hw1_p5

February 1, 2026

1 CS 498 HW1 Problem 5

```python
import numpy as np
from IPython.display import display, Math
import sympy as sp

def latexPrint(A):
    display(Math(sp.latex(sp.Matrix(A))))
    return sp.latex(sp.Matrix(A))

A = np.array([[1,1],[1,3]])
b = np.array([5, 11])
```

The active constraint is (1) and (2) since

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

This also shows (1),(2),(3),(4) are all set with positivity obvious

1.1 Possible c

Definition Normal Cone Let $\Omega \subset \mathbb{R}^n$ be convex, the normal cone to Ω at x is defined as:

$$N_\Omega(x) = \{x \in \mathbb{R}^n : \langle x, y - x \rangle \leq 0, \forall y \in \Omega\}$$

For a point x to be a maximizer, we have to have $\nabla f(x) \in N_\Omega(x)$, in our case:

$$N_\Omega(2,3) = \{(2,3) + \alpha(1,1) + \beta(1,3) : \alpha, \beta > 0\}$$

Proposition Let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, x \in \mathbb{R}^n$. Let Ω be the region given by $Ax \leq b$. Suppose

$$Ax \leq b$$

then the *normal cone* of Ω defines all possible $c \in \mathbb{R}^n$ such that the LP $\Pi = (A, b, c)$ has a solution at x

A simple explanation: for x to be a maximizer, towards any v we should have $\nabla f \cdot D_v f(x) \leq 0$. If we assume star-convexity, then the resulting possible c is equivalent to the definition of normal cone as above.

```python
[109]:  import matplotlib.pyplot as plt

        x = np.linspace(0, 10, 500)
        f1 = (5 - x)
        f2 = (11-x)/3
        f =np.where(f1<f2, f1, f2)
        fig, ax = plt.subplots()
        ax.plot(x, f)
        ax.plot(x, f2)
        ax.fill_between(x, y1=0, y2=f, facecolor='green', alpha=.3)

        g1 = x + 1
        g2 = 3*x -3
        ax.fill_between(x, g1, g2, where=x>2.0, facecolor='grey', alpha=.3)

        plt.text(2.6, 4.5, r'$N_{\Omega}(2,3)$', fontsize=14)
        plt.text(1.5, 1.5, r'$\Omega$', fontsize=14)
        plt.ylim(0,10)


        def random_from_normal_cone(point, lower_bound_func, upper_bound_func,␣
         ↪interval_len=10.0, number=10):
            x,y=point
            x_ = x+ interval_len * np.random.rand(number)
            y_ = lower_bound_func(x_) + (upper_bound_func(x_) - lower_bound_func(x_)) *␣
         ↪np.random.rand(number)
            return x_, y_

        number=10
        x, y = random_from_normal_cone((2,3), lambda x : x+1, lambda x : 3*x-3,␣
         ↪number=number)
        for i in range(number):
            ax.plot(x[i],y[i], marker='o', color='black', alpha=1, markersize=1)
        plt.show()
```
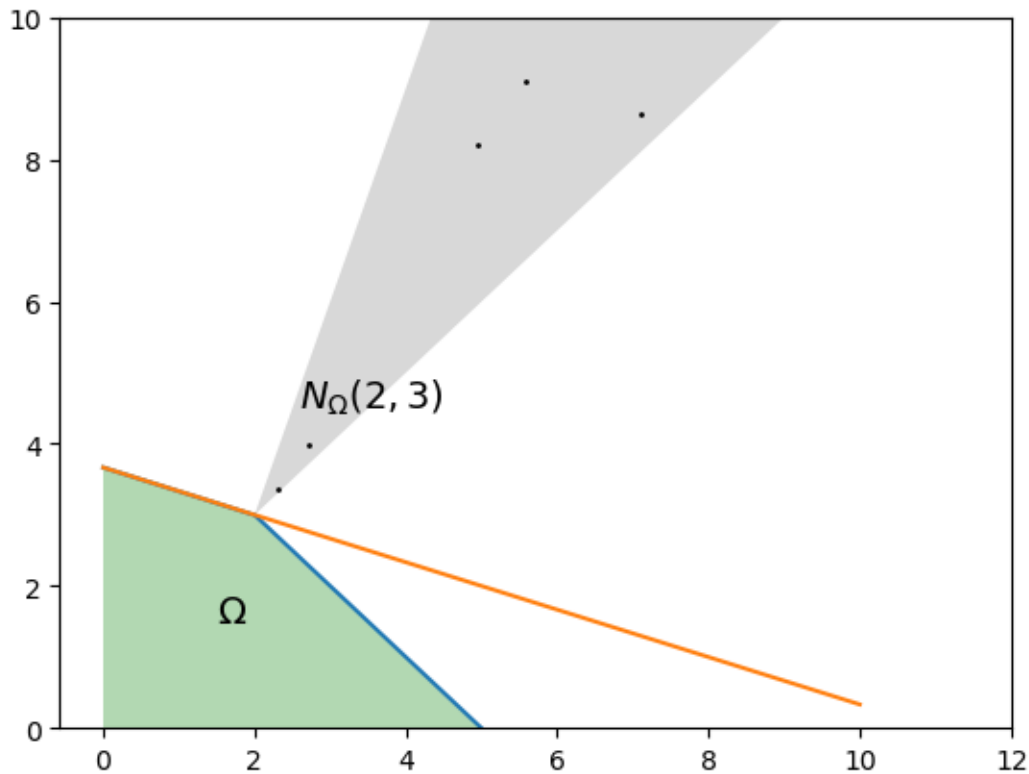
Let's check against gurobi:

```
[122]: from hw1_p1_lp import build_model, solve_with_gurobi
       cs = np.column_stack([x,y])

       x_opts = []
       for i in range(number):
           x_opt, _ =solve_with_gurobi(A, b, cs[i])
           x_opts.append(x_opt)

       if ((x_opts - np.array([2.,3.])) == 0).all() :
           print("unit test passed!")
       else:
           print(r"there are points in the cone that does not give rise to LP $\Pi$ at␣
       ↪(2,3)!")
```

Gurobi Optimizer version 13.0.1 build v13.0.1rc0 (mac64[arm] - Darwin 24.2.0
24C101)

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 2 rows, 2 columns and 4 nonzeros (Max)

3

```
Model fingerprint: 0x293dbda6
Model has 2 linear objective coefficients
Coefficient statistics:
  Matrix range     [1e+00, 3e+00]
  Objective range  [9e+00, 2e+01]
  Bounds range     [0e+00, 0e+00]
  RHS range        [5e+00, 1e+01]

Presolve time: 0.01s
Presolved: 2 rows, 2 columns, 4 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    3.0277251e+31   3.000000e+30   3.027725e+01      0s
       2    8.1903712e+01   0.000000e+00   0.000000e+00      0s

Solved in 2 iterations and 0.01 seconds (0.00 work units)
Optimal objective  8.190371222e+01
Gurobi Optimizer version 13.0.1 build v13.0.1rc0 (mac64[arm] - Darwin 24.2.0
24C101)

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 2 rows, 2 columns and 4 nonzeros (Max)
Model fingerprint: 0xea925092
Model has 2 linear objective coefficients
Coefficient statistics:
  Matrix range     [1e+00, 3e+00]
  Objective range  [3e+00, 4e+00]
  Bounds range     [0e+00, 0e+00]
  RHS range        [5e+00, 1e+01]

Presolve time: 0.00s
Presolved: 2 rows, 2 columns, 4 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    6.7164867e+30   3.000000e+30   6.716487e+00      0s
       2    1.7427632e+01   0.000000e+00   0.000000e+00      0s

Solved in 2 iterations and 0.00 seconds (0.00 work units)
Optimal objective  1.742763156e+01
Gurobi Optimizer version 13.0.1 build v13.0.1rc0 (mac64[arm] - Darwin 24.2.0
24C101)

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 2 rows, 2 columns and 4 nonzeros (Max)
```

```
Model fingerprint: 0xdecac820
Model has 2 linear objective coefficients
Coefficient statistics:
  Matrix range      [1e+00, 3e+00]
  Objective range   [1e+01, 2e+01]
  Bounds range      [0e+00, 0e+00]
  RHS range         [5e+00, 1e+01]

Presolve time: 0.00s
Presolved: 2 rows, 2 columns, 4 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    2.6613864e+31   3.000000e+30   2.661386e+01      0s
       2    6.8397185e+01   0.000000e+00   0.000000e+00      0s

Solved in 2 iterations and 0.00 seconds (0.00 work units)
Optimal objective  6.839718481e+01
Gurobi Optimizer version 13.0.1 build v13.0.1rc0 (mac64[arm] - Darwin 24.2.0
24C101)

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 2 rows, 2 columns and 4 nonzeros (Max)
Model fingerprint: 0xd5f71acc
Model has 2 linear objective coefficients
Coefficient statistics:
  Matrix range      [1e+00, 3e+00]
  Objective range   [7e+00, 9e+00]
  Bounds range      [0e+00, 0e+00]
  RHS range         [5e+00, 1e+01]

Presolve time: 0.00s
Presolved: 2 rows, 2 columns, 4 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    1.5745087e+31   3.000000e+30   1.574509e+01      0s
       2    4.0133552e+01   0.000000e+00   0.000000e+00      0s

Solved in 2 iterations and 0.00 seconds (0.00 work units)
Optimal objective  4.013355170e+01
Gurobi Optimizer version 13.0.1 build v13.0.1rc0 (mac64[arm] - Darwin 24.2.0
24C101)

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 2 rows, 2 columns and 4 nonzeros (Max)
```

```
Model fingerprint: 0x12c2047e
Model has 2 linear objective coefficients
Coefficient statistics:
  Matrix range     [1e+00, 3e+00]
  Objective range  [2e+00, 3e+00]
  Bounds range     [0e+00, 0e+00]
  RHS range        [5e+00, 1e+01]

Presolve time: 0.00s
Presolved: 2 rows, 2 columns, 4 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    5.6569832e+30   3.000000e+30   5.656983e+00      0s
       2    1.4663865e+01   0.000000e+00   0.000000e+00      0s

Solved in 2 iterations and 0.00 seconds (0.00 work units)
Optimal objective  1.466386527e+01
Gurobi Optimizer version 13.0.1 build v13.0.1rc0 (mac64[arm] - Darwin 24.2.0
24C101)

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 2 rows, 2 columns and 4 nonzeros (Max)
Model fingerprint: 0xc2bbb219
Model has 2 linear objective coefficients
Coefficient statistics:
  Matrix range     [1e+00, 3e+00]
  Objective range  [6e+00, 9e+00]
  Bounds range     [0e+00, 0e+00]
  RHS range        [5e+00, 1e+01]

Presolve time: 0.00s
Presolved: 2 rows, 2 columns, 4 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    1.4687391e+31   3.000000e+30   1.468739e+01      0s
       2    3.8475263e+01   0.000000e+00   0.000000e+00      0s

Solved in 2 iterations and 0.00 seconds (0.00 work units)
Optimal objective  3.847526345e+01
Gurobi Optimizer version 13.0.1 build v13.0.1rc0 (mac64[arm] - Darwin 24.2.0
24C101)

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 2 rows, 2 columns and 4 nonzeros (Max)
```

```
Model fingerprint: 0x0bab164b
Model has 2 linear objective coefficients
Coefficient statistics:
  Matrix range     [1e+00, 3e+00]
  Objective range  [8e+00, 1e+01]
  Bounds range     [0e+00, 0e+00]
  RHS range        [5e+00, 1e+01]

Presolve time: 0.00s
Presolved: 2 rows, 2 columns, 4 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    2.2557727e+31   3.000000e+30   2.255773e+01      0s
       2    5.9950654e+01   0.000000e+00   0.000000e+00      0s

Solved in 2 iterations and 0.00 seconds (0.00 work units)
Optimal objective  5.995065355e+01
Gurobi Optimizer version 13.0.1 build v13.0.1rc0 (mac64[arm] - Darwin 24.2.0
24C101)

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 2 rows, 2 columns and 4 nonzeros (Max)
Model fingerprint: 0x24eebf4f
Model has 2 linear objective coefficients
Coefficient statistics:
  Matrix range     [1e+00, 3e+00]
  Objective range  [1e+01, 3e+01]
  Bounds range     [0e+00, 0e+00]
  RHS range        [5e+00, 1e+01]

Presolve time: 0.00s
Presolved: 2 rows, 2 columns, 4 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    3.7936844e+31   3.000000e+30   3.793684e+01      0s
       2    1.0282285e+02   0.000000e+00   0.000000e+00      0s

Solved in 2 iterations and 0.00 seconds (0.00 work units)
Optimal objective  1.028228529e+02
Gurobi Optimizer version 13.0.1 build v13.0.1rc0 (mac64[arm] - Darwin 24.2.0
24C101)

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 2 rows, 2 columns and 4 nonzeros (Max)
```

```
Model fingerprint: 0xc08c7c1b
Model has 2 linear objective coefficients
Coefficient statistics:
  Matrix range     [1e+00, 3e+00]
  Objective range  [1e+01, 2e+01]
  Bounds range     [0e+00, 0e+00]
  RHS range        [5e+00, 1e+01]

Presolve time: 0.00s
Presolved: 2 rows, 2 columns, 4 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    2.7922627e+31   3.000000e+30   2.792263e+01      0s
       2    7.3825694e+01   0.000000e+00   0.000000e+00      0s

Solved in 2 iterations and 0.00 seconds (0.00 work units)
Optimal objective  7.382569351e+01
Gurobi Optimizer version 13.0.1 build v13.0.1rc0 (mac64[arm] - Darwin 24.2.0
24C101)

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 2 rows, 2 columns and 4 nonzeros (Max)
Model fingerprint: 0x4d66b7a5
Model has 2 linear objective coefficients
Coefficient statistics:
  Matrix range     [1e+00, 3e+00]
  Objective range  [5e+00, 8e+00]
  Bounds range     [0e+00, 0e+00]
  RHS range        [5e+00, 1e+01]

Presolve time: 0.00s
Presolved: 2 rows, 2 columns, 4 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    1.3145064e+31   3.000000e+30   1.314506e+01      0s
       2    3.4483889e+01   0.000000e+00   0.000000e+00      0s

Solved in 2 iterations and 0.00 seconds (0.00 work units)
Optimal objective  3.448388862e+01
unit test passed!
```