Problem: Given an image $(u, v) \in [0, W) \times [0, H)$ sampled from 360 view by $\lambda(u) = 2\pi\frac{u}{W} - \pi$ $\phi(v) = \frac{\pi}{2} - \pi\frac{v}{H}$

1 Part 1

General Strategy outline:

- First, we want to transform the image into pixel on the viewing sphere in 3D space obtaining (λ_w, ϕ_w).
- Then, rotate the sphere to position relative to the camera, obtaining (λ_c, ϕ_c).
- Sample (u', v') within latitude range and longtitude range $(\pi/4, -\pi/4)$ from (λ_c, ϕ_c).

1.1 Transforming image into 3D Dome

The affine transformation $\tau : (u, v) \to (\lambda_w, \phi_w)$ can be seen as linear transformation in homogenous P^2 space, denoted by matrix multiplication:

$$\begin{bmatrix} \frac{2\pi}{W} & 0 & -\pi \\ 0 & \frac{-\pi}{H} & \frac{\pi}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = S \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_w \\ \phi_w \\ 1 \end{bmatrix}$$

Computing the inverse allows us to sample points on the 3D sphere using the information in the image by

$$\begin{bmatrix} u(\lambda_w) \\ v(\phi_w) \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{W}{2\pi} & 0 & \frac{W}{2} \\ 0 & \frac{-H}{\pi} & \frac{H}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_w \\ \phi_w \\ 1 \end{bmatrix} = S^{-1} \begin{bmatrix} \lambda_w \\ \phi_w \\ 1 \end{bmatrix}$$

1.2 Rotating sphere to camera perspective

Applying rotation matrices on the world coordinates to obtain the camera view, we have:

$$\begin{aligned} \begin{bmatrix} \lambda_c \\ \phi_c \\ 1 \end{bmatrix} &= P^{-1} R_{yaw}(30^\circ) R_{pitch}(10^\circ) R_{roll}(25^\circ) P \begin{bmatrix} \lambda_w \\ \phi_w \\ 1 \end{bmatrix} \\ &= R \begin{bmatrix} \lambda_w \\ \phi_w \\ 1 \end{bmatrix} \end{aligned}$$

Where R is the camera rotation matrix and P is the matrix transforming polar coordinates to cartesian coordinates. We let x be the horizonal aspect of the view, y be the vertical, and z be the depth. According to the rotations conventions, we have

$$R_{yaw}(30^\circ) = \begin{bmatrix} \cos 30^\circ & 0 & -\sin 30^\circ \\ 0 & 1 & 0 \\ \sin 30^\circ & 0 & cos 30^\circ \end{bmatrix} \qquad \text{y axis}$$

$$R_{pitch}(10^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 10^\circ & -\sin 10^\circ \\ 0 & \sin 10^\circ & \cos 10^\circ \end{bmatrix} \qquad \text{x axis}$$

$$R_{roll}(25^\circ) = \begin{bmatrix} \cos 25^\circ & -\sin 25^\circ & 0 \\ \sin 25^\circ & \cos 25^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{z axis}$$

1.3 Sampling from Camera coordinates

We sample $(u_c, v_c) \in [0, W) \times [0, H)$ with $\lambda_c(u_c) = \frac{\pi}{2W} u_c - \frac{\pi}{4}$, $\phi_c(v_c) = \frac{\pi}{2H} v_c - \frac{\pi}{4}$. This way, the range of sampled angles for $\lambda_c, \phi_c \in (-\frac{\pi}{4}, \frac{\pi}{4})$, which corresponds to a 90° viewing angle.

1.4 Putting it together

. Therefore, we can construct the perspective view image (new) from the original image (old) by composing transformations of each step. Giving us

$$New(u_c, v_c) = Old(S^{-1}R^{-1}\begin{bmatrix}\frac{\pi}{2W}u_c - \frac{\pi}{4}\\ \frac{\pi}{2H}v_c - \frac{\pi}{4}\\ 1\end{bmatrix})$$

2 Part 2

Starting from any orientation, you can yaw 90, -90, and 180 to obtain 3 faces. And pitch up 90 and down 90 to obtain the other 2. Therefore, let R be as defined in part 1. Then the question is to find r, p, y such that $R(r,p,y) = R_\Delta R(25, 10, 30)$ where $R_\Delta \in \{R_{yaw}(90), R_{yaw}(-90), R_{yaw}(180), R_{pitch}(90), R_{pitch}(-90)\}$

3 Part 3: Sampling Jacobian and solid-angle element

Solid-angle element on the unit sphere. The catesian position of unit sphere points are parametrized by

$$\boldsymbol{p}(\lambda,\phi) = \begin{bmatrix}\cos\phi\cos\lambda\\ \sin\phi\\ \cos\phi\sin\lambda\end{bmatrix}.$$

longitude $\lambda \in [-\pi, \pi)$ and latitude $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ via Then

$$\partial_\lambda \boldsymbol{p} = \begin{bmatrix}-\cos\phi\sin\lambda\\ 0\\ \cos\phi\cos\lambda\end{bmatrix}, \qquad \partial_\phi \boldsymbol{p} = \begin{bmatrix}-\sin\phi\cos\lambda\\ \cos\phi\\ -\sin\phi\sin\lambda\end{bmatrix}.$$

The differential area (and hence solid angle on the unit sphere) is

$$d\Omega \;=\; \|\partial_\lambda \boldsymbol{p} \times \partial_\phi \boldsymbol{p}\|\, d\lambda\, d\phi \;=\; \cos\phi\; d\lambda\, d\phi.$$

Thus the solid-angle element maps from ERM angles as

$$\boxed{d\Omega = \cos\phi\; d\lambda\, d\phi}.$$

From image pixels (u, v) to solid angle. With the equirectangular mapping

$$\lambda(u) = 2\pi\frac{u}{W} - \pi, \qquad \phi(v) = \frac{\pi}{2} - \pi\frac{v}{H},$$

the Jacobian from (u, v) to (λ, ϕ) is constant:

$$\frac{\partial(\lambda,\phi)}{\partial(u,v)} = \begin{vmatrix}\frac{\partial\lambda}{\partial u} & \frac{\partial\lambda}{\partial v}\\ \frac{\partial\phi}{\partial u} & \frac{\partial\phi}{\partial v}\end{vmatrix} = \begin{vmatrix}\frac{2\pi}{W} & 0\\ 0 & -\frac{\pi}{H}\end{vmatrix} = -\frac{2\pi^2}{WH}.$$

Taking the absolute value for area,

$$d\lambda\, d\phi = \frac{2\pi^2}{WH}\, du\, dv,$$

so a pixel area $du\, dv$ subtends solid angle

$$\boxed{d\Omega \;=\; \cos\phi(v)\;\frac{2\pi^2}{WH}\; du\, dv}.$$

Equivalently, the per-pixel solid angle is

$$\boxed{\Delta\Omega(u,v) \;\approx\; \cos\bigl(\phi(v)\bigr)\,\frac{2\pi^2}{WH}}.$$

Hence each ERM texel covers *larger* solid angle near the equator ($\phi \approx 0$) and *shrinks* to zero near the poles ($|\phi| \to \frac{\pi}{2}$).

Implications for importance sampling. Uniformly picking ERM pixels (uniform in u, v) is *not* uniform over directions: the induced PDF on latitude is proportional to $\cos\phi$. To sample directions uniformly on the sphere, draw

$$\lambda \sim \mathrm{Unif}[-\pi,\pi), \qquad \phi \;\text{ with PDF }\; f_\phi(\phi) = \tfrac{1}{2}\cos\phi \;\Rightarrow\; \phi = \arcsin(2\xi-1),\; \xi \sim \mathrm{Unif}[0,1].$$

For environment-map *importance sampling*, weight texels by $l(u,v)\cos\phi(v)$ (radiance/luminance times $\cos\phi$) when building the sampling distribution, so that selection probability is proportional to contribution per unit solid angle.

Implications for antialiasing near the poles. Because horizontal arc length on the sphere scales as $\cos\phi$ for a given $d\lambda$, ERM *oversamples* near the poles and *undersamples* near the equator in terms of solid angle per texel. When resampling an ERM into a perspective view, the reconstruction footprint in texture space should therefore be *anisotropic*, using a larger longitudinal filter support at high latitudes:

$$\text{effective longitudinal footprint} \;\propto\; \frac{1}{\cos\phi}.$$

Practically: use MIP/anisotropic filtering with an LOD bias that increases toward the poles (in the u/λ direction), or average $\sim 1/\cos\phi$ more ERM texels horizontally at high $|\phi|$. This preserves energy (solid angle) and reduces aliasing/over-blur artifacts that otherwise appear when extracting views near the poles.