Homework 1

Anonymous

September 24, 2020

Problem 1. *Let $d \in N$ and $V := \{0,1\}^d$; thus V is the set of all 0-1 sequences of length d. The graph on V in which two such sequences form an edge if and only if they differ in exactly one position is called the d-dimensional cube. Determine the average degree, number of edges, diameter, girth and circumference of this graph.*

Solution. Denote the d-dimensional cube as graph G.

- **Average degree.** For any vertex in d-dimensional cube, there are exactly d vertices with exactly one different digit. Thus the degree of each vertex is $d(G) = d$.
- **Number of edges.** $||G|| = \frac{1}{2} \times |G| \times d(G) = \frac{1}{2} \times 2^d \times d = d2^{d-1}$.
- **Diameter.** It's obvious that a shortest path connecting two vertices in G can be constructed by flipping the different digits sequencially. With the fact that two vertices have at most d different digits, the diameter of G is diam(G)= d.
- **Girth.**

Case 1. $d = 1$. G has no cycle, $g(G) = \infty$

Case 2. $d \geq 2$. Note that every cycle has an even length because digits should be flipped over and over again to return back. Only by flipping one digit will generate a path but not a cycle. Thus at least 2 digits should be flipped to generate a 4-cycle.

Finally, we have

$$g(G) = \begin{cases} \infty, & d = 1 \\ 4, & d \geq 2 \end{cases}$$

- **Circumference.** To show that the circumference of G is 2^d with $d \geq 2$, we can prove that G has a Hamiltonian cycle by induction.
  - **Base case:** The 2-dimensional cube is a 4-cycle. It's itself a Hamiltonian cycle.

– **Inductive step:** Assume that an n-dimensional cube have a Hamiltonian cycle, say $v_1, v_2, \ldots, v_{2^n}, v_1$. Then the $n+1$-dimensional cube H_{n+1} is formed from two n-dimensional cubes, say H_n with Hamiltonian cycle $v_1, v_2, \ldots, v_{2^n}, v_1$ and I_n with Hamiltonian cycle $w_1, w_2, \ldots, w_{2^n}, w_1$, where we connect v_i to w_i for $i = 1, 2, \ldots, 2^n$ to form H_{n+1}. Then the cycle $v_1, v_2, \ldots, v_{2^n}, w_{2^n}, \ldots, w_2, w_1, v_1$ is the new Hamiltonian cycle we want.

Finally, we have

$$\text{circumference of } G = \begin{cases} 0, & d = 1 \\ 2^d, & d \geq 2 \end{cases}$$

□

Problem 2. *Let G be a graph containing a cycle C, and assume that G contains a path of length at least k between two vertices of C. Show that G contains a cycle of length at least $\sqrt{k}$.*

Solution. Assume that the path, say P and C share n vertices.

Case 1. $n \geq \sqrt{k}$. C has a length of at least $\sqrt{k}$, which is what we want.

Case 2. $n < \sqrt{k}$. The shared vertices slice P into $n-1$ segments. The minimum value of the maximum length of $n-1$ segments is

$$\frac{k}{n-1} > \frac{k}{\sqrt{k}-1} > \sqrt{k}+1$$

Because both ends of the longest slice is on C, the endpoints have a path in C. Combining the path with P, we get a cycle with length of at least $\sqrt{k}+2$.

□

Problem 3. *Is the bound in Proposition 1.3.2 (Diestel) best possible?*

Solution. Yes. Take K^3 as an example. □

Problem 4. *Let v_0 be a vertex in a graph G, and $D_0 := \{v_0\}$. For $n = 1, 2, \cdots$ inductively define $D_n := N_G(D_0 \cup \cdots \cup D_{n-1})$. Show that $D_n = \{v | d(v_0, v) = n\}$ and $D_{n+1} \subseteq N(D_n) \subseteq D_{n-1} \cup D_{n+1}$ for all $n \in N$.*

Solution.

- $\boldsymbol{D_n = \{v | d(v_0, v) = n\}}$ **:** We prove that $D_n = \{v | d(v_0, v) = n\}$ by induction.

  – **Base case:** $D_0 = \{v_0\} = \{v | d(v_0, v) = 0\}$.

- **Inductive step:** Assume that $D_i = \{v|d(v_0, v) = i\}$ for $i = 1, 2, \ldots, n$. For any vertex $u \in D_{n+1}$, we have $d(v_0, u) \geq n+1$, otherwise $u \in \bigcup_{i=0}^{n} D_i$, which is contradict with $D_{n+1} := N_G(D_0 \cup \cdots \cup D_n)$. Also by the definaion of D_{n+1}, we have $\exists w \in \bigcup_{i=0}^{n} D_i$ s.t. $d(u, w) = 1$. Thus $d(v_0, u) \leq d(v_0, w) + d(w, u) \leq n+1$. So far we proved that $D_{n+1} \subseteq \{v|d(v_0, v) = n+1\}$.
  Consider any vertex $v \in \{u|d(v_0, u) = n+1\}$, there exists a path $v_0, u_1, u_2, \ldots, u_n, v$. Obviously $d(v_0, u_n) \leq n$ and thus $u_n \in \bigcup_{i=0}^{n} D_i$. This means $\{v|d(v_0, v) = n+1\} \subseteq D_{n+1}$.
  In all, $D_{n+1} = \{v|d(v_0, v) = n+1\}$.

- $\boldsymbol{D_{n+1} \subseteq N(D_n)}$ **:** $\forall v \in D_{n+1}$, there exists a path $v_0, u_1, u_2, \ldots, u_n, v$. Obviously $d(v_0, u_n) = n$, otherwise $d(v_0, v) \leq n$. This means $u_n \in D_n$ and there exists an edge (u_n, v). Thus $D_{n+1} \subseteq N(D_n)$ for all $n \in N$.

- $\boldsymbol{N(D_n) \subseteq D_{n-1} \cup D_{n+1}}$ **:** $\forall v \in N(D_n), \exists u \in D_n$ s.t. $(u, v) \in E(G)$. Thus $d(v_0, v) \leq d(v_0, u) + d(u, v) = n+1$ and $d(v_0, v) \geq d(v_0, u) - d(v, u) = n-1$. Noticing that $v \notin D_n$, we can conclude that $N(D_n) \subseteq D_{n-1} \cup D_{n+1}$ for all $n \in N$.

□

Problem 5. *Let G be a graph. If $\delta(G) \geq d \geq 2$ and $g(G) \geq g \in N$ then $|G| \geq n_0(d, g)$. (The defination of n_0 is presented in Page 9 (Diestel)).*

Solution. Case 1. $g =: 2r+1$ is odd. Let v be any vertex in G, and for $i \in \mathbb{N}$

$$D_i = \{v|d(v_0, v) = i\}.$$

Since v cannot be in any cycle of length of lesser than $2r+1$, which equals to $\forall u, w \in D_i, 0 \leq i \leq r-1$, there is no edge connecting $N(u) \cap D_{i+1}$ and $N(w) \cap D_{i+1}$. Hence $\forall u \in D_i, 0 \leq i \leq r-1$, u is connected to exactly one vertex in D_{i-1} and at least $\delta - 1$ vertices in D_{i+1}. Hence $|D_0| = 1$ and $|D_i| \geq \delta(\delta-1)^{i-1}$ for $1 \leq i \leq r$. Thus

$$|G| \geq \bigcup_{i=0}^{r-1} |D_i| = 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^{i-1}$$

Case 2. $g =: 2r$ is even. Let $uv \in E(G)$, and for $i \in \mathbb{N}$,

$$D_i^u = \{w : \hat{d}(u, w) = i\}, \quad D_i^v = \{w : \hat{d}(v, w) = i\},$$

where

$$\hat{d}(x, w) := d_{G-uv}(x, w).$$

Similarly, $\forall x \in \{u, v\}, \forall y, z \in D_i^x$, $N(x) \cap D_{i+1}^x$ is disjoint from $N(y) \cap D_{i+1}^y$ for $i = \{1, 2, \ldots, r-2\}$. Also $\forall w \in D_i^x, 0 \leq i \leq r-2$, w is connected to exactly one vertex in D_{i-1}^x and at least $\delta - 1$ vertices in D_{i+1}^x. Hence $|D_0^x| = 1$ and $|D_i^x| \geq \delta(\delta-1)^{i-1}$ for $1 \leq i \leq r-1$. Note that $(\bigcup_{i=0}^{r-1} D_i^u) \cap (\bigcup_{i=0}^{r-1} D_i^v) = \emptyset$, otherwise there exist a cycle in G of length at most $2r-1$. Hence

$$|G| \geq \bigcup_{i=0}^{r-1} D_i^u + \bigcup_{i=0}^{r-1} D_i^v = 2\sum_{i=0}^{r-1}(\delta-1)^i.$$

□

Problem 6. *Show that graphs of girth at least 5 and order n have a minimum degree of $o(n)$. In other words,show that there is a function $f : N \to N$ such that $f(n)/n \to 0$ as $n \to \infty$ and $\delta(G) \leq f(n)$ for all such graph G.*

Solution. From *Problem 5* we have

$$n \geq 1 + \delta(G)\sum_{i=0}^{1}(\delta(G)-1)^i = (\delta(G))^2 + 1$$

Then

$$\delta(G) \leq \sqrt{n-1} = o(n)$$

□

Problem 7. *Show that the components of a graph partition its vertex set.*

Solution. Given a graph G and its components G_i for $i \in \{1, 2, \ldots, n\}$.

- $\boldsymbol{G_i \cap G_j = \emptyset, i, j \in \{1, 2, \ldots, n\}}$**.** Otherwise we can get a new component $G_i \cup G_j$.
- $\boldsymbol{\bigcup_{i=1}^n G_i = G}$**.** Otherwise vertices in $G \backslash \bigcup_{i=1}^n G_i$ can either be added into some G_i or make up new component(s).

□

Problem 8. *Show that every 2-connected graph contains a cycle.*

Solution. Let G be a 2-connected graph. Then $\delta(G) \geq 2$, since if $d(v) \leq 1$ for some $v \in V(G)$, v will be either a cutvertex or isolated, which is contradict with G to be 2-connected. From *Proposition (1.3.1, D)*, we know that G contains a cycle. □

Problem 9. *Determine $\kappa(G)$ and $\lambda(G)$ for $G = P^m, C^n, K^n, K_{m,n}$ and the d-dimensional cube; $d, m, n > 2$.*

Solution. Given graphs are all connected.

- $\boldsymbol{\kappa(P^m) = 1}$**.** For $m \leq 2$ it's clear and for $m > 2$ every interior vertex is a cutvertex.
- $\boldsymbol{\lambda(P^m) = 1}$**.** Every edge is a bridge.
- $\boldsymbol{\kappa(C^n) = 2}$**.** By deleting any vertex we will get P^{n-1}. Hence we need to delete at least 2 vertices to get a disconnected graph.
- $\boldsymbol{\lambda(C^n) = 2}$**.** By deleting any edge we will get P^n. Hence we need to delete at least 2 edges to get a disconnected graph.
- $\boldsymbol{\kappa(K^n) = n - 1}$**.** By deleting any vertex we will get K^{n-1}. Hence we need to delete at least $n-1$ vertices to get a disconnected graph.
- $\boldsymbol{\lambda(K^n) = n - 1}$**.** From *Proposition (1.4.2, D)* we have $n - 1 = \kappa(K^n) \leq \lambda(K^n) \leq \delta(K^n) = n - 1$.
- $\boldsymbol{\kappa(K_{m,n}) = \min\{m, n\}}$**.** For complete bipartite graph $K_{m,n} = (V_1, V_2, E)$ with $|V_1| = m, |V_2| = n$, by deleting any p vertices in V_1 and q vertices in V_2 we will get $K_{m-p,n-q}$. Hence we need to delete at least $\min\{m, n\}$ vertices to get a disconnected graph.
- $\boldsymbol{\lambda(K_{m,n}) = \min\{m, n\}}$**.** From *Proposition (1.4.2, D)* we have $\min\{m, n\} = \kappa(K_{m,n}) \leq \lambda(K_{m,n}) \leq \delta(K_{m,n}) = \min\{m, n\}$.
- $\boldsymbol{\kappa(Q_d) = d, \lambda(Q_d) = d}$**.** (Prove using *Balinski's theorem*)
  - **Proof of** $\boldsymbol{\kappa(Q_d) \geq d}$**.** If S is a set of fewer than d vertices to be removed from Q_d, we can add one more vertex v_0 to S and find a linear function f that has value 0 on $S \cup \{v_0\}$ but is not identically zero on the whole space. Then any remained vertex at which f is non-negative (including v_0) can be connected to the vertex with the maximum value of f, while any remained vertex at which f is non-positive (including v_0) can be connected to the vertex with the minimum value of f. Thus the entire remaining graph is connected.
  - From *Proposition (1.4.2, D)* we have $d = \kappa(Q_d) \leq \lambda(Q_d) \leq \delta(Q_d) = d$.

□

Problem 10. *If* $\delta(G) \geq \frac{n-2}{2}$, *then* G *need not be connected.*

Solution. Take $G = K^{\frac{n}{2}} \cup K^{\frac{n}{2}}$ for even n and then G is what we want. □

Problem 11. *Find the relation between 'G has a cutvertex.' and 'G has a bridge'.*

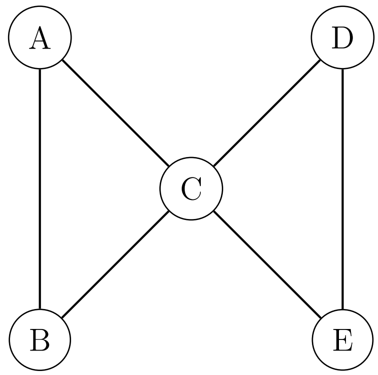

Figure 1: An example

Solution. 'G has a cutvertex.'$\Leftarrow$ 'G has a bridge'.
For any bridge $uw \in E(G)$, u and w are both cutvertex.
In some case, G has a cutvertex but no bridge. For example, in figure 1, C is a cutvertex but there is no bridge. $\square$