Lecture 7: Coloring

Shuai Li

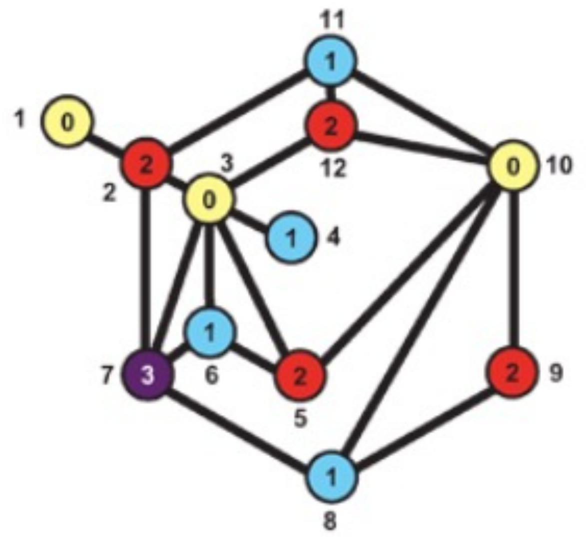
John Hopcroft Center, Shanghai Jiao Tong University

https://shuaili8.github.io

https://shuaili8.github.io/Teaching/CS445/index.html

Motivation: Scheduling and coloring

- University examination timetabling
  - Two courses linked by an edge if they have the same students
- Meeting scheduling
  - Two meetings are linked if they have same member



Definitions

- Given a graph G and a positive integer k, a k-coloring is a function $K: V(G) \longrightarrow \{1, \dots, k\}$ from the vertex set into the set of positive integers less than or equal to k. If we think of the latter set as a set of k "colors," then K is an assignment of one color to each vertex.
- We say that K is a proper k-coloring of G if for every pair u, v of adjacent vertices, $K(u) \neq K(v)$ — that is, if adjacent vertices are colored differently. If such a coloring exists for a graph G, we say that G is k-colorable
- In a proper coloring, each color class is an independent set. Then G is k-colorable $\Leftrightarrow$ $V(G)$ is the union of k independent sets

Chromatic number

- Given a graph G, the chromatic number of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable
- Examples

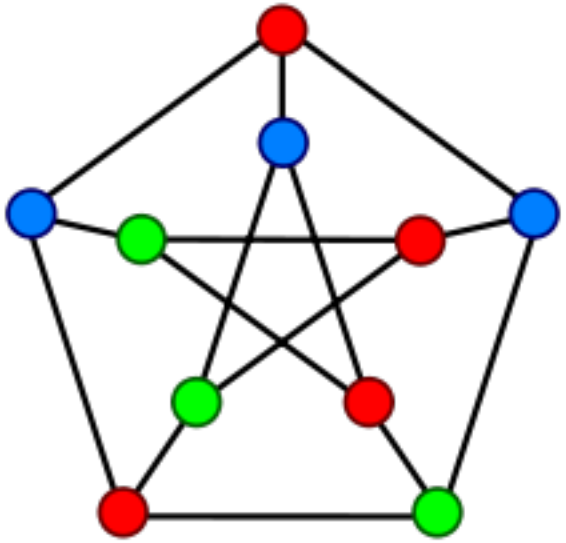
$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd,} \end{cases}$$

$$\chi(P_n) = \begin{cases} 2 & \text{if } n \geq 2, \\ 1 & \text{if } n = 1, \end{cases}$$

$$\chi(K_n) = n,$$

$\chi(E_n) = 1,$ ←Empty graph

$$\chi(K_{m,n}) = 2.$$



- (Ex5, S1.6.1, H) A graph G of order at least two is bipartite $\Leftrightarrow$ it is 2-colorable

Theorem (1.2.18, W, Kőnig 1936)
A graph is bipartite $\Leftrightarrow$ it contains no odd cycle

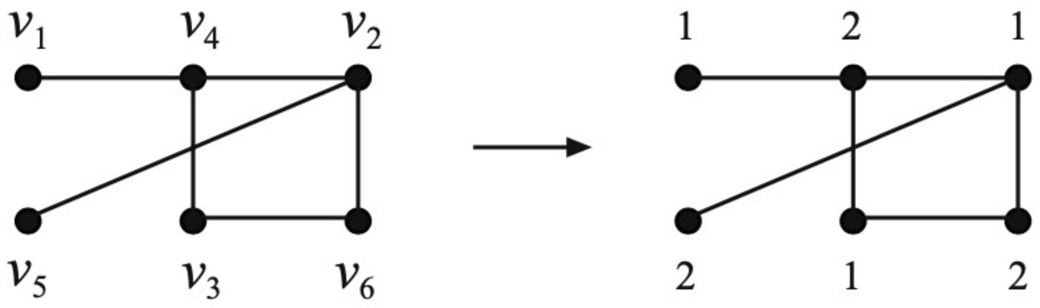
Bounds on Chromatic number

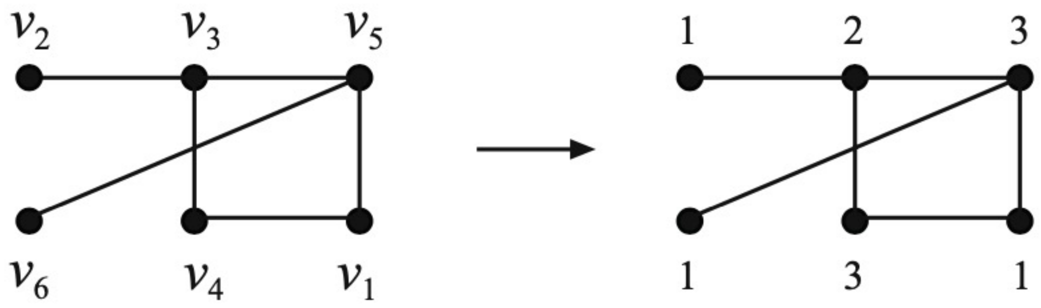
- Theorem (1.41, H) For any graph G of order n, $\chi(G) \leq n$
- It is tight since $\chi(K_n) = n$
- $\chi(G) = n \Leftrightarrow G = K_n$

Greedy algorithm

- First label the vertices in some order—call them $v_1, v_2, \dots, v_n$
- Next, order the available colors $(1,2,\dots,n)$ in some way
  - Start coloring by assigning color 1 to vertex v_1
  - If v_1 and v_2 are adjacent, assign color 2 to vertex v_2; otherwise, use color 1
  - To color vertex v_i, use the first available color that has not been used for any of v_i's previously colored neighbors

Examples: Different orders result in different number of colors



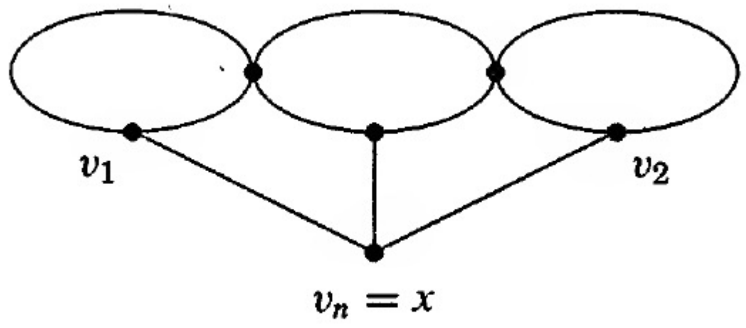


Bound of the greedy algorithm

- Theorem (1.42, H) For any graph G, $\chi(G) \leq \Delta(G) + 1$
  The equality is obtained for complete graphs and odd cycles

Brooks’s theorem

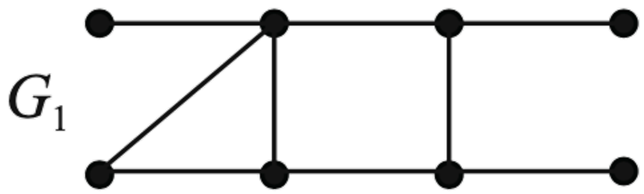
- Theorem (1.43, H; 5.1.22, W; 5.2.4, D; Brooks 1941)
  If G is a connected graph that is neither an odd cycle or a complete graph, then $\chi(G) \leq \Delta(G)$

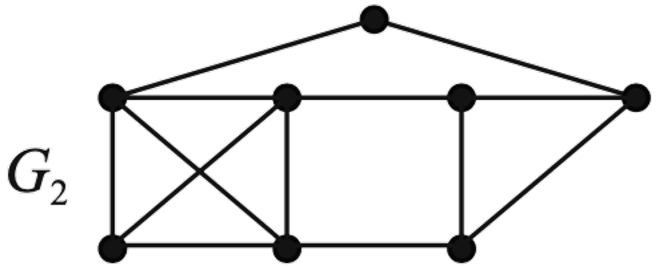


- $\Rightarrow$The Petersen graph is 3-colorable

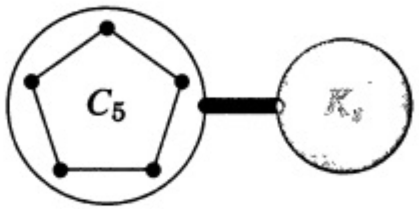
Chromatic number and clique number

- The clique number $\omega(G)$ of a graph is defined as the order of the largest complete graph that is a subgraph of G
- Example: $\omega(G_1) = 3, \omega(G_2) = 4$





- Theorem (1.44, H; 5.1.7, W) For any graph G, $\chi(G) \geq \omega(G)$
- Example (5.1.8, W) For $G = C_{2r+1} \vee K_s$, $\chi(G) > \omega(G)$



Chromatic number and independence number

- Theorem (1.45, H; 5.1.7, W; Ex6, S1.6.2, H) For any graph G of order n,

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G)$$

The independence number of a graph G, denoted as $\alpha(G)$, is the largest size of an independent set

In a proper coloring, each color class is an independent set. Then G is k-colorable $\Leftrightarrow$ $V(G)$ is the union of k independent sets

Extremal problems for k-chromatic graphs

- Proposition (5.2.5, W) Every k-chromatic graph with n vertices has at least $\binom{k}{2}$ edges
  - Equality holds for a complete graph plus isolated vertices.

In a proper coloring, each color class is an independent set. Then G is k-colorable $\Leftrightarrow V(G)$ is the union of k independent sets

- The Turán graph $T_{n,r}$ is the complete r-partite graph with n vertices whose partite sets differ by at most 1 vertex
  - Every partite set has size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$
- Lemma (5.2.8, W) Among simple r-partite (that is, r-colorable) graphs with n vertices, the Turán graph is the unique graph with the most edges
- Turán's Theorem (5.2.9, W; Turán 1941) Among the n-vertex simple K_{r+1}-free graphs, $T_{n,r}$ has the maximum number of edges

Summary

- Coloring

Shuai Li

https://shuaili8.github.io

Questions?