# Lecture 8: Matchings (2)

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https://shuaili8.github.io/Teaching/CS445/index.html

# General regular graph

- Corollary (2.1.5, D) Every regular graph of positive even degree has a 2-factor
  - A $k$-regular spanning subgraph is called a $k$-factor
  - A perfect matching is a 1-factor

# Application to SDR

- Given some family of sets $X$, a system of distinct representatives for the sets in $X$ is a 'representative' collection of distinct elements from the sets of $X$

$$S_1 = \{2, 8\},$$
$$S_2 = \{8\},$$
$$S_3 = \{5, 7\},$$
$$S_4 = \{2, 4, 8\},$$
$$S_5 = \{2, 4\}.$$

The family $X_1 = \{S_1, S_2, S_3, S_4\}$ does have an SDR, namely $\{2, 8, 7, 4\}$. The family $X_2 = \{S_1, S_2, S_4, S_5\}$ does not have an SDR.

- Theorem(1.52, H) Let $S_1, S_2, \dots, S_k$ be a collection of finite, nonempty sets. This collection has SDR $\Leftrightarrow$ for every $t \in [k]$, the union of any $t$ of these sets contains at least $t$ elements

# König Theorem
# Augmenting Path Algorithm

# Vertex cover

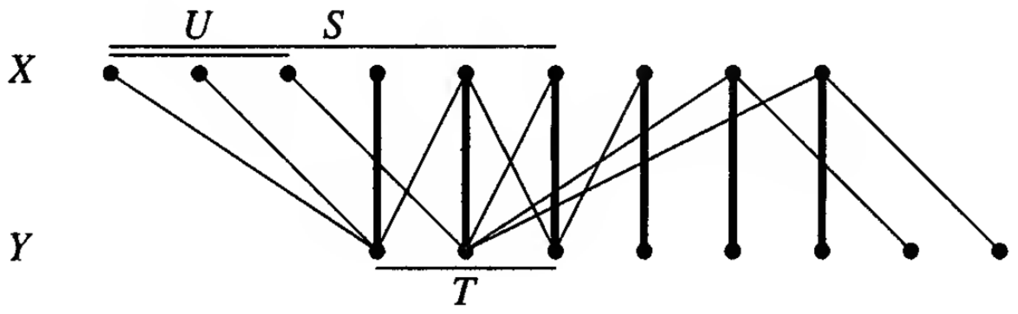
- A set $U \subseteq V$ is a (vertex) cover of $E$ if every edge in $G$ is incident with a vertex in $U$
- Example:
  - Art museum is a graph with hallways are edges and corners are nodes
  - A security camera at the corner will guard the paintings on the hallways
  - The minimum set to place the cameras?

# König-Egeváry Theorem (Min-max theorem)

- Theorem (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931) Let $G$ be a bipartite graph. The maximum size of a matching in $G$ is equal to the minimum size of a vertex cover of its edges

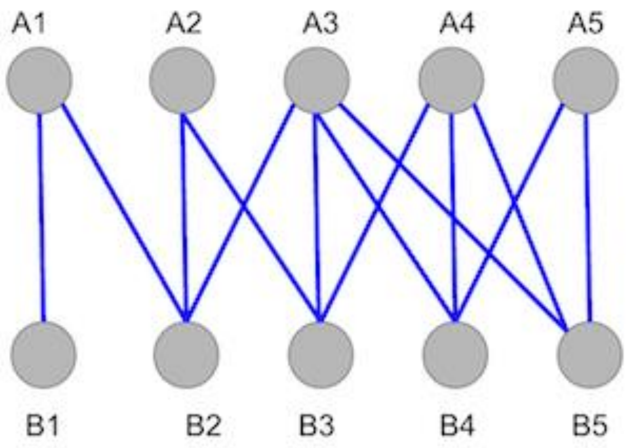
> Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching $M$ in a graph $G$ is a maximum matching in $G \Leftrightarrow G$ has no $M$-augmenting path

# Augmenting path algorithm

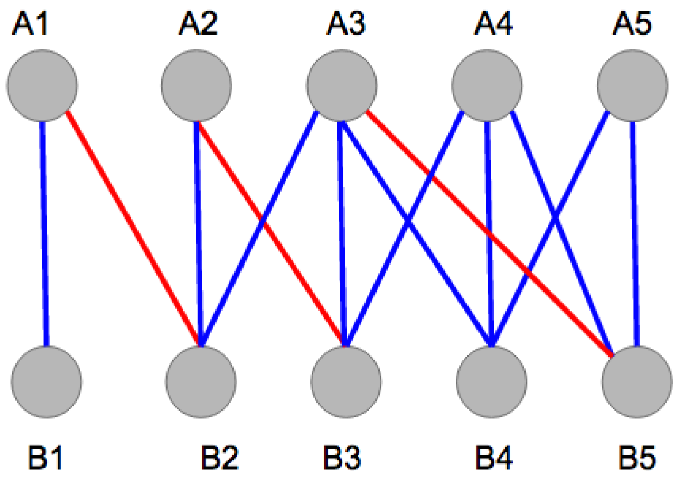



- **Input**: $G = B(X, Y)$, a matching $M$ in $G$
  $U = \{M$–unsaturated vertices in $X\}$
- **Idea**: Explore $M$-alternating paths from $U$ letting $S \subseteq X$ and $T \subseteq Y$ be the sets of vertices reached
- **Initialization**: $S = U, T = \emptyset$ and all vertices in $S$ are unmarked
- **Iteration**:
  - If S has no unmarked vertex, stop and report $T \cup (X - S)$ as a minimum cover and $M$ as a maximum matching
  - Otherwise, select an unmarked $x \in S$ to explore
    - Consider each $y \in N(x)$ such that $xy \notin M$
      - If $y$ is unsaturated, terminate and report an $M$-augmenting path from $U$ to $y$
      - Otherwise, $yw \in M$ for some $w$
        - include $y$ in $T$ (reached from $x$) and include $w$ in $S$ (reached from $y$)
    - After exploring all such edges incident to $x$, mark $x$ and iterate.

# Example




Red: A random matching



# Theoretical guarantee for Augmenting path algorithm

- Theorem (3.2.2, W) Repeatedly applying the Augmenting Path Algorithm to a bipartite graph produces a matching and a vertex cover of equal size

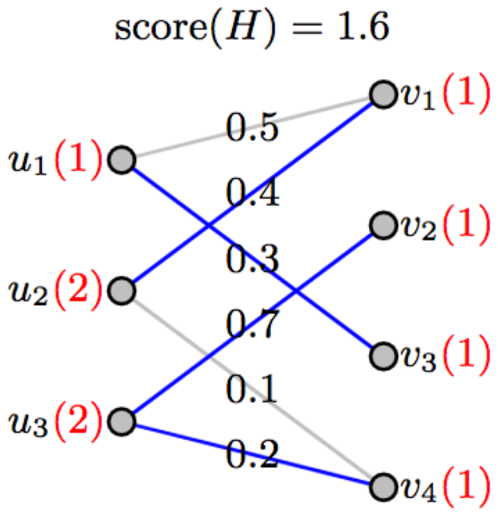
# Weighted Bipartite Matching
# Hungarian Algorithm

# Weighted bipartite matching

- The maximum weighted matching problem is to seek a perfect matching $M$ to maximize the total weight $w(M)$
- Bipartite graph
  - W.l.o.g. Assume the graph is $K_{n,n}$ with $w_{i,j} \geq 0$ for all $i, j \in [n]$
  - Optimization:

$$\max \sum_{i,j} a_{i,j} w_{i,j}$$
$$s.t.\; a_{i,1} + \cdots + a_{i,n} \leq 1 \text{ for any } i$$
$$a_{1,j} + \cdots + a_{n,j} \leq 1 \text{ for any } j$$
$$a_{i,j} \in \{0,1\}$$

  - Integer programming
  - General IP problems are NP-Complete



# (Weighted) cover

- A (weighted) cover is a choice of labels $u_1, \dots, u_n$ and $v_1, \dots, v_n$ such that $u_i + v_j \geq w_{i,j}$ for all $i, j$
  - The cost $c(u, v)$ of a cover $(u, v)$ is $\sum_i u_i + \sum_j v_j$
  - The minimum weighted cover problem is that of finding a cover of minimum cost
- Optimization problem

$$\min \sum_i u_i + \sum_j v_j$$
$$s.t.\, u_i + v_j \geq w_{i,j} \text{ for any } i, j$$
$$u_i, v_j \geq 0 \text{ for any } i, j$$

# Duality

(IP)

$$\max \sum_{i,j} a_{i,j} w_{i,j}$$

$$s.t.\, a_{i,1} + \cdots + a_{i,n} \leq 1 \text{ for any } i$$

$$a_{1,j} + \cdots + a_{n,j} \leq 1 \text{ for any } j$$

$$a_{i,j} \in \{0,1\}$$

(Linear programming)

$$\max \sum_{i,j} a_{i,j} w_{i,j}$$

$$s.t.\, a_{i,1} + \cdots + a_{i,n} \leq 1 \text{ for any } i$$

$$a_{1,j} + \cdots + a_{n,j} \leq 1 \text{ for any } j$$

$$a_{i,j} \geq 0$$

(Dual)

$$\min \sum_i u_i + \sum_j v_j$$

$$s.t.\, u_i + v_j \geq w_{i,j} \text{ for any } i, j$$

$$u_i, v_j \geq 0$$

- Weak duality theorem
  - For each feasible solution $a$ and $(u, v)$

$$\sum_{i,j} a_{i,j} w_{i,j} \leq \sum_i u_i + \sum_j v_j$$

thus $\max \sum_{i,j} a_{i,j} w_{i,j} \leq \min \sum_i u_i + \sum_j v_j$

# Duality (cont.)

- Strong duality theorem
  - If one of the two problems has an optimal solution, so does the other one and that the bounds given by the weak duality theorem are tight

$$\max \sum_{i,j} a_{i,j} w_{i,j} = \min \sum_{i} u_i + \sum_{j} v_j$$

- Lemma (3.2.7, W) For a perfect matching $M$ and cover $(u, v)$ in a weighted bipartite graph $G$, $c(u, v) \geq w(M)$
  $c(u, v) = w(M) \Leftrightarrow$ $M$ consists of edges $x_i y_j$ such that $u_i + v_j = w_{i,j}$
  In this case, $M$ and $(u, v)$ are optimal.

# Equality subgraph

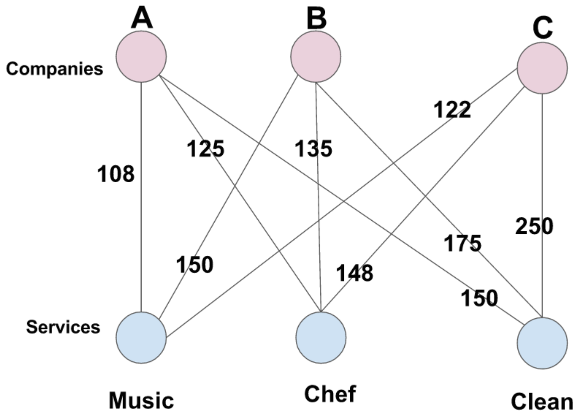
- The equality subgraph $G_{u,v}$ for a cover $(u, v)$ is the spanning subgraph of $K_{n,n}$ having the edges $x_i y_j$ such that $u_i + v_j = w_{i,j}$
  - So if $(u, v)$ is optimal, then $M$ consistes the edges in $G_{u,v}$

# Hungarian algorithm

- **Input**: Weighted $K_{n,n} = B(X, Y)$
- **Idea**: Iteratively adjusting the cover $(u, v)$ until the equality subgraph $G_{u,v}$ has a perfect matching
- **Initialization**: Let $(u, v)$ be a cover, such as $u_i = \max_j w_{i,j}$, $v_j = 0$

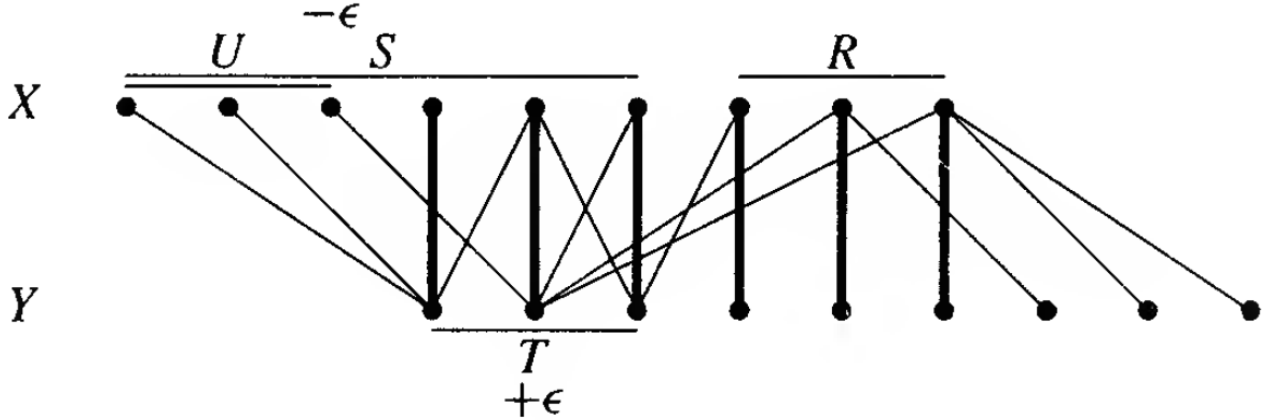
(Dual)

$$\min \sum_i u_i + \sum_j v_j$$
$$s.t.\, u_i + v_j \geq w_{i,j} \text{ for any } i, j$$
$$u_i, v_j \geq 0$$

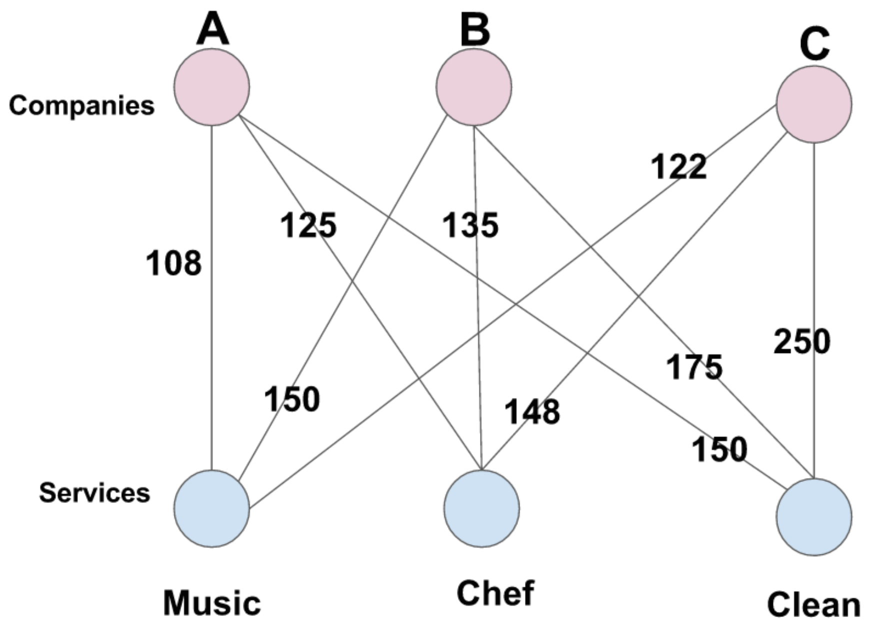


# Hungarian algorithm (cont.)

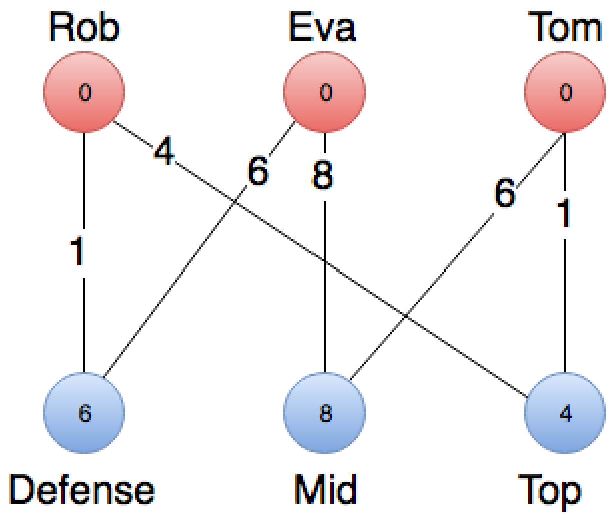
- **Iteration**: Find a maximum matching $M$ in $G_{u,v}$
  - If $M$ is a perfect matching, stop and report $M$ as a maximum weight matching
  - Otherwise, let $Q$ be a vertex cover of size $|M|$ in $G_{u,v}$
    - Let $R = X \cap Q, T = Y \cap Q$
    $$\epsilon = \min\{u_i + v_j - w_{i,j} : x_i \in X - R, y_j \in Y - T\}$$
    - Decrease $u_i$ by $\epsilon$ for $x_i \in X - R$ and increase $v_j$ by $\epsilon$ for $y_j \in T$
  - Form the new equality subgraph and repeat



# Example



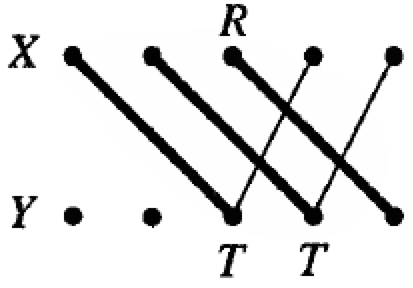
A
B
C
Companies
108
125
135
122
150
148
175
250
150
Services
Music
Chef
Clean

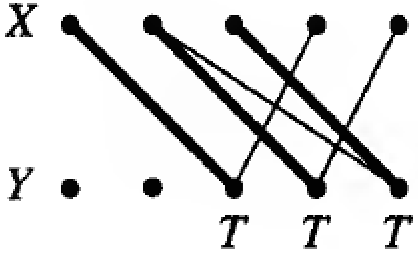
Rob
Eva
Tom
0
0
0
4
6
8
6
1
1
6
8
4
Defense
Mid
Top

# Example 2

$$
\begin{pmatrix} 4 & 1 & 6 & 2 & 3 \\ 5 & 0 & 3 & 7 & 6 \\ 2 & 3 & 4 & 5 & 8 \\ 3 & 4 & 6 & 3 & 4 \\ 4 & 6 & 5 & 8 & 6 \end{pmatrix}
\rightarrow
\begin{array}{c} \\ 6 \\ 7 \\ 8 \\ 6 \\ 8 \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array} \\
\begin{pmatrix} 2 & 5 & \underline{0} & 4 & 3 \\ 2 & 7 & 4 & \underline{0} & 1 \\ 6 & 5 & 4 & 3 & \underline{0} \\ 3 & 2 & 0 & 3 & 2 \\ 4 & 2 & 3 & 0 & 2 \end{pmatrix} \\
\begin{array}{ccccc} & & T & T & \end{array}
\end{array}
\begin{array}{c} \\ \\ \\ R \\ \\ \\ \\ \end{array}
$$




$$
\begin{array}{c} \\ 5 \\ 6 \\ 8 \\ 5 \\ 7 \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 0 & 1 & 1 & 0 \end{array} \\
\begin{pmatrix} 1 & 4 & \underline{0} & 4 & 2 \\ 1 & 6 & 4 & \underline{0} & 0 \\ 6 & 5 & 5 & 4 & \underline{0} \\ 2 & 1 & 0 & 3 & 1 \\ 3 & 1 & 3 & 0 & 1 \end{pmatrix} \\
\begin{array}{ccccc} & & T & T & T \end{array}
\end{array}
$$




$$
\rightarrow
\begin{array}{c} \\ 4 \\ 5 \\ 7 \\ 4 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 0 & 2 & 2 & 1 \end{array} \\
\begin{pmatrix} 0 & 3 & \underline{0} & 4 & 2 \\ \underline{0} & 5 & 4 & 0 & 0 \\ 5 & 4 & 5 & 4 & \underline{0} \\ 1 & \underline{0} & 0 & 3 & 1 \\ 2 & 0 & 3 & \underline{0} & 1 \end{pmatrix}
\end{array}
$$

Optimal value is the same
But the solution is not unique

# Theoretical guarantee for Hungarian algorithm

- Theorem (3.2.11, W) The Hungarian Algorithm finds a maximum weight matching and a minimum cost cover

# Back to (unweighted) bipartite graph

- The weights are binary $0,1$
- Hungarian algorithm always maintain integer labels in the weighted cover, thus the solution will always be $0,1$
- The vertices receiving label $1$ must cover the weight on the edges, thus cover all edges
- So the solution is a minimum vertex cover