Modelling the water-to-air ratio of the Virginia Class submarine to achieve neutral buoyancy

International Baccalaureate Math IA

1 Introduction

Submarines, being a means for people to travel deep underwater, are meticulously engineered to find their use in various fields: one of the most important defense systems of a nation's navy, leading means for scientists to conduct research and observations underwater or even in tourist industries. Submarines can serve all of these purposes thanks to their ability to remain at certain depths underwater, which is related to the submarine's buoyancy.

Submarines can be designed to be of different shapes and sizes depending on their purposes, and computing their optimal designs has become increasingly less complex with the help of newly developed computing software that considers the many possible factors that can influence the performance of these submarines. For the submarines to remain at certain depths underwater, they must achieve neutral buoyancy, which is the equilibrium between the submarine's buoyancy force and weight that enables it to neither sink or rise (Principles of Ship Performances, 9). The specific computing method of this neutral buoyancy is based on Archimedes Principle. However, the seemingly easy calculation with Archimedes' Principle (See Eq 1) is rather taken as granted by students in Physics and Engineering. The specific detail as to how the buoyancy force is resulted from the pressure differences between the top and bottom surfaces of an object is not explored at all, considering the complex mathematics. Inspired by the application and importance of neutral buoyancy of submarines, it leads to the question that this paper attempts to address: how can the water-to-air ratio of a Virginia Class submarine be modeled to achieve neutral buoyancy? The aim is to model the Virginia Class Submarine with relatively simple geometries and compute the relative water and air volumes needed for the submarine to achieve neutral buoyancy.

2 Background Information

Submarines are generally divided into two hulls, which are the watertight pressure hull and the outer hull, where the shape of the outer hull defines the amount of area exposed to the ocean pressure and the general shape of the submarine. The outer hull of the Virginia Class Submarine particularly, has a relatively cylindrical body and rather spherical front and narrow end, as seen in Figure 1 (New River): We model the outer hull of this submarine based on

Figure 1: Model of the Virginia Class Submarine outer hull, showcasing its cylindrical body, spherical front and narrow end.

simpler geometries to avoid complications, as more complex geometries can lead to the need for computer software to resolve. The cylindrical body of the submarine is characterized by a hollow cylinder of length $2L$, with radius r. The two ends are simplified to be hemispheres also with radius r. The overall shape has a thickness of t, representative of the layer of steel as shown in Figure 2.

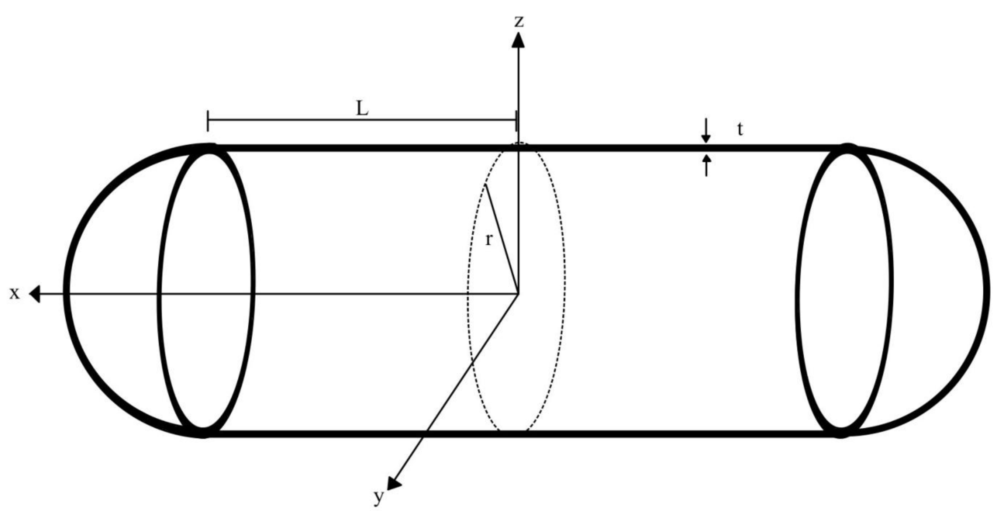

Figure 2: Simplified model of the Virginia Class Submarine outer hull

The specific coordinate system chosen is as follows: the x axis is defined to be along the length of the submarine as shown in Figure 2, the y axis defined to be radially outwards and the z axis defined to be along the depths or height of the submarine. This outer hull is the geometry that defines how pressure acts along the surface of the submarine, which is explained later. Although no literature information is found about the approximate geometry of the pressure hull, we assume it to be of a cylindrical shape that is encapsulated inside the outer hull as shown in Figure 3,

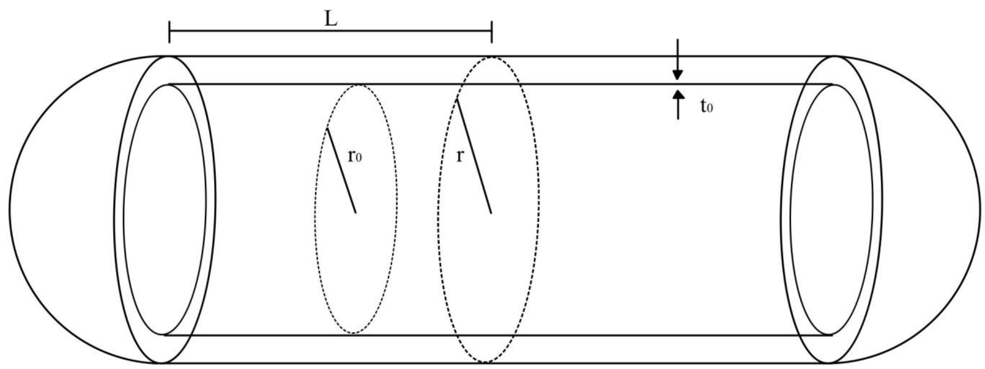

Figure 3: Model of the Virginia Class Submarine pressure hull

Figure 3 shows that the pressure hull has the same length as the cylinder component of the outer hull, which is $2L$, whereas the radius is r_0. The pressure hull has a different thickness (typically thicker) as it has to withstand the hydrostatic pressure to ensure the pressure inside the hull is at atmospheric pressure. The pressure hull has a thickness of t_0 as shown in Figure 3.

For a submarine to achieve neutral buoyancy under water, its buoyant force must be at equilibrium with its weight. The buoyant force is primarily caused by the difference in pressure between the top surface and bottom surfaces of the submarine as pressure is a function of depth. It is usually characterized by the simple equation of the Archimedes' Principle, where it states that the buoyant force is the weight of the fluid displaced, given by:

$$F_B = \rho g V \tag{1}$$

Where ρ is the density of the fluid, g is the gravitational acceleration, and V is the volume of the fluid displaced, which is also equivalent to the volume of the object placed in the fluid. We must first investigate the relationship between pressure and depth by considering the gradient of a 3 dimensional pressure field $\nabla p(x, y, z)$. The gradient of a $\mathbb{R}^3$ function $f(x, y, z)$ is defined with the gradient operator ∇, where applying it to a scalar function gives (Web Formulas):

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \tag{2}$$

Where $\hat{\mathbf{i}}$ is the unit vector in the positive x direction, $\hat{\mathbf{j}}$ in the y direction, and $\hat{\mathbf{k}}$ in the z direction. Therefore applying Eq 2 to our pressure field gives:

$$\nabla p(x, y, z) = \frac{\partial p}{\partial x}\hat{\mathbf{i}} + \frac{\partial p}{\partial y}\hat{\mathbf{j}} + \frac{\partial p}{\partial z}\hat{\mathbf{k}} \tag{3}$$

However, for a uniform gravitational field, we assume that the rate of change in pressure in the x and y directions is negligible, and thus goes to zero. This leaves us with only the pressure gradient in the z direction. With the hydrostatic equation given by $\nabla p(x, y, z) = -\rho\vec{g}$ (Lima 4), we can equate this expression with Eq 3 to get:

$$\frac{dp}{dz} = -\rho\vec{g}$$

Where rearranging dz to the right hand side (RHS) and integrating gives:

$$p(z) = p_0 - \rho g z \tag{4}$$

Where $p(z)$ is the pressure as a function of depth, which we apply to our coordinate system later. We derive the buoyant force that encompasses the pressure function by considering a sphere (for simplicity) that is completely submerged in a fluid as shown in Figure 4.

Figure 4 shows how the pressure acting on the bottom surface is greater relative to that of the top surface, as indicated with the magnitude of the arrows. It is also noted that pressure acts in a normal direction to that of a surface, where if we consider an infinitesimally small mass element dA and its corresponding unit normal vector $\hat{\mathbf{n}}$, then we can express the mass element as a vector $d\vec{A}$, such that $d\vec{A} = dA\hat{n}$. Using this, we can express the differential force

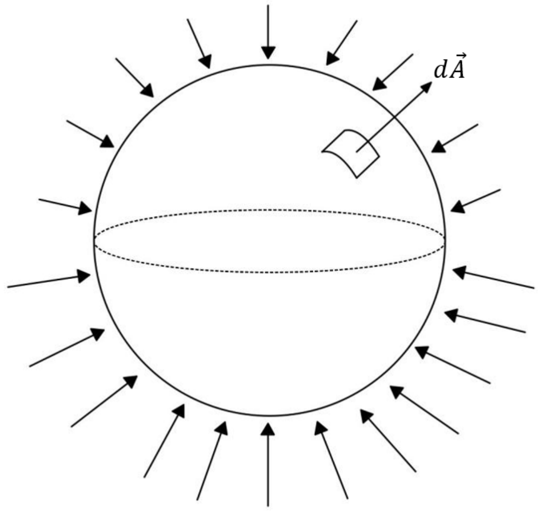

Figure 4: The distribution of pressure along the surface of sphere, with an infinitesimally area element as a vector

$d\vec{F_B}$ acting on the differential area dA as:

$$d\vec{F_B} = -p(z)dA\hat{\mathbf{n}} \tag{5}$$

Where the negative sign comes from the pressure pointing radially inwards, towards the area element, which is opposite to that of the unit normal vector of the differential mass element. Therefore, with Eq 5, upon integrating over the entire top and bottom surface, and taking the difference in the magnitude of the forces, gives us the buoyant force:

$$\vec{F_B} = \iint_{S_B} -p(z)dA\hat{\mathbf{n}} + \iint_{S_T} p(z)dA\hat{\mathbf{n}} \tag{6}$$

Where the surface S_B means the bottom surface, where the pressure acts at a greater magnitude, and the surface S_T refers to the top surface of the object. Taking the difference between the forces acting on the bottom surface and the top surface results in the buoyant force.

3 Buoyant force of the Cylinder

As mentioned that our model of the Virginia Class Submarine consists of a cylinder of length $2L$ and radius r, it can be redrawn as shown in Figure 5:

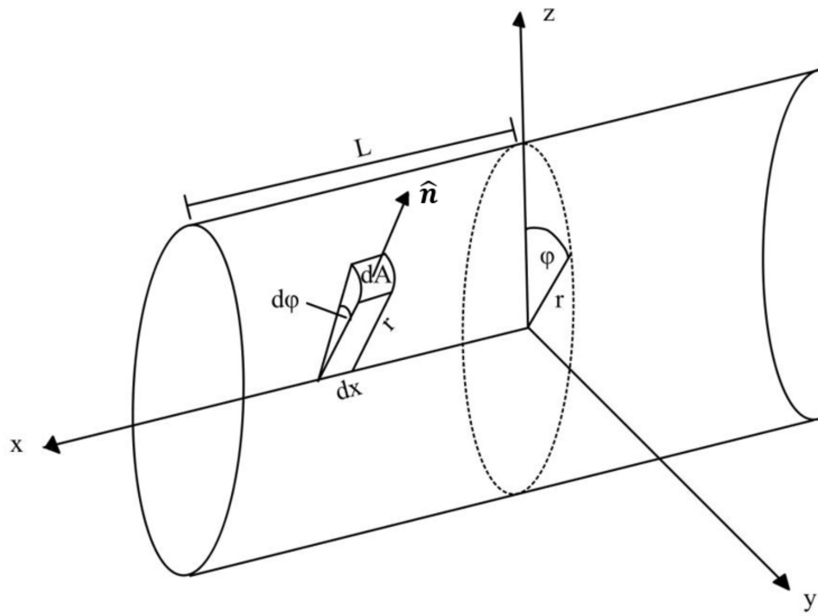

Figure 5: Defining the variables for the cylinder

Figure 5 shows that the the angle made between the z axis and the radius r is defined to be φ. This allows us to parametrically represent our cylinder with a vector function. The Cartesian equation that describes a cylinder is given by:

$$y^2 + z^2 = r^2$$

Where parameterizing the above equation, such that it becomes a function of φ gives us the following change in coordinates: $x = x$, $y = r\cos\varphi$, and $z = r\sin\varphi$, derived based on Figure 5. As we only want to compute the top surface first. The boundaries that define the surface is given by: $-L \leq x \leq L$ and $-\pi/2 \leq \varphi \leq \pi/2$. Therefore, if we express the changed coordinate system, and describe the cylinder parametrically in a vector equation $\vec{\psi}(x, \varphi)$:

$$\vec{\psi}(x, \varphi) = x\hat{\mathbf{i}} + r\cos\varphi\hat{\mathbf{j}} + r\sin\varphi\hat{\mathbf{k}} \tag{7}$$

We first compute the unit normal vector of the differential mass element given by (Suvillan, 1030) for a vector equation of $\vec{r}(u, v)$:

$$\hat{\mathbf{n}} = \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||} \tag{8}$$

Where it is a cross product between the two vectors $\vec{r}_u$ and $\vec{r}_v$ ($\vec{r}_u = \frac{\partial \vec{r}}{\partial u}$ and $\vec{r}_v = \frac{\partial \vec{r}}{\partial v}$), divided by the magnitude of the cross product. We can apply this to our vector equation of the top half of the cylinder $\vec{\psi}$, where the computed partial derivatives with respect to x and φ are as follows:

$$\vec{\psi}_x = \frac{\partial \vec{\psi}}{\partial x} = \hat{\mathbf{i}}$$
$$\vec{\psi}_\varphi = \frac{\partial \vec{\psi}}{\partial \varphi} = -r \sin\varphi \hat{\mathbf{j}} + r \cos\varphi \hat{\mathbf{k}}$$

We can compute the cross product between these vectors by using matrices:

$$\vec{\psi}_x \times \vec{\psi}_\varphi = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & -r\sin\varphi & r\cos\varphi \end{vmatrix} \tag{9}$$

Using the Cofactor method of matrices (Paul's Online Notes), Eq 9 simplifies to:

$$\vec{\psi}_x \times \vec{\psi}_\varphi = -r \cos\varphi \hat{\mathbf{j}} - r \sin\varphi \hat{\mathbf{k}}$$

The magnitude of this cross product is:

$$||\vec{\psi}_x \times \vec{\psi}_\varphi|| = \sqrt{(-r\cos\varphi)^2 + (-r\sin\varphi)^2} = r$$

Since we are only interested in the buoyant force, which acts along the z axis, we only take the z component of the resulted cross product, as the force components that act at other different angles get cancelled out. Therefore the unit normal vector following Eq 8 that acts along the z direction is:

$$\hat{\mathbf{n}}_z = \frac{-r\cos\varphi\hat{\mathbf{k}}}{r} = -\sin\varphi\hat{\mathbf{k}} \tag{10}$$

From Figure 5, we can see the differential area element dA can be characterized by multiplying its differential arc length, $rd\varphi$ with its differential length dx to get:

$$dA = rd\varphi dx \tag{11}$$

Therefore, we can combine Eq 11 and Eq 10 in the differential force equation to get:

$$d\vec{F_{ct}} = -p(z)r(-\sin\varphi)\ d\varphi dx\hat{\mathbf{k}}$$

Where $p(z)$ in this case is the pressure at the depth of the surface of the top cylinder. Considering the submarines are often submerged at some depths, we assume the depth of the submarine to the x axis is h_0 (see Figure 5). This gives us the pressure function of $p(h_0 - r\sin\varphi)$. Substituting into the above equation gives:

$$\vec{F_{ct}} = \iint_{ct} \rho g(h_0 - r\sin\varphi)r\sin\varphi d\varphi dx\hat{\mathbf{k}} \tag{12}$$

Where the integration upper and lower limits are given by the boundaries of x and φ mentioned previously. Substituting the upper and lower limits simplifies the integration to:

$$\vec{F_{ct}} = \int_{x=-L}^{L}\int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho g r h_0 \sin\varphi\ d\varphi dx - \int_{x=-L}^{L}\int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho g r^2 \sin^2\varphi\ d\varphi dx\ \hat{\mathbf{k}}$$

The second term can be integrated by a trigonometric identity of $sin^2\varphi = \frac{1-\cos 2\varphi}{2}$, which results in the buoyant force of the top cylindrical surface:

$$\begin{aligned} \vec{F_{ct}} &= -\rho g\pi r^2 L\ \hat{\mathbf{k}} \\ &= \rho g\pi r^2 L\ \hat{\mathbf{k}} \end{aligned} \tag{13}$$

Where the negative sign is removed due to the conventional coordinate system defined in our model - which has the positive $\hat{\mathbf{k}}$ is pointing upwards. Similarly, we can compute the hydrostatic force along the bottom surface of the cylinder by simply changing the pressure function from Eq 12. The new pressure function is now given by: $p(h_0 + r\sin\varphi)$, and due to the symmetry in the cylinder, the limits of integration do not change for the bottom half of the cylinder. This gives us the force equation:

$$\vec{F_{cb}} = \iint_{cb} \rho g(h_0 + r\sin\varphi)r\sin\varphi\ d\varphi dx\ \hat{\mathbf{k}}$$

Using the same integration upper and lower limits, this gives us:

$$\vec{F_{cb}} = \rho g \pi r^2 L\ \hat{\mathbf{k}} \tag{14}$$

Therefore, the buoyant force on the entire cylindrical surface is given by the sum of the forces of the bottom surface and top surface. Using Eq 13 and Eq 14 gives us:

$$\begin{aligned} \vec{F_c} &= \vec{F_{cb}} + VecF_{ct} \\ &= 2\rho g \pi r^2 L\ \hat{\mathbf{k}} \end{aligned} \tag{15}$$

We use this information later on after we compute the buoyant force on the hemispheres. From Eq 15, we see that it obeys Archimedes' principle, characterized by Eq. 1: $F_B = \rho g V$, where V the volume here is the volume of the cylinder pir^2L.

4 Buoyant force of the hemispheres

As mentioned in the Background Information, our submarine has two hemispheres attached at the two ends with radius r, where it can be redrawn as a sphere as shown in Figure 6 due to the symmetry: Figure 6 shows how instead of expressing the coordinates in Carte-

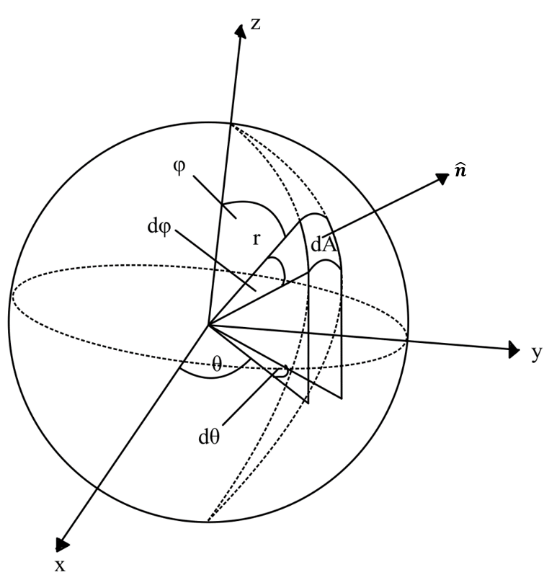

Figure 6: A sphere defined in spherical coordinates

sian coordinate, we express them in spherical coordinates. The relationship between these coordinate systems are given by (Math Insight):

$$\begin{aligned} x &= r \sin\varphi \cos\theta \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\varphi \end{aligned} \tag{16}$$

Therefore, for the Cartesian function of a sphere: $x^2 + y^2 + z^2 = r^2$, it can be represented parametrically as a vector equation, $\vec{\alpha}$ using spherical coordinates, just like the cylinder:

$$\vec{\alpha}(\theta, \varphi) = r \cos\theta \sin\varphi \hat{\mathbf{i}} + r \sin\theta \sin\varphi \hat{\mathbf{j}} + r \cos\varphi \hat{\mathbf{k}}$$

Where the two parameters θ and φ are bounded depending on the hemispheres that we are dealing with. For the top hemisphere, these parameters are bounded by: $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi/2$, whereas the bottom hemisphere has the same boundary for θ but is bounded by a different φ: $\pi/2 \leq \varphi \leq \pi$. We can compute the force exerted on the top hemispheres through the unit normal vector. Calculating the vector equation's partial derivatives with respect to its two parameters:

$$\begin{aligned} \vec{\alpha}_\theta &= \frac{\partial \vec{\alpha}}{\partial \theta} = -r \sin\varphi \sin\theta \hat{\mathbf{i}} + r \cos\theta \sin\varphi \hat{\mathbf{j}} \\ \vec{\alpha}_\varphi &= \frac{\partial \vec{\alpha}}{\partial \varphi} = r \cos\theta \cos\varphi \hat{\mathbf{i}} + r \sin\theta \cos\varphi \hat{\mathbf{j}} - r \sin\varphi \hat{\mathbf{k}} \end{aligned}$$

Computing the cross product between the two vector quantities:

$$\vec{\alpha}_\theta \times \vec{\alpha}_\varphi = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -r \sin\varphi \sin\theta & r \cos\theta \sin\varphi & 0 \\ r \cos\theta \cos\varphi & r \sin\theta \cos\varphi & -r \sin\varphi \end{vmatrix} \tag{17}$$

Using the similar method of the Cofactor method of matrices, we obtain the simplified expression:

$$\vec{\alpha}_\theta \times \vec{\alpha}_\varphi = -r^2 \sin^2\varphi \cos\theta \hat{\mathbf{i}} - r^2 \sin^2\varphi \sin\theta \hat{\mathbf{j}} - r^2 \sin\varphi \cos\varphi \hat{\mathbf{k}}$$

Therefore, the magnitude of the above expression is:

$$\begin{aligned}
||\vec{\alpha}_\theta \times \vec{\alpha}_\varphi|| &= \sqrt{(-r^2 \sin^2\varphi \cos\theta)^2 + (r^2 \sin^2\varphi \sin\theta)^2 + (r^2 \sin\varphi \cos\varphi)^2} \\
&= \sqrt{r^4(\sin^4\varphi \cos^2\theta + \sin^4\varphi \sin^2\theta + \sin^2\varphi \cos^2\varphi)} \\
&= r^2\sqrt{(cos^2\theta + \sin^2\theta)\sin^4\varphi + sin^2\varphi \cos^2\varphi}
\end{aligned}$$

Where applying the trigonometric identity of $\sin^2 x + \cos^2 x = 1$ simplifies the magnitude of the cross product to:

$$||\vec{\alpha}_\theta \times \vec{\alpha}_\varphi|| = r^2 \sin\varphi$$

We come to the unit normal vector of the hemisphere as a function of the parameter φ, and taking its z component only (as explained previously due to our interest in the direction of the buoyant force which is along the z axis):

$$\hat{\mathbf{n}}_z = \frac{\vec{\alpha}_\theta \times \vec{\alpha}_\varphi}{||\vec{\alpha}_\theta \times \vec{\alpha}_\varphi||} = \frac{-r^2 \sin\varphi \cos\varphi\, \hat{\mathbf{k}}}{r^2 \sin\varphi} = -\cos\varphi\, \hat{\mathbf{k}}$$

We can also derive the differential area element based on Figure 6, its "length" is characterized by the arc length, which is $rd\varphi$ and its "width" changes with the angle φ as the area element is bounded by the two dotted lines, where the smaller the angle φ is, the smaller the 'width" is. Taking this into account, the "width" can be characterized with $r \sin\varphi d\theta$. Combining the "length" and "width" gives us the differential area element:

$$\begin{aligned}
dA &= rd\varphi\, r\sin\varphi d\theta \\
&= r^2 \sin\varphi d\theta d\varphi
\end{aligned} \tag{18}$$

Finally, we can present all of these quantities in the differential force equation that acts along the top surface of the hemisphere:

$$d\vec{F_{st}} = -p(z)\hat{\mathbf{n}}_z dA$$

The pressure function for the top surface is given by $p(h_0 - r\cos\varphi)$ as seen from Figure 6. Thus, combining all of the previously derived expressions, we get the following integral:

$$\vec{F_{st}} = -\iint_{st} \rho g(h_0 - r\cos\varphi)(-\cos\varphi)\, r^2 \sin\varphi d\theta d\varphi\ \hat{\mathbf{k}}$$

Simplifying the expression and substituting the lower and upper limits of integration gives us:

$$\vec{F_{st}} = \int_{\varphi=0}^{\pi/2}\int_{\theta=0}^{2\pi} \rho g h_0 r^2 \cos\varphi \sin\varphi d\theta d\varphi - \int_{\varphi=0}^{\pi/2}\int_{\theta=0}^{2\pi} \rho g r^3 \cos^2\varphi \sin\varphi d\theta d\varphi\ \hat{\mathbf{k}}$$

Integrating the above expression through substitution gives us:

$$\vec{F_{st}} = -\rho g \pi r^2 \left(-h_0 - \frac{2r}{3}\right)\ \hat{\mathbf{k}} \tag{19}$$

Where the negative sign comes from the force pointing downwards (against our defined coordinate system of $+\hat{\mathbf{k}}$ as upwards) as a result of pressure acting normally downwards on the top surface. Similarly to the case of the cylinder, we compute the hydrostatic force of the bottom surface of the hemisphere by using a different pressure function. The boundary that defines the bottom hemispherical surface is also different, where the parameter φ is defined to be $\pi/2 \leq \varphi \leq \pi$, and that the pressure function is defined to be $p(h_0 + r\cos\varphi)$. Making these changes to the integration gives us:

$$\vec{F_{sb}} = \rho g \pi r^2 \left(-h_0 + \frac{2r}{3}\right) \tag{20}$$

We can compute the buoyant force of the sphere by summing the hydrostatic force acting along the top and bottom surfaces of the sphere:

$$\begin{aligned} \vec{F_s} &= \vec{F_{sb}} + \vec{F_{st}} \\ &= \frac{4}{3}\rho g \pi r^3 \hat{\mathbf{k}} \end{aligned} \tag{21}$$

As seen from Eq. 21, this again agrees with Eq. 1, where the buoyant force only encompasses gravitational acceleration (g), density of fluid (ρ), and the volume of the displaced fluid or volume of the object. In this case, the volume of the object is a sphere, which is $\frac{4}{3}\rho g\pi r^3$, enabling us to understand that computing the hydrostatic forces along the top and bottom surfaces thus in fact give us the buoyant force.

5 Water to air ratio

To compute the water to ratio, we need to find the final buoyant force of our entire submarine model, which includes the hemispheres at the ends and the cylindrical body. It can be written as as the sum of the magnitudes of the individual hydrostatic forces derived above:

$$F_B = F_c + F_s$$

Using Eq. 21 and Eq. 15, we get:

$$F_B = 2\rho g\pi r^2 \left(\frac{3L + 2r}{3}\right) \tag{22}$$

Eq. 22 is the summary equation of the previous computations regarding the pressure difference between the top and bottom surfaces of our submarine model. We use this to calculate the neutral buoyancy of our submarine. But first, we must identify the different weights in our submarine. We recall a more sophisticated diagram as shown in Figure 7

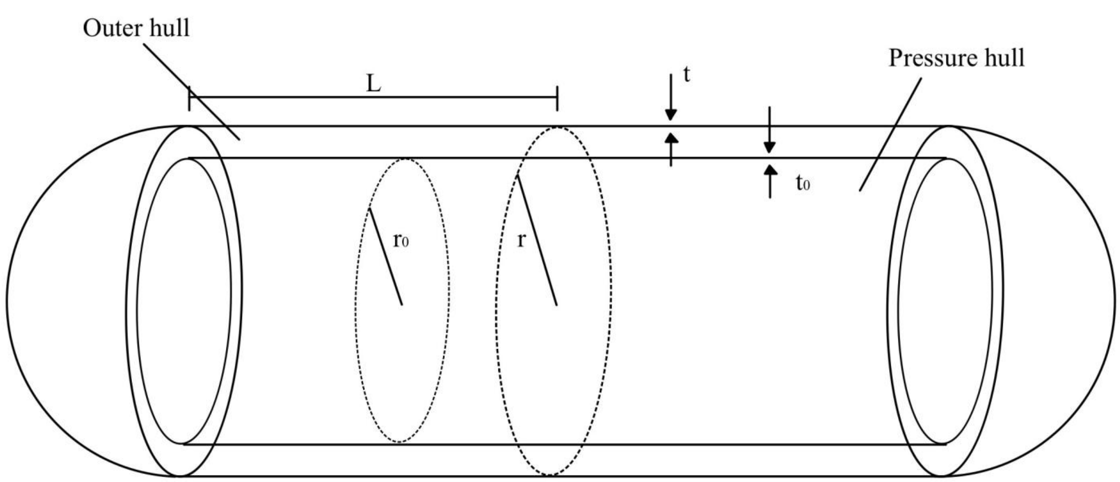

Figure 7: The model of the submarine showcasing our outer hull and pressure hull, with their corresponding thicknesses

As seen from Figure 7, there is a certain thickness associated with the outer hull and pressure hull, and we have defined that the pressure hull has the same length as that of the cylindrical component of the outer hull. We identify that there are three major weights associated with our model submarine. First being the weight of the pressure hull - which includes the weight of the pressure hull shell and the weight of the air inside - denoted as F_p. Second being the weight of the outer hull - which is only the weight of the outer hull shell - denoted F_{out}. Lastly, the weight of the water to air ratio in the ballast tank - defined to be the region between the inner wall of the outer hull and the outer wall of the pressure hull - denoted to have a volume of $V_{between}$.

Based on Figure 7, we can see that the weight of the pressure hull is:

$$F_p = g\rho_s \pi 2L((r_0 + t_0)^2 - r_0^2) + \rho_{air}\pi 2L r_0^2$$

Where $\rho_s = 7850kg/m^3$ (All about density) is the density of steel, the assumed material of our submarine, $\rho_{air} = 1.25kg/m^3$ (USNavyMuseum) is the density of air. No literature value was found regarding r_0 and t_0, which are the radius and thickness of the pressure hull respectively, so we make an assumption that $r_0 = 4m$ and $t_0 = 0.05m$ based on the two literature values provided by the US navy (presented later). The length of the Virginia Class Submarine is given to be $L = 377ft \approx 114.91m$ (Naval Technology). We can substitute all of these values to get: $F_p = 3.38 \times 10^7 N$. The weight of the outer hull is defined to be:

$$m_{out} = m_{sphere} + m_{cylinder} = \rho_s \frac{4}{3}((r+t)^3 - r^3) + \rho_s \pi 2L((r+t)^2 - r^2)$$

Where here we have the radius of the outer hull to be given by the US navy as $r = 34ft \approx 10.3632m$ (Naval Technology), and the thickness of the outer hull assumed to be $t = 0.003m$ as the outer hull does not need to withstand a lot of pressure, instead it has the same pressure as the surrounding water. Substituting all of these values into the above equation gives: $F_out = 2.04 \times 10^6 N$. Lastly, we compute $V_{between}$ as follows:

$$V_{between} = \frac{4}{3}\pi r^3 + 2\pi r^2 L - 2\pi L(r_0 + t_0)^2$$

Substituting the values again gives us: $V_{between} = 2.9 \times 10^4 m^3$ Finally, we combine all of these weights and the volume for air and water to coexist and have it equated with our buoyant force:

$$F_B = F_{out} + F_p + gV_{between}(\rho_w \lambda + \rho_{air}(1 - \lambda))$$

Where $\rho_w = 997 kg/m^3$ is the density of water and the factor λ and $1 - \lambda$ are associated with the ratio of the volume between the outer and pressure hull that allows the submarine to be in neutral buoyancy. In this case, λ is the ratio of water required as it is multiplied to the density of water. We can rearrange this equation such that λ is on the LHS:

$$\lambda = \frac{\frac{F_B - F_{out} - F_p}{gV_{between}} - \rho_{air}}{\rho_w - \rho_{air}} \tag{23}$$

Substituting all of the numbers into Eq 23 give us the following ratio:

$$\lambda_w = 0.0806$$

$$\lambda_{air} = 0.9194$$

Where λ_w is the ratio of $V_{between}$ needed to be water and λ_{air} is that needed to be air. It can be clearly seen from the magnitude of these numbers that the submarine needs a lot more air to achieve neutral buoyancy than water.

6 Evaluation and Conclusion

In this investigation, we explored the water to air ratio of the Virginia Class Submarine to achieve neutral buoyancy through modelling it with simpler geometries (two hemispheres and a cylinder). We were able to derive the buoyant force of our submarine model through the summation of the hydrostatic forces acting along the surfaces. The particular changes in coordinate systems for both geometries - spherical coordinate for the hemispheres - and cylindrical coordinate for the cylinder, were used to simplify our model further. The derived buoyant force was then combined with the weights of the different components of our modelled submarine to result in the water to air ratio needed to achieve neutral buoyancy.

Our model suggests that there must a great proportion of air occupying the volume between the outer and pressure hulls in order for the submarine to be in neutral buoyancy. Although this could not be verified with literature values, there were many assumptions made along our development of the model. We assumed the geometry of the submarine to be cylindrical in the middle and have almost spherical ends. However, this is not necessarily the case for the Virginia Class Submarine, as they have a more narrowed end. This problem could be addressed by perhaps replacing one of the hemisphere on the end with an elliptical parabolic, which has a more sharp end. Another assumption made was the dimensions of the submarine. Due to insufficient literature values provided by the US Navy, primarily due to them being of particular interest of other nations' naval system, we made educated assumptions about the thicknesses and radii. We also assumed that the additional components of the submarine (i.e., periscopes, propellers, machinery in the pressure hull, etc.) were not to be taken into account due to the complication our model could result in. The more practical applications of weight distributions and constant loss in weight with the loss of food, defects, etc. were also not taken into account, but must be precisely monitored in real submarines.

Overall, this paper had provided insights into the basis of Archimedes' Principle, and the practical application to modelling the water to air ratio of a Virginia Class Submarine.

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