STRATHMORE UNIVERSITY

SCHOOL OF FINANCE AND APPLED ECONOMICS

Bachelor of Business Science- Actuarial Science, Finance & Financial Economics

BSA 3109/BSF 3224/BSA 3144: STOCHASTIC MODELS FOR ACTUARIAL AND FINANCE APPLICATIONS & STOCHASTIC ANALYSIS FOR FINANCIAL APPLICATIONS

Topic One: Review of introduction to Stochastic Modelling

Syllabus objectives

At the end of the topic the learner should be able to recall the basic concepts covered in Introduction to Stochastic Modelling including:

1. Definition of a stochastic process, filtration and martingales
2. Properties of Conditional Expectations; linearity, measurability, independence, double expectation, constants and tower property.

A stochastic Process

A stochastic process is a sequence of random variables over time or simply a collection of random variables indexed by time. It is a statistical process with random variables that depend on time (which is also a variable parameter).It is sequence of values of some quantity in which the future values cannot be predicted with certainty.

Example of a stochastic Process: The share price over time, S_t

Let the current (at $t = 0$) price of a share price be S_0. At time $t = 1$, this price could go up to a random price S_u or go down to a random price S_d or even stay at S_0.The price of the share S_t is therefore a stochastic process because it represents a sequence of values in which future values cannot be predicted with certainty.

A martingale

Mathematical Definition

A stochastic process X_t is a martingale if $E[X_{t+1}|F_t] = X_t$ (Discrete-time martingale) or for $s < t$, if $E[X_t|F_s] = X_s$ (Continuous-time martingale) and F_t is the filtration. Filtration is the history of a process up to time t.

For instance X_t could represent the amount of money we have gained at time t in a gambling game. We know what has happened in the past up to time t this information is what we are calling a filtration,F_t. A martingale is therefore a model of fair game with the expected gain in a single play being 0 (a zero drift). This means that even with the information of past events, the best expected estimate of the next value is the current value.

X_t could also represent the current price of a share price at time t . We know all the prices up to time t, but the best expected estimate of the next price, even with all the collected information, is the current price. This, however, does not mean that the value does not change. It can change but its expected value remains the same.

Simple word definition

A martingale is a process whose current value is the best estimate of its expected future value. We can also say is a model of a fair game where knowledge of past events never helps predict the mean of the future winnings. The expected gain in a single play is 0 (a zero drift).

A martingale is a sequence of random variables (i.e., a stochastic process) for which, the expectation of the next value in the sequence is equal to the present observed value even given knowledge of all prior observed values.

The conditional expectation $E[Y|X]$

Definition: The conditional expectation of Y given X $=$ x . is the mean of the conditional distribution of Y given X $=$ x . This conditional expectation is denoted by $E[\mathrm{Y}|\mathrm{X}]$.

Properties of conditional Expectations

This is a short list of important rules for manipulating and calculating conditional expectations.

1. Constance property:

   a) If c is a constant then $E[c|Z] = c$
   b) If c is a constant then $E[cX|Z] = \mathrm{c}E[X|Z]$

2. Measurability/ adaptedness:

   If whenever we are given Y we can know what X is, then we can say that X is Y-measurable or that X is adapted to Y

   a) If X is Y-measurable then $E[X|Y] = X$
   b) If X is Y-measurable then $E[XY|Z] = YE[Z]$

3. Independence property:

   If X is independent of Y then $E[X\ |Y] = E[X]$

4. Double Expectation:

$$E\big[E[X|Y]\big] = E[X]$$

5. Tower property of conditional Expectation:

   If $H \subset G$,then $E[E[X|H]|G] = E[X|H]$

6. Linearity:

$$E[(Y + X)|Z] = E[Y|Z] + E[X|Z]$$

Application of conditional Expectations to manipulation of Martingales

1. Measurability/ adaptedness

   c) If X_n is F_n-measurable then $E[X_n|F_n] = X_n$

   d) If X_n is F_n-measurable then $E[X_nY|F_n] = X_nE[Y|F_n]$

2. Independence property

   If X_{n+1} is independent of F_n then $E[X_{n+1}\ |F_n] = E[X_{n+1}]$

3. Double Expectation

   $E\big[E[X_{n+1}|F_n]\big] = E[X_{n+1}]$

4. Tower property of conditional Expectation

   $$E[E[X_{n+1}|F_n]|F_{n+1}] = E[X_{n+1}|F_n]$$

   This must be the case because $F_0 \subset F_1 \subset F_2 \ldots . \subset F_\infty$

   Linearity: $E[Y_n + X_{n+1}|F_n] = E[Y_n|F_n] + E[Y_n|F_n] + E[X_{n+1}|F_n]$