



## 10.1 Properties of Markov Chains

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In this section, we will study a concept that utilizes a mathematical model that combines probability and matrices to analyze what is called a *stochastic process*, which consists of a sequence of trials satisfying certain conditions. The sequence of trials is called a *Markov Chain* which is named after a Russian mathematician called Andrei Markov (1856-1922).


# Andrei Markov (1856-1922)

<http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Markov.html>

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- Markov is particularly remembered for his study of Markov chains, sequences of random variables in which the future variable is determined by the present variable but is independent of the way in which the present state arose from its predecessors. This work launched the theory of stochastic processes





**Transition probability matrix:** Rows indicate the current state and column indicate the transition . For example, given the current state of A, the probability of going to the next state A is s. Given the current state A', the probability of going from this state to A is r. Notice that the rows sum to 1. We will call this matrix **P**.

		Next State	
		A	A'
Current State	A	$s$	$1-s$
	A'	$r$	$1-r$



## Initial State distribution matrix:

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- This is the initial probabilities of being in state A as well as not A , A' . Notice again that the row probabilities sum to one, as they should.

$$S_0 = \begin{matrix} & \begin{matrix} A & A' \end{matrix} \\ \begin{bmatrix} t & 1-t \end{bmatrix} \end{matrix}$$



## First and second state matrices:

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- If we multiply the Initial state matrix by the transition matrix, we obtain the first state matrix.

$$S_1 = S_o P$$

- If the first state matrix is multiplied by the transition matrix we obtain the second state matrix:

$$S_2 = S_1 P = S_o P \cdot P = S_o P^2$$



## Kth – State matrix

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- If this process is repeated we will obtain the following expression: The entry in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column indicates the probability of the system moving from the  $i^{\text{th}}$  state to the  $j^{\text{th}}$  state in  $k$  observations or trials.

$$S_k = S_{k-1}P = S_0P^k$$

**An example:** An insurance company classifies drivers as low-risk if they are accident-free for one year. Past records indicate that 98% of the drivers in the low-risk category (L) will remain in that category the next year, and 78% of the drivers who are not in the low-risk category ( L') one year will be in the low-risk category the next year.

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1. Find the transition matrix, P

$$P = \begin{matrix} & \begin{matrix} L & L' \end{matrix} \\ \begin{matrix} L \\ L' \end{matrix} & \begin{bmatrix} 0.98 & 0.02 \\ 0.78 & 0.22 \end{bmatrix} \end{matrix}$$

1. If 90% of the drivers in the community are in the low-risk category this year, what is the probability that a driver chosen at random from the community will be in the low-risk category the next year? The year after next ? (answer 0.96, 0.972 from matrices)

$$\begin{matrix} & \begin{matrix} L & L' \end{matrix} \\ S_0 & = \begin{bmatrix} 0.90 & 0.10 \end{bmatrix} \end{matrix}$$

$$S_1 = S_0 P \rightarrow$$

$$\begin{matrix} & \begin{matrix} L & L' \end{matrix} \\ S_1 & = \begin{bmatrix} 0.96 & 0.04 \end{bmatrix} \end{matrix}$$

$$S_2 = S_1 P = S_0 P \cdot P = S_0 P^2$$

$$\begin{matrix} & \begin{matrix} L & L' \end{matrix} \\ S_2 & = \begin{bmatrix} 0.972 & 0.028 \end{bmatrix} \end{matrix}$$



## Finding the kth State matrix

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- Use the formula  $S_k = S_0 P^k$  to find the 4<sup>th</sup> state matrix for the previous problem.

$$S_4 = S_0 P^4$$

- $[0.90 \quad 0.10] \begin{bmatrix} 0.98 & 0.02 \\ 0.78 & 0.22 \end{bmatrix}^4 =$

$$[.97488 \quad 0.02512]$$

- after four states, the percentage of low-risk drivers has increased to .97488





## 10.2 Regular Markov Chains

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In this section, we will study what happens to the entries in the  $k$ th state matrix as the number of trials increases. We wish to determine the long-run behavior of the both the state matrices and the powers of the transition matrix  $P$ .



## Stationary Matrix

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- When we computed the fourth state matrix of a previous problem we saw that the numbers appeared to approaching fixed values. Recall,

$$S_4 = S_0 P^4$$

$$\begin{bmatrix} 0.90 & 0.10 \end{bmatrix} \begin{bmatrix} 0.98 & 0.02 \\ 0.78 & 0.22 \end{bmatrix}^4$$

$$\begin{bmatrix} .97488 & 0.02512 \end{bmatrix}$$

- We would find if we calculated the 5<sup>th</sup> , 6<sup>th</sup> and and kth state matrix, we would find that they approach a limiting matrix of  **$\begin{bmatrix} 0.975 & 0.025 \end{bmatrix}$**  This final matrix is called a **stationary matrix**.
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## Stationary Matrix

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- The **stationary matrix** for a Markov chain with transition matrix  $P$  has the property that  $\mathbf{SP} = \mathbf{S}$
- To prove that the matrix  $\begin{bmatrix} 0.975 & 0.025 \end{bmatrix}$  is the stationary matrix, we need to show that  $\mathbf{SP} = \mathbf{S}$

- $\begin{bmatrix} 0.975 & 0.025 \end{bmatrix} \begin{bmatrix} 0.98 & 0.02 \\ 0.78 & 0.22 \end{bmatrix} = \begin{bmatrix} 0.975 & 0.025 \end{bmatrix}$

- Upon multiplication, we find the above statement to be true, so the stationary matrix is  $\begin{bmatrix} 0.975 & 0.025 \end{bmatrix}$



## Stationary matrix concept

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- What this means is that in the long-run the system will be at a **steady state** . Later states will change very little, if at all. This means that in the long run , the number of low-risk drivers will be 0.975 and the number of drivers which are not low-risk will be 0.025.
- The question of whether or not every Markov chain has a unique stationary matrix can be answered- it is no. However, if a Markov chain is **regular** , then it will have a unique stationary matrix and successive state matrices will always approach this stationary matrix.



## Regular Markov Chains

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- A transition matrix **P** is **regular** if some power of P has **only positive entries**. A Markov chain is a **regular Markov chain** if its transition matrix is regular. For example, if you take successive powers of the matrix D, the entries of D will always be positive (or so it appears). So D would be **regular**.

- $D = \begin{bmatrix} 0.3 & 0.7 \\ 1 & 0 \end{bmatrix}$

$$D^2 = \begin{bmatrix} .79 & .210 \\ 0.30 & 0.70 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} .447 & .553 \\ .79 & .21 \end{bmatrix}$$



## Properties of Regular Markov chains

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- If a Markov chain is regular (that is, it has a transition matrix,  $P$ , that is regular (successive powers of this matrix  $P$  contains only positive entries)) then
- **there is a unique stationary matrix  $S$  that can be found by solving the equation**
- **$SP = S$**



Finding the stationary matrix. We will find the stationary matrix for the matrix we have just identified as regular:

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1. Solve  $\mathbf{SP} = \mathbf{S}$ , where
- Solve the system below:

$$\mathbf{S} = \begin{bmatrix} s_1 & s_2 \end{bmatrix} \quad \begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \end{bmatrix}$$

$$\begin{bmatrix} 0.3 & 0.7 \\ 1 & 0 \end{bmatrix} = \mathbf{P} \quad \begin{bmatrix} 0.3s_1 + s_2 & 0.7s_1 + 0 \end{bmatrix} = \begin{bmatrix} s_1 & s_2 \end{bmatrix}$$

$$\rightarrow 0.3s_1 + s_2 = s_1 \rightarrow$$

$$s_2 = 0.7s_1 \rightarrow$$

$$\text{if } s_1 + s_2 = 1 \rightarrow$$

$$s_1 + 0.7s_1 = 1 \rightarrow s_1 \approx 0.5882 \rightarrow s_2 \approx 0.4118$$

- Thus,  $\mathbf{S} = [0.5882 \quad 0.4118]$



## Limiting matrix $P$ to the power $k$

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- According to theorem 1 of this section, the state matrices  $S_k$  will approach the stationary matrix  $S$  and the matrices given by successive powers of  $P$  approach a limiting matrix  $P^*$  where each row of  $P^*$  is equal to the stationary matrix  $S$ .

- $S = \begin{bmatrix} 0.5882 & 0.4118 \end{bmatrix}$

$$P^* = \begin{bmatrix} 0.5881 & 0.4112 \\ 0.5881 & 0.4112 \end{bmatrix}$$





## 10.3 Absorbing Markov Chains

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In this section, the concepts of absorbing Markov chains will be discussed and illustrated through examples. We will see that the powers of the transition matrix for an absorbing Markov chain will approach a limiting matrix. We must introduce some terminology first.



## Absorbing state and Absorbing Chains

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- A state in a Markov chain is called an **absorbing state** if once the state is entered, it is impossible to leave.
- This concept will be illustrated by an example: For the following transition matrix, we determine that B is an absorbing state since the probability from going from state B to state B is one (this means that once the system enters state B it does not leave since the probability of moving from state B to states A or C is zero as indicated by the second row.

	<i>A</i>	<i>B</i>	<i>C</i>
<i>A</i>	0.5	0	0.5
<i>B</i>	0	1	0
<i>C</i>	0	0.5	0.5



## Theorem 1: Absorbing states and transition matrices

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- A state in a Markov chain is **absorbing** if and only if the row of the transition matrix corresponding to the state has a 1 on the main diagonal and zeros elsewhere.
- To ensure that the transition matrices for Markov chains with one or more absorbing states have **limiting matrices** it is necessary that the chain satisfies the following definition:
  - A Markov chain is an **absorbing chain** if:
  - A) there is at least one absorbing state.
  - B) It is possible to go from each non-absorbing state to at least one absorbing state in a finite number of steps.



## Recognizing Absorbing Markov chains

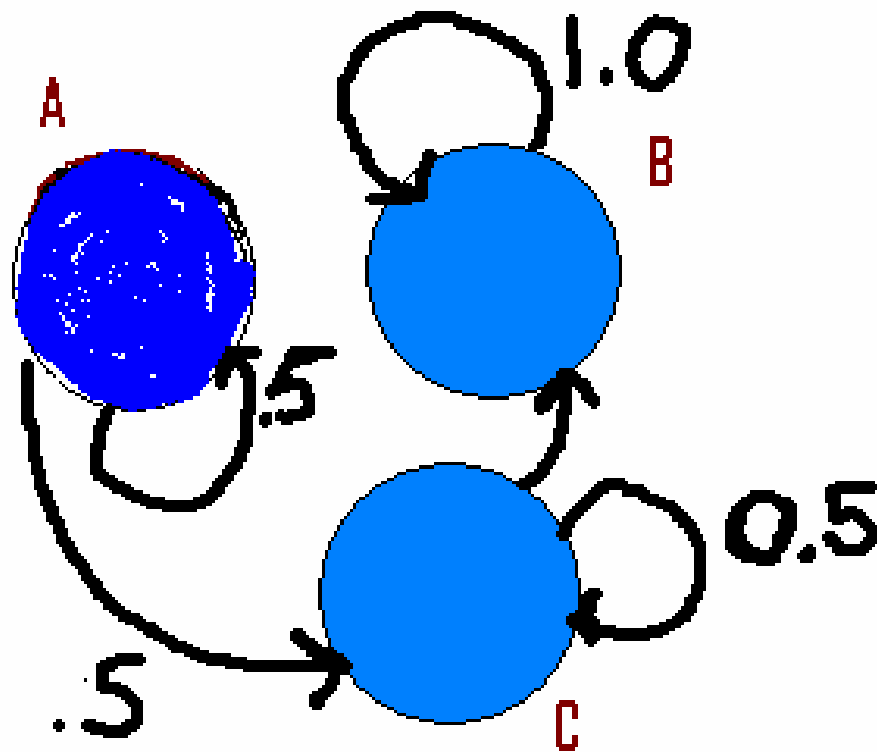
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- We will use a transition diagram to determine whether  $P$  is the transition matrix for an absorbing Markov chain.

$$P = \begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} \end{matrix}$$

We see that B is an absorbing state

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In a transition matrix, if all the absorbing states precede all the non-absorbing states, the matrix is said to be in **standard form**. Standard forms are very useful in determining limiting matrices for absorbing Markov chains.

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- The matrix below is in standard form since the absorbing states A and B precede the non-absorbing state C. The general standard form matrix P is listed on the right in which the matrix P is partitioned into four sub-matrices **I**, **O**, **R** and **Q** where I is an identity matrix and O is the zero matrix.

$$\begin{array}{c} A \quad B \quad C \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \end{array}$$

$$P = \left[ \begin{array}{c|c} I & O \\ \hline R & Q \end{array} \right]$$



## Limiting Matrix

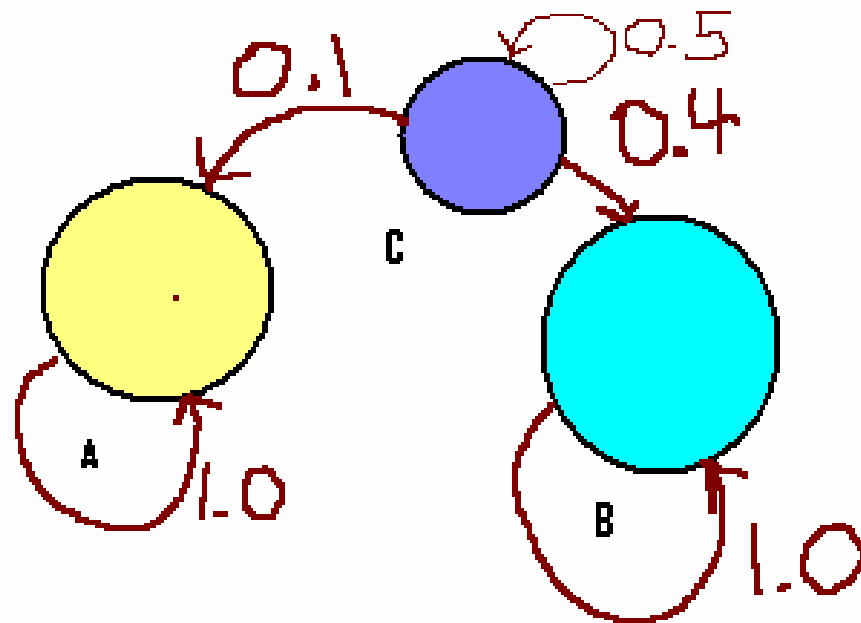
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- **We are now ready to discuss the long-run behavior of absorbing Markov chains. This will be illustrated with an application.**
- *Two competing real estate companies are trying to buy all the farms in a particular area for future housing development. Each year, 10% of the farmers sell to company A each year, 40% sell to company B and the remainder continue farming. Neither company ever sells any of the farms they purchase.*
- A) Draw a transition diagram for this Markov process and determine whether the associated Markov chain is absorbing.
- B) Write a transition matrix in standard form
- C) If neither company owns any farms at the beginning of this competitive buying process, estimate the percentage of farms that each company will purchase in the long run.

## Transition Diagram

We see that states A and B are absorbing. We also see that the associated Markov chain is absorbing, since there are two absorbing states A and B and it is possible to go from the non-absorbing state C to either A or B in one step.

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We use the transition diagram to write a transition matrix that is in standard form:

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$$P = \begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \end{matrix}$$

## Initial State Matrix; Successive state matrices

- At the beginning of this process, all of the farmers are in state C since no land has been sold initially. Therefore, we have the following initial state matrix: The successive state matrices are given as well. Do the state matrices have a limit? To find out, we compute the 10<sup>th</sup> state matrix and look at the product. We conjecture the limit of the state matrices is

○ .

$$S_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad P = \begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \end{matrix}$$

$$S_1 = S_0 P$$

$$S_2 = S_1 P = S_0 P \cdot P = S_0 P^2$$

- [ 0.2    0.8 ]**

$$S_{10} = S_0 P^{10} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}^{10} = \begin{bmatrix} 0.1998 & 0.7992 \end{bmatrix}$$



## Theorem 2: Limiting Matrices for Absorbing Markov chains

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- If a standard form  $P$  for an absorbing Markov chain is partitioned as

$$P = \left[ \begin{array}{c|c} I & 0 \\ \hline R & Q \end{array} \right]$$

- then  $P^k$  approaches a limiting matrix  $\bar{P}$  as  $k$  increases where

$$\bar{P} = \left[ \begin{array}{c|c} I & 0 \\ \hline FR & 0 \end{array} \right]$$

- The matrix  $F$  is given by  $F = (I - Q)^{-1}$
- and is called the **fundamental matrix for  $P$**
- The identity matrix used to form the fundamental matrix  $F$  must be the same size as the matrix  $Q$ .



## Using the Formulas to find the limiting matrix

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- We identify the display the appropriate matrices:
- Use the formula to find the matrix F. Q is a 1x1 matrix.
- Next, determine FR
- Display the limiting matrix. Notice that the last row is the same the kth state matrix.

$$\text{○} \cdot \mathbf{P} = \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0.1 & 0.4 & 0.5 \end{array} \right]$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{O} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{R} = [0.1 \quad 0.4]$$
$$\mathbf{Q} = [0.5]$$

$$\mathbf{F} = [\mathbf{I} - \mathbf{Q}]^{-1} = [[1] - [.5]]^{-1} = 2$$

$$\mathbf{FR} = 2[0.1 \quad 0.4] = [0.2 \quad 0.8]$$

$$\overline{\mathbf{P}} = \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.2 & 0.8 & 0 \end{array} \right]$$