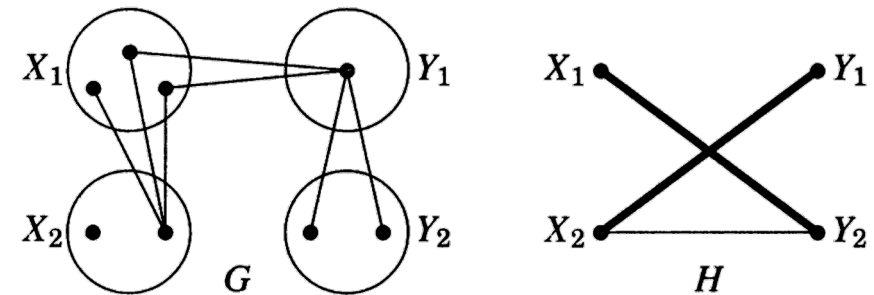
- The **Turán graph** $T_{n,r}$ is the complete r -partite graph with n vertices whose partite sets differ by at most 1 vertex
 - Every partite set has size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$
- **Lemma** (5.2.8, W) Among simple r -partite (that is, r -colorable) graphs with n vertices, the Turán graph is the unique graph with the **most** edges
- **Turán's Theorem** (5.2.9, W; Turán 1941) Among the n -vertex simple K_{r+1} -free graphs, $T_{n,r}$ has the maximum number of edges

Color-critical

- If $\chi(H) < \chi(G) = k$ for every proper subgraph H , then G is **color-critical** or **k -critical**
- K_2 is the only 2-critical graph
 K_1 is the only 1-critical graph
- (5.2.12, W) A graph with no isolated vertices is color-critical $\Leftrightarrow \chi(G - e) < \chi(G)$ for every edge $e \in E(G)$
- **Proposition** (5.2.13, W) Let G be a k -critical graph
 - (a) For every $v \in V(G)$, there is a proper coloring such that v has a unique color and other $k - 1$ colors all appear on $N(v)$
 $\Rightarrow \delta(G) \geq k - 1$
 - (b) For every $e \in E(G)$, every proper $(k - 1)$ -coloring of $G - e$ gives the same color to the two endpoints of e

Color-critical has edge-connectivity

- **Theorem** (5.2.16, W; Dirac 1953) Every k -critical graph is $(k - 1)$ -edge-connected
- **Lemma** (5.2.15, W; Kainen) Let G be a graph with $\chi(G) > k$ and let X, Y be a partition of $V(G)$. If $G[X]$ and $G[Y]$ are k -colorable, then the edge cut $[X, Y]$ has at least k edges

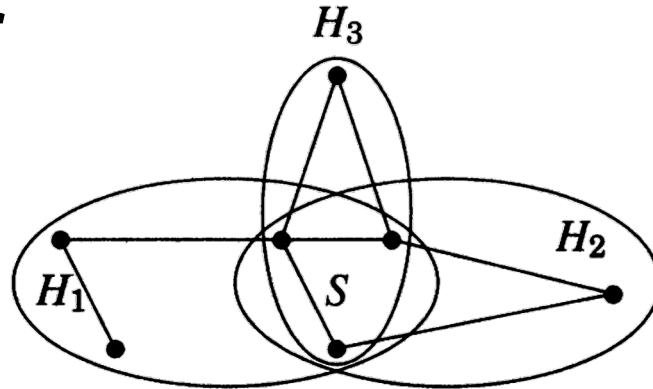


Theorem (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931)
Let G be a bipartite graph. The **maximum** size of a matching in G is equal to the **minimum** size of a vertex cover of its edges

Remark (4.1.8, W) Every minimal disconnecting set of edges is an edge cut

Color-critical and vertex cut set

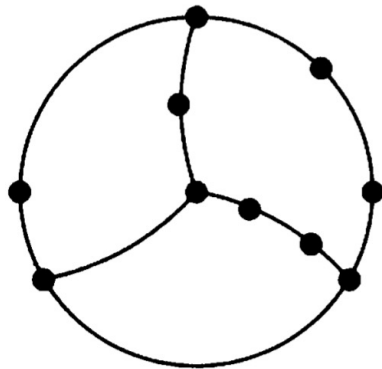
- Let S be a set of vertices in a graph G . An **S -lobe** of G is an induced subgraph of G whose vertex set consists of S and the vertices of a component in $G - S$



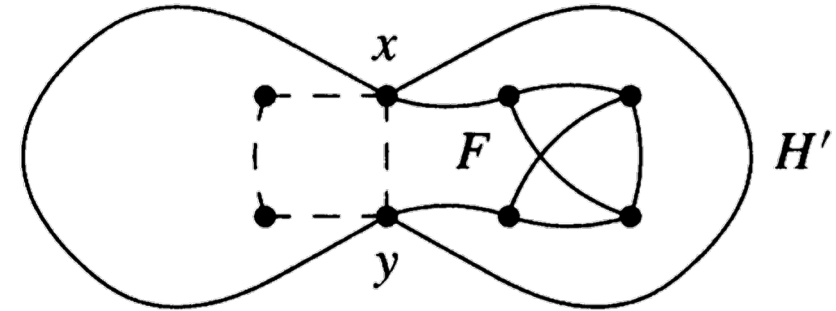
- Proposition** (5.2.18, W) If G is k -critical, then G has no clique cutset. In particular, if G has a cutset $S = \{x, y\}$, then x, y are non-adjacent and G has an S -lobe H such that $\chi(H + xy) = k$

Chromatic number 4 has a K_4 -subdivision

- **Theorem** (5.2.20, W; Dirac 1952) Every graph with chromatic number at least 4 contains a K_4 -subdivision



a subdivision of K_4



Proposition (5.2.18, W) If G is k -critical, then G has no clique cutset. In particular, if G has a cutset $S = \{x, y\}$, then x, y are non-adjacent and G has an S -lobe H such that $\chi(H + xy) = k$

Lemma (4.2.3, W; Expansion Lemma) If G is a k -connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G , then G' is k -connected



Hajós' conjecture

- Hajós' conjecture [1961]: Every k -chromatic graph contains a subdivision of K_k
- $k = 2$: Every 2-chromatic graph has a nontrivial path
- $k = 3$: Every 3-chromatic graph has a cycle
- It is open for $k = 5, 6$
- **Exercise** (Ex W) It is false for $k \geq 7$

Chromatic Polynomials

Definition and examples

- It is brought up by George David Birkhoff in 1912 in an attempt to prove the four color theorem
- Define $\chi(G; k)$ to be the number of different colorings of a graph G using at most k colors
- Examples:
 - How many different colorings of K_4 using 4 colors?
 - $4 \times 3 \times 2 \times 1$
 - $\chi(K_4; 4) = 24$
 - How many different colorings of K_4 using 6 colors?
 - $6 \times 5 \times 4 \times 3$
 - $\chi(K_4; 6) = 360$
 - How many different colorings of K_4 using 2 colors?
 - 0
 - $\chi(K_4; 2) = 0$

Examples

- If $k \geq n$

$$\chi(K_n; k) = k(k-1) \cdots (k-n+1)$$

- If $k < n$

$$\chi(K_n; k) = 0$$

- G is k -colorable $\Leftrightarrow \chi(G) \leq k \Leftrightarrow \chi(G; k) > 0$
- $\chi(G) = \min\{k \geq 1: \chi(G; k) > 0\}$

Chromatic recurrence

- $G - e$ and G/e

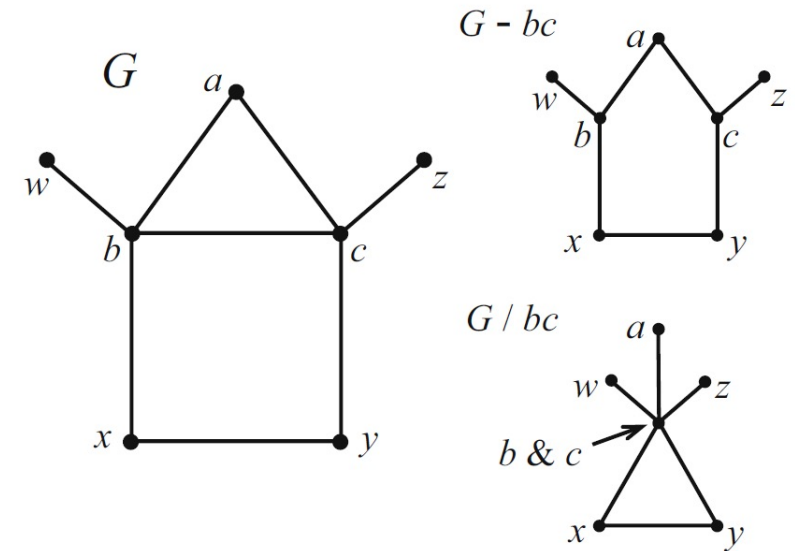


FIGURE 1.98. Examples of the operations.

- **Theorem** (1.48, H; 5.3.6, W) Let G be a graph and e be any edge of G . Then

$$\chi(G; k) = \chi(G - e; k) - \chi(G/e; k)$$

Use chromatic recurrence to compute $\chi(G; k)$

- Example: Compute $\chi(P_3; k) = k^4 - 3k^3 + 3k^2 - k$
- Check: $\chi(P_3; 1) = 0, \chi(P_3; 2) = 2$



FIGURE 1.102. Two 2-colorings of P_3

- Example: What is $\chi(K_n - e; k)$?

More examples

- Path P_{n-1} has $n - 1$ edges (n vertices)

$$\chi(P_{n-1}; k) = k(k - 1)^{n-1}$$

- Any tree T on n vertices

$$\chi(T; k) = k(k - 1)^{n-1}$$

- Cycle C_n

$$\chi(C_n; k) = (k - 1)^n + (-1)^n(k - 1)$$

- When n is odd, $\chi(C_n; 2) = 0, \chi(C_n; 3) > 0$
- When n is even, $\chi(C_n; 2) > 0$

Properties of chromatic polynomials

- **Theorem** (1.49, H; Ex 3, S1.6.4, H) Let G be a graph of order n
 - $\chi(G; k)$ is a polynomial in k of degree n
 - The leading coefficient of $\chi(G; k)$ is 1
 - The constant term of $\chi(G; k)$ is 0
 - If G has i components, then the coefficients of k^0, \dots, k^{i-1} are 0
 - G is connected \Leftrightarrow the coefficient of k is nonzero
 - The coefficients of $\chi(G; k)$ alternate in sign
 - The coefficient of the k^{n-1} term is $-|E(G)|$
 - A graph G is a tree $\Leftrightarrow \chi(G; k) = k(k-1)^{n-1}$

\Leftrightarrow (**Theorem** 1.10, 1.12, H) T is connected with $n - 1$ edges
 - A graph G is complete $\Leftrightarrow \chi(G; k) = k(k-1) \cdots (k-n+1)$

Simplicial elimination ordering

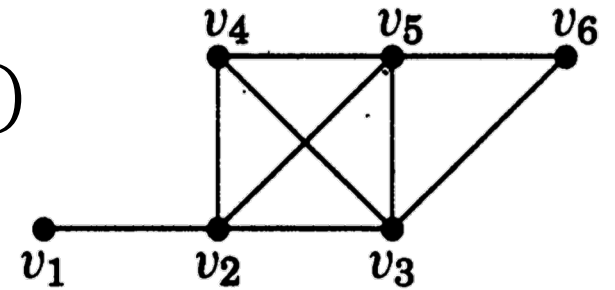
- Roots for the chromatic polynomials?
Fundamental theorem of algebra
- A vertex of G is **simplicial** if its neighborhood in G induces a clique
- A **simplicial elimination ordering** is an ordering v_n, \dots, v_1 for deletion of vertices s.t. each vertex v_i is a simplicial vertex of the graph reduced by $\{v_1, \dots, v_i\}$
- Chromatic polynomials
If we have colored v_1, \dots, v_{i-1} , then there are $k - d(i)$ ways to color v_i where $d(i) = |N(v_i) \cap \{v_1, \dots, v_{i-1}\}|$. Thus

$$\chi(G; k) = \prod_{i=1}^n (k - d(i))$$

Nice factorization property!

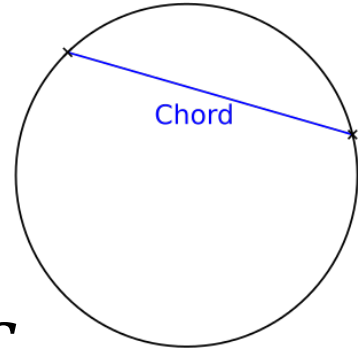
Examples

- In a tree, a simplicial elimination ordering is a successive deletion of leaves
- Example (5.3.13, W) v_6, \dots, v_1 is a simplicial elimination ordering. The values $d(i)$ are 0,1,1,2,3,2. Thus the chromatic polynomial is $k(k-1)(k-1)(k-2)(k-3)(k-2)$

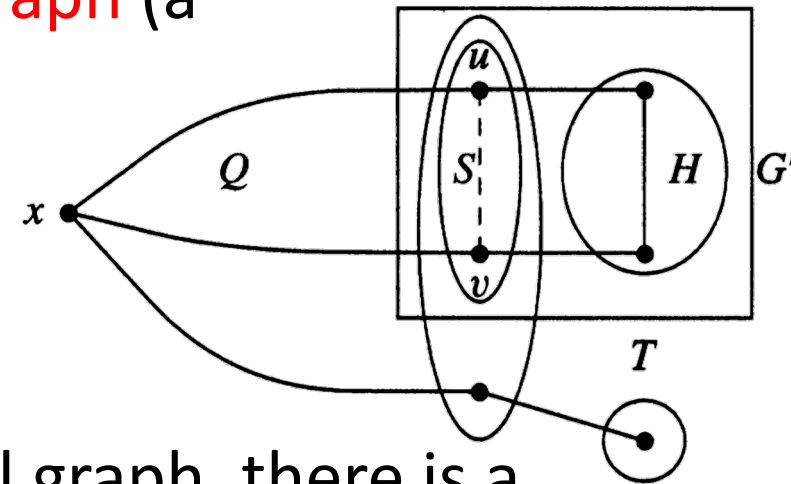


- **Exercise** There exists some graph without simplicial elimination ordering but has a nice factorization form for chromatic polynomial

Chordal graphs



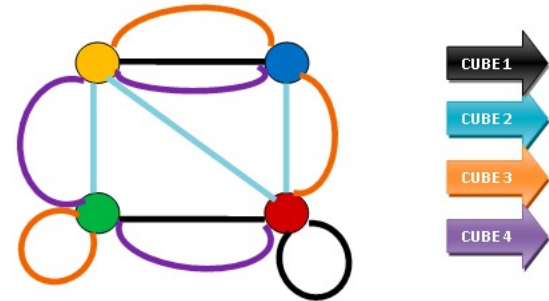
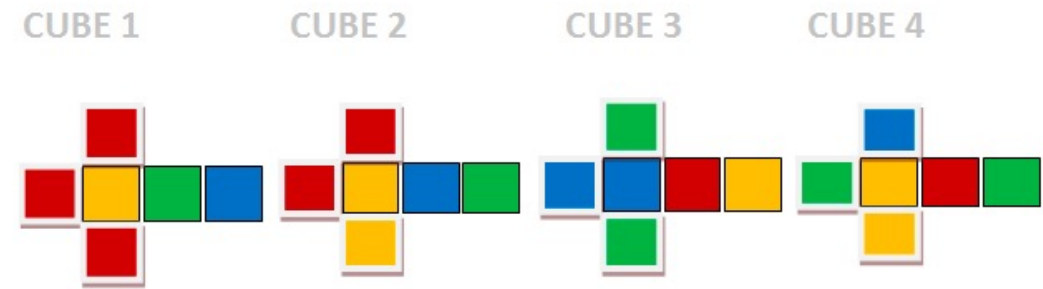
- A **chord** of a cycle C is an edge not in C whose endpoints lie in C
- A **chordless cycle** in G is a cycle of length at least 4 that has no chord
- **Theorem** (5.3.17, W; Dirac 1961) A simple graph has a simplicial elimination ordering \Leftrightarrow it is a **chordal graph** (a simple graph without chordless cycle)
- TONCAS!
- Further $\chi(C_n; k) = (k - 1)^n + (-1)^n(k - 1)$ does not have a degree-1 decomposition
- **Lemma** (5.3.16, W) For every vertex x in a chordal graph, there is a simplicial vertex of G among the vertices farthest from x



Proof Using Coloring

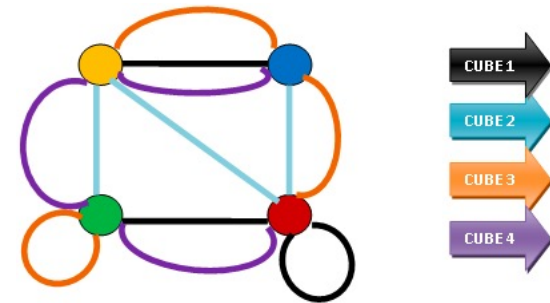
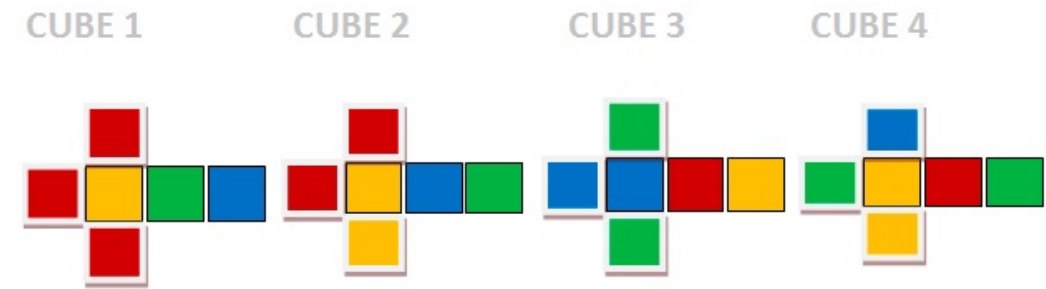
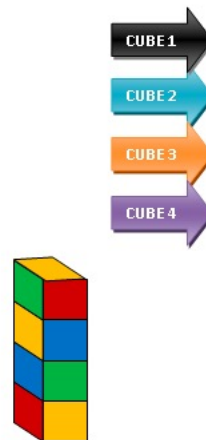
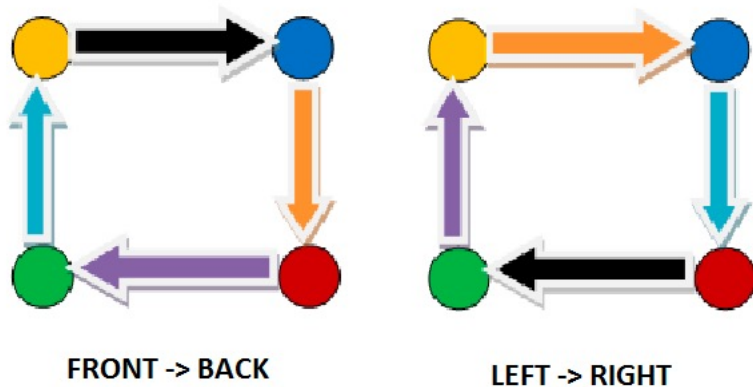
Example -- Instant Insanity 四色方柱问题 (1.2, L)

- **Problem** make a stack of these cubes so that all four colors appear on each of the four sides of the stack
- An edge indicates that the two adjacent colors occur on opposite faces of the cube
- **Problem** necessary to find two subgraphs s.t.
 - are regular of degree 2
 - four edges from each cube
 - no edge in common



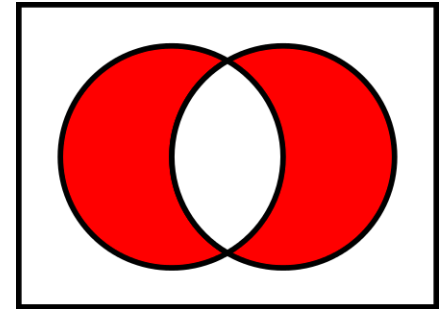
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An example about sets (1E, L)

- Let A_1, \dots, A_n be n distinct subsets of the n -set $N := \{1, \dots, n\}$. Show that there is an element $x \in N$ such that the sets $A_i \setminus \{x\}$, $1 \leq i \leq n$, are all distinct
- **Proof** Consider a graph with vertices A_1, \dots, A_n .
 - An edge of 'color' x between A_i and A_j iff $A_i \Delta A_j = \{x\}$
 - Then the problem is equivalent to find y s.t. no color y
 - Notice that a cycle in this graph must have even length and each color appears even times
 - Then we can remove an edge if there is an edge with same color
 - Thus the number of colors remain the same and no cycle exists
 - By tree property, the number of edges is at most $n - 1$



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Summary

- Coloring, proper coloring, chromatic #
- Brooks's theorem
- Chromatic # vs. clique/independence #
- Turán graph
- Color-critical, w/ vertex/edge-connectivity
- Chromatic number 4 has a K_4 -subdivision
- Chromatic polynomials, chromatic recurrence, path/trees/cycles, properties
- Simplicial elimination ordering, chordal graph, TONCAS
- Examples of proof with coloring

Questions?