Lecture 7: Coloring

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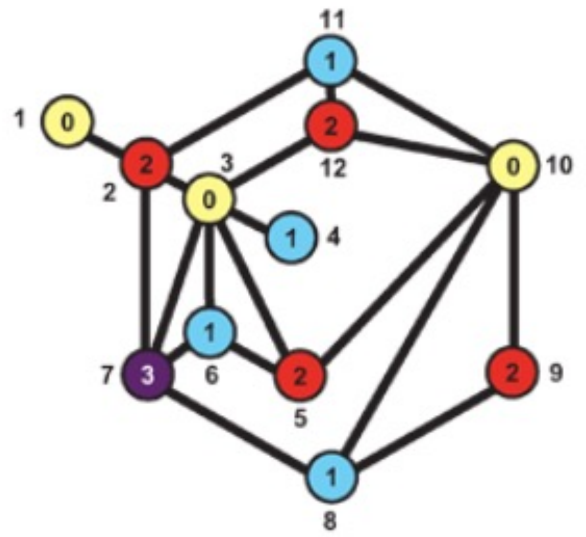
Motivation: Scheduling and coloring

- University examination timetabling
  - Two courses linked by an edge if they have the same students
- Meeting scheduling
  - Two meetings are linked if they have same member



Definitions

- Given a graph G and a positive integer k, a k-coloring is a function $K: V(G) \longrightarrow \{1, \ldots, k\}$ from the vertex set into the set of positive integers less than or equal to k. If we think of the latter set as a set of k "colors," then K is an assignment of one color to each vertex.
- We say that K is a proper k-coloring of G if for every pair u, v of adjacent vertices, $K(u) \neq K(v)$ — that is, if adjacent vertices are colored differently. If such a coloring exists for a graph G, we say that G is k-colorable
- In a proper coloring, each color class is an independent set. Then G is k-colorable $\Leftrightarrow V(G)$ is the union of k independent sets

Chromatic number

- Given a graph G, the chromatic number of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable. G is said to be k-chromatic
- Examples

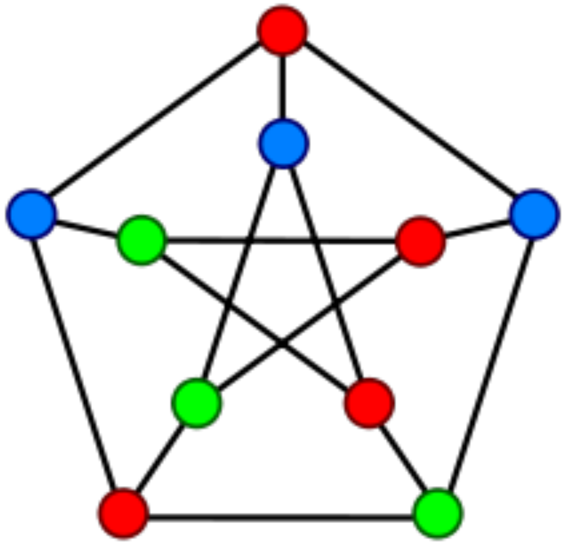
$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd,} \end{cases}$$

$$\chi(P_n) = \begin{cases} 2 & \text{if } n \geq 2, \\ 1 & \text{if } n = 1, \end{cases}$$

$$\chi(K_n) = n,$$

$\chi(E_n) = 1,$ ←Empty graph

$$\chi(K_{m,n}) = 2.$$



- (Ex5, S1.6.1, H) A graph G of order at least two is bipartite $\Leftrightarrow$ it is 2-colorable

Theorem (1.2.18, W, Kőnig 1936)
A graph is bipartite $\Leftrightarrow$ it contains no odd cycle

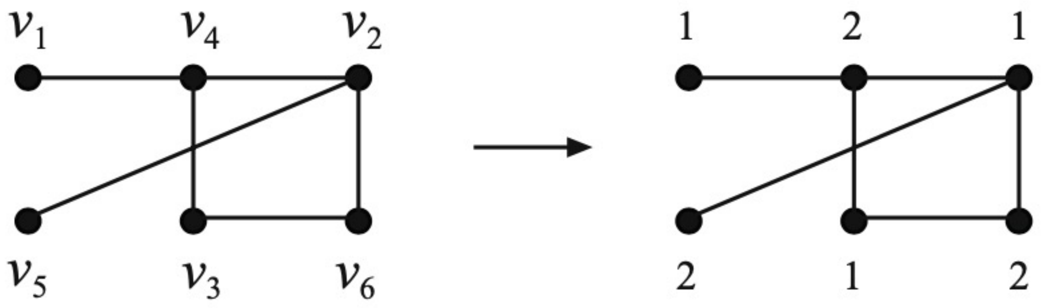
Bounds on Chromatic number

- Theorem (1.41, H) For any graph G of order n, $\chi(G) \leq n$
- It is tight since $\chi(K_n) = n$
- $\chi(G) = n \Leftrightarrow G = K_n$

Greedy algorithm

- First label the vertices in some order—call them $v_1, v_2, \ldots, v_n$
- Next, order the available colors $(1,2, \ldots, n)$ in some way
  - Start coloring by assigning color 1 to vertex v_1
  - If v_1 and v_2 are adjacent, assign color 2 to vertex v_2; otherwise, use color 1
  - To color vertex v_i, use the first available color that has not been used for any of v_i's previously colored neighbors

Examples: Different orders result in different number of colors

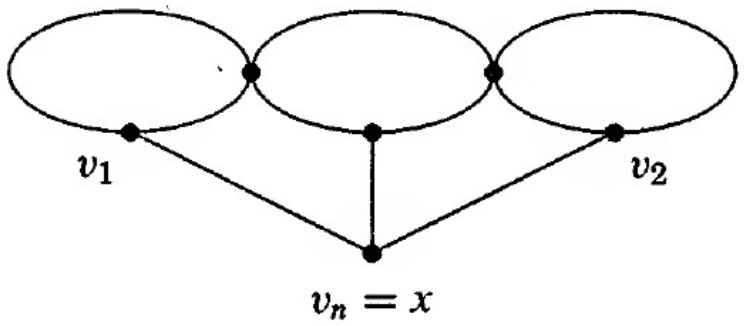


Bound using the greedy algorithm

- Theorem (1.42, H) For any graph G, $\chi(G) \leq \Delta(G) + 1$
  The equality is obtained for complete graphs and odd cycles

Brooks's theorem

- Theorem (1.43, H; 5.1.22, W; 5.2.4, D; Brooks 1941)
  If G is a connected graph that is neither an odd cycle or a complete graph, then $\chi(G) \leq \Delta(G)$

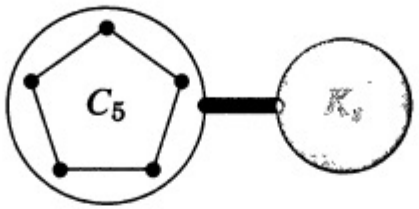


- $\Rightarrow$The Petersen graph is 3-colorable

Chromatic number and clique number

- The clique number $\omega(G)$ of a graph is defined as the order of the largest complete graph that is a subgraph of G
- Example: $\omega(G_1) = 3, \omega(G_2) = 4$

- Theorem (1.44, H; 5.1.7, W) For any graph G, $\chi(G) \geq \omega(G)$
- Example (5.1.8, W) For $G = C_{2r+1} \vee K_s$, $\chi(G) > \omega(G)$



Chromatic number and independence number

- Theorem (1.45, H; 5.1.7, W; Ex6, S1.6.2, H) For any graph G of order n,

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G)$$

The independence number of a graph G, denoted as $\alpha(G)$, is the largest size of an independent set

In a proper coloring, each color class is an independent set. Then G is k-colorable $\Leftrightarrow$ $V(G)$ is the union of k independent sets

Extremal problems for k-chromatic graphs

- Proposition (5.2.5, W) Every k-chromatic graph with n vertices has at least $\binom{k}{2}$ edges
  - Equality holds for a complete graph plus isolated vertices.

> In a proper coloring, each color class is an independent set. Then G is k-colorable $\Leftrightarrow V(G)$ is the union of k independent sets

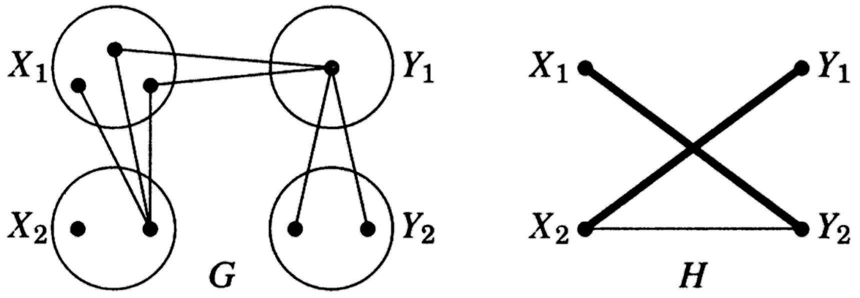
- The Turán graph $T_{n,r}$ is the complete r-partite graph with n vertices whose partite sets differ by at most 1 vertex
  - Every partite set has size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$
- Lemma (5.2.8, W) Among simple r-partite (that is, r-colorable) graphs with n vertices, the Turán graph is the unique graph with the most edges
- Turán's Theorem (5.2.9, W; Turán 1941) Among the n-vertex simple K_{r+1}-free graphs, $T_{n,r}$ has the maximum number of edges

Color-critical

- If $\chi(H) < \chi(G) = k$ for every proper subgraph H, then G is color-critical or k-critical
- K_2 is the only 2-critical graph
  K_1 is the only 1-critical graph
- (5.2.12, W) A graph with no isolated vertices is color-critical $\Leftrightarrow$ $\chi(G - e) < \chi(G)$ for every edge $e \in E(G)$
- Proposition (5.2.13, W) Let G be a k-critical graph
  (a) For every $v \in V(G)$, there is a proper coloring such that v has a unique color and other $k - 1$ colors all appear on $N(v)$
  $\Rightarrow \delta(G) \geq k - 1$
  (b) For every $e \in E(G)$, every proper $(k - 1)$-coloring of $G - e$ gives the same color to the two endpoints of e

Color-critical has edge-connectivity

- Theorem (5.2.16, W; Dirac 1953) Every k-critical graph is $(k-1)$-edge-connected
- Lemma (5.2.15, W; Kainen) Let G be a graph with $\chi(G) > k$ and let X, Y be a partition of $V(G)$. If $G[X]$ and $G[Y]$ are k-colorable, then the edge cut $[X, Y]$ has at least k edges

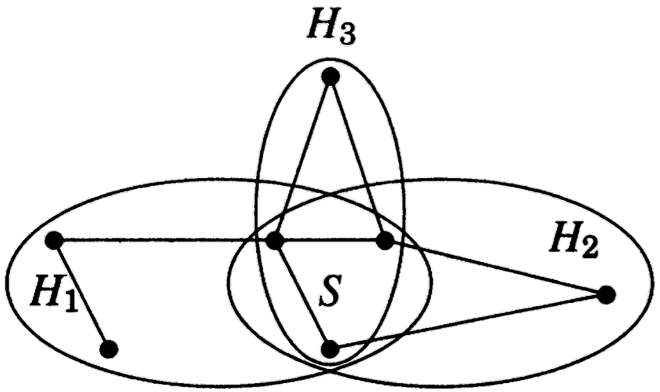


Theorem (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931) Let G be a bipartite graph. The maximum size of a matching in G is equal to the minimum size of a vertex cover of its edges

Remark (4.1.8, W) Every minimal disconnecting set of edges is an edge cut

Color-critical and vertex cut set

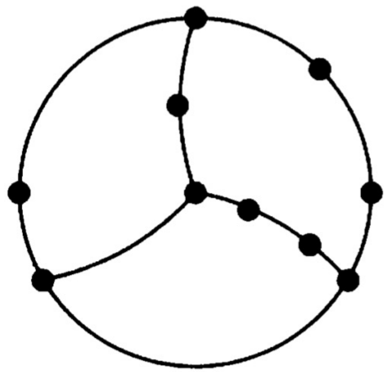
- Let S be a set of vertices in a graph G. An S-lobe of G is an induced subgraph of G whose vertex set consists of S and the vertices of a component in $G - S$



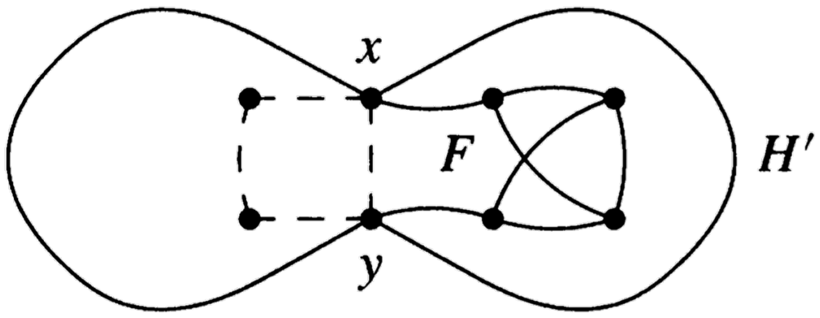
- Proposition (5.2.18, W) If G is k-critical, then G has no clique cutset. In particular, if G has a cutset $S = \{x, y\}$, then x, y are non-adjacent and G has an S-lobe H such that $\chi(H + xy) = k$

Chromatic number 4 has a K_4-subdivision

- Theorem (5.2.20, W; Dirac 1952) Every graph with chromatic number at least 4 contains a K_4-subdivision

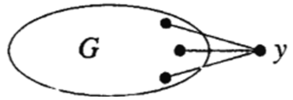


a subdivision of K_4



Proposition (5.2.18, W) If G is k-critical, then G has no clique cutset. In particular, if G has a cutset $S=\{x,y\}$, then x,y are non-adjacent and G has an S-lobe H such that $\chi(H+xy)=k$

Lemma (4.2.3, W; Expansion Lemma) If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected



Hajós' conjecture

- Hajós' conjecture [1961]: Every k-chromatic graph contains a subdivision of K_k
- $k = 2$: Every 2-chromatic graph has a nontrivial path
- $k = 3$: Every 3-chromatic graph has a cycle
- It is open for $k = 5,6$
- Exercise It is false for $k \geq 7$

Chromatic Polynomials

Definition and examples

- It is brought up by George David Birkhoff in 1912 in an attempt to prove the four color theorem
- Define $\chi(G; k)$ to be the number of different colorings of a graph G using at most k colors
- Examples:
  - How many different colorings of K_4 using 4 colors?
    - $4 \times 3 \times 2 \times 1$
    - $\chi(K_4; 4) = 24$
  - How many different colorings of K_4 using 6 colors?
    - $6 \times 5 \times 4 \times 3$
    - $\chi(K_4; 6) = 360$
  - How many different colorings of K_4 using 2 colors?
    - 0
    - $\chi(K_4; 2) = 0$

Examples

- If $k \geq n$

$$\chi(K_n; k) = k(k-1)\cdots(k-n+1)$$

- If $k < n$

$$\chi(K_n; k) = 0$$

- G is k-colorable $\Leftrightarrow$ $\chi(G) \leq k$ $\Leftrightarrow \chi(G; k) > 0$
- $\chi(G) = \min\{k \geq 1 \colon \chi(G; k) > 0\}$

Chromatic recurrence

- $G - e$ and G/e

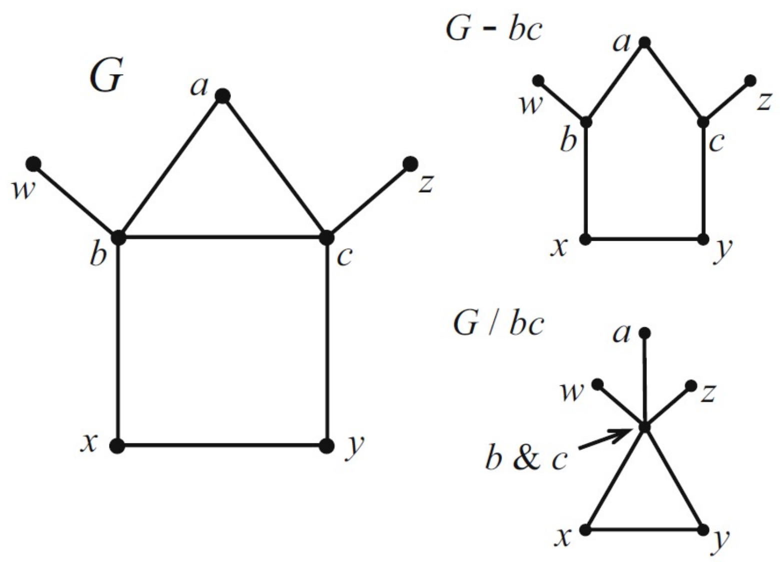


FIGURE 1.98. Examples of the operations.

- Theorem (1.48, H; 5.3.6, W) Let G be a graph and e be any edge of G. Then

$$\chi(G; k) = \chi(G - e; k) - \chi(G/e; k)$$

Use chromatic recurrence to compute $\chi(G;k)$

- Example: Compute $\chi(P_3;k) = k^4 - 3k^3 + 3k^2 - k$
- Check: $\chi(P_3;1) = 0, \chi(P_3;2) = 2$

FIGURE 1.102. Two 2-colorings of P_3

More examples

- Path P_{n-1} has $n-1$ edges (n vertices)
$$\chi(P_{n-1}; k) = k(k-1)^{n-1}$$
- Any tree T on n vertices
$$\chi(T; k) = k(k-1)^{n-1}$$
- Cycle C_n
$$\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$$
  - When n is odd, $\chi(C_n; 2) = 0, \chi(C_n; 3) > 0$
  - When n is even, $\chi(C_n; 2) > 0$

Properties of chromatic polynomials

- Theorem (1.49, H; Ex 3, S1.6.4, H) Let G be a graph of order n
  - $\chi(G; k)$ is a polynomial in k of degree n
  - The leading coefficient of $\chi(G; k)$ is 1
  - The constant term of $\chi(G; k)$ is 0
    - If G has i components, then the coefficients of $k^0, \ldots, k^{i-1}$ are 0
    - G is connected $\Leftrightarrow$ the coefficient of k is nonzero
  - The coefficients of $\chi(G; k)$ alternate in sign
  - The coefficient of the k^{n-1} term is $-|E(G)|$
    - A graph G is a tree $\Leftrightarrow \chi(G; k) = k(k-1)^{n-1}$

      $\Leftrightarrow$ (Theorem 1.10, 1.12, H) T is connected with $n - 1$ edges

    - A graph G is complete $\Leftrightarrow \chi(G; k) = k(k-1)\cdots(k-n+1)$

Summary

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- Coloring, proper coloring, chromatic #
- Brooks's theorem
- Chromatic # vs. clique/independence #
- Turán graph

Questions?