

# First order mean-motion resonance

Revisiting the second fundamental model of resonance

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## Abstract

In this document, I go over the analytical study of a first order mean-motion resonance between a pair of planets and I revisit the second fundamental model of resonance by Jacques Henrard and Anne Lemaître. I also give a compact analytical expression of the first integral of motion, independent from the total angular momentum and the scaling factor, that only exists at first order in eccentricity in the averaged problem.

## 1 Introduction

I consider two planets of masses  $m_1$  and  $m_2$  orbiting a star of mass  $m_0$ . The problem is assumed planar. I focus on the case where the pair of planets is close to a first order mean motion resonance  $p : p + 1$ . That is, the orbital periods  $T_j = 2\pi/n_j$  are near the equality  $pT_2 \approx (p + 1)T_1$  for some integer  $p$ . In this work, subscripts 1 and 2 refer to the inner and outer planet, respectively.

## 2 Hamiltonian of the problem

Denoting  $\mathbf{r}_j$  and  $\mathbf{v}_j$  the heliocentric position and barycentric speed of planet  $j$ , respectively, the Hamiltonian of the problem can be written

$$H = \sum_{j=1}^2 \left( \frac{\tilde{r}_j^2}{2\beta_j} - \frac{\mathcal{G}m_0m_j}{r_j} \right) + \frac{\tilde{\mathbf{r}}_1 \cdot \tilde{\mathbf{r}}_2}{m_0} - \frac{\mathcal{G}m_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \quad (1)$$

where  $\tilde{\mathbf{r}}_j = \beta_j \mathbf{v}_j$  is conjugated to  $\mathbf{r}_j$  and  $\beta_j = m_0m_j / (m_0 + m_j)$ . I make use of the Poincaré canonical coordinates  $(\Lambda_j, D_j; \lambda_j, -\varpi_j)$ , where  $\lambda_j$  is the mean longitude,  $\varpi_j$  is the longitude of the periapsis,  $\Lambda_j = \beta_j \sqrt{\mu_j a_j}$  is conjugated to  $\lambda_j$  and  $D_j = \Lambda_j (1 - \sqrt{1 - e_j^2})$  is conjugated to  $-\varpi_j$ .  $a_j$  and  $e_j$  are the semi-major axis and eccentricity, while  $\mu_j = \mathcal{G}(m_0 + m_j)$ . In these coordinates, the Keplerian part of the Hamiltonian, due to star–planet interactions, reads

$$H_K(\Lambda_j) = \sum_{j=1}^2 \left( \frac{\tilde{r}_j^2}{2\beta_j} - \frac{\mathcal{G}m_0m_j}{r_j} \right) = - \sum_{j=1}^2 \frac{\beta_j^3 \mu_j^2}{2\Lambda_j^2}, \quad (2)$$

whereas the perturbative part, due to planet–planet interactions, can be written (Laskar and Robutel, 1995)

$$H_P(\Lambda_j, D_j; \lambda_j, -\varpi_j) = \sum_{\mathbf{k} \in \mathbb{Z}^2} \left( \sum_{\mathbf{q} \in \mathbb{N}^4} \Xi_{\mathbf{k}, \mathbf{q}}(\Lambda_j) X_1^{q_1} X_2^{q_2} \bar{X}_1^{q_3} \bar{X}_2^{q_4} \right) e^{i(k_1 \lambda_1 + k_2 \lambda_2)}, \quad (3)$$

where  $X_j = \sqrt{2D_j/\Lambda_j} \exp(i\varpi_j)$  and the upper bar denotes the complex conjugated. Because  $|X_j| = e_j + \mathcal{O}(e_j^3)$  and since the eccentricities are expected to remain small, this serie expansion is generally truncated to some degree in eccentricity.

The coefficient  $\Xi_{\mathbf{k}, \mathbf{q}}$  depends on  $\Lambda_1$  and  $\Lambda_2$  only through the ratio  $a_1/a_2 = \alpha_{12}$ . In fact, for most choices of the tuples  $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$  and  $\mathbf{q} = (q_1, q_2, q_3, q_4) \in \mathbb{N}^4$ , this coefficient vanishes. Indeed,  $H_P$  is invariant by any rotation of angle  $\psi$  along an axis perpendicular to the orbital plane. That is,  $H_P(\lambda_j, \varpi_j) = H_P(\lambda_j + \psi, \varpi_j + \psi)$  for all  $\psi$ . Injecting into Eq. (3), this yields the d'Alembert relation

$$\Xi_{\mathbf{k}, \mathbf{q}} \neq 0 \Rightarrow k_1 + k_2 + p_1 + p_2 - p_3 - p_4 = 0. \quad (4)$$

For most choices of  $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$ , the angle  $k_1 \lambda_1 + k_2 \lambda_2$  is fast circulating, and the corresponding term averages out to zero over timescales much longer than the orbital periods. Only those such that  $npk_1 = n(p+1)k_2$  for some integer  $n$  are slow and do not average out. Therefore, in order to only retain the secular (*i.e.* long-term) dynamics, I only keep such values of  $\mathbf{k}$  in  $H_P$ . After getting rid of all fast-circulating terms, d'Alembert rule shows that only two terms of degree one in eccentricity remain in Eq. (3). That is, there exist  $f_1(\alpha_{12})$  and  $f_2(\alpha_{12})$  such that

$$H_P = \frac{\mathcal{G}m_1m_2}{a_2} \left( f_1(\alpha_{12}) \sqrt{\frac{2D_1}{\Lambda_1}} \cos[p\lambda_1 - (p+1)\lambda_2 + \varpi_1] + \right. \\ \left. f_2(\alpha_{12}) \sqrt{\frac{2D_2}{\Lambda_2}} \cos[p\lambda_1 - (p+1)\lambda_2 + \varpi_2] \right) + \mathcal{O}(e_j^2). \quad (5)$$

## 2.1 Transformation to relevant angle

The Hamiltonian  $H = H_K + H_P$  currently has four degrees of freedom that are  $(\Lambda_j; \lambda_j)$  and  $(D_j; -\varpi_j)$  for  $j \in \{1, 2\}$ . A linear change of variable adapted to the mean motion resonance  $p : p + 1$  allows two degrees of freedom to be lost. I define (*e.g.* Delisle, 2017)

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -p & p+1 & 0 & 0 \\ -p & p+1 & 1 & 0 \\ -p & p+1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ -\varpi_1 \\ -\varpi_2 \end{pmatrix}. \quad (6)$$

The change of variable is made canonical by transforming the actions according to

$$\begin{pmatrix} \Gamma \\ G \\ D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} p+1 & p & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ D_1 \\ D_2 \end{pmatrix}. \quad (7)$$

The inverse transformation reads

$$\begin{aligned} \lambda_1 &= (p+1) \varphi_1 + \varphi_2, \\ \lambda_2 &= p \varphi_1 + \varphi_2, \\ \varpi_j &= \varphi_2 - \sigma_j, \\ \Lambda_1 &= \Gamma - p \epsilon', \\ \Lambda_2 &= -\Gamma + (p+1) \epsilon', \end{aligned} \quad (8)$$

where  $\epsilon' = G + D_1 + D_2$ . The action variable  $G$  is the total angular momentum of the system and is a conserved quantity. According to the Hamilton equation  $dG/dt = -\partial H/\partial \varphi_2$ , the angle  $\varphi_2$  should be absent from the Hamiltonian to guarantee the conservation of  $G$ . Because the angle  $\varphi_1$  is fast-circulating in the  $p : p + 1$  resonance and since all fast-circulating angles have been removed from the Hamiltonian, the angle  $\varphi_1$  should also be absent from the Hamiltonian and its conjugated action  $\Gamma$ , often called *scaling factor* (Michtchenko *et al.*, 2008; Delisle, 2017; Petit *et al.*, 2020) is also a conserved quantity of the model.

Unlike  $G$  which is conserved even in the true system,  $\Gamma$  is only conserved in the averaged model. Another conserved quantity exists and allows for one more degree of freedom to be lost (Sect. 3). For now, I am left with the two degrees of freedom  $(D_1, D_2; \sigma_1, \sigma_2)$ . Since  $\Gamma$  and  $G$  are two parameters of the model, it is customary to consider the single parameter  $g = G/\Gamma$  instead. Furthermore, in order to work with dimensionless action variables while still maintaining the form of Hamilton equations, I perform the rescaling

$$g = \frac{G}{\Gamma}, \quad 1 = \frac{\Gamma}{\Gamma}, \quad d_j = \frac{D_j}{\Gamma}, \quad \mathcal{H} = \frac{H}{\Gamma}, \quad \epsilon = \frac{\epsilon'}{\Gamma} = g + d_1 + d_2. \quad (9)$$

After this rescaling, the Hamiltonian depends on exactly four parameters that are  $g$ ,  $m_1$ ,  $m_2$  and  $p$ . I explain in Sect. 4 how to reduce the dependency to one parameter only.

$p$	1	2	3	4	5
$f_1$	1.1904936978	2.0252226899	2.8404318567	3.6496182441	4.4561427851
$-f_2$	0.4283898341	2.4840051833	3.2832567218	4.0837053718	4.8847062975

Table 1 — Coefficients  $f_j$  appearing in Eq. (5) evaluated at  $\alpha_{12} = (p/(p+1))^{2/3}$ .

## 2.2 Expression of the perturbative part

The system is expected to remain close to the commensurability  $pn_1 \approx (p+1)n_2$ , where  $n_j = 2\pi/T_j$ . Therefore, I define nominal mean motions  $n_{j,0}$  such that  $pn_{1,0} = (p+1)n_{2,0}$  and associated nominal semi-major axes  $a_{j,0}$  and  $\Lambda_{j,0}$  with

$$\mu_j = n_{j,0}^2 a_{j,0}^3 \text{ and } \Lambda_{j,0} = \beta_j \sqrt{\mu_j a_{j,0}}. \quad (10)$$

I expand the perturbative part of the Hamiltonian to order zero in  $\Delta\Lambda_j = \Lambda_j - \Lambda_{j,0}$  and the Keplerian part to order two because both expansions generate a remainder of the same size (due to the smallness of the perturbation with respect to the Keplerian part). For the perturbation, a expansion to order zero means that  $\Lambda_j$  is simply evaluated at  $\Lambda_{j,0}$ . Similarly,  $\alpha_{12} = a_1/a_2$  is evaluated at  $a_{1,0}/a_{2,0} = (p/(p+1))^{2/3} (1 + \mathcal{O}(m_j/m_0))$ . Because the perturbative part is already of size  $\mathcal{O}(m_j/m_0)$  relative to the Keplerian part, quantities of the form  $1 + \mathcal{O}(m_j/m_0)$  are assumed to be unity in the perturbation. The coefficients  $f_1(\alpha_{12})$  and  $f_2(\alpha_{12})$  in Eq. (5) take the form (*e.g.* Petit, 2021)

$$\begin{aligned} f_1(\alpha_{12}) &= \left( p + 1 + \frac{\alpha_{12}}{2} \frac{\partial}{\partial \alpha_{12}} \right) b_{1/2}^{(p+1)}(\alpha_{12}), \\ f_2(\alpha_{12}) &= - \left( p + \frac{1}{2} + \frac{\alpha_{12}}{2} \frac{\partial}{\partial \alpha_{12}} \right) b_{1/2}^{(p)}(\alpha_{12}) + \alpha_{12}^{-1/2} \delta_{p,1}, \end{aligned} \quad (11)$$

where the Laplace coefficient  $b_s^{(l)}(\alpha)$  is given by (Brouwer and Clemence, 1961)

$$\frac{1}{2} b_s^{(l)}(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(l\lambda)}{(1 + \alpha^2 - 2\alpha \cos \lambda)^s} d\lambda. \quad (12)$$

The term  $\alpha_{12}^{-1/2} \delta_{p,1}$ , that vanishes for all first-order MMR except the 1 : 2, comes from the term  $\tilde{\mathbf{r}}_1 \cdot \tilde{\mathbf{r}}_2$  in Eq. (1). Once evaluated at  $\alpha_{12} = (p/(p+1))^{2/3}$ , the coefficients  $f_j(\alpha_{12}) = f_j$  depend only on  $p$ . I give them in Table 1 for  $1 \leq p \leq 5$ . Defining the constants  $C_1$  and  $C_2$  as  $C_j = \Gamma/\Lambda_{j,0}$ , I get

$$C_1 = \frac{p+1}{p} + \frac{m_2}{m_1} \left( \frac{p+1}{p} \right)^{1/3} + \mathcal{O}\left(\frac{m_j}{m_0}\right), \quad C_2 = 1 + \frac{m_1}{m_2} \left( \frac{p+1}{p} \right)^{2/3} + \mathcal{O}\left(\frac{m_j}{m_0}\right), \quad (13)$$

and truncating to first order in eccentricity, the perturbative part of the Hamiltonian now reads<sup>1</sup>

$$\mathcal{H}_P = \frac{H_P}{\Gamma} = \frac{m_1 n_{2,0}}{m_0 C_2} \left( f_1 \sqrt{2C_1 d_1} \cos \sigma_1 + f_2 \sqrt{2C_2 d_2} \cos \sigma_2 \right). \quad (14)$$

<sup>1</sup>A term of order 0 in eccentricity proportional to  $b_{1/2}^{(0)}(\alpha_{12})$  exists in Eq. (5), but it only depends on  $\Lambda_j$  and becomes constant after the evaluation  $\Lambda_j = \Lambda_{j,0}$ . It does not affect the dynamics and I disregard it.

## 2.3 Expression of the Keplerian part

The Keplerian part is expanded to second order in  $\Delta\Lambda_j = \Lambda_j - \Lambda_{j,0}$ . I obtain

$$H_K = \sum_{j \in \{1,2\}} \left( n_{j,0} \Delta\Lambda_j - \frac{3}{2} n_{j,0} \frac{\Delta\Lambda_j^2}{\Lambda_{j,0}} \right). \quad (15)$$

Substituting  $\Lambda_j - \Lambda_{j,0}$  for  $\Delta\Lambda_j$  and getting rid of constant terms, I get

$$H_K = \sum_{j \in \{1,2\}} \left( 4n_{j,0} \Lambda_j - \frac{3}{2} \frac{n_{j,0}}{\Lambda_{j,0}} \Lambda_j^2 \right). \quad (16)$$

Converting to the new variables and rescaling by  $\Gamma$ , this yields

$$\mathcal{H}_K = \frac{H_K}{\Gamma} = -\frac{3}{2} n_{1,0} p \epsilon [(C_1 + C_2)(p\epsilon - 2) + C_2 \epsilon], \quad (17)$$

where I recall that  $\epsilon = g + d_1 + d_2$ .

## 3 Reduction to one degree of freedom

The rescaled Hamiltonian  $\mathcal{H} = \mathcal{H}_K + \mathcal{H}_P$ , while compact, still has two degrees of freedom. However, there exists yet another conserved quantity, besides  $G$  and  $\Gamma$ , that allows one more degree of freedom to be lost. I first define the cartesian canonical coordinates

$$\begin{aligned} u_j &= \sqrt{2d_j} \cos \sigma_j, \\ v_j &= \sqrt{2d_j} \sin \sigma_j. \end{aligned} \quad (18)$$

In these coordinates, the Hamilton equations are  $\dot{u}_j = -\partial\mathcal{H}/\partial v_j$  and  $\dot{v}_j = \partial\mathcal{H}/\partial u_j$ . The expression of  $\mathcal{H}_K$  is unchanged but  $\epsilon = g + \frac{1}{2}(u_1^2 + v_1^2 + u_2^2 + v_2^2)$ . The perturbative part now reads

$$\mathcal{H}_P = \frac{m_1 n_{2,0}}{m_0 C_2} (f_1 C_1^{1/2} u_1 + f_2 C_2^{1/2} u_2) = \alpha_1 u_1 + \alpha_2 u_2. \quad (19)$$

Let  $U$  and  $V$  be two complex numbers and  $\phi$  an angle defined as

$$U = u_1 + iu_2, \quad V = v_1 + iv_2, \quad \phi = \arctan \frac{\alpha_2}{\alpha_1}. \quad (20)$$

Following Henrard *et al.*, 1986, I now perform the canonical rotation

$$U = e^{i\phi} \tilde{U}, \quad V = e^{i\phi} \tilde{V}, \quad (21)$$

where  $\tilde{U} = x_1 + ix_2$  and  $\tilde{V} = y_1 + iy_2$ . The Keplerian part is still unchanged but  $\epsilon = g + \frac{1}{2}(x_1^2 + y_1^2 + x_2^2 + y_2^2)$ . As for the perturbative part, using the identities  $\cos(\arctan z) = 1/\sqrt{1+z^2}$  and  $\sin(\arctan z) = z/\sqrt{1+z^2}$ , it now reads

$$\mathcal{H}_P = \frac{m_1 n_{2,0}}{m_0 C_2} \sqrt{f_1^2 C_1 + f_2^2 C_2} x_1. \quad (22)$$

I go back to polar canonical coordinates by writing

$$\begin{aligned} x_1 &= \sqrt{2R} \cos r, & y_1 &= \sqrt{2R} \sin r, \\ x_2 &= \sqrt{2S} \cos s, & y_2 &= \sqrt{2S} \sin s. \end{aligned} \quad (23)$$

Once again, the Keplerian part is unchanged in these coordinates but  $\epsilon = g + R + S$ . The perturbative part reads

$$\mathcal{H}_P = \frac{m_1 n_{2,0}}{m_0 C_2} \sqrt{f_1^2 C_1 + f_2^2 C_2} \sqrt{2R} \cos r. \quad (24)$$

The Hamiltonian does not depend on the angle  $s$ , which means that its conjugated action  $S$  is a first integral. I explicitly give the value of this new conserved quantity as a function of  $e_1$ ,  $e_2$ ,  $\varpi_1$  and  $\varpi_2$  in Sect. 5. I am now reduced to the single degree of freedom  $(R; r)$ .

## 4 Reduction to one parameter

By substituting  $g + S + R$  for  $\epsilon$  in the Keplerian part (Eq. (17)) and by removing constant terms, the Hamiltonian  $\mathcal{H} = \mathcal{H}_K + \mathcal{H}_P$  can be written under the form

$$\mathcal{H}(R; r) = \alpha R - \beta R^2 + \gamma \sqrt{2R} \cos r, \quad (25)$$

where

$$\begin{aligned} \alpha &= -3n_{1,0}p[(g + S)(pC_1 + (p + 1)C_2) - C_1 - C_2], \\ \beta &= \frac{3}{2}n_{1,0}p(pC_1 + (p + 1)C_2), \\ \gamma &= \frac{m_1 n_{2,0}}{m_0 C_2} \sqrt{f_1^2 C_1 + f_2^2 C_2}. \end{aligned} \quad (26)$$

This Hamiltonian is the so-called second fundamental model of resonance described by Jacques Henrard and Anne Lemaître (Henrard and Lemaitre, 1983). While simple enough with only one degree of freedom, it still depends on the three parameters  $\alpha$ ,  $\beta$  and  $\gamma$ . However, there exists a classical method to reduce the Hamiltonian (25) to one single parameter  $\delta$ . Following Henrard and Lemaitre, 1983, I define

$$\sigma = r, \quad \Sigma = \frac{R}{K}, \quad \Pi = \frac{\mathcal{H}}{\omega}, \quad (27)$$

and I choose  $K$  and  $\omega$  in such a way that<sup>2</sup> the coefficient in front of  $\Sigma^2$  in  $\Pi$  be  $-1$  and that in front of  $\sqrt{2R} \cos r$  be  $2$ . This gives me  $K = (2\beta/\gamma)^{-2/3}$  and  $\omega = \beta(2\beta/\gamma)^{-4/3}$ . Equations of motions are no longer in canonical form as

$$\frac{d\Sigma}{dt} = -\frac{\omega}{K} \frac{\partial \Pi}{\partial \sigma}, \quad \text{and} \quad \frac{d\sigma}{dt} = \frac{\omega}{K} \frac{\partial \Pi}{\partial \Sigma}. \quad (28)$$

However, the canonical form can be recovered by merely rescaling time. Writing  $\tau =$

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<sup>2</sup>Choices like putting a 2 in front of  $\sqrt{2R} \cos r$  or putting a  $3\delta$  in front of  $\Sigma$  may seem arbitrary. Here, they allow for the bifurcation to be at  $\delta = 1$  while having the nicest form for the Hamiltonian (30).

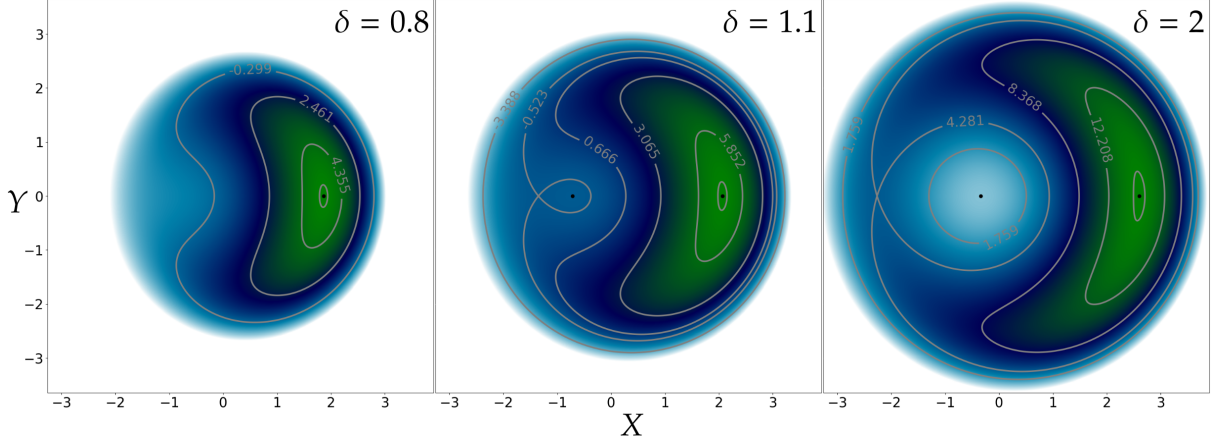


Figure 1 — Phase Space of the Hamiltonian (32) for  $\delta \in \{0.8, 1.1, 2\}$ . Elliptic fixed points are shown with a black point, whereas the hyperbolic fixed point is identified as the intersection of a level curve (the separatrix) with itself.

$\omega t/K$ , we obtain  $d\Sigma/d\tau = -\partial \Pi / \partial \sigma$  and  $d\sigma/d\tau = \partial \Pi / \partial \Sigma$ . Since  $R = \mathcal{O}(e_j^2)$  and  $\beta/\gamma = \mathcal{O}(m_0/m_j)$ , the new action  $\Sigma$  has order of magnitude  $\Sigma = \mathcal{O}(e_j^2) \mathcal{O}(m_0/m_j)^{2/3}$ . Defining the last parameter  $\delta$  as

$$\delta = \alpha \left( \frac{4}{27\beta\gamma^2} \right)^{1/3}, \quad (29)$$

the one-parameter and one-degree-of-freedom Hamiltonian of the second fundamental model of resonance reads

$$\Pi(\Sigma; \sigma) = 3\delta\Sigma - \Sigma^2 + 2\sqrt{2\Sigma} \cos \sigma. \quad (30)$$

Using the cartesian canonical coordinates

$$X = \sqrt{2\Sigma} \cos \sigma, \quad Y = \sqrt{2\Sigma} \sin \sigma, \quad (31)$$

the Hamiltonian is written

$$\Pi(X; Y) = \frac{3}{2}\delta(X^2 + Y^2) - \frac{1}{4}(X^2 + Y^2)^2 + 2X, \quad (32)$$

or equivalently

$$\Pi(X; Y) = 2X - \frac{1}{4}(X^2 + Y^2 - 3\delta)^2. \quad (33)$$

The equilibria of  $\Pi$  are located at  $Y = 0$  and  $X$  the roots of

$$X^3 - 3\delta X - 2 = 0. \quad (34)$$

The equilibrium values of  $X$  are then

$$\begin{aligned} X_{\text{eq}} &\in \{u + v, \quad ju + j^2v, \quad j^2u + jv\}, \text{ where} \\ u &= \left(1 + \sqrt{1 - \delta^3}\right)^{1/3}, \\ v &= \left(1 - \sqrt{1 - \delta^3}\right)^{1/3}, \end{aligned} \quad (35)$$

and  $j = \exp(2i\pi/3)$ . With the constraint that  $X_{\text{eq}}$  is a real number, there is a bifurcation at  $\delta = 1$ . When  $\delta < 1$ , only one (elliptic) fixed point exists, whereas for  $\delta > 1$ , three fixed points exist, one of them being hyperbolic from which a separatrix emanates and defines the mathematical resonance. In Fig. 1, I plot the phase space of the Hamiltonian (32) for different values of  $\delta$ , showing the bifurcation at  $\delta = 1$ . By linearizing the equations of motions in the vicinity of the fixed points, it can be shown that a fixed point is hyperbolic if, and only if,  $\delta < X^2 < 3\delta$ .

## 5 Explicit expression of the transformations towards one degree of freedom

Working around the transformations of Sect. 3, the variable  $R$  reads

$$R = \frac{f_1^2 C_1 d_1 + f_2^2 C_2 d_2 + 2f_1 f_2 \sqrt{C_1 d_1 C_2 d_2} \cos(\sigma_1 - \sigma_2)}{f_1^2 C_1 + f_2^2 C_2}, \quad (36)$$

whereas the first integral  $S$  is given by

$$S = \frac{f_1^2 C_1 d_2 + f_2^2 C_2 d_1 - 2f_1 f_2 \sqrt{C_1 d_1 C_2 d_2} \cos(\sigma_1 - \sigma_2)}{f_1^2 C_1 + f_2^2 C_2}. \quad (37)$$

Using the equations of motions  $\dot{d}_j = -\partial\mathcal{H}/\partial\sigma_j$  and  $\dot{\sigma}_j = \partial\mathcal{H}/\partial d_j$  derived from the Hamiltonian  $\mathcal{H}_K + \mathcal{H}_P$  given by Eqs. (14) and (17), I verified that,

$$\frac{dS}{dt} = \{\mathcal{H}, S\} = \frac{\partial S}{\partial d_1} \dot{d}_1 + \frac{\partial S}{\partial d_2} \dot{d}_2 + \frac{\partial S}{\partial \sigma_1} \dot{\sigma}_1 + \frac{\partial S}{\partial \sigma_2} \dot{\sigma}_2 = 0, \quad (38)$$

proving in another way that  $S$  is indeed a conserved quantity. If  $S$  is a conserved quantity, then so is  $\Upsilon$ , defined as

$$\Upsilon = -2 \frac{f_1^2 C_1 + f_2^2 C_2}{f_1 f_2} S. \quad (39)$$

Using the relations  $\sqrt{2C_j d_j} = e_j + \mathcal{O}(e_j^3)$  and  $\sigma_1 - \sigma_2 = \varpi_2 - \varpi_1$ , and the ratio

$$\frac{e_{1,\text{eq}}}{e_{2,\text{eq}}} = \frac{f_1}{-f_2} \frac{m_2}{m_1} \left( \frac{p+1}{p} \right)^{1/3}, \quad (40)$$

between the equilibrium values of  $e_j$  at the fixed point (the one that exists for all values of  $\delta$ ),  $\Upsilon$  can be written

$$\Upsilon = \frac{e_{2,\text{eq}}}{e_{1,\text{eq}}} e_1^2 + \frac{e_{1,\text{eq}}}{e_{2,\text{eq}}} e_2^2 + 2e_1 e_2 \cos(\varpi_1 - \varpi_2). \quad (41)$$

This gives, for resonances 1 : 2 to 5 : 6, respectively,

$$\begin{aligned} \Upsilon_{1:2} &= 0.28561 \frac{m_1}{m_2} e_1^2 + 3.50132 \frac{m_2}{m_1} e_2^2 + 2e_1 e_2 \cos(\varpi_1 - \varpi_2), \\ \Upsilon_{2:3} &= 1.07148 \frac{m_1}{m_2} e_1^2 + 0.93329 \frac{m_2}{m_1} e_2^2 + 2e_1 e_2 \cos(\varpi_1 - \varpi_2), \\ \Upsilon_{3:4} &= 1.05021 \frac{m_1}{m_2} e_1^2 + 0.95219 \frac{m_2}{m_1} e_2^2 + 2e_1 e_2 \cos(\varpi_1 - \varpi_2), \\ \Upsilon_{4:5} &= 1.03873 \frac{m_1}{m_2} e_1^2 + 0.96271 \frac{m_2}{m_1} e_2^2 + 2e_1 e_2 \cos(\varpi_1 - \varpi_2), \\ \Upsilon_{5:6} &= 1.03154 \frac{m_1}{m_2} e_1^2 + 0.96943 \frac{m_2}{m_1} e_2^2 + 2e_1 e_2 \cos(\varpi_1 - \varpi_2). \end{aligned} \quad (42)$$



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