DDC

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Nested Fixed Point Theorem (Rust, J., 1987)

Setup of *Regenerative Optimal Stopping Problem*

- Zurcher chooses discrete actions i_t each period, $i_t = 1(replace), i_t = 0(keep)$
- We observe regenerative process $\{i_t, x_t\}$, the likelihood of the process can be written as $l(i_1, ..., i_t, x_1, ..., x_t|\theta)$
- Define differentiable cost function $c(x_t, \theta_1)$, the flow utility will be

$$u(x_t, i_t, \theta_1) = \begin{cases} -c(x_t, \theta_1) & i_t = 0 \\ -s - c(0, \theta_1) & i_t = 1 \end{cases}$$

- The question is also called Markov Decision Problem (MDP), because the state (milage of the engine x_t) has Markov property

$$p(x_{t+1}|x_t, i_t, \theta_2) = \begin{cases} \theta_2 \exp\{\theta_2(x_{t+1} - x_t)\} & i_t = 0, x_{t+1} \geq x_t \\ \theta_2 \exp\{\theta_2(x_{t+1})\} & i_t = 0, x_{t+1} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Nested Fixed Point Theorem (Rust, J., 1987)

Now we solve the problem

- First we can write out the Bellman equation

$$V_\theta(x_t) = \max_{i_t} \mathbb{E}\left[u(x_t, i_t, \theta_1) + \beta \int_0^\infty V_\theta(x_{t+1}) dG(x_{t+1}|x_t, i_t, \theta_2)\right]$$

- In Rust (1986), he proved there is an optimal stationary, Markovian replacement policy $\Pi = \{f, f, ...\}$

$$i_t^* = f(x_t, \theta) = \begin{cases} 1 & x > \gamma(\theta_1, \theta_2) \\ 0 & x \leq \gamma(\theta_1, \theta_2) \end{cases}$$

  where

$$s(1-\beta) = \int_0^{\gamma(\theta_1,\theta_2)} [1 - \beta \exp\{-\theta_2(1-\beta)x'\}] \frac{\partial c(x', \theta_1)}{\partial x'} dx'$$

- Then we can form the likelihood to use MLE and solve for θ

Nested Fixed Point Theorem (Rust, J., 1987)

It looks good, but there are several drawbacks

- Assuming the monthly mileage $(x_{t+1} - x_t)$ has i.i.d. exponential does't reflect real data, but using other flexible distribution will ruin previous explicit solution
- The cutoff replacement rule doesn't fit the data, in the data, there are replacement in many values of x_t, potential explanation: x_t is just one dimension of the quality of the engine, there are unobserved characteristics ε_t that affect the replacement decision
- So the decision rule should be something like

$$i_t = f(x_t, \theta) + \varepsilon_t$$

  where ε_t is observed by the agent but not the econometrician.
- However this solution makes the estimation very challenging: the structural model assume optimization, but the solution means agent deviates the optimal behavior randomly.

Nested Fixed Point Theorem (Rust, J., 1987)

- Given Markovian state variables (x_t, ε_t) embodied by the transition probability $p(\theta_2, \theta_3)$—also relate to the optimal control f

$$p(x_{t+N}, \varepsilon_{t+N}, ..., x_{t+1}, \varepsilon_{t+N} | x_t, \varepsilon_t) = \prod_{i=1}^{N-1} p(x_{i+1}, \varepsilon_{i+1} | x_i, \varepsilon_i, f_i(x_i, \varepsilon_i), \theta_2, \theta_3)$$

- Agent chooses sequence of decision rule $\Pi = \{f_1, f_2, ...\}$ to maximize utility

$$V_\theta(x_0, \varepsilon_0) = \max_{\Pi} \mathbb{E} \left\{ \sum_{j=1}^{\infty} \beta^j [u(x_j, f_j, \theta_1) + \varepsilon_j(f_j)] | x_0, \varepsilon_0, \theta_2, \theta_3 \right\}$$

$$V_\theta(x_t, \varepsilon_t) = \max_{i_t} [u(x_t, i_t, \theta_1) + \varepsilon_t(i_t) + \beta \mathbb{E} V_\theta(x_{t+1}, i_t, \varepsilon_{t+1})]$$

where

$$\mathbb{E} V_\theta(x_{t+1}, i_t, \varepsilon_{t+1}) = \int_{x'} \int_{\varepsilon'} V_\theta(x', \varepsilon') dG(x', \varepsilon' | x_t, \varepsilon_t, i_t, \theta_2, \theta_3)$$

- The optimal control thus is

$$f_t(x_t, \varepsilon_t, \theta) = \arg\max_{i_t} [u(x_t, \theta_1) + \varepsilon_t(i_t) + \beta \mathbb{E} V_\theta(x_{t+1}, i_t, \varepsilon_{t+1})]$$

Nested Fixed Point Theorem (Rust, J., 1987)

$$V_\theta(x_t, \varepsilon_t) = \max_{i_t}[u(x_t, i_t, \theta_1) + \varepsilon_t(i_t) + \beta \mathbb{E} V_\theta(x_{t+1}, i_t, \varepsilon_{t+1})]$$

$$\mathbb{E} V_\theta(x_{t+1}, i_t, \varepsilon_{t+1}) = \int_{x'} \int_{\varepsilon'} V_\theta(x', \varepsilon') dp(x', \varepsilon' | x_t, \varepsilon_t, i_t, \theta_2, \theta_3)$$

$$f_t(x_t, \varepsilon_t, \theta) = \arg\max_{i_t}[u(x_t, \theta_1) + \varepsilon_t(i_t) + \beta \mathbb{E} V_\theta(x_{t+1}, i_t, \varepsilon_{t+1})]$$

The optimal control is very hard to solve

- The unobservable ε_t's support usually is unbounded, when do grid search, it creates difficulty to find fixed point of $V_\theta(x_t, \varepsilon_t)$
- Since $\mathbb{E} V_\theta(x_{t+1}, i_t, \varepsilon_{t+1})$ is an unknown function, in every iteration, we need to numerically integrate V_θ over finite grid approximation of p to get $\mathbb{E} V_\theta$. And then integrate the Bellman equation to get $P(i_t | x_t, \theta)$

Under some condition, Rust propose a MLE algorithm to solve the dynamic discrete choice (DDC) problem.

Nested Fixed Point Theorem (Rust, J., 1987)

conditional independence assumption:

$$p(x_{t+1}, \varepsilon_{t+1} | x_t, \varepsilon_t, i_t, \theta_2, \theta_3) = q(\varepsilon_{t+1} | x_{t+1}, \theta_2) p(x_{t+1} | x_t, i_t, \theta_3)$$

under this assumption

- Let $G([u(x, \theta_1) + \beta \mathbb{E} V_\theta(x)] | x, \theta_2)$ denote the social surplus function corresponding to the density $q(\varepsilon | x, \theta_2)$

$$G = \int_\varepsilon \max_i [u(x, i, \theta_1) + \beta \mathbb{E} V_\theta(x, i)] dq(\varepsilon | x, \theta_2)$$

  Then

$$P(i | x, \theta) = \frac{\partial G([u(x, \theta_1) + \beta \mathbb{E} V_\theta(x)] | x, \theta_2)}{\partial u(x, i, \theta_1)}$$

  We can use the following contraction mapping to solve $\mathbb{E} V_\theta$

$$\mathbb{E} V_\theta(x, i) = \int_{x'} G([u(x', \theta_1) + \beta \mathbb{E} V_\theta(x')] | x', \theta_2) dp(x' | x, i, \theta_3)$$

Nested Fixed Point Theorem (Rust, J., 1987)

Example: $q(\varepsilon|x,\theta_2)$ is given by a multivariate EV distribution

$$q(\varepsilon|x,\theta_2) = \prod_{i\in C} e^{-\varepsilon(i)+\theta_2} e^{-e^{-\varepsilon(i)+\theta_2}}$$

Now let's solve the model

$$\begin{aligned}
G &= \int_{\varepsilon} \max_i [u(x,i,\theta_1) + \beta \mathbb{E}V_\theta(x,i)] dq(\varepsilon|x,\theta_2) \\
&= \ln \left\{ \sum_{i\in C} \exp[u(x,i,\theta_1) + \beta \mathbb{E}V_\theta(x,i)] \right\} \\
P(i|x,\theta) &= \frac{\exp[u(x,i,\theta_1) + \beta \mathbb{E}V_\theta(x,i)]}{\sum_{i\in C} \exp[u(x,i,\theta_1) + \beta \mathbb{E}V_\theta(x,i)]} \\
\mathbb{E}V_\theta(x,i) &= \int_{x'} \ln \left\{ \sum_{i\in C} \exp[u(x',i,\theta_1) + \beta \mathbb{E}V_\theta(x',i)] \right\} dp(x'|x,i,\theta_3)
\end{aligned}$$

Nested Fixed Point Theorem (Rust, J., 1987)

under this assumption (cont.)

- The likelihood function of the sample can be written as

$$l(x_1, ..., x_T, i_1, ..., i_T | x_0, i_0, \theta) = \prod_{t=1}^{T} P(i_t | x_t, \theta) p(x_t | x_{t-1}, i_{t-1}, \theta_3)$$

  Then use MLE to solve for θ

Example (cont.)

- The likelihood function of the sample can be written as

$$l = \prod_{t=1}^{T} \frac{\exp[u(x_t, i_t, \theta_1) + \beta \mathbb{E} V_\theta(x_t, i_t)]}{\sum_{i \in C} \exp[u(x_t, i_t, \theta_1) + \beta \mathbb{E} V_\theta(x_t, i_t)]} p(x_t | x_{t-1}, i_{t-1}, \theta_3)$$

  Then use MLE to solve for θ

CCP (Hotz, V. J., & Miller, R. A., 1993)

Despite Nested Fixed Point Theorem is very general in estimating structural DDC model, but it's computationally heavy. Hotz and Miller proposed alternative method (Conditional Choice Probability) to do the estimation.