

Group Target Tracking with the Gaussian Mixture Probability Hypothesis Density Filter

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Abstract

The Probability Hypothesis Density (PHD) filter was originally devised to address non-conventional tracking problems such as group target processing, tracking in high target density, tracking closely spaced targets and detecting targets of interest in a dense multi-target background. The intention was to track overall group behaviour, and then attempt to track individual targets and then attempt to detect and track individual targets only as the quantity and quality of the data permits. Despite this, most practical implementations of the PHD filter have been applied to standard multi-target tracking problems and there have been few implementations of the PHD filter for tackling groups of targets. In this work, we investigate some practical strategies for group tracking with the Gaussian mixture implementation of the PHD filter.

1. INTRODUCTION

The purpose of multiple target tracking algorithms is to detect, track and identify targets from sequences of noisy observations of the targets provided by one or more sensors. This problem is complicated by the fact that these observations tend to have many false alarms and targets may not always give rise to observations. A common assumption in the target tracking literature is that each target produces only one observation per scan and these targets move independently. In practise, this is not always true, since targets may produce multiple observations, known as extended objects, and targets may move in common formation, known as group targets. If the formation of the group is constrained, these two scenarios have some similarities and can be formulated in a similar manner.

From 1997-1999, Salmond and Gordon [1], [2], [3] investigated methods for group and extended object tracking by allowing individual components to move independently but with a 'bulk' component that describes the evolution of the group. Techniques based on Multiple Hypothesis Tracking [4] were used to identify the targets in clutter and these were tested in scenarios for tracking rigid objects and groups of point targets. The number of targets in these examples was known and fixed, and this was only used on tracking single groups. In 2005, Gilholm and Salmond [5] developed a spatial distribution model for tracking extended objects in clutter, where the number of observations from the target is assumed to be Poisson distributed. Based on this approach Poisson

likelihood models for group and extended object tracking were developed [6], [7]. In 2007, Mahler [8], and Vo, Vo and Cantoni [9], derived methods for tracking single extended objects based on random-set techniques.

A random-set filtering approach for tracking group-targets was proposed in 2001 by Mahler [10], [11] using an extended first-order Bayes filter for force aggregation. The principle of this was to extend Mahler's Probability Hypothesis Density filter (PHD) [12] to incorporate group dynamics and group likelihoods. Mahler also proposed an alternative PHD update for this approach for cluster tracking [13]. There have been no practical implementations of these techniques yet developed, although the techniques proposed in this paper could in principle be formulated within this framework. In 2003, Mahler and Zajic [14] describe a particle filter implementation of the PHD filter for tracking a 'bulk' of targets i.e. they do not attempt to determine individual target states from the PHD filter but observe that the characteristics of a group of closely spaced targets can be found as peaks of the intensity function.

In this paper, we develop the first methods for group tracking using the Gaussian mixture PHD filter [15], [16] using a similar approach to the 'bulk PHD filter'. In our approach, we explicitly identify group targets and their constituent members by creating a graph of connected components. This allows us to constrain the evolution of the group.

In the next section we present the multiple target Bayes filter and its first-order approximation, known as the PHD filter. In section 3, we describe the Gaussian mixture PHD filter and methods for tracking. Section 4 is the main contribution of the paper where we present our implementation of the Gaussian mixture PHD filter group target tracking and discuss two techniques for modelling group transitions: the virtual leader-follower model [8] and the Cucker-Smale flocking model [17], [18]. Simulations of these techniques are given in section 5 and we conclude in section 6.

2. RANDOM SET FILTERING

The multiple-target tracking framework based on random-sets was proposed by Mahler [8] to unify the problems of detecting, identifying, classifying and tracking targets within a unified Bayesian paradigm. In this section, we describe the multiple-target generalisation of a Bayes filter and its first-order approximation.

A. The Multiple Target Bayes Filter

The optimal multi-target Bayes filter propagates the multi-target posterior density $p_k(\cdot|Z_{1:k})$ conditioned on the sets of observations up to time k , $Z_{1:k}$, with the following recursion

$$p_{k|k-1}(X_k|Z_{1:k-1}) = \int f_{k|k-1}(X_k|X)p_{k-1}(X|Z_{1:k-1})\mu_s(dX), \quad (1)$$

$$p_k(X_k|Z_{1:k}) = \frac{g_k(Z_k|X_k)p_{k|k-1}(X_k|Z_{1:k-1})}{\int g_k(Z_k|X)p_{k|k-1}(X|Z_{1:k-1})\mu_s(dX)}, \quad (2)$$

where the dynamic model is governed by the transition density $f_{k|k-1}(X_k|X_{k-1})$ and multi-target likelihood $g_k(Z_k|X_k)$ and μ_s takes the place of the Lebesgue measure, as described in [19].

The function $g_{k|k}(Z_k|X_k)$ is the joint multi-target likelihood function, or global density, of observing the set of measurements, Z , given the set of target states, X , which is the total probability density of association between measurements in Z and parameters in X . The parameters for this density are the set of observations, $Z = \{z_1, \dots, z_k\}$, the unknown set of target states, $X = \{x_1, \dots, x_k\}$, the sensor noise distribution, or observation noise, clutter models, and the detection profile of sensor.

The complexity of computing the multi-target Bayes recursion grows exponentially with the number of targets and is thus not practical for more than a few targets. The Probability Hypothesis Density filter [20] was proposed as a practical suboptimal alternative to computing the full multiple-target posterior distribution by propagating the first-order moment statistic.

B. The Probability Hypothesis Density Filter

Let v_k and $v_{k|k-1}$ denote the respective intensities for the multi-target prediction and update recursion. The prediction equation is given by

$$v_{k|k-1}(x) = \int p_{S,k}(\zeta)f_{k|k-1}(x|\zeta)v_{k-1}(\zeta)d\zeta + \gamma_k(x), \quad (3)$$

where $f_{k|k-1}(\cdot|\zeta)$ is the single target transition density at time k , $p_{S,k}(\zeta)$ is the probability of target survival at time k , $\gamma_k(\cdot)$ is the intensity of spontaneous births at time k . The update equation is given by

$$v_k(x) = [1 - p_{D,k}(x)]v_{k|k-1}(x) + \sum_{z \in Z_k} \frac{p_{D,k}(x)g_k(z|x)v_{k|k-1}(x)}{\kappa_k(z) + \int p_{D,k}(\xi)g_k(z|\xi)v_{k|k-1}(\xi)d\xi}, \quad (4)$$

where Z_k is the measurement set at time k , $g_k(\cdot|x)$ is the single target measurement likelihood at time k , $p_{D,k}(x)$ is the probability of target detection at time k , and $\kappa_k(\cdot)$ is the intensity of clutter measurements at time k .

Practical implementations of this filter include a spectral compression technique [21], particle filtering [22], [23], [19], and Gaussian mixture versions [15], [24].

3. THE GAUSSIAN MIXTURE PHD FILTER

A closed-form solution to the PHD filter was derived by Vo and Ma [15] under linear assumptions on the system and observation equations and Gaussian process and observation noises, called the Gaussian Mixture PHD (GM-PHD) filter. The multiple target states in the GM-PHD filter are estimated by taking the Gaussian components with highest weights. This leads naturally to an extension of the GM-PHD filter which allows the evolution of individual target states to be determined over time, which ensures the continuity of the target tracks [16]. This has been used for tracking objects in forward-scan sonar images [25].

The assumptions on the Gaussian mixture PHD filter are as follows: Each target follows a linear Gaussian dynamical and observation model i.e.

$$f_{k|k-1}(x|\zeta) = \mathcal{N}(x; F_{k-1}\zeta, Q_{k-1}), \quad (5)$$

$$g_k(z|x) = \mathcal{N}(z; H_k x, R_k), \quad (6)$$

where $\mathcal{N}(\cdot; m, P)$ denotes a Gaussian density with mean m and covariance P , F_{k-1} is the state transition matrix, Q_{k-1} is the process noise covariance, H_k is the observation matrix, and R_k is the observation noise covariance. The survival and detection probabilities are state independent, i.e. $p_{S,k}(x) = p_{S,k}$ and $p_{D,k}(x) = p_{D,k}$. The PHD for the birth of new targets is a Gaussian mixture of the form

$$\gamma_k(x) = \sum_{i=1}^{J_{\gamma,k}} w_{\gamma,k}^{(i)} \mathcal{N}(x; m_{\gamma,k}^{(i)}, P_{\gamma,k}^{(i)}), \quad (7)$$

where $J_{\gamma,k}$, $w_{\gamma,k}^{(i)}$, $m_{\gamma,k}^{(i)}$, $P_{\gamma,k}^{(i)}$, $i = 1, \dots, J_{\gamma,k}$, are given model parameters that determine the birth intensity.

GM-PHD Initialisation

At time $k = 0$, the initial intensity, v_0 , is the sum of J_0 Gaussians,

$$v_0(x) = \sum_{i=1}^{J_0} w_0^{(i)} \mathcal{N}(x; m_0^{(i)}, P_0^{(i)}). \quad (8)$$

where each Gaussian $\mathcal{N}(x; m_0^{(i)}, P_0^{(i)})$ has mean state vector $m_0^{(i)}$, covariance $P_0^{(i)}$ and weight $w_0^{(i)}$. In addition, an identifying label, $L_0^{(i)}$ is assigned to each Gaussian component i , to enable track continuity,

$$L_0 = \{L_0^{(1)}, \dots, L_0^{(J_0)}\}. \quad (9)$$

GM-PHD Prediction

The predicted intensity from time $k - 1$ to time k is the Gaussian mixture,

$$v_{k|k-1}(x) = v_{S,k|k-1}(x) + \gamma_k(x), \quad (10)$$

where $\gamma_k(x)$ is the birth PHD and

$$v_{S,k|k-1}(x) = p_{S,k} \sum_{j=1}^{J_{k-1}} w_{k-1}^{(j)} \mathcal{N}(x; m_{S,k|k-1}^{(j)}, P_{S,k|k-1}^{(j)}), \quad (11)$$

$$m_{S,k|k-1}^{(j)} = F_{k-1} m_{k-1}^{(j)}, \quad (12)$$

$$P_{S,k|k-1}^{(j)} = Q_{k-1} + F_{k-1} P_{k-1}^{(j)} F_{k-1}^T. \quad (13)$$

The labels from the previous time step are concatenated with new labels assigned to the birth components,

$$L_{k|k-1} = L_{k-1} \cup \{L_{\gamma_k}^{(1)}, \dots, L_{\gamma_k}^{(J_{\gamma_k})}\}. \quad (14)$$

GM-PHD Update

The posterior intensity at time k is given by Gaussian Mixture,

$$v_k(x) = (1 - p_{D,k}) v_{k|k-1}(x) + \sum_{z \in Z_k} v_{D,k}(x; z) \quad (15)$$

where

$$v_{D,k}(x; z) = \sum_{j=1}^{J_{k|k-1}} w_k^{(j)}(z) \mathcal{N}(x; m_{k|k}^{(j)}(z), P_{k|k}^{(j)}), \quad (16)$$

$$w_k^{(j)}(z) = \frac{p_{D,k} w_{k|k-1}^{(j)} q_k^{(j)}(z)}{\kappa_k(z) + p_{D,k} \sum_{\ell=1}^{J_{k|k-1}} w_{k|k-1}^{(\ell)} q_k^{(\ell)}(z)}, \quad (17)$$

$$q_k^{(j)}(z) = \mathcal{N}(z; H_k m_{k|k-1}^{(j)}, R_k + H_k P_{k|k-1}^{(j)} H_k^T), \quad (18)$$

$$m_{k|k}^{(j)}(z) = m_{k|k-1}^{(j)} + K_k^{(j)}(z - H_k m_{k|k-1}^{(j)}), \quad (19)$$

$$P_{k|k}^{(j)} = [I - K_k^{(j)} H_k] P_{k|k-1}^{(j)}, \quad (20)$$

$$K_k^{(j)} = P_{k|k-1}^{(j)} H_k^T (H_k P_{k|k-1}^{(j)} H_k^T + R_k)^{-1}. \quad (21)$$

There are $(1 + |Z_k|)$ Gaussian components for each prediction term. We assign the same label as its related prediction component to form the set,

$$L_k = L_{k|k-1}^{v_{k|k-1}} \cup L_{k|k-1}^{z_1} \cup \dots \cup L_{k|k-1}^{z_{|Z_k|}}. \quad (22)$$

GM-PHD Pruning and Merging

Components falling below a given threshold are pruned and the remaining components are reweighted accordingly. If the distance between the means of the Gaussian components, defined by the covariance matrix, falls within a merging threshold τ_m , then they are merged. More specifically, starting with the component with the largest weights $w_k^{(j)}$, we merge components in set $M_k^{(j)}$,

$$M_k^{(j)} := \{i : (m_k^{(i)} - m_k^{(j)})^T (P_k^{(i)})^{-1} (m_k^{(i)} - m_k^{(j)}) \leq \tau_m\}. \quad (23)$$

with

$$\begin{aligned} \tilde{w}_k^{(\ell)} &= \sum_{i \in M_k^{(j)}} w_k^{(i)}. \\ \tilde{x}_k^{(\ell)} &= \frac{1}{\tilde{w}_k^{(\ell)}} \sum_{i \in M_k^{(j)}} w_k^{(i)} x_k^{(i)}. \\ \tilde{P}_k^{(\ell)} &= \frac{1}{\tilde{w}_k^{(\ell)}} \sum_{i \in M_k^{(j)}} w_k^{(i)} (P_k^{(i)} + (\tilde{m}_k^{(\ell)} - m_k^{(i)})(\tilde{m}_k^{(\ell)} - m_k^{(i)})^T). \end{aligned} \quad (24)$$

In the case where two or more components have the same label $L_k^{(i)}$, we assign this label to the component with the largest weight $\tilde{w}_k^{(i)}$ and reassign new labels to the other components.

GM-PHD State Estimation and Track Continuity

The target states are determined by taking the Gaussians with weights above a threshold and those which have previously defined to be a target, so that the set of live tracks from time k is

$$\hat{L}_k = \{L_k^{(i)} : w_k^{(i)} > \tau_E\} \quad (25)$$

and the set of target state estimates is

$$\hat{X}_k = \{(m_k^{(i)}, P_k^{(i)}) : L_k^{(i)} \in \hat{L}_k, j = 1, \dots, k\}. \quad (26)$$

4. GROUP TRACKING IMPLEMENTATION

In our implementation of the Gaussian mixture PHD filter for tracking groups, we form group targets by creating a graph of connected components of target state estimates. Edges between the state estimates are formed when the position or velocities fall within some distance criterion. The groups formed by this process can then be used to constrain the motion of the constituent Gaussians within the respective groups. Gaussians within the mixture that are not identified as target states are predicted with the usual Kalman prediction.

Let \hat{X}_k be the set of estimated target state means and covariances at time k ,

$$\hat{X}_k = \{(\hat{x}_k^{(i)}, \hat{P}_k^{(i)})\}_{i=1}^{\hat{N}_k}. \quad (27)$$

Create a graph of connected components such that there is an edge between the i^{th} and j^{th} component if they fall within some distance on the positions or velocities, eg.

$$d_p^{(i,j)} := (I_p \hat{x}_k^{(i)} - I_p \hat{x}_k^{(j)}) (\hat{P}_k^{(i)})^{-1} (I_p \hat{x}_k^{(i)} - I_p \hat{x}_k^{(j)})^T < \tau_p, \quad (28)$$

or

$$d_v^{(i,j)} := (I_v \hat{x}_k^{(i)} - I_v \hat{x}_k^{(j)}) (\hat{P}_k^{(i)})^{-1} (I_v \hat{x}_k^{(i)} - I_v \hat{x}_k^{(j)})^T < \tau_v, \quad (29)$$

where I_p and I_v are identities on the positions and velocities respectively (and other elements zero). Thus, the set of estimated target states is partitioned into group states,

$$\hat{X}_k = \bigcup_{g=1}^{\hat{G}_k} \hat{X}_k^{(g)}, \quad (30)$$

where \hat{G}_k is the estimated number of groups and $\hat{X}_k^{(g)}$ contains the estimated target states within group g . These group states can then be used to constrain the Markov transition to evolve as a group.

Track identities for individual targets can be maintained with the Gaussian mixture PHD filter by labelling each Gaussian [16]. We can extend this concept to group target tracking by labelling the groups of state estimates and checking if the constituent components of a group in the previous time-step correspond to those in the current time-step. A simple procedure for doing this is as follows:

At time $k - 1$, each group g has a label $T_{k-1}^{(g)}$. If there is a group g' at time k that contains at least half of the tracks from group g , then assign $T_k^{g'} := T_{k-1}^g$, otherwise a new group label is defined for $T_k^{g'}$.

In the next two subsections, we describe techniques for modelling group transitions.

A. Virtual Leader-Follower Model

The virtual leader-follower type of co-ordinated multi-target motion was first applied by Salmond and Gordon for group and extended object tracking [2], [3]. The underlying assumption is that the deterministic state of any target is a translational offset of the centroid of the group. Salmond and Gordon used this for tracking a rigid object, with a fixed number of vertices (targets). Mahler generalised this model to allow for the introduction of targets into the group and for targets to disappear [8]. We use this model here to constrain the evolution of the different groups, although the number of targets within the groups and the number of groups can vary.

Define $\Delta x_k^{(i)}$ as the offset from individual target i to the centroid of a group,

$$\Delta x_k^{(i)} := x_k^{(i)} - \bar{x}_k, \quad (31)$$

where $\hat{X}_k = \{(\hat{x}_k^{(i)}, \hat{P}_k^{(i)})\}_{i=1}^{\hat{N}_k}$ is the set of target states at time k and \bar{x}_k is the centroid, or virtual leader,

$$\bar{x}_k := \frac{1}{\hat{N}_k} \sum_{i=1}^{\hat{N}_k} x_k^{(i)}. \quad (32)$$

We use the dead reckoning, or constant velocity, linear motion model for group targets described in [8] (pp480-481) on a state-space consisting of 2D positions and velocities. The positions are calculated according to

$$p_{k|k-1}^{(i)} = p_{k-1}^{(i)} + \Delta t \bar{v}_{k-1}, \quad (33)$$

and the velocities with

$$v_{k|k-1}^{(i)} = v_{k-1}^{(i)}, \quad (34)$$

where Δt is the time difference between time k and time $k - 1$.

B. Cucker-Smale Flocking Model

The science of flocking is motivated by studying the behaviour and motion of flocks of birds. Complicated flocking patterns can be explained by assuming that birds follow three simple rules on the separation, alignment and cohesion of the birds [26]. Each bird avoids colliding with neighbouring birds (separation), steers towards the average heading (alignment) and towards the average position (cohesion). In recent work by Cucker and Smale developed a model to describe the emergence of a flock and gave conditions under which this behaviour can spontaneously emerge [17], [18]. The model is parameterised by a constant β capturing the rate of decay of the influence between birds in a flock as they separate in space. It is postulated that each bird adjusts its velocity by adding to it a weighted average of the differences of its velocity with those of other birds. The position of the birds are assumed to follow the usual constant velocity model,

$$p_{k|k-1}^{(i)} = p_{k-1}^{(i)} + \Delta t v_{k-1}^{(i)}, \quad (35)$$

but the velocities are adjusted according to their neighbours' velocities

$$v_{k|k-1}^{(i)} = v_{k-1}^{(i)} + \sum_{j=1}^{\hat{N}_{k-1}} a_{k|k-1}^{(i,j)} (v_{k-1}^{(j)} - v_{k-1}^{(i)}), \quad (36)$$

where the weights $a_{k|k-1}^{(i,j)}$ quantify the way the birds influence each other. More specifically, these form the adjacency matrix

$$a_{k|k-1}^{(i,j)} = \frac{H}{(1 + \|p_{k-1}^{(i)} - p_{k-1}^{(j)}\|^2)^\beta}, \quad (37)$$

for some fixed $H > 0$ and $\beta \geq 0$. Cucker and Smale found conditions under which the birds' velocities converge to a common one and the distance between the birds remains bounded. We introduce this model for the predicted group motion and present some preliminary results with the GM-PHD filter in the next section.

5. SIMULATED RESULTS

In this section, we present some results for group tracking with the Gaussian mixture PHD filter using simulated scenarios. We demonstrate the performance of the tracking with the co-ordinated tracking motions proposed in the previous section and also test the grouping strategy without constraining the motion as groups. Due to shortage of space and for clarity of presentation, in the first two figures we limit the results presented to the virtual leader-follower model. In figure 1, we have an example of two groups of targets over 100 time steps. In the top picture, we can clearly see the motion of 2 groups with individual targets moving in parallel with the same prediction model and the groups formed at each time step. The middle and lower figures show close-ups of these results to better illustrate the performance. Figure 2 illustrates the continuity of the track identities by connecting the centroids of the groups over time. If all the targets are not identified at each time step, then this value can fluctuate somewhat, and so it is not the best indicator of the performance of the group tracking.

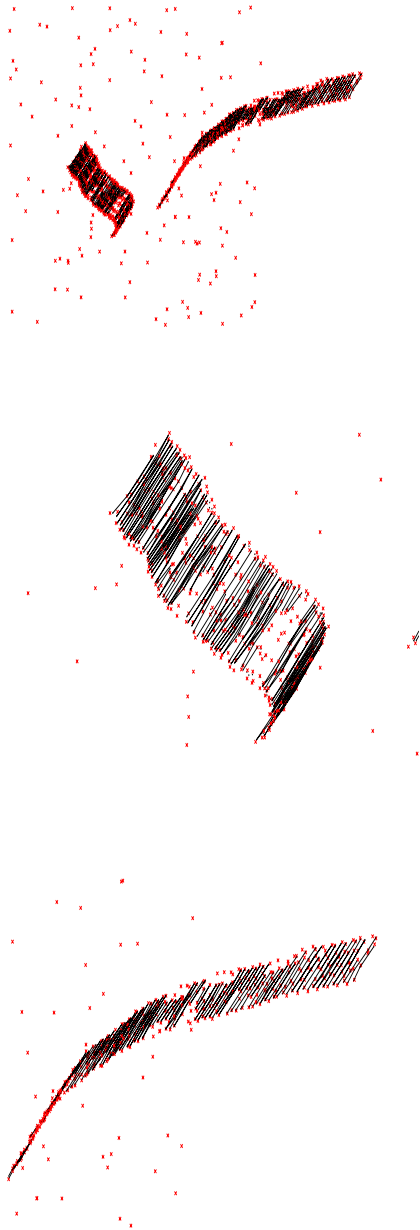


Fig. 1: Example of groups as connected components: both estimated groups (top); left group (middle); right group (bottom).

The variation in the number of targets estimated could be improved by using the Gaussian mixture implementation of the CPHD filter [27], [28]. Figure 3 shows the estimated number of groups for the three different types of group motion. The virtual leader-follower seems to estimate the correct number of groups better here, although the motion model used favours this approach. It is expected that further investigation of the flocking model for different parameter values and situations could yield improved performance.

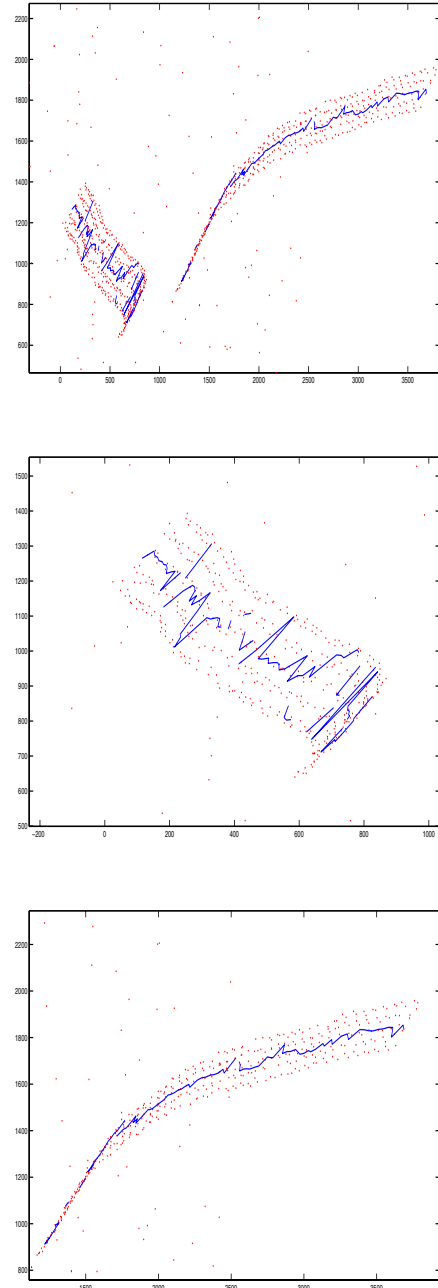


Fig. 2: Estimated centroids of groups: both estimated groups (top); left group (middle); right group (bottom).

6. CONCLUSIONS

In this paper, we have investigated methods for multi-group multi-target tracking with the Gaussian mixture PHD filter and presented some preliminary results. The main idea in this work is to group the state estimates as connected components of a graph, by forming edges if the positions and velocities are within a distance determined by the state covariances, and using a co-ordinated multi-target motion model for the group dynamics. We have shown that it is possible to maintain

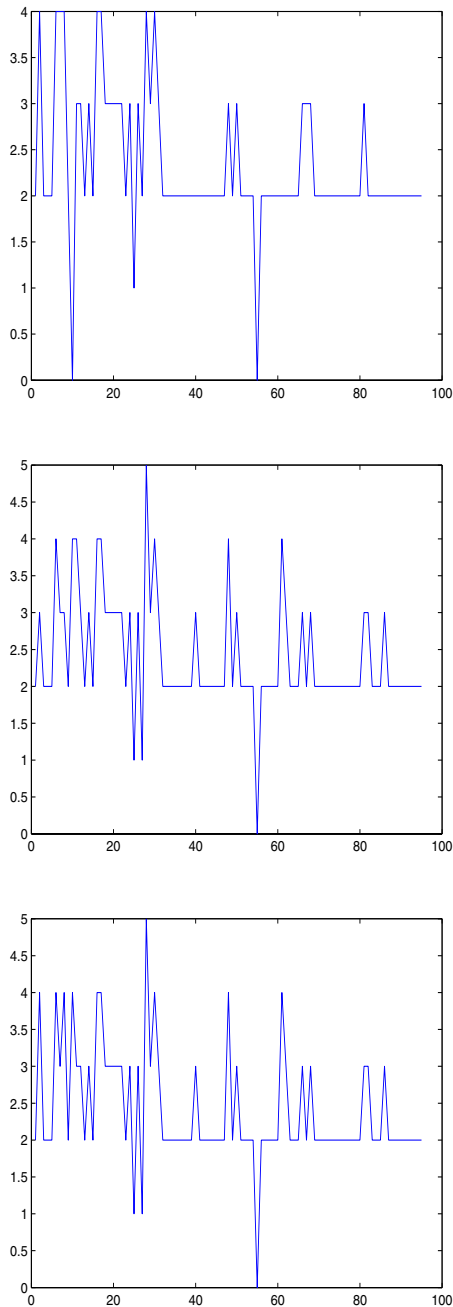


Fig. 3: Estimated number of groups over time: virtual leader-follower motion (top); flock motion (middle); no group motion constraint (bottom).

group-track identities over time using track continuity ideas developed for implementations of the PHD filter. Future work will include testing these methods on real data to validate the techniques in practical situations.

ACKNOWLEDGMENTS

The authors would like to thank Ronald Mahler for introducing them to the Cucker-Smale flocking model. This work was funded by the DIF-DTC Tracking Cluster.

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