1. If $2 \le p' \le p$ and $2 \le q' \le q$, then prove that $R(p', q') \le R(p, q)$. Also, prove that equality holds if and only if $p' = p$ and $q' = q$.

   **Solution:** (Assume the two colors are blue and red.) Known from the definition of $R(p, q) = r$, there exists a 2-coloring of a complete graph $K_{r-1}$ satisfying that there is no red $K_p$ or blue $K_q$. Now exchange the colors in the coloring (,which is to turn blue into red and to turn red into blue). We would produce another 2-coloring of $K_{r-1}$ which does not contain blue $K_p$ or red $K_q$. Thus, according to the definition, $R(q, p) \ge r = R(p, q)$. Similarly, we can get $R(p, q) \ge R(q, p)$.

2. Use Theorem 1.64 to prove that $R(3, 5) = 14$.

   **Solution:** Using Theorem 1.64,

$$R(3,5) \le R(2,5) + R(3,4) \le 5 + (4^2 + 3)/2 \le 14 \tag{1}$$

   Now we need to prove that there is a 2-coloring of $K_{13}$ that does not contain monochromatic $K_3$ or $K_5$. Figure 1 is an example.

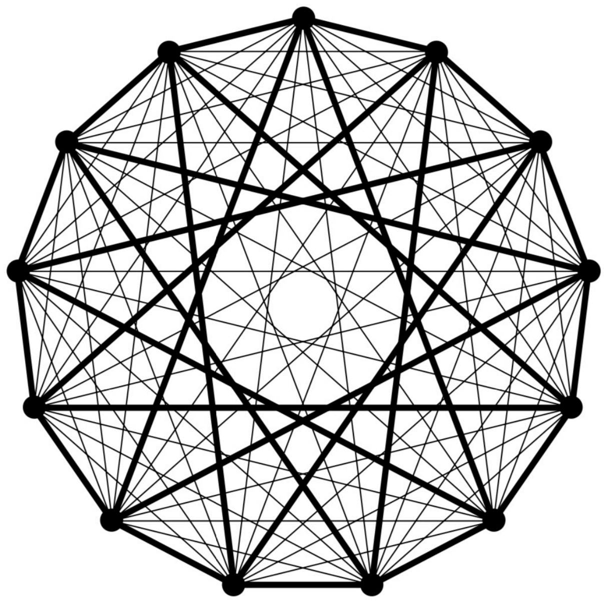

Figure 1: A 2-coloring of $K_{13}$

3. Use Theorem 1.64 to prove that $R(4, 4) = 18$.

   **Solution:** Using Theorem 1.64,

$$R(4,4) \le R(3,4) + R(3,4) \le 2 * \lfloor (4^2 + 3)/2 \rfloor \le 18 \tag{2}$$

   Now we need to prove that there is a 2-coloring of $K_{17}$ that does not contain monochromatic $K_4$ . Figure 2 is an example.

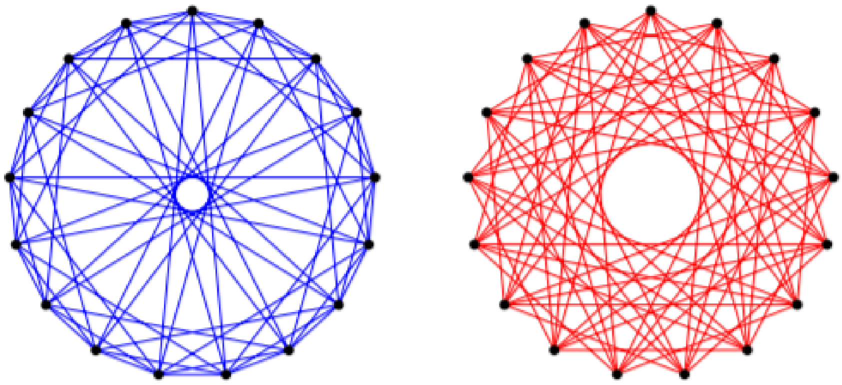

Figure 2: A 2-coloring of $K_{17}$

4. Find $R(P_3, P_3)$.

   **Solution:** (!!!The definition of $P_n$ here is a path containing n edges.)

   We can construct a 2-coloring of $K_4$ that all of the edges except the edges connected to one specific vertex are red, and the remaining edges are blue. Then there is no monochromatic $P_3$.

   Claim: $R(P_3, P_3) = 5$

   Now consider a two coloring of the edges of $K_5$. There is a color used at least twice at vertex Red. So assume that edges (1,2),(1,3) are Red. If (2,4) is Red then (4,2,1,3) is Red and if (2,5) is Red then (5,2,1,3) is Red. So we can assume that (2,4),(2,5) are Blue. But then if (3,4) is Red we have (4,3,1,2) is Red and if (3,4) is Blue then (3,4,2,5) is Blue.

5. Find $R(P_3, C_4)$.

   **Solution:**

   First of all, based on Theorem 1.68 (Harris)

$$R(P_3, C_4) \geq (2-1)(4-1) + 1 = 4 \tag{3}$$

   We can construct a 2-coloring of $K_4$ that all of the edges except the edges connected to one specific vertex are red, and the remaining edges are blue. Then there is no $P_3$ and $C_4$.

   Then we claim $R(P_3, C_4) = 5$. Now consider a two coloring of the edge of K5. We can assume from r(P3, P3) = 5 that there is a Blue P3, say (1,2,3,4) and that (1,4) is Red. Consider the edges (2,5),(4,5). If they are both Red then (1,4,5,2) is Red. If they are both Blue then (2,3,4,5,2) is Blue. A similar argument deals with the case where (1,5) and (3,5) have the same color. Assume next that (4,5) is Blue and (2,5) is Red. If (1,5) is Red then (2,5,1,4) is Red. So assume that (1,5) is Blue and (3,5) is Red. If now (1,3) is Red then so is (4,1,3,5) and if (1,3) is Blue then so is (1,3,4,5,1). Finally, suppose that (4,5) is Red and (2,5) is Blue. If (3,5) is Red then (3,5,4,1) is Red. So assume that (3,5) is Blue and (1,5) is Red. If (1,3) is Red then so is (3,1,5,4) and if (1,3) is Blue then so is (1,3,5,2,1).

6. Find $R(C_4, C_4)$.

   **Solution:** Using Theorem 1.71 (Harris),

$$R(C_4, C_4) \geq R(2K_2, 2K_2) = 5 \tag{4}$$

As $K_5$ can be decomposed into 2 $C_5$, $R(C_4, C_4) \geq 6$. Now consider a two coloring of the edge of K6 with all the vertices labeled as integers from 1 to 6. Consider the edges incident with 1. At least 3 must be the same color. Assume therefore that the edges (1,2),(1,3),(1,4) are all Red. It follows that neither of vertices 5 and 6 can have 2 Red edges joining them to 1,2,3. Suppose that edges (2,5),(3,5) are Blue. Then at most one of (6,2),(6,3) can be Blue. Otherwise we have the Blue (2,5,3,6,2). So assume that (3,6),(4,6) are Blue and (2,6),(4,5) are Red. Suppose now that (5,6) is Blue. If (2,3) is Blue then so is (5,6,3,2,5). If (3,4) is Blue then so is (5,6,4,3,5). But now if (2,3) and (3,4) are Red then (1,2,3,4,1) is Red. So we can assume that (5,6) is Red. If (2,4) is Red then (2,4,6,5,2) is Red. Suppose then that (2,4) Blue. If for example (2,3) is Blue then so is (2,3,6,4,2). So assume now that (2,3) and (3,4) both Red and then (1,2,3,4,1) is Red.

7. Prove that $R(K_{1,3}, K_{1,3}) = 6$.

   **Solution:** Using the theorem 1.70 (Harris), with $K_{1,3}$ considered as a tree of order 4, we can know
   $$R(K_{1,3}, K_{1,3}) = 4 + 3 - 1 = 6 \tag{5}$$

8. Prove that $R(2K_3, K_3) = 8$.

   **Solution:** First Figure 3 shows that $R(2K_3, K_3) > 7$.

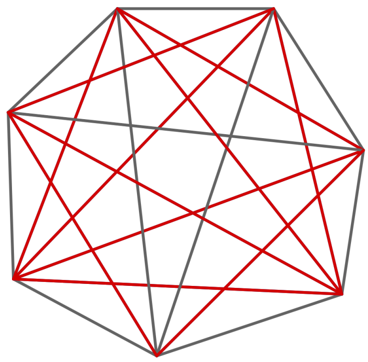

Figure 3: .

We need to find two disjointed red $K_3$ or a blue $K_3$ in any 2-coloring of $K_8$. Since $K_8$ would always contain a monochromatic triangle $R(3,3) = 6$, if the triangle is blue, then the 2-coloring satisfies the condition. If the triangle is red, we need to find another red trianle or blue triangle. Consider $K_6$ which is a subgraph of $K_8$ and does not contain the triangle we mentioned. Then there must be a blue or red triangle in this $K_6$. If the trianle happens to be blue, than we find a blue triangle. If the triangle is red, and the red triangle does not share a vertex with the former triangle, then we find two disjointed red trianles. The remaining part would be too difficult. See proof in Page9-11 of `https://www.ams.org/journals/tran/1975-209-00/S0002-9947-1975-0409255-0/S0002-9947-1975-0409255-0.pdf`.

9. Prove that any 2-coloring of $K_6$ produces two monochromatic $K_3$ and a monochromatic $C_4$.

   **Solution:** $C_4$ is proved in Problem 6. The remaining part of the proof is written in another file.