1 Introduction

In this paper, we examine tunnelling solutions to the fundamental equation in quantum mechanics: the Schrödinger equation for a single particle.

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \Psi + V\Psi \tag{1}$$

Here $\Psi = \Psi(\mathbf{x})$ is the wave function describing a quantum mechanical particle, t is time, $V = V(\mathbf{x}, t)$ is the potential with which the particle interacts. The constant m is the mass of the particle, and $\hbar$ is Planck's constant divided by 2π [**?**].

We consider the following scenario. A particle is confined to a well in $\mathbb{R}^2$. It passes by a ring-shaped well which is connected to a second well leaving the system (see figure 1). Such a situation is the problem of interest in characterizing quantum ring resonators **TODO: citation**.

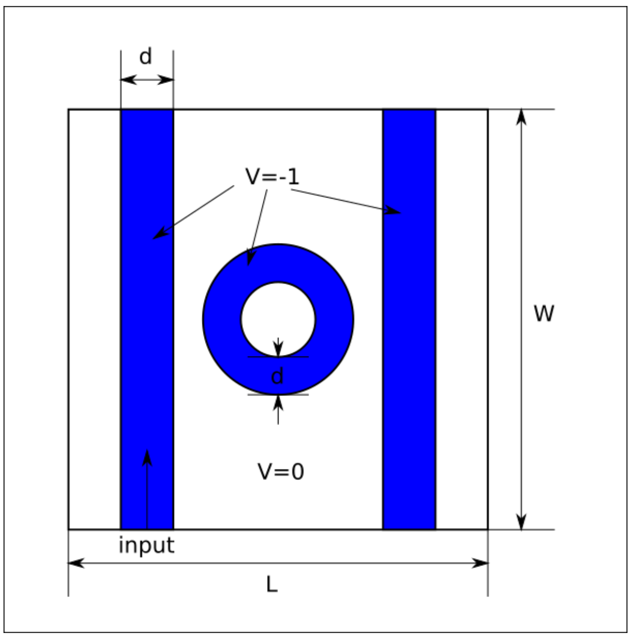

Figure 1: Sketch of Ring resonator. The domain excluding the shaded regions is Ω_0. The two shaded rectangles are Ω_1 and Ω_2 from left to right. Finally the center ring is Ω_3. The entire domain is $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \Omega_3$.

We will solve this problem using finite elements, and using the Python library FEniCS, which will take care of the heavy lifiting in constructing the finite element representation, as well as solving the resulting linear algebra system.

Let Ω_0 be the region without color, a rectangle of width W and length L, excluding the colored regions. The two long rectangles, in order from left to right are Ω_1 and Ω_2 respectively. Finally, the ring in the center is Ω_3.

1.1 Nondimensionalization

Nondimensionalizing the problem is essential since $\hbar \approx 10^{-34} J \cdot s$ in S.I. units. Let the spacial scaling be d, the width of each of the long potential wells. This induces a time scaling of $\frac{2md^2}{\hbar}$. Finally there is an energy scaling of V_0, which characterizes the strength of the potential. To summarize, if $\tilde{\mathbf{x}}, \tilde{t}, \tilde{V}$ all have units, then

$$\tilde{\mathbf{x}} = d \cdot \mathbf{x}, \quad \tilde{t} = \frac{2md^2}{\hbar} t, \quad \tilde{V}(\tilde{\mathbf{x}}) = V_0 V(\mathbf{x}). \tag{2}$$

Let $\nu = \dfrac{2md^2V_0}{\hbar^2}$. The nondimensionalized problem then reads

$$i\frac{\partial \Psi}{\partial t} = -\nabla^2\Psi + \nu V(\mathbf{x})\Psi. \tag{3}$$

Here, the form of $V(\mathbf{x})$ is

$$V(\mathbf{x}) = \begin{cases} -1 & \mathbf{x} \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \\ 0 & \mathbf{x} \in \Omega_0 \end{cases}. \tag{4}$$

1.2 Boundary and Initial Conditions

An input is specified on the lower portion of the left-most shaded rectangle. This will be denoted by the region Γ_{in}. All other boundaries of Ω which are also boundaries of Ω_i for $i = 1, 2$ will be denoted Γ_{out}. Finally, the rest of the boundary will be denonted Γ_D.

On Γ_{in}, there is a specified Dirichlet boundary condition: $f(\mathbf{x}, t)$, and on Γ_D the Dirichlet condition is zero. Lastly, on Γ_{out} we specify an outflow boundary condition. In quantum mechanics, the momentum operator, in this dimensionless system: $\hat{\mathbf{p}} = -\dfrac{i\hbar}{d}\nabla$. The outflow condition relates $\partial_t\psi$ with the momentum. All together the boundary conditions are

$$\text{Boundary Conditions} = \begin{cases} \Psi(\mathbf{x}, t) = 0 & \mathbf{x} \in \Gamma_D \\ \Psi(\mathbf{x}, t) = f(\mathbf{x}, t) & \mathbf{x} \in \Gamma_{\text{in}} \\ \dfrac{\partial \Psi}{\partial t} = -2i\nabla\Psi & \mathbf{x} \in \Gamma_{\text{out}} \end{cases} \tag{5}$$

Where $f(\mathbf{x}, t)$ is a prespecified function.

Finally for simplicity, we take an initial condition of an empty system: i.e.

$$\Psi(\mathbf{x}, 0) = 0.$$