Time-consistent lifetime portfolio selection under smooth ambiguity

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Abstract

This paper studies the optimal consumption, life insurance and investment problem for an income earner with uncertain lifetime under smooth ambiguity model. We assume that risky assets have unknown market prices that result in ambiguity. The individual forms his belief, that is, the distribution of market prices, according to available information. His ambiguity attitude, which is similar to the risk attitude described by utility function U, is represented by an ambiguity preference function ϕ. Under the smooth ambiguity model, the problem becomes time-inconsistent. We derive the extended Hamilton-Jacobi-Bellman (HJB) equation for the equilibrium value function and equilibrium strategy. Then, we obtain the explicit solution for the equilibrium strategy when both U and ϕ are power functions. We find that a more risk- or ambiguity-averse individual will consume less, buy more life insurance and invest less. Moreover, we find that the Tobin-Markowitz separation theorem is no longer applicable when ambiguity attitude is taken into consideration. The investment strategy will change with the characteristics of the decision maker, such as risk attitude, ambiguity attitude and age.

Keywords: Smooth ambiguity, Time-consistent strategy, Life insurance, Optimal portfolio.

JEL: G11, G22, C61

1 Introduction

Life insurance has long been the focus of actuarial research. Richard[24] first studied the optimal consumption, life insurance and investment problem for an individual with an uncertain lifetime and determinate income in a continuous-time modelling framework based on the classic work of Merton[22]. And this model has been extended in many ways to study the lifetime portfolio management problem for individuals. For example, Pliska and Ye [25] have extended Richard's model to a planning horizon and studied the optimal life insurance for a wage earner before retirement date. Liu et.al [19] introduced the hyperbolic discount function and habit formation of decision makers in the portfolio management problem, which is a time-inconsistent problem. Jensen[13] analysed

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the optimal portfolio management problem under recursive utility. In those studies, the individual are assumed to have clear knowledge about the movement of risky assets with known parameters, i.e. the expected return and volatility are constants. There are also literatures consider the parametric form of the return and volatility. For example, Liang and Zhao[18] assume that short-term interest rate follows Vasicek model and volatility follows Heston model. Shen and Wei [27] assume that all the parameters can be adapted random variables. However, the parameters in those stochastic process are still given constants.

This assumption is unreasonable because the individual does not have access to all information about the financial market; therefore, the individual cannot observe the "right" values for the parameters in the model. Sometimes, even though the individual has correct model of the movement of assets' price, the individual may assign multiple probabilities on the parameters and not know which is the "right" one. Such uncertainty is called ambiguity, and Knight[16] emphasized that ambiguity may be important for economic decision making. Ellsberg[4] also showed that people behave differently when considering ambiguity. Ambiguity is a common phenomenon in our daily life. For example, an investor makes investment decisions according to some experts' opinions; however, those experts report different opinions about the financial market. The individual forms his belief according to his experience about how reliable each expert is. In other words, the belief is a distribution of possible values for the unknown market parameters. The individual bets on which expert's report is correct and assigns a higher probability to the event that he believes is more likely to happen. In this paper, we allow for Knightian uncertainty about the market price in the risky asset price model, which means the individual considers a distribution, instead of a known value, of market price.

Four approaches can be used to address ambiguity in the financial market. The first is to find the optimal strategy in the worst case, which is called the max-min expected utility (MEU) model[8]. For example, Seifried[26] derived the optimal investment in the worst case via a martingale approach. Fei[5] studied the Merton problem for a decision maker with insider information within the MEU framework. For more discussions about the max-min problem, one can refer to Hansen and Sargent[10]. Because the MEU model cannot describe the ambiguity attitude of the decision maker, the framework has been extended to the second model, which is called the α-MEU model[7]. The α-MEU model considers a linear combination of the best and worst cases and uses the parameter α to describe the ambiguity attitude (see Li et al.[17] and Fei[6]). The third approach is the penalty-based multiple-priors utility model[20](see[11], [12]), which introduces a penalty term into the optimization problem to describe the disparity between the alternative model and the reference model. We use the fourth model, that is, the smooth ambiguity model introduced by Klibanoff et al.[15]. Here, preferences over a decision maker's act f are represented by a function with the following double expectation form, i.e. $V(f) = \int_\Delta \phi(\int_S U(f)d\pi)d\Pi$.

A main advantage of smooth ambiguity is the ability to separate risk attitude and ambiguity attitude. Π measures the likelihood of a particular π as the "right" probability, which is the decision maker's belief. The ambiguity attitude is captured by ambiguity preference function ϕ, which is similar to the way we address risk attitude represented by utility function U. When ϕ is concave, the decision maker is ambiguity averse. When ϕ is convex, the decision maker is ambiguity loving. A linear ϕ indicates ambiguity neutrality. Therefore, we can study the effect of changes in ambiguity attitude (ϕ) by keeping ambiguity (Π) fixed. Such separation is not possible in the MEU model, where we consider only the worst case. For decision makers with the same belief but different ambiguity attitudes, those decision makers are expected to make different decisions based on their ambiguity attitude. However, under the MEU model, such decision makers may make the same investment decisions regardless of their ambiguity attitude. Although the MEU model has been extended to the α-MEU model, which uses α to describe ambiguity attitude, this model still considers only two cases, the best and worst, and ignores other possible probabilities for π.

In the fourth model, the double expectation in $V(f)$ makes the dynamic control problem difficult to solve. In the finance, when the volatility is uncertain, the PDE of Black-Scholes was fully nonlinear. Peng[23] have proposed an non-linear G-expectation to solve such kind of nonlinear PDE. Due to the non-linear ambiguity preference function, the continuous-time dynamic control problem under the smooth ambiguity model is a time-inconsistent problem. Recently, Guan et al.[9] applied an extended Hamilton-Jacobi-Bellman (HJB) equation based on Björk et al.[1] to study the problem of maximizing the utility of the final wealth of an insurer who makes reinsurance and investment decisions under smooth ambiguity model. We follow the method in Guan et al.[9] to derive the extended HJB equation for the portfolio management problem under a smooth ambiguity model.

Many potential choices exist for the pair (U, ϕ), such as the exponential-power case (Chen et al.[2]), exponential-exponential case (Taboga[28]) and exponential-power case (Guan et al.[9]). Unlike the utility function, which is defined on $[0, +\infty)$, the ambiguity preference function must be defined according to the range of U. As the utility U might be negative, we may need the domain of ϕ to include a negative interval. The exponential function $\phi(x)$ can be defined on $\mathbb{R}$, and exponential function $U(x)$ can avoid negative utility. Thus, no more discussion is needed in the three cases. However, most literature on the portfolio management problem uses the CRRA utility function, i.e., a power function for U, because the function is easy to analyse. When we set $U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \gamma > 0, \gamma \neq 1$, no suitable power function exists for $\phi(x)$. Ju and Miao[14] discussed and solved the problem by limiting $0 < \gamma < 1$. Here, to demonstrate the influences of both ambiguity attitude and risk attitude, we solve the problem by limiting $\gamma > 1$ and assuming $\phi(x) = \frac{1-(-x)^{\alpha}}{\alpha}, \alpha \neq 0$. Specifically, we set $\phi(x) \triangleq -ln(-x)$ when $\alpha = 0$. Therefore, for $\alpha > 1$, $\alpha < 1$ and $\alpha = 1$, the decision maker is ambiguity averse, ambiguity loving and ambiguity neutral, respectively.

In this paper, we consider an individual's consumption, life insurance and investment strategy before a fixed T or before death. The time T can be regarded as retirement time. The financial market consists of a risk-free asset and two risky assets. The market prices for the two risky assets are unknown, but their distributions are given. Meanwhile, the individual receives a deterministic income during $[0, T]$ and before death. The individual evaluates his strategy on the basis of the smooth ambiguity model. We derive the extended HJB equation for the equilibrium value function and equilibrium strategy. Moreover, we present an example of the power-power case (U, ϕ) and show some interesting results of the equilibrium strategy. Then, we use a numerical experiment to illustrate the influences of risk aversion and ambiguity aversion on the equilibrium strategy.

The remainder of this paper is organized as follows. In Section 2, we present the problem and derive the extended HJB equation to solve the problem. In Section 3, we derive the equilibrium strategy for an individual with power-power case (U, ϕ) and analyse the strategy. Section 4 presents the sensitive analysis of the effects of risk and ambiguity aversion. Section 5 provides the conclusion.

2 The Model

We consider the optimal consumption, life insurance purchase and investment problem in a financial market for an individual during time $[0, T]$. $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbf{P})$ is a filtered complete probability space, and $\mathscr{F}_t$ is the information of the market available up to time t. $[0, T]$ is a fixed time horizon. All the processes introduced below are assumed to be adapted to $\{\mathscr{F}_t\}_{t\in[0,T]}$.

2.1 The wealth process

We assume that the financial market consists of one risk-free asset and two risky assets. The price for the risk-free asset evolves according to the equation:

$$dS_0(t) = S_0(t)rdt,$$

where $r > 0$ is a constant that represents the risk-free interest rate in the market.

The prices for the two risky assets are characterized by the following geometric Brownian motions:

$$dS_1(t) = S_1(t)[rdt + \sigma_1(\lambda_1 dt + dB_1(t))],$$

$$dS_2(t) = S_2(t)[rdt + \sigma_2(\lambda_2 dt + dB_2(t))].$$

(B_1, B_2) is standard Brownian motion in $\mathbb{R}^2$ defined on the probability space $(\Omega, \mathscr{F}, P)$. We suppose that B_1 and B_2 are independent. As we allow Knightian uncertainty, $\lambda = (\lambda_1, \lambda_2)$ (market price) is a random vector with distribution function $F^\lambda(\lambda_1, \lambda_2)$. The support set of $F^\lambda(\lambda_1, \lambda_2)$ is $\Lambda \subset \mathbb{R}^2$. Both λ_1 and λ_2 can be discrete or continuous. Although (σ_1, σ_2) can also be ambiguous, we set (σ_1, σ_2)(volatility) as a constant vector for ease of the analysis. For convenience, we also assume that (λ_1, λ_2) is independent of other stochastic processes in the financial market.

At time 0, the individual has an uncertain lifetime denoted by τ, which is a random variable. Assume that τ is independent of (λ_1, λ_2) and Brownian motion (B_1, B_2). Given that the hazard function for mortality is $\mu(t)$, the probability that $\tau > s$ given $\tau > t$ can be written as:

$$P(\tau > s | \tau > t) = e^{-\int_t^s \mu(v)dv}.$$

The life insurance company sells life insurance within an infinitesimal time horizon to the insured with a premium rate $\eta(t)$; therefore, the price for per unit insured is $\eta(t)$. The individual buys life insurance at a rate of $p(t)$. In other words, the individual will leave a legacy $X(t) + \frac{p(t)}{\eta(t)}$ at time t to inheritors if he or she dies immediately after purchasing life insurance.

At time t, the total wealth of the individual is denoted by $X(t)$. In addition to the life insurance expenditure $p(t)$, the individual invests a portion of the wealth $(\theta_1(t), \theta_2(t))$ in risky assets, allocates the remaining wealth $X(t) - \theta_1(t) - \theta_2(t)$ to the risk-free asset and chooses the level of consumption $c(t)$. Additionally, the individual receives deterministic income at the rate of $i(t)$. Thus, the individual's wealth satisfies:

$$\begin{aligned}
dX(t) =& \theta_1(t)\frac{dS_1(t)}{S_1(t)} + \theta_2(t)\frac{dS_2(t)}{S_2(t)} + [X(t) - \theta_1(t) - \theta_2(t)]\frac{dS_0(t)}{S_0(t)} - c(t)dt - p(t)dt + i(t)dt \\
=& \theta_1(t)[rdt + \sigma_1(\lambda_1 dt + dB_1(t))] + \theta_2(t)[rdt + \sigma_2(\lambda_2 dt + dB_2(t))] + [X(t) - \theta_1(t) - \theta_2(t)]rdt \\
& - c(t)dt - p(t)dt + i(t)dt \\
=& [rX(t) + \theta_1(t)\sigma_1\lambda_1 + \theta_2(t)\sigma_2\lambda_2 - c(t) - p(t) + i(t)]dt + \theta_1(t)\sigma_1 dB_1(t) + \theta_2(t)\sigma_2 dB_2(t).
\end{aligned}$$

2.2 The value function

We consider a risk-averse and ambiguity-averse individual who evaluates the strategy by means of the smooth ambiguity model from Klibanoff[15]. The individual's utility consists of three components: consumption, wealth at retirement time T and bequest. Given that the wealth at time t is $X(t)=x$, for a strategy $\mathbf{u}\triangleq\{c(t),p(t),\theta_1(t),\theta_2(t)\}_{0\le t\le T}$ and $s>t$, we denote the objective function as

$$
\begin{aligned}
J(t,x;\mathbf{u}) &= E^{\lambda}\phi\Big\{E^{P_\lambda}_{t,x}\Big[\int_t^{\tau\wedge T} e^{-\beta(s-t)}U(c(s))ds+e^{-\beta(T-t)}U(X(T))1_{\{\tau>T\}}\\
&\quad+e^{-\beta(\tau-t)}U\left(X(\tau)+\frac{p(\tau)}{\eta(\tau)}\right)1_{\{\tau\le T\}}\Big]\Big\}\\
&= E^{\lambda}\phi\Big\{E^{P_\lambda}_{t,x}\Big[\int_t^{T} e^{-\int_t^s\beta+\mu(u)du}U(c(s))ds+\int_t^T\mu(s)e^{-\int_t^s\beta+\mu(u)du}U\left(X(s)+\frac{p(s)}{\eta(s)}\right)ds\\
&\quad+e^{-\int_t^T\beta+\mu(u)du}U(X(T))\Big]\Big\}.
\end{aligned}
$$

$U(x)$ is a strictly increasing and concave continuous function defined on $[0,+\infty)$, and $\phi(x)$ is a strictly increasing continuous function defined on the range of U. The individual's risk attitude is represented by $U(x)$, and his ambiguity attitude is formed by $\phi(x)$. P_λ is the conditional probability given (λ_1,λ_2). The conditional expectation under the probability measure P_λ is denoted by E^{P_λ}. Under the conditional expectation E^{P_λ}, (λ_1,λ_2) can be regarded as constant. The expectation over λ is denoted by E^{λ}. β is the discount rate, which discounts the utility in time $s>t$ to time t.

We first define the admissible set for the consumption-investment-life insurance policy.

Definition 2.1. *An $\mathbb{R}^4$-valued stochastic process $\mathbf{u}=\{c(t),p(t),\theta_1(t),\theta_2(t)\}_{0\le t\le T}$ is called an admissible strategy process if the following conditions hold:*
(i) It is progressively measurable with respect to the σ-filtration $\{\mathscr{F}_t\}_{0\le t\le T}$.
(ii) It is independent of the value of (λ_1,λ_2).
(iii) $c(t)\ge 0$ for all t a.s.; $X(t)+\frac{p(t)}{\eta(t)}\ge 0$ for all t a.s.; $X(T)\ge 0$ a.s.
(iv) $E[\int_0^t\theta_i(s)^2ds]<+\infty$ for any $0\le t\le T$ and $i=1,2$.
(v) $|J(t,x;\mathbf{u})|<\infty$ for all $t\in[0,T]$.

Set $\mathscr{A}$ as the collection of admissible strategies. Then, we have the admissible strategy $\mathbf{u}=\{c(t),p(t),\theta_1(t),\theta_2(t)\}_{0\le t\le T}$, which satisfies $\mathbf{u}\in\mathscr{A}$.

The strategy is independent of the value of (λ_1,λ_2), which means that the individual does not know the correct value of (λ_1,λ_2) and thus cannot make the decision dependent on this value. However, the individual has subjective information about the distribution of (λ_1,λ_2), so the decision can be dependent on the distribution of (λ_1,λ_2) instead of the value of (λ_1,λ_2) in the non-ambiguity case. It is obvious that $\mathbf{u}=\{0,0,0,0\}_{0\le t\le T}$ is an admissible strategy. Therefore, $\mathcal{A}$ is non-empty.

2.3 The extended HJB equation

As stated previously, the double expectation in the smooth ambiguity model makes the optimization problem time-inconsistent. In a time-inconsistent problem, dynamic programming cannot be applied to obtain a time-consistent optimal strategy. We must derive the equilibrium strategy for the time-inconsistent problem via Nash equilibrium theory. To obtain an equilibrium strategy, we follow Björk et al.[1] and Guan et al.[9] to derive the extended HJB equation for this problem.

Definition 2.2. *An admissible strategy* $\mathbf{u}^* = \{c^*(t), p^*(t), \theta_1^*(t), \theta_2^*(t)\}_{0\le t\le T}$ *is called an equilibrium strategy policy if there exists a map* $F = (F_1, F_2, F_3, F_4) : [0, T] \times \mathbb{R} \to [0, +\infty) \times \mathbb{R}^3$ *for any* t, x,

$$\lim_{h\to 0^+} \inf \frac{J(t, x; \mathbf{u}^*) - J(t, x; \mathbf{u}_h)}{h} \ge 0,$$

where
(1)$c^*(s) = F_1(s, X^*(s))_{t\le s\le T}$, $p^*(s) = F_2(s, X^*(s))_{t\le s\le T}$, $\theta_1^*(s) = F_3(s, X^*(s))_{t\le s\le T}$, $\theta_2^*(s) = F_4(s, X^*(s))_{t\le s\le T}$*;*
(2) $X^*(s)_{t\le s\le T}$ *is the wealth under the equilibrium strategy, which satisfies*

$$dX^*(s) = [rX^*(s)+\theta_1^*(s)\sigma_1\lambda_1+\theta_2^*(s)\sigma_2\lambda_2-c^*(s)-p^*(s)+i(s)]dt+\theta_1^*(s)\sigma_1 dB_1(s)+\theta_2^*(s)\sigma_2 dB_2(s)) \quad X^*(t) = x;$$

(3) $\mathbf{u}_h(s)$ *is another strategy defined by* $\mathbf{u}_h(s) = \begin{cases} \mathbf{u}^*(s) & t+h < s \le T \\ u & t \le s \le t+h \end{cases}$ *and* $u = \{c, p, \theta_1, \theta_2\}_{t\le s\le t+h}$ *is any strategy that makes* $\mathbf{u}_h$ *an admissible strategy.*

We define the value function by $V(t, x) = J(t, x; \mathbf{u}^*)$. To drive the extended HJB equation, we must first define the following variational operator:

$$\begin{aligned} A_\lambda^u f(t, x) &= \lim_{h\to 0^+} \frac{E_{t,x}^{P_\lambda}[f(t+h, X^u(t+h)) - f(t, x)]}{h} \\ &= f_t + f_x(rx + \theta_1\sigma_1\lambda_1 + \theta_2\sigma_2\lambda_2 - c - p + i(t)) + \frac{1}{2} f_{xx}(\theta_1^2\sigma_1^2 + \theta_2^2\sigma_2^2). \end{aligned}$$

The following theorem presents the conditions for the equilibrium value function and strategy when the solution is classic.

Theorem 2.1. *Assume* ϕ *is continuously differentiable. If there exists a class of functions* $g(t, x, \lambda_1, \lambda_2) \in \mathbf{C}^{1,2,0,0}([0, T], \mathbb{R}, \Lambda)$ *satisfying the following conditions for* $\forall (t, x) \in ([0, T], \mathbb{R})$,

$$\sup_{\mathbf{u}\in\mathscr{A}} E^\lambda \left[\phi'[g(t, x, \lambda_1, \lambda_2)][U(c) + \mu(t)U\left(x + \frac{p}{\eta(t)}\right) - (\beta + \mu(t))g(t, x, \lambda_1, \lambda_2) + A_\lambda^{\mathbf{u}} g(t, x, \lambda_1, \lambda_2)]\right] = 0, \quad (2.1)$$

$$U(c^*(t)) + \mu(t)U\left(x + \frac{p^*(t)}{\eta(t)}\right) - (\beta + \mu(t))g(t, x, \lambda_1, \lambda_2) + A_\lambda^{\mathbf{u}^*} g(t, x, \lambda_1, \lambda_2) = 0, \quad (2.2)$$

$$g(T, x, \lambda_1, \lambda_2) = U(x), \quad (2.3)$$

where

$$\mathbf{u}^* \triangleq \arg\sup_{\mathbf{u}\in\mathscr{A}} E^\lambda \left[\phi'[g(t, x, \lambda_1, \lambda_2)][U(c) + \mu(t)U\left(x + \frac{p}{\eta(t)}\right) - (\beta + \mu(t))g + A_\lambda^{\mathbf{u}} g(t, x, \lambda_1, \lambda_2)]\right], \quad (2.4)$$

the value function satisfies $V(t, x) = E^\lambda[\phi(g(t, x, \lambda_1, \lambda_2))]$ *and* $\mathbf{u}^*$ *is the equilibrium strategy.*

This theorem is proved in Appendix 1. Let

$$\begin{aligned} g(t, x, \lambda_1, \lambda_2) =& E_{t,x}^{P_\lambda}\left[\int_t^T e^{-\int_t^s \beta+\mu(u)du} U(c^*(s))ds + \int_t^T \mu(s)e^{-\int_t^s \beta+\mu(u)du} U\left(X^*(s) + \frac{p^*(s)}{\eta(s)}\right) ds \right. \\ & \left. + e^{-\int_t^T \beta+\mu(u)du} U(X^*(T))\right], \end{aligned} \quad (2.5)$$

and from Appendix 1, we know that $g(t, x, \lambda_1, \lambda_2)$ satisfies the condition in Theorem 2.1.

If λ is a constant, Eq. (2.1) becomes

$$\sup_{\mathbf{u}\in\mathscr{A}} \phi'[g(t,x,\lambda_1,\lambda_2)][U(c)+\mu(t)U\left(X(t)+\frac{p}{\eta(t)}\right)-(\beta+\mu(t))g+A^{\mathbf{u}}_{\lambda}g(t,x,\lambda_1,\lambda_2)]=0,$$

and we have $V(t,x)=\phi(g(t,x,\lambda_1,\lambda_2))$.

As $\phi(x)$ is a strictly increasing function, $\phi'(x)>0$. Eq.(2.1) reduces to

$$\sup_{\mathbf{u}\in\mathscr{A}}[U(c)+\mu(t)U\left(X(t)+\frac{p}{\eta(t)}\right)-(\beta+\mu(t))g+A^{\mathbf{u}}_{\lambda}g(t,x,\lambda_1,\lambda_2)]=0,$$

which is the HJB equation for $\max_{\mathbf{u}\in\mathscr{A}} g(t,x,\lambda_1,\lambda_2)$.

Because $V(t,x)=\phi(g(t,x,\lambda_1,\lambda_2))$ increases with $g(t,x,\lambda_1,\lambda_2)$, the optimal control problem $\max_{\mathbf{u}\in\mathscr{A}} V(t,x)$ is equivalent to $\max_{\mathbf{u}\in\mathscr{A}} g(t,x,\lambda_1,\lambda_2)$. Thus, the equilibrium strategy in this case is exactly the same as the optimal control strategy in Richard[22].

3 Example

In this section, we set $U(x)=\frac{x^{1-\gamma}}{1-\gamma},\gamma>1$ to describe a risk-averse decision maker. The larger γ is, the more risk averse the individual is. As we limit $\gamma>1$, we have $U(x)<0$. Hence, we set $\phi(x)\triangleq\frac{1-(-x)^{\alpha}}{\alpha},\alpha\neq 0$ to describe the ambiguity attitude. Specifically, when we restrict $\alpha\to 0$, we find that $\phi(x)\to -ln(-x)$. Thus, we set $\phi(x)\triangleq -ln(-x)$ when $\alpha=0$. Moreover, when $\alpha=1$, $\phi(x)=1+x$, which means the decision maker is ambiguity neutral. The individual shows more ambiguity aversion with increasing α.

3.1 Equilibrium strategy

According to the first-order conditions in extended HJB equation (2.1), we have

$$\begin{cases} c(t)=\left[\dfrac{E^{\lambda}[\phi'(g)g_x]}{E^{\lambda}[\phi'(g)]}\right]^{-\frac{1}{\gamma}},\\ p(t)=\eta(t)\left\{\left[\dfrac{E^{\lambda}[\phi'(g)g_x]\eta(t)}{E^{\lambda}[\phi'(g)]\mu(t)}\right]^{-\frac{1}{\gamma}}-x\right\},\\ \theta_1(t)=-\dfrac{E^{\lambda}[\phi'(g)g_x\lambda_1]}{E^{\lambda}[\phi'(g)g_{xx}]\sigma_1},\\ \theta_2(t)=-\dfrac{E^{\lambda}[\phi'(g)g_x\lambda_2]}{E^{\lambda}[\phi'(g)g_{xx}]\sigma_2}. \end{cases} \tag{3.1}$$

Assume $g(t,x,\lambda_1,\lambda_2)=A(t,\lambda_1,\lambda_2)\frac{[x+B(t)]^{1-\gamma}}{1-\gamma}$, where $A(T,\lambda_1,\lambda_2)=1$, $B(T)=0$. Then, we have

$$\begin{cases} g_x=A(t,\lambda_1,\lambda_2)[x+B(t)]^{-\gamma},\\ g_{xx}=-\gamma A(t,\lambda_1,\lambda_2)[x+B(t)]^{-\gamma-1},\\ g_t=A_t\dfrac{[x+B(t)]^{1-\gamma}}{1-\gamma}+A[x+B(t)]^{-\gamma}B'(t). \end{cases} \tag{3.2}$$

We can obtain the equilibrium strategy by substituting (3.2) into (3.1) as follows:

$$
\begin{cases}
c^*(t) = [x+B(t)]\Big[\dfrac{E^\lambda[A(t,\lambda_1,\lambda_2)^\alpha]}{E^\lambda[A(t,\lambda_1,\lambda_2)^{\alpha-1}]}\Big]^{-\frac{1}{\gamma}},\\
p^*(t) = \eta(t)\Big\{[x+B(t)]\Big[\dfrac{E^\lambda[A(t,\lambda_1,\lambda_2)^\alpha]\eta(t)}{E^\lambda[A(t,\lambda_1,\lambda_2)^{\alpha-1}]\mu(t)}\Big]^{-\frac{1}{\gamma}} - x\Big\},\\
\theta_1^*(t) = \dfrac{[x+B(t)]E^\lambda[A(t,\lambda_1,\lambda_2)^\alpha\lambda_1]}{E^\lambda[A(t,\lambda_1,\lambda_2)^\alpha]\gamma\sigma_1},\\
\theta_2^*(t) = \dfrac{[x+B(t)]E^\lambda[A(t,\lambda_1,\lambda_2)^\alpha\lambda_2]}{E^\lambda[A(t,\lambda_1,\lambda_2)^\alpha]\gamma\sigma_2}.
\end{cases}
$$

To simplify the expression, we define

$$
\begin{cases}
m(t) = \Big[\dfrac{E^\lambda[A(t,\lambda_1,\lambda_2)^\alpha]}{E^\lambda[A(t,\lambda_1,\lambda_2)^{\alpha-1}]}\Big]^{-\frac{1}{\gamma}},\\
q_1(t) = \dfrac{E^\lambda[A(t,\lambda_1,\lambda_2)^\alpha\lambda_1]}{E^\lambda[A(t,\lambda_1,\lambda_2)^\alpha]},\\
q_2(t) = \dfrac{E^\lambda[A(t,\lambda_1,\lambda_2)^\alpha\lambda_2]}{E^\lambda[A(t,\lambda_1,\lambda_2)^\alpha]}.
\end{cases}
\tag{3.3}
$$

Then, the equilibrium strategy can be expressed as

$$
\begin{cases}
c^*(t) = [x+B(t)]m(t),\\
p^*(t) = \eta(t)\{m(t)[\dfrac{\eta(t)}{\mu(t)}]^{-\frac{1}{\gamma}}[x+B(t)] - x\},\\
\theta_1^*(t) = \dfrac{[x+B(t)]q_1(t)}{\gamma\sigma_1},\\
\theta_2^*(t) = \dfrac{[x+B(t)]q_2(t)}{\gamma\sigma_2}.
\end{cases}
\tag{3.4}
$$

Substituting (3.4) into (2.2), we obtain the ODE for $A(t,\lambda_1,\lambda_2)$ and $B(t)$ as

$$
A(t,\lambda_1,\lambda_2)(1-\gamma)\Big\{r + \frac{q_1(t)\lambda_1}{\gamma} + \frac{q_2(t)\lambda_2}{\gamma} - m(t) - \eta(t)m(t)[\frac{\eta(t)}{\mu(t)}]^{-\frac{1}{\gamma}} + \eta(t) - \frac{1}{2}(\frac{q_1^2(t)}{\gamma} + \frac{q_2^2(t)}{\gamma}) - \frac{\beta+\mu(t)}{1-\gamma}\Big\} +
$$
$$
m(t)^{1-\gamma} + \mu(t)m(t)^{1-\gamma}[\frac{\eta(t)}{\mu(t)}]^{\frac{\gamma-1}{\gamma}} + A_t = 0, \quad A(T,\lambda_1,\lambda_2) = 1,
$$

and

$$
-[r+\eta(t)]B(t) + B'(t) + i(t) = 0, \quad B(T) = 0.
$$

Thus, we have expressions for $A(t,\lambda_1,\lambda_2)$ and $B(t)$ that satisfy

$$
B(t) = \int_t^T i(y)e^{-\int_t^y r+\eta(u)du}dy,
$$

and

$$A(t,\lambda_1,\lambda_2)=$$

$$\int_t^T e^{-\int_t^y(\gamma-1)\left\{r+\frac{q_1(v)\lambda_1}{\gamma}+\frac{q_2(v)\lambda_2}{\gamma}-m(v)-\eta(v)m(v)[\frac{\eta(v)}{\mu(v)}]^{-\frac{1}{\gamma}}+\eta(v)-\frac{1}{2}(\frac{q_1^2(v)}{\gamma}+\frac{q_2^2(v)}{\gamma})-\frac{\beta+\mu(v)}{1-\gamma}\right\}dv}[m(y)^{1-\gamma}+\mu(y)m(y)^{1-\gamma}\left(\frac{\eta(y)}{\mu(y)}\right)^{\frac{\gamma-1}{\gamma}}]dy$$
$$+e^{-\int_t^T(\gamma-1)\left\{r+\frac{q_1(v)\lambda_1}{\gamma}+\frac{q_2(v)\lambda_2}{\gamma}-m(v)-\eta(v)m(v)[\frac{\eta(v)}{\mu(v)}]^{-\frac{1}{\gamma}}+\eta(v)-\frac{1}{2}(\frac{q_1^2(v)}{\gamma}+\frac{q_2^2(v)}{\gamma})-\frac{\beta+\mu(v)}{1-\gamma}\right\}dv}. \tag{3.5}$$

3.2 Economic analysis of the equilibrium strategy

The equilibrium strategy shows that the individual will make consumption, life insurance purchase and investment decisions based on $x+B(t)$. $B(t)$ is the discount accumulated value of future income, which is called "human capital" in many studies (e.g. [24], [3]). To distinguish this value from the current wealth x, we call $x+B(t)$ the total capital at time t. Following the equilibrium strategy, we obtain

$$\begin{aligned}
&d[X^*(t)+B(t)]\\
=&dX^*(t)+dB(t)\\
=&[X^*(t)+B(t)]\big[[r+\eta(t)+\frac{q_1(t)\lambda_1}{\gamma}+\frac{q_2(t)\lambda_2}{\gamma}-m(t)-\eta(t)m(t)[\frac{\eta(t)}{\mu(t)}]^{-\frac{1}{\gamma}}]dt+\frac{q_1(t)}{\gamma}dB_1(t)+\frac{q_2(t)}{\gamma}dB_2(t)\big].
\end{aligned}$$

Thus, given $X^*(0)>0$, we have $X^*(t)+B(t)>0$ for all t. From (2.5), we know that $g(t,x,\lambda_1,\lambda_2)<0$, which means $A(t,\lambda_1,\lambda_2)>0$. By definition of $m(t)$, $q_1(t)$ and $q_2(t)$ from (3.3), we know these values are all positive. Therefore, the equilibrium consumption $c^*(t)$, legacy $X^*(t)+\frac{p^*(t)}{\eta(t)}$ and investment in risky assets $\theta_1^*(t)$ and $\theta_2^*(t)$ are all positive. Furthermore, $X^*(T)=X^*(T)+B(T)>0$.

The proportion of consumption to total capital is determined by $m(t)$. According to (3.3), $m(t)$ depends on the distributions of λ_1 and λ_2 rather than the exact values of λ_1 and λ_2. Moreover, $m(t)$ considers all possibilities of λ_1 and λ_2, in contrast to the MEU model which considers only the extreme value. In fact, Klibanoff[15] showed that the result of the MEU model is actually a special case of the smooth ambiguity model when the ambiguity aversion tends to infinity, i.e., $\alpha\to+\infty$. Therefore, only an individual with extreme ambiguity aversion will make the decision by considering how to maximize his utility in the worst case. However, such behaviour is a special case that is not common in the real world, and an individual is unlikely to implement this strategy.

From (3.4), the legacy that the individual will leave if he or she dies at time t is $x+\frac{p(t)}{\eta(t)}=m(t)[\frac{\eta(t)}{\mu(t)}]^{-\frac{1}{\gamma}}[x+B(t)]$. Thus, the proportion of legacy to total capital depends on both $m(t)$ and the loading for life insurance $\frac{\eta(t)}{\mu(t)}$. Generally, life insurance companies charge a loading for their product for the purpose of safety and profit, which means $\eta(t)\geq\mu(t)$. A higher loading means that the life insurance product is more expensive, which leads to the purchase of less life insurance.

In the non-ambiguity case, the investments for the two risky assets are $\theta_1^*(t)=\frac{x+B(t)}{\gamma}\frac{\lambda_1}{\sigma_1}$ and $\theta_2^*(t)=\frac{x+B(t)}{\gamma}\frac{\lambda_2}{\sigma_2}$, which implies that the investor can invest in a market fund consisting of the two risky assets. The share of the two risky assets depends on their market performance $\lambda_1,\sigma_1,\lambda_2,\sigma_2$. This result can be easily extended to n risky assets and can be summarized according to the Tobin–Markowitz separation theorem. In general, the Tobin–Markowitz separation theorem states that it is equivalent for a rational individual to invest in n risky assets or a market fund consisting of an appropriate mix of those risky assets.

However, if the individual is uncertain about the right value of the market price for those risky assets, the market fund no longer works. The investments are then $\theta_1^*(t) = \frac{[x+B(t)]}{\gamma}\frac{q_1(t)}{\sigma_1}$ and $\theta_2^*(t) = \frac{[x+B(t)]}{\gamma}\frac{q_2(t)}{\sigma_2}$. Unlike the non-ambiguity case, $q_1(t)$ and $q_2(t)$ depend on not only the market performance of the two risky assets (λ and σ) but also the characteristics of the decision maker, such as the risk attitude (γ), ambiguity attitude (α) and mortality rate ($\mu(t)$). Therefore, if the investor wants to select a fund to replace the risky assets, he needs a custom-built fund. In other words, a manager who serves several clients should provide investment advice depending on the conditions of each client.

Because of the complexity of the equilibrium strategy, we cannot identify the effects of risk aversion attitude and ambiguity aversion attitude from the above mathematical expression. Therefore, we discuss the impact of γ and α by means of a numerical simulation.

4 Numerical result

In this section, we apply numerical simulation to discuss how the risk attitude and ambiguity attitude of the individual affect the equilibrium strategy. And we analyse the influence of market performance of risk assets on the equilibrium investment strategy. Unless otherwise stated, the basic values of the parameters we adopt for the simulation are defined in Table 1.

Table 1: Values of the parameters

Text interpretation	Symbol	Value
risk-free interest rate	r	0.03
discount rate	β	0.04
mortality hazard function	$\mu(t)$	$\frac{e^{\frac{t-86.3}{9.5}}}{9.5}$
premium rate	$\eta(t)$	$1.1\mu(t)$
income	$i(t)$	$50000e^{0.04t}$
retirement time	T	40
volatility of risky assets	σ_1,σ_2	2.5%
distribution of λ_1	$N(u_1, \Sigma_1)$	$N(0.1, 0.3)$
distribution of λ_2	$N(u_2, \Sigma_2)$	$N(0.1, 0.5)$
initial wealth	$X(0)$	20000
risk aversion	γ	2
ambiguity aversion	α	2

The basic values of the parameters represent an ambiguity-averse and risk-averse individual. We change the value of γ or α in the following sections to observe how the equilibrium strategy changes with changes in risk or ambiguity attitude. To analyse the influence of market performance of risk assets on the investment strategy, we will change the volatility σ_1, σ_2 and the value of expectation of market prices μ_1, μ_2.

4.1 Effect of ambiguity aversion

In this section, we take $\alpha = 0, 1, 2$ and set $\gamma = 2$ to observe the effect of ambiguity attitude on the consumption, life insurance and investment decision. As stated previously, $\alpha = 0, 1, 2$ represent ambiguity loving, ambiguity neutrality and ambiguity aversion, respectively.

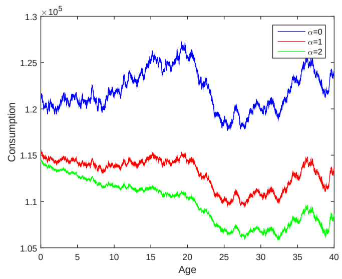

Figure 1: Effect of α on consumption

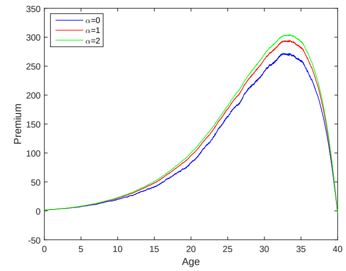

Figure 2: Effect of α on premium

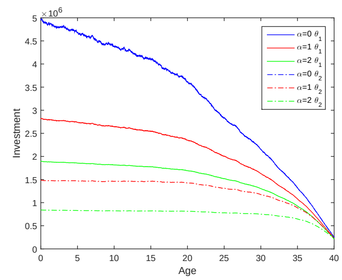

Figure 3: Effect of α on investment

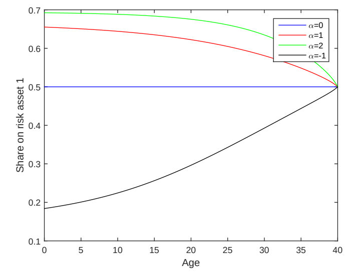

Figure 4: Effect of α on investment share

Figure 1 shows the consumption for the three types of individuals. Apparently, consumption decreases as α increases during the whole decision period. Thus, an individual who is more ambiguity averse will spend less to consume throughout his whole life.

Figure 2 shows the life insurance purchase. The spending on life insurance increases with time at first and then decreases during the last approximately 7 years. Moreover, the premium increases as α increases, which means that if the decision maker is more averse to ambiguity, he will purchase more life insurance to prevent future uncertainty.

Figure 3 illustrates the risky-asset investment strategy for the three types of decision makers. The investments in both risky assets decrease over time because total wealth decreases. Furthermore, the investment decreases as α increases. Hence, as the ambiguity aversion tends to the extreme ($\alpha \to +\infty$), the investment will tend to zero, which is the optimal investment in the worst case, i.e., the strategy stated in the MEU model.

We assume that the two risky assets have the same volatilities ($\sigma_1 = \sigma_1$) and that their market prices λ_1 and λ_2 have the same expectation ($u_1 = u_2$). The only difference between the two risky assets is that λ_2 is more widely distributed than λ_1, which means $\Sigma_2 > \Sigma_1$. Therefore, the individual will feel that risky asset 2 has more uncertainty than risky asset 1. Figure 3 shows that both ambiguity-averse and ambiguity-neutral decision makers prefer risky asset 1 to risky asset 2. By contrast, in the case $\alpha = 0$, $\theta_1^*(t)$ and $\theta_2^*(t)$ coincide. As stated in (3.3), we obtain $q_1(t) = E^{\lambda}[\lambda_1]$ and $q_2(t) = E^{\lambda}[\lambda_2]$ when $\alpha = 0$. It is reasonable that the spread of λ_1 and λ_2 has no effect on investment decisions $\theta_1^*(t)$ and $\theta_2^*(t)$ when $\alpha = 0$ from the equilibrium strategy (3.4). Because we set $\mu_1 = \mu_2$, we have $\theta_1^*(t) = \theta_2^*(t)$. Unlike the risk-lover, who prefers risk assets with higher volatility, the ambiguity-loving individual ($\alpha = 0$) does not show a preference for the asset with greater ambiguity (Σ) and the ambiguity-neutral individual ($\alpha = 1$) prefers the asset with less ambiguity.

Figure 4 shows the share of risky asset 1 in the total risky asset investment $(\theta_1(t)/(\theta_1(t) + \theta_2(t)))$. We also include the results for $\alpha = -1$ in figure 4. Thus, figure 4 shows that an individual prefers the asset with greater ambiguity when $\alpha < 0$ and the asses with less ambiguity when $\alpha > 0$. The individual appears to be ambiguity neutral when $\alpha = 0$ instead of $\alpha = 1$. This puzzle is explained in [14], which stated that the decision maker's preference for ambiguity is described by the following utility function:

$$v^{-1}\Big(\int_{\Delta} v\Big(U^{-1}\Big(\int_{S} U(f)d\pi\Big)\Big)d\Pi\Big),$$

where v is an increasing function. When we define $\phi \triangleq v \circ U^{-1}$, the result is equivalent to the smooth ambiguity model. In other words, $v = \phi \circ U$, so the individual shows ambiguity aversion when v is a concave transformation of U. Therefore, an ambiguity-averse individual is more averse to ambiguity than to risk. Thus, even an ambiguity-neutral person will prefer assets with less ambiguity because the individual will believe that greater ambiguity represents higher risk.

From figure 4, we can also see that the share of risky asset 1 is decreasing with time and tends to 0.5 when $\alpha > 0$ because the two risky assets' market prices have the same expectations. Moreover, if $\alpha < 0$, the share of risky asset 1 increases with time to 0.5. A more ambiguity-averse individual will put more weight in risky asset 1.

In the next section, we can see that the investment fund's structure will also change in the same way with a change in risk aversion γ. Thus, the structure of the risky investment depends on the risk attitude, ambiguity attitude, age and other characteristics of the investor. A market fund whose structure depends on only the market performance of each risky asset in Tobin–Markowitz separation theorem cannot satisfy investors with a different risk attitude, ambiguity attitude and age: an investment manager should provide investment advice according to the characteristics of each client.

4.2 Effect of risk aversion

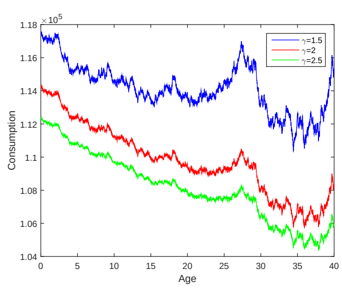

Figure 5: Effect of γ on consumption

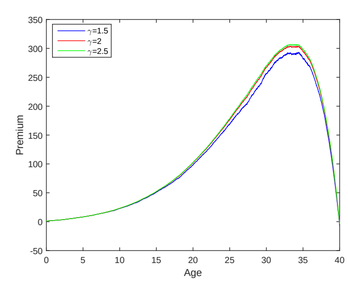

Figure 6: Effect of γ on premium

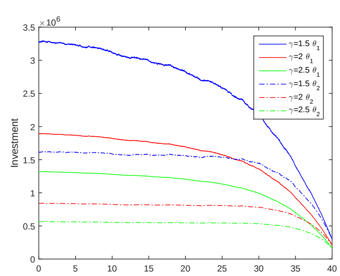

Figure 7: Effect of γ on investment

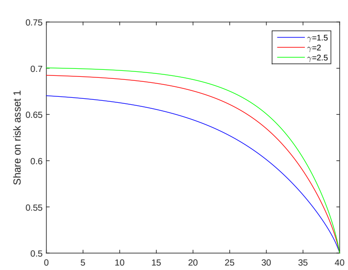

Figure 8: Effect of γ on investment share

Comparing Figure 5 to 8 with the results of the last section, we find that the equilibrium consumption, life insurance purchase and investment show the same trend when we increase γ as we increase α; that is, risk aversion and ambiguity aversion work in the same direction. Thus, if we ignore ambiguity aversion, risk aversion would be overestimated. This result coincides with that of Klibanoff[15] and indicates that the smooth ambiguity model can explain the equity premium puzzle of [21] to some extent.

4.3 Effect of market performance on risky investment

We have discussed the effect of spread (Σ) in the distribution of market price. Now, we are interested in the effect of other market performance indicators (σ and μ) of the risky assets in the investment strategy.

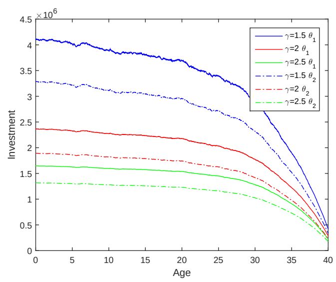

Figure 9: Effects of σ and γ on risk investment

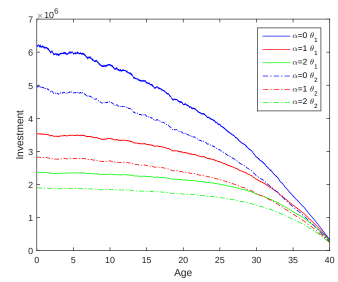

Figure 10: Effects of σ and α on risk investment

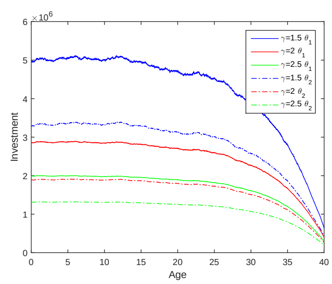

Figure 11: Effects of μ and γ on risk investment

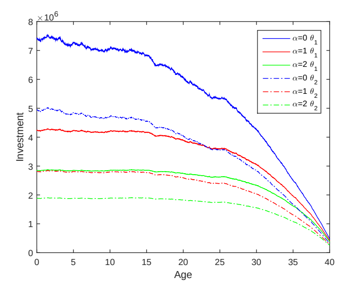

Figure 12: Effects of μ and α on risk investment

In figure 9 and figure 10, we assign the same distribution $N(0.1, 0.3)$ to λ_1 and λ_2 and set $\sigma_1 = 2\%$ and $\sigma_2 = 2.5\%$. Thus, risky asset 2 is more risky than risky asset 1. Figure 9 shows that the investment in risky asset 1 is larger than that in risky asset 2 and both investments decrease as γ increases. It is reasonable that a more risk-adverse individual will prefer less risk assets more. Figure 10 shows that a risk-averse individual prefers the less risky asset regardless of the ambiguity attitude.

From (2.1), we know that if λ_1 and λ_2 have the same distributions, the ratio $\frac{\theta_1^*(t)}{\theta_2^*(t)} = \frac{\sigma_2}{\sigma_1}$ is a constant which will not change with the value of α and γ

To determine the effect of μ, we set $\mu_1 = 0.15, \mu_2 = 0.1$, $\Sigma_1 = \Sigma = 0.3$ and $\sigma_1 = \sigma_2 = 2.5\%$ in figure 11 and 12. In this scenario, risky asset 1 has a higher return than that of risky asset 2 with no additional risk or ambiguity. Therefore, it would be reasonable for the investor to prefer risky asset 1. Indeed, figure 11 and 12 shows that the investor tends to invest more in risky asset 1.

5 Conclusion

In this paper, we study the portfolio management problem in Richard[24] under smooth ambiguity model. Different from many previous works, we assume that the values of market prices λ_1 and λ_2 for risky assets are not known by the individual. Instead, the individual knows the distribution of the market price. The individual has an uncertain lifetime and makes his decision before retirement time T or early death. At time t, the individual will consume $c(t)$, buy life insurance $p(t)$ and invest in two risky assets $\theta_1(t)$ and $\theta_2(t)$ while receiving income $i(t)$. The decision maker will evaluate his decision using the smooth ambiguity model, where we use $U(x)$ to describe the risk attitude and $\phi(x)$ to describe the ambiguity attitude. Because of the double expectation in the smooth ambiguity model, the portfolio management problem is time-inconsistent. Following the method in Guan et al.[9], we derive the extended HJB equation for the equilibrium value function and equilibrium strategy. Then, we solve the equilibrium strategy in the power-power (U, ϕ) case by setting $U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \gamma > 1$ and $\phi(x) = \frac{1-(-x)^{\alpha}}{\alpha}, \alpha \neq 0$. Specifically, we set $\phi(x) = -ln(-x)$ when $\alpha = 0$. Thus, the individual is ambiguity averse if $\alpha > 1$, ambiguity neutral if $\alpha = 1$ and ambiguity loving if $\alpha < 1$.

The equilibrium strategy is based on the total capital of the individual, which consists of current wealth and human capital. Instead of depending on the market price in the non-ambiguity case or the extreme value in the MEU model, the equilibrium strategy depends on the distributions of λ_1 and λ_2. Therefore, the individual can consider all possible values of market price based on the available information. Furthermore, we find that the Tobin-Markowitz separation theorem is not applicable in this scenario: individuals with different risk or ambiguity attitudes will invest in different funds consisting of the available risky assets instead of the market fund, as in the Tobin-Markowitz separation theorem.

We analyse the effect of risk aversion degree γ and ambiguity attitude α on the equilibrium strategy via numerical simulation. The results show that the equilibrium investment decrease with time, whereas the equilibrium life insurance purchase increases at first and then decreases. As the individual become more ambiguity averse or risk averse, he will consume less, buy more life insurance and invest less in risky assets. Additionally, the numerical results support the analysis with respect to the Tobin-Markowitz separation theorem. An individual prefers the asset with less ambiguity when $\alpha > 0$ and the asset with more ambiguity when $\alpha < 0$. As time passes, the share of the asset with less ambiguity in the total investment will decrease if $\alpha > 0$ and increase if $\alpha < 0$. However, when α or γ increases, the investor will place more weight on the asset with less ambiguity. Furthermore, the investor will also be more interested in the risky asset with less volatility and higher expected return.

Appendix 1

In this Appendix, we will prove Theorem 2.1.

Proof. We assume

$$g(t,x,\lambda_1,\lambda_2) = E_{t,x}^{P_\lambda}\left[\int_t^T e^{-\int_t^s \beta+\mu(u)du} U(c^*(s))ds + \int_t^T \mu(s) e^{-\int_t^s \beta+\mu(u)du} U(X^*(s) + \frac{p^*(s)}{\eta(s)})ds + e^{-\int_t^T \beta+\mu(u)du} U(X^*(T))\right].$$

Here, $\mathbf{u}^*(t) = \{c^*(t), p^*(t), \theta_1^*(t), \theta_2^*(t)\}_{0\leq t\leq T}$ is defined as (2.4).

Step 1

We have $g(T,x,\lambda_1,\lambda_2) = E_{T,x}^{P_\lambda}[U(X^*(T))] = U(x)$. Thus, $g(t,x,\lambda_1,\lambda_2)$ satisfies the condition (2.3).

$$
\begin{aligned}
& g(t,x,\lambda_1,\lambda_2)e^{-\int_0^t \beta+\mu(u)du} \\
=& E_{t,x}^{P_\lambda}\Big[\int_t^T e^{-\int_0^s \beta+\mu(u)du}U(c^*(s))ds + \int_t^T \mu(s)e^{-\int_0^s \beta+\mu(u)du}U(X^*(s)+\frac{p^*(s)}{\eta(s)})ds + e^{-\int_0^T \beta+\mu(u)du}U(X^*(T))\Big] \\
=& E_{t,x}^{P_\lambda}\Big[\int_t^{t+h} e^{-\int_0^s \beta+\mu(u)du}U(c^*(s))ds + \int_{t+h}^T e^{-\int_0^s \beta+\mu(u)du}U(c^*(s))ds + \int_t^{t+h} \mu(s)e^{-\int_0^s \beta+\mu(u)du}U(X^*(s)+\frac{p^*(s)}{\eta(s)})ds + \\
& \int_{t+h}^T \mu(s)e^{-\int_0^s \beta+\mu(u)du}U(X^*(s)+\frac{p^*(s)}{\eta(s)})ds + e^{-\int_0^T \beta+\mu(u)du}U(X^*(T))\Big] \\
=& E_{t,x}^{P_\lambda}\Big[\int_{t+h}^T e^{-\int_0^s \beta+\mu(u)du}U(c^*(s))ds + \int_{t+h}^T \mu(s)e^{-\int_0^s \beta+\mu(u)du}U(X^*(s)+\frac{p^*(s)}{\eta(s)})ds + e^{-\int_0^T \beta+\mu(u)du}U(X^*(T))\Big] + \\
& e^{-\int_0^t \beta+\mu(u)du}U(c^*(t))h + \mu(t)e^{-\int_0^t \beta+\mu(u)du}U(X^*(t)+\frac{p^*(t)}{\eta(t)})h \\
=& e^{-\int_0^{t+h} \beta+\mu(u)du}E_{t,x}^{P_\lambda}\Big\{E_{t+h,X^{\mathbf{u}^*}(t+h)}^{P_\lambda}\Big[\Big[\int_{t+h}^T e^{-\int_{t+h}^s \beta+\mu(u)du}U(c^*(s))ds + \int_{t+h}^T \mu(s)e^{-\int_{t+h}^s \beta+\mu(u)du}U(X^*(s)+\frac{p^*(s)}{\eta(s)})ds + \\
& e^{-\int_{t+h}^T \beta+\mu(u)du}U(X^*(T))\Big]\Big\} + e^{-\int_0^t \beta+\mu(u)du}U(c^*(t))h + \mu(t)e^{-\int_0^t \beta+\mu(u)du}U(X^*(t)+\frac{p^*(t)}{\eta(t)})h \\
=& e^{-\int_0^{t+h} \beta+\mu(u)du}E_{t,x}^{P_\lambda}[g(t+h,X^{\mathbf{u}^*}(t+h),\lambda_1,\lambda_2)] + e^{-\int_0^t \beta+\mu(u)du}U(c^*(t))h + \mu(t)e^{-\int_0^t \beta+\mu(u)du}U(X^*(t)+\frac{p^*(t)}{\eta(t)})h.
\end{aligned}
$$

Therefore, we have

$$
g(t,x,\lambda_1,\lambda_2) = e^{-[\beta+\mu(t)]h}E_{t,x}^{P_\lambda}[g(t+h,X^{\mathbf{u}^*}(t+h),\lambda_1,\lambda_2)] + U(c^*(t))h + \mu(t)U(X^*(t)+\frac{p^*(t)}{\eta(t)})h.
$$

According to the definition of $A_\lambda^u f(t,x)$, we have

$$
\begin{aligned}
& A_\lambda^{\mathbf{u}^*}g(t,x,\lambda_1,\lambda_2) \\
=& \lim_{h\to 0}\frac{E_{t,x}^{P_\lambda}[g(t+h,X^{\mathbf{u}^*}(t+h),\lambda_1,\lambda_2) - g(t,x,\lambda_1,\lambda_2)]}{h} \\
=& \lim_{h\to 0}\frac{E_{t,x}^{P_\lambda}[g(t+h,X^{\mathbf{u}^*}(t+h),\lambda_1,\lambda_2)] - e^{-[\beta+\mu(t)]h}E_{t,x}^{P_\lambda}[g(t+h,X^{\mathbf{u}^*}(t+h),\lambda_1,\lambda_2)] - U(c^*(t))h - \mu(t)U(X^*(t)+\frac{p^*(t)}{\eta(t)})h}{h} \\
=& \lim_{h\to 0}\frac{(1-e^{-[\beta+\mu(t)]h})E_{t,x}^{P_\lambda}[g(t+h,X^{\mathbf{u}^*}(t+h),\lambda_1,\lambda_2)] - U(c^*(t))h - \mu(t)U(X^*(t)+\frac{p^*(t)}{\eta(t)})h}{h} \\
=& [\beta+\mu(t)]g(t,x,\lambda_1,\lambda_2) - U(c^*(t)) - \mu(t)U(X^*(t)+\frac{p^*(t)}{\eta(t)})
\end{aligned}
$$

Given $X(t)=x$, $g(t,x,\lambda_1,\lambda_2)$ satisfies

$$
U(c^*(t)) + \mu(t)U(x+\frac{p^*(t)}{\eta(t)}) - (\beta+\mu(t))g + A_\lambda^{\mathbf{u}^*}g(t,x,\lambda_1,\lambda_2) = 0,
$$

so $g(t,x,\lambda_1,\lambda_2)$ satisfies (2.2).

Step 2:

Define $F(t,x) = E^\lambda[\phi(g(t,x,\lambda_1,\lambda_2))] = J(t,x;\mathbf{u}^*)$ and $\mathbf{u}_h(s) = \begin{cases} \mathbf{u}^*(s) & t+h < s \le T \\ u & t \le s \le t+h \end{cases}$, where $u = \{c,p,\theta_1,\theta_2\}$.

First, we obtain

$$
\begin{aligned}
&E^{\lambda}E^{P_{\lambda}}_{t,x}[J(t+h,X^u(t+h);\mathbf{u}^*)]-F(t,x)\\
=&E^{\lambda}\big\{E^{P_{\lambda}}_{t,x}[J(t+h,X^u(t+h);\mathbf{u}^*)]-F(t,x)\big\}\\
=&E^{\lambda}\big\{E^{P_{\lambda}}_{t,x}[F(t+h,X^u(t+h))]-F(t,x)\big\}.
\end{aligned}
\tag{5.1}
$$

Second, we assume there exists a random vector $\hat{\lambda}=(\hat{\lambda}_1,\hat{\lambda}_2)$ that has the same distribution as $\lambda=(\lambda_1,\lambda_2)$ and is independent of λ, so we have

$$
\begin{aligned}
&J(t,x;\mathbf{u}_h)-E^{\lambda}E^{P_{\lambda}}_{t,x}[J(t+h,X^u(t+h);\mathbf{u}^*)]\\
=&E^{\lambda}\big\{J(t,x;\mathbf{u}_h)-E^{P_{\lambda}}_{t,x}[J(t+h,X^u(t+h);\mathbf{u}^*)]\big\}\\
=&E^{\lambda}\Big\{E^{\hat{\lambda}}\big\{\phi E^{P_{\hat{\lambda}}}_{t,x}\big[\int_t^{t+h}e^{-\int_t^s\beta+\mu(u)du}U(c)ds+\int_{t+h}^{T}e^{-\int_t^s\beta+\mu(u)du}U(c^*(s))ds+\int_t^{t+h}\mu(s)e^{-\int_t^s\beta+\mu(u)du}U(X(s)+\frac{p}{\eta(s)})ds+\\
&\int_{t+h}^{T}\mu(s)e^{-\int_t^s\beta+\mu(u)du}U(X(s)+\frac{p^*(s)}{\eta(s)})ds+e^{-\int_t^T\beta+\mu(u)du}U(X^{\mathbf{u}_h}(T))\big]\big\}-E^{P_{\lambda}}_{t,x}[E^{\hat{\lambda}}[\phi(g(t+h,X^u(t+h),\hat{\lambda}_1,\hat{\lambda}_2))]]\Big\}\\
=&E^{\lambda}\Big\{E^{\hat{\lambda}}\big\{\phi\big[U(c)h+\mu(t)U(X(t)+\frac{p}{\eta(t)})h+e^{-\int_t^{t+h}\beta+\mu(u)du}E^{P_{\hat{\lambda}}}_{t,x}E^{P_{\hat{\lambda}}}_{t+h,X^u(t+h)}[\int_{t+h}^{T}e^{-\int_{t+h}^s\beta+\mu(u)du}U(c^*(s))ds+\\
&\int_{t+h}^{T}\mu(s)e^{-\int_{t+h}^s\beta+\mu(u)du}U(X(s)+\frac{p^*(s)}{\eta(s)})ds+e^{-\int_{t+h}^T\beta+\mu(u)du}U(X^{\mathbf{u}^*}(T))]\big]\big\}-E^{P_{\lambda}}_{t,x}[E^{\hat{\lambda}}[\phi(g(t+h,X^u(t+h),\hat{\lambda}_1,\hat{\lambda}_2))]]\Big\}\\
=&E^{\lambda}\Big\{E^{\hat{\lambda}}\big\{\phi\big[U(c)h+\mu(t)U(X(t)+\frac{p}{\eta(t)})h+e^{-\int_t^{t+h}\beta+\mu(u)du}E^{P_{\hat{\lambda}}}_{t,x}[g(t+h,X^u(t+h),\hat{\lambda}_1,\hat{\lambda}_2)]\big]\big\}-\\
&E^{\hat{\lambda}}[E^{P_{\lambda}}_{t,x}[\phi(g(t+h,X^u(t+h),\hat{\lambda}_1,\hat{\lambda}_2))]]\Big\}\\
=&E^{\lambda}E^{\hat{\lambda}}\Big\{\phi\big[U(c)h+\mu(t)U(X(t)+\frac{p}{\eta(t)})h+e^{-\int_t^{t+h}\beta+\mu(u)du}E^{P_{\hat{\lambda}}}_{t,x}[g(t+h,X^u(t+h),\hat{\lambda}_1,\hat{\lambda}_2)]\big]-\\
&E^{P_{\lambda}}_{t,x}[\phi(g(t+h,X^u(t+h),\hat{\lambda}_1,\hat{\lambda}_2))]\Big\}\\
=&E^{\lambda}E^{\hat{\lambda}}\Big\{\phi\big[U(c)h+\mu(t)U(X(t)+\frac{p}{\eta(t)})h+e^{-\int_t^{t+h}\beta+\mu(u)du}E^{P_{\hat{\lambda}}}_{t,x}[g(t+h,X^u(t+h),\hat{\lambda}_1,\hat{\lambda}_2)]\big]-\phi[g(t,x,\hat{\lambda}_1,\hat{\lambda}_2)]+\\
&\phi[g(t,x,\hat{\lambda}_1,\hat{\lambda}_2)]-E^{P_{\lambda}}_{t,x}[\phi(g(t+h,X^u(t+h),\hat{\lambda}_1,\hat{\lambda}_2))]\Big\}.
\end{aligned}
\tag{5.2}
$$

Combining (5.1) and (5.2), we obtain

$$
\begin{aligned}
&\lim_{h\to 0}\frac{J(t,x;\mathbf{u}_h)-J(t,x;\mathbf{u}^*)}{h}\\
=&\lim_{h\to 0}\frac{E^{\lambda}E^{P_\lambda}_{t,x}[J(t+h,X^u(t+h);\mathbf{u}^*)]-F(t,x)+J(t,x;\mathbf{u}_h)-E^{\lambda}E^{P_\lambda}_{t,x}[J(t+h,X^u(t+h);\mathbf{u}^*)]}{h}\\
=&\lim_{h\to 0}\frac{E^{\lambda}E^{P_\lambda}_{t,x}[J(t+h,X^u(t+h);\mathbf{u}^*)]-F(t,x)}{h}+\frac{J(t,x;\mathbf{u}_h)-E^{\lambda}E^{P_\lambda}_{t,x}[J(t+h,X^u(t+h);\mathbf{u}^*)]}{h}\\
=&\lim_{h\to 0}\frac{E^{\lambda}\{E^{P_\lambda}_{t,x}[F(t+h,X^u(t+h))]-F(t,x)\}}{h}-\lim_{h\to 0}\frac{E^{\lambda}E^{\hat\lambda}\Big\{E^{P_\lambda}_{t,x}[\phi(g(t+h,X^u(t+h),\hat\lambda_1,\hat\lambda_2))]-\phi[g(t,x,\hat\lambda_1,\hat\lambda_2)]\Big\}}{h}+\\
&\lim_{h\to 0}\frac{E^{\lambda}E^{\hat\lambda}\Big\{\phi\big[U(c)h+\mu(t)U(X(t)+\frac{p}{\eta(t)})h+e^{-\int_t^{t+h}\beta+\mu(u)du}E^{P_{\hat\lambda}}_{t,x}[g(t+h,X^u(t+h),\hat\lambda_1,\hat\lambda_2)]\big]-\phi[g(t,x,\hat\lambda_1,\hat\lambda_2)]\Big\}}{h}\\
=&E^{\lambda}[A^u_{\lambda}F(t,x)]-E^{\lambda}E^{\hat\lambda}\{A^u_{\lambda}\phi[g(t,x,\hat\lambda_1,\hat\lambda_2)]\}+\\
&\lim_{h\to 0}\frac{E^{\lambda}E^{\hat\lambda}\Big\{\phi\big[U(c)h+\mu(t)U(X(t)+\frac{p}{\eta(t)})h+[1-(\beta+\mu(t))h]E^{P_{\hat\lambda}}_{t,x}[g(t+h,X^u(t+h),\hat\lambda_1,\hat\lambda_2)]\big]-\phi[g(t,x,\hat\lambda_1,\hat\lambda_2)]\Big\}}{h}\\
=&E^{\lambda}[A^u_{\lambda}F(t,x)]-E^{\lambda}E^{\hat\lambda}\{A^u_{\lambda}\phi[g(t,x,\hat\lambda_1,\hat\lambda_2)]\}+\lim_{h\to 0}\frac{E^{\lambda}E^{\hat\lambda}\{\phi\big[E^{P_{\hat\lambda}}_{t,x}[g(t+h,X^u(t+h),\hat\lambda_1,\hat\lambda_2)]\big]-\phi[g(t,x,\hat\lambda_1,\hat\lambda_2)]\}}{h}+\\
&E^{\lambda}E^{\hat\lambda}\{\phi'\big[g(t,x,\hat\lambda_1,\hat\lambda_2)\big][U(c)+\mu(t)U(X(t)+\frac{p}{\eta(t)})-(\beta+\mu(t))g(t,x,\hat\lambda_1,\hat\lambda_2)]\}\\
=&E^{\lambda}[A^u_{\lambda}F(t,x)]-E^{\lambda}E^{\hat\lambda}\{A^u_{\lambda}\phi[g(t,x,\hat\lambda_1,\hat\lambda_2)]\}+\lim_{h\to 0}\frac{E^{\lambda}E^{\hat\lambda}\{\phi\big[A^u_{\hat\lambda}g(t,x,\hat\lambda_1,\hat\lambda_2)h+g(t,x,\hat\lambda_1,\hat\lambda_2)\big]-\phi[g(t,x,\hat\lambda_1,\hat\lambda_2)]\}}{h}+\\
&E^{\lambda}E^{\hat\lambda}\{\phi'\big[g(t,x,\hat\lambda_1,\hat\lambda_2)\big][U(c)+\mu(t)U(X(t)+\frac{p}{\eta(t)})-(\beta+\mu(t))g(t,x,\hat\lambda_1,\hat\lambda_2)]\}\\
=&E^{\lambda}[A^u_{\lambda}F(t,x)]-E^{\lambda}E^{\hat\lambda}\{A^u_{\lambda}\phi[g(t,x,\hat\lambda_1,\hat\lambda_2)]\}\\
&E^{\lambda}E^{\hat\lambda}\{\phi'\big[g(t,x,\hat\lambda_1,\hat\lambda_2)\big][U(c)+\mu(t)U(X(t)+\frac{p}{\eta(t)})-(\beta+\mu(t))g(t,x,\hat\lambda_1,\hat{}\lambda_2)+A^u_{\hat\lambda}g(t,x,\hat\lambda_1,\hat\lambda_2)]\}.
\end{aligned}
$$

There are two facts: (a) $F(t,x)=E^{\lambda}[\phi(g(t,x,\lambda_1,\lambda_2))]$, (b) $\hat\lambda$ and λ are i.i.d., so we have $F(t,x)=E^{\hat\lambda}[\phi(g(t,x,\hat\lambda_1,\hat\lambda_2))]$. As such,

$$E^{\lambda}E^{\hat\lambda}[A^u_{\lambda}\phi(g(t,x,\hat\lambda_1,\hat\lambda_2))]=E^{\lambda}A^u_{\lambda}E^{\hat\lambda}[\phi(g(t,x,\hat\lambda_1,\hat\lambda_2))]=E^{\lambda}[A^u_{\lambda}F(t,x)].$$

Then,

$$
\begin{aligned}
&\lim_{h\to 0}\frac{J(t,x;\mathbf{u}_h)-J(t,x;\mathbf{u}^*)}{h}\\
=&E^{\lambda}E^{\hat\lambda}\{\phi'\big[g(t,x,\hat\lambda_1,\hat\lambda_2)\big][U(c)+\mu(t)U(X(t)+\frac{p}{\eta(t)})-(\beta+\mu(t))g(t,x,\hat\lambda_1,\hat\lambda_2)+A^u_{\hat\lambda}g(t,x,\hat\lambda_1,\hat\lambda_2)]\}\\
=&E^{\hat\lambda}\{\phi'\big[g(t,x,\hat\lambda_1,\hat\lambda_2)\big][U(c)+\mu(t)U(X(t)+\frac{p}{\eta(t)})-(\beta+\mu(t))g(t,x,\hat\lambda_1,\hat\lambda_2)+A^u_{\hat\lambda}g(t,x,\hat\lambda_1,\hat\lambda_2)]\}
\end{aligned}
$$

We omit E^{λ} here because the function inside has no relationship with λ. Furthermore, since λ and $\hat\lambda$ have the same distribution, we can replace $\hat\lambda$ with λ. so the HJB equation can be reduced to

$$\sup_{\mathbf{u}\in\mathscr{A}}E^{\lambda}\big[\phi'[g(t,x,\lambda_1,\lambda_2)][U(c)+\mu(t)U\left(x+\frac{p}{\eta(t)}\right)-(\beta+\mu(t))g(t,x,\lambda_1,\lambda_2)+A^{\mathbf{u}}_{\lambda}g(t,x,\lambda_1,\lambda_2)]\big]=0$$

According to the definition of $\mathbf{u}^*$ and condition (2.1), we have

$$E^{\lambda}\big\{\phi'\big[g(t,x,\lambda_1,\lambda_2)\big]\big[U(c)+\mu(t)U(X(t)+\frac{p}{\eta(t)})-(\beta+\mu(t))g(t,x,\lambda_1,\lambda_2)+A_{\lambda}^{u}g(t,x,\lambda_1,\lambda_2)\big]\big\}\leq 0,$$

which means

$$\lim_{h\to 0}\frac{J(t,x;\mathbf{u}^*)-J(t,x;\mathbf{u}_h)}{h}\geq 0.$$

Thus, $\mathbf{u}^*$ is the equilibrium strategy satisfying Definition 2.1, and $V(t,x)=J(t,x;\mathbf{u}^*)=E^{\lambda}[\phi(g(t,x,\lambda_1,\lambda_2))]$.
□

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