Solution to Homework 6

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Problem 1. Suppose $G := K_n - e$. By chromatic recurrence, we have $c_{K_n}(i) = c_G(i) - c_{K/e}(i)$. But K/e is exactly K_{n-1}, thus

$$\begin{aligned} c_G(i) &= c_{K_n}(i) + c_{K_{n-1}}(i) \\ &= i(i-1)\cdots(i-n+1) + i(i-1)\cdots(i-n+2) \\ &= i(i-1)\cdots(i-n+3)(i-n+2)^2 \end{aligned}$$

Problem 2.

(a) K_1 is the only 1-critical graph. P_1 is the only 2-critical graph.

(b) The triangle K_3 is a good example.

(c) Suppose G is not connected. Let C be the component that requires at least $\chi(G) = k$ colours to dye. (Such component exists, of course.) Take $v \in G - C$. Clearly, $\chi(G - v) = k$ still, a contradiction. $\square$

(d) Suppose for the sake of contradiction that $\delta(G) < k - 1$. That is, there exists a vertex $v : |N(v)| < k - 1$. We take an arbitrary optimal colouring σ° in graph $G^\circ := G - v$. By definition of k-critical graphs, $\chi(G^\circ) \leq k - 1$, thus σ° uses at most $k - 1$ colours, say $[k-1]$. Now we take $c \in [k-1] \setminus \sigma(N(v))$, which always exists by our choice of v. Extend σ° to σ by requiring $\sigma(v) := c$. Clearly σ dyes the graph G with only $k - 1$ colours, contradicting $\chi(G) = k$. $\square$

(e) By (d) we have $\delta(G) \geq 2$ for any 3-critical graphs, so every vertices in G lie on some cycle. Clearly G must contain an odd cycle; otherwise G is bipartite and $\chi(G) = 2$. But G never contains more than one cycle: If there were two, say C_1 and C_2 where C_1 is odd, then removing $v \in C_2 \setminus C_1$ will not influence C_1 and its chromatic number, contradicting the fact that G is 3-critical. Therefore, G is 3-critical graphs iff it is an odd cycle.

(f) Suppose there is a cut vertex v in G. Removing v would result in several components $C_1, \ldots, C_r$. By defintion of k-critical graphs and the independence of these components, we have $\chi(C_i) = k - 1$ for all i.

Claim. $\chi(C_i + v) \geq k$ for some i.

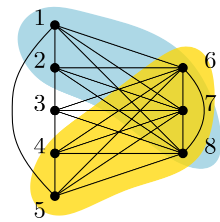
proof. Otherwise, we would have $\chi(C_i + v) = k - 1$ for all i. Then, for each $C_i + v$ we find its optimal colouring $\sigma_i : C_i + v \to [k-1]$. It is always possible to assume $\forall i, \sigma_i(v) = \sigma_1(v)$ by permuting the colours in σ_i. But this induces a colouring $\sigma : G \to [k-1]$ by letting $\sigma := \bigcup_{i=1}^{r} \sigma_i$. Thus $\chi(G) = k - 1$, a contradiction. $\square$

We fix an i that satisfies the claim, and select an arbitrary vertex $u \notin C_i + v$. Removing u from G gives us a graph $H \supseteq C_i + v$, so $\chi(H) \geq k$, contradicting that G is k-critical. $\square$

Problem 3. *Proof.* $k^4 - 4k^3 + 3k = k^2(k-1)(k-3)$ has two *discrete* positive roots 1 and 3, which is absurd: Any chromatic polynomial restricted in $\mathbb{N}$ should be non-decreasing. $\square$

Problem 4. The statement is false. Consider a K_4 glued at the end of P_n. We denote the graph as G. The average degree $d(G) = 2m/n = 2(n+5)/n = 2 + \frac{10}{n}$. Take $n > 10$, then $d < 3$. However, $\chi(G) = 4 > 1 + d(G)$.

Problem 5. *Proof.* The graph G contains two 5-cliques, as illustrated by the diagram below. It doesn't have a 6-clique by enumeration. Therefore $\omega(G) = 5$. Let f be a colouring. Without loss of generality we assume $f(6) = 1, f(7) = 2, f(8) = 3, f(4) = 4, f(5) = 5$. Then $f(\{1,2\}) = \{4,5\}$. But we notice an edge between 1 and 5, thus $f(1) = 4$ and $f(2) = 5$. But this implies that $f(3)$ must use a colour apart from $\{1,2,3,4,5\}$. Therefore, $\chi(G) = 6$.



$\square$

Problem 6. *Proof.* We may safely assume that G is connected; otherwise we may conduct the same argument on its components. Now, prove by induction on n.

For $n \leq 2$, clearly $\chi(G) = n = \tau(G)$. For $n \geq 3$, we choose a longest path $P = v_1 v_2 \dots v_k$ in G, where $k = \tau(G)$. Define $G' := G - v_k$, and observe that $\tau(G') \leq k$. By induction hypothesis, we have $\chi(G') \leq k$. Let σ be the optimal colouring on G', and $[k]$ be its palette. We are going to extend it to G. Since P is a longest path, we konw $N(v_k) \subseteq P - v_k$ and hence $|N(v_k)| < k$. Therefore, we are always able to pick a colour $c \in [k] \setminus \sigma(N(v_k))$. Then, we simply extend $\sigma(v_k) := c$, which proves $\chi(G) \leq k = \tau(G)$. $\square$

Problem 7. It breaks down because planarity does not guarantee a vertex $v : \deg(v) \leq 4$. (It does guarantees a vertex $v : \deg(v) \leq 5$, though.) The core of the inductive proof in Five Colour Theorem resides on such vertex. Let us sketch a "proof" for Four Colour Theorem in the same way:

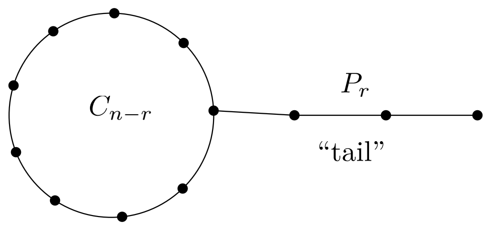
1. Find a vertex $v : \deg(v) \leq 5$. By I.H., colour the rest of the graph, i.e. $G - v$, within four colours and obtain a colouring σ.
2. If $|\sigma(N(v))| \leq 3$, then we are done.
3. If $|\sigma(N(v))| = 4$, then we try to switch the colours of $N(v)$.

But the last step cannot be done, because there might be repeating colours in $\sigma(N(v))$ and it is simply impossible to switch things around easily.

Problem 8 (Chromatic uniqueness)**.** *Proof.* Let $p_n(x)$ be the polynomial defined in the problem. Expanding it by binomial theorem,

$$
\begin{aligned}
p_n(x) &= \left(\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} x^i\right) + (-1)^n (x-1) \\
&= (-1)^{n-1}(n-1)x + \sum_{i=2}^{n} \binom{n}{i} (-1)^{n-i} x^i \\
&=: \sum_{i=1}^{n-1} a_i x^i
\end{aligned}
$$

By the properties of chromatic polynomials, G is connected because $a_1 = (-1)^{n-1}(n-1) \neq 0$. Also, the number of edges $m = |a_{n-1}| = \binom{n}{n-1} = n$. Therefore, G is a connected graph with *exactly one cycle*.



Finally, we have to exclude the possibility that G contains a "tail" outside the cycle (see the figure above). Suppose it does, i.e. $G = C_{n-r} \cup P_r$ ($r \geq 1$). Then we can dye G with x colours by the following procedure: (1) Colour all the vertices on C_{n-r} by x colours, which has $p_{n-r}(x)$ choices. (2) After that, colour all the vertices on P_r, which has $(x-1)^r$ choices. Hence, $p_n(x) = (x-1)^r p_{n-r}(x) = (x-1)^n + (-1)^{n-r}(x-1)^{1+r}$, contradicting the definition of $p_n(x)$. $\square$