1. Find the minimum size of a maximal matching in each of the following graphs.

   (a) C_{10}

   (b) C_{11}

   (c) C_n

   Solution:

   (a) 4

   (b) 4

   (c) $\lceil \frac{n}{3} \rceil$

   Notice: The problem is asking for the minimum size of a maximal matching.

2. The matching graph $M(G)$ of a graph G has the maximum matchings of G as its vertices, and two vertices M_1 and M_2 of $M(G)$ are adjacent if M_1 and M_2 differ in only one edge. Show that each cycle C_n, $n = 4, 5$ is the matching graph of some graph.

   Solution:

   (a) C_4: $P_3 \cup P_3$

   (b) C_5: C_5

   You can find all the maximum matching and verify the conclusion.

3. For the graphs of Figure 1, with matchings M as shaded, find

   (a) an M-alternating path that is not M-augmenting;

   (b) an M-augmenting path if one exists; and, if so, use it to obtain a bigger matching.

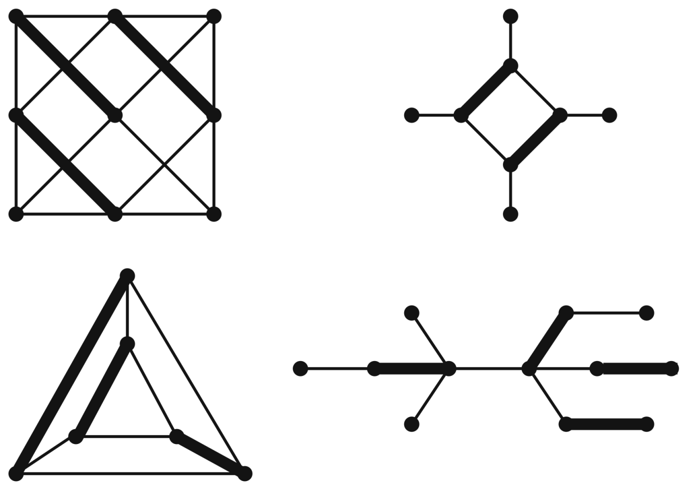

Figure 1:

Solution: Notice: The left-bottom figure does not have a bigger matching.

4. For each of the following families of sets, determine whether the conditionof Theorem 1.52 is met. If so, then find an SDR. If not, then show how thecondition is violated.

   (a) {1,2,3},{2,3,4},{3,4,5},{4,5},{1,2,5}
   (b) {1,2,4},{2,4},{2,3},{1,2,3}
   (c) {1,2},{2,3},{1,2,3},{2,3,4},{1,3},{3,4}
   (d) {1,2,5},{1,5},{1,2},{2,5}
   (e) {1,2,3},{1,2,4},{1,3,4},{1,2,3,4},{2,3,4}

   Solution:

   (a) 1,2,3,4,5 (Not the only answer)
   (b) 4,2,3,1 (Not the only answer)
   (c) There are not enough elements
   (d) There are not enough elements
   (e) There are not enough elements

5. Let G be a bipartite graph. Show that G has a matching of size at least $|E(G)|/\Delta(G)$

   Solution: Any vertex is adjacent to at most $\Delta(G)$ edges. Thus, an edge cover must have size at least $\frac{|E(G)|}{\Delta(G)}$. By König–Egerváry Theorem, we can know the maximum matching in the graph must have size at least $\frac{|E(G)|}{\Delta(G)}$

6. Let M_1 and M_2 be matchings in a bipartite graph G with partite sets X and Y. If $S \subseteq X$ is saturated by M_1 and $T \subseteq Y$ is saturated by M_2, show that there exists a matching in G that saturates $S \cup T$.

   Solution: Problem 5 in http://www.sfu.ca/~mdevos/345/homework5_sol.pdf.

7. Let G be a bipartite graph with partite sets X and Y. Let δX denote the minimum degree of the vertices in X, and let ΔY denote the maximum degree of the vertices in Y. Prove that if $\delta X \geq \Delta Y$, then there exists a matching in G that saturates X.

   Solution: For any $S \subseteq X$, we can know that $\delta X|S| \leq \Delta Y|N(S)|$. As long as $\delta X \geq \Delta Y$, $|S| \leq |N(S)|$. Then based on Hall's Theorem, there exists a matching in G that saturates X.

8. Use the König–Egerváry Theorem to prove Hall's Theorem.

   Solution: That X can be matched into Y (imply $|Y| \geq |X|$) is equivalent to the size of maximum matching of graph G is $|X|$. Based on the König–Egerváry Theorem, the minimum number of vertices in an edge cover of G is $|X|$. We only need to prove the sub-conclusion is equivalent to the Hall's condition.

   For each $S \subseteq X, T \subseteq Y$, the sub-conclusion is equivalent to the fact that if there is no lines between S and T, then $|S| + |T| \leq |X| \Leftrightarrow |T| \leq |X| - |S|$. We simply choose $T = Y - N(S)$, and there is no lines between S and the chosen T. Then we can know the sub-conclusion is equivalent to $|Y| - |N(S)| \leq |X| - |S| \Leftrightarrow |S| \leq |N(S)| + |X| - |Y| \leq |N(S)|$. Prove done.

9. Let k be some fixed integer, $1 \leq k \leq n$, and let G be some subgraph of $K_{n,n}$ with more than $(k-1)n$ edges. Prove that G has a matching of size at least k.

   Solution: Use the conclusion of Problem 5, we easily get the conclusion. (As $\Delta(G) \leq n$, $|E(G)|/\Delta(G) > k-1$)