Lecture 6: Markov Decision Processes

Shuai Li

John Hopcroft Center, Shanghai Jiao Tong University

https://shuaili8.github.io

https://shuaili8.github.io/Teaching/CS410/index.html

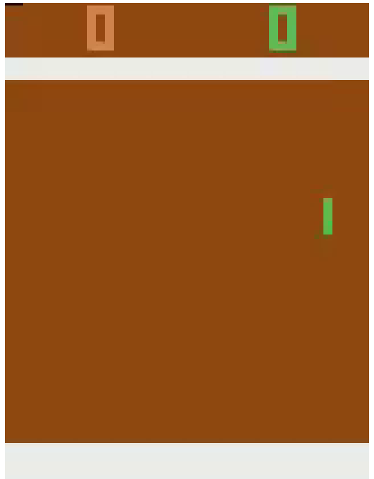
Part of slide credits: CMU AI & http://ai.berkeley.edu

Recent Progress by Deep Reinforcement Learning

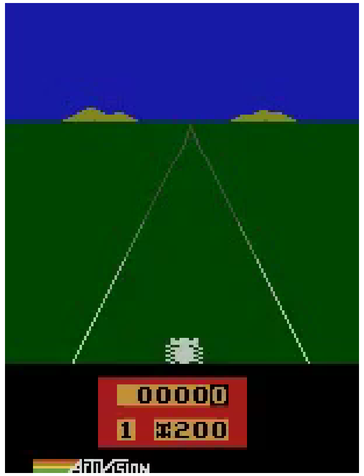
Deep Reinforcement Learning

2013

Atari (DQN)
[Deepmind]

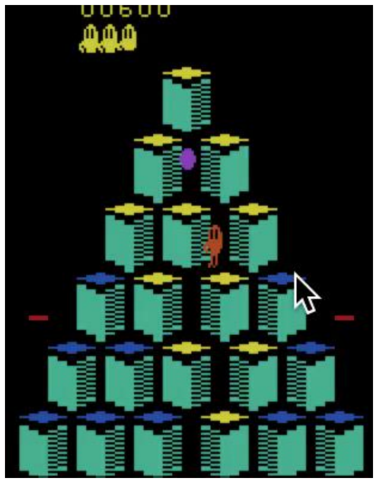


Pong



Enduro

Beamrider

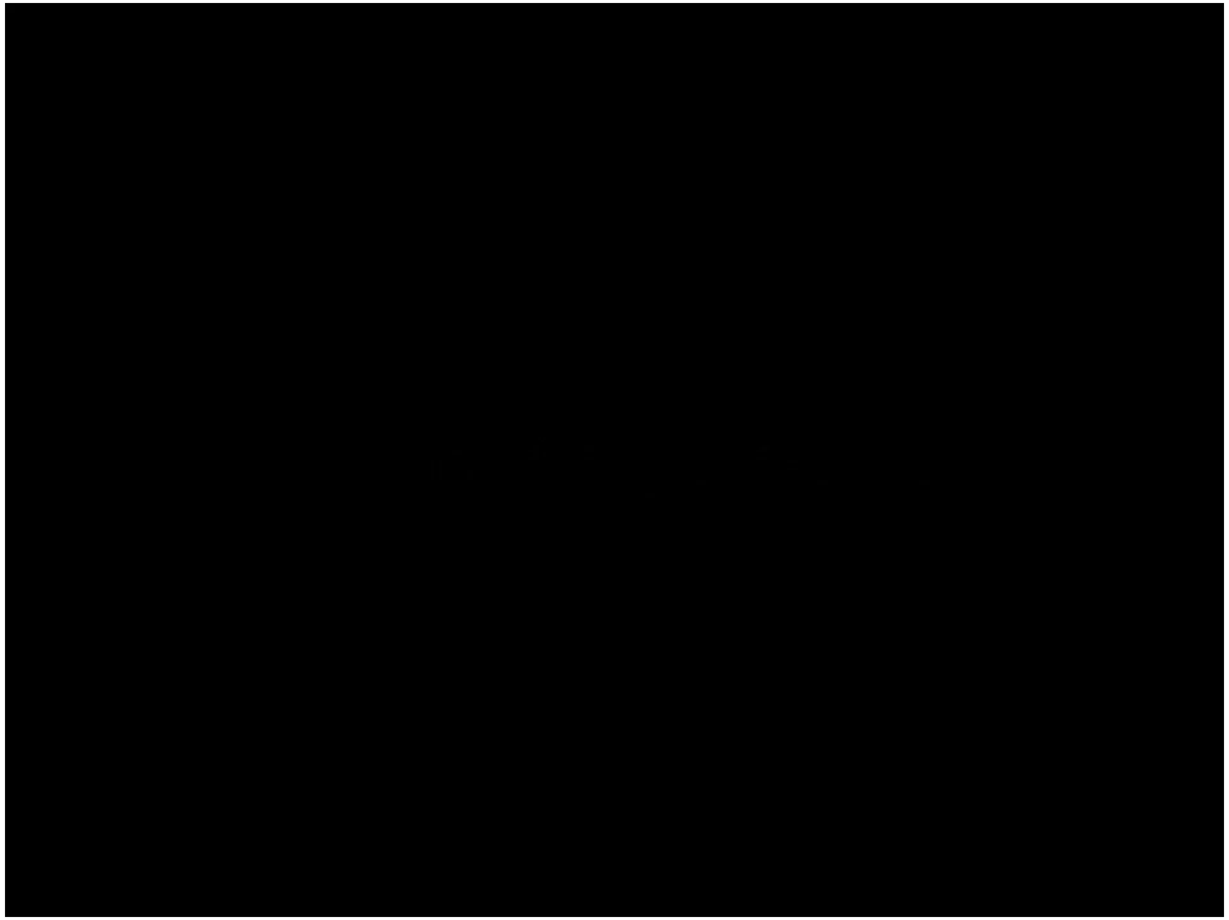


Q*bert

Deep Reinforcement Learning 2

2013 **Atari (DQN)**
[Deepmind]

2015 **Human-level control**
[Deepmind]



Trained separate DQN agents for 50 different Atari games, without any prior knowledge of the game rules

Mnih, V., Kavukcuoglu, K., Silver, D., Rusu, A. A., Veness, J., Bellemare, M. G., ... & Petersen, S. (2015). Human-level control through deep reinforcement learning. *Nature*, *518*(7540), 529.

Deep Reinforcement Learning 3

2013 **Atari (DQN)**
[Deepmind]

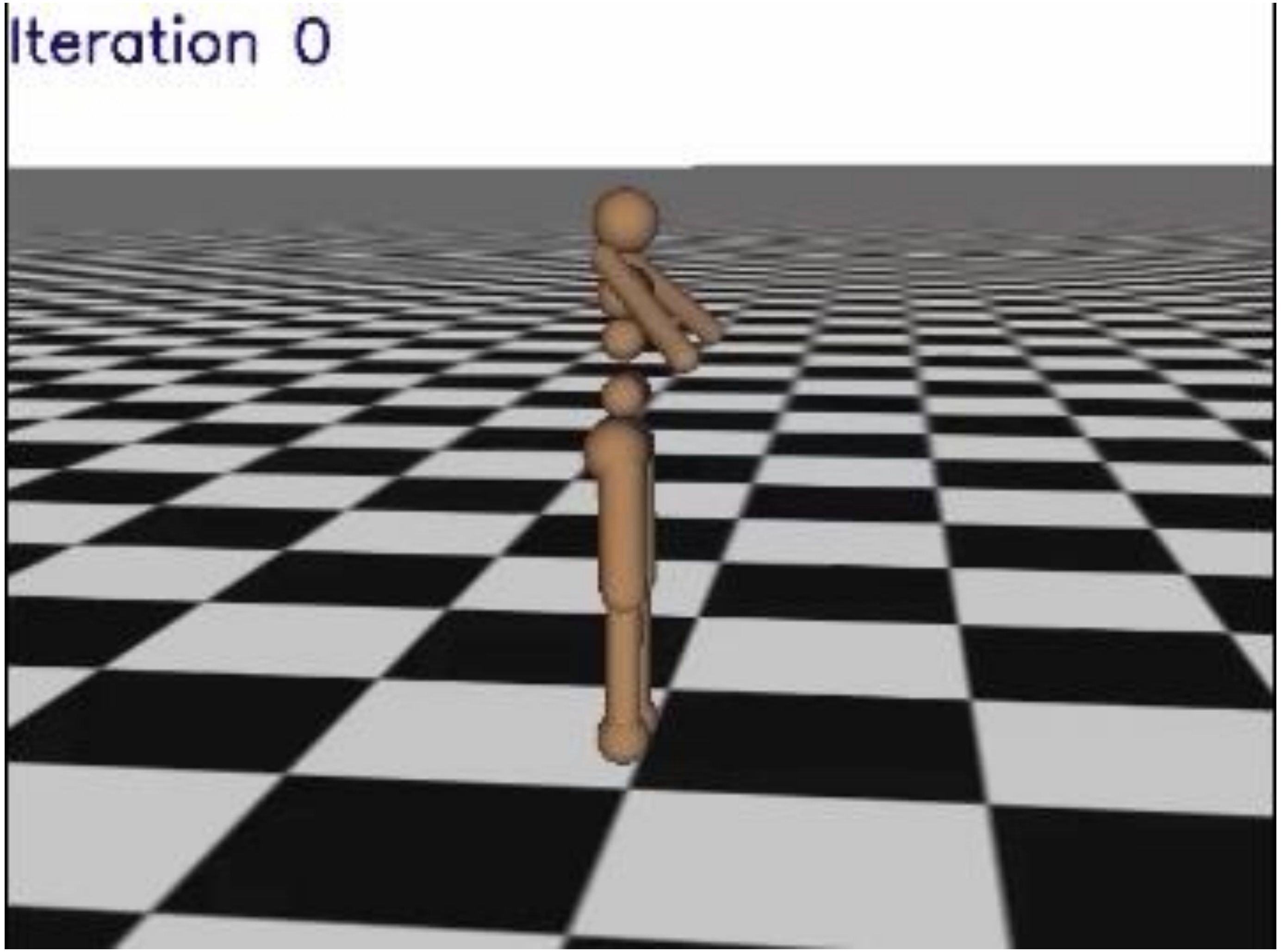
2015 **Human-level control**
[Deepmind]

AlphaGo
[Deepmind]

AlphaGo Silver et al, Nature 2015
AlphaGoZero Silver et al, Nature 2017
AlphaZero Silver et al, 2017
Tian et al, 2016; Maddison et al, 2014; Clark et al, 2015

Deep Reinforcement Learning 4

2013	**Atari (DQN)** **[Deepmind]**
2015	**Human-level control** **[Deepmind]**
	AlphaGo **[Deepmind]**
2016	**3D locomotion (TRPO+GAE)** **[Berkeley]**



[Schulman, Moritz, Levine, Jordan, Abbeel, ICLR 2016]

Deep Reinforcement Learning 5

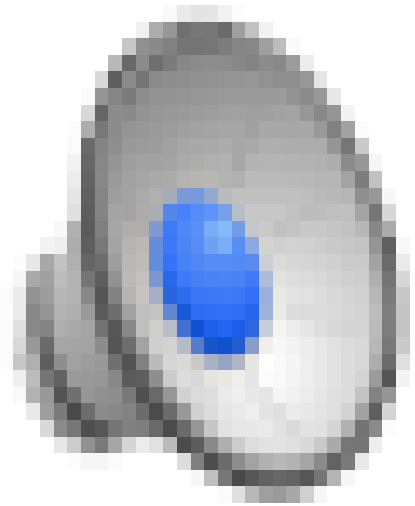
2013 **Atari (DQN)**
[Deepmind]

2015 **Human-level control**
[Deepmind]

AlphaGo
[Deepmind]

2016 **3D locomotion (TRPO+GAE)**
[Berkeley]

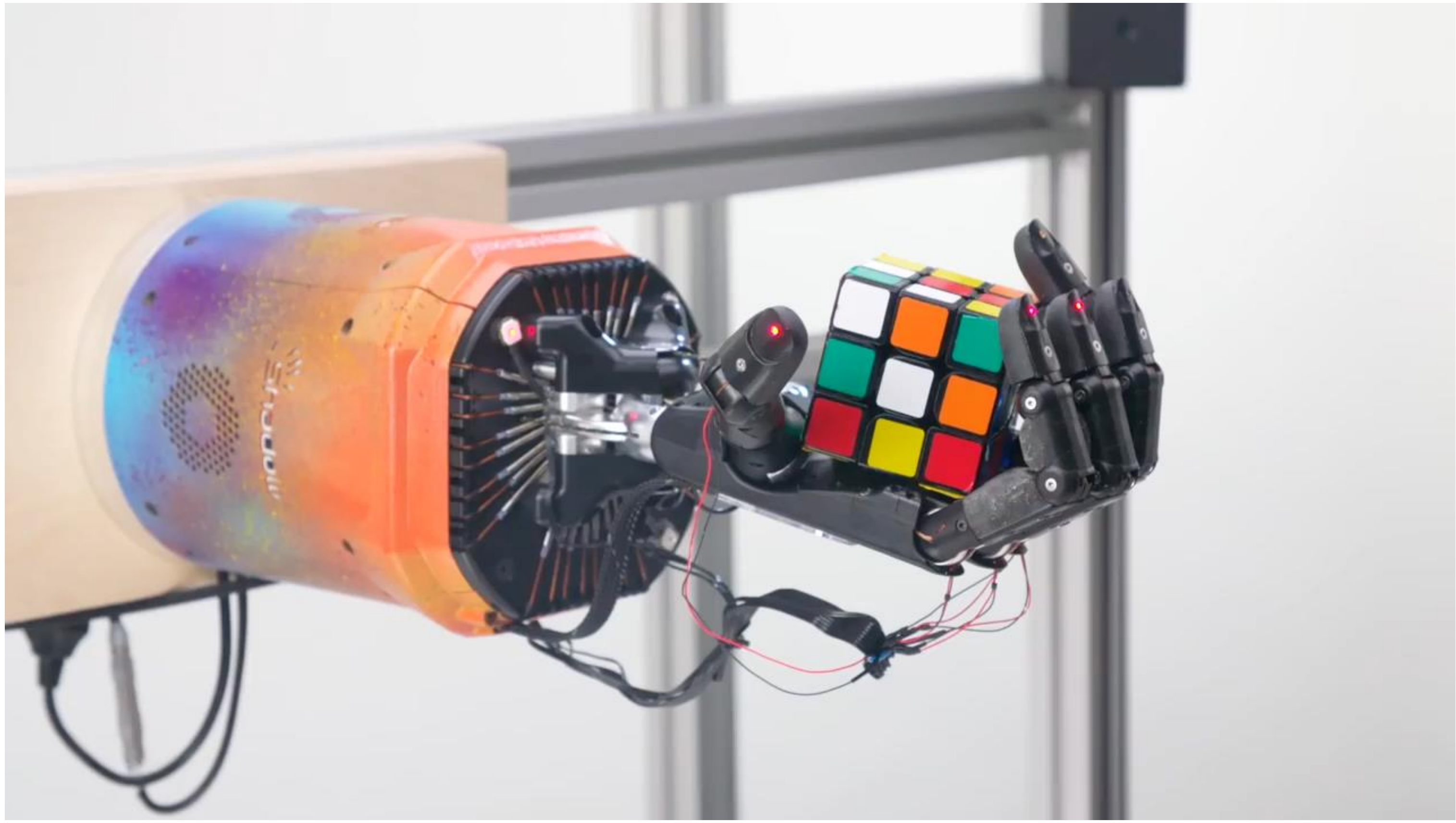
Real Robot Manipulation (GPS)
[Berkeley]



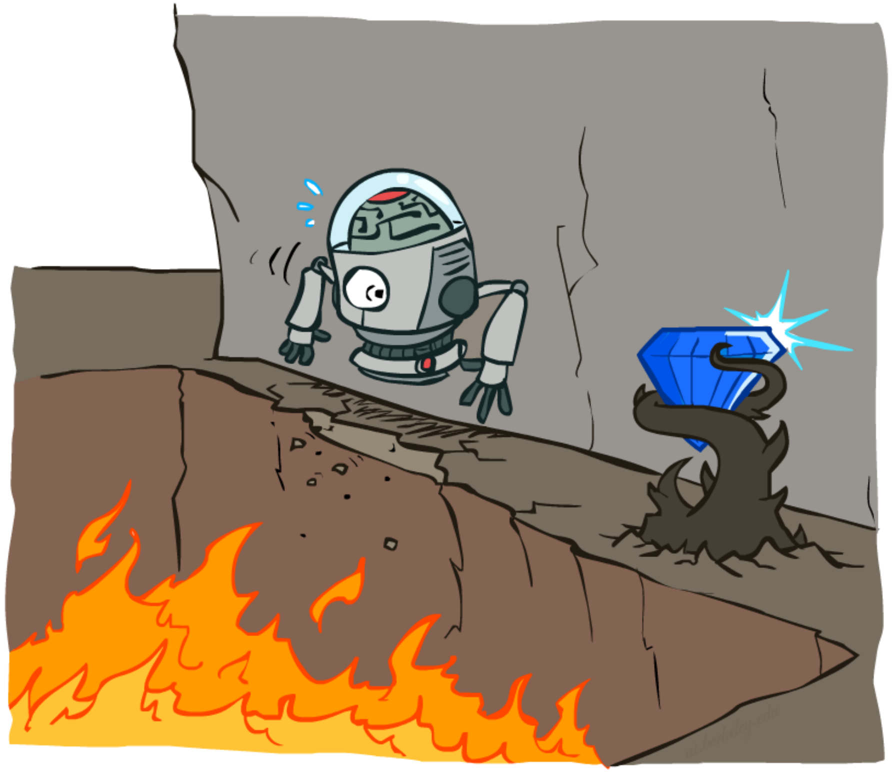
[Levine*, Finn*, Darrell, Abbeel, JMLR 2016]

Deep Reinforcement Learning 6

2013	**Atari (DQN)** **[Deepmind]**
2015	**Human-level control** **[Deepmind]**
	AlphaGo **[Deepmind]**
2016	**3D locomotion (TRPO+GAE)** **[Berkeley]**
	Real Robot Manipulation (GPS) **[Berkeley]**
2019	**Rubik's Cube (PPO+DR)** **[OpenAI]**

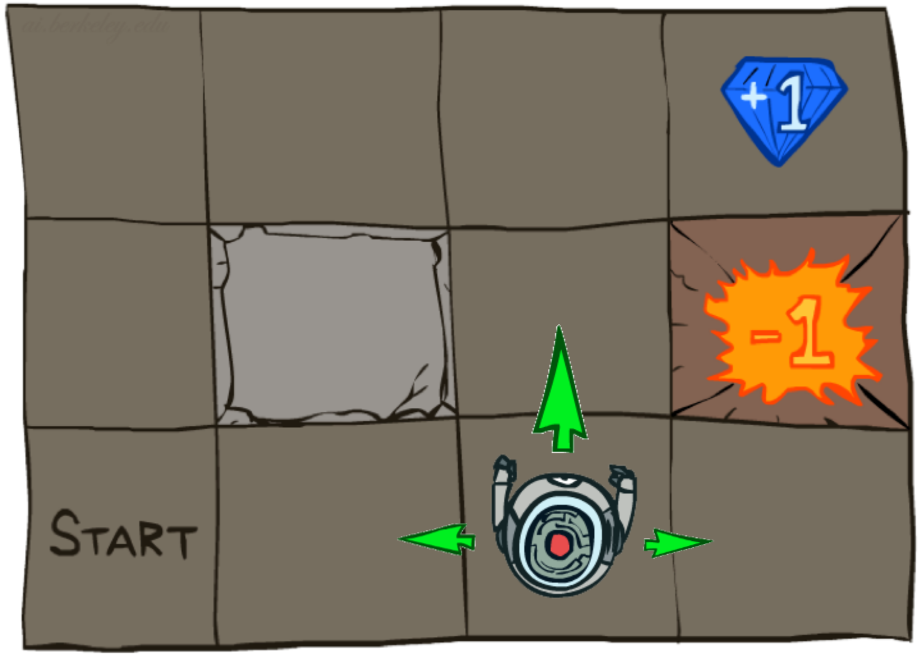


OpenAI



Non-Deterministic Search

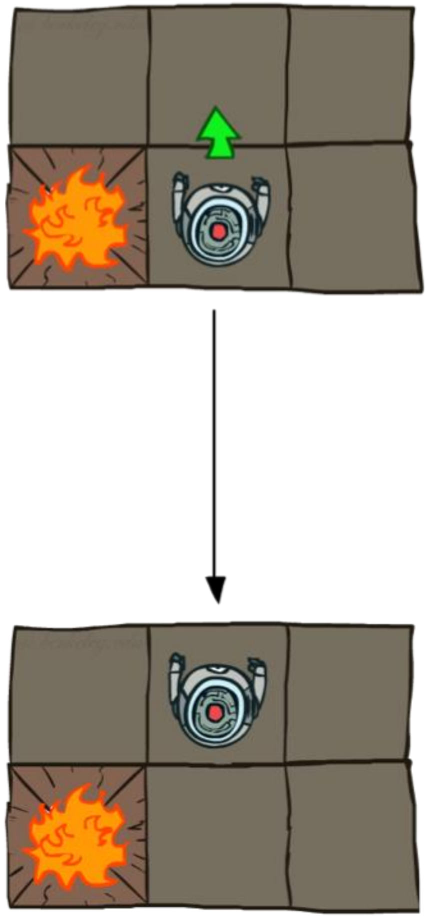
Example: Grid World



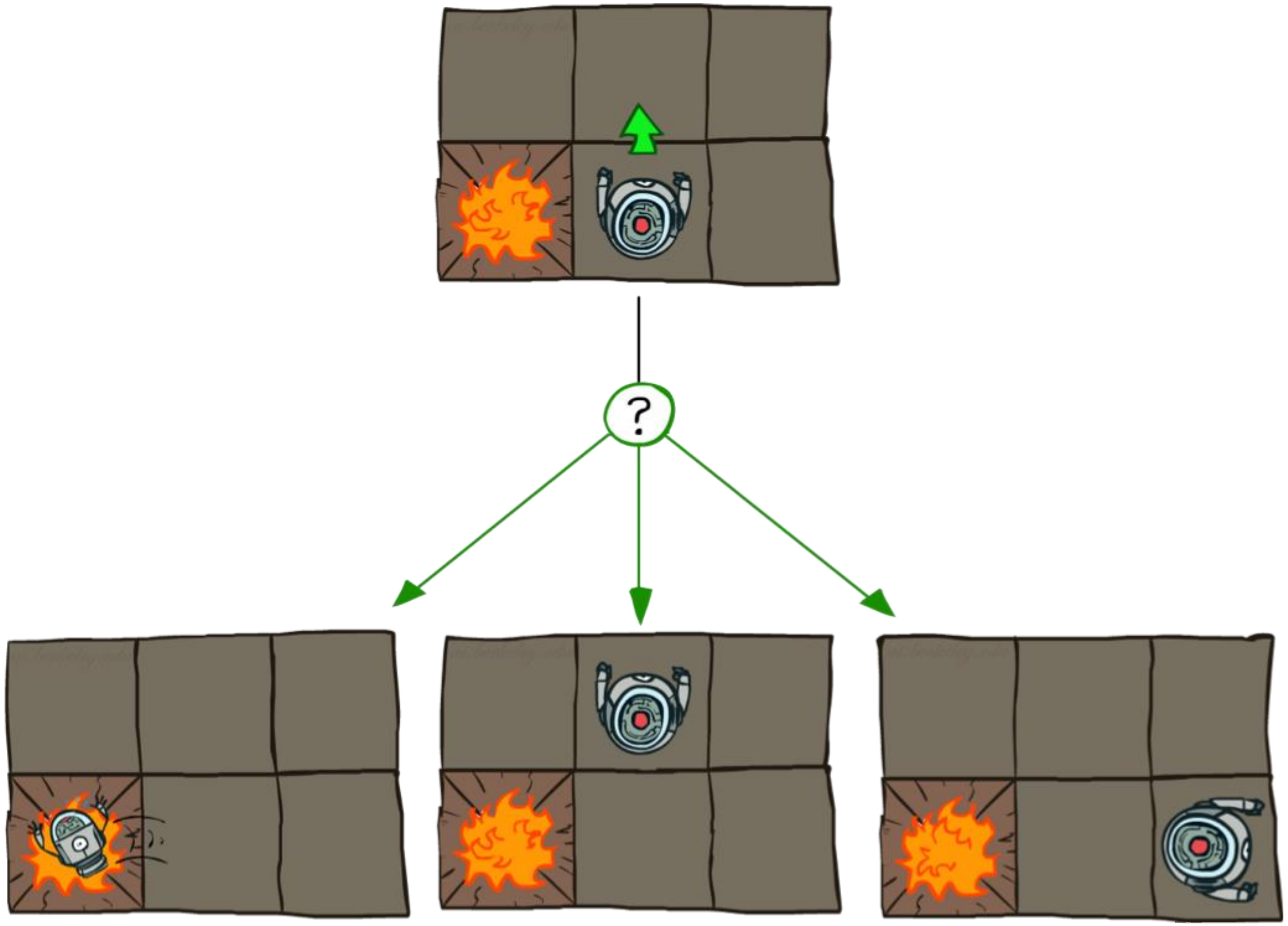
- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent's path
- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards
  - Small "living" reward each step (can be negative)
  - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards

Grid World Actions

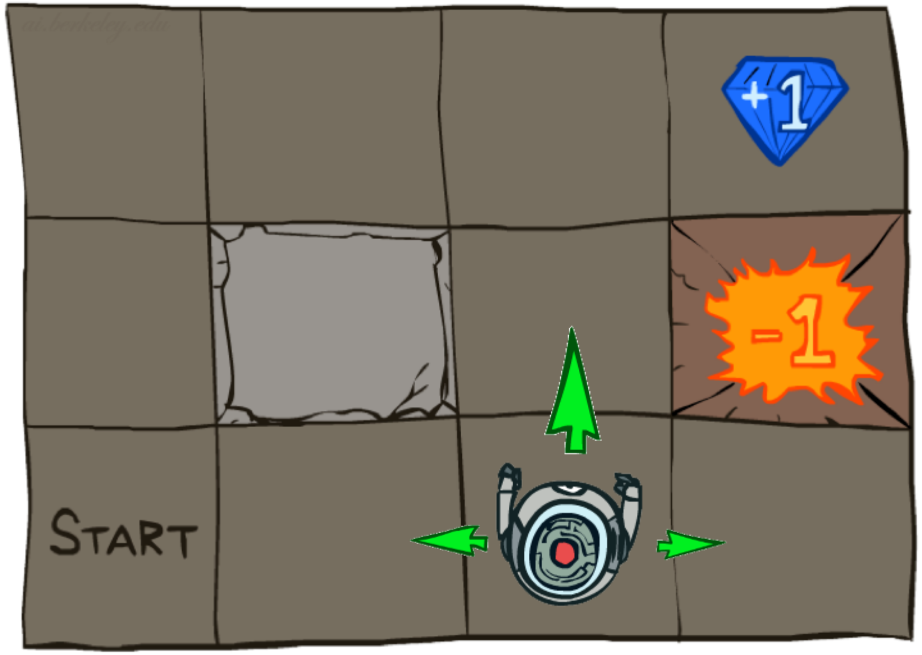
Deterministic Grid World



Stochastic Grid World



Markov Decision Processes

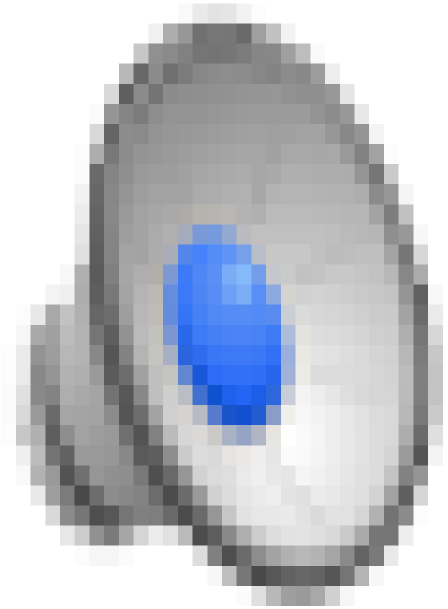


- An MDP is defined by:
  - A set of states $s \in S$
  - A set of actions $a \in A$
  - A transition function $T(s, a, s')$
    - Probability that a from s leads to s', i.e., $P(s' | s, a)$
    - Also called the model or the dynamics
  - A reward function $R(s, a, s')$
    - Sometimes just $R(s)$ or $R(s')$
  - A start state
  - Maybe a terminal state

- MDPs are non-deterministic search problems
  - One way to solve them is with expectimax search
  - We'll have a new tool soon

[Demo – gridworld manual intro (L8D1)]

Video of Demo Gridworld Manual Intro



What is Markov about MDPs?

- "Markov" generally means that given the present state, the future and the past are independent
- For Markov decision processes, "Markov" means action outcomes depend only on the current state

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \ldots S_0 = s_0)$$

$$=$$

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

- This is just like search, where the successor function could only depend on the current state (not the history)

Andrey Markov
(1856-1922)

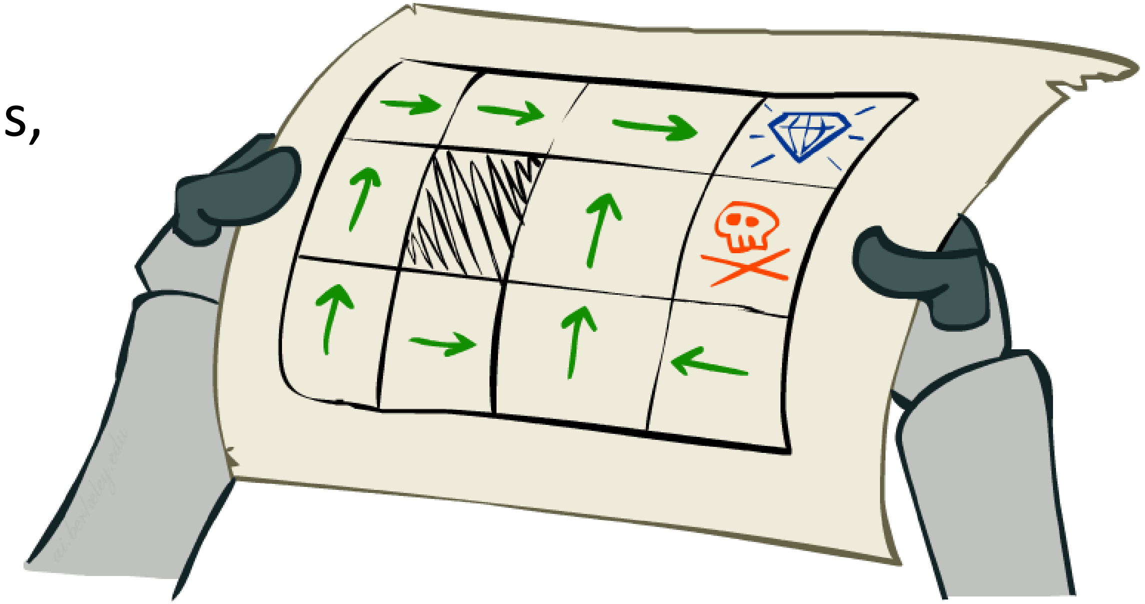
Policies

- In deterministic single-agent search problems, we wanted an optimal plan, or sequence of actions, from start to a goal

- For MDPs, we want an optimal

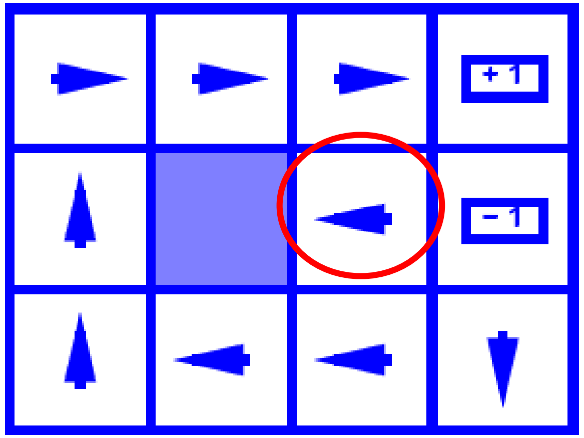
  policy $\pi^*: S \rightarrow A$

  - A policy π gives an action for each state
  - An optimal policy is one that maximizes expected utility if followed
  - An explicit policy defines a reflex agent

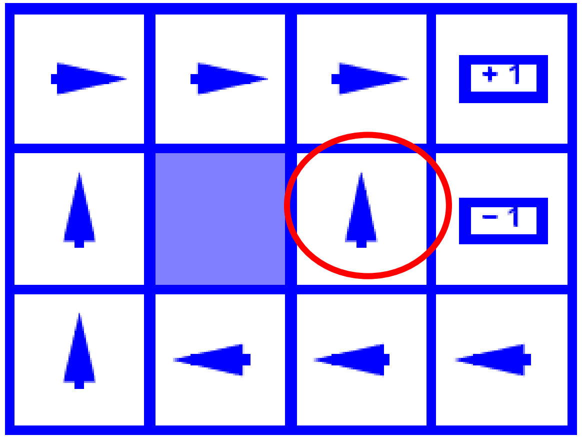


Optimal policy when $R(s, a, s') = -0.03$ for all non-terminals s

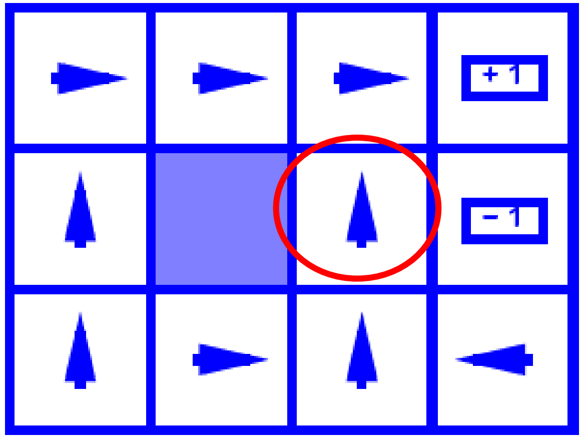
Optimal Policies



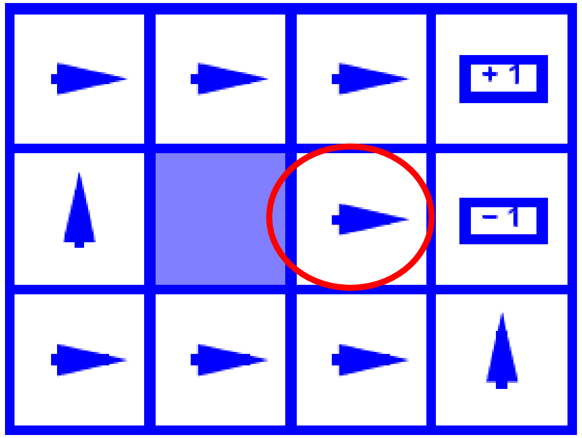
R(s) = -0.01



R(s) = -0.03

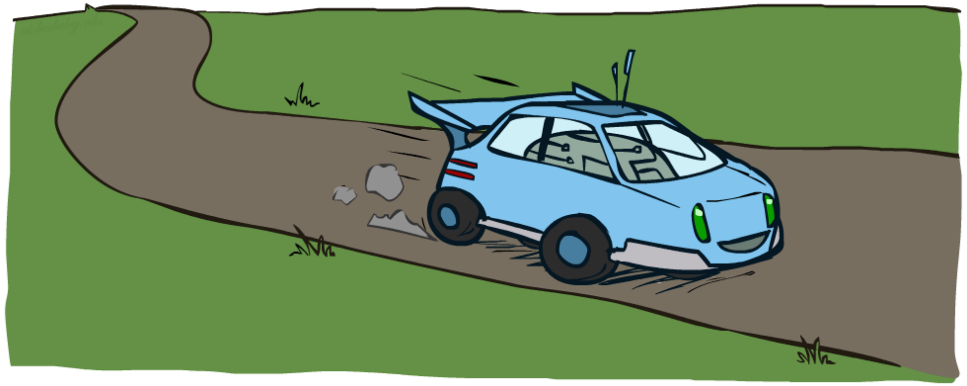


R(s) = -0.4

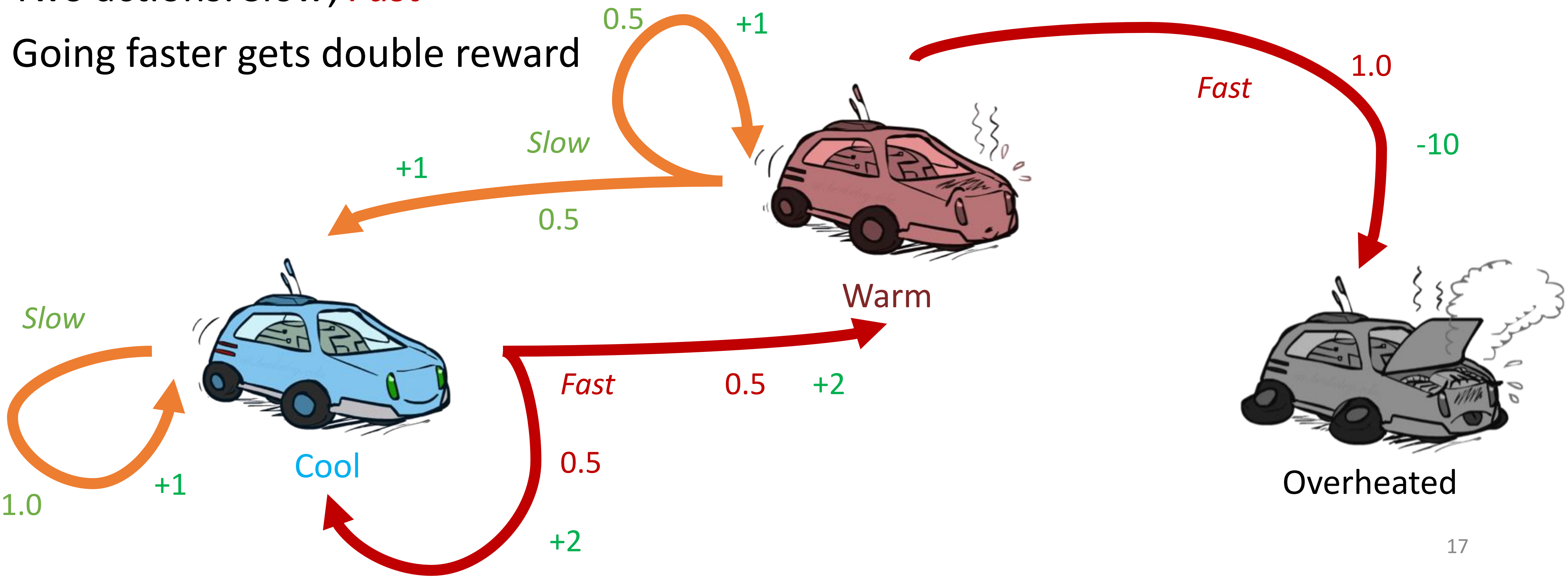


R(s) = -2.0

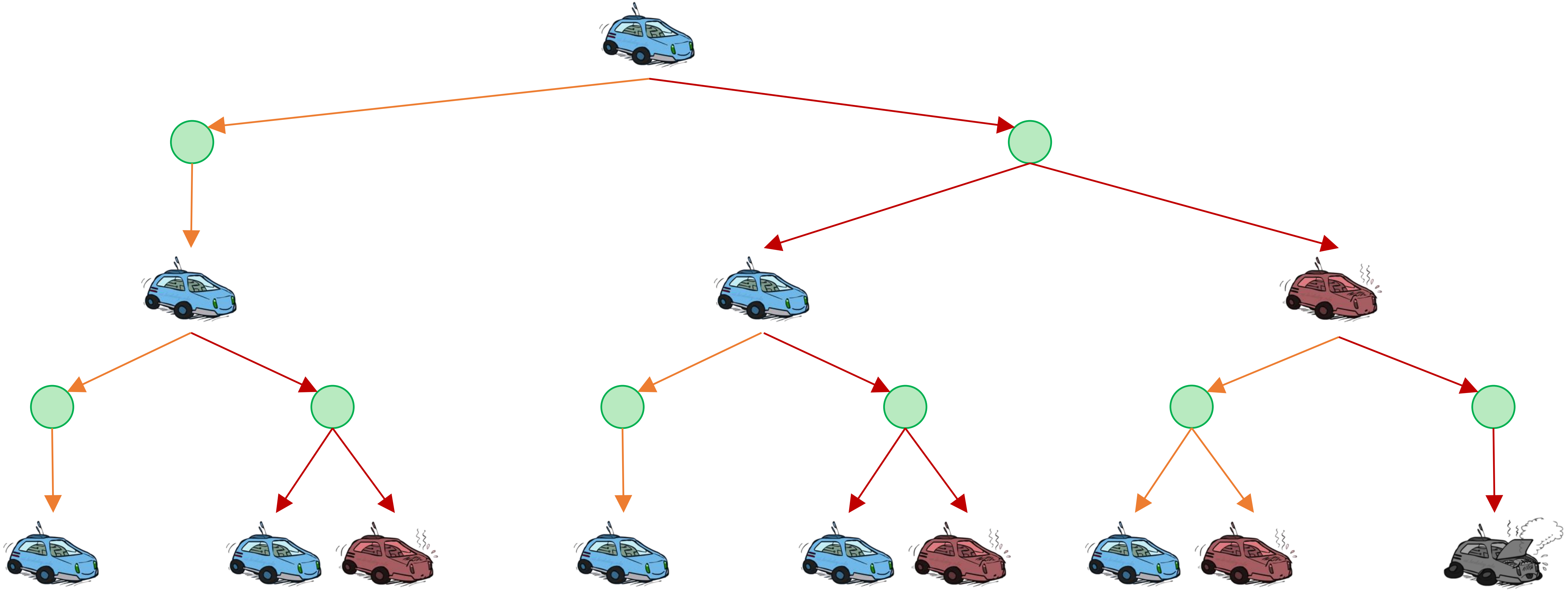
Example: Racing



- A robot car wants to travel far, quickly
- Three states: Cool, Warm, Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward

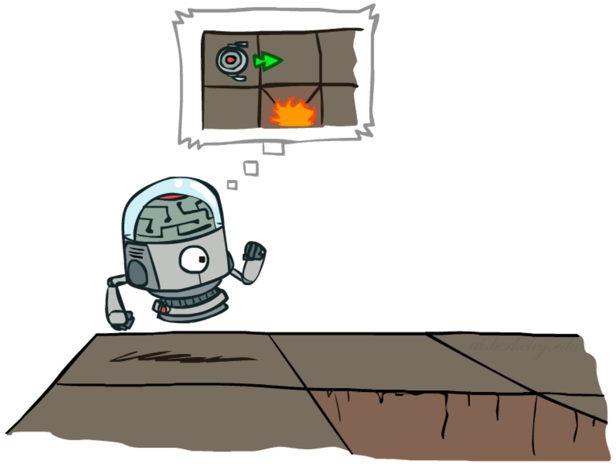


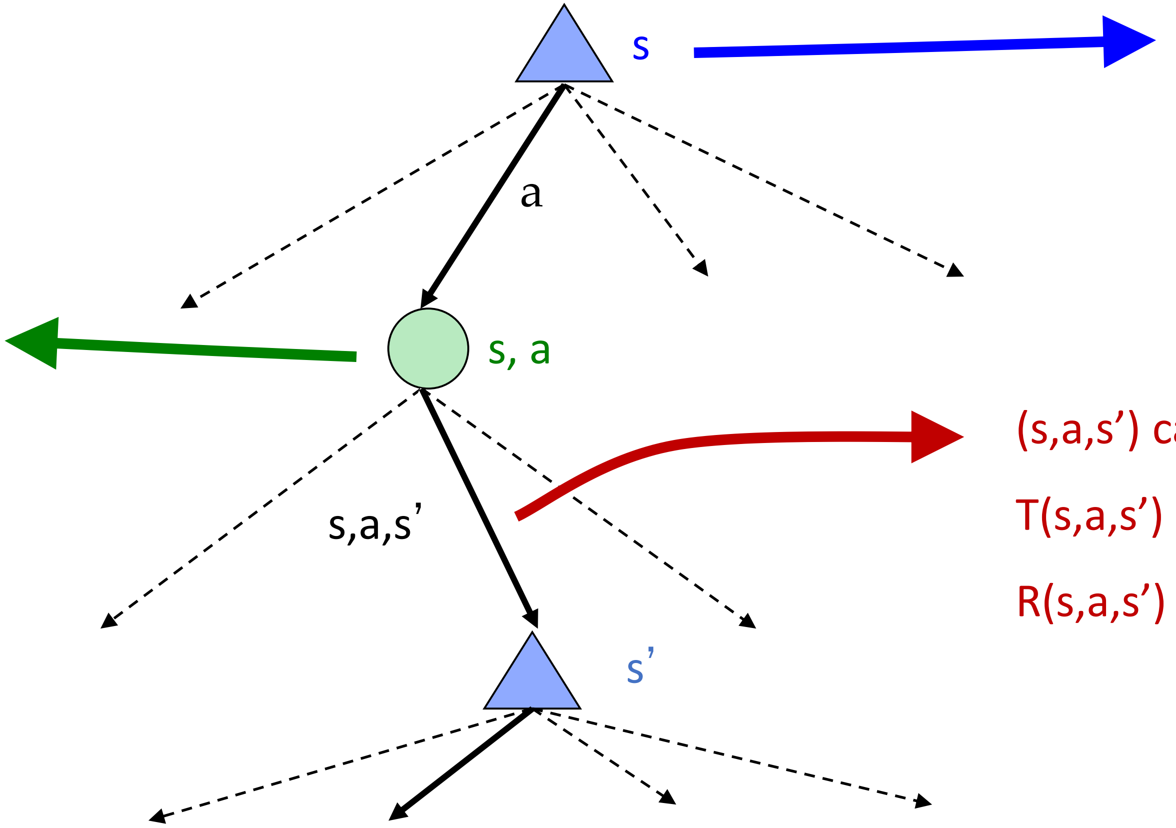
Example: Racing - Search Tree



MDP Search Trees

- Each MDP state projects an expectimax-like search tree



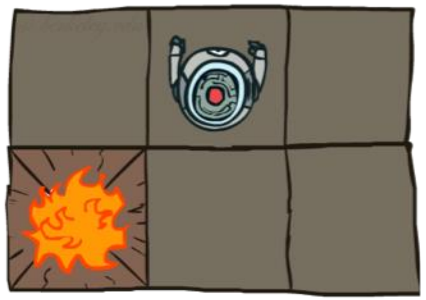


s is a *state*

(s,a,s') called a *transition*

T(s,a,s') = P(s'|s,a)

R(s,a,s')



Utilities of Sequences

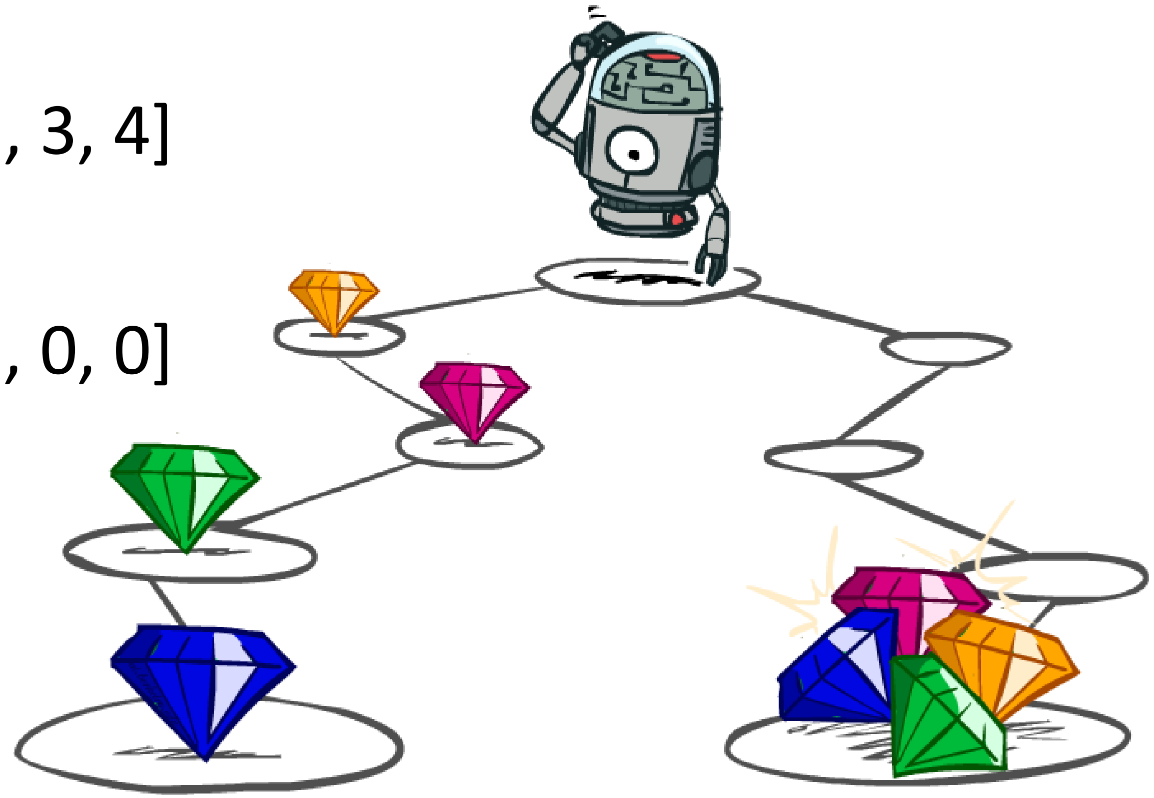
- What preferences should an agent have over reward sequences?

- More or less?

  [1, 2, 2] or [2, 3, 4]

- Now or later?

  [0, 0, 1] or [1, 0, 0]



Utilities of Sequences: Discounting

- It's reasonable to maximize the sum of rewards
- It's also reasonable to prefer rewards now to rewards later
- One solution: values of rewards decay exponentially

1

Worth Now

γ

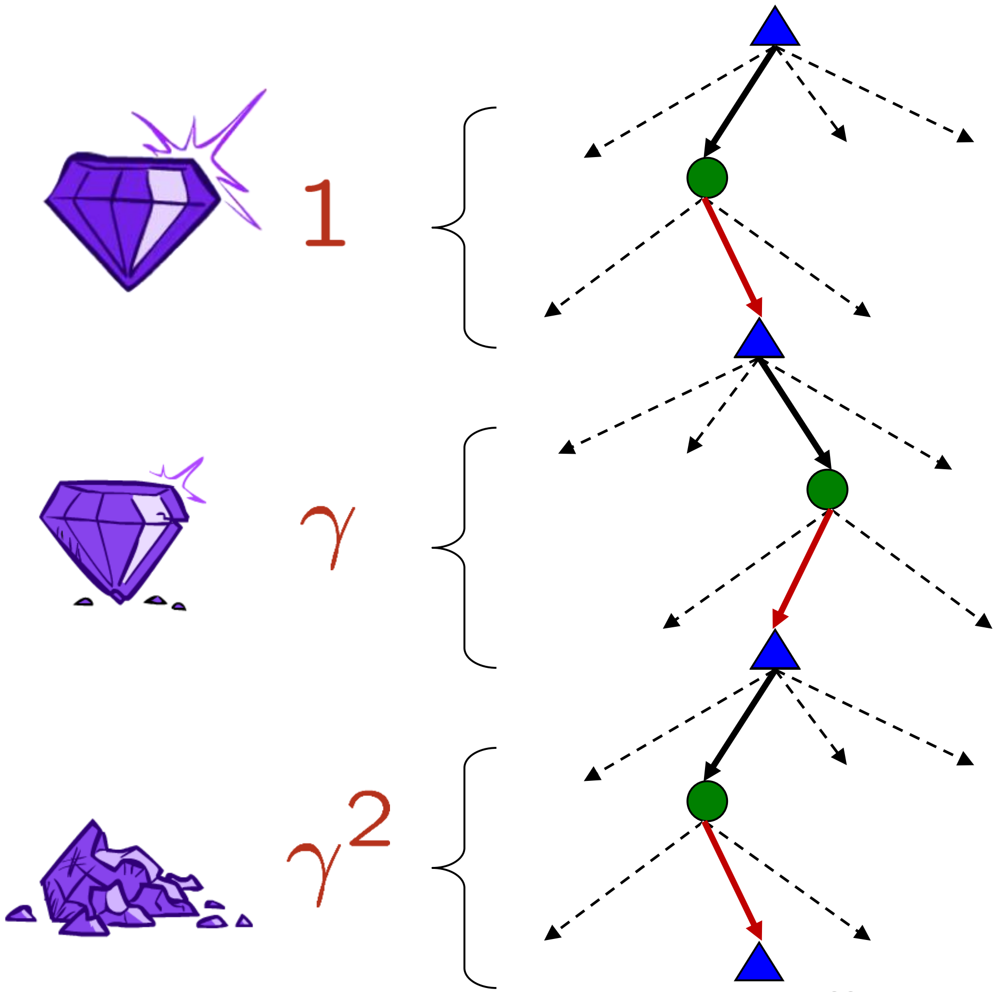
Worth Next Step

γ^2

Worth In Two Steps

Utilities of Sequences: Discounting 2

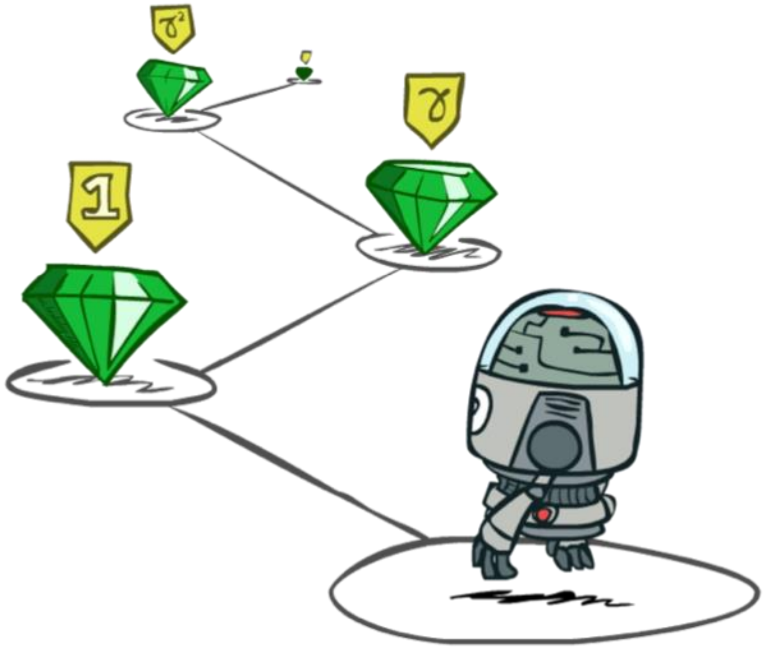
- How to discount?
  - Each time we descend a level, we multiply in the discount once
- Why discount?
  - Reward now is better than later
  - Can also think of it as a 1-gamma chance of ending the process at every step
  - Also helps our algorithms converge
- Example: discount of 0.5
  - U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3
  - U([1,2,3]) < U([3,2,1])



Utilities of Sequences: Stationary Preferences

- Theorem: if we assume stationary preferences:

$$[a_1, a_2, \ldots] \succ [b_1, b_2, \ldots]$$
$$\Updownarrow$$
$$[r, a_1, a_2, \ldots] \succ [r, b_1, b_2, \ldots]$$



- Then: there are only two ways to define utilities
  - Additive utility: $U([r_0, r_1, r_2, \ldots]) = r_0 + r_1 + r_2 + \cdots$
  - Discounted utility: $U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots$

Quiz: Discounting

- Given:

10				1
a	b	c	d	e

  - Actions: East, West, and Exit (only available in exit states a, e)
  - Transitions: deterministic

- Quiz 1: For γ = 1, what is the optimal policy?

10	<-	<-	<-	1

- Quiz 2: For γ = 0.1, what is the optimal policy?

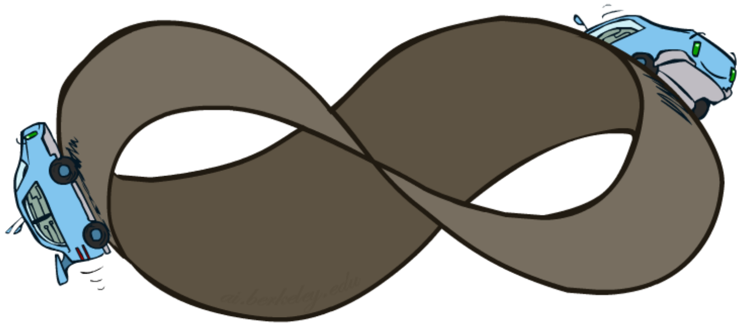
10	<-	<-	->	1

- Quiz 3: For which γ are West and East equally good when in state d?

$1\gamma=10\ \gamma^3$

Infinite Utilities?!

- Problem: What if the game lasts forever? Do we get infinite rewards?
- Solutions:
  - Finite horizon: (similar to depth-limited search)
    - Terminate episodes after a fixed T steps (e.g. life)
    - Gives nonstationary policies (π depends on time left)



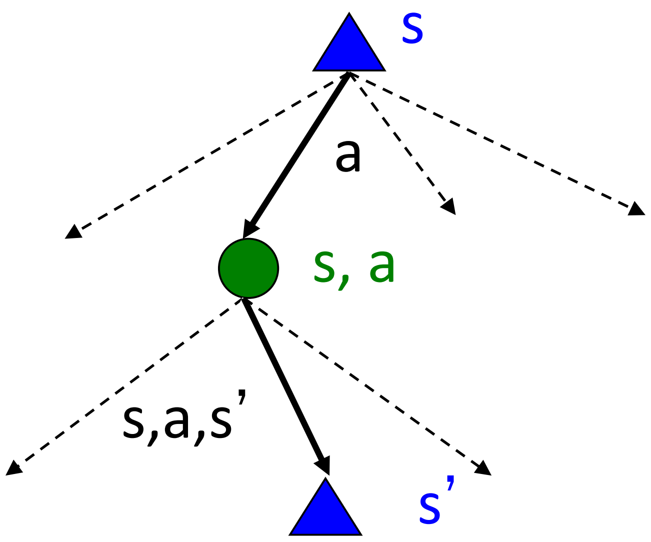
  - Discounting: use $0 < \gamma < 1$

$$U([r_0, \dots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1-\gamma)$$

    - Smaller γ means smaller "horizon" – shorter term focus
  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like "overheated" for racing)

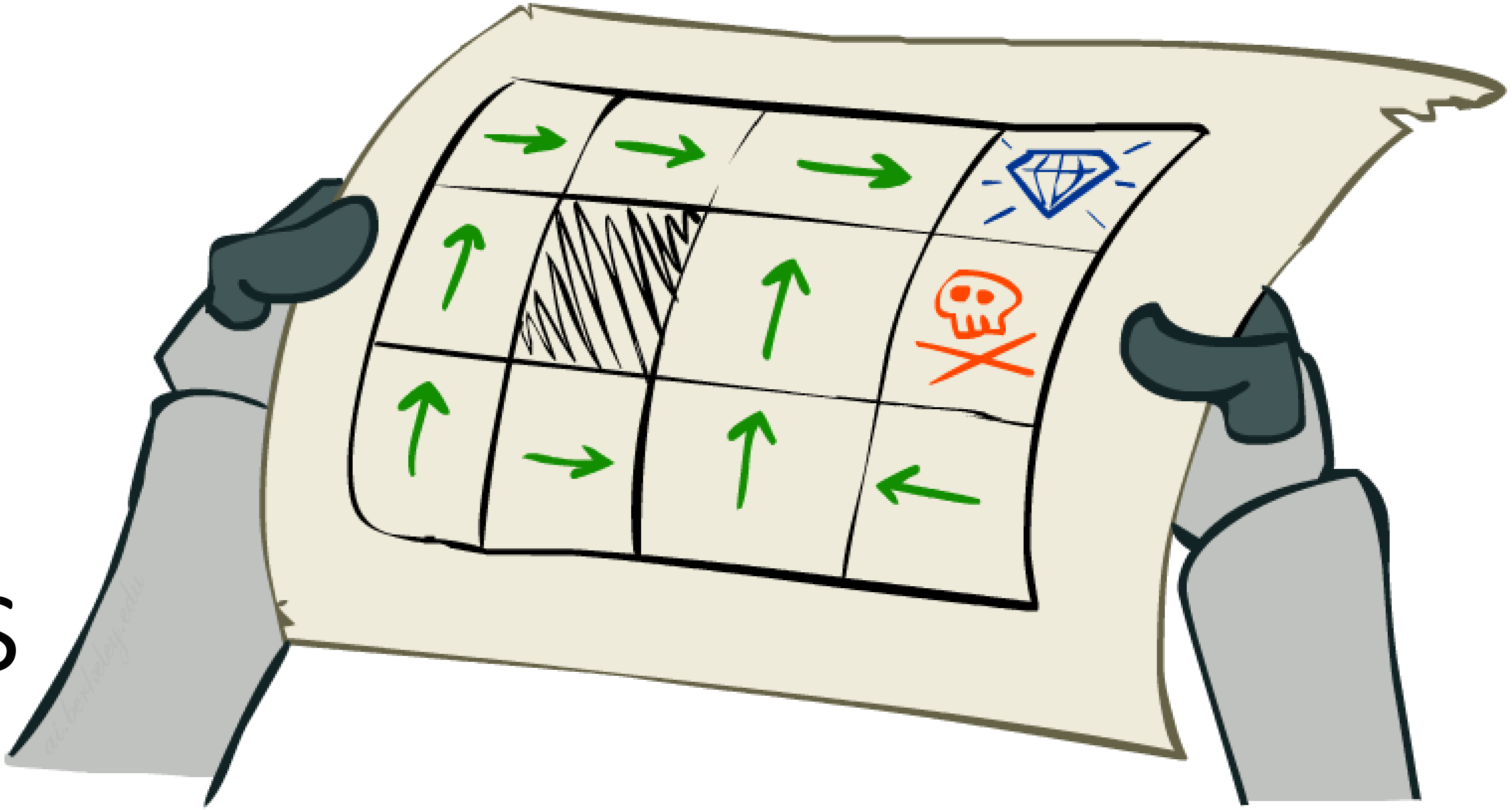
Recap: Defining MDPs

- Markov decision processes:
  - Set of states S
  - Start state s_0
  - Set of actions A
  - Transitions P(s'|s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount γ)

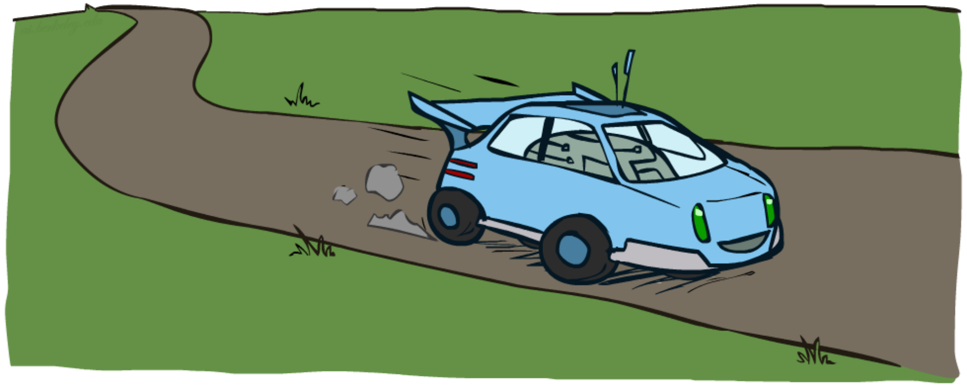


- MDP quantities so far:
  - Policy = Choice of action for each state
  - Utility = sum of (discounted) rewards

Solving MDPs



Recall: Racing MDP



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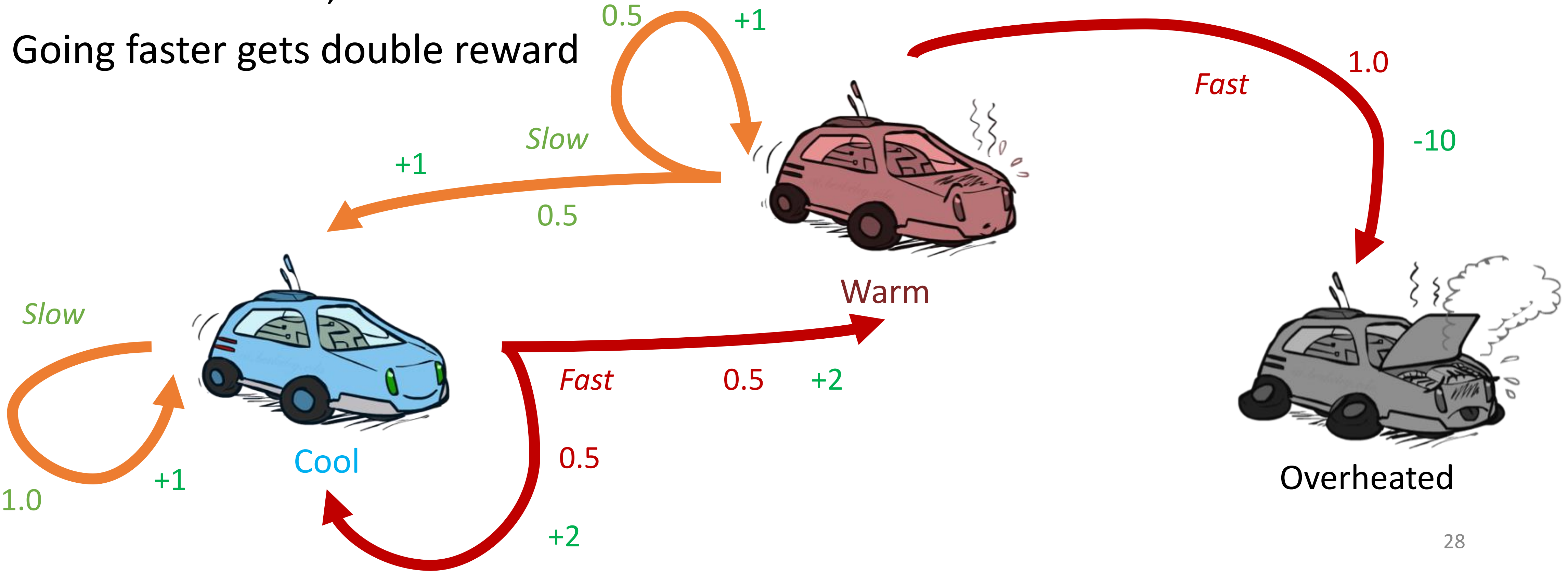
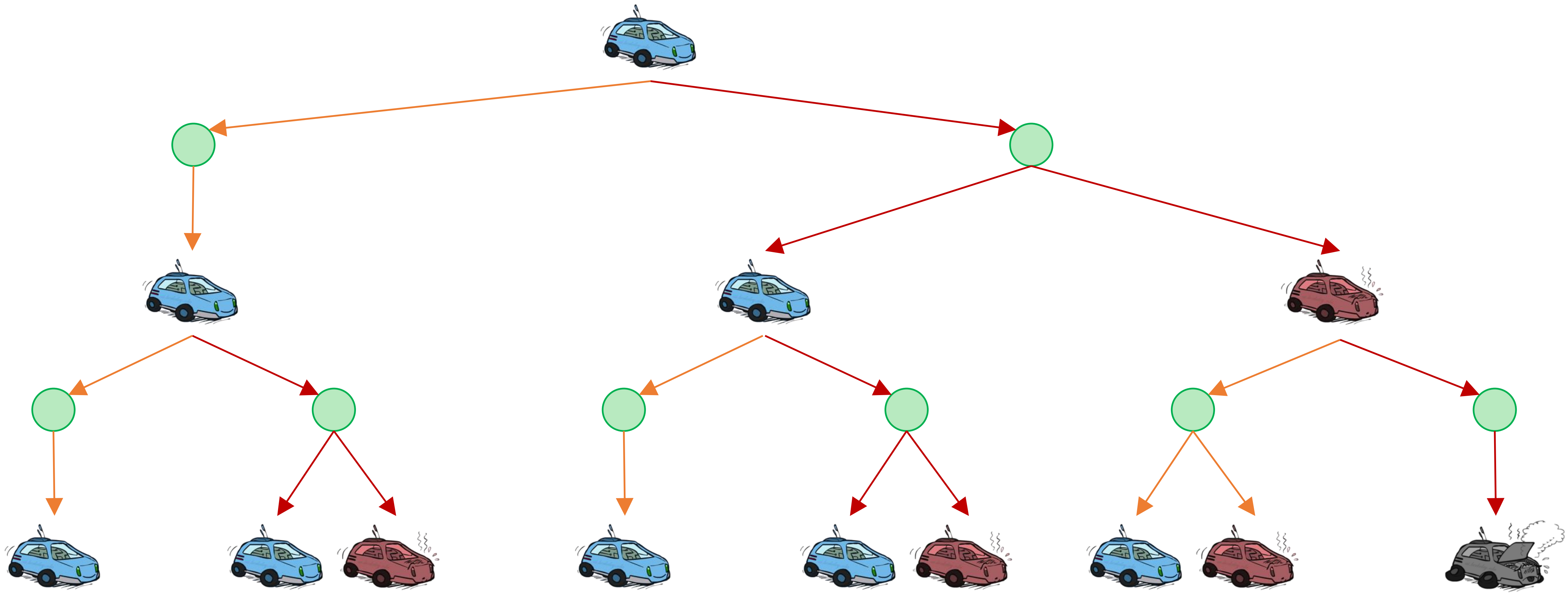
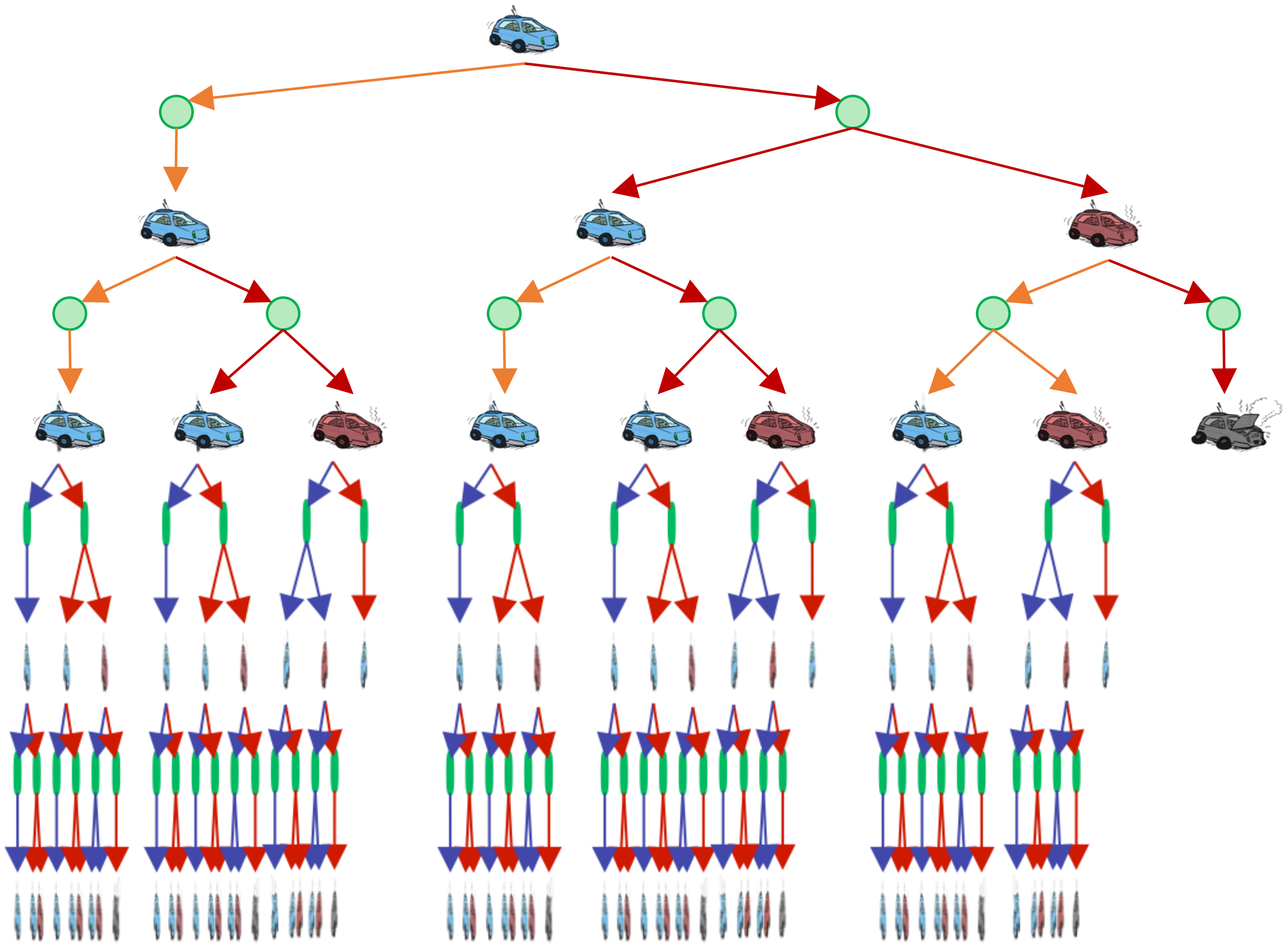


Racing Search Tree

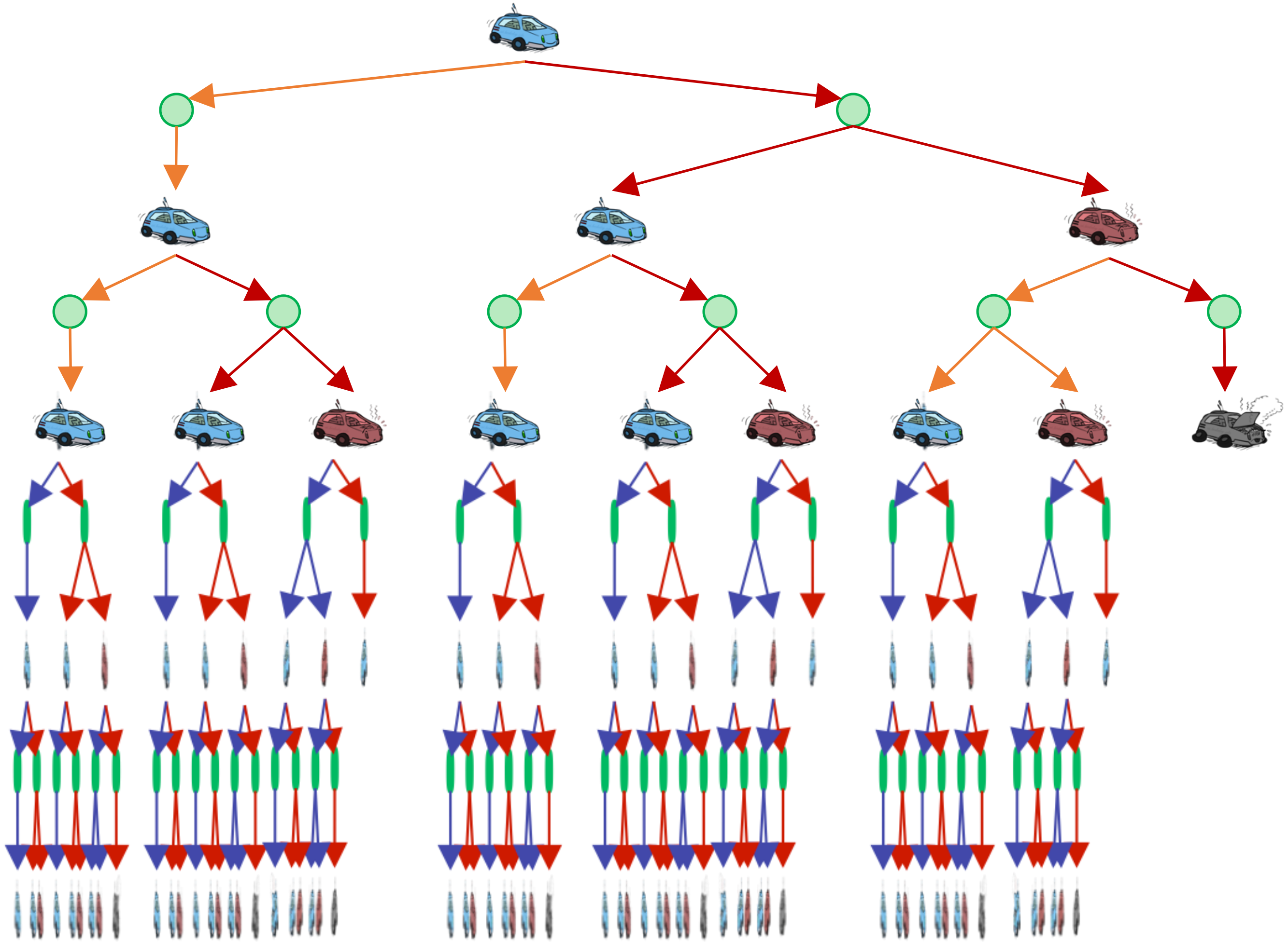


Racing Search Tree 2



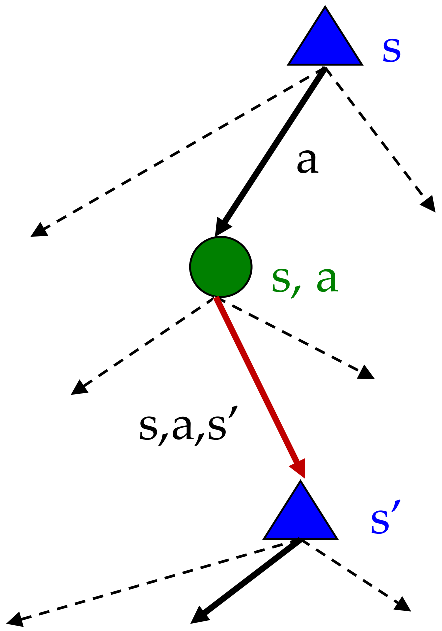
Racing Search Tree 3

- We're doing way too much work with expectimax!
- Problem: States are repeated
  - Idea: Only compute needed quantities once
- Problem: Tree goes on forever
  - Idea: Do a depth-limited computation, but with increasing depths until change is small
  - Note: deep parts of the tree eventually don't matter if $\gamma < 1$

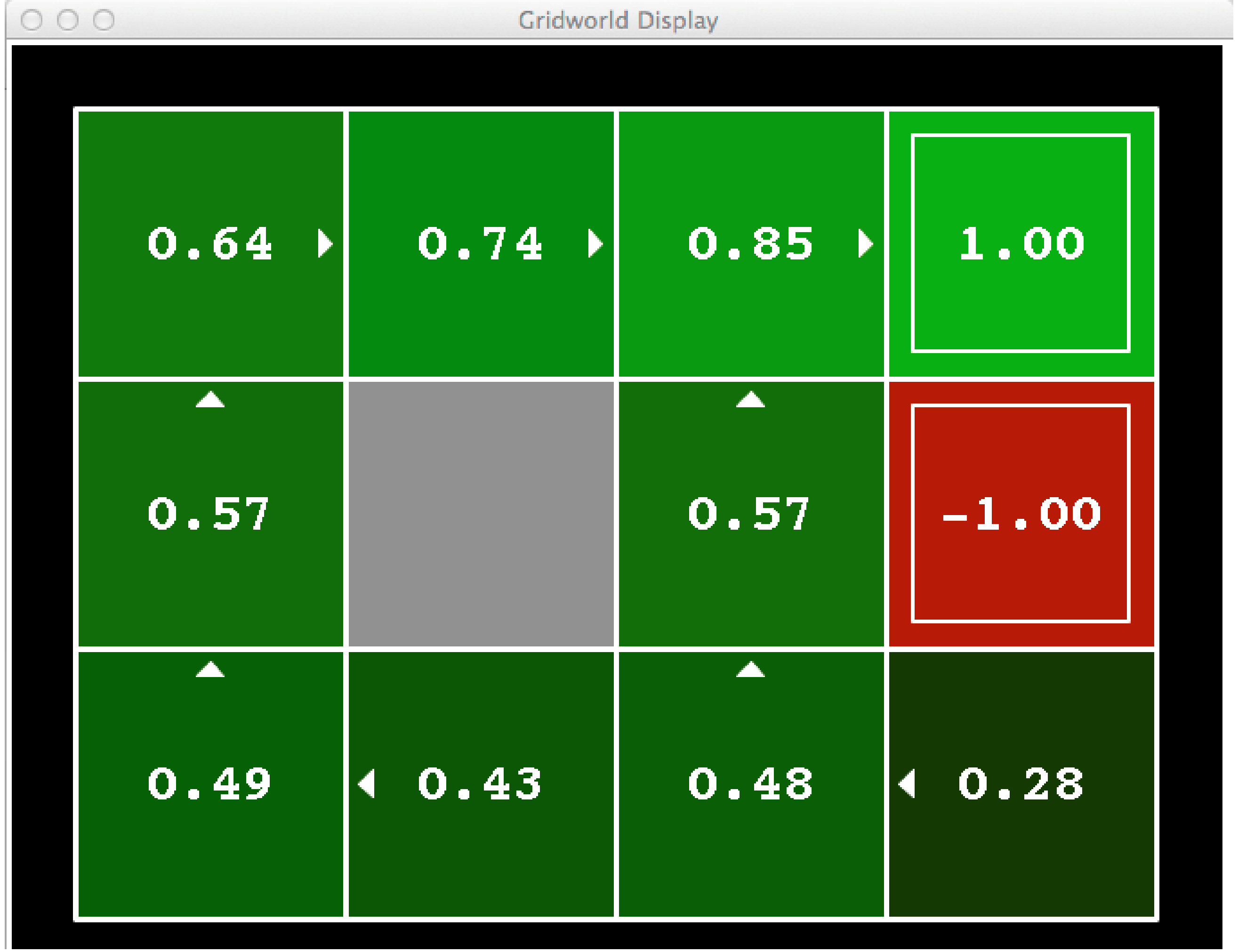


Optimal Quantities

- The value (utility) of a state s:
  - V*(s) = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a):
  - Q*(s,a) = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
  - π*(s) = optimal action from state s

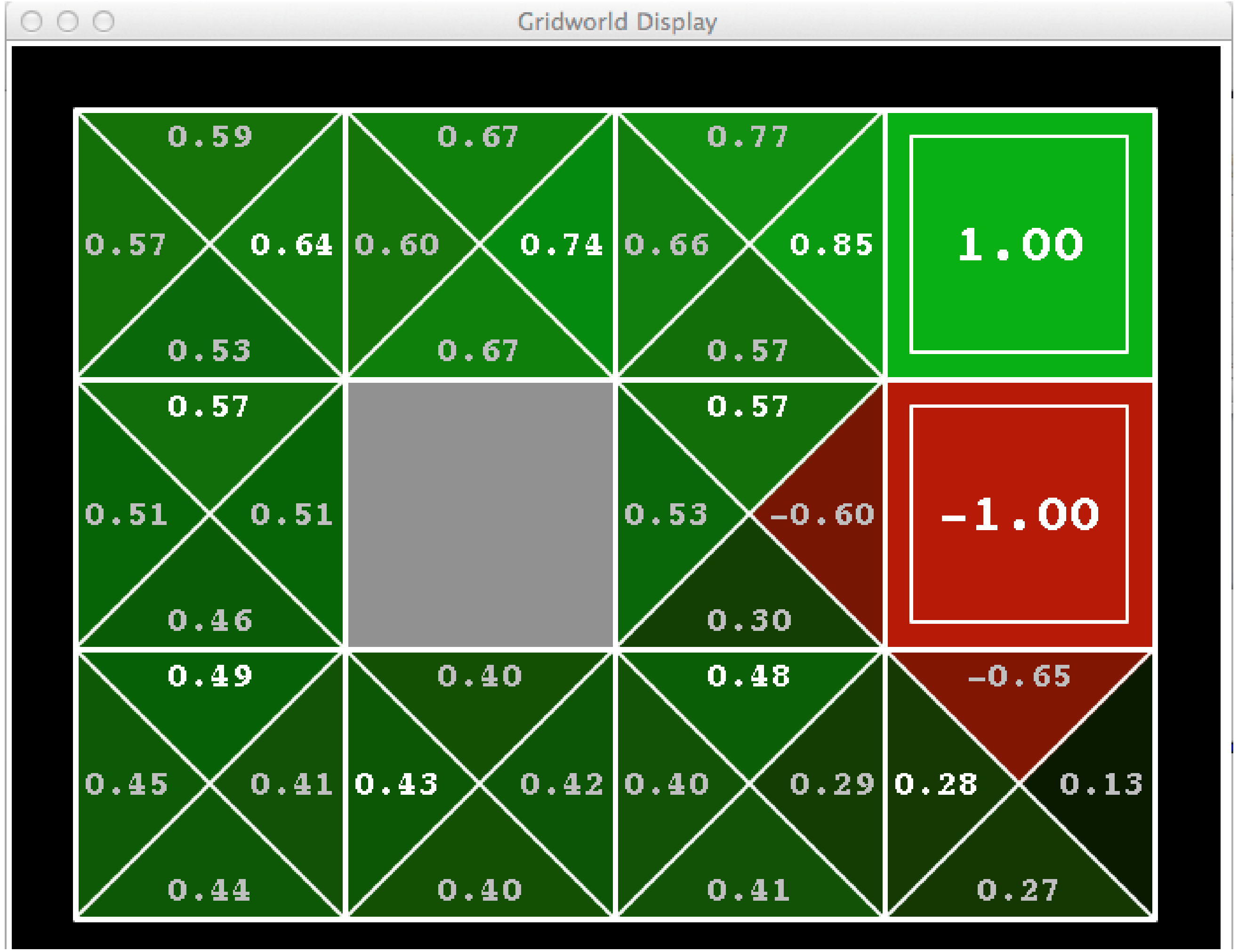


Gridworld V* Values



Noise = 0.2
Discount = 0.9
Living reward = 0

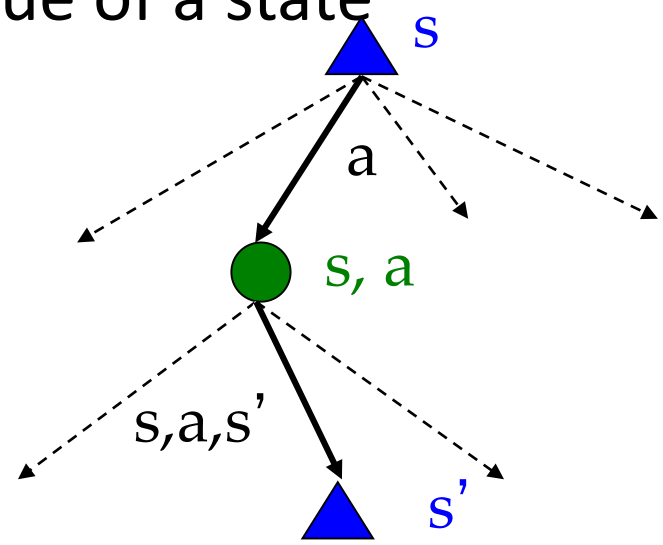
Gridworld Q* Values



Noise = 0.2
Discount = 0.9
Living reward = 0

Values of States

- Fundamental operation: compute the (expectimax) value of a state
  - Expected utility under optimal action
  - Average sum of (discounted) rewards
  - This is just what expectimax computed!
- Recursive definition of value:

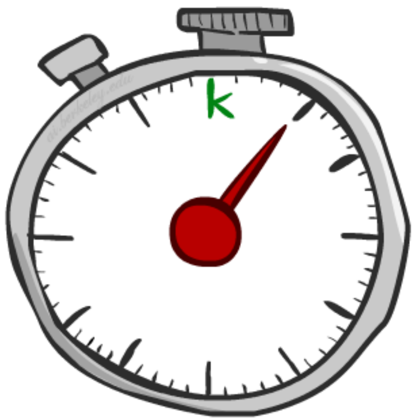


$$V^*(s) = \max_a Q^*(s,a)$$

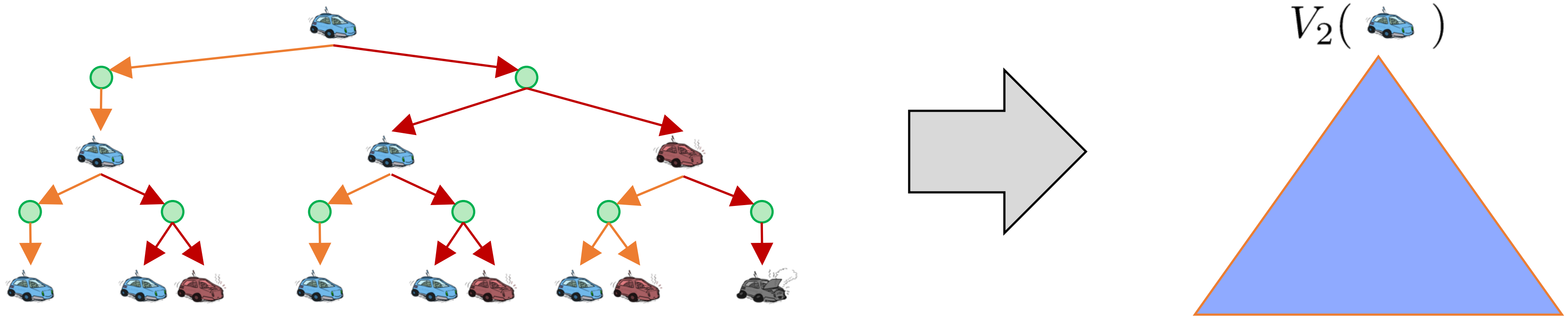
$$Q^*(s,a) = \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

Time-Limited Values

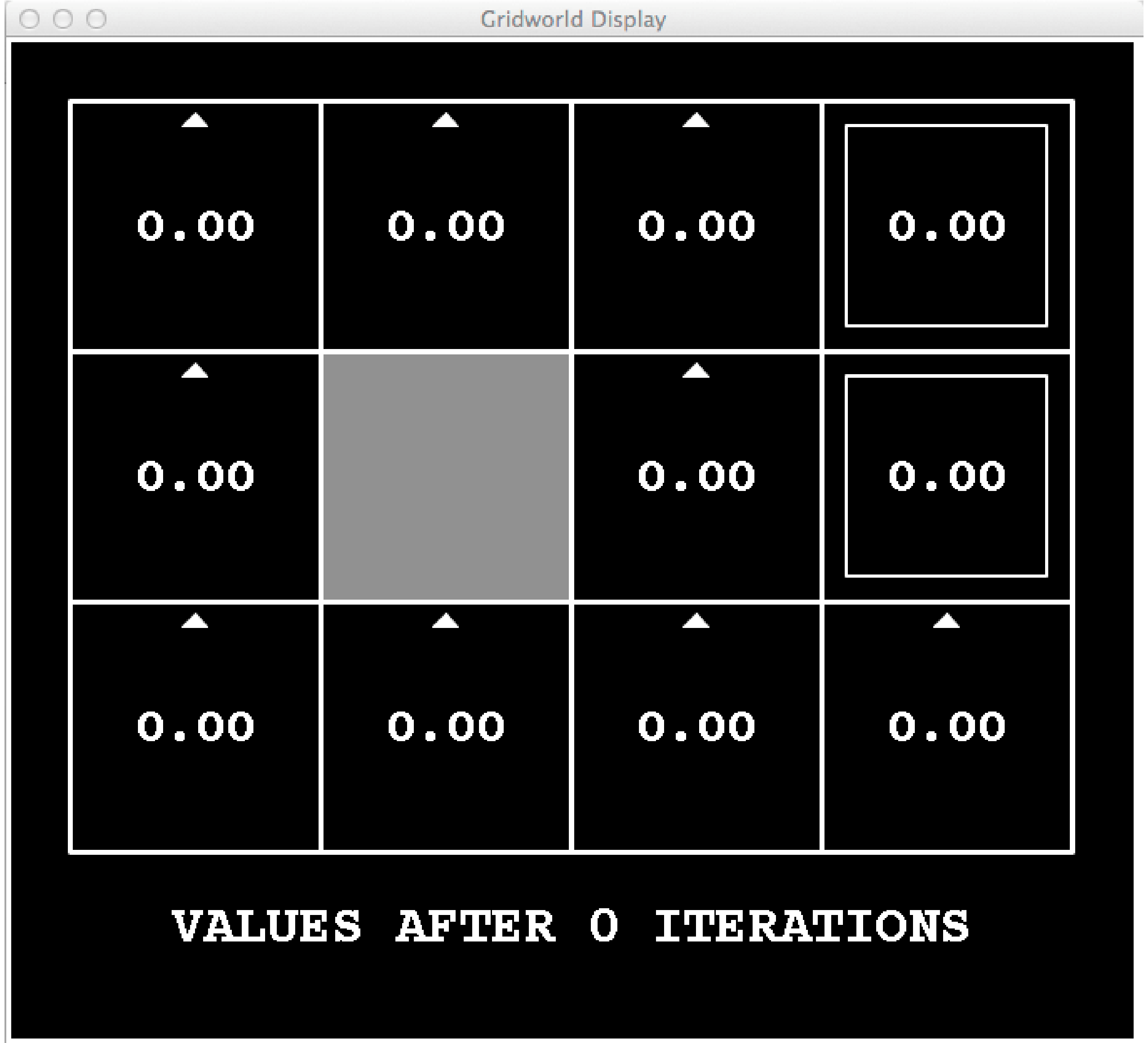


- Key idea: time-limited values
- Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps
  - Equivalently, it's what a depth-k expectimax would give from s



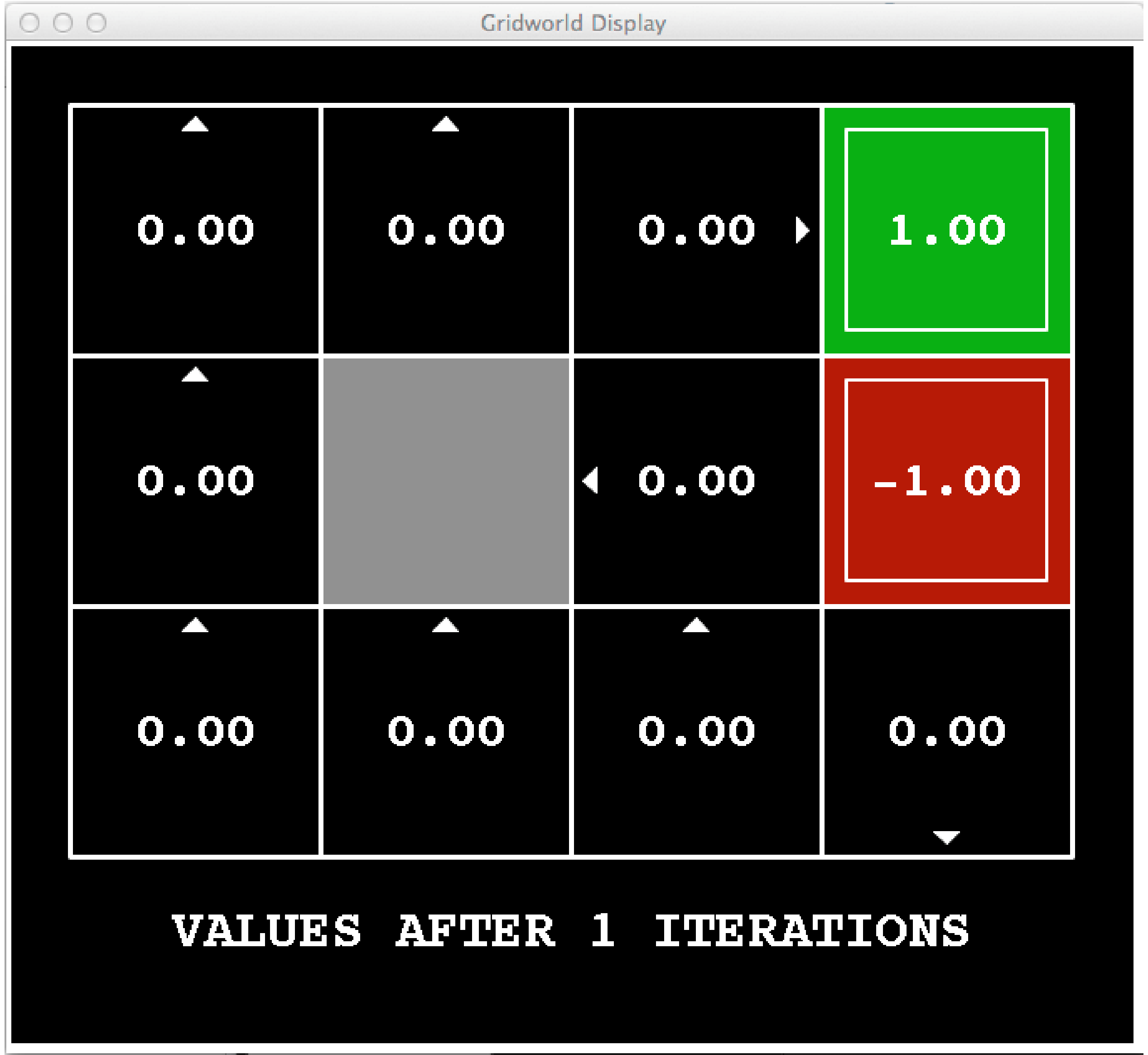
[Demo – time-limited values (L8D4)]

Gridworld: k=0



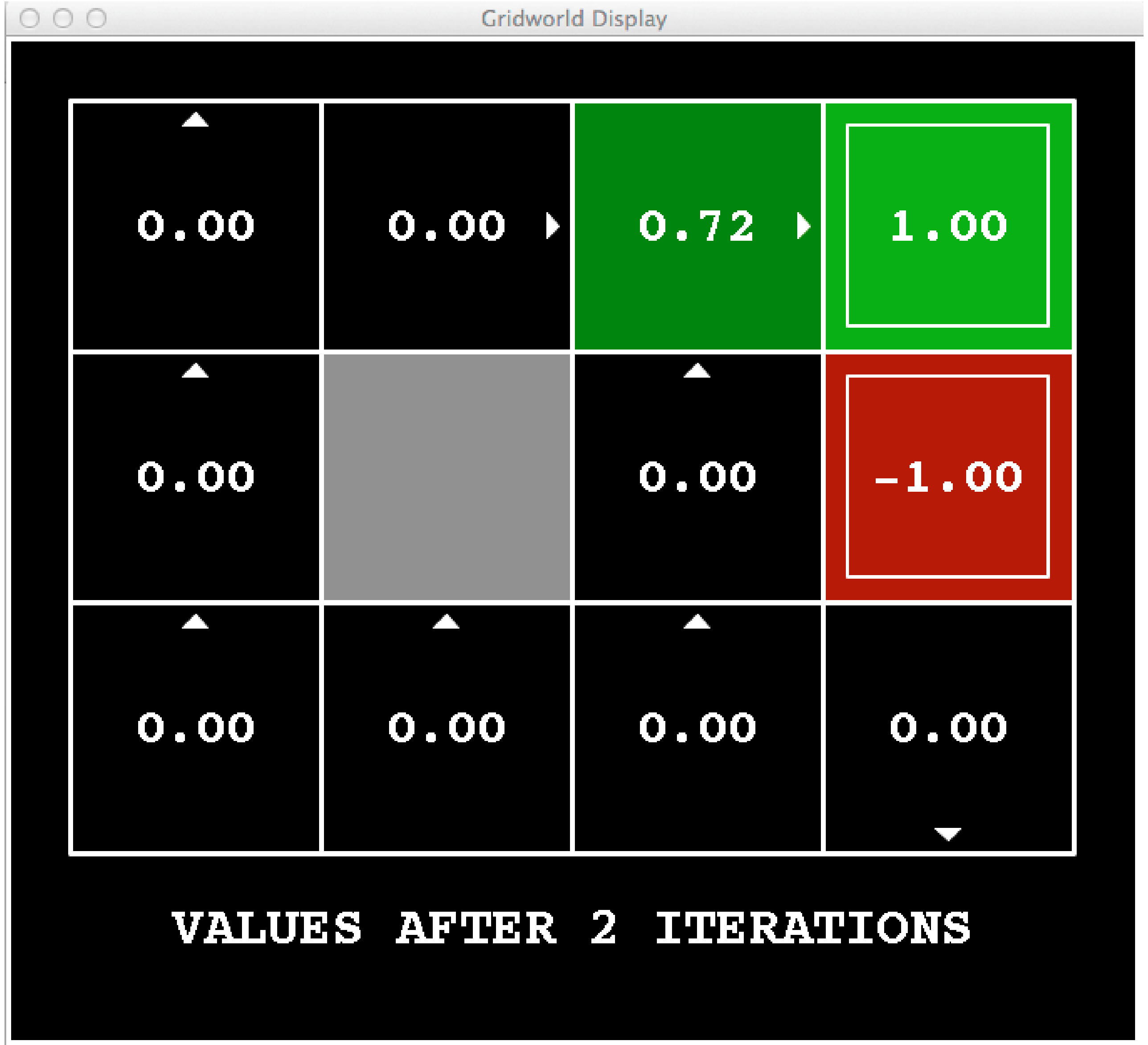
Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld: k=1



Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld: k=2

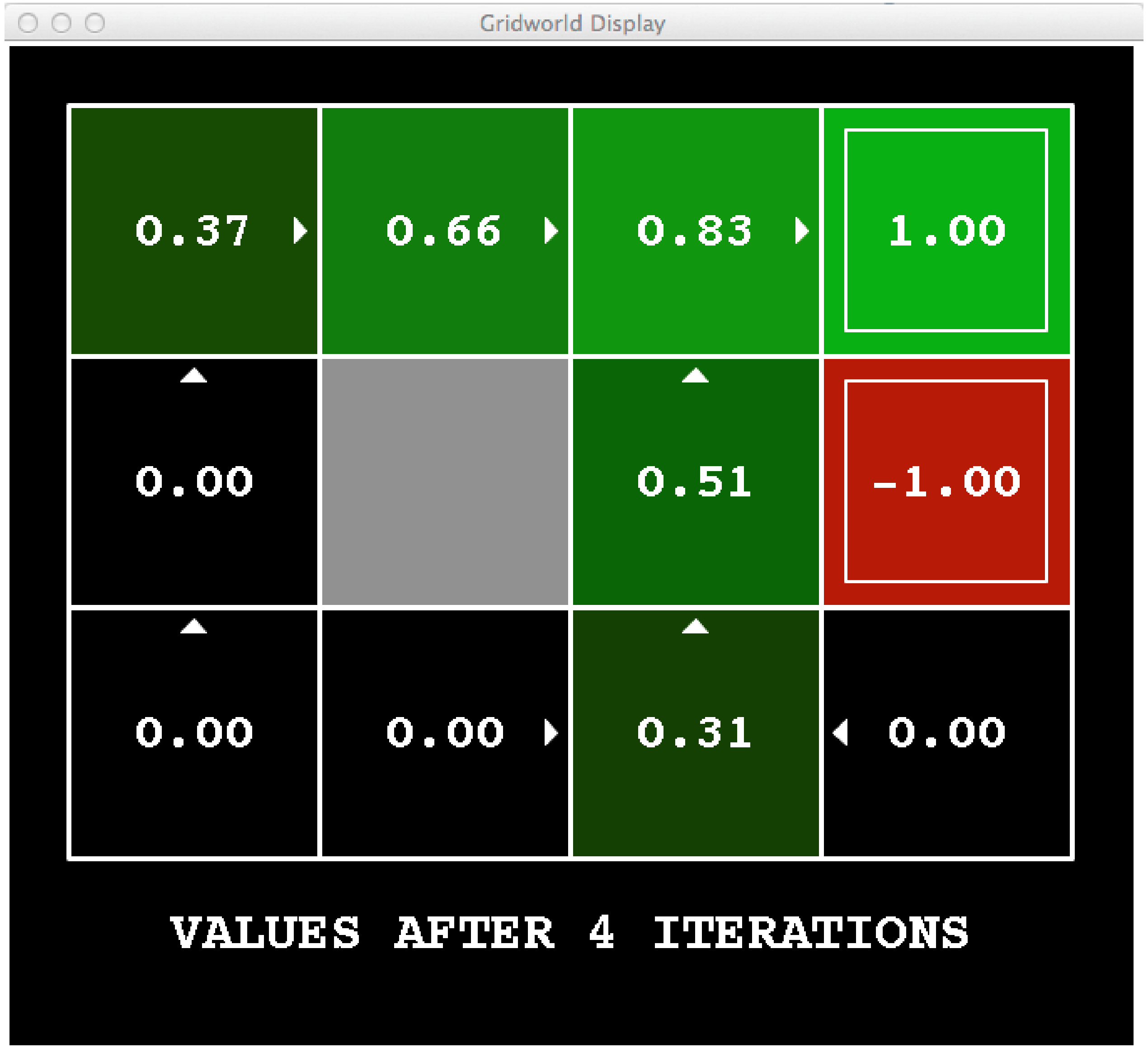


Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld: k=3

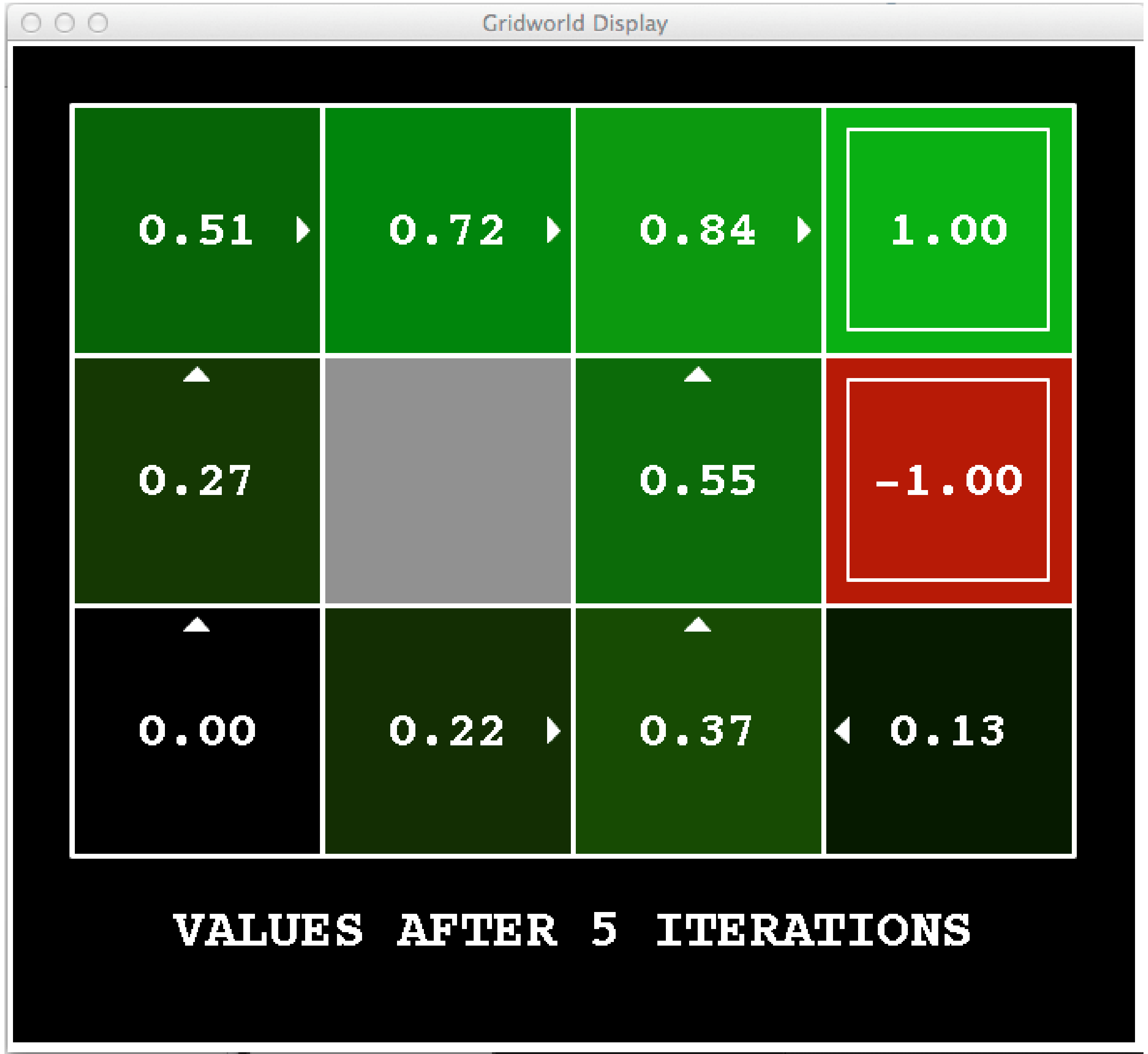
Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld: k=4



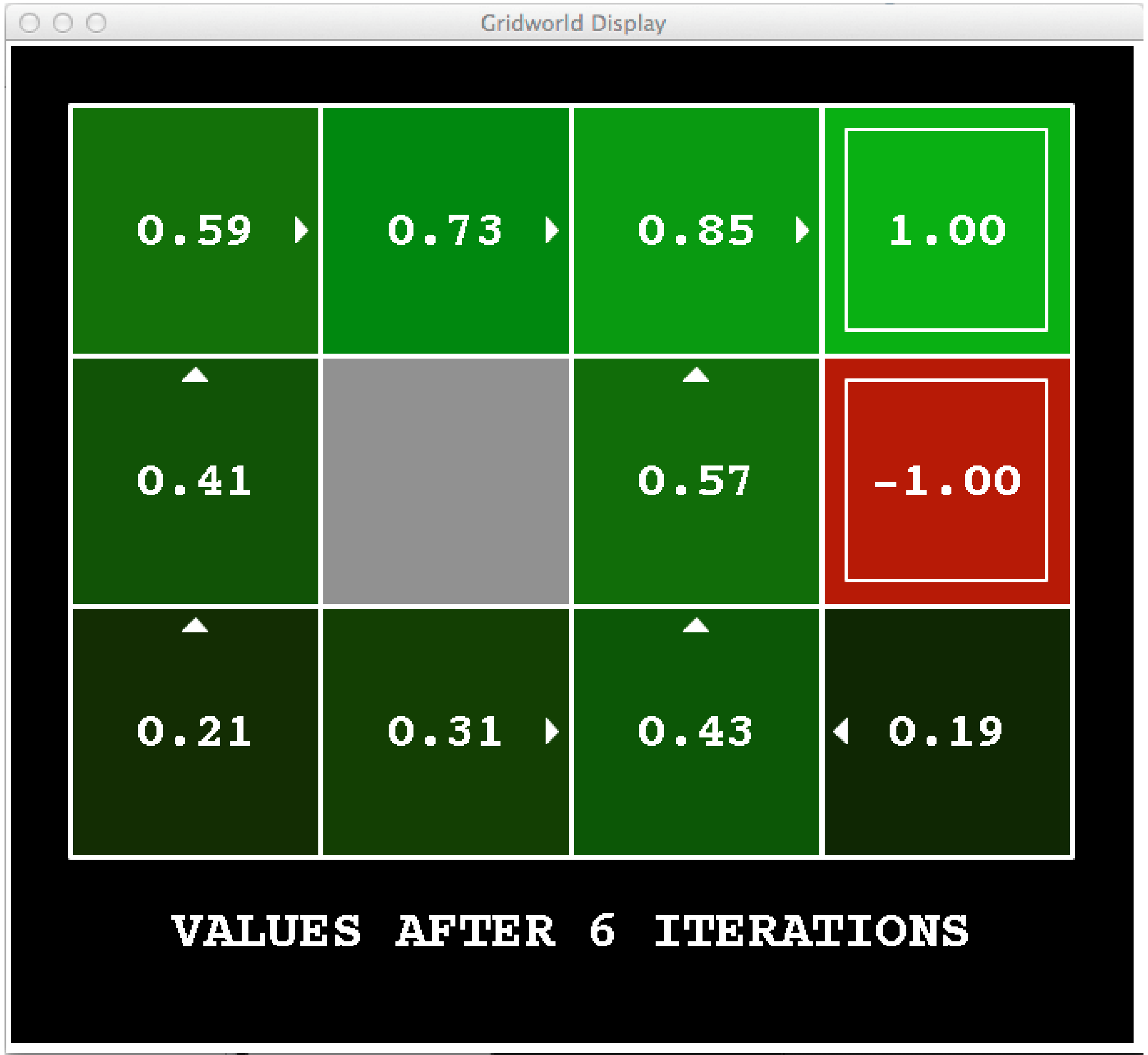
Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld: k=5



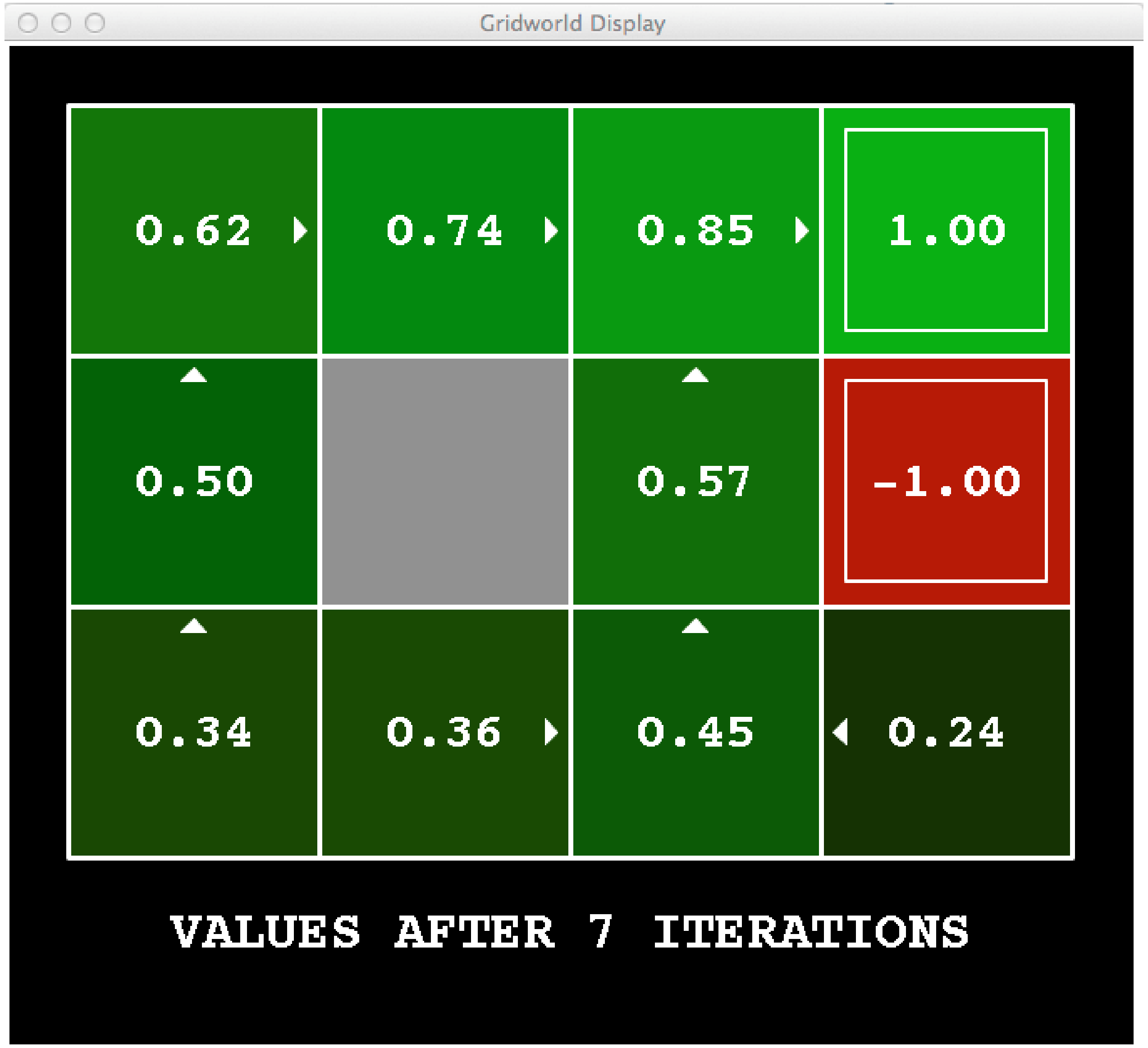
Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld: k=6



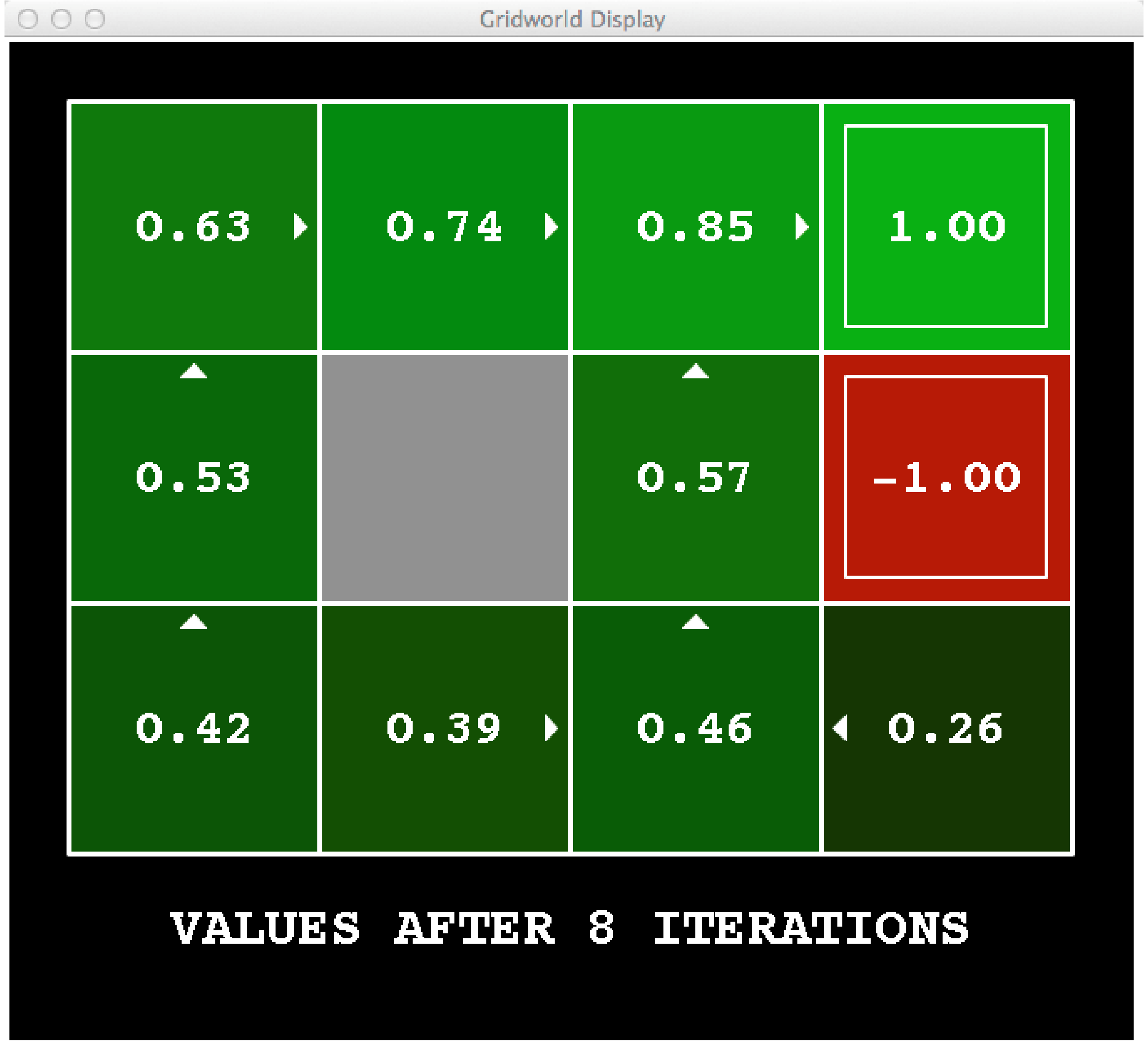
Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld: k=7



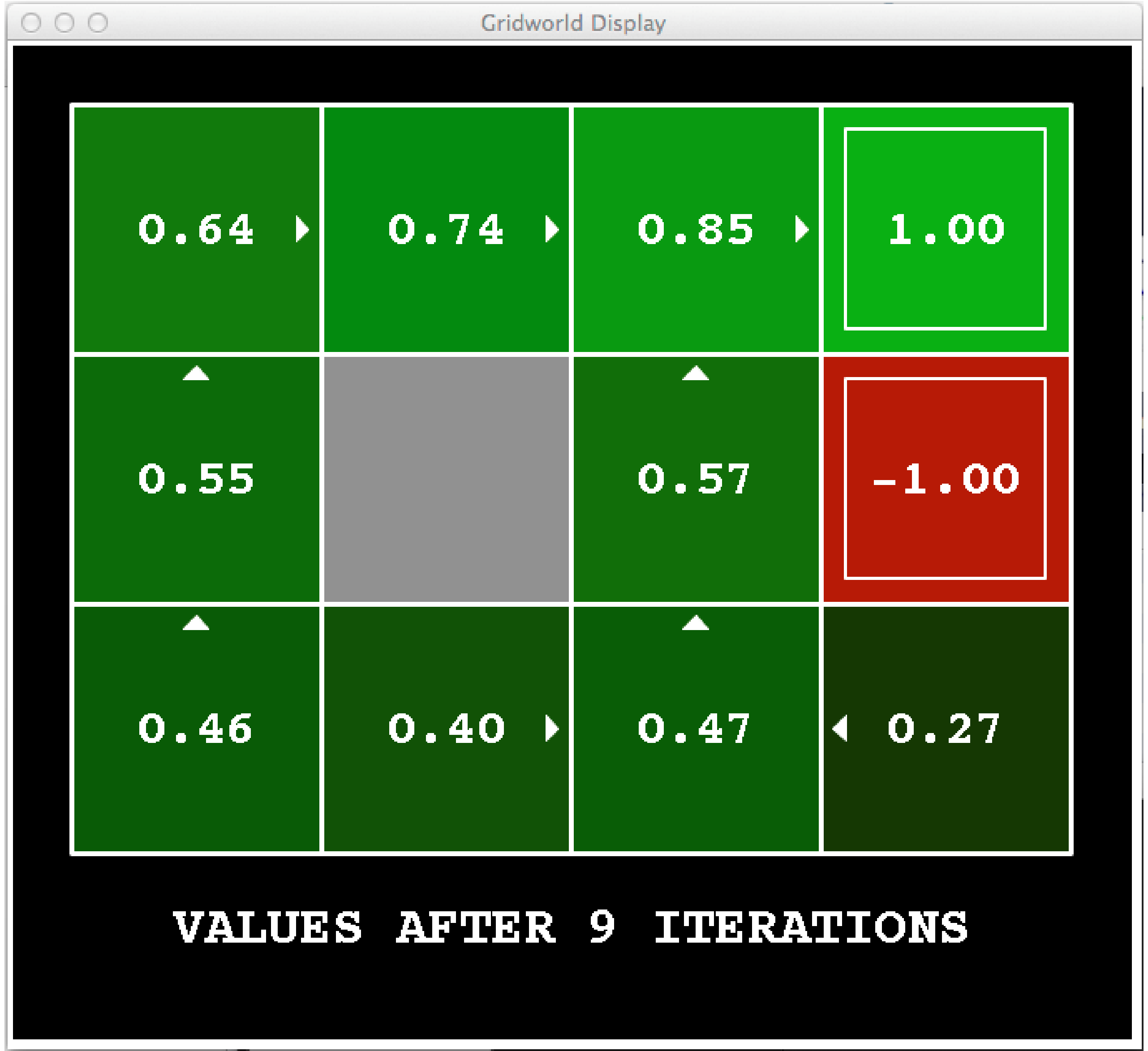
Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld: k=8



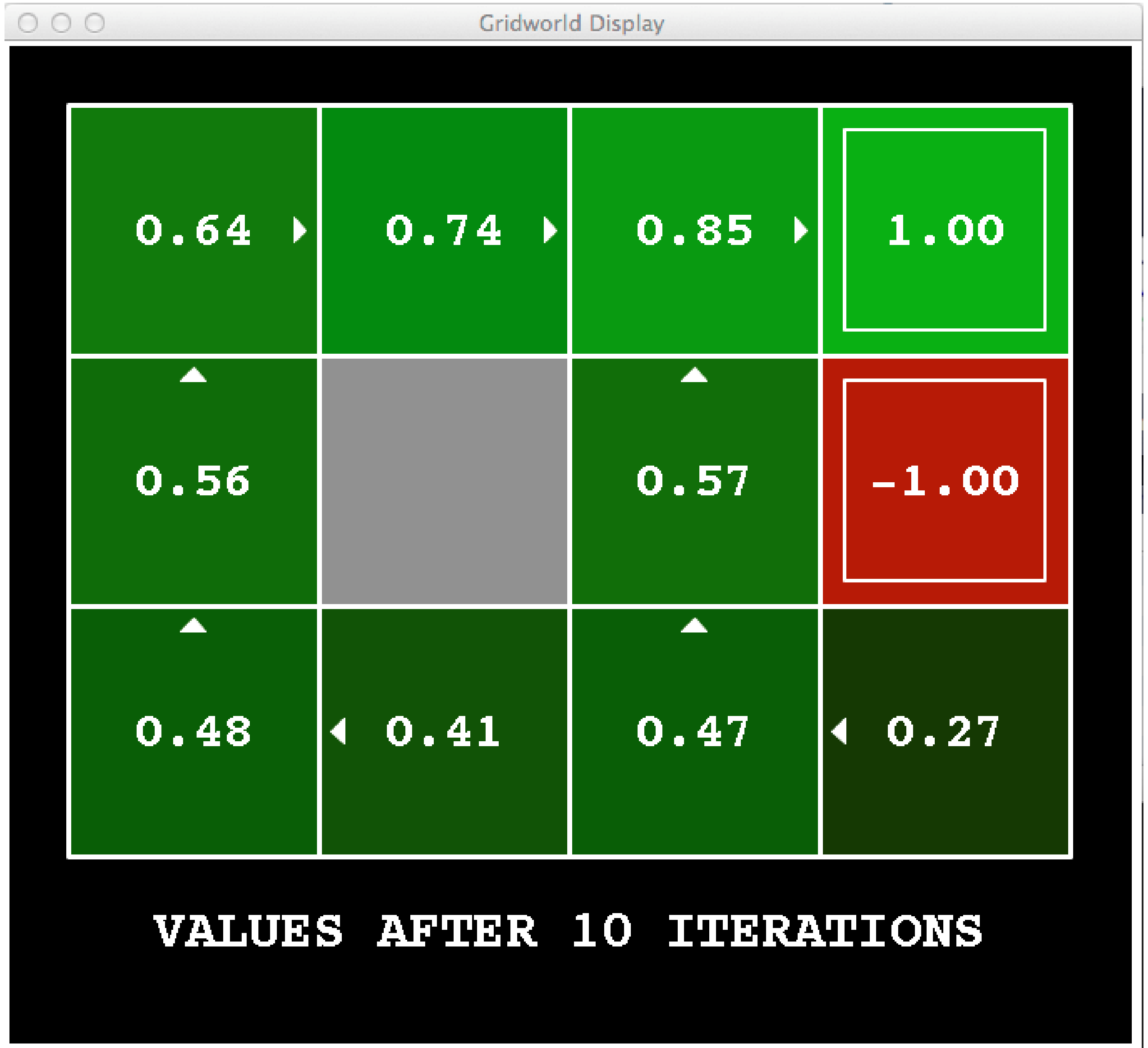
Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld: k=9



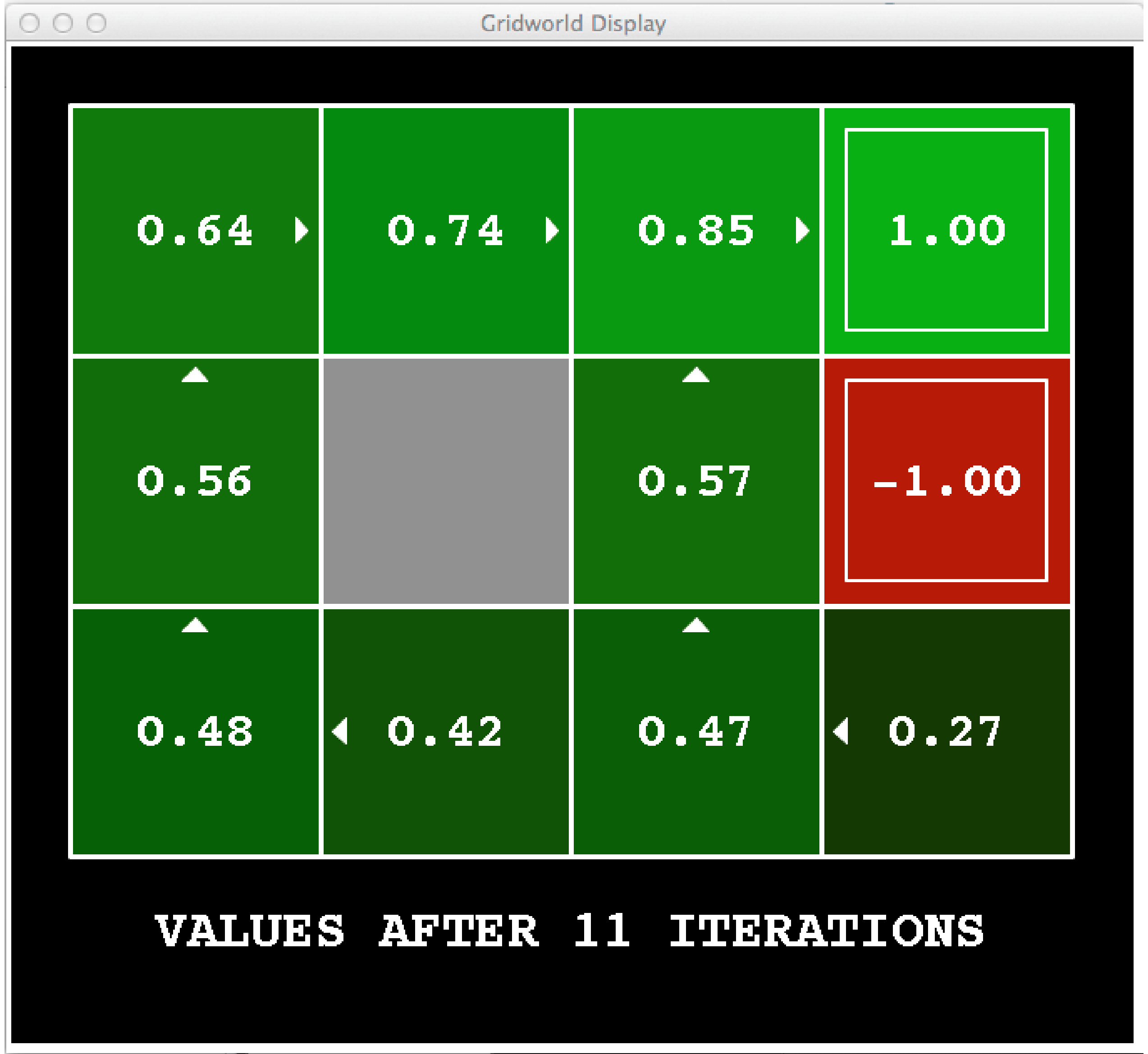
Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld: k=10



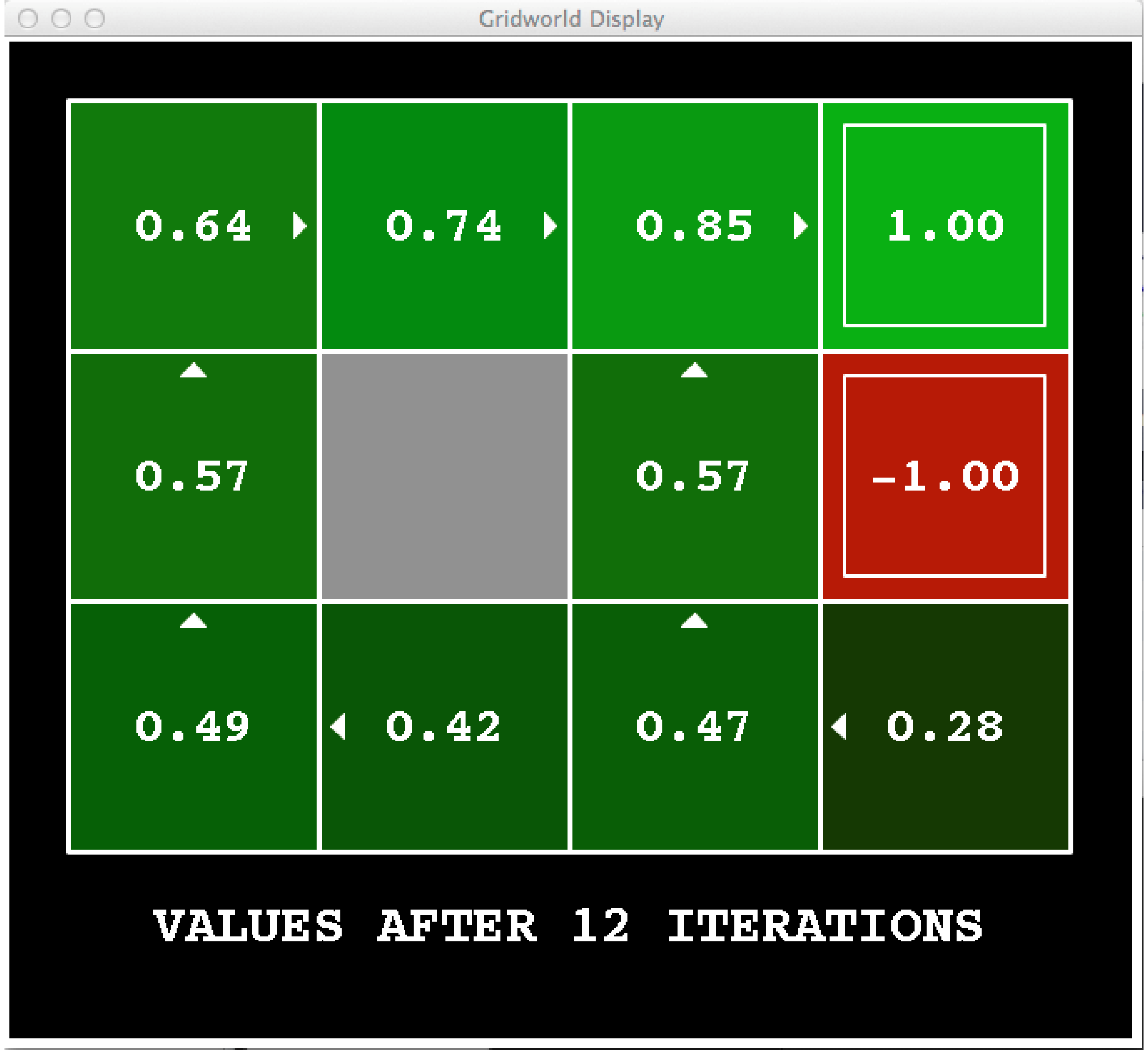
Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld: k=11



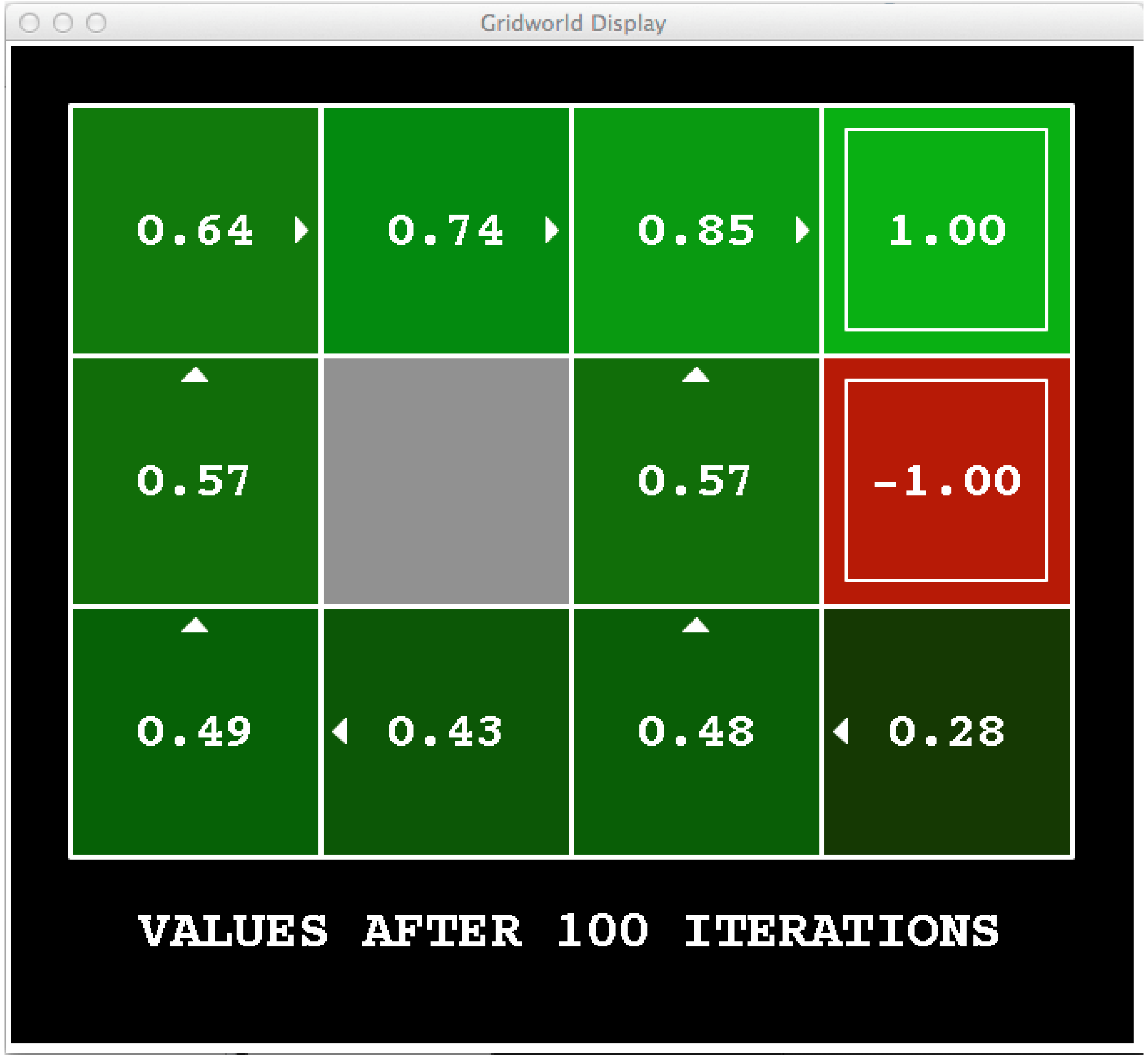
Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld: k=12



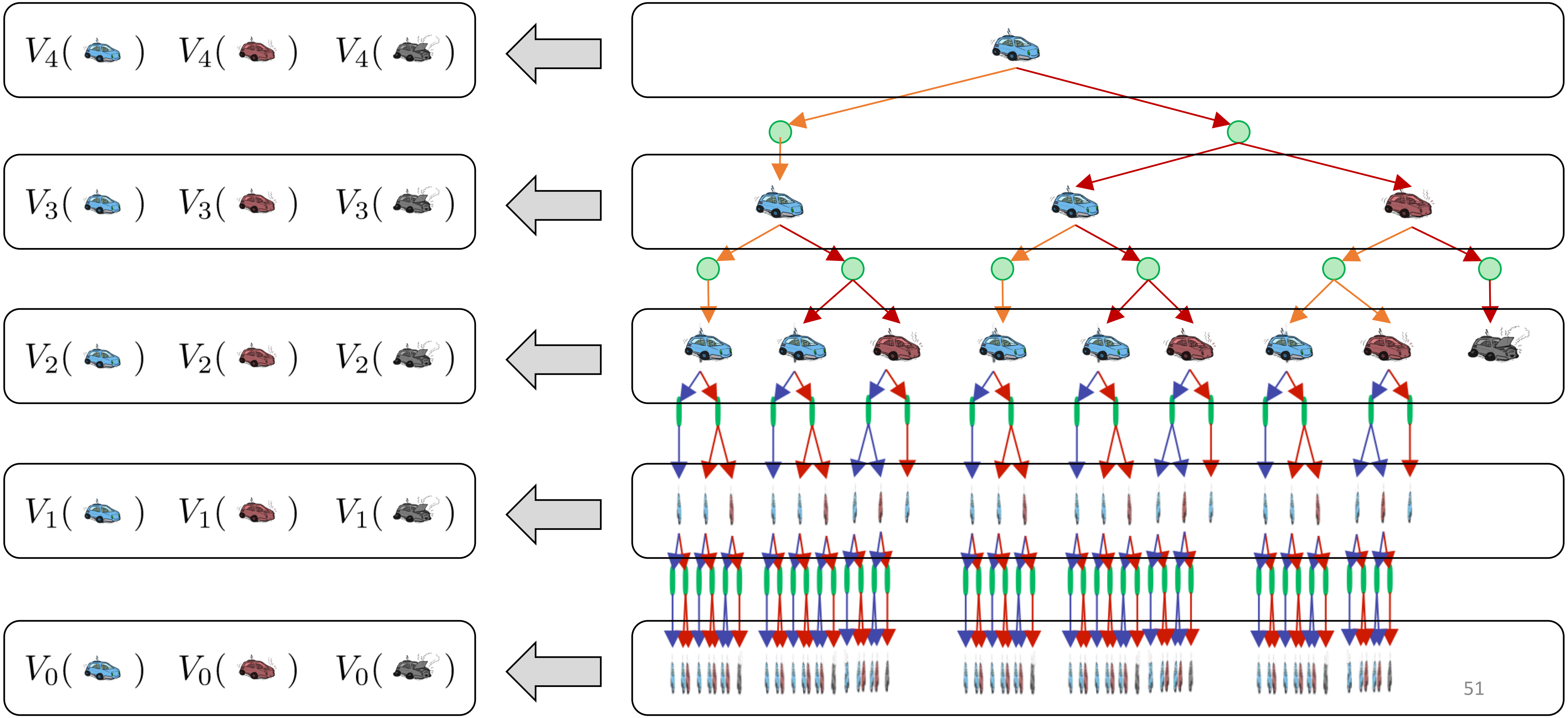
Noise = 0.2
Discount = 0.9
Living reward = 0

Gridworld: k=100

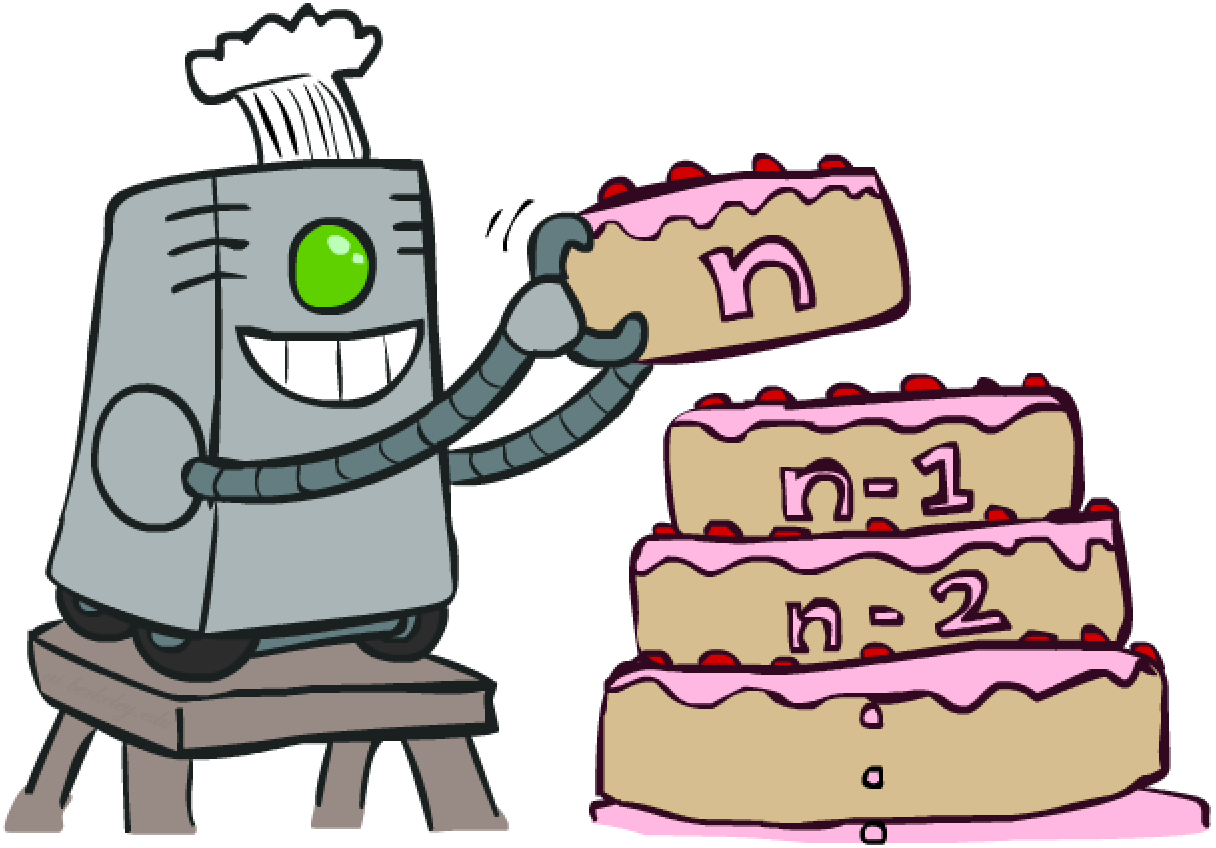


Noise = 0.2
Discount = 0.9
Living reward = 0

Time-Limited Values: Computing



Value Iteration

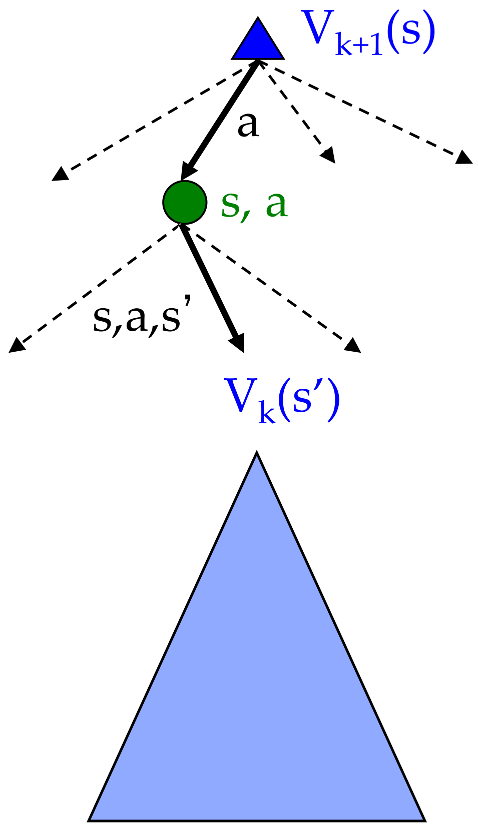


Value Iteration

- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero
- Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$

- Repeat until convergence, which yields V*
- Complexity of each iteration: $O(S^2A)$
- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do

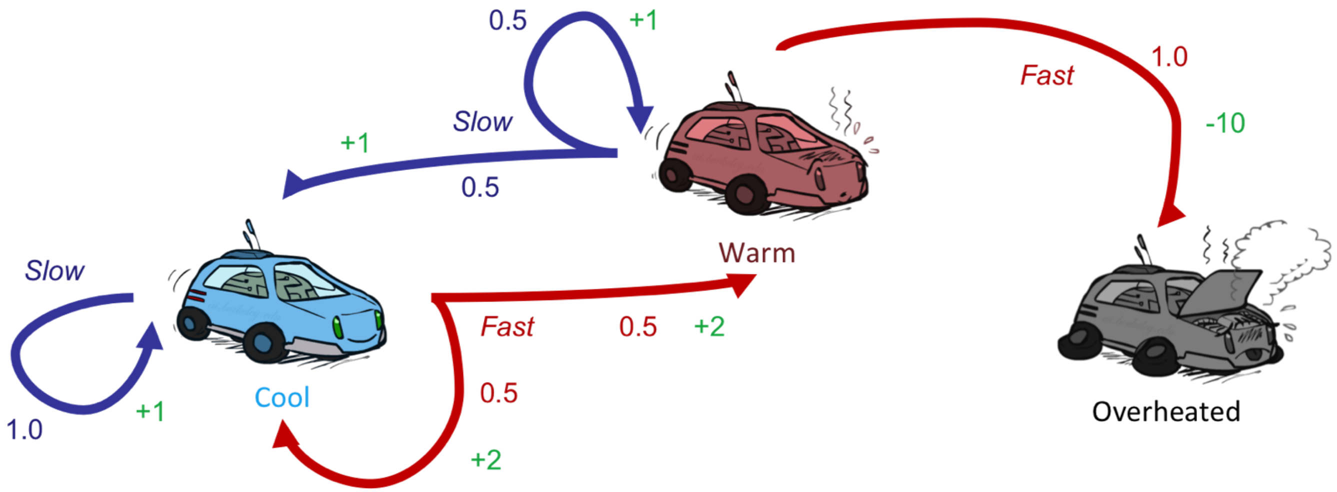


Example

V_2

V_1
S: 1
F: .5*2+.5*2=2

V_0
0 0 0

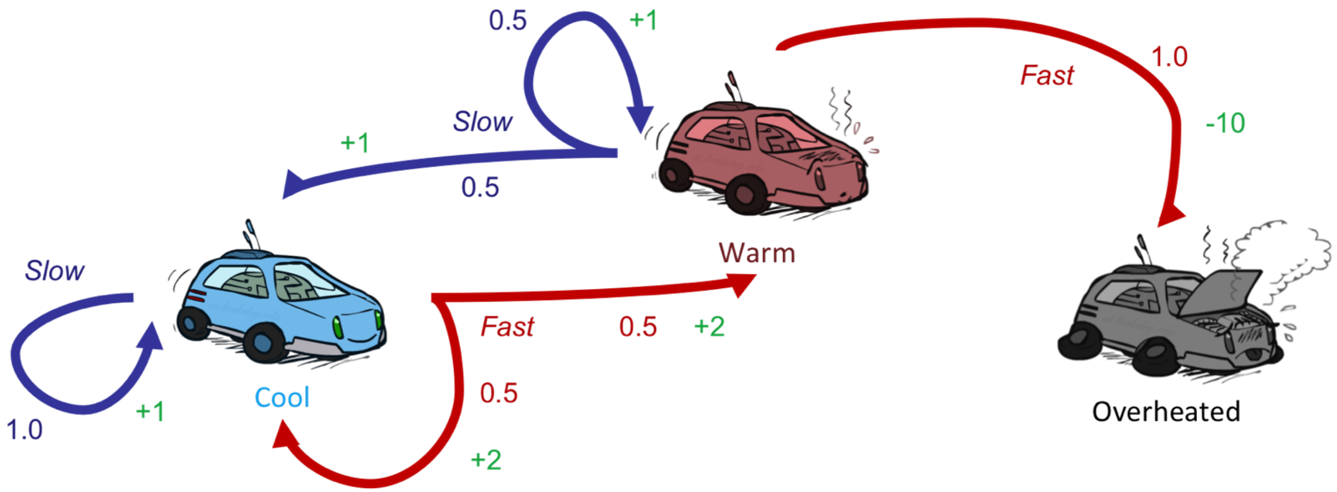


Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

Example 2

V_2			
V_1	2	S: .5*1+.5*1=1 F: -10	
V_0	0	0	0

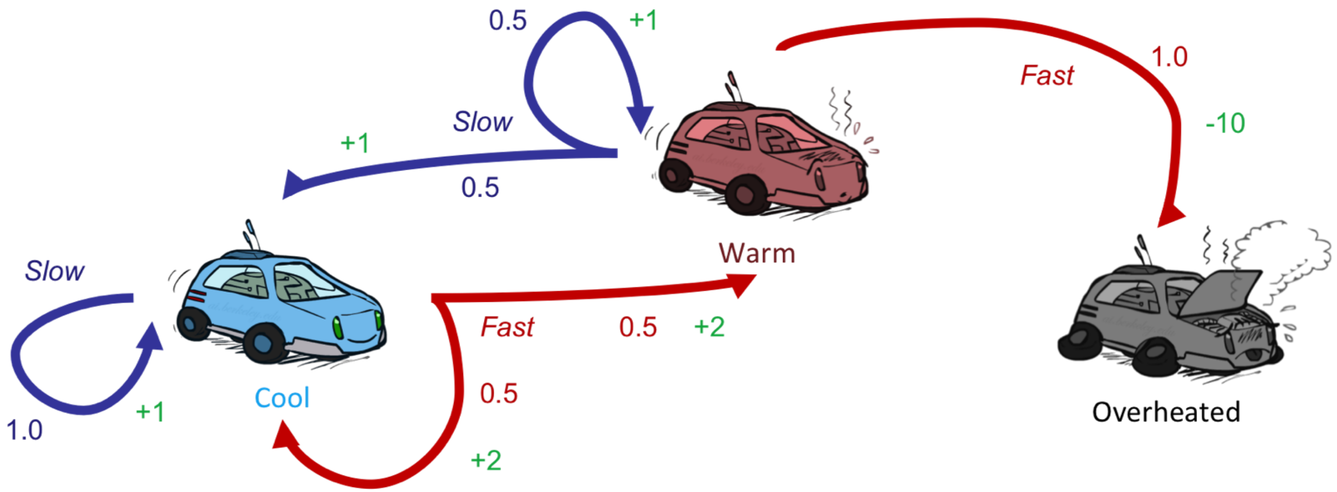


Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

Example 3

V_2			
V_1	2	1	0
V_0	0	0	0

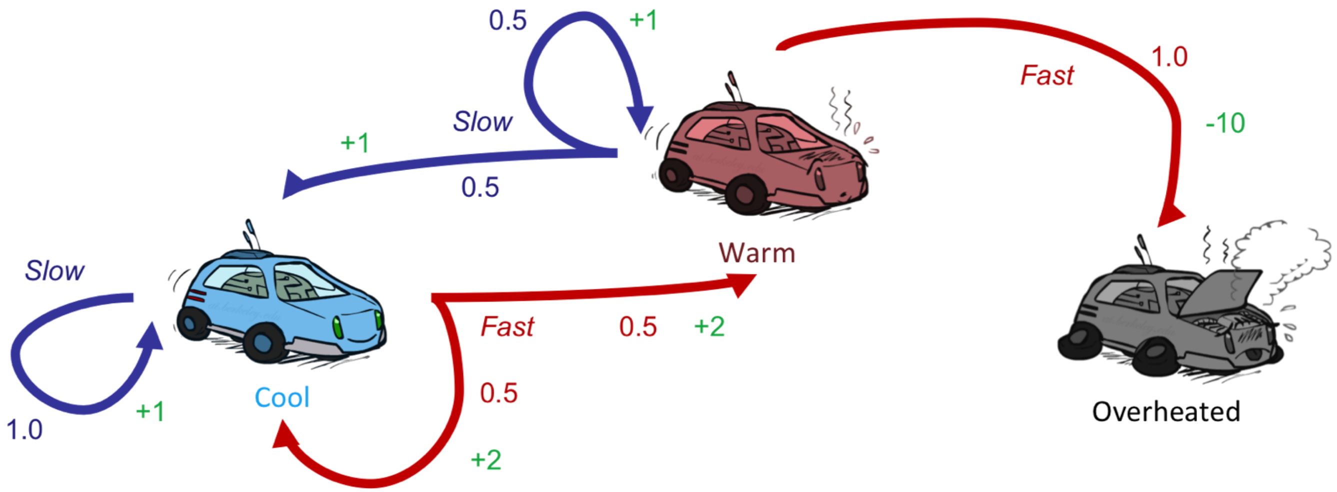


Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

Example 4

V_2	S: 1+2=3 F: .5*(2+2)+.5*(2+1)=3.5		
V_1	2	1	0
V_0	0	0	0

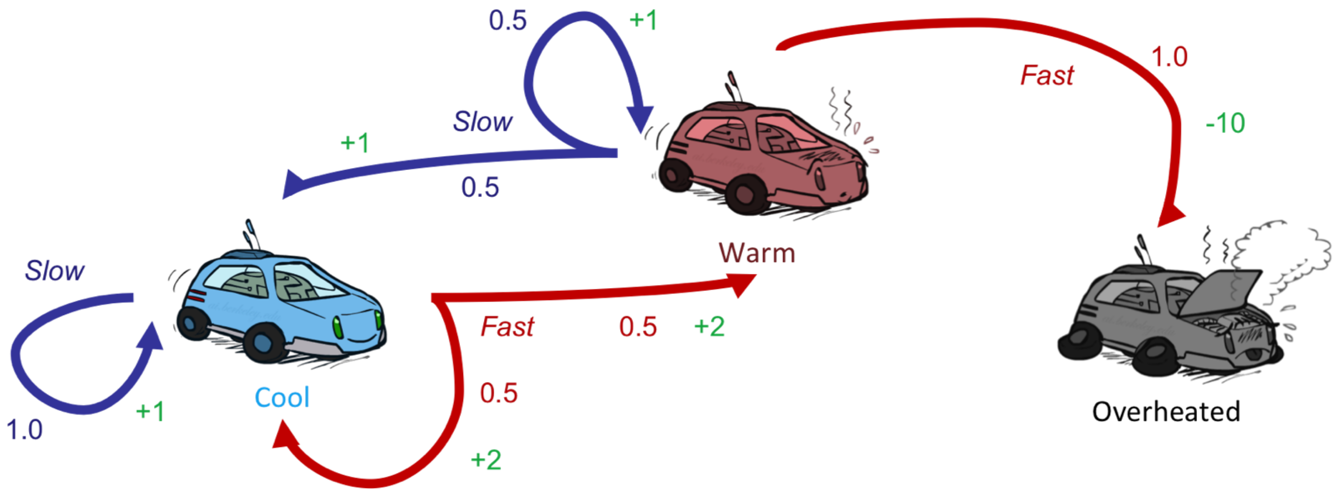


Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

Example 5

V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0

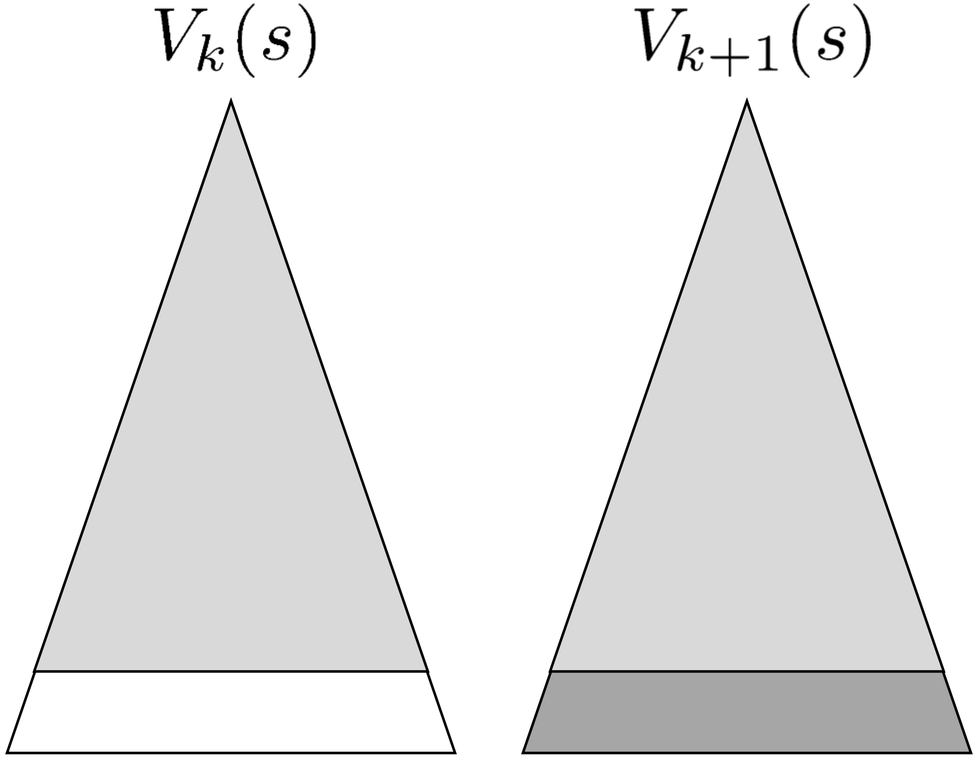


Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

Convergence

- How do we know the V_k vectors are going to converge?
- Case 1: If the tree has maximum depth M, then V_M holds the actual untruncated values
- Case 2: If the discount is less than 1
- Proof Sketch:
  - For any state V_k and V_{k+1} can be viewed as depth k+1 expectimax results in nearly identical search trees
  - The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
  - That last layer is at best all R_{MAX}
  - It is at worst R_{MIN}
  - But everything is discounted by γ^k that far out
  - So V_k and V_{k+1} are at most γ^k max|R| different
  - So as k increases, the values converge



Summary

- Definition

Shuai Li

https://shuaili8.github.io

Questions?