Numerical Simulation of Wave Scattering Off Antennae

Jerome Troy

University of Delaware

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Overview

1. Motivation and Background
2. Numerical Methods
3. Results
4. Conclusions

The Wave Equation

Scalar Waves (eg. sound waves)

$$c^2\nabla^2\psi = \frac{\partial^2\psi}{\partial t^2}$$

- $\psi(\boldsymbol{x}, t)$ - the amplitude of the wave
- $\boldsymbol{x} \in \mathbb{R}^2$ - position
- $t > 0$ - time
- c - the speed of wave propagation

Letting $t \mapsto \frac{1}{c}t$ effectively sets $c = 1$

The Antenna Problem

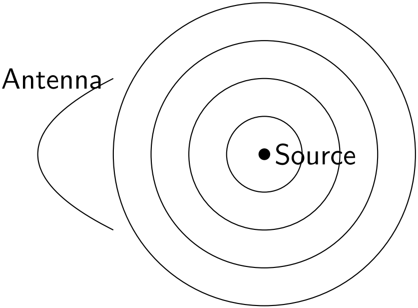
Waves emminate from a source

These waves reflect off the antenna in such a way to be directed towards a target

The problem is how to calculate the scattering pattern off an antenna

Setting reflection at the antenna surface (S_A)

$$\psi|_{S_A} = 0$$



Discretization

Problem: we cannot simulate an infinite domain!
Will let domain be large enough to minimize reflections off boundaries
$x \in [-X, X]\,, \quad y \in [-Y, Y]$
Number of nodes: M_x, M_y in x and y directions respectively
$x_i = -X + ih_x, \quad h_x = \frac{2X}{M_x}, \quad i = 0, 1, ..., M_x$, and
$y_j = -Y + jh_y, \quad h_y = \frac{2Y}{M_y}, \quad j = 0, 1, ..., M_y$

$$\mathbb{R}^{M_x+1 \times M_y+1} \ni \Psi_{ij}(t) \approx \psi(x_i, y_j, t) \implies \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Psi = \nabla^2\Psi$$

Differentiation Matrices!

$$\nabla^2\Psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi \mapsto D_{xx}\Psi + \Psi D_{yy}^T$$

Solving

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Psi = D_{xx}\Psi + \Psi D_{yy}^T, \quad \Psi|_{S_A} = 0$$

Building Differentiation Matrices (continued)

Put this all together:

$$D_{xx} = \frac{2}{h^2} \begin{bmatrix} 1 & -\frac{5}{2} & 2 & -\frac{1}{2} & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 \\ & & \frac{1}{2} & -2 & \frac{5}{2} & -1 \end{bmatrix}$$

For D_{yy} we wish to operate on the rows, so we need to use the transpose!

Verlet Integration

How do we solve

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} u = f(t, u), \quad u(0) = u_0, \quad u'(0) = v_0, \quad t \geq 0$$

Störmer Verlet Integration:
$t_k = k\tau$, τ time step size, then let $u_k \approx u(t_k)$
Define:

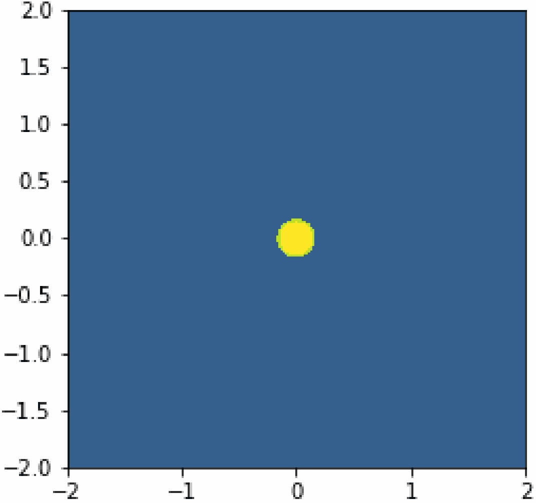
$$u_1 = u_0 + \tau v_0 + \frac{\tau^2}{2} f(t_0, u_0)$$

Then iterate via:

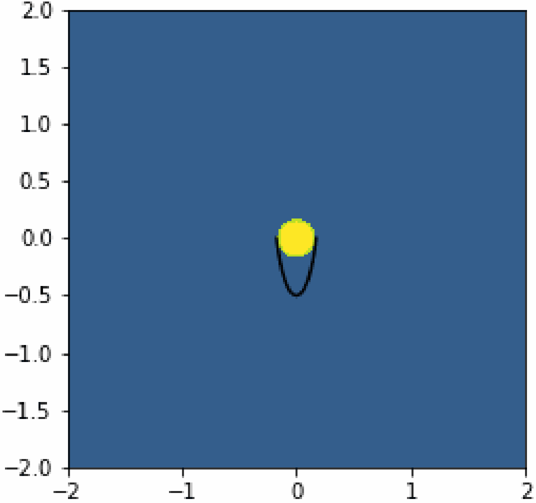
$$u_{k+1} = 2u_k - u_{k-1} + \tau^2 f(t_k, u_k), \quad k \geq 1$$

Verlet's method can be shown to be $O(\tau^2)$ [1]
It also has the added benefit of conserving energy

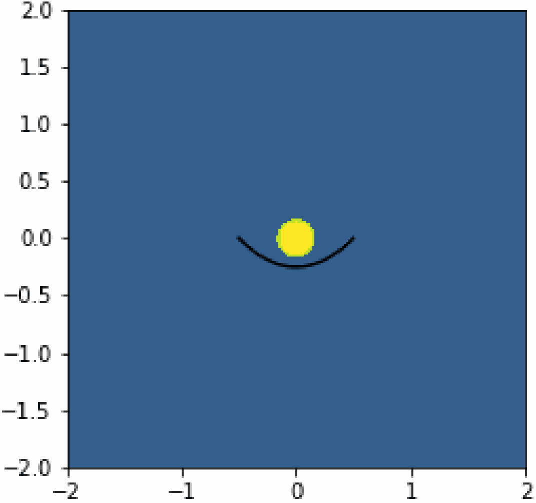
Test Case - No Reflector

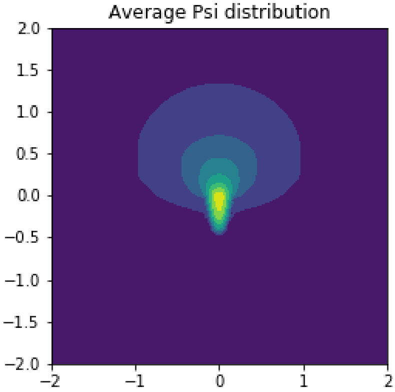


Elliptical Reflector Dish

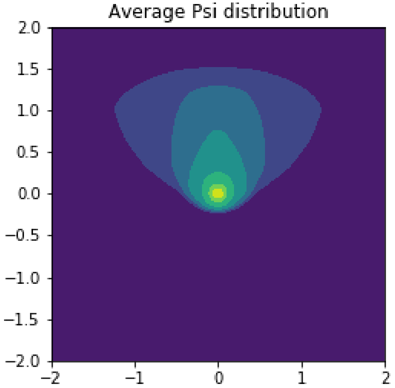


Parabolic Reflector





(a) Elliptic reflector



(b) Parabolic reflector

Conclusions

- Störmer Verlet Integration has good accuracy for 2D wave equation
- This allows us to solve the equation without factorizing ∇^2 as in the Maxwell Equations [2]
- Verified parabolic columnation of energy from focus

Thank You!

Questions?

References

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T. Driscoll and R. Braun, *Fundamentals of Numerical Computation*, ch. 12. Advection Equations, pp. 548–552.
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