1. Prove that if G is 2-edge-connected and G' is obtained from G by subdividing an edge of G, then G' is 2-edge-connected. Use this to prove that every graph having a closed-ear decomposition is 2-edge-connected.

4.2.2. *If G is 2-edge-connected and G' is obtained from G by subdividing an edge of G, then G' is 2-edge-connected.* Let G' be obtained by subdividing an edge e, introducing a new vertex w. A graph is 2-edge-connected if and only if every edge lies on a cycle. This holds for G. If also holds for G', since every cycle in G containing e can be replaced with a cycle using the two edges incident to w instead of e.

Every graph having a closed-ear decomposition is 2-edge-connected. A cycle is 2-edge-connected; we show that adding ears and closed ears preserves 2-edge-connectedness. An ear or closed ear can be added by adding an edge joining existing endpoints or a double edge joining an old vertex to a new vertex, following by subdividing to lengthen the ear.

We have shown that subdivision preserves 2-edge-connectedness. The other operations preserve old cycles. When we add an edge, the new edge form a cycle with a path joining its endpoints. When we add two edges with the same endpoints, together they form a cycle. Hence the additions also preserve 2-edge-connectedness.

2. Prove or disprove: If P is a u,v-path in a 2-connected graph G, then there is a u,v-path Q that is internally disjoint from P.

4.2.4. *If P is a u,v-path in a 2-connected graph G, then there need not be a u,v-path internally disjoint from P.* The graph $G = K_4 - uv$ with $V(G) = \{u, v, x, y\}$ is 2-connected (connected and no cut-vertex), but it has no u,v-path internally disjoint from the u,v-path P that visits vertices u, x, y, v in order.

3. Use results of this section to prove that a simple graph G is 2-connected if and only if G can be obtained from C_3 by a sequence of edge additions and edge subdivisions.

4.2.6. *A simple graph G is 2-connected if and only if G can be obtained from C_3 by a sequence of edge additions and edge subdivisions.* We have shown that edge addition and edge subdivision preserve 2-connectedness, so the condition is sufficient. For necessity, observe that every 2-connected graph has an ear decomposition. The initial cycle arises from C_3 by edge subdivisions, and then each ear addition consists of an edge addition followed by edge subdivisions.

4. Let xy be an edge in a digraph G. Prove that $\kappa(G - xy) \geq \kappa(G) - 1$

4.2.7. *If xy is an edge in a digraph G, then $\kappa(G - xy) \geq \kappa(G) - 1$.* Since every separating set of G is a separating set of $G - xy$, we have $\kappa(G - xy) \leq \kappa(G)$. Equality holds unless $G - xy$ has a separating set S that is smaller than $\kappa(G)$ and hence is not a separating set of G. Since $G - S$ is strongly connected, $G - xy - S$ has two induced subdigraphs $G[X]$ and $G[Y]$ such that $X \cup Y = V(G)$ and xy is the only edge from X to Y.

If $|X| \geq 2$, then $S \cup \{x\}$ is a separating set of G, and $\kappa(G) \leq \kappa(G - xy) + 1$. If $|Y| \geq 2$, then again the inequality holds. In the remaining case, $|S| = n(G) - 2$. Since we have assumed that $|S| < \kappa(G)$, $|S| = n(G) - 2$ implies that $\kappa(G) \geq n(G) - 1$, which holds only when each ordered pair of distinct vertices is the head/tail for some edge. Thus $\kappa(G - xy) = n(G) - 2 = \kappa(G) - 1$, as desired.

5. Prove that a simple graph G is 2-connected if and only if for every triple (x, y, z) of distinct vertices, G has an x, z-path through y.

4.2.8. *A graph is 2-connected if and only if for every ordered triple (x, y, z) of vertices, there is an x, z-path through y.* If G is 2-connected, then for any $y \in V(G)$ and set $U = \{x, z\}$, there is a y, U-fan. The two paths of such a fan together form an x, z-path through y. Conversely, if the condition holds, then clearly G is connected. Furthermore, G has no cut-vertex, because for any vertex x and any pair y, z, the condition as stated implies that $G - x$ has an y, z-path.

6. For a connected graph G with at least three vertices, prove that the following statements are equivalent.

   (a) G is 2-edge-connected.

   (b) Every edge of G appears in a cycle.

   (c) G has a closed trail containing any specified pair of edges.

   (d) G has a closed trail containing any specified pair of vertices.

4.2.11. *For a connected graph G with at least three vertices, the following are equivalent.*

A) G is 2-edge-connected.

B) Every edge of G appears in a cycle.

C) G has a closed trail containing any specified pair of edges.

D) G has a closed trail containing any specified pair of vertices.

A$\Leftrightarrow$B. A connected graph is 2-edge-connected if and only if it has no cut-edges. Cut-edges are precisely the edges belonging to no cycles.

A$\Rightarrow$D. By Menger's Theorem, a 2-edge-connected graph G has two edge-disjoint x, y-paths, where $x, y \in V(G)$. Following one path and returning on the other yields a closed trail containing x and y. (Without using Menger's Theorem, this can be proved by induction on $d(x, y)$.)

D$\Rightarrow$B. Let xy be an edge. D yields a closed trail containing x and y. This breaks into two trails with endpoints x and y. At least one of them, T, does not contain the edge xy. Since T is an x, y-walk, it contains an x, y-path. Since T does not contain xy, this path completes a cycle with xy.

B$\Rightarrow$C. Choose $e, f \in E(G)$; we want a closed trail through e and f. Subdivide e and f to obtain a new graph G', with x, y being the new vertices. Subdividing an edge does not destroy paths or cycles, although it may lengthen them. Thus G' is connected and has every edge on a cycle, because G has these properties. Because we have already proved the equivalence of B and D, we know that G' has a closed trail containing x and y. Replacing the edges incident to x and y on this trail with e and f yields a closed trail in G containing e and f.

C$\Rightarrow$D. Given a pair of vertices, choose edges incident to them. A closed trail containing these edges is a closed trail containing the original vertices.

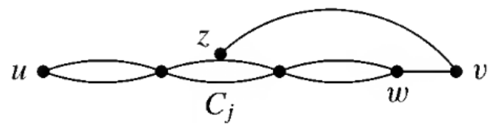
7. A u, v-necklace is a list of cycles $C_1, \cdots, C_k$ such that $u \in C_1, v \in C_k$, consecutive cycles share one vertex, and nonconsecutive cycles are disjoint. Use induction on $d(u, v)$ to prove that a graph G is 2-edge-connected if and only if for all $u, v \in V(G)$ there is a u, v-necklace in G.

4.2.14. *A graph G is 2-edge-connected if and only if for all $u, v \in V(G)$ there is a u, v-necklace in G,* where a *u, v-necklace* is a list of cycles $C_1, \ldots, C_k$ such that $u \in C_1$, $v \in C_k$, consecutive cycles share one vertex, and non-consecutive cycle are disjoint. The condition is sufficient, because a u, v-necklace has two edge-disjoint u, v-paths, and these cannot both be cut by deleting a single edge. Conversely, suppose that G is 2-edge-connected. We obtain a u, v-necklace.

Proof 1 (induction on $d(u, v)$). Basis step ($d(u, v) = 1$): A u, v-path in $G - uv$ combines with the edge uv to form a u, v-necklace in G.

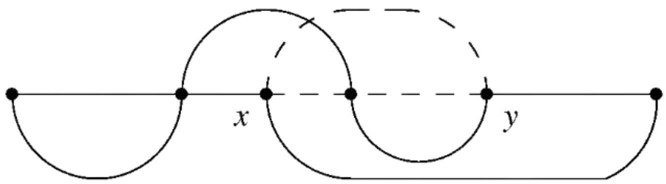
Induction step ($d(u, v) > 1$). Let w be the vertex before v on a shortest u, v-path; note that $d(u, w) = d(u, v) - 1$. By the induction hypothesis, G has a u, w-necklace. If v lies on this u, w-necklace, then the cycles up to the one containing v form a u, v-necklace.

Otherwise, let R be a u, v-path in $G - wv$; this exists since G is 2-edge-connected. Let z be the last vertex of R on the u, w-necklace; let C_j be the last cycle containing z in the necklace. The desired u, v-necklace consists of the cycles before C_j in the u, w-necklace together with a final cycle containing v. The final cycle consists of the remainder of R from z to v, the edge vw, a path from w to C_j in the u, v-necklace, and the path on C_j from there to z that contains the vertex of $C_j \cap C_{j-1}$. The choice of z guarantees that this is a cycle.



Comment. There is also a proof by induction on the number of ears in an ear decomposition, but showing that all pairs still have necklaces when an open ear is added still involves a discussion like that above. Another inductive proof involves showing that the union of a necklace from u to w and an necklace from w to v contains a necklace from u to v.

Proof 2 (extremality). Since G is 2-edge-connected, there exist two edge-disjoint u, v-paths. Among all such pairs of paths, choose a pair P_1, P_2 whose lengths have minimum sum. Let S be the set of common vertices of P_1 and P_2. If the vertices of S occur in the same order on P_1 and P_2, then $P_1 \cup P_2$ is a u, v-necklace. Otherwise, let x, y be the first vertices of P_1 in S that occur in the opposite order on P_2, with x before y in P_1 and after y in P_2. In the figure, P_1 is the straight path. Form two new u, v-paths: Q_1 consists of the portion of P_1 up to x and the portion of P_2 after x, and Q_2 consists of the portion of P_2 up to y and the portion of P_1 after y. Neither of Q_1, Q_2 uses any portion of P_1 or P_2 between x and y, so we have found edge-disjoint u, v-paths with shorter total length. This contradiction completes the proof.



8. Let G be a 2-connected graph. Prove that if T_1, T_2 are two spanning trees of G, then T_1 can be transformed into T_2 by a sequence of operation in which a leaf is removed and reattached using another edge of G.

4.2.16. *If G is a 2-connected graph, and T_1 and T_2 are two spanning trees of G, then T_1 transforms into T_2 by a sequence of operations in which a leaf is removed and reattached using another edge of G.* Let T be a largest tree contained in both T_1 and T_2; this is nonempty, since each single vertex is such a tree. We use induction on the number of vertices of G omitted by T. If none are omitted, then $T_1 = T_2$ and the sequence has length 0. If one vertex is omitted, then it is a leaf in both T_1 and T_2, and a single reattachment suffices.

Otherwise, for $i \in \{1, 2\}$ let $x_i y_i$ be an edge of T_i with $x_i \in V(T)$ and $y_i \notin V(T)$. If $y_1 \neq y_2$, then enlarge $T + x_1y_1 + x_2y_2$ to a spanning tree T' of G. Since T' shares more with T_1 than T does, the induction hypothesis yields a sequence of leaf exchanges that turns T_1 into T'. Similarly, it yields a sequence that turns T' into T_2. Together, they complete the desired transformation.

Hence we may assume that $y_1 = y_2$ (this may be necessary even when T omits many vertices of G). We generate another edge x_3y_3 with $x_3 \in V(T)$ and $y_3 \in V(G - y_1)$ (this is possible since G is 2-connected). Now enlarge $T + x_1y_1 + x_3y_3$ to a spanning tree T' and $T + x_2y_2 + x_3y_3$ to a spanning tree T''. For each pair (T_1, T'), (T', T''), or (T'', T_2), there is now a common subtree consisting of T and one additional edge. Hence we can use the induction hypothesis to turn T_1 into T', then T' into T'', and finally T'' into T_2, completing the desired transformation.

(Note: Induction also yields the statement that the common subtree T is never changed during the transformation.)

9. Prove that if G is 2-connected, then $G - xy$ is 2-connected if and only if x and y lie on a cycle in $G - xy$. Conclude that a 2-connected graph is minimally 2-connected if and only if every cycle is an induced subgraph. (A 2-connected graph is minimally 2-connected if for every $e \in E(G)$, the graph $G - e$ is not 2-connected.)

4.2.35. *A 2-connected graph is minimally 2-connected if and only if no cycle has a chord.* Suppose that G is 2-connected. We show that $G - xy$ is 2-connected if and only if x and y lie on a cycle in $G - xy$. If $G - xy$ is not 2-connected, then there is a vertex v whose deletion separates x and y, and thus all x, y-paths in $G - xy$ pass through v and $G - xy$ has no cycle containing x and y. Conversely, if $G - xy$ is 2-connected, then every pair of vertices (including x, y) lies on a cycle.

If a cycle in G has a chord x, y, then this argument shows that $G - xy$ is still 2-connected, and hence G is not minimally 2-connected. If no cycle has a chord, then for any edge xy, the graph $G - xy$ has no cycle containing x and y, and so $G - xy$ is not 2-connected.