

# Lecture: Ramsey Theory

Shuai Li

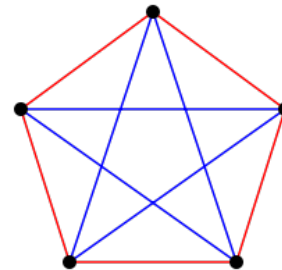
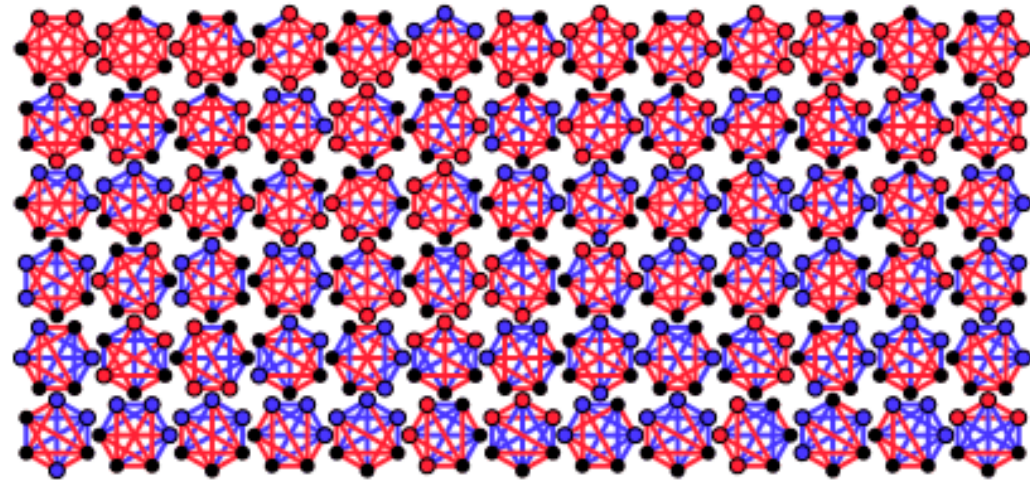
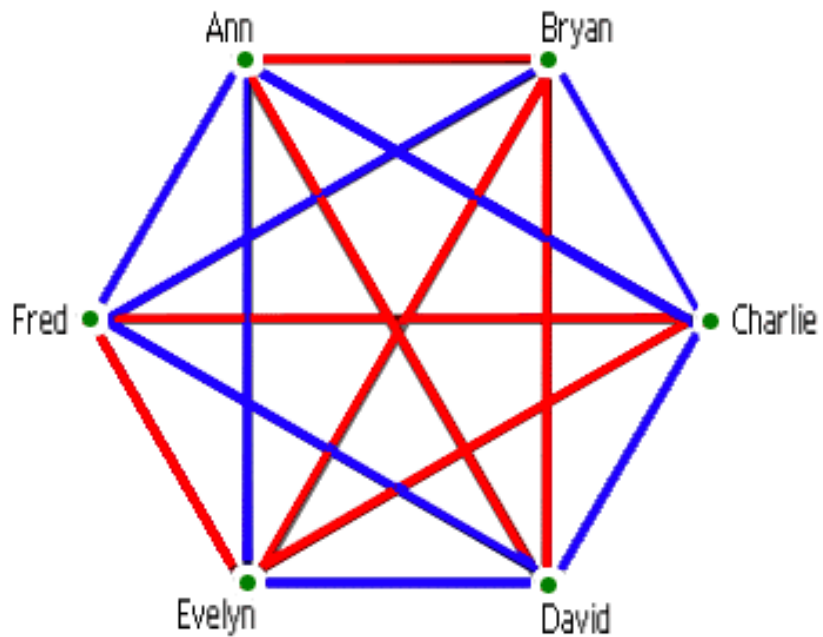
John Hopcroft Center, Shanghai Jiao Tong University

<https://shuaili8.github.io>

<https://shuaili8.github.io/Teaching/CS445/index.html>

# The friendship riddle

- Does every set of six people contain three mutual acquaintances or three mutual strangers?



$$R(3,3)=6$$

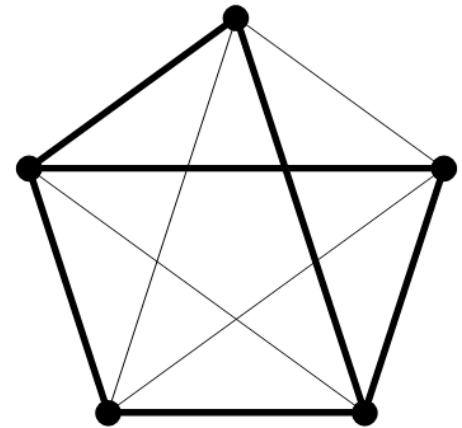
$$R(3,4)=R(4,3)=9$$

$$R(3,5)=R(5,3)=14$$

$$R(3,6)=R(6,3)=18$$

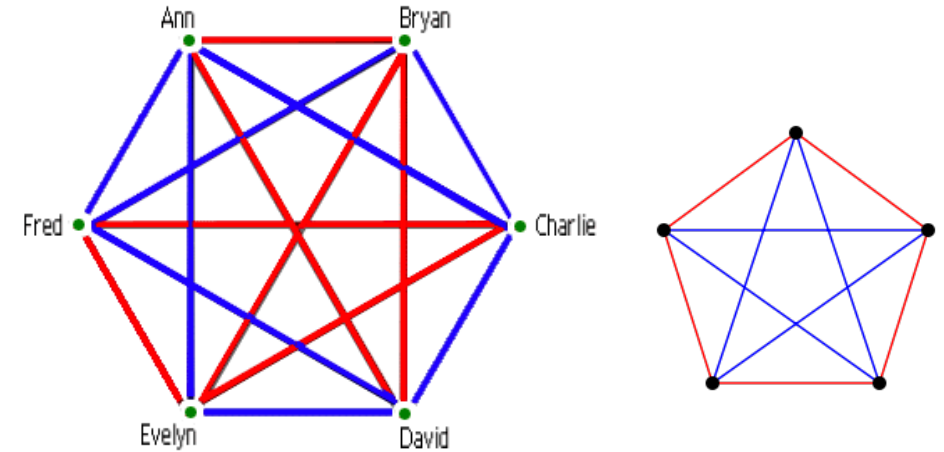
# (classical) Ramsey number

- A **2-coloring of the edges** of a graph  $G$  is any assignment of one of two colors of each of the edges of  $G$
- Let  $p$  and  $q$  be positive integers. The (classical) **Ramsey number** associated with these integers, denoted by  $R(p, q)$ , is defined to be the smallest integer  $n$  such that every 2-coloring of the edges of  $K_n$  either contains a red  $K_p$  or a blue  $K_q$  as a subgraph
- It is a typical problem of extremal graph theory



# Examples

- $R(1,3) = 1$
- (Ex2, S1.8.1, H)  $R(1, k) = 1$
- $R(2,4) = 4$
- (Ex3, S1.8.1, H)  $R(2, k) = k$
- **Theorem** (1.61, H)  $R(3,3) = 6$

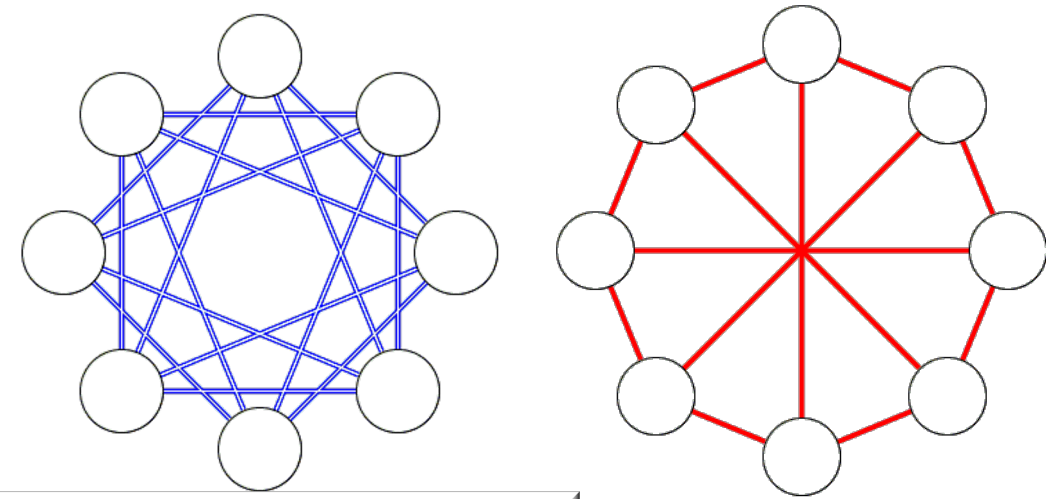


# Examples (cont.)

- **Theorem** (1.62, H)  $R(3,4) = 9$

**Theorem** A finite graph  $G$  has an even number of vertices with odd degree

- (Ex4, S1.8.1, H)  $R(p, q) = R(q, p)$



Values / known bounding ranges for Ramsey numbers  $R(r, s)$  (sequence [A212954](#) in the [OEIS](#))

$r \backslash s$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2		2	3	4	5	6	7	8	9	10
3			6	9	14	18	23	28	36	40–42
4				18	25 <sup>[10]</sup>	36–41	49–61	59 <sup>[14]</sup> –84	73–115	92–149
5					43–48	58–87	80–143	101–216	133–316	149 <sup>[14]</sup> –442
6						102–165	115 <sup>[14]</sup> –298	134 <sup>[14]</sup> –495	183–780	204–1171
7							205–540	217–1031	252–1713	292–2826
8								282–1870	329–3583	343–6090
9									565–6588	581–12677
10										798–23556

# Bounds on Ramsey numbers

- **Theorem** (1.64, H; 2.28, H) If  $q \geq 2, q \geq 2$ , then

$$R(p, q) \leq R(p - 1, q) + R(p, q - 1)$$

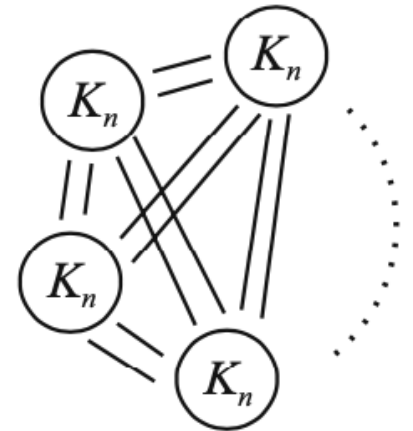
Furthermore, if both terms on the RHS are even, then the equality is strict

**Theorem** A finite graph  $G$  has an even number of vertices with odd degree

- **Theorem** (1.63, H; 2.29, H)  $R(p, q) \leq \binom{p + q - 2}{p - 1}$
- **Theorem** (1.65, H) For integer  $q \geq 3$ ,  $R(3, q) \leq \frac{q^2 + 3}{2}$
- **Theorem** (1.66, H; Erdos and Szekeres 1935)  
If  $p \geq 3$ ,  $R(p, p) > \lfloor 2^{p/2} \rfloor$

# Graph Ramsey Theory

- Given two graphs  $G$  and  $H$ , define the graph **Ramsey number  $R(G, H)$**  to be the smallest value of  $n$  such that any 2-coloring of the edges of  $K_n$  contains either a red copy of  $G$  or a blue copy of  $H$ 
  - The classical Ramsey number  $R(p, q)$  would in this context be written as  $R(K_p, K_q)$
- Theorem** (1.67, H) If  $G$  is a graph of order  $p$  and  $H$  is a graph of order  $q$ , then  $R(G, H) \leq R(p, q)$
- Theorem** (1.68, H) Suppose the order of the largest component of  $H$  is denoted as  $C(H)$ .  
Then  $R(G, H) \geq (\chi(G) - 1)(C(H) - 1) + 1$



# Graph Ramsey Theory (cont.)

- **Theorem** (1.69, H)  $R(T_m, K_n) = (m - 1)(n - 1) + 1$

**Theorem** (1.45, H; Ex6, S1.6.2, H) For any graph  $G$  of order  $n$ ,

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G)$$

A graph  $G$  is called  $k$ -critical if  $\chi(G) = k$  and  $\chi(G - v) < k$  for each vertex  $v$  of  $G$ .

- (a) Find all 1-critical and 2-critical graphs.
- (b) Give an example of a 3-critical graph.
- (c) If  $G$  is  $k$ -critical, then show that  $G$  is connected.
- (d) If  $G$  is  $k$ -critical, then show that  $\delta(G) \geq k - 1$ .
- (e) Find all of the 3-critical graphs. Hint: Use part (d).

**Theorem** (1.16, H) Let  $T$  be a tree of order  $k + 1$  with  $k$  edges. Let  $G$  be a graph with  $\delta(G) \geq k$ . Then  $G$  contains  $T$  as a subgraph