STRATHMORE UNIVERSITY

SCHOOL OF FINANCE AND APPLIED ECONOMICS

Bachelor of Business Science – Actuarial Science, Finance & Financial Economics

BSM 2215: INTRODUCTION TO STOCHASTIC MODELLING

Stochastic Processes

Syllabus objectives

The student should be able to:

1. Define a stochastic process giving examples of quantities that can be modelled using Stochastic process.
2. Explain what it means for a stochastic process to be stationary.
3. Describe each of the following stochastic processes explaining their defining characteristics and their applications in modelling:
   - Random walk-general random walk, simple random walk and simple symmetric random walk.
   - Standard Brownian Motion
   - Poisson process
   - Compound Poisson Process

Introduction

We start by defining a stochastic process in general terms as well as giving examples of finance and Actuarial quantities that can be modelled using stochastic process. We also define Stationarity which is a characteristic of the stochastic processes we are going to study in this topic, we explain what they mean. The last part of this topic describes the various examples of stochastic processes, particularly Random walk (General random walk, simple random walk and simple symmetric random walk), Standard Brownian Motion, Poisson process and Compound Poisson Process. We also explain their defining characteristics as well as their applications in modelling.

Definition of a Stochastic Process

It is sequence of values of some quantity in which the future values cannot be predicted with certainty. For instance Let the current (at $\mathbf{t = 0}$) price of a share price be $\mathbf{S_0}$. At time $\mathbf{t = 1}$, this price could go up to a random price $\mathbf{S_u}$ or go down to a random price $\mathbf{S_d}$or even stay at the current price $\mathbf{S_0}$.The price of the share is therefore a stochastic process because it represents a sequence of values in which future values cannot be predicted with certainty.

Stationarity

A stochastic process X is said to be Stationary if the distribution $\mathrm{X}_{\mathrm{t}_1+\mathrm{k}}$ and $\mathrm{X}_{\mathrm{t}_2+\mathrm{k}}$ are identical. i.e. their statistical properties remain unchanged (variance, mean e.t.c.) as time elapses. The statistical properties only depend on the time lag.

Random walk

Let X_1, X_2…X_∞ be a sequence of independent, identically distributed discrete random variables. Define $S_n = X_1 + X_2 + \cdots + X_n$ with the initial condition e $S_0 = 0$. The sequence S_n is called a general random walk.

Simple random walk

In a special case where the steps $X_i's$ can only take values of either +1 or -1, the process is known as simple random walk.

Simple symmetric Random walk

In more special case where the the values of the steps can be +1 or -1 with equal probabilities (0.5). I.e.

$$X_i = \begin{cases} +1 & with\ \ a\ probability\ of\ 0.5 \\ -1 & with\ \ a\ probability\ of\ 0.5 \end{cases}$$

such a process is called simple symmetric random walk.

It is equally likely to step upwards or downwards.

Standard Brownian Motion

Standard Brownian Motion can be viewed as the continuous version of the simple symmetric random walk. If we reduce the size of the step progressively from 1 unit until it is infinitesimal, the simple symmetric random walk becomes a standard Brownian motion.

Definition of a standard Brownian Motion

A continuous stochastic process W_t is a standard Brownian Motion if it has the following defining characteristics:

- The initial value is zero i.e. $W_0 = 0$
- It has independent increments i.e. for $u < s < t,\ W_t - W_s$ does not depend on W_u.
- It has Gaussian(Normally distributed) increments i.e. $W_t - W_s \sim N(0, t - s)$
- It has stationary increments i.e. the distribution of $W_t - W_s$ only depend on the time lag $t - s$
- W_t has a continuous sample path i.e. W_t is a continuous function of time, $t \to W_t$

Remark: For $s < t$ W_t can be decomposed as $W_t = W_s + W_t - W_s$

Other Properties of standard Brownian Motion

Rescaling Property

If W_t is a standard Brownian motion and c is a positive constant, then the rescaled process $W^*{}_t = \frac{1}{c} W_{ct}$ is also a standard Brownian Motion.

Proof:

$$E\left[\frac{1}{c} W_{ct}\right] = \frac{1}{c} E[W_{ct}] = 0$$

$$Var\left[\frac{1}{c} W_{ct}\right] = \frac{1}{c^2} Var[W_{ct}] = \frac{1}{c^2} \times ct = t$$

Time-inversion property

If W_t is a standard Brownian motion and c is a positive constant, then the rescaled process $W^*{}_t = tW_{\frac{1}{t}}$ is also a standard Brownian Motion.

Proof:

$$E\left[tW_{\frac{1}{t}}\right] = \frac{1}{c} E\left[W_{\frac{1}{t}}\right] = 0$$

$$Var\left[tW_{\frac{1}{t}}\right] = t^2 Var\left[W_{\frac{1}{t}}\right] = t^2 \times \frac{1}{t} = t$$

$Cov(W_t , W_s) = min\ (s, t)$

Proof:

$$Cov(W_t , W_s) = Cov(W_s + W_t - W_s , W_s) = Cov(W_s , W_s) + Cov(W_t - W_s, W_s)$$
$$Cov(W_s , W_s) = s = \min (s, t)$$

Applications of Standard Brownian Motion

Standard motion is rarely used to model anything directly. A standard Brownian motion is certain to become negative eventually. That means that it cannot be used to model the share prices, for example, since we do not expect the share prices to be negative. However, it forms the key building block for most continuous-time stochastic models which are modelled as diffusion (Ito processes). Examples of these are Geometric Brownian motion (used to model share prices) and Ornstein-Uhlenbeck (used to model interest rates). Some martingales are constructed from standard Brownian motion. Martingales are useful in the pricing and hedging of financial derivative.

Poisson process

An integer-valued process $\mathrm{N_t}$ with intensity $\boldsymbol{\lambda}$ is called a Poisson process if it has the following properties:

- $N_0 = 0$
- N_t has independent increments
- N_t has Poisson distributed increments with intensity $\lambda(t - s)$. I.e.

$$P[N_t - N_s = n] = \frac{[\lambda(t - s)]^n e^{-\lambda(t-s)}}{n!}$$

Proposition:

Let N be a Poisson process with intensity λ, then the increments have the following moment generating function:

$$M_{N_t-N_s}(r) = E\left[e^{r(N_t-N_s)}\right] = e^{[\lambda(t-s)(e^r-1)]}$$

The Poisson process is known as a counting process because it can be used in counting the occurrence of events for instance claim in insurance company, car accident and arrival of customers at a service point.

Compound Poisson

The Poisson process alone is inadequate in modelling, for instance it can only count the number of claims but no information on the amounts/size of claims. However, the insurance company is interested in both the number and size of the claims. Let X_i denote independent identically distributed claim sizes. We also assume further that X_i are independent on N_t. We can then define a compound process S as follows:

$$S_t = X_1 + X_2 + \cdots + X_{N_t}$$

Remark: A compound Poisson process has independent and stationary increments too.

Proposition:

The Moment generating function(MGF) of the increment of a compound Poisson process $S_t - S_s$ is given by:

$$M_{S_t - S_s}(r) = E\left[e^{r(S_t - S_s)}\right] = M_{N_{t-s}}(\ln M_X(r)) = e^{[\lambda(t-s)(M_X(r)-1)]}$$

Proof:

$$M_{S_t - S_s}(r) = E\left[e^{r(S_t - S_s)}\right] = E\left[e^{r\sum_{N_s+1}^{N_t} Y_i}\right] = E[e^{r\sum_{i=1}^{N_{t-s}} Y_i}]$$

$$= E\left[E\left[e^{r\sum_{i=1}^{N_{t-s}} X_i}|N_{t-s}\right]\right] = E\left[E\left[e^{r\sum_{i=1}^{N_{t-s}} X_i}|N_{t-s}\right]\right] = E\left[E\left[e^{rX_1 + rX_2 + \cdots + rX_{N_{t-s}}}\right]\right]$$

$$E\left[E[e^{rX_1}]E[e^{rX_1}]E[e^{rX_2}] \ldots \times E\left[e^{rX_{N_{t-s}}}\right]\right]$$

$$E[M_X(r) \times M_X(r) \times \ldots \times M_X(r)] = E[\left(M_X(r)\right)^{N_{t-s}}]$$

$$E\left[\left(M_X(r)\right)^{N_{t-s}}\right] = E\left[e^{\ln\left(M_X(r)\right)^{N_{t-s}}}\right] = E\left[e^{N_{t-s}\ln M_X(r)}\right] = M_{N_{t-s}}(\ln M_X(r))$$

$$= e^{[\lambda(t-s)(M_X(r)-1)]}$$