How to Solve Your Horrible Equation using FEniCS

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April 29, 2020

What is FEniCS?

- Finite Element Computational Software
- Gives intuitive implementations for the finite element method to solve PDEs
- Written in Python - easy to understand
- Mirrors mathematical notation
- Github for codes used: `https://github.com/JeromeTroy/fenics-hgss.git`

A Brief Introduction to Finite Elements

- How it works:

$$Lu = f, \quad \text{some boundary conditions}$$

- Discretize domain, Ω, into some mesh
- Choose a *Test function* v such that $v = 0$ on the (Diriclet part of) boundary

$$\int_\Omega vLu\,dx = \int_\Omega vf\,dx$$

- Integrate by parts, boundary condition on v makes boundary terms vanish
- Reduces order of derivatives in equation through integration
- FEniCS wraps functional analysis parts of finite elements nicely so they are all taken care of

An Example

- Consider Laplace's Equation in a Circle of radius 1, with Dirichlet Boundary Condition:

$$\begin{aligned} \nabla^2 u = \partial_x^2 u + \partial_y^2 u = f(\mathbf{x}), \quad & \mathbf{x} \in \Omega := \left\{ \mathbf{x} \in \mathbb{R}^2 : 0 \leq ||\mathbf{x}|| < 1 \right\} \\ & u(\mathbf{x} \in \partial\Omega) = 1 \end{aligned} \tag{1}$$

- Let v be a piecewise $C^1(\Omega)$ test function such that $v(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \partial\Omega$
- Variational form

$$\int_\Omega v \nabla^2 u \, d\mathbf{x} = \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, d\sigma - \int_\Omega \nabla v \cdot \nabla u \, d\mathbf{x} = \int_\Omega v f(\mathbf{x}) \, d\mathbf{x}$$

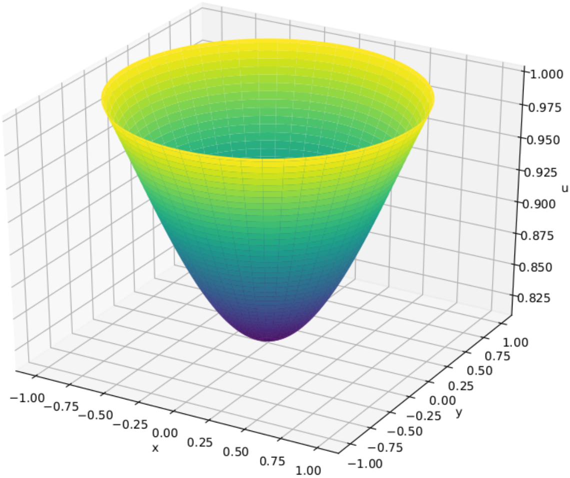
$$\text{BC on } v \implies A(u,v) := - \int_\Omega \nabla v \cdot \nabla u \, d\mathbf{x} = \int_\Omega v f(\mathbf{x}) \, d\mathbf{x} =: F(v; f)$$

The Code

```
# imports
from fenics import *
from mshr import *   # meshing utility
import matplotlib.pyplot as plt    # plotting

Ω = Circle(Point(0, 0), 1.0)

N = 100   # N is like 1 / Δx
mesh = generate_mesh(Ω, N)

V = FunctionSpace(mesh, "CG", 1)

u = TrialFunction(V)
v = TestFunction(V)

bc = DirichletBC(V, Constant(1.0), "on_boundary")

f_expr = Expression("1 - pow(x[0], 2) - pow(x[1], 2)", degree=1)
f = project(f, V)  # have to evaluate on function space

# variational form
A = -dot(grad(v), grad(u)) * dx
F = v * f * dx

# solve
u_sol = Function(V)
solve(A == F, u_sol, bc)
```

Solution using FEniCS

Example: $f(\mathbf{x}) = 1 - ||\mathbf{x}||_2^2$



Norm-2 Error: 0.002392

A Time Dependent Example

- Consider the heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} = \nabla \cdot (\kappa \nabla u), \quad & \mathbf{x} \in \Omega \\ & u(\mathbf{x}, t = 0) = u_0(\mathbf{x}) \end{aligned} \tag{2}$$

- Cylindrical Domain:

$$\Omega = \{x = (r\cos\theta, r\sin\theta, z) : 0 \leq \theta \leq 2\pi,\ 0 \leq r < 1,\ 0 < z < h\}$$

- Mixed boundary conditions:

$$u(r = 1, \theta, z, t) = u(r, \theta, z = 0, t) = 0, \quad \frac{\partial u}{\partial z}(r, \theta, z = h, t) = 0$$

Variational Form and Solution

- Split into many stationary problems: discretize time (Crank Nicolson time discretization)

$$t_n = n\Delta t, \quad u_n(\mathbf{x}) = u(\mathbf{x}, t_n)$$
$$\left.\frac{\partial u}{\partial t}\right|_{t_n} \approx \frac{u_{n+1} - u_n}{\Delta t} = \frac{1}{2}\left(\nabla \cdot (\kappa \nabla u_n) + \nabla \cdot (\kappa \nabla u_{n+1})\right)$$
$$\implies u_{n+1} - \frac{\Delta t}{2}\nabla \cdot (\kappa \nabla u_{n+1}) = u_n + \frac{\Delta t}{2}\nabla \cdot (\kappa \nabla u_n)$$

- Variational Form

$$\int_\Omega \left(v u_{n+1} + \frac{\Delta t}{2}\kappa \nabla v \cdot \nabla u_{n+1}\right) dx = \int_\Omega \left(v u_n - \frac{\Delta t}{2}\kappa \nabla v \cdot \nabla u_n\right) dx$$
$$A(v, u_{n+1}) = F(v, u_n)$$

- Solve $A(v, u_{n+1}) = F(v, u_n)$ subject to the boundary conditions for $n = 1, ..., N$

Heat Equation Solution

Linear Heat Equation Solution

Nonlinear Heat Equation

- $\kappa \to \kappa(u) = 1 - \frac{1-k}{2}\left[1 - \tanh\left(\frac{u-\tau}{\eta}\right)\right]$
- $0 \le k \le 1, \quad 0 \le \tau \le 1, \quad \eta > 0$
- Mimics a phase change where $\kappa = 1$ for high u, then $\kappa = \frac{1+k}{2} \le 1$ for low u
- η controls rate of switching from high to low κ, τ is phase change temperature
- Variational Form:

$$F(v, u_n; \Delta t) = \int_\Omega \left(v u_n - \frac{\Delta t}{2}\kappa(u_n)\nabla v \cdot \nabla u \right) dx$$

$$A(v, u_{n+1}; \Delta t) = \int_\Omega \left(v u_{n+1} + \frac{\Delta t}{2}\kappa(u_{n+1})\nabla v \cdot \nabla u_{n+1} \right) dx$$

- Solve $A(v, u_{n+1}) = F(v, u_n)$ subject to same boundary conditions

Comparison of Linear and Nonlinear Heat Equations

Comparison Solutions

A Nonlinear (Horrible) Example

- The catenary problem

$$\min_{u} \int_{-1}^{1} u\sqrt{1+(\epsilon u')^2}\,dx, \quad \text{subject to} \quad \int_{-1}^{1} \sqrt{1+(\epsilon u')^2}\,dx = \ell \tag{3}$$

With $u(\pm 1) = 0$.

- FEniCS applied to Lagrange Multiplier Problem

$$\min_{u,\lambda} \mathcal{F}(u;\lambda) = \int_{-1}^{1} u\sqrt{1+(\epsilon u')^2}\,dx - \lambda \int_{-1}^{1} \left[\frac{\ell}{2} - \sqrt{1+(\epsilon u')^2}\right] dx \tag{4}$$

- Note $\mathcal{F} : C^1[-1,1] \times \mathbb{R} \to \mathbb{R}$
- Let $X := C^1[-1,1] \times \mathbb{R}$
- For simplicity, let

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1+(\epsilon u')^2}$$

Horrible Problem Made Easy

- Some notation to simplify
  - $X := C^1[-1,1] \times \mathbb{R}$ mixed space
  - $y \in X$, $y = (u, \lambda)$
- Minimize $\mathcal{F}(y) = \mathcal{F}((u,\lambda)) = \int_{-1}^{1} u \frac{\mathrm{d}s}{\mathrm{d}x} - \lambda \left[\frac{\ell}{2} - \frac{\mathrm{d}s}{\mathrm{d}x} \right] dx$

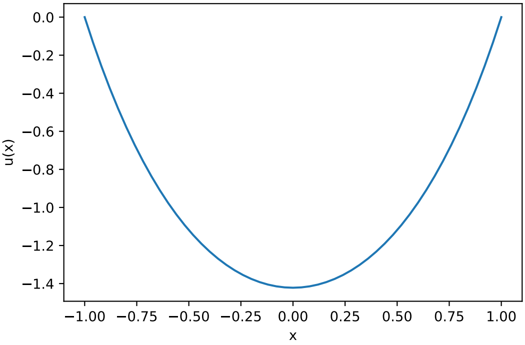


Figure: Solution to the Catenary Problem using FEniCS

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