Lecture 6: Circuits (2) and Matchings

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https://shuaili8.github.io

https://shuaili8.github.io/Teaching/CS445/index.html

Wiener index

- In a communication network, large diameter may be acceptable if most pairs can communicate via short paths. This leads us to study the average distance instead of the maximum
- Wiener index $D(G) = \sum_{u,v \in V(G)} d_G(u,v)$
- Theorem (2.1.14, W) Among trees with n vertices, the Wiener index $D(T)$ is minimized by stars and maximized by paths, both uniquely
- Over all connected n-vertex graphs, $D(G)$ is minimized by K_n and maximized by paths
  - (Corollary 2.1.16, W) If G is a connected n-vertex graph, then $D(G) \leq D(P_{n-1})$
    - (Lemma 2.1.15, W) If H is a subgraph of G, then $d_G(u,v) \leq d_H(u,v)$

Hamiltonian path/circuits

- A path P is Hamiltonian if $V(P) = V(G)$
  - Any graph contains a Hamiltonian path is called traceable
- A cycle C is called Hamiltonian if it spans all vertices of G
  - A graph is called Hamiltonian if it contains a Hamiltonian circuit
- In the mid-19th century, Sir William Rowan Hamilton tried to popularize the exercise of finding such a closed path in the graph of the dodecahedron

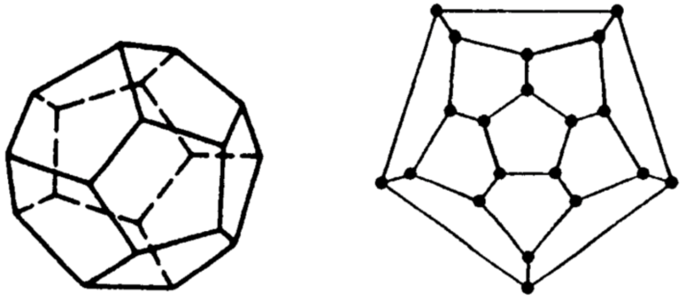
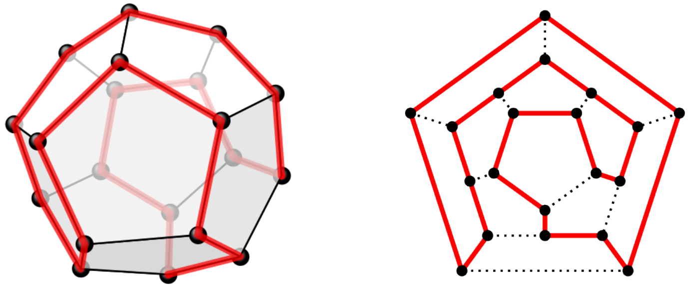


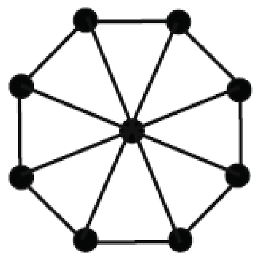
Figure 1.9



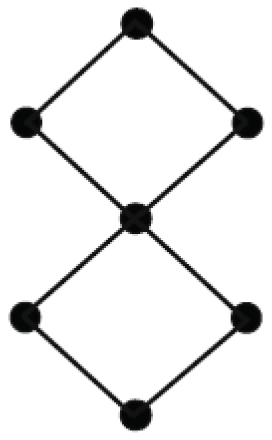
Degree parity is not a criterion

Theorem (1.2.26, W) A graph G is Eulerian $\Leftrightarrow$ it has at most one nontrivial component and its vertices all have even degree

- Hamiltonian graphs
  - all even degrees C_{10}
  - all odd degrees K_{10}
  - a mixture G_1
- non-Hamiltonian graphs
  - all even G_2
  - all odd $K_{5,7}$
  - mixed P_9



G_1



G_2

Example

- The Petersen graph has a Hamiltonian path but no Hamiltonian cycle

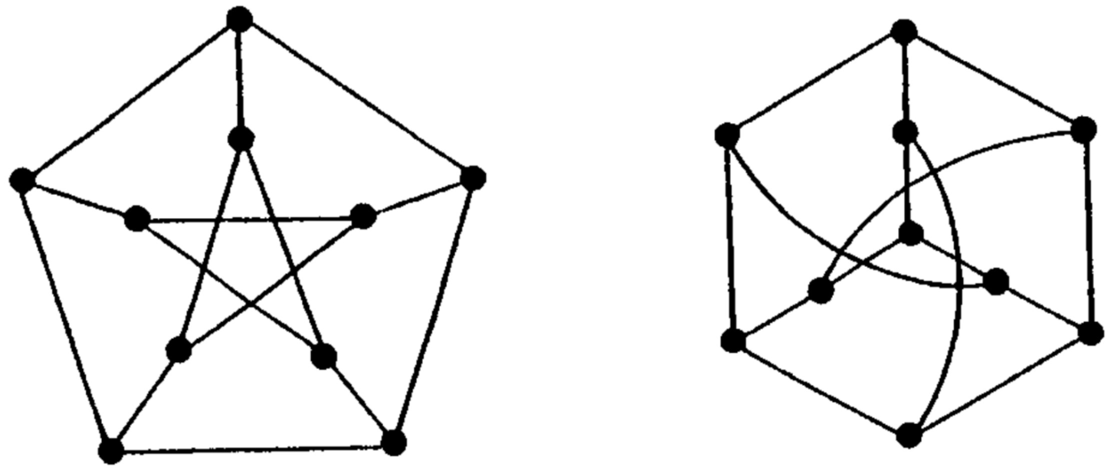
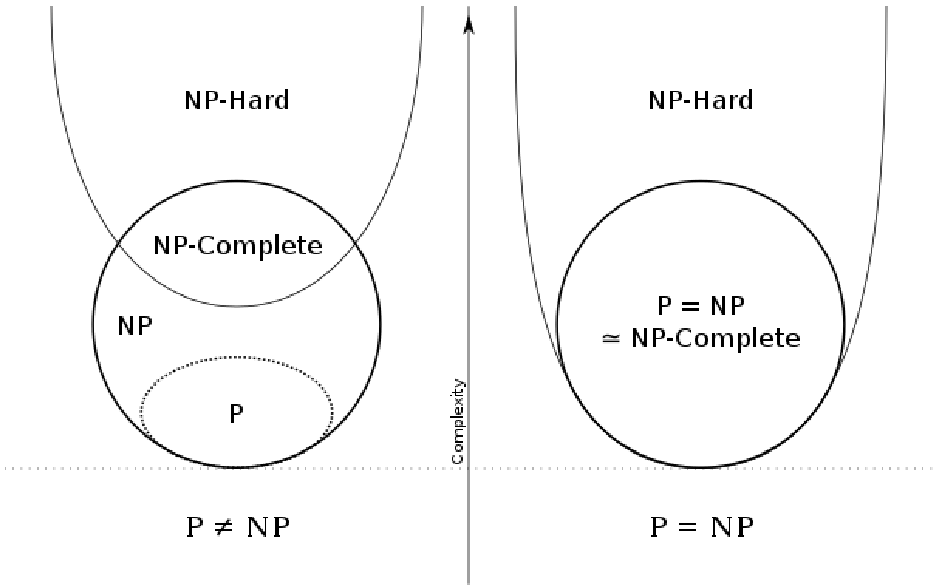


Figure 1.4

- Determining whether such paths and cycles exist in graphs is the Hamiltonian path problem, which is NP-complete

P, NP, NPC, NP-hard

- P The general class of questions for which some algorithm can provide an answer in polynomial time
- NP The class of questions for which an answer can be *verified* in polynomial time
- NP-Complete
    1. c is in NP
    2. Every problem in NP is reducible to c in polynomial time
- NP-hard
    - ~~c is in NP~~
    - Every problem in NP is reducible to c in polynomial time



Large minimal degree implies Hamiltonian

- Theorem (1.22, H, Dirac) Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian

Proposition (1.3.15, W) If $\delta(G) \geq \frac{n-1}{2}$, then G is connected

(Ex16, S1.1.2, H) (1.3.16, W)
If $\delta(G) \geq \frac{n-2}{2}$, then G need not be connected

- The bound is tight
  (Ex12b, S1.4.3, H) $G = K_{r,r+1}$ is not Hamiltonian
- The condition is not necessary
  - C_n is Hamiltonian but with small minimum (and even maximum) degree

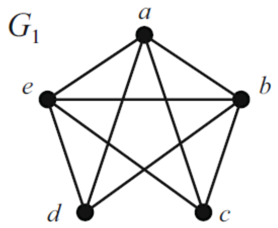
Generalized version

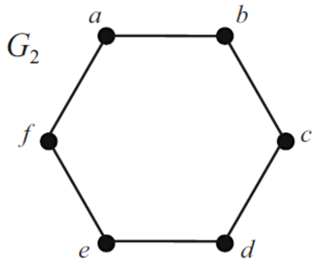
- Exercise (Theorem 1.23, H, Ore; Ex3, S1.4.3, H) Let G be a graph of order $n \geq 3$. If $\deg(x) + \deg(y) \geq n$ for all pairs of nonadjacent vertices x, y, then G is Hamiltonian

Theorem (1.22, H, Dirac) Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian

Independence number & Hamiltonian

- A set of vertices in a graph is called independent if they are pairwise nonadjacent
- The independence number of a graph G, denoted as $\alpha(G)$, is the largest size of an independent set
- Example: $\alpha(G_1) = 2, \alpha(G_2) = 3$
- Theorem (1.24, H) Let G be a connected graph of order $n \geq 3$. If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian
- The result is tight: $\kappa(G) \geq \alpha(G) - 1$ is not enough
  - $K_{r,r+1}$: $\kappa = \mathrm{r}, \alpha = r + 1$
  - Peterson graph: $\kappa = 3, \alpha = 4$ (Ex4, S1.4.3, H)





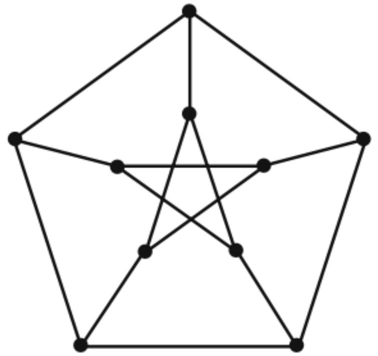
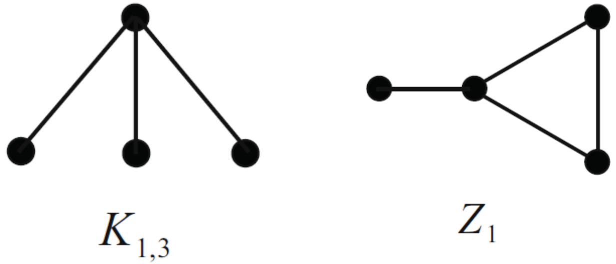


FIGURE 1.63. The Petersen Graph.

Pattern-free & Hamiltonian



- G is H-free if G doesn't contain a copy of H as induced subgraph
- Theorem (1.25, H) If G is 2-connected and $\{K_{1,3}, Z_1\}$-free, then G is Hamiltonian

(Ex14, S1.1.2, H) $\kappa(G) \geq 2$ implies G has at least one cycle

- The condition 2-connectivity is necessary
- (Ex2, S1.4.3, H) If G is Hamiltonian, then G is 2-connected

Matchings

Definitions

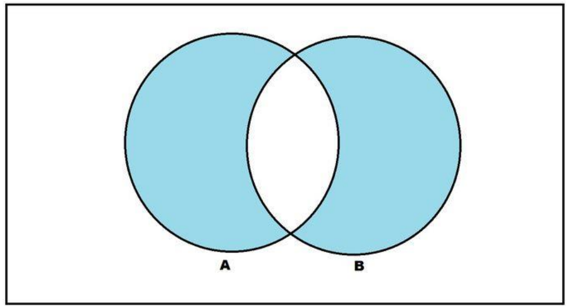
- A matching is a set of independent edges, in which no pair shares a vertex
- The vertices incident to the edges of a matching M are M-saturated; the others are M-unsaturated
- A perfect matching in a graph is a matching that saturates every vertex
- Example (3.1.2, W) The number of perfect matchings in $K_{n,n}$ is $n!$
- Example (3.1.3, W) The number of perfect matchings in K_{2n} is

$$f_n = (2n-1)(2n-3)\cdots 1 = (2n-1)!!$$

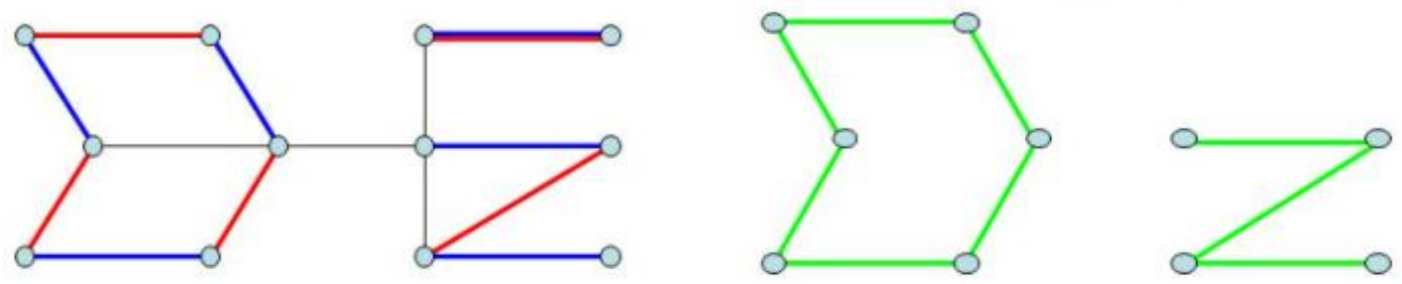
Maximal/maximum matchings 极大/最大

- A maximal matching in a graph is a matching that cannot be enlarged by adding an edge
- A maximum matching is a matching of maximum size among all matchings in the graph
- Example: P_3, P_5

Symmetric difference of matchings

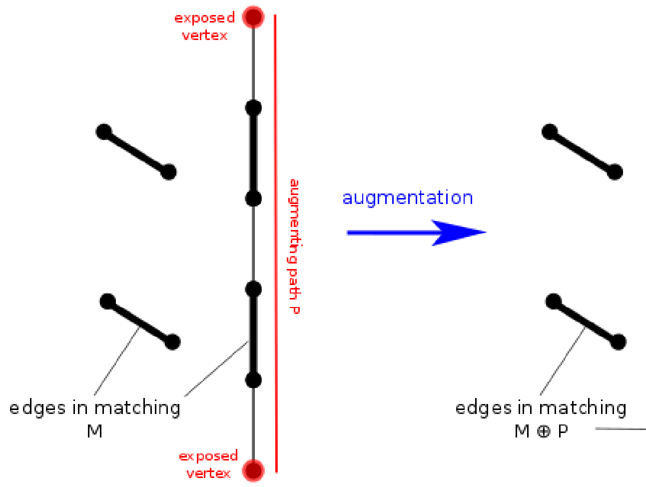


- The symmetric difference of M, M' is $M \Delta M' = (M - M') \cup (M' - M)$
- Lemma (3.1.9, W) Every component of the symmetric difference of two matchings is a path or an even cycle



Equivalent condition for maximum matching

- Given a matching M, an M-alternating path is a path that alternates between edges in M and edges not in M
- An M-alternating path whose endpoints are M-unsaturated is an M-augmenting path
- Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching M in a graph G is a maximum matching in G $\Leftrightarrow$ G has no M-augmenting path



Hall's theorem (TONCAS)

- Theorem (3.1.11, W; 1.51, H; 2.1.2, D; Hall 1935) Let G be a bipartite graph with partition X, Y.
  G contains a matching of $X \Leftrightarrow |N(S)| \geq |S|$ for all $S \subseteq X$

> Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching M in a graph G is a maximum matching in $G \Leftrightarrow G$ has no M-augmenting path

- Exercise. Read the other two proofs in Diestel.
- Corollary (3.1.13, W; 2.1.3, D) Every k-regular ($k > 0$) bipartite graph has a perfect matching

Application to SDR

- Given some family of sets X, a system of distinct representatives for the sets in X is a 'representative' collection of distinct elements from the sets of X

$$S_1 = \{2, 8\},$$
$$S_2 = \{8\},$$
$$S_3 = \{5, 7\},$$
$$S_4 = \{2, 4, 8\},$$
$$S_5 = \{2, 4\}.$$

The family $X_1 = \{S_1, S_2, S_3, S_4\}$ does have an SDR, namely $\{2, 8, 7, 4\}$. The family $X_2 = \{S_1, S_2, S_4, S_5\}$ does not have an SDR.

- Theorem(1.52, H) Let $S_1, S_2, \dots, S_k$ be a collection of finite, nonempty sets. This collection has SDR $\Leftrightarrow$ for every $t \in [k]$, the union of any t of these sets contains at least t elements

König-Egeváry Theorem (Min-max theorem)

- A set $U \subseteq V$ is a (vertex) cover of E if every edge in G is incident with a vertex in U
- Theorem (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931) Let G be a bipartite graph. The maximum size of a matching in G is equal to the minimum size of a vertex cover of its edges

> Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching M in a graph G is a maximum matching in $G \Leftrightarrow G$ has no M-augmenting path

Perfect matchings

- K_{2n}, C_{2n}, P_{2n} have perfect matchings

Corollary (3.1.13, W; 2.1.3, D) Every k-regular ($k > 0$) bipartite graph has a perfect matching

- Theorem(1.58, H) If G is a graph of order $2n$ such that $\delta(G) \geq n$, then G has a perfect matching

Theorem (1.22, H, Dirac) Let G be a graph of order $n \geq 3$. If $\delta(G) \geq n/2$, then G is Hamiltonian

Tutte's Theorem (TONCAS)

- Let $q(G)$ be the number of connected components with odd order
- Theorem (1.59, H) Let G be a graph of order $n \geq 2$. G has a perfect matching $\Leftrightarrow q(G - S) \leq |S|$ for all $S \subseteq V$

Petersen’s Theorem

- Theorem (1.60, H) Every bridgeless, 3-regular graph contains a perfect matching