

# Lecture 5: Matchings

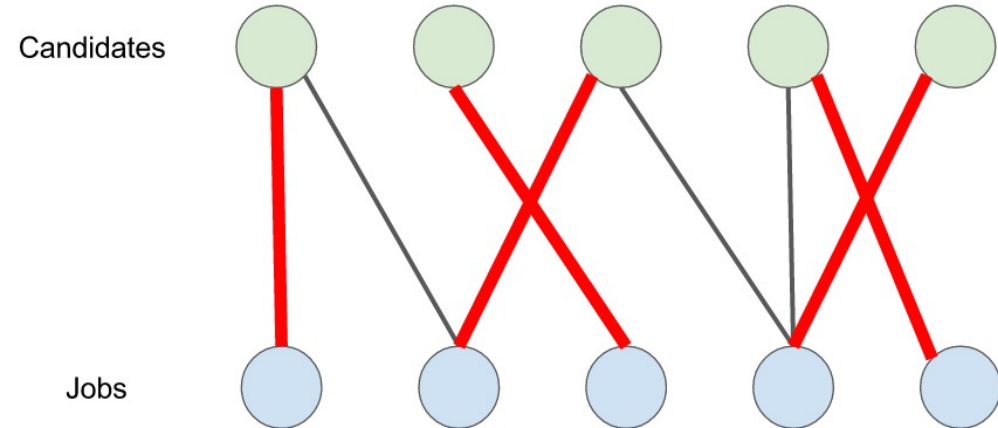
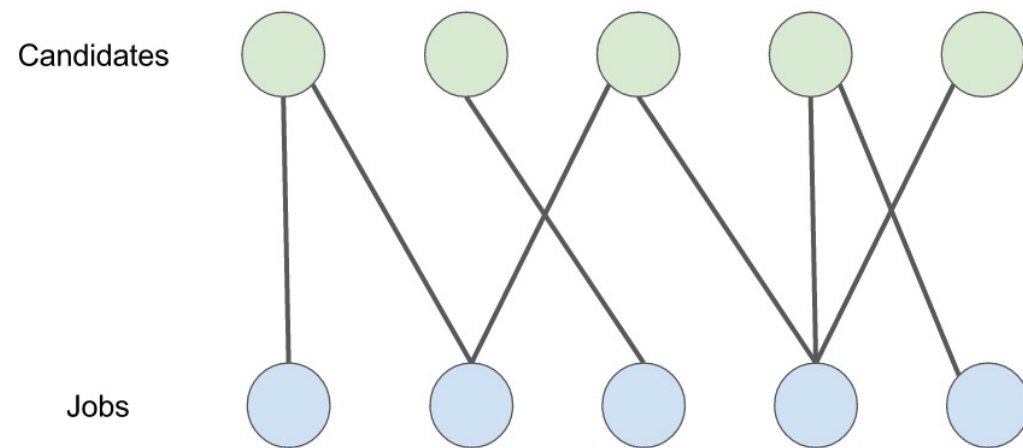
Shuai Li

John Hopcroft Center, Shanghai Jiao Tong University

<https://shuaili8.github.io>

<https://shuaili8.github.io/Teaching/CS445/index.html>

# Motivating example

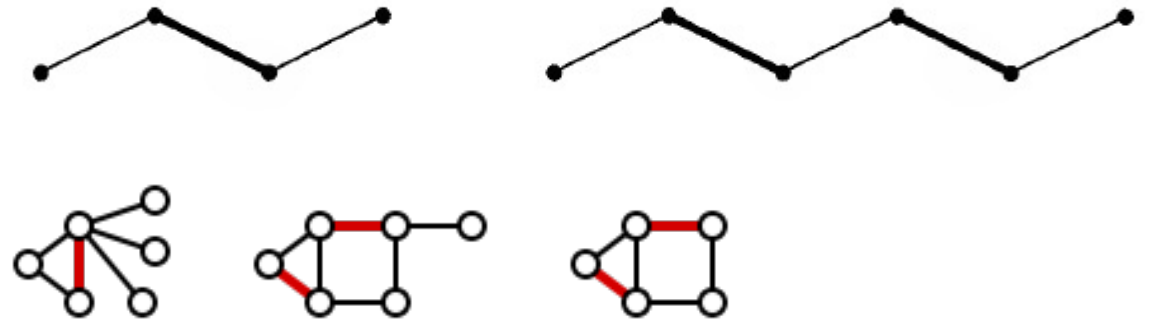


# Definitions

- A **matching** is a set of independent edges, in which no pair of edges shares a vertex
- The vertices incident to the edges of a matching  $M$  are  **$M$ -saturated** (饱和的); the others are  **$M$ -unsaturated**
- A **perfect matching** in a graph is a matching that saturates every vertex
- **Example** (3.1.2, W) The number of perfect matchings in  $K_{n,n}$  is  $n!$
- **Example** (3.1.3, W) The number of perfect matchings in  $K_{2n}$  is
$$f_n = (2n - 1)(2n - 3) \cdots 1 = (2n - 1)!!$$

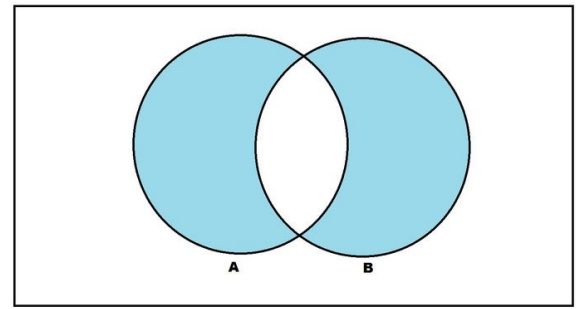
# Maximal/maximum matchings 极大/最大

- A **maximal matching** in a graph is a matching that cannot be enlarged by adding an edge
- A **maximum matching** is a matching of maximum size among all matchings in the graph
- Example:  $P_3, P_5$

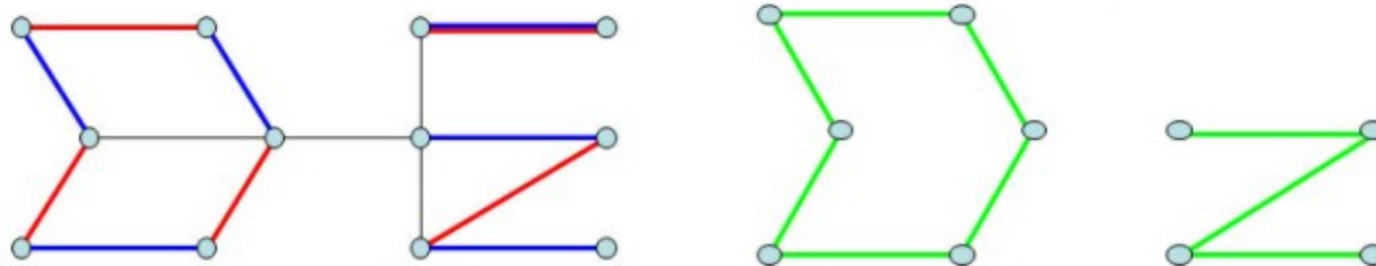


- Every maximum matching is maximal, but not every maximal matching is a maximum matching

# Symmetric difference of matchings



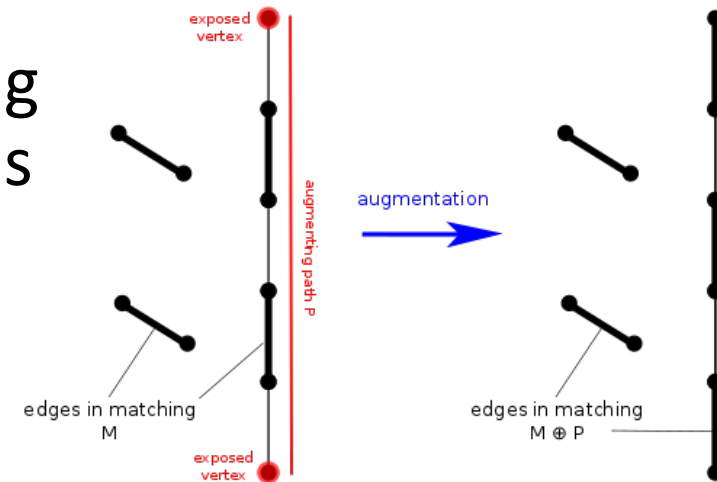
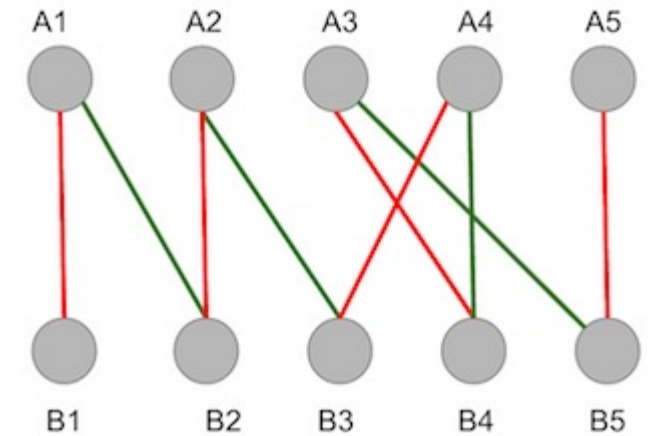
- The **symmetric difference** of  $M, M'$  is  $M \Delta M' = (M - M') \cup (M' - M)$
- **Lemma** (3.1.9, W) Every component of the symmetric difference of two matchings is a path or an even cycle



# Maximum matching and augmenting path

- Given a matching  $M$ , an  $M$ -alternating path is a path that alternates between edges in  $M$  and edges not in  $M$
- An  $M$ -alternating path whose endpoints are  $M$ -unsaturated is an  $M$ -augmenting path
- Theorem** (3.1.10, W; 1.50, H; Berge 1957) A matching  $M$  in a graph  $G$  is a **maximum** matching in  $G \iff G$  has no  $M$ -augmenting path

**Lemma** (3.1.9, W) Every component of the symmetric difference of two matchings is a path or an even cycle



# Hall's theorem (TONCAS)

- **Theorem** (3.1.11, W; 1.51, H; 2.1.2, D; Hall 1935) Let  $G$  be a bipartite graph with partition  $X, Y$ .

$G$  contains a matching of  $X \Leftrightarrow |N(S)| \geq |S|$  for all  $S \subseteq X$

**Theorem** (3.1.10, W; 1.50, H; Berge 1957) A matching  $M$  in a graph  $G$  is a **maximum** matching in  $G \Leftrightarrow G$  has no  $M$ -augmenting path

- **Exercise**. Read the other two proofs in Diestel.
- **Corollary** (3.1.13, W; 2.1.3, D) Every  $k$ -regular ( $k > 0$ ) bipartite graph has a perfect matching

# General regular graph

- **Corollary** (2.1.5, D) Every regular graph of positive even degree has a 2-factor
  - A  $k$ -regular spanning subgraph is called a  **$k$ -factor**
  - A perfect matching is a 1-factor

**Theorem** (1.2.26, W) A graph  $G$  is Eulerian  $\iff$  it has at most one nontrivial component and its vertices all have even degree

**Corollary** (3.1.13, W; 2.1.3, D) Every  $k$ -regular ( $k > 0$ ) bipartite graph has a perfect matching



# Application to SDR

- Given some family of sets  $X$ , a **system of distinct representatives** for the sets in  $X$  is a 'representative' collection of distinct elements from the sets of  $X$

$$S_1 = \{2, 8\},$$

$$S_2 = \{8\},$$

$$S_3 = \{5, 7\},$$

$$S_4 = \{2, 4, 8\},$$

$$S_5 = \{2, 4\}.$$

The family  $X_1 = \{S_1, S_2, S_3, S_4\}$  does have an SDR, namely  $\{2, 8, 7, 4\}$ . The family  $X_2 = \{S_1, S_2, S_4, S_5\}$  does not have an SDR.

- Theorem**(1.52, H) Let  $S_1, S_2, \dots, S_k$  be a collection of finite, nonempty sets. This collection has SDR  $\Leftrightarrow$  for every  $t \in [k]$ , the union of any  $t$  of these sets contains at least  $t$  elements

**Theorem** (3.1.11, W; 1.51, H; 2.1.2, D; Hall 1935) Let  $G$  be a bipartite graph with partition  $X, Y$ .

$G$  contains a matching of  $X \Leftrightarrow |N(S)| \geq |S|$  for all  $S \subseteq X$

# König Theorem

## Augmenting Path Algorithm

# Vertex cover

- A set  $U \subseteq V$  is a **(vertex) cover** of  $E$  if every edge in  $G$  is incident with a vertex in  $U$
- Example:
  - Art museum is a graph with hallways are edges and corners are nodes
  - A security camera at the corner will guard the paintings on the hallways
  - The minimum set to place the cameras?

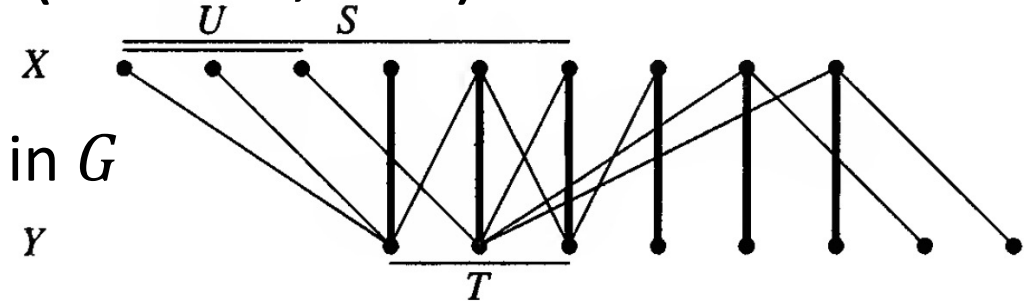
# König-Egeváry Theorem (Min-max theorem)

- **Theorem** (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931)  
Let  $G$  be a bipartite graph. The **maximum** size of a matching in  $G$  is equal to the **minimum** size of a vertex cover of its edges

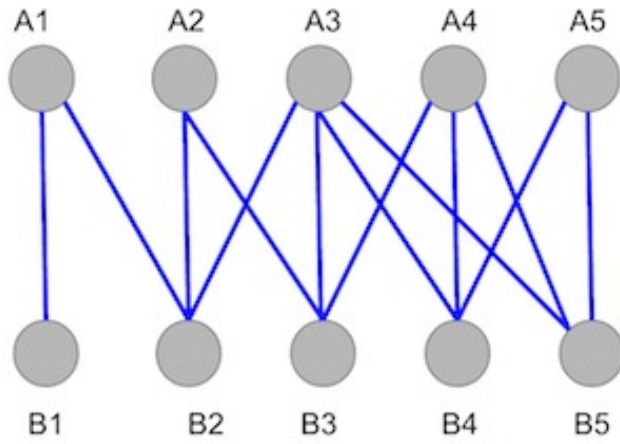
**Theorem** (3.1.10, W; 1.50, H; Berge 1957) A matching  $M$  in a graph  $G$  is a **maximum** matching in  $G \Leftrightarrow G$  has no  $M$ -augmenting path

# Augmenting path algorithm (3.2.1, W)

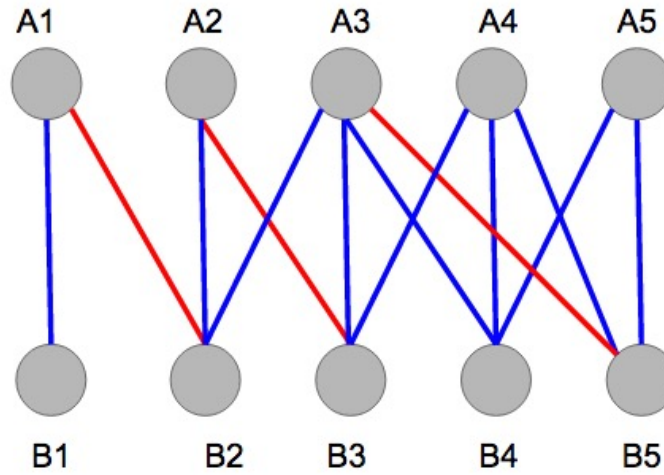
- **Input:**  $G$  is Bipartite with  $X, Y$ , a matching  $M$  in  $G$   
 $U = \{M\text{-unsaturated vertices in } X\}$
- **Idea:** Explore  $M$ -alternating paths from  $U$   
 letting  $S \subseteq X$  and  $T \subseteq Y$  be the sets of vertices reached
- **Initialization:**  $S = U, T = \emptyset$  and all vertices in  $S$  are unmarked
- **Iteration:**
  - If  $S$  has no unmarked vertex, stop and report  $T \cup (X - S)$  as a minimum cover and  $M$  as a maximum matching
  - Otherwise, select an unmarked  $x \in S$  to explore
    - Consider each  $y \in N(x)$  such that  $xy \notin M$ 
      - If  $y$  is unsaturated, terminate and report an  $M$ -augmenting path from  $U$  to  $y$
      - Otherwise,  $yw \in M$  for some  $w$ 
        - include  $y$  in  $T$  (reached from  $x$ ) and include  $w$  in  $S$  (reached from  $y$ )
    - After exploring all such edges incident to  $x$ , mark  $x$  and iterate.



# Example



Red: A random matching



# Theoretical guarantee for Augmenting path algorithm

- **Theorem** (3.2.2, W) Repeatedly applying the Augmenting Path Algorithm to a bipartite graph produces a matching and a vertex cover of equal size

# Weighted Bipartite Matching

## Hungarian Algorithm

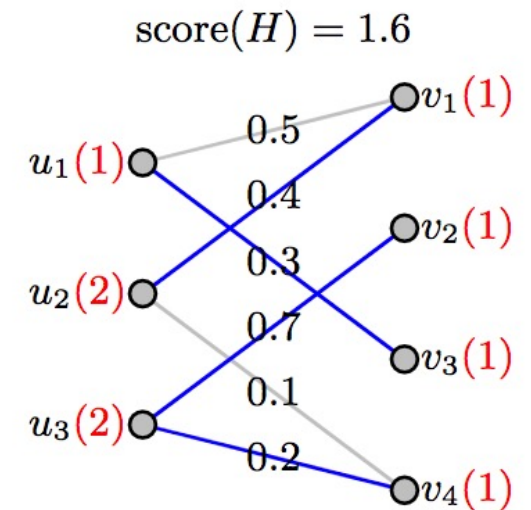


# Weighted bipartite matching

- The **maximum weighted matching problem** is to seek a perfect matching  $M$  to maximize the total weight  $w(M)$
- Bipartite graph
  - W.l.o.g. Assume the graph is  $K_{n,n}$  with  $w_{i,j} \geq 0$  for all  $i, j \in [n]$
  - Optimization:

$$\begin{aligned} \max w(M) &= \sum_{i,j} a_{i,j} w_{i,j} \\ \text{s.t. } a_{i,1} + \dots + a_{i,n} &\leq 1 \text{ for any } i \\ a_{1,j} + \dots + a_{n,j} &\leq 1 \text{ for any } j \\ a_{i,j} &\in \{0,1\} \end{aligned}$$

- Integer programming
- General IP problems are NP-Complete



# (Weighted) cover

- A (weighted) **cover** is a choice of labels  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  such that  $u_i + v_j \geq w_{i,j}$  for all  $i, j$ 
  - The **cost**  $c(u, v)$  of a cover  $(u, v)$  is  $\sum_i u_i + \sum_j v_j$
  - The **minimum weighted cover problem** is that of finding a cover of minimum cost
- Optimization problem

$$\begin{aligned} \min c(u, v) &= \sum_i u_i + \sum_j v_j \\ \text{s.t. } u_i + v_j &\geq w_{i,j} \text{ for any } i, j \\ u_i, v_j &\geq 0 \text{ for any } i, j \end{aligned}$$

# Duality

(IP)

$$\begin{aligned} \max \sum_{i,j} a_{i,j} w_{i,j} \\ \text{s.t. } a_{i,1} + \dots + a_{i,n} \leq 1 \text{ for any } i \\ a_{1,j} + \dots + a_{n,j} \leq 1 \text{ for any } j \\ a_{i,j} \in \{0,1\} \end{aligned}$$

(Linear programming)

$$\begin{aligned} \max \sum_{i,j} a_{i,j} w_{i,j} \\ \text{s.t. } a_{i,1} + \dots + a_{i,n} \leq 1 \text{ for any } i \\ a_{1,j} + \dots + a_{n,j} \leq 1 \text{ for any } j \\ a_{i,j} \geq 0 \end{aligned}$$

(Dual)

$$\begin{aligned} \min \sum_i u_i + \sum_j v_j \\ \text{s.t. } u_i + v_j \geq w_{i,j} \text{ for any } i,j \\ u_i, v_j \geq 0 \end{aligned}$$

- Weak duality theorem

- For each feasible solution  $a$  and  $(u, v)$

$$\sum_{i,j} a_{i,j} w_{i,j} \leq \sum_i u_i + \sum_j v_j$$

$$\text{thus } \max \sum_{i,j} a_{i,j} w_{i,j} \leq \min \sum_i u_i + \sum_j v_j$$

# Duality (cont.)

- Strong duality theorem
  - If one of the two problems has an optimal solution, so does the other one and that the bounds given by the weak duality theorem are tight

$$\max \sum_{i,j} a_{i,j} w_{i,j} = \min \sum_i u_i + \sum_j v_j$$

- **Lemma** (3.2.7, W) For a perfect matching  $M$  and cover  $(u, v)$  in a weighted bipartite graph  $G$ ,  $c(u, v) \geq w(M)$ .  
 $c(u, v) = w(M) \Leftrightarrow M$  consists of edges  $x_i y_j$  such that  $u_i + v_j = w_{i,j}$   
In this case,  $M$  and  $(u, v)$  are optimal.

# Equality subgraph

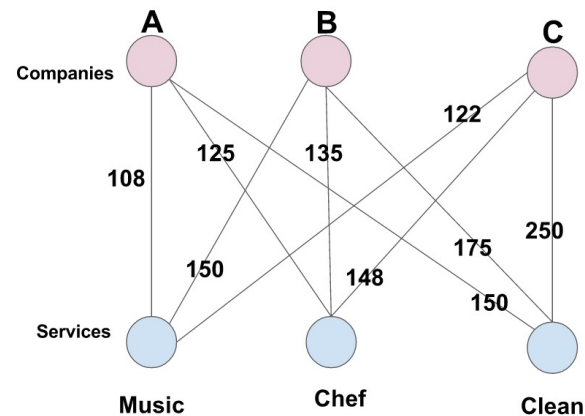
- The **equality subgraph**  $G_{u,v}$  for a cover  $(u, v)$  is the **spanning** subgraph of  $K_{n,n}$  having the edges  $x_i y_j$  such that  $u_i + v_j = w_{i,j}$ 
  - So if  $c(u, v) = w(M)$  for some perfect matching  $M$ , then  $M$  is composed of edges in  $G_{u,v}$
  - And if  $G_{u,v}$  contains a perfect matching  $M$ , then  $(u, v)$  and  $M$  (whose weights are  $u_i + v_j$ ) are both optimal

# Hungarian algorithm

- **Input:** Weighted  $K_{n,n} = B(X, Y)$
- **Idea:** Iteratively adjusting the cover  $(u, v)$  until the equality subgraph  $G_{u,v}$  has a perfect matching
- **Initialization:** Let  $(u, v)$  be a cover, such as  $u_i = \max_j w_{i,j}$ ,  $v_j = 0$

(Dual)

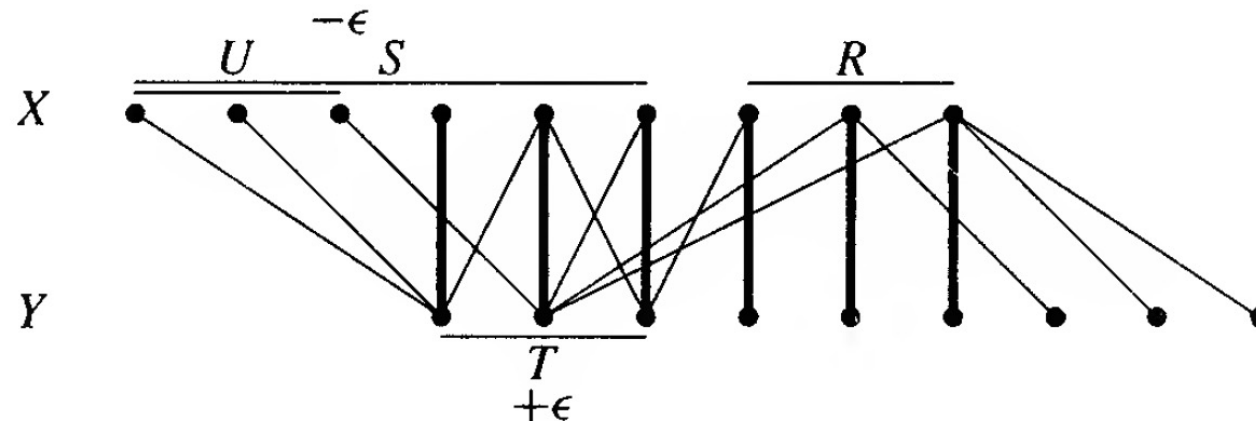
$$\begin{aligned} \min \quad & \sum_i u_i + \sum_j v_j \\ \text{s.t.} \quad & u_i + v_j \geq w_{i,j} \text{ for any } i, j \\ & u_i, v_j \geq 0 \end{aligned}$$



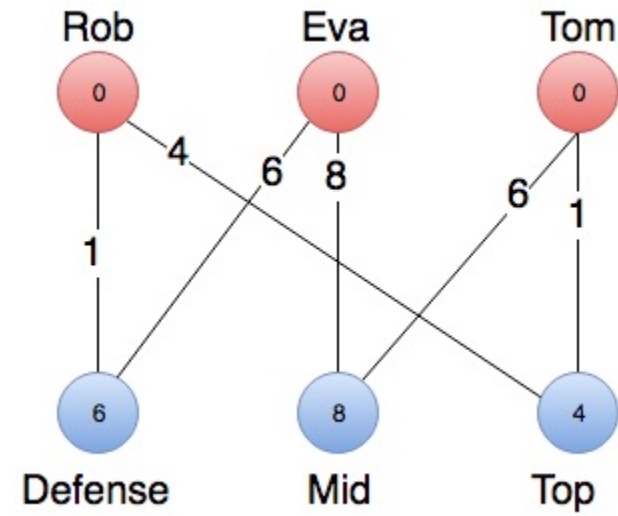
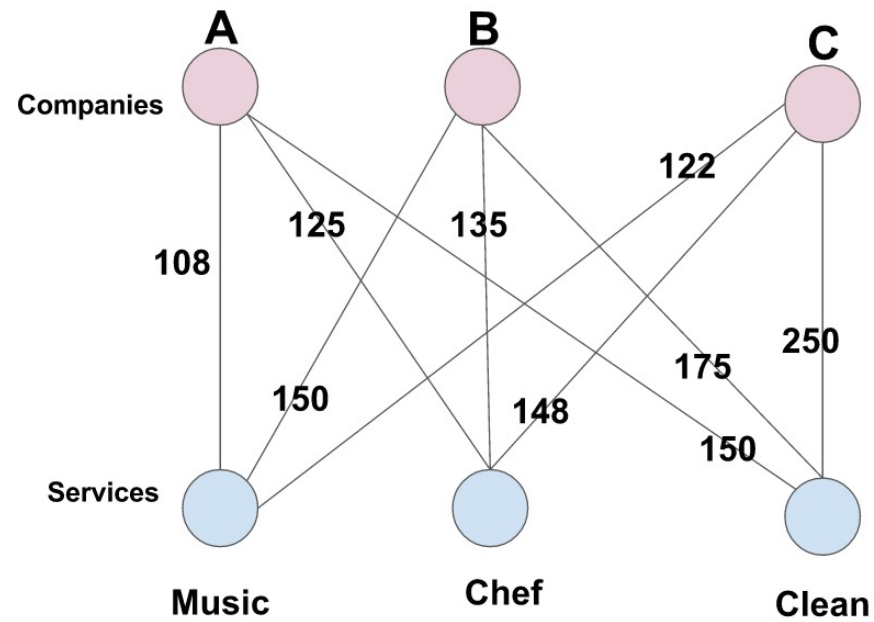
# Hungarian algorithm (cont.)

- **Iteration:** Find a maximum matching  $M$  in  $G_{u,v}$ 
  - If  $M$  is a perfect matching, stop and report  $M$  as a maximum weight matching
  - Otherwise, let  $Q$  be a vertex cover of size  $|M|$  in  $G_{u,v}$ 
    - Let  $R = X \cap Q, T = Y \cap Q$ 

$$\epsilon = \min\{u_i + v_j - w_{i,j} : x_i \in X - R, y_j \in Y - T\}$$
      - Decrease  $u_i$  by  $\epsilon$  for  $x_i \in X - R$  and increase  $v_j$  by  $\epsilon$  for  $y_j \in T$
  - Form the new equality subgraph and repeat



# Example

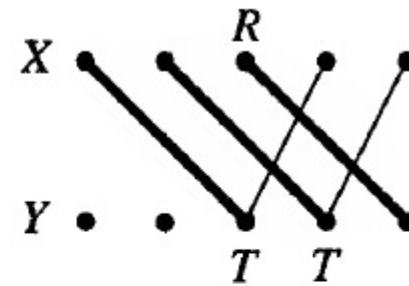




## Example 2: Excess matrix

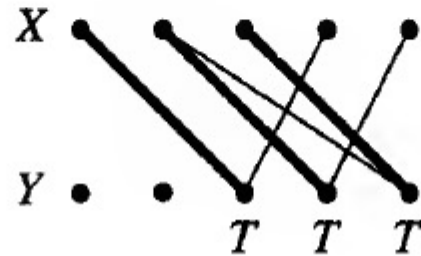
$$\begin{pmatrix} 4 & 1 & 6 & 2 & 3 \\ 5 & 0 & 3 & 7 & 6 \\ 2 & 3 & 4 & 5 & 8 \\ 3 & 4 & 6 & 3 & 4 \\ 4 & 6 & 5 & 8 & 6 \end{pmatrix} \rightarrow \begin{matrix} & 0 & 0 & 0 & 0 & 0 \\ 6 & 2 & 5 & \underline{0} & 4 & 3 \\ 7 & 2 & 7 & 4 & \underline{0} & 1 \\ 8 & 6 & 5 & 4 & 3 & \underline{0} \\ 6 & 3 & 2 & 0 & 3 & 2 \\ 8 & 4 & 2 & 3 & 0 & 2 \end{matrix} \begin{matrix} \\ \\ \\ \\ R \\ \end{matrix}$$

$T \quad T$



$$\begin{matrix} & 0 & 0 & 1 & 1 & 0 \\ 5 & 1 & 4 & \underline{0} & 4 & 2 \\ 6 & 1 & 6 & 4 & \underline{0} & 0 \\ 8 & 6 & 5 & 5 & 4 & \underline{0} \\ 5 & 2 & 1 & 0 & 3 & 1 \\ 7 & 3 & 1 & 3 & 0 & 1 \end{matrix}$$

$T \quad T \quad T$



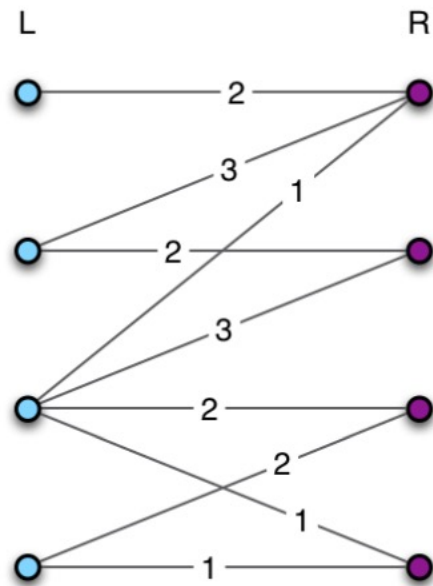
$$\rightarrow \begin{matrix} & 0 & 0 & 2 & 2 & 1 \\ 4 & 0 & 3 & \underline{0} & 4 & 2 \\ 5 & \underline{0} & 5 & 4 & 0 & 0 \\ 7 & 5 & 4 & 5 & 4 & \underline{0} \\ 4 & 1 & \underline{0} & 0 & 3 & 1 \\ 6 & 2 & 0 & 3 & \underline{0} & 1 \end{matrix}$$

Optimal value is the same  
But the solution is not unique

# Theoretical guarantee for Hungarian algorithm

- **Theorem** (3.2.11, W) The Hungarian Algorithm finds a maximum weight matching and a minimum cost cover

# Example 3



# Back to (unweighted) bipartite graph

- The weights are binary 0,1
- Hungarian algorithm always maintain integer labels in the weighted cover, thus the solution will always be 0,1
- The vertices receiving label 1 must cover the weight on the edges, thus cover all edges
- So the solution is a minimum vertex cover

# Stable Matchings

# Stable matching

- A family  $(\leq_v)_{v \in V}$  of linear orderings  $\leq_v$  on  $E(v)$  is a set of **preferences** for  $G$
- A matching  $M$  in  $G$  is **stable** if for any edge  $e \in E \setminus M$ , there exists an edge  $f \in M$  such that  $e$  and  $f$  have a common vertex  $v$  with  $e <_v f$ 
  - **Unstable**: There exists  $xy \in E \setminus M$  but  $xy', x'y \in M$  with  $xy' <_x xy$   
 $x'y <_y xy$

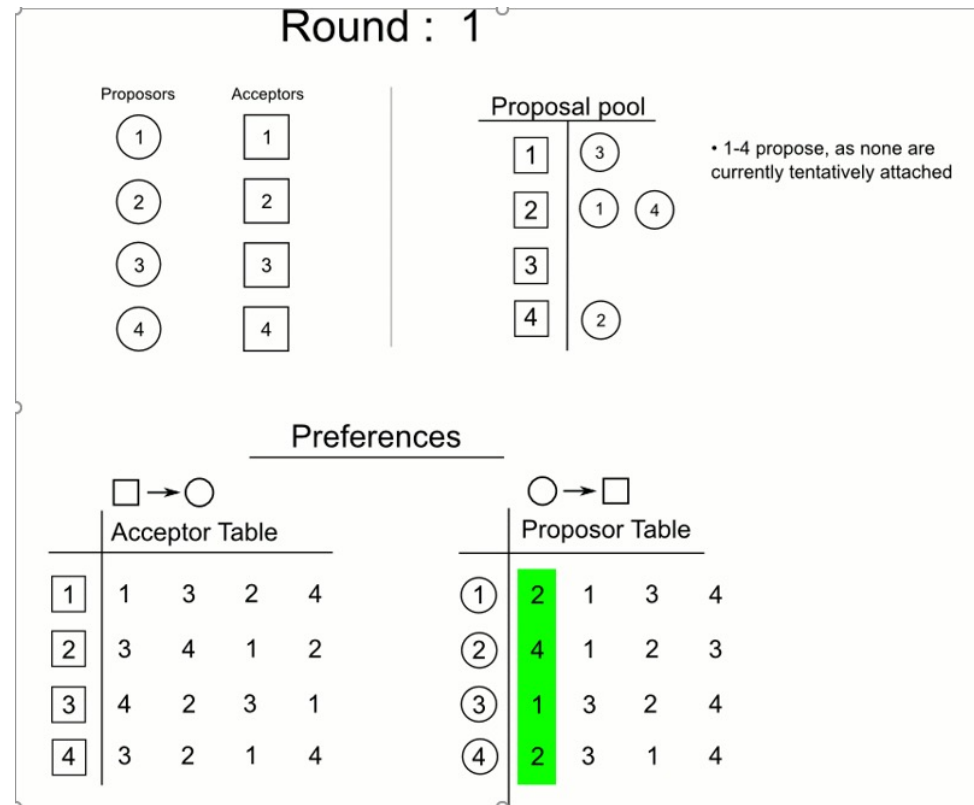
**3.2.16. Example.** Given men  $x, y, z, w$ , women  $a, b, c, d$ , and preferences listed below, the matching  $\{xa, yb, zd, wc\}$  is a stable matching. ■

Men $\{x, y, z, w\}$	Women $\{a, b, c, d\}$
$x : a > b > c > d$	$a : z > x > y > w$
$y : a > c > b > d$	$b : y > w > x > z$
$z : c > d > a > b$	$c : w > x > y > z$
$w : c > b > a > d$	$d : x > y > z > w$

# Gale-Shapley Proposal Algorithm

- **Input:** Preference rankings by each of  $n$  men and  $n$  women
- **Idea:** Produce a stable matching using proposals by maintaining information about who has proposed to whom and who has rejected whom
- **Iteration:** Each man proposes to the highest woman on his preference list who has not previously rejected him
  - If each woman receives exactly one proposal, stop and use the resulting matching
  - Otherwise, every woman receiving more than one proposal rejects all of them except the one that is highest on her preference list
  - Every woman receiving a proposal says “maybe” to the most attractive proposal received

# Example





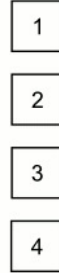
# Example (gif)

Round : 1

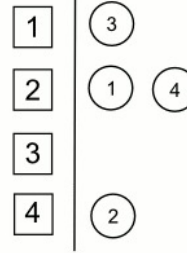
Proposors



Acceptors



Proposal pool



• 1-4 propose, as none are currently tentatively attached

Preferences

□ → ○

Acceptor Table

1	1	3	2	4
2	3	4	1	2
3	4	2	3	1
4	3	2	1	4

○ → □

Proposor Table

1	2	1	3	4
2	4	1	2	3
3	1	3	2	4
4	2	3	1	4

# Theoretical guarantee for the Proposal Algorithm

- **Theorem** (3.2.18, W, Gale-Shapley 1962) The Proposal Algorithm produces a stable matching
- Who proposes matters (jobs/candidates)
- When the algorithm runs with women proposing, every woman is at least as happy as when men do the proposing
  - And every man is at least as unhappy

**3.2.16. Example.** Given men  $x, y, z, w$ , women  $a, b, c, d$ , and preferences listed below, the matching  $\{xa, yb, zd, wc\}$  is a stable matching. ■

Men $\{x, y, z, w\}$	Women $\{a, b, c, d\}$
$x : a > b > c > d$	$a : z > x > y > w$
$y : a > c > b > d$	$b : y > w > x > z$
$z : c > d > a > b$	$c : w > x > y > z$
$w : c > b > a > d$	$d : x > y > z > w$

# Matchings in general graphs

# Perfect matchings

- $K_{2n}, C_{2n}, P_{2n}$  have perfect matchings
- **Corollary** (3.1.13, W; 2.1.3, D) Every  $k$ -regular ( $k > 0$ ) bipartite graph has a perfect matching
- **Theorem** (1.58, H) If  $G$  is a graph of order  $2n$  such that  $\delta(G) \geq n$ , then  $G$  has a perfect matching

**Theorem** (1.22, H, Dirac) Let  $G$  be a graph of order  $n \geq 3$ . If  $\delta(G) \geq n/2$ , then  $G$  is Hamiltonian

# Tutte's Theorem (TONCAS)

- Let  $q(G)$  be the number of connected components with odd order
- **Theorem** (1.59, H; 2.2.1, D; 3.3.3, W)  
Let  $G$  be a graph of order  $n \geq 2$ .  $G$  has a perfect matching  $\Leftrightarrow q(G - S) \leq |S|$  for all  $S \subseteq V$

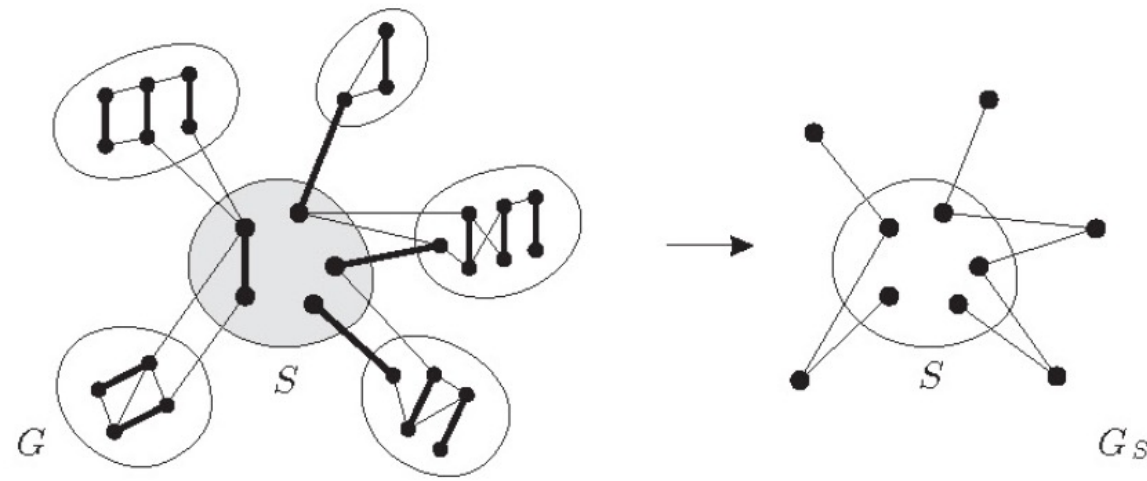


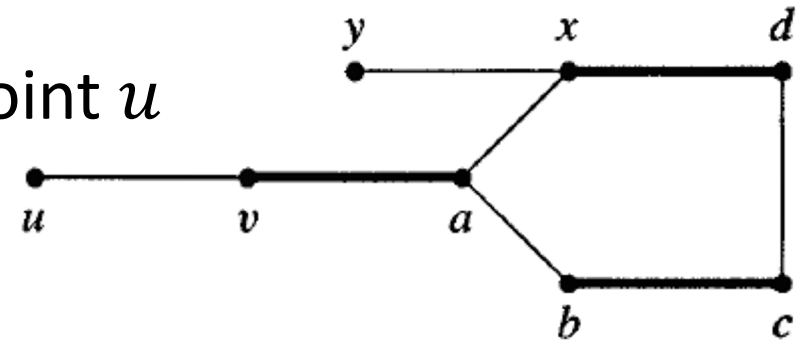
Fig. 2.2.1. Tutte's condition  $q(G - S) \leq |S|$  for  $q = 3$ , and the contracted graph  $G_S$  from Theorem 2.2.3.

# Petersen's Theorem

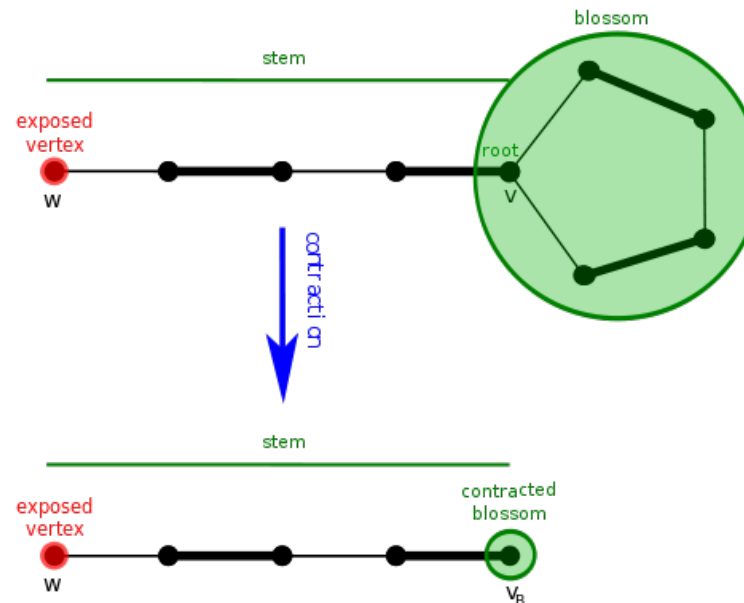
- **Theorem** (1.60, H; 2.2.2, D; 3.3.8, W)  
Every bridgeless, 3-regular graph contains a perfect matching

# Find augmenting paths in general graphs

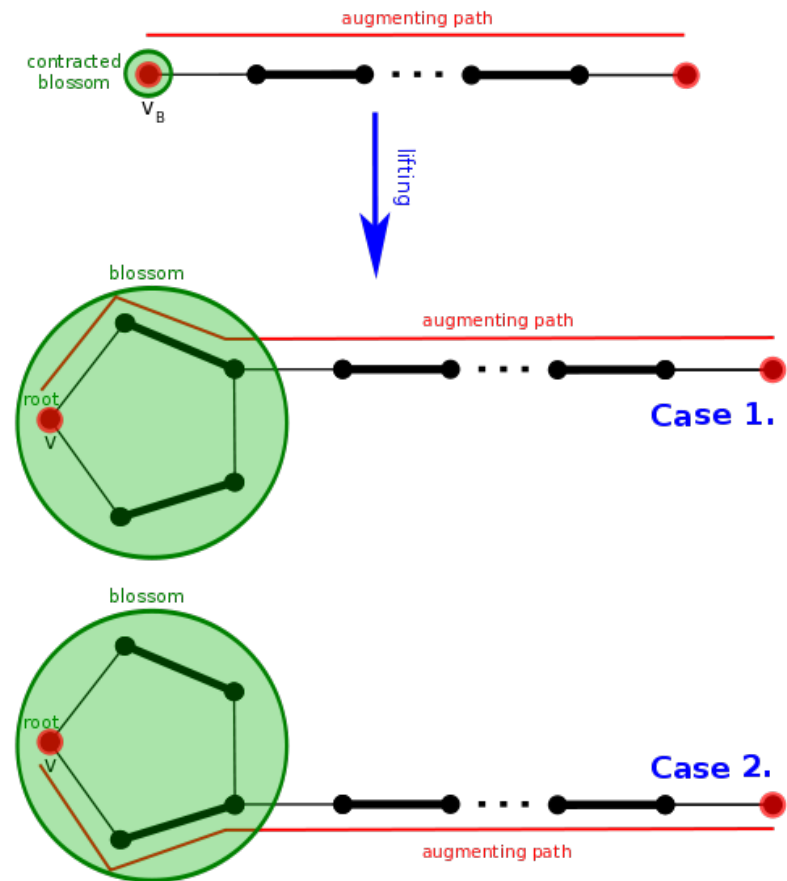
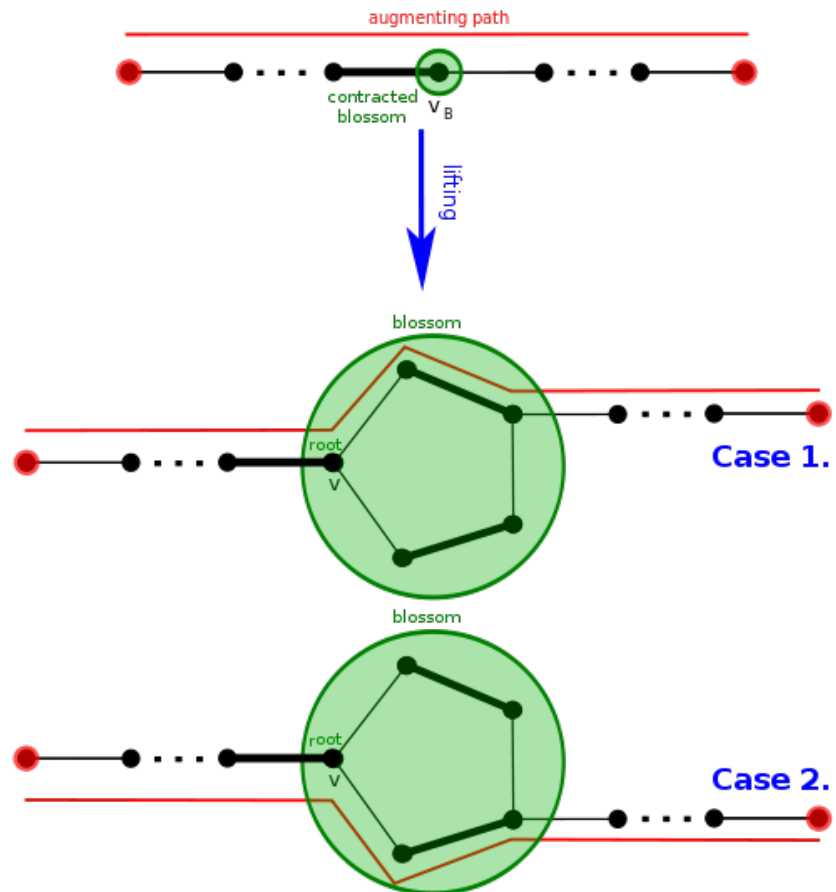
- Different from bipartite graphs
- Example: How to explore from  $M$ -unsaturated point  $u$



- Flower/stem/blossom



# Lifting

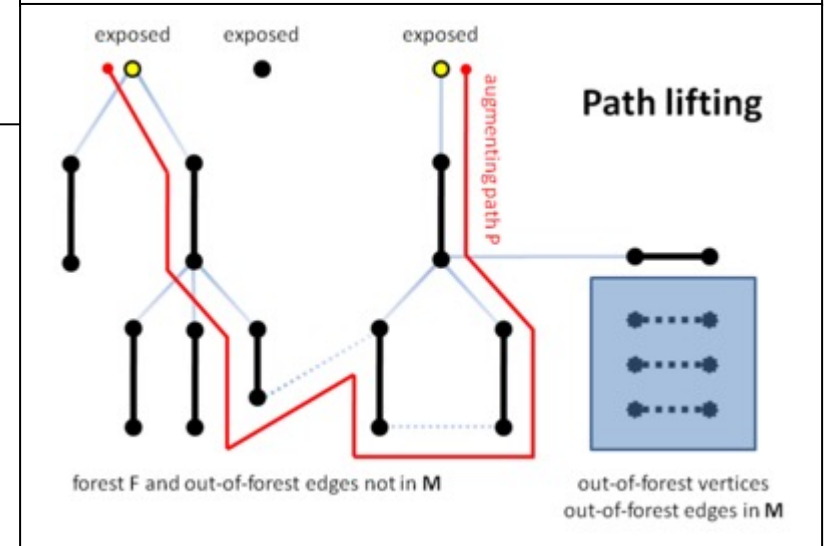
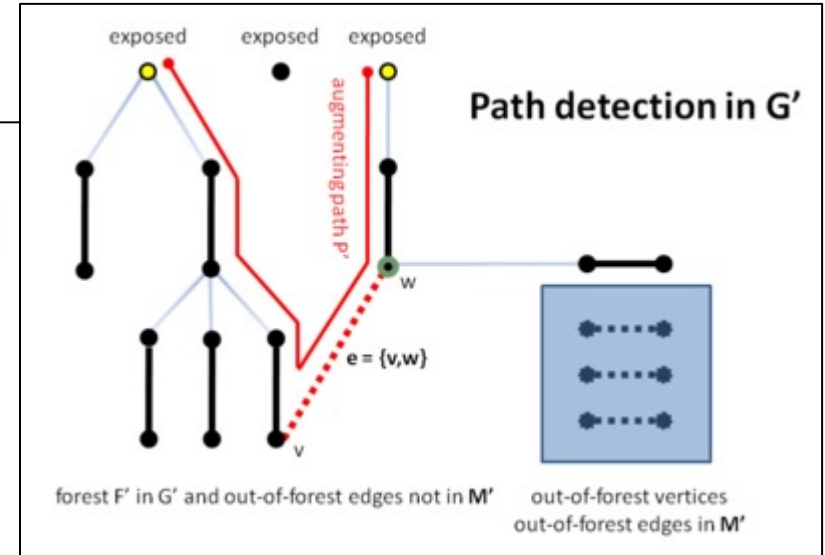
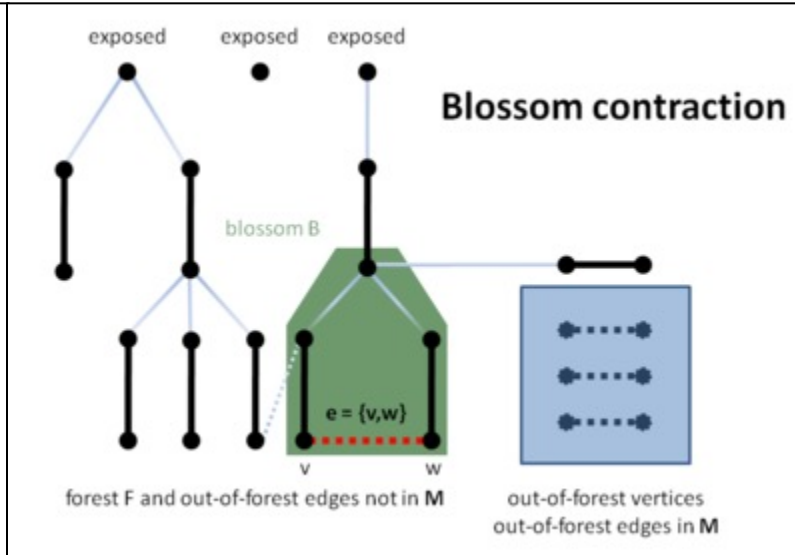
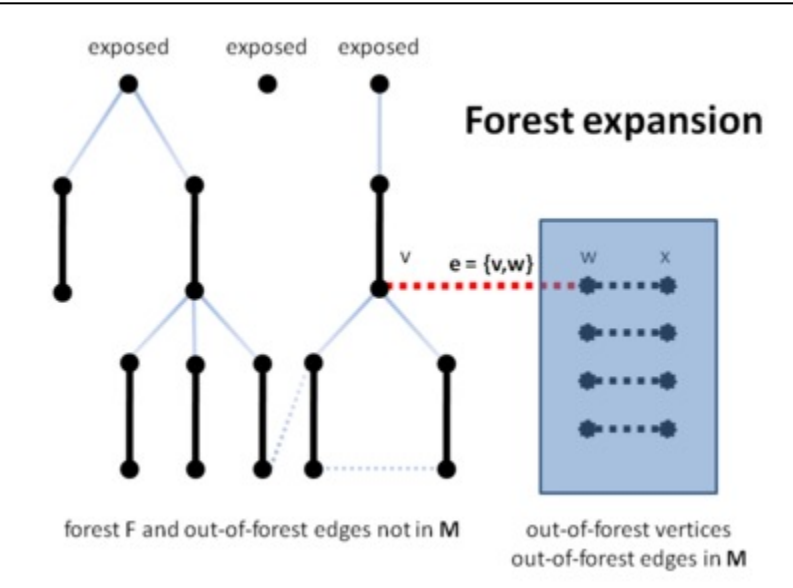




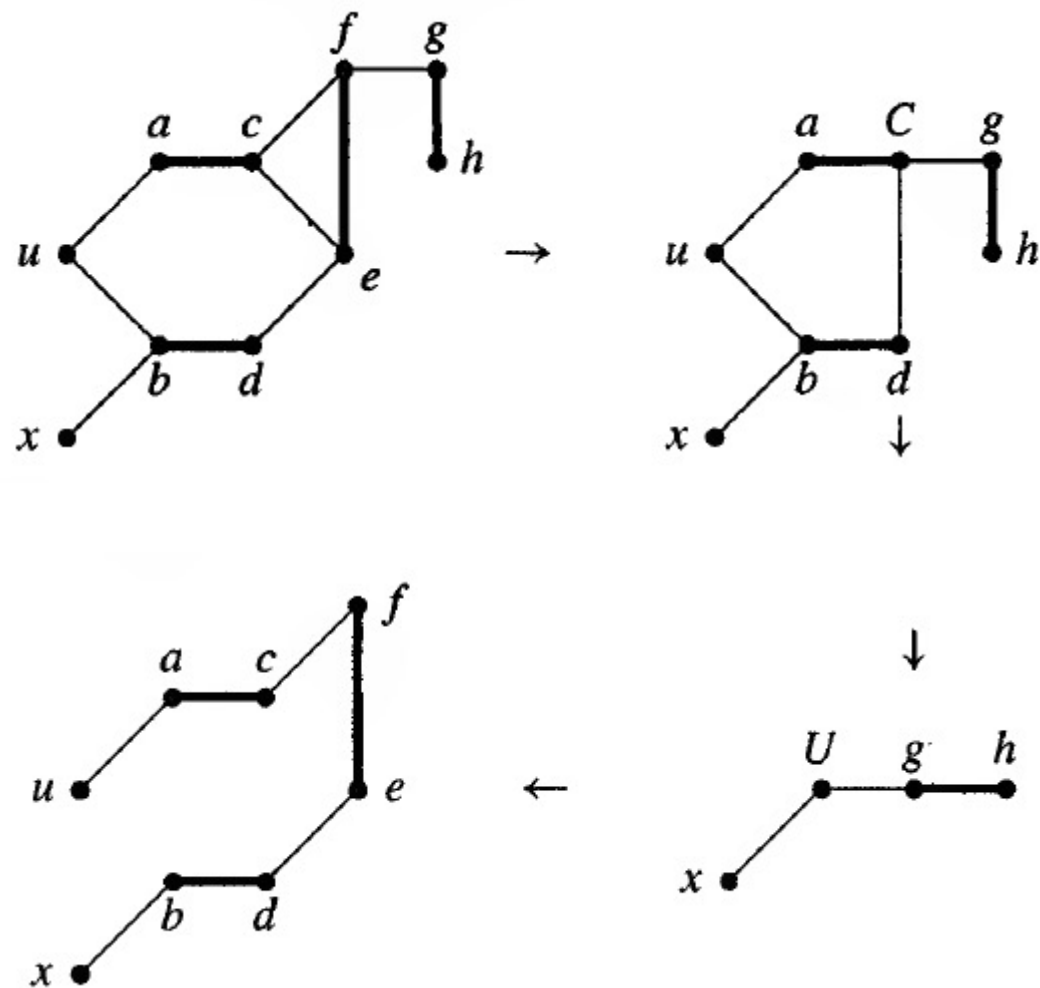
# Edmonds' blossom algorithm (3.3.17, W)

- **Input:** A graph  $G$ , a matching  $M$  in  $G$ , an  $M$ -unsaturated vertex  $u$
- **Idea:** Explore  $M$ -alternating paths from  $u$ , recording for each vertex the vertex from which it was reached, and **contracting blossoms** when found
  - Maintain sets  $S$  and  $T$  analogous to those in Augmenting Path Algorithm, with  $S$  consisting of  $u$  and the vertices reached along saturated edges
  - Reaching an unsaturated vertex yields an augmentation.
- **Initialization:**  $S = \{u\}$  and  $T = \emptyset$
- **Iteration:** If  $S$  has no unmarked vertex, stop; there is no  $M$ -augmenting path from  $u$ 
  - Otherwise, select an unmarked  $v \in S$ . To explore from  $v$ , successively consider each  $y \in N(v)$  s.t.  $y \notin T$ 
    - If  $y$  is unsaturated by  $M$ , then trace back from  $y$  (expanding blossoms as needed) to report an  $M$ -augmenting  $u, y$ -path
    - **If  $y \in S$ , then a blossom has been found. Suspend the exploration of  $v$  and contract the blossom**, replacing its vertices in  $S$  and  $T$  by a single new vertex in  $S$ . Continue the search from this vertex in the smaller graph.
    - Otherwise,  $y$  is matched to some  $w$  by  $M$ . Include  $y$  in  $T$  (reached from  $v$ ), and include  $w$  in  $S$  (reached from  $y$ )
  - After exploring all such neighbors of  $v$ , mark  $v$  and iterate

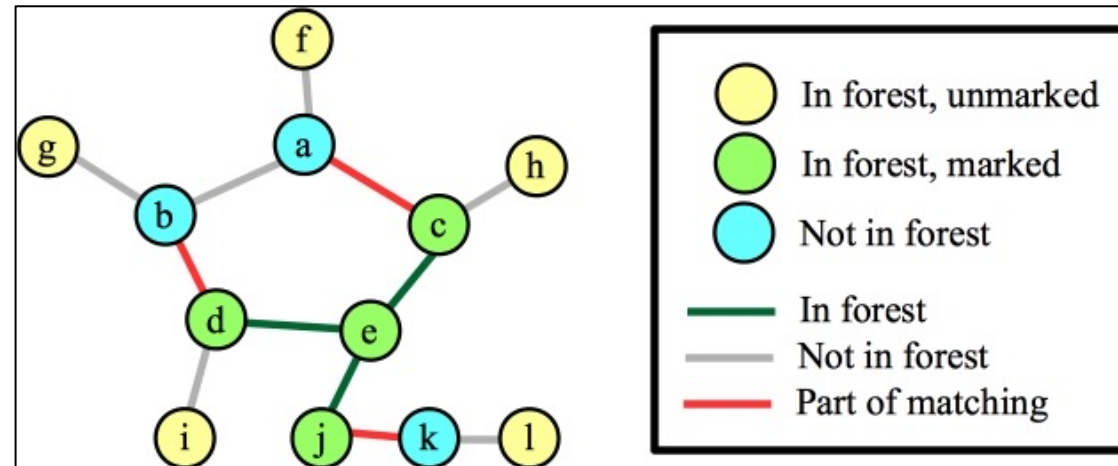
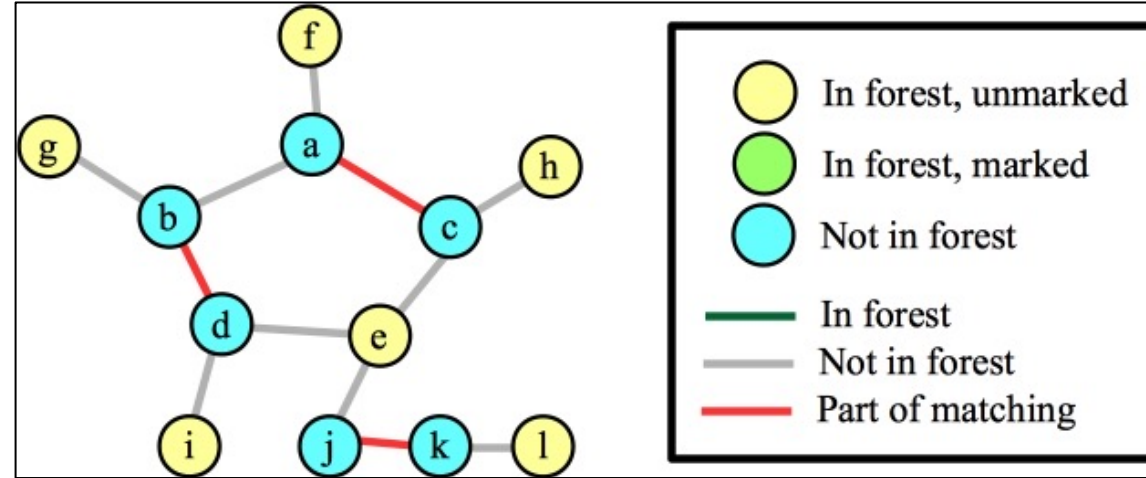
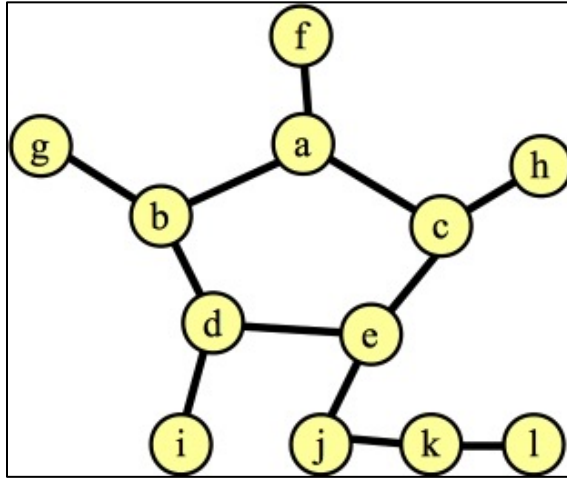
# Illustration



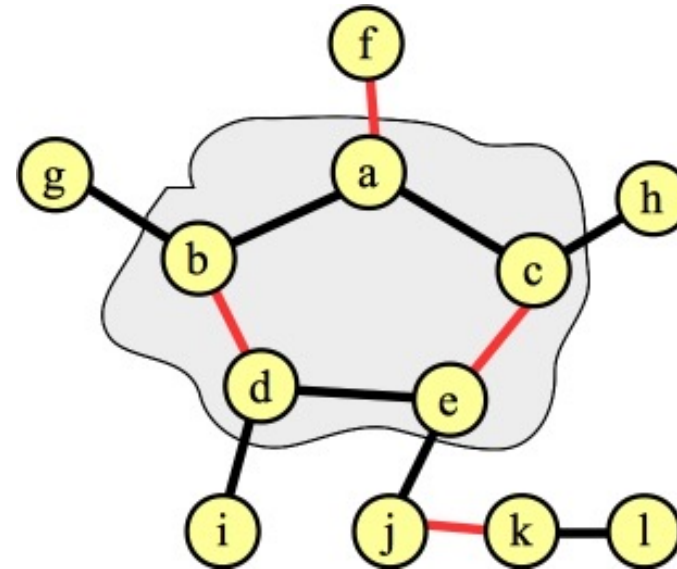
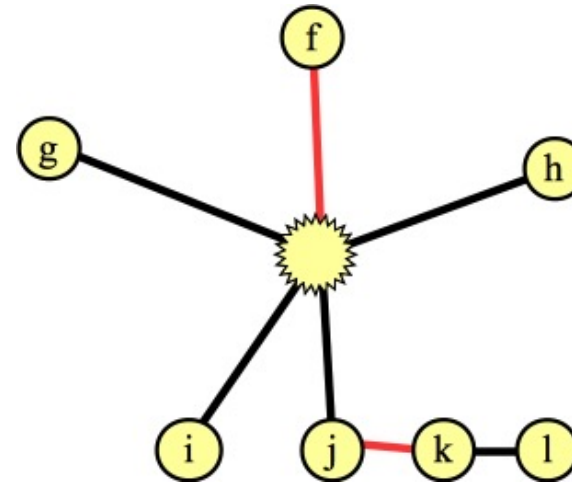
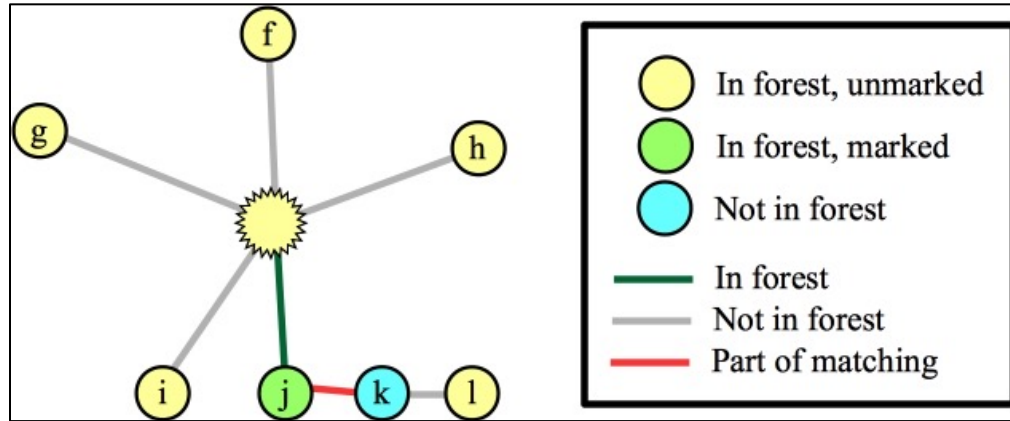
# Example



# Example 2



## Example 2 (cont.)



# Summary

Shuai Li

<https://shuaili8.github.io>

- Matching
  - M-alternating path, M-augmenting path
  - Hall's Thm
  - Berge Thm: A matching  $M$  in a graph  $G$  is a maximum iff  $G$  has no  $M$ -augmenting path
  - König Thm: For bipartite graph, the maximum size of a matching is equal to the minimum size of a vertex cover of its edges
  - Augmenting Path Algorithm

## Questions?