1. Find planar representations for each of the planar graphs in Figure 1

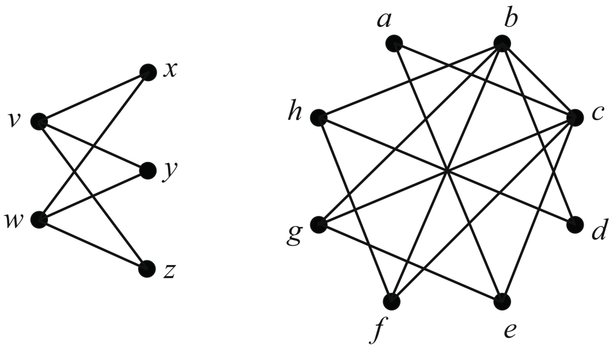

Figure 1: .

2. Give planar representations of the graph in Figure 2 such that each of the following is the exterior region.

   (a) R_1

   (b) R_2

   (c) R_3

   (d) R_4

   (e) R_5

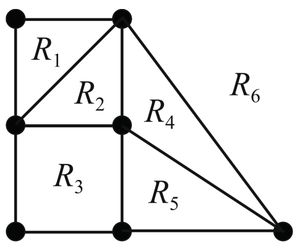

Figure 2: .

3. Draw a planar graph in which every vertex has degree exactly 5.

   Solution: Shown in Figure 3

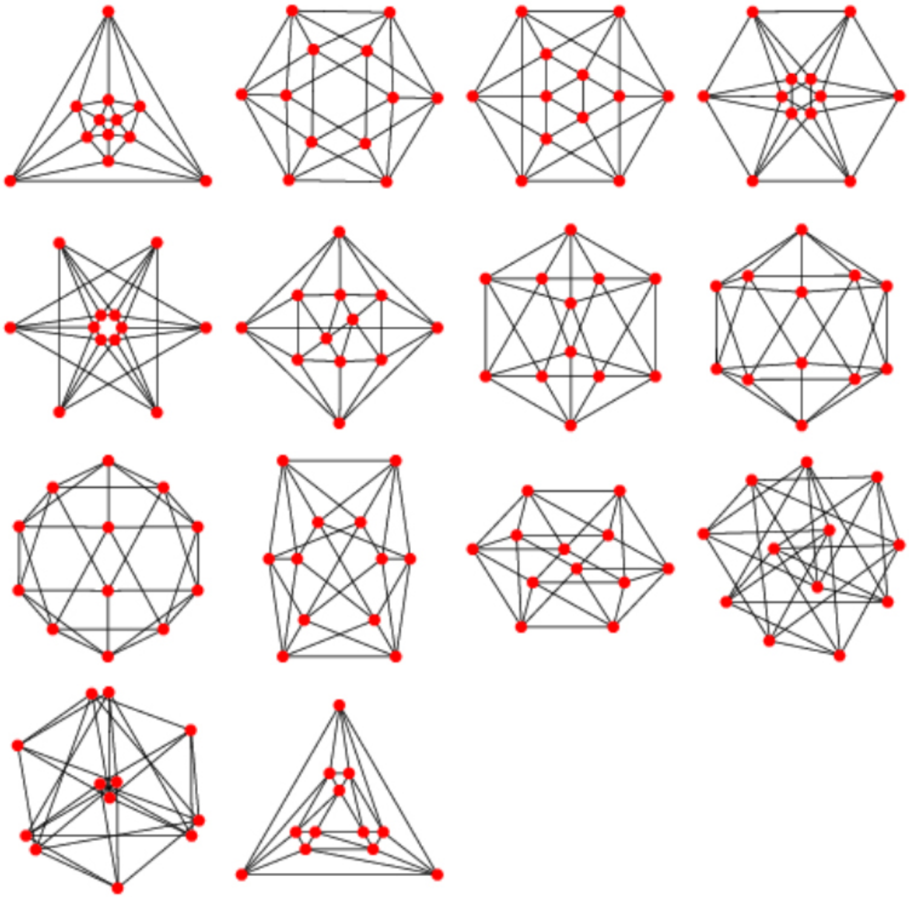

Figure 3: Problem 3

4. Suppose that e is a bridge of a planar graph G. Prove that e does not bound a region in any planar representation of G.

   Solution: If e bounds a region in a planar representation of G, then e is in a circle (otherwise, there would not be a region). Thus, e cannot be a bridge, contradictory to the conditon.

5. Fáry and Wagner proved independently that every planar graph has a planar representation in which every edge is a straight linesegment. Find such a representation for the graph in Figure 4

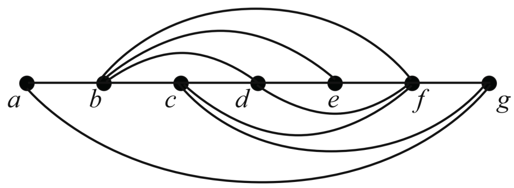

Figure 4: .

Solution: Shown in Figure 5

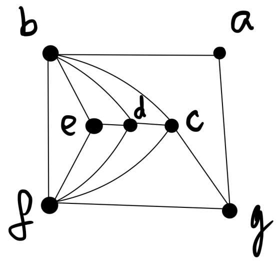

Figure 5: Problem 5

6. If planar graphs G_1 and G_2 each have n vertices, q edges, and r regions,must the graphs be isomorphic? Justify your answer with a proof or a counterexample.

   Solution: Shown in Figure 6

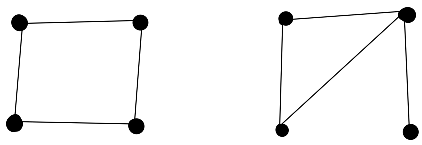

Figure 6: Problem 6

7. Let G be of order $n \geq 11$. Show that at least one of G and $\bar{G}$ is nonplanar. (Given a graph G, the complement of G, denoted by $\bar{G}$, is the graph whose vertex set is the same that of G, and whose edge set consists of all the edges that are not present in G.)

   Solution: Assume that there exists G of order $n \neq 11$ and both G and $\bar{G}$ are planar. We can know according to Theorem 1.33 that the edges of both graph cannot be more than $3n-6$. However, the summation of G and $\bar{G}$ is $n(n-1)/2$, which implies $n(n-1) \leq 4(3n-6) \Rightarrow n < 11$, contradictory to the condition.

8. A planar graph G is called **maximal planar** if the addition of any edge to G creates a nonplanar graph.

   (a) Show that every region of a maximal planar graph is a triangle.

   (b) If a maximal planar graph has order n, how many edges and regions does it have?

   Solution:

   (a) Otherwise we can always add an edge in a region that is not a triangle.

   (b) Consider $C = \sum_R b(R)$, where $b(R)$ denotes the number of boundaries of region R and the sum is over all regions. Since each region is bounded by 3 edges, we have $C = 3r$. Since every edge of G can be on the boundary of exactly 2 regions. If e is the boundary of only one region, than the both side of the edge must be in the same region, which is contradictory to the fact of every region is a triangle. Thus, we have $C = 2q = 3r$. With $n - q + r = 2$, we can deduce that $r = 2n-4$ and $q = 3n-6$.

9. Use Kuratowski's Theorem to prove that the Petersen graph (Figure 7) is nonplanar.

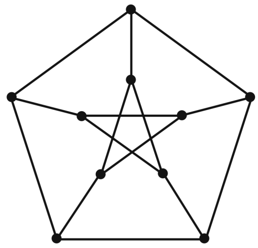

Figure 7: The Petersen Graph.

Solution: Peterson Graph contains a subdivision of K_5.

10. Determine all complete multiparitite graphs (of the form $K_{r_1,\cdots,r_n}$) that are planar.

Solution: In the solution, we only consider $n > 1$. First of all, based on Kuratowski's Theorem, n must be less than 5 or we can choose a point from different independent sets respectively and form a K_5. Then there should not be two independent sets that contains more than 3 vertices at the same time. Thus, we enumerate all the possible conditions. Assume $r_1 \leq \cdots \leq r_n$, using I_n to denote the independent set containing r_n vertices.

- n = 2
  We can testify that both $K_{1,m}$ and $K_{2,m}$ where m can be any positive integer, are planar.
- n = 3
  $K_{1,1,n}$ is planar, easy to draw. Considering $K_{1,2,n}$, when n is 1 or 2, it is easy to confirm that the graph is planar. When n is more than 3, choose arbitrarily 3 points from I_3 and consider union I_{12} of I_1 and I_2. The I_3 and I_{12} with all the edges connecting the vertices contain a subvision of $K_{3,3}$, not planar. Consider $K2, 2, n$, based on the discussion on $K_{1,2,n}$, n must be 2. We can prove by drawing the planar presentation that $K_{2,2,2}$ is planar.
- n = 4
  $K_{1,1,1,n}$ cannot be planar when $n > 2$. Since the induced subgraph of $\bigcup_{i=1}^{3} I_i$ and 3 vertices in I_4 contains a subdivision of $K_{3,3}$. Thus n must be 2. Considering $K_{1,1,2,n}$, when $n > 1$, it contains $K_{3,3}$. The other conditions would contain the discussed conditions which is proved not planar.

To sum up, $K_{1,n}$, $K_{2,n}$, $K_{1,1,n}$, $K_{1,2,2}$, $K_{2,2,2}$, $K_{1,1,1,2}$ are all the possible complete multipartite graphs that are planar.