Lecture: Coloring (3) and Planarity

Shuai Li

John Hopcroft Center, Shanghai Jiao Tong University

https://shuaili8.github.io

https://shuaili8.github.io/Teaching/CS445/index.html

Use chromatic recurrence to compute $c_G(k)$

- Example: Compute $c_{P_3}(k) = k^4 - 3k^3 + 3k^2 - k$
- Check: $c_{P_3}(1) = 0, c_{P_3}(2) = 2$

FIGURE 1.102. Two 2-colorings of P_4.

More examples

- Path P_{n-1} has $n-1$ edges (n vertices)

$$c_{P_{n-1}}(k) = k(k-1)^{n-1}$$

- Any tree T on n vertices

$$c_T(k) = k(k-1)^{n-1}$$

- Cycle C_n

$$c_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$$

  - When n is odd, $c_{C_n}(2) = 0, c_{C_n}(3) > 0$
  - When n is even, $c_{C_n}(2) > 0$

Properties of chromatic polynomials

- Theorem (1.49, H; Ex 3, S1.6.4, H) Let G be a graph of order n
  - $c_G(k)$ is a polynomial in k of degree n
  - The leading coefficient of $c_G(k)$ is 1
  - The constant term of $c_G(k)$ is 0
    - If G has i components, then the coefficients of $k^0, \ldots, k^{i-1}$ are 0
    - G is connected $\Leftrightarrow$ the coefficient of k is nonzero
  - The coefficients of $c_G(k)$ alternate in sign
  - The coefficient of the k^{n-1} term is $-|E(G)|$
    - A graph G is a tree $\Leftrightarrow c_G(k) = k(k-1)^{n-1}$

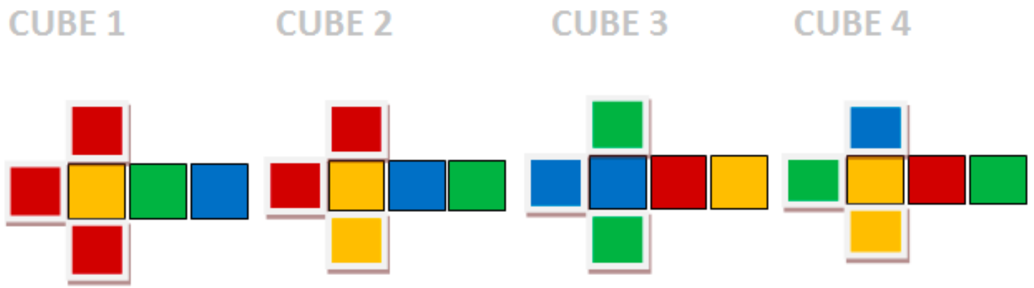
      $\Leftrightarrow$ (Theorem 1.10, 1.12, H) T is connected with $n-1$ edges

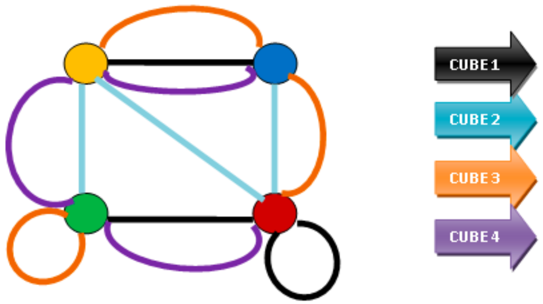
    - A graph G is complete $\Leftrightarrow c_G(k) = k(k-1)\cdots(k-n+1)$

Proof Using Coloring

Example -- Instant Insanity 四色方柱问题 (1.2, L)

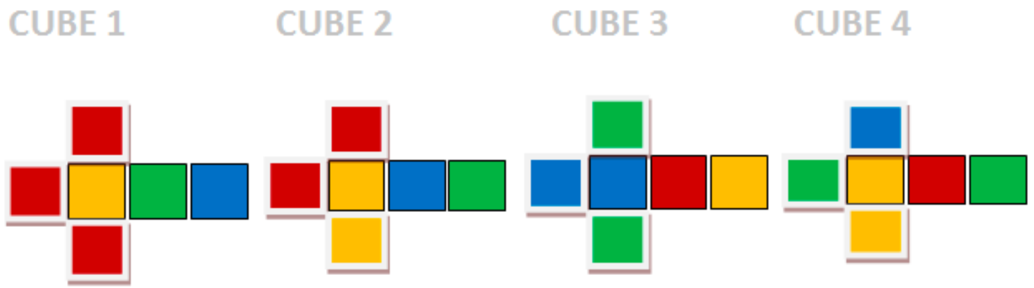
- Problem make a stack of these cubes so that all four colors appear on each of the four sides of the stack
- An edge indicates that the two adjacent colors occur on opposite faces of the cube
- Problem necessary to find two subgraphs s.t.
  - are regular of degree 2
  - four edges from each cube
  - no edge in common

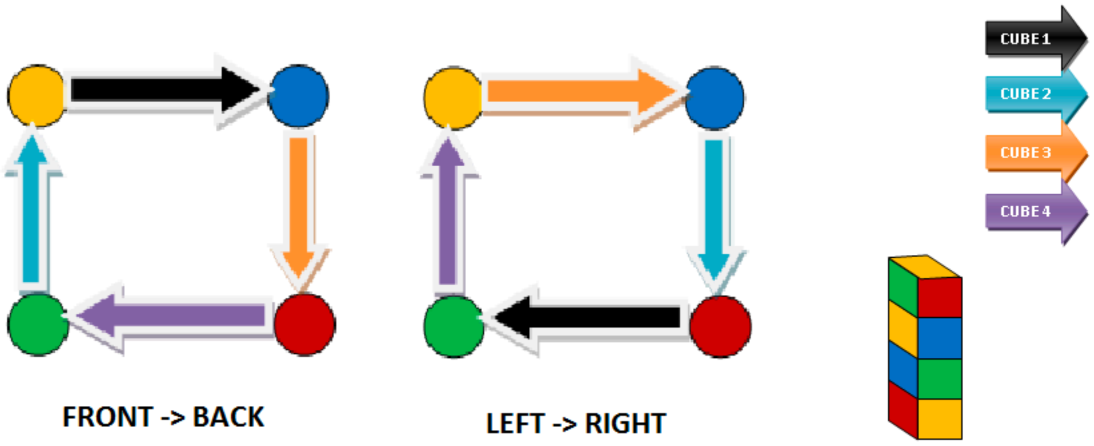


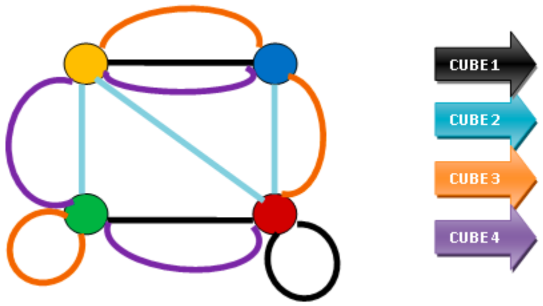


Example -- Instant Insanity 四色方柱问题 (1.2, L)

- Problem necessary to find two subgraphs s.t.
    - are regular of degree 2
    - four edges from each cube
    - no edge in common

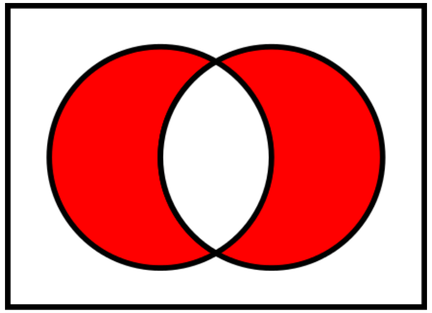






An example about sets (1E, L)

- Let $A_1, \dots, A_n$ be n distinct subsets of the n-set $N := \{1, \dots, n\}$. Show that there is an element $x \in N$ such that the sets $A_i \setminus \{x\}, 1 \le i \le n,$ are all distinct
- Proof Consider a graph with vertices $A_1, \dots, A_n$.
  - An edge of `color' x between A_i and A_j iff $A_i \Delta A_j = \{x\}$
  - Then the problem is equivalent to find y s.t. no color y
  - Notice that a cycle in this graph must have even length and each color appears even times
  - Then we can remove an edge if there is an edge with same color
  - Thus the number of colors remain the same and no cycle exists
  - By tree property, the number of edges is at most $n - 1$



Planarity

Motivation

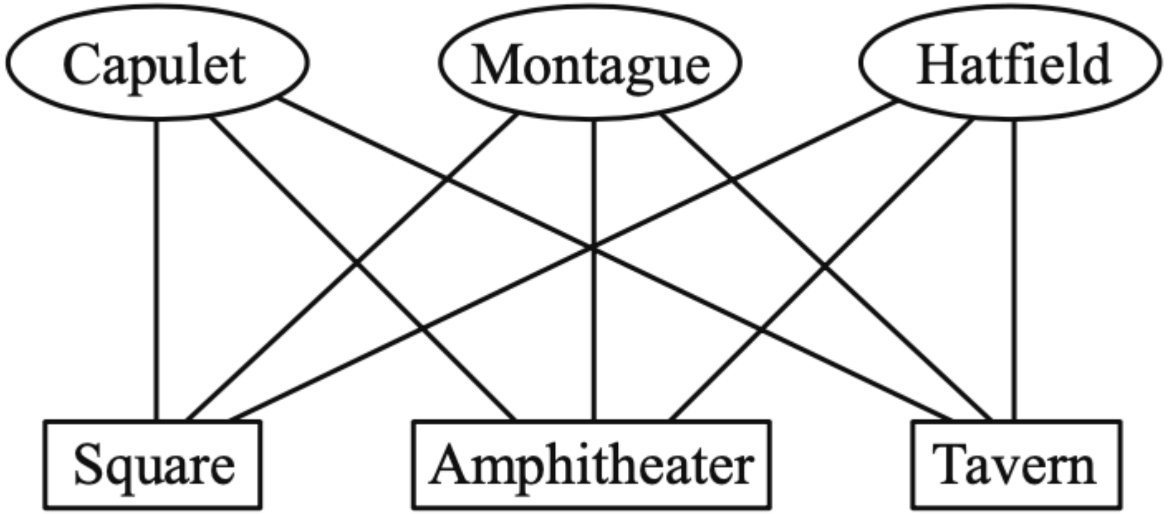


FIGURE 1.72. Original routes.

Definition and examples

- A graph G is said to be planar if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices
- If G has no such representation, G is called nonplanar
- A drawing of a planar graph G in the plane in which edges intersect only at vertices is called a planar representation (or a planar embedding) of G

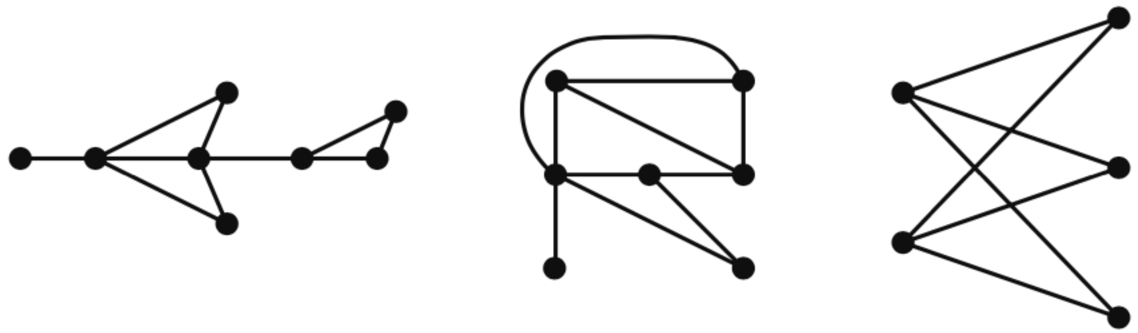
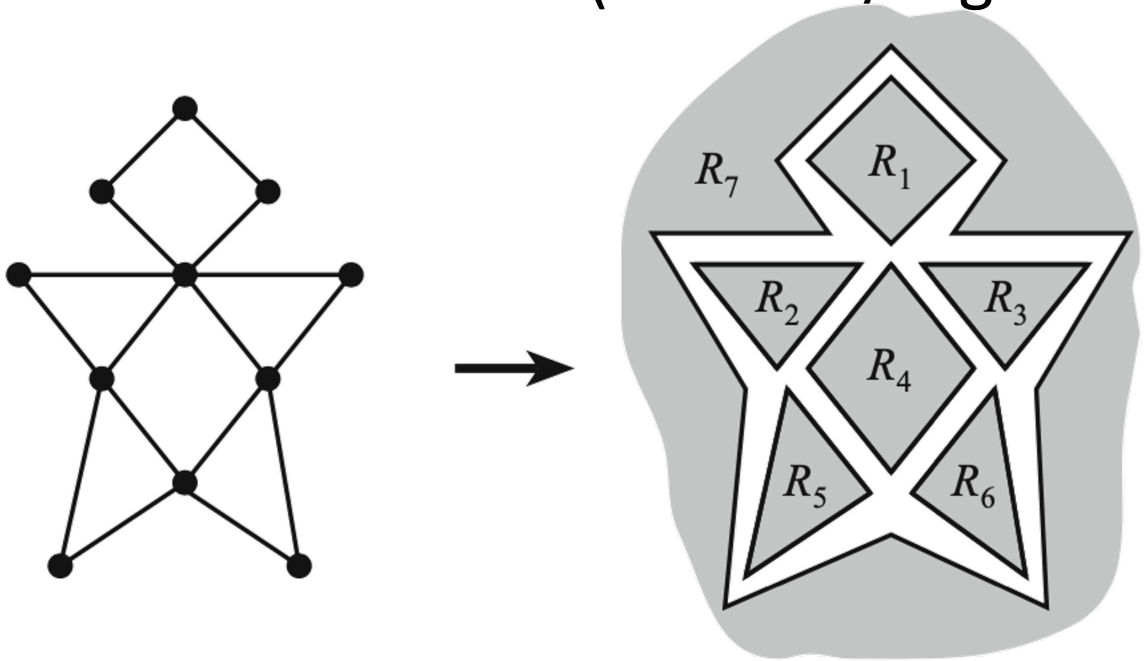


FIGURE 1.73. Examples of planar graphs.

Region

- Given a planar representation of a graph G, a region is a maximal section of the plane in which any two points can be joined by a curve that does not intersect any part of G
- The region R_7 is called the exterior (or outer) region



An edge bounds a region

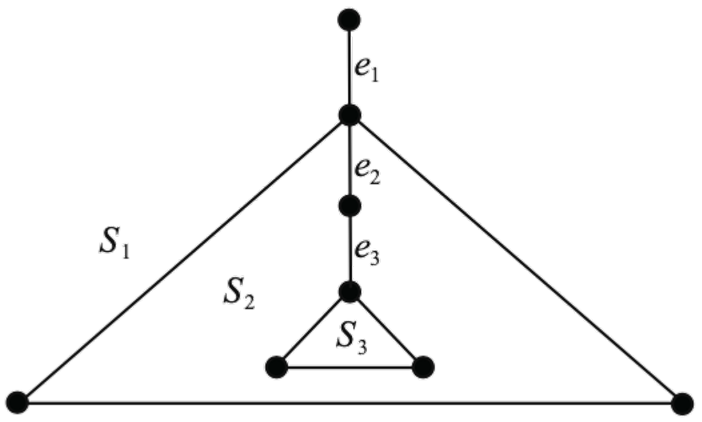
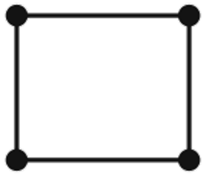


FIGURE 1.76. Edges e_1, e_2, and e_3 touch one region only.

- An edge can come into contact with either one or two regions
- Example:
  - Edge e_1 is only in contact with one region S_1
  - Edge e_2, e_3 are only in contact with S_2
  - Each of other edges is in contact with two regions
- An edge e bounds a region R if e comes into contact with R and with a region different from R
- The bounded degree $b(R)$ is the number of edges that bound the region
  - Example: $b(S_1) = b(S_3) = 3, b(S_2) = 6$

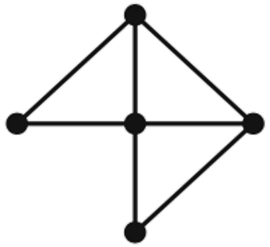
The relationship between numbers of vertices, edges and regions

- The number of vertices n
- The number of edges m
- The number of regions r

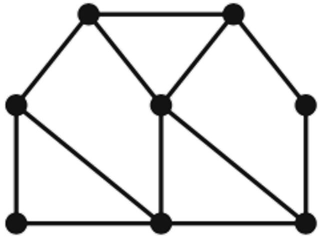


$n = 4$
$m = 4$
$r = 2$

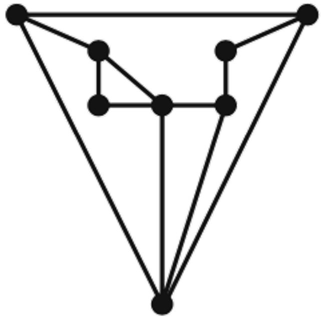
$n = 7$
$m = 9$
$r = 4$



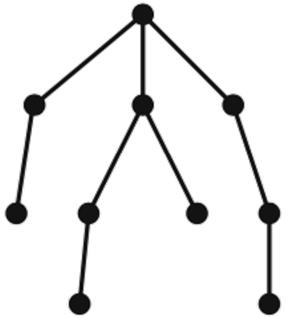
$n = 5$
$m = 7$
$r = 4$



$n = 8$
$m = 12$
$r = 6$



$n = 8$
$m = 12$
$r = 6$



$n = 10$
$m = 9$
$r = 1$

Euler’s formula

- Theorem (1.31, H; Euler 1748) If G is a connected planar graph with n vertices, m edges, and r regions, then
$$n - m + r = 2$$
  - Need Lemma: Every tree is planar

$K_{3,3}$ is nonplanar

- Theorem (1.32, H) $K_{3,3}$ is nonplanar

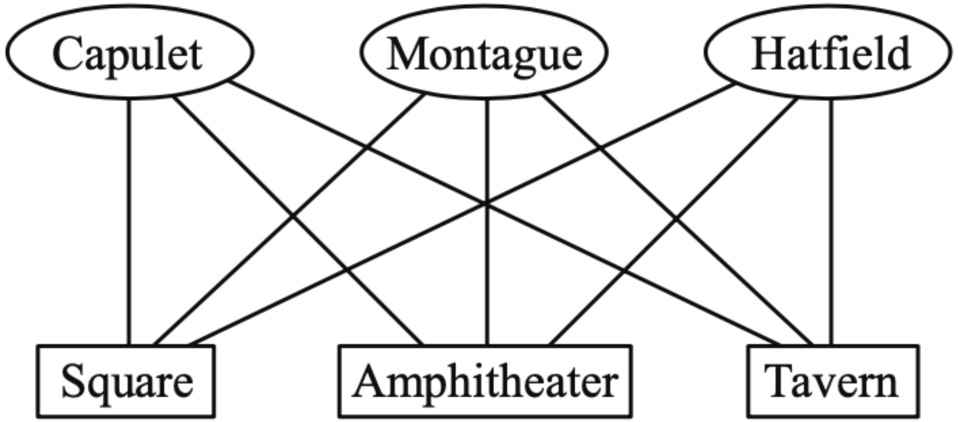


FIGURE 1.72. Original routes.

Upper bound for m

- Theorem (1.33, H) If G is a planar graph with $n \geq 3$ vertices and m edges, then $m \leq 3n - 6$. Furthermore, if equality holds, then every region is bounded by 3 edges.
- Corollary (1.34, H) K_5 is nonplanar
- Theorem (1.35, H) If G is a planar graph , then $\delta(G) \leq 5$