1 Finite Dimensional Approximation

To build a finite dimensional version for our problem to apply finite elements, first we make the following approximation. Let $\Omega \subset \mathbb{R}^2$ be bounded with $\Omega_0, \Omega_1 \subset \Omega$ and with

$$\max\{\mathrm{diam}(\Omega_0), \mathrm{diam}(\Omega_1)\} \ll \mathrm{diam}(\Omega).$$

We will then approximation the solution Ψ and its initial condition, Ψ_0 on Ω, by applying the boundary condition

$$\Psi_{(0)}\big|_{\partial\Omega} = 0.$$

Next let $\mathcal{T}$ be a triangulation for Ω, and on this triangulation we **TOOD: specify finite element in form** (K, P, N). Let V be the finite dimensional restriction of H_0^2 onto this triangulation, and let $\{\phi_\ell\}_{\ell=0}^{M}$ be a basis for V.

Suppose that

$$\Psi_0(\mathbf{x}) = \sum_{\ell=0}^{M} \alpha_\ell^{(0)} \phi_\ell(\mathbf{x}), \quad \Psi(\mathbf{x}, t) = \sum_{\ell=0}^{M} \alpha_\ell(t)\phi_\ell(\mathbf{x}), \quad \alpha_\ell(0) = \alpha_\ell^{(0)}.$$

The variational form of the problem then becomes:

$$i\sum_{k=0}^{M} \langle \phi_\ell \phi_k \rangle_{L^2(\Omega)} \alpha_k' = \sum_{k=0}^{M} \langle \nabla\phi_\ell, \nabla\phi_k \rangle_{L^2(\Omega)} \alpha_k + \nu \langle \phi_\ell, V\phi_k \rangle_{L^2(\Omega)} \alpha_k \tag{1}$$

$$\sum_{k=0}^{M} \langle \nabla\phi_\ell, \nabla\phi_k \rangle_{L^2(\Omega)} \alpha_k^{(0)} + \nu \left\langle \phi_\ell, V^{(0)}\phi_k \right\rangle_{L^2(\Omega)} \alpha_k = \epsilon_{\min} \langle \phi_\ell, \phi_k \rangle_{L^2(\Omega)} \alpha_k \tag{2}$$

Finally, let

$$M_{ij} = \langle \phi_i, \phi_j \rangle_{L^2(\Omega)}, \quad S_{ij} = \langle \nabla\phi_i, \nabla\phi_j \rangle_{L^2(\Omega)}, \quad V_{ij} = \langle \phi_i, V\phi_j \rangle_{L^2(\Omega)}.$$

Similarly for $V_{ij}^{(0)}$. The problem then reads in matrix form

$$iM\alpha' = S\alpha + \nu V\alpha \tag{3}$$

$$S\alpha^{(0)} + \nu V^{(0)}\alpha^{(0)} = \epsilon_{\min} M\alpha^{(0)} \tag{4}$$