CS445: Combinatorics — Homework 9

Shanghai Jiao Tong University, Fall 2020 — Due Time: 2020.12.11 8:00AM

1. For a connected graph G with blocks $B_1, \cdots, B_k$, prove that $n(G) = (\sum_{i=1}^{k} n(B_i)) - k + 1$.

Solution:

4.1.6. *A connected graph with blocks $B_1, \ldots, B_k$ has $\left(\sum_{i=1}^{k} n(B_i)\right) - k + 1$ vertices.* We use induction on k. Basis step: $k = 1$. A graph that is a single block B_1 has $n(B_1)$ vertices.

Induction step: $k > 1$. When G is not 2-connected, there is a block B that contains only one of the cut-vertices; let this vertex be v, and index the blocks so that $B_k = B$. Let $G' = G - (V(B) - \{v\})$. The graph G' is connected and has blocks $B_1, \ldots, B_{k-1}$. By the induction hypothesis, $n(G') = \left(\sum_{i=1}^{k-1} n(B_i)\right) - (k-1) + 1$. Since we deleted $n(B_k) - 1$ vertices from G to obtain G', the number of vertices in G is as desired.

2. Prove that $\lambda(G) = \kappa(G)$ when G is a simple graph with $\Delta(G) \leq 3$.

Solution:

4.1.11. *$\kappa' = \kappa$ when $\Delta(G) \leq 3$.* Let S be a minimum vertex cut ($|S| = \kappa(G)$). Since $\kappa(G) \leq \kappa'(G)$ always, we need only provide an edge cut of size $|S|$. Let H_1 and H_2 be two components of $G - S$. Since S is a minimum vertex cut, each $v \in S$ has a neighbor in H_1 and a neighbor in H_2. Since $\Delta(G) \leq 3$, v cannot have two neighbors in H_1 and two in H_2. For each such v, delete the edge to a member of $\{H_1, H_2\}$ in which v has only one neighbor. These $\kappa(G)$ edges break all paths from H_1 to H_2 except in the case drawn below, where a path can come into S via v_1 and leave via v_2. Here we simply choose the edge to H_1 for each v_i.

3. Use Proposition 4.1.12 to prove that the Petersen graph has an edge cut of size m if and only if $3 \leq m \leq 12$.(Hint: Consider $|[S, \bar{S}]|$) for $1 \leq |S| \leq 5$.)

Solution:

4.1.16. *The Petersen graph has an edge cut of size m if and only if $3 \le m \le 12$.* Since the graph has 10 vertices, we consider edge cuts of the form $[S, \overline{S}]$ for $1 \le |S| \le 5$. Since $\left|[S, \overline{S}]\right| = \sum_{v \in S} d(v) - 2e(G[S]) = 3|S| - 2e(G[S])$, we consider the number of edges in $G[S]$. Since the girth is 5, all induced subgraphs with at most four vertices are forests.

The independent sets with up to four vertices yield cuts of sizes $3, 6, 9, 12$. Deleting two adjacent vertices and their neighbors leaves $2K_2$, so there induced subgraphs with two to four vertices that have one edge, yielding cuts of sizes $4, 7, 10$. Deleting the vertices of a P_3 and their neighbors leaves $2K_1$, so there are induced subgraphs with three to five vertices that have two edges, yielding cuts of sizes $5, 8, 11$.

Let $e(S)$ denote $e(G[S])$. An edge cut of size less than 3 requires $3|S| - 2e(S) \le 2$, or $e(S) \ge (3/2)|S| - 1$. Since $e(S) \le |S| - 1$ when $|S| \le 4$, we combine the two inequalities to obtain $|S| \le 0$, which is impossible. (For $|S| = 5$, $e(S) \le |S|$ yields $|S| \le 2$, again a contradiction.)

Similarly, an edge cut of size more than 12 requires $2e(S) \le 3|S| - 13$. With $|S| \le 5$, this yields $2e(S) \le 2$, but there is no 5-vertex induced subgraph with only one edge.

4. Prove that $\kappa(G) = \delta(G)$ if G is simple and $\delta(G) \ge n(G) - 2$. Prove that this is best possible for each $n \ge 4$ by constructing a simple n-vertex graph with minimum degree $n - 3$ and connectivity less than $n - 3$.

   Solution:

   4.1.19. *a) If $\delta(G) \ge n - 2$ for a simple n-vertex graph G, then $\kappa(G) = \delta(G)$.* If $\delta = n - 1$, then $G = K_n$, which has connectivity $n - 1$. If $\delta = n - 2$, then when u and v are nonadjacent the other $n - 2$ vertices are all common neighbors of u and v. It is necessary to delete all common neighbors of some pair of vertices to separate the graph, so $\kappa \ge n - 2 = \delta$.

   b) Construction of graphs with $\delta = n - 3$ and $\kappa < \delta$. For any $n \ge 4$, let $G = K_n - E(C_4)$; i.e., G is formed by deleting the edges of a 4-cycle from a clique. The subgraph induced by these four vertices is $2K_2$, so deleting the other $n - 4$ vertices of G disconnects the graph. However, G has 4 vertices of degree $n - 3$ and $n - 4$ of degree $n - 1$, so $\kappa(G) < \delta(G)$.

5. A **cactus** is a connected graph in which every block is an edge or a cycle (e.g. Figure 1). Prove that the maximum number of edges in a simple n-vertex cactus is $\lfloor 3(n-1)/2 \rfloor$. (Hint: $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x + y \rfloor$.)

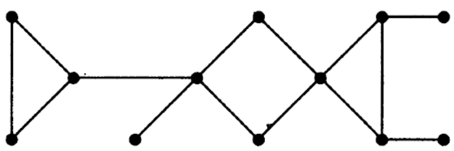

Figure 1: .

Solution:

4.1.31. *The maximum number of edges in a simple n-vertex cactus G is* $\lfloor 3(n-1)/2 \rfloor$. A cactus is a connected graph in which every block is an edge or a cycle. The bound is achieved by a set of $\lfloor (n-1)/2 \rfloor$ triangles sharing a single vertex, plus one extra edge to a leaf if n is even.

Proof 1 (induction on the number of blocks). Let k be the number of blocks. If $k = 1$, then $e(G) = n(G) - 1$ if $n(G) \le 2$, and $e(G) = n(G)$ if $n(G) > 2$. In either case, $e(G) \le \lfloor 3(n(G)-1)/2 \rfloor$.

A graph that has more than one block is not a single block, so it has a cut-vertex v. Let S be the vertex set of one component of $G - v$. Let $G_1 = G[S \cup \{v\}]$, and let $G_2 = G - S$. Both G_1 and G_2 are cacti, and every block of G is a block in exactly one of $\{G_1, G_2\}$. Thus each has fewer blocks than G, and we can apply the induction hypothesis to obtain $e(G_i) \le \lfloor 3(n(G_i)-1)/2 \rfloor$.

If $|S| = m$, then $n(G_1) = m + 1$ and $n(G_2) = n(G) - m$, since v belongs to both graphs. We thus have

$$e(G) = e(G_1) + e(G_2) \le \left\lfloor \tfrac{3(m+1-1)}{2} \right\rfloor + \left\lfloor \tfrac{3(n(G)-m-1)}{2} \right\rfloor \le \left\lfloor \tfrac{3(n(G)-1)}{2} \right\rfloor.$$

Proof 2 (summing over blocks). Let G be a simple n-vertex cactus with k blocks that are cycles and l blocks that are single edges. When we describe G by starting with one block and iteratively adding neighboring blocks, each time we add a block the number of vertices increases by one less than the number of vertices in the block, since one of those vertices (the shared cut-vertex) was already in the graph. If the blocks are $B_1, \ldots, B_{k+l}$, then $n(G) = (\sum n(B_i)) - (k + l - 1)$.

On the other hand, $e(G) = \sum e(B_i)$. We have $e(B_i) = n(B_i)$ if B_i is a cycle, and $e(B_i) = n(B_i) - 1$ if B_i is an edge. Therefore,

$$e(G) = \sum e(B_i) = (\sum n(B_i)) - l = n(G) + k - 1.$$

This implies that we maximize the number of edges by maximizing k, the number of blocks that are cycles. Viewing the cactus again as grown by adding blocks, observe that we add at least two vertices every time we add a block that is a cycle, since cycles have at least three vertices. Starting from a single vertex, the maximum number of cycles we can form is thus $\lfloor (n-1)/2 \rfloor$. This bound on k yields $e(G) \le \lfloor 3(n-1)/2 \rfloor$.

Proof 4 (spanning trees). An n-vertex cactus is a connected graph, so it has a spanning tree with $n - 1$ edges. Each additional edge completes a cycle using at least two edges in the tree. Each edge of the tree is used in at most one such cycle. Hence there are at most $(n-1)/2$ additional edges, and the total number of edges is at most $n + \lfloor (n-1)/2 \rfloor$.

6. Prove that every vertex of a graph has even degree if and only if every block is Eulerian.

Solution:

4.1.32. *Every vertex of G has even degree if and only if every block of G is Eulerian. Sufficiency.* If every block is Eulerian, then each vertex receives even degree from each block containing it. The blocks partition the edges, so the total degree at each vertex is even.

Necessity. Since every block is connected, it suffices to show that each vertex has even degree in each block. Certainly this holds for a vertex appearing in only one block. For a cut-vertex v, let G' be the subgraph consisting of one component of $G - v$ together with its edges to v. Each block containing v appears in one such subgraph. Every vertex of G' other than v has even degree in G', since it retains all of its incident edges from G. By the Degree-Sum Formula, also v has even degree in G'. Hence v has even degree in the block of G containing v that is contained in G'.

7. Prove that a connected graph is k-edge-connected if and only if each of its blocks is k-edge-connected.

Solution:

4.1.33. *A connected graph is k-edge-connected if and only if each of its blocks is k-edge-connected.* We show that a set F of edges is a disconnecting set in a graph G if and only if it disconnects some block. If deleting F leaves each block of G connected, then the full graph remains connected. If deleting F disconnects some block B, then the remainder of G cannot contain a path between distinct components of $B - F$, because then B would not be a maximal subgraph having no cutvertex.

With this claim, the edge-connectivity of G is the minimum of the edge-connectivities of its blocks, which yields the desired statement.

8. Let H be the block-cutpoint graph of a graph G that has a cut-vertex.

   (a) Prove that H is a forest.

   (b) Prove that G has at least two blocks each of which contains exactly one cut-vertex of G.

   (c) Prove that a graph G with k components has exactly $k + \sum_{v \in V(G)}(b(v) - 1)$ blocks, where $b(v)$ is the number of blocks containing v.

   (d) Prove that every graph has fewer cut-vertices than blocks.

Solution:

4.1.34. *The block-cutpoint tree.* Given a graph G with connectivity 1, let $B(G)$ be the bipartite graph whose partite sets correspond to the blocks and the cut-vertices of G, with $x \leftrightarrow B$ if B is a block of G containing x.

a) $B(G)$ is a tree. If $G = K_2$, then $B(G) = K_1$. Otherwise G has at least two blocks, and every cut-vertex belongs to a block. Hence to show $B(G)$ is connected it suffices to establish a B, B'-path in $B(G)$, where B, B' are blocks of G. Since G is connected, G has a u, v-path, for any choice of vertices $u \in B$, $v \in B'$. This path visits some sequence of blocks from B to B', moving from one to the next via a cut-vertex of G belonging to both of them. This describes a B, B'-path in $B(G)$.

We prove by contradiction that $B(G)$ also has no cycles and hence is a tree. Suppose x is a cut-vertex of G on a cycle C in $B(G)$. Let B, B' be the neighbors of x on C. The B, B' path $C - x$ provides a route from $B - x$ to $B' - x$ without using x. This is impossible, since when B, B' are two blocks of G containing cut-vertex x, every path between $B - x$ and $B' - x$ in G must pass through x.

b) If G is not a block, then at least two blocks of G each contain exactly one cut-vertex of G. Each cut-vertex of G belongs to at least two blocks of G. Hence the leaves of $B(G)$ all arise from blocks of G, not cut-vertices of G. If G is not a block, then $B(G)$ has at least two leaves, and the leaves of $B(G)$ are the desired blocks in G.

c) G has exactly $k + \sum_{v \in V(G)}(b(v) - 1)$ blocks, where k is the number of components of G and $b(v)$ is the number of blocks containing v.

Proof 1 (explicit count). Since we can count the blocks separately in each component, it suffices to show that a connected graph has $1 + \sum(b(v) - 1)$ blocks. Select a block in a connected graph G and view it as a root; this corresponds to the 1 in the formula. Each vertex v in this block leads us to $b(v) - 1$ new blocks. For each new block, each vertex v other than the one that leads us there leads us to $b(v) - 1$ new blocks. This process stops when we have counted $b(v) - 1$ for each vertex of G.

This tree-like exploration gives the desired count of blocks as long as two facts hold: 1) no two blocks intersect in more than one vertex, and 2) no block can be reached in more than one way from the root. These guarantee that we don't count blocks more than once. If either happens, we get a cycle of blocks, $B_1, \ldots, B_n, B_1$, with $n \geq 2$, so that successive blocks share a vertex. Then there is no vertex whose deletion will disconnect the subgraph that is the union of these blocks, which is impossible since blocks are maximal subgraphs with no cut-vertex.

Proof 2 (induction on the number of blocks). We need only prove the formula for connected graphs, since both the number of blocks and the value of the formula are sums over the components of G. If G is a block, then every vertex of G appears in one block, and the formula holds.

If G has a cutvertex, then by part (a) this component has a block B containing only one cutvertex, u. Delete all vertices of $B - u$ to obtain a graph G'. The blocks of G' are the blocks of G other than u, u appears in one less block than before, and all other terms of the formula are the same except that for G' we have left out the value 0 for the other vertices of B. The induction hypothesis now yields

$$\begin{aligned}\#blocks(G) = \#blocks(G') + 1 &= [1 + \sum_{v \in V(G')} (b_{G'}(v) - 1)] + 1 \\ &= 1 + \sum_{v \in V(G')} (b(v) - 1).\end{aligned}$$

d) Every graph has fewer cut-vertices than blocks. In the formula of part (c), there is a positive contribution for each cut-vertex. Thus the number of blocks is bigger than the number of cut-vertices, each yielding a term that contributes at least one to the sum.

9. Apply DFS algorithm to find the blocks in Figure 1. Show your computation steps.

Solution: