1. Characterize when $K_{r_1,r_2,\cdots,r_k}$ has a perfect matching.

   Solution:

   (a) $r = \sum_{i=1}^{k} r_i$ is even.

   (b) $r_i \le r - r_i \Leftrightarrow r_i \le \frac{r}{2}$

2. Prove that every tree has at most one perfect matching.

   Solution: Assume that a tree T has two different perfect matching M_1 and M_2. We consider $H = M_1 \cup M_2$. In H, every point is of degree 0 or 2. Thus, we can either find a circle or a single point in H. As $H \subseteq T$, which means T contains circle or single independent point, contradictory to the definition of trees.

3. Use Tutte's Theorem to prove Hall's Theorem.

   Solution: A graph bipartite graph G can be separated into two components containing vertices set X and Y. Consider $|X|+|Y|$, if the value is odd, then add a point into Y. Connect all the points in Y to each other (form a complete graph) and we use G' to denote the new graph. X can be matched into Y $\Leftrightarrow$ G' has perfect matching $\Leftrightarrow \forall S \subseteq X, \Omega(G'-N(S)) \le N(S)$. Consider $G'-N(S)$, there are at least $|S|$ components of order 1 (which are S actually). Thus, $|S| \le \Omega(G'-N(S)) \le |N(S)|$, which is the Hall's condition.

4. Show how to use the Hungarian Algorithm to test for the existence of a perfect matching in a bipartite graph.

   Solution: Set the weights of the edge equally and apply the algorithm. If the sum of maximum matching equal to half of total weight, then there is a perfect matching.

5. Give an example of the stable matching problem with two men and two women in which there is more than one stable matching.

   Solution: men:x,y women: a,b

   $$x : a > b \qquad a : y > x$$
   $$y : b > a \qquad b : x > y$$

6. Determine the stable matchings resulting from the Proposal Algorithm run with men proposing and with women proposing, given the preference lists below.

Men$\{u, v, w, x, y, z\}$	Women$\{a, b, c, d, e, f\}$
$u : a > b > d > c > f > e$	$a : z > x > y > u > v > w$
$v : a > b > c > f > e > d$	$b : y > z > w > x > v > u$
$w : c > b > d > a > f > e$	$c : v > x > w > y > u > z$
$x : c > a > d > b > e > f$	$d : w > y > u > x > z > v$
$y : c > d > a > b > f > e$	$e : u > v > x > w > y > z$
$z : d > e > f > c > b > a$	$f : u > w > x > v > z > y$

   Solution:

- Men proposing: ax, bw, cv, dy, ez, fu
- Women proposing: uf, vc, wd, xe, yb, za

7. Find a transveral of maximum total sum (weight) in each matrix below. Prove that there is no larger weight transveral by exhibiting a solution to the dual problem. Explain why this proves that there is no larger transveral. (Definition: A transveral of an n-by-n matrix consists of n positions, one in each row and each column.)

   (a)
$$\begin{pmatrix} 4 & 4 & 4 & 3 & 6 \\ 1 & 1 & 4 & 3 & 4 \\ 1 & 4 & 5 & 3 & 5 \\ 5 & 6 & 4 & 7 & 9 \\ 5 & 3 & 6 & 8 & 3 \end{pmatrix} \tag{1}$$

   (b)
$$\begin{pmatrix} 7 & 8 & 9 & 8 & 7 \\ 8 & 7 & 6 & 7 & 6 \\ 9 & 6 & 5 & 4 & 6 \\ 8 & 5 & 7 & 6 & 4 \\ 7 & 6 & 5 & 5 & 5 \end{pmatrix} \tag{2}$$

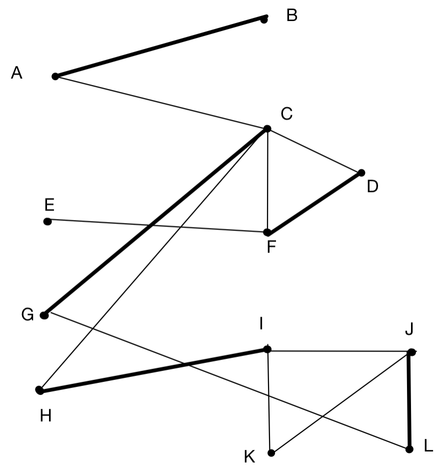
   (c)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 7 & 2 \\ 1 & 3 & 4 & 4 & 5 \\ 3 & 6 & 2 & 8 & 7 \\ 4 & 1 & 3 & 5 & 4 \end{pmatrix} \tag{3}$$

   Solution: Use the Hungarian algorithm, and the max weights are 29, 36 and 28 separately.

8. Find a minimum-weight transveral in the matrix below, and use duality to prove that the solution is optimal. (Hint: Use a transformation of the problem.)
$$\begin{pmatrix} 4 & 5 & 8 & 10 & 11 \\ 7 & 6 & 5 & 7 & 4 \\ 8 & 5 & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & 7 \\ 4 & 5 & 7 & 9 & 8 \end{pmatrix} \tag{4}$$

   Solution: Use Hungarian algorithm to find the maximum matching of $13 - matrix$, the total weight will be 35. Use $13 * 6 - 35 = 30$ to get the solution of minimum transveral.

9. Apply Edmonds' Blossom Algorithm on the following graph. (The matching is the combination of bold lines in the graph. Let the initial M-unsaturated vertex be E.)



Solution: Notice: there is no blossom when applying the algorithm.

The augmenting path is EFDCGLJK.