# Research Practicum 2

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# Most Efficient IV (Chamberlain, G., 1987)

- In linear regression, $y = X\beta + u$, and $\mathbb{E}[uu'] = \Omega$
- $X$ serves as IV for itself
- The IV(OLS) condition $\mathbb{E}[u_i|X_i] = 0$. Within GMM framework, we have the moment condition $\mathbb{E}[X_i u_i] = 0$ or say $\mathbb{E}[f(X_i)u_i] = 0$
- And thus the GMM estimates is

$$\begin{aligned}
\hat{\beta}^{GMM} &= \arg\min_{\beta}[\frac{1}{n}\sum_{i=1}^{n} f(X_i)u_i]' W [\frac{1}{n}\sum_{i=1}^{n} f(X_i)u_i] \\
&= \arg\min_{\beta}[\frac{1}{n}\sum_{i=1}^{n} f(X_i)(y_i - X_i\beta)]' W [\frac{1}{n}\sum_{i=1}^{n} f(X_i)(y_i - X_i\beta)] \\
&:= \arg\min_{\beta}[\frac{1}{n}\sum_{i=1}^{n} \psi(X_i, \beta)]' W [\frac{1}{n}\sum_{i=1}^{n} \psi(X_i, \beta)]
\end{aligned}$$

- From Chris's class we know the best $W$ is $\mathbb{E}[\psi(X_i, \beta)\psi'(X_i, \beta)]^{-1}$

# Most Efficient IV (Chamberlain, G., 1987)

- With the best $W$, we get the limiting distribution of $\hat{\beta}^{GMM}$

$$\sqrt{n}(\hat{\beta}^{GMM} - \beta) \sim N(a_n, \Lambda_0)$$

- Where

$$\Lambda_0 = \left(\mathbb{E}\left[\frac{\partial\psi(X,\beta)}{\partial\beta'}\right]'\mathbb{E}[\psi(X,\beta)\psi'(X,\beta)]^{-1}\mathbb{E}\left[\frac{\partial\psi(X,\beta)}{\partial\beta'}\right]\right)^{-1}$$

- In linear regression case

$$\Lambda_0 = \left(\mathbb{E}\Big[f(X)X'\Big]'\mathbb{E}[f(X)(y - X\beta)(y - X\beta)'f(X)']^{-1}\mathbb{E}\Big[f(X)X'\Big]\right)^{-1}$$

# Most Efficient IV (Chamberlain, G., 1987)

- Question: How to choose $f(Z)$ such that the estimator is most efficient?
- Define the conditional moment restrictions: $\mathbb{E}[\rho(Z,\beta)|Z] = 0$
- Chamberlain said and proved:

$$f(Z) = D(Z)'\Omega^{-1}$$

  where $D(Z) = \mathbb{E}[\frac{\partial\rho(Z,\beta)}{\partial\beta'}|Z],\ \Omega = \mathbb{E}[\rho(Z,\beta)\rho'(Z,\beta)|Z]$
- **Example:** In linear regression the conditional moment restrictions $\mathbb{E}[y - X\beta|X] = 0,\ \rho(X,\beta) = y - X\beta$

$$f(X) = \mathbb{E}[-X]'\mathbb{E}[(y - X\beta)(y - X\beta)']^{-1} = -\frac{X}{\sigma^2(X)}$$

  The moment condition we should use is

$$\mathbb{E}[\frac{X}{\sigma^2(X)}(y - X\beta)] = 0$$

# Good IV in BLP (Reynaert, M. and Verboven, F., 2014)

Demand Set up

- $u_{ijt} = x_{jt}\beta_i - \alpha p_{jt} + \xi_{jt} + \varepsilon_{ijt}$
- $\beta_i$ is the random coefficient (preference) $\beta_i^k = \beta^k + \sigma^k \nu_i^k$, where $\sigma^k$ is with zero mean and unit variance.
- $u_{ijt} = x_{jt}\beta - \alpha p_{jt} + \xi_{jt} + \sum_k x_{jt}^k \sigma^k \nu_i^k + \varepsilon_{ijt} := \delta_{jt} + \mu_{jt}(\nu_i) + \varepsilon_{ijt}$
- Then we get market share

$$s_{jt}(\delta_{jt}, \sigma) = \int \frac{\exp\left(\delta_{jt} + \mu_{jt}(v_i)\right)}{1 + \sum_{l=1}^{J} \exp\left(\delta_{lt} + \mu_{lt}(v_i)\right)} dP_\nu(\nu)$$

- The market share can be computed by Monte Carlo or polynomial-based integration

# Good IV in BLP (Reynaert, M. and Verboven, F., 2014)

Supply Set up

- The marginal cost $c_{jt} = x_{jt}\gamma_1 + w_{jt}\gamma_2 + \omega_{jt}$
- Under the perfect competition assumption $p_{jt} = c_{jt}$, with different assumption, we can back out the proper $c_{jt}$
  e.g. firm set price of their product set $M \subset J$ to maximize

$$\pi_t = \sum_{m \in M_t} s_{mt}(p_{mt})[p_{mt} - c_{mt}]$$

$$s_{mt}(p^*) + \frac{\partial s_{mt}(p^*)}{\partial p^*}(p^* - c_{mt}) = 0$$

# Good IV in BLP (Reynaert, M. and Verboven, F., 2014)

Estimation

- Identification Assumption: $\mathbb{E}[\xi_{jt}|X_t, W_t] = 0,\ \mathbb{E}[\omega_{jt}|X_t, W_t] = 0$
- Let $Z_t = [X_t\ W_t], \rho_{jt} = [\xi_{jt}\ \omega_{jt}]'$, use GMM, the moment condition is

$$\mathbb{E}[f_{jt}(Z_t)\rho_{jt}] = 0$$

- From demand side $\xi_t = \delta_t - X_t\beta + \alpha p_t$
- From supply side $\omega_t = c_t - X_t\gamma_1 - W_t\gamma_2$

# Good IV in BLP (Reynaert, M. and Verboven, F., 2014)

Estimation ($\Theta = \{\beta, \alpha, \sigma, \gamma\}$)

- Recall that

$$s_{jt}(\delta_{jt}, \sigma) = \int \frac{\exp\left(\delta_{jt} + \mu_{jt}(v_i)\right)}{1 + \sum_{l=1}^{J} \exp\left(\delta_{lt} + \mu_{lt}(v_i)\right)} dP_\nu(\nu)$$

- We can compute $\delta_{jt} = s_{jt}^{-1}(s_{jt}, \sigma) := \delta_{jt}(s_{jt}, \sigma)$ by BLP's contraction
- Then we can write $\xi_t = \rho_1(\Theta),\ \omega_t = \rho_2(\Theta),\ \rho(\Theta) = [\rho_1(\Theta)\ \rho_2(\Theta)]'$
- Now we choose $\Theta$

$$\min_{\Theta} \rho(\Theta)' f(Z)' A f(Z) \rho(\Theta)$$

- This is actually hard, there is a fixed point problem ($\delta_{jt}(s_{jt}, \sigma)$) inside the optimization problem, so called nested fixed-point(NFXP)

# Good IV in BLP (Reynaert, M. and Verboven, F., 2014)

Estimation ($\Theta = \{\beta, \alpha, \sigma, \gamma\}$)

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