Basic Finite Elements

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1 The Problem

Let $\Omega \subset \mathbb{R}^2$ be a compact polygonal domain. Let $\Gamma_N, \Gamma_D \subset \partial\Omega$ with

$$\Gamma_N \cap \Gamma_D = \emptyset, \quad \Gamma_N \cup \Gamma_D = \partial\Omega.$$

Let $c \in \mathbb{R}$ be fixed and $f : \Omega \to \mathbb{R}$ be a continuous function. The problem in question is

$$-\nabla^2 u + cu = f \quad x \in \Omega \tag{1}$$

$$u|_{\Gamma_D} = u_D \quad x \in \Gamma_D \tag{2}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_N} = g \quad x \in \Gamma_N. \tag{3}$$

The goal is to solve this using the finite element method.

2 Triangular Mesh

To do this, we first partition Ω into a triangular mesh. Let $\mathcal{T} = \{K_\ell\}_{\ell=0}^{N-1}$ where

$$K_\ell^\circ \cap K_j^\circ = \emptyset, \quad \bigcup_{\ell=0}^{N-1} K_\ell = \Omega.$$

Each K_ℓ is a triangle.
As a reference let $\hat{K}$ be the triangle with vertices

$$\hat{\mathbf{z}}_0 = \mathbf{0}, \quad \hat{\mathbf{z}}_1 = (1,0)^T, \quad \hat{\mathbf{z}}_2 = (0,1)^T.$$

Then for each ℓ we define an affine transformation of the form

$$\mathbf{F}_\ell : \hat{K} \to K_\ell, \quad \hat{\mathbf{x}} \to B_\ell \hat{\mathbf{x}} + \mathbf{z}_0^{(\ell)}.$$

Where

$$B_\ell = \begin{bmatrix} x_1^{(\ell)} - x_0^{(\ell)} & x_2^{(\ell)} - x_0^{(\ell)} \\ y_1^{(\ell)} - y_0^{(\ell)} & y_2^{(\ell)} - y_0^{(\ell)} \end{bmatrix}.$$

In this way, K_ℓ has vertices $\mathbf{z}_{0,1,2}^{(\ell)}$ and the affine transformation has

$$\mathbf{F}_\ell(\hat{\mathbf{z}}_j) = \mathbf{z}_j^{(\ell)}.$$

Thus for any triangle K_ℓ in the triangulation, we can always map back to the original reference triangle $\hat{K}$.

2.1 Continuous Piecewise Linear Functions

A finite element is of the form $(K, \mathcal{P}, \mathcal{N})$. For this program, we will use $K = K_\ell$, then $\mathcal{P}$ will be the set of all linear polynomials defined on K, and $\mathcal{N}$ will be the degrees of freedom defined by

$$\mathcal{N} = \{N_0, N_1, N_2\}, \quad N_i(f) = f(\mathbf{z}_i).$$

Using affine equivalence, these can be mapped back to the reference element: $(\hat{K}, \hat{\mathcal{P}}, \hat{\mathcal{N}})$. Then $\hat{\mathcal{N}}$ induces a dual basis for $\hat{\mathcal{P}}$ of the form

$$\hat{\mathcal{B}} = \{\hat{\varphi}_0, \hat{\varphi}_1, \hat{\varphi}_2\} \tag{4}$$

$$\hat{\varphi}_0(x, y) = 1 - x - y \tag{5}$$

$$\hat{\varphi}_1 = x \tag{6}$$

$$\hat{\varphi}_2 = y \tag{7}$$

Using the affine transformation, we can map this basis on the reference triangle to that of any triangle using

$$\varphi_i^{(\ell)}(\mathbf{x}) = \hat{\varphi}_i(\mathbf{F}_\ell^{-1}(\mathbf{x})).$$

These basis functions can then be assembled together to build global basis functions. The global basis functions will be $\{\phi_\ell\}_{\ell=0}^{N-1}$. Each of which are defined on all of Ω. These are designed so that

$$\phi_\ell(\mathbf{z}_j) = \delta_{\ell j}.$$

Point evaluation of these functions is a bit complex, but proceeds as follows. Given $\mathbf{x} \in \Omega$, let $0 \leq \ell \leq N-1$ so that $\mathbf{x} \in K_\ell$. Then if $K_\ell \subset \operatorname{supp}(\phi)_i$ we have an index $k_{\ell i}$ which indicates for triangle K_ℓ that the node $\mathbf{z}_i$ occupies vertex $\mathbf{z}_{k_{\ell i}}^{(\ell)}$. Point evaluation is then given by

$$\phi_i(\mathbf{x}) = \varphi_{k_{\ell i}}^{(\ell)}(\mathbf{x}).$$

In this way, we can write ϕ_i as follows. Let

$$\mathcal{S}_i = \{\ell : K_\ell \subset \operatorname{supp}(\phi)_i\}.$$

Then

$$\phi_i(\mathbf{x}) = \sum_{\ell \in \mathcal{S}_i} \varphi_{k_{\ell i}}^{(\ell)}(\mathbf{x}) \mathbb{1}(\mathbf{x} \in K_\ell).$$

3 Assembling Matrix Forms

To place the problem into its FEM discretized form, we build an approximation to u of the form

$$u_h(\mathbf{x}) = \sum_{i=0}^{N-1} u_i \phi_i(\mathbf{x}).$$

Let $V_h = H^1(\Omega)$. This is the trial function space, where u_h will live. The test function space will be

$$V_{h,D} = H_D^1(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}.$$

We will then apply the variational form. Doing so for $u \in H^1(\Omega), v \in H_D^1(\Omega)$ gives

$$\int_\Omega \nabla v \cdot \nabla u \, d^2x + c \int_\Omega vu \, d^2x - \int_{\Gamma_N} vg \, d\sigma = \int_\Omega vf \, d^2x.$$

We next use the approximation for u:

$$\sum_{i=0}^{N-1} u_i \int_\Omega \nabla v \cdot \nabla \phi_i \, d^2x + c \sum_{i=0}^{N-1} u_i \int_\Omega v\phi_i \, d^2x = \int_\Omega vf \, d^2x + \int_{\Gamma_N} vg \, d\sigma.$$

Next, since $v \in H^1(\Omega)$ we must require that for $0 \le j \le N-1$ that

$$\sum_{i=0}^{N-1} u_i \int_\Omega \nabla\phi_j \cdot \nabla\phi_i \, d^2x + c \sum_{i=0}^{N-1} u_i \int_\Omega \phi_j \phi_i \, d^2x = \int_\Omega \phi_j f \, d^2x + \int_{\Gamma_N} \phi_j g \, d\sigma.$$

Thus it is imperitive that we compute the following matrices and vectors.

- Mass matrix, M:

$$M_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, d^2x.$$

- Stiffness matrix, S:

$$S_{ij} = \int_\Omega \phi_i \phi_j \, d^2x.$$

- Load vector, $\mathbf{b}$:

$$b_i = \int_\Omega \phi_i f \, d^2x.$$

- Traction vector, $\mathbf{t}$:

$$t_i = \int_{\Gamma_N} \phi_i g \, d\sigma.$$

3.1 Mass Matrix

3.2 Stiffness Matrix

3.3 Load Vector

We can employ a cheat with the load vector. First replace f by its interpolant built out of the ϕ_i:

$$If := \sum_{i=0}^{N-1} f(\mathbf{z}_i)\phi_i.$$

Then

$$b_i = \sum_{j=0}^{N-1} f(\mathbf{z}_j) \int_\Omega \phi_i \phi_j \, d^2x = (S\mathbf{f})_i.$$

That is $\mathbf{b} = S\mathbf{f}$ where $f_i = f(\mathbf{z}_i)$. While this is not exact, it drastically saves computational cost, and the method is only accurate to the first derivative anyway.

3.4 Traction Vector

On Γ_D the values of u are known. Let $\mathcal{I}_D$ be the set of indices indicating

$$\ell \in \mathcal{I}_D \implies \mathbf{z}_\ell \in \Gamma_D.$$

Then the sum can be split to examine only the free indices:

$$\sum_{\substack{i=0 \\ i \notin \mathcal{I}_D}}^{N-1} u_i \int_\Omega \nabla v \cdot \nabla \phi_i \, d^2x + c \sum_{\substack{i=0 \\ i \notin \mathcal{I}_D}}^{N-1} u_i \int_\Omega v \phi_i \, d^2x = \int_\Omega v f \, d^2x + \int_{\Gamma_N} v g \, d\sigma - \sum_{i \in \mathcal{I}_D} u_D(\mathbf{z}_i) \int_\Omega .$$