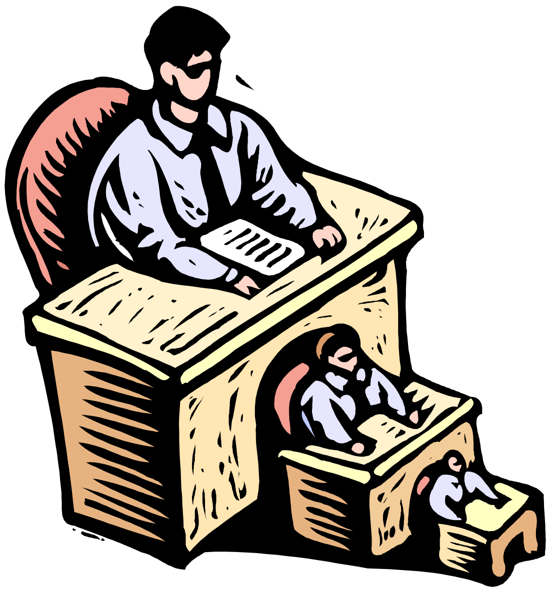
Recursion



Definition

- Programming technique

  A function can call itself

- One of the central ideas of computer science

- It's super effective!

  "The power of recursion evidently lies in the possibility of defining an infinite set of objects by a finite statement. In the same manner, an infinite number of computations can be described by a finite recursive program, even if this program contains no explicit repetitions." – *Niklaus Wirth*

  "I'm lovin' it" – *Charles Ponzi*

World's Simplest Recursion Program

```
def count (index):
        print (index)
        if index < 2:
                count(index+1)
        return

if __name__ == '__main__':
        count(0)
```

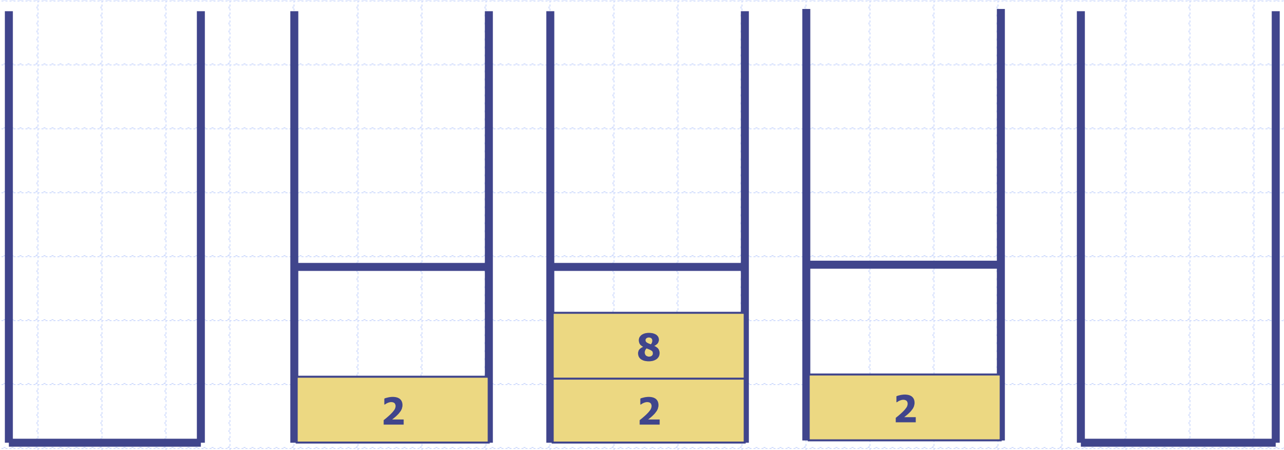
This is where the recursion occurs. You can see that the count() function calls itself.

Visualizing Recursion

- To understand how recursion works, it helps to visualize what's going on.
- To help visualize, we will use a common concept called the *Stack*.
- A stack basically operates like a container of trays in a cafeteria. It has only two operations:
  - Push: you can push something onto the stack.
  - Pop: you can pop something off the top of the stack.
- Let's see an example stack in action.

Stacks

The diagram below shows a stack over time.
We perform two pushes and one pop.



Time: 0
Empty Stack

Time 1:
Push "2"

Time 2:
Push "8"

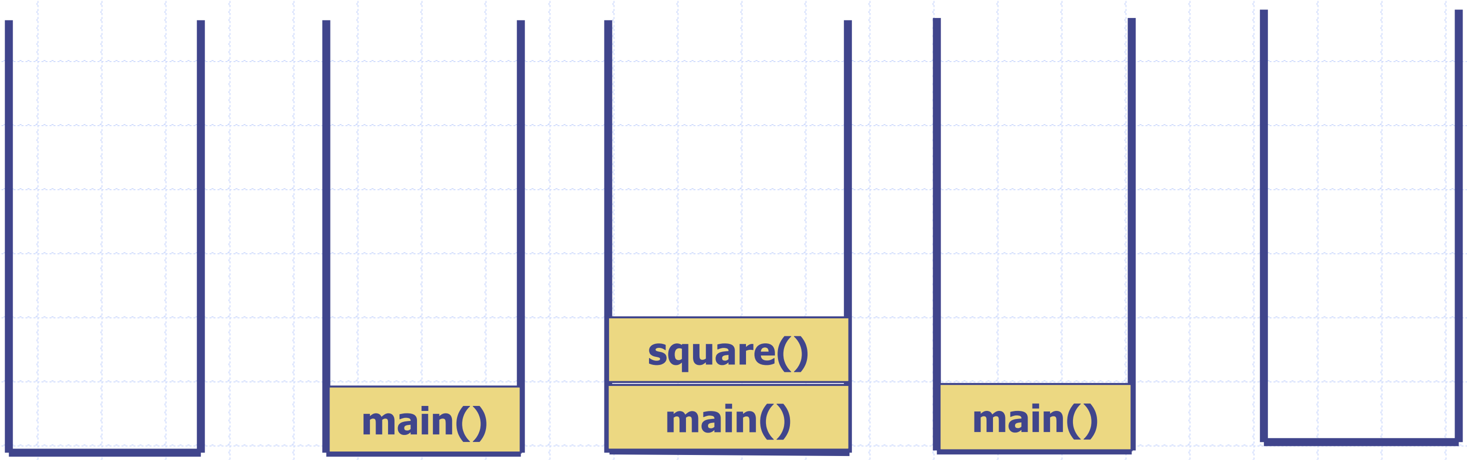
Time 3:
Pop: Gets 8

Time 4:
Pop: Gets 2

Stacks and Methods

- When you run a program, the computer creates a stack for you.
- Each time you invoke a method, the method is placed on top of the stack.
- When the method returns or exits, the method is popped off the stack.
- The diagram on the next page shows a sample stack for a simple program.

Stacks and Methods



Time: 0
Empty Stack

Time 1:
Push: main()

Time 2:
Push: square()

Time 3:
Pop: square()
returns a value.
method exits.

Time 4:
Pop: main()
returns a value.
method exits.

Stacks and Recursion

- Each time a method is called, you *push* the method on the stack.
- Each time the method returns or exits, you *pop* the method off the stack.
- If a method calls itself recursively, you just push another copy of the method onto the stack.
- We therefore have a simple way to visualize how recursion really works.

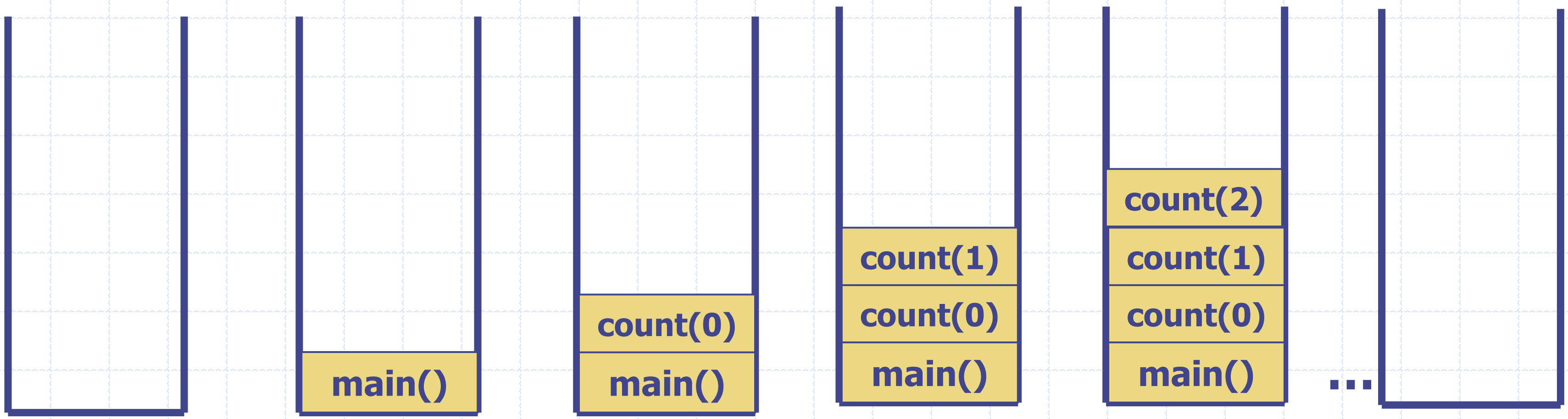
Back to the Simple Recursion Program

- Here's the code again. Now, that we understand stacks, we can visualize the recursion.

```
def count (index):
        print (index)
        if index < 2:
                count(index+1)
        return

if __name__ == '__main__':
        count(0)
```

Stacks and Recursion in Action



Time: 0
Empty Stack

Time 1:
Push: main()

Time 2:
Push:
count(0)

Time 3:
Push:
count(1)

Time 4:
Push:
count(2)

Times 5-8:
Pop everything

Inside count(0):
print (index)
→ 0
if index < 2:
count(index+1)

Inside count(1):
print (index)
→ 1
if index < 2:
count(index+1)

Inside count(2):
print (index)
→ 2
if index < 2:
count(index+1)

This condition now fails!

Hence, recursion stops, and we proceed to pop all functions off the stack.

Recursion, Variation 1

What will the following program do?

```
def count (index):
    print (index)
    if index < 2:
        count(index+1)
    return

if __name__ == '__main__':
    count(3)
```

Recursion, Variation 2

What will the following program do?

```
def count (index):
        if index < 2:
                count(index+1)
        print (index)
        return

if __name__ == '__main__':
        count(0)
```

Note that the print statement has been moved to the end of the method.

Recursion Example #2

Recursion Example #2

```
def upAndDown (n):
        print("level:",n)
        if n < 4:
                upAndDown(n+1)
        print("LEVEL:",n)
        return

upAndDown(1)
```

Recursion occurs here.

Determining the Output

- Suppose you were given this problem on the final exam, and your task is to "determine the output."
- How do you figure out the output?
- Answer: Use Stacks to Help Visualize

Stack Short-Hand

- Rather than draw each stack like we did last time, you can try using a short-hand notation.

time	stack	output
time 0:	empty stack	
time 1:	f(1)	Level: 1
time 2:	f(1), f(2)	Level: 2
time 3:	f(1), f(2), f(3)	Level: 3
time 4:	f(1), f(2), f(3), f(4)	Level: 4
time 5:	f(1), f(2), f(3)	LEVEL: 4
time 6:	f(1), f(2)	LEVEL: 3
time 7:	f(1)	LEVEL: 2
time 8:	empty	LEVEL: 1

Factorials

- Computing factorials are a classic problem for examining recursion.
- A factorial is defined as follows:

  n! = n * (n-1) * (n-2) …. * 1;
- For example:

  1! = 1 (Base Case)

  2! = 2 * 1 = 2

  3! = 3 * 2 * 1 = 6

  4! = 4 * 3 * 2 * 1 = 24

  5! = 5 * 4 * 3 * 2 * 1 = 120

Iterative Approach

```
def findFactorialIterative(n)
        if n<0:
                return 0
        factorial = 1
        while n>0:
                factorial = factorial*n
                n = n-1
        return factorial

print(findFactorialInterative(5))
```

This is an iterative solution to finding a factorial.
It's iterative because we have a simple for loop.
Note that the for loop goes from n-1 to 1.

Factorials

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  3! = 3 * 2 * 1 = 6

  4! = 4 * 3 * 2 * 1 = 24

  5! = 5 * 4 * 3 * 2 * 1 = 120

If you study this table closely, you will start to see a pattern.
The pattern is as follows:
You can compute the factorial of any number (n) by taking n and multiplying it by the factorial of (n-1).

For example:
5! = 5 * 4!
(which translates to 5! = 5 * 24 = 120)

Seeing the Pattern

- Seeing the pattern in the factorial example is difficult at first.
- But, once you see the pattern, you can apply this pattern to create a recursive solution to the problem.
- Divide a problem up into:
  - What it can do (usually a base case)
  - What it cannot do
    - What it cannot do resembles original problem
    - The function launches a new copy of itself (recursion step) to solve what it cannot do.

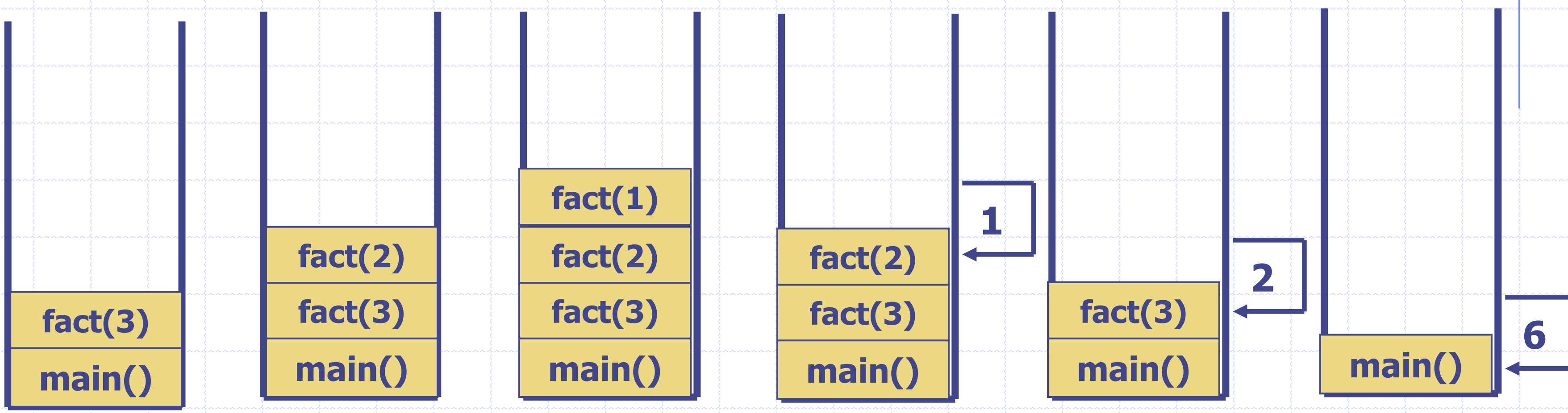
Recursive Solution

```
def findFactorialR(n)
        if n<0:
                return 0
        elif n==0 or n==1:
                return 1

        return n*findFactorialR(n-1)

print(findFactorialR(5))
```

Finding the factorial of 3



Time 2:
Push: fact(3)

Time 3:
Push: fact(2)

Time 4:
Push: fact(1)

Time 5:
Pop: fact(1)
returns 1.

Time 6:
Pop: fact(2)
returns 2.

Time 7:
Pop: fact(3)
returns 6.

Inside findFactorialR(3):

return 3 * factorial (2)

Inside findFactorialR(2):
return 2 * factorial (1)

Inside findFactorialR(1):
elif n==0 or n==1:
return 1;

Recursion vs. Iteration

- Iteration
  - Uses repetition structures (`for`, `while` or `do…while`)
  - Repetition through explicitly use of repetition structure
  - Terminates when loop-continuation condition fails
  - Controls repetition by using a counter
- Recursion
  - Uses selection structures (`if`, `if…else`)
  - Repetition through repeated method calls
  - Terminates when base case is satisfied
  - Controls repetition by dividing problem into simpler one

Recursion vs. Iteration (cont.)

- Recursion
  - More overhead than iteration
  - More memory intensive than iteration
  - Can also be solved iteratively
  - Often can be implemented with only a few lines of code

Characteristics of a Recursive Method

- Calls itself to solve a smaller problem

  Simplifies the initial problem conceptually

- Base case
  - Smallest problem to be solved
  - Result is returned to the calling method **(Terminal condition)**

- Induces overhead
  - Transfer of the control to the beginning of the method
  - Storage of all return points, intermediate arguments, return values

Recursive Binary Search

Binary search can also be a recursion

- Method calls itself with new starting and ending values
- Base case: starting value > end value

Binary Search

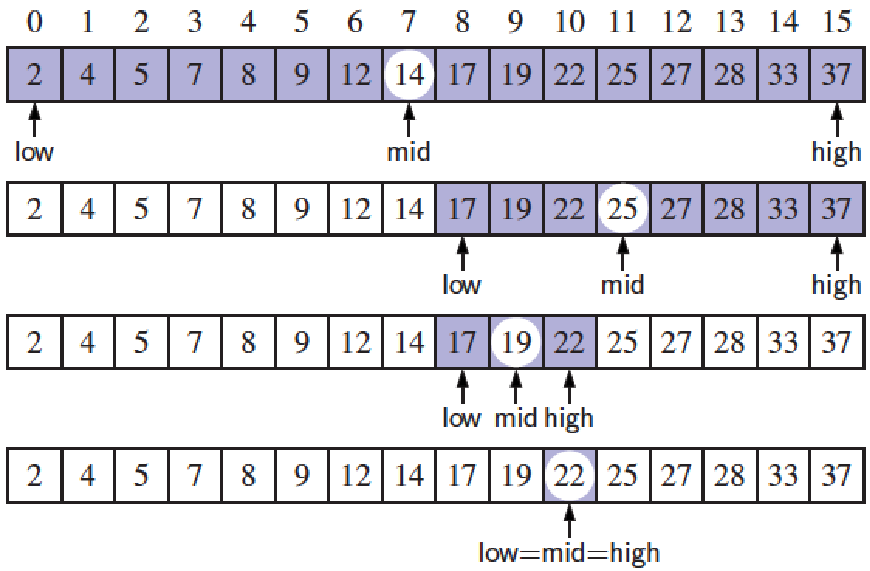
- Search for an integer, target, in an ordered list.

```python
def binary_search(data, target, low, high):
  """Return True if target is found in indicated portion of a Python list.

  The search only considers the portion from data[low] to data[high] inclusive.
  """
  if low > high:
    return False                                    # interval is empty; no match
  else:
    mid = (low + high) // 2
    if target == data[mid]:                         # found a match
      return True
    elif target < data[mid]:
      # recur on the portion left of the middle
      return binary_search(data, target, low, mid − 1)
    else:
      # recur on the portion right of the middle
      return binary_search(data, target, mid + 1, high)
```

Visualizing Binary Search

- We consider three cases:
  - If the target equals data[mid], then we have found the target.
  - If target < data[mid], then we recur on the first half of the sequence.
  - If target > data[mid], then we recur on the second half of the sequence.



Analyzing Binary Search

- Runs in O(log n) time.
  - The remaining portion of the list is of size high – low + 1.
  - After one comparison, this becomes one of the following:

$$(\mathsf{mid}-1)-\mathsf{low}+1=\left\lfloor\frac{\mathsf{low}+\mathsf{high}}{2}\right\rfloor-\mathsf{low}\leq\frac{\mathsf{high}-\mathsf{low}+1}{2}$$

$$\mathsf{high}-(\mathsf{mid}+1)+1=\mathsf{high}-\left\lfloor\frac{\mathsf{low}+\mathsf{high}}{2}\right\rfloor\leq\frac{\mathsf{high}-\mathsf{low}+1}{2}.$$

  - Thus, each recursive call divides the search region in half; hence, there can be at most log n levels.

Linear Recursion

- Test for base cases
  - Begin by testing for a set of base cases (there should be at least one).
  - Every possible chain of recursive calls must eventually reach a base case, and the handling of each base case should not use recursion.
- Recur once
  - Perform a single recursive call
  - This step may have a test that decides which of several possible recursive calls to make, but it should ultimately make just one of these calls
  - Define each possible recursive call so that it makes progress towards a base case.

Example of Linear Recursion

Algorithm LinearSum(*A, n*):

Input:

A integer array *A* and an integer *n* = 1, such that *A* has at least *n* elements

Output:

The sum of the first *n* integers in *A*

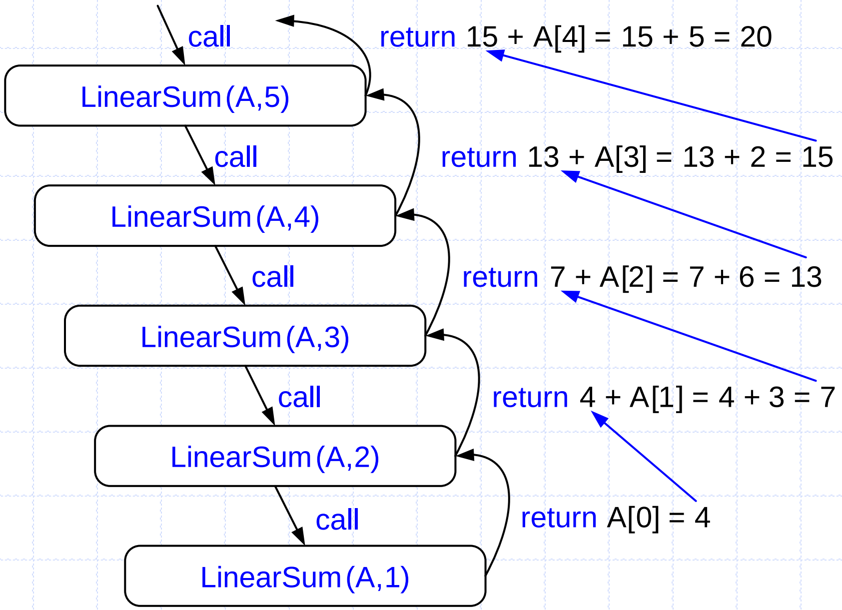
if *n* = 1 **then**

  return *A*[0]

else

  return LinearSum(*A, n* - 1) + *A*[*n* - 1]

Example recursion trace:



Reversing an Array

Algorithm ReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

if $i < j$ **then**

Swap $A[i]$ and $A[j]$

ReverseArray(A, $i + 1$, $j - 1$)

return

Defining Arguments for Recursion

- In creating recursive methods, it is important to define the methods in ways that facilitate recursion.
- This sometimes requires we define additional paramaters that are passed to the method.
- For example, we defined the array reversal method as ReverseArray(*A*, *i*, *j*), not ReverseArray(*A*).
- Python version:

```
def reverse(S, start, stop):
  """Reverse elements in implicit slice S[start:stop]."""
  if start < stop - 1:                          # if at least 2 elements:
    S[start], S[stop-1] = S[stop-1], S[start]   # swap first and last
    reverse(S, start+1, stop-1)                 # recur on rest
```

Computing Powers

- The power function, $p(x,n)=x^n$, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, n-1) & \text{else} \end{cases}$$

- This leads to an power function that runs in O(n) time (for we make n recursive calls).
- We can do better than this, however.

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x > 0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } x > 0 \text{ is even} \end{cases}$$

- For example,

$$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$$

$$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$$

$$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$$

$$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128.$$

Recursive Squaring Method

Algorithm Power(x, n):

Input: A number x and integer $n = 0$

Output: The value x^n

if $n = 0$ **then**

return 1

if n is odd **then**

$y = $ Power(x, $(n - 1)/2$)

return $x \cdot y \cdot y$

else

$y = $ Power(x, $n/2$)

return $y \cdot y$

Analysis

Algorithm Power(x, n):

Input: A number x and integer $n = 0$

Output: The value x^n

if $n = 0$ **then**

  return 1

if n is odd **then**

  $y = \text{Power}(x, (n - 1)/2)$

  return $x \cdot y \cdot y$

else

  $y = \text{Power}(x, n/2)$

  return $y \cdot y$

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in O(log n) time.

It is important that we use a variable twice here rather than calling the method twice.

Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its last step.
- The array reversal method is an example.
- Such methods can be easily converted to non-recursive methods (which saves on some resources).
- Example:

Algorithm IterativeReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

while $i < j$ **do**

Swap $A[i]$ and $A[j]$

$i = i + 1$

$j = j - 1$

return

Binary Recursion

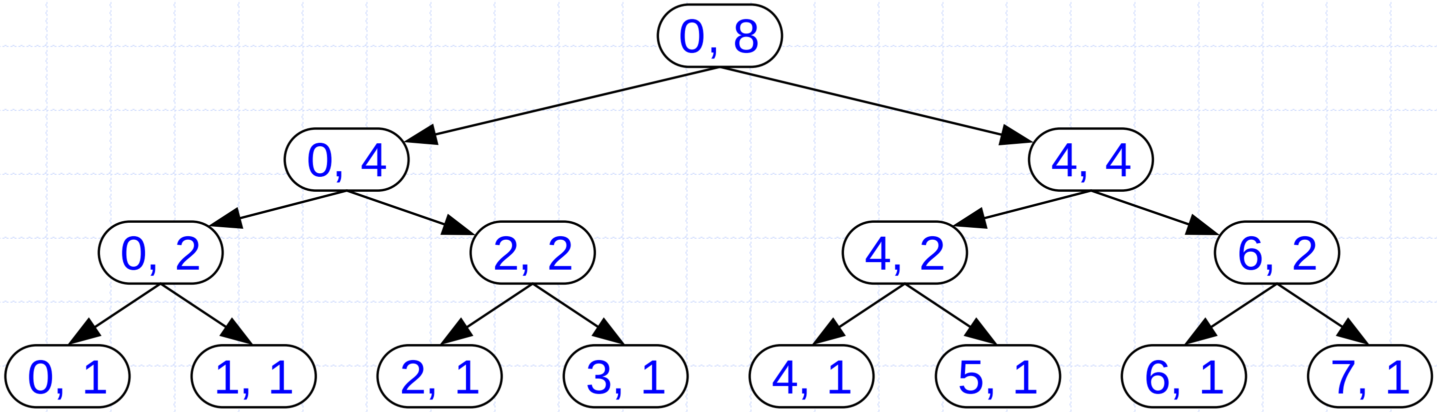
- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.

Binary Recusive Method

- Problem: add all the numbers in an integer array A:

Algorithm BinarySum(A, i, n):
    Input: An array A and integers i and n
    Output: The sum of the n integers in A starting at index i
    if $n = 1$ **then**
    return $A[i]$
    return BinarySum(A, i, $n/2$) + BinarySum(A, $i + n/2$, $n/2$)

- **Example trace:**



Computing Fibonacci Numbers

- Fibonacci numbers are defined recursively:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_i = F_{i-1} + F_{i-2} \quad \text{for } i > 1.$$

- Recursive algorithm (first attempt):

Algorithm BinaryFib(k):

Input: Nonnegative integer k

Output: The kth Fibonacci number F_k

if $k = 0$ **then**

return k

else if k = 1 then

return k

else

return BinaryFib(k - 1) + BinaryFib(k - 2)

Analysis

- Let n_k be the number of recursive calls by BinaryFib(k)
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67$.
- Note that n_k at least doubles every other time
- That is, $n_k > 2^{k/2}$. It is exponential!

A Better Fibonacci Algorithm

- Use linear recursion instead

```
Algorithm LinearFibonacci(k):
  Input: A nonnegative integer k
  Output: Pair of Fibonacci numbers (F_k , F_{k−1})
 if k = 1 then
     return (k, 0)
 else
     (i, j) = LinearFibonacci(k − 1)
     return (i +j, i)
```

- LinearFibonacci makes $k-1$ recursive calls

Multiple Recursion

- Multiple recursion:
  - makes potentially many recursive calls
  - not just one or two

Algorithm for Multiple Recursion

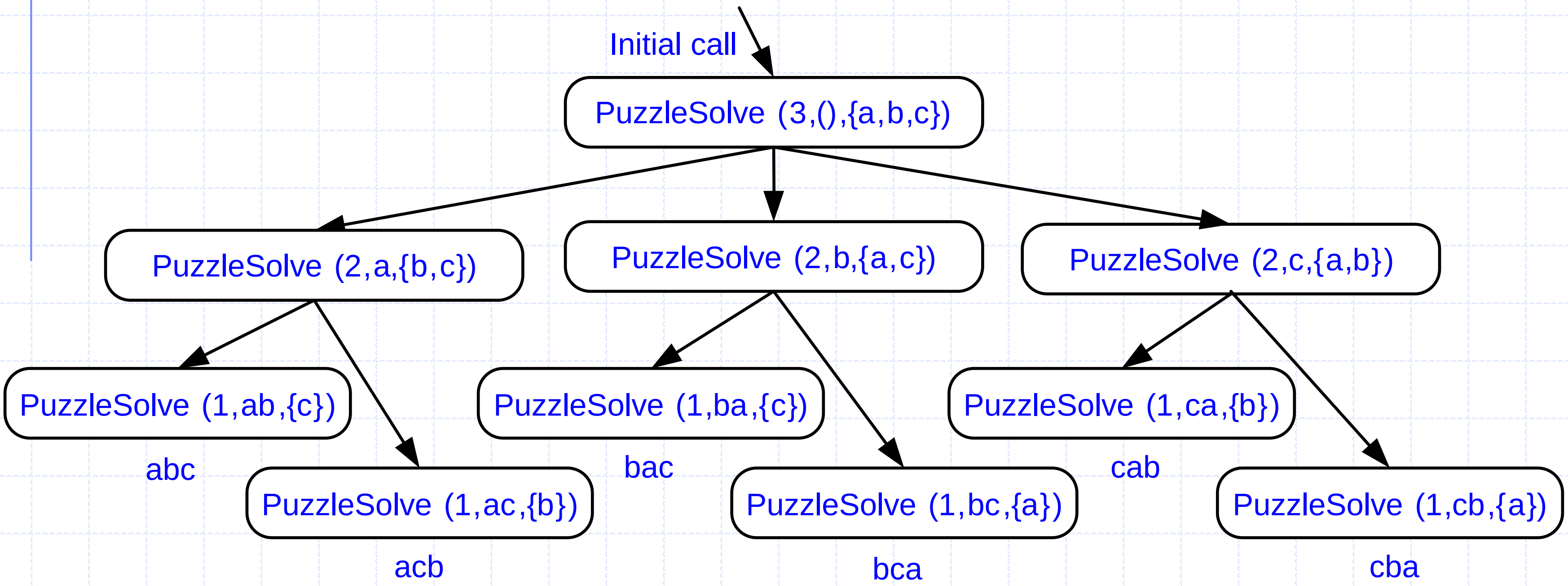
Algorithm PuzzleSolve(k,S,U):

Input: Integer k, sequence S, and set U (universe of elements to test)

Output: Enumeration of all k-length extensions to S using elements in U without repetitions

```
for all e in U do
    Remove e from U        {e is now being used}
    Add e to the end of S
    if k = 1 then
        Test whether S is a configuration that solves the puzzle
        if S solves the puzzle then
            return "Solution found: " S
    else
        PuzzleSolve(k - 1, S,U)
    Add e back to U        {e is now unused}
    Remove e from the end of S
```

Visualizing PuzzleSolve



Divide-and-Conquer

- Recursive binary search is an example of divide-and-conquer
- Idea
  - Divide the bigger problem into smaller problems
  - Solve each smaller problem separately
  - If smaller problem still too big, then solve its divisions
  - Continue process until smaller problem is a base case

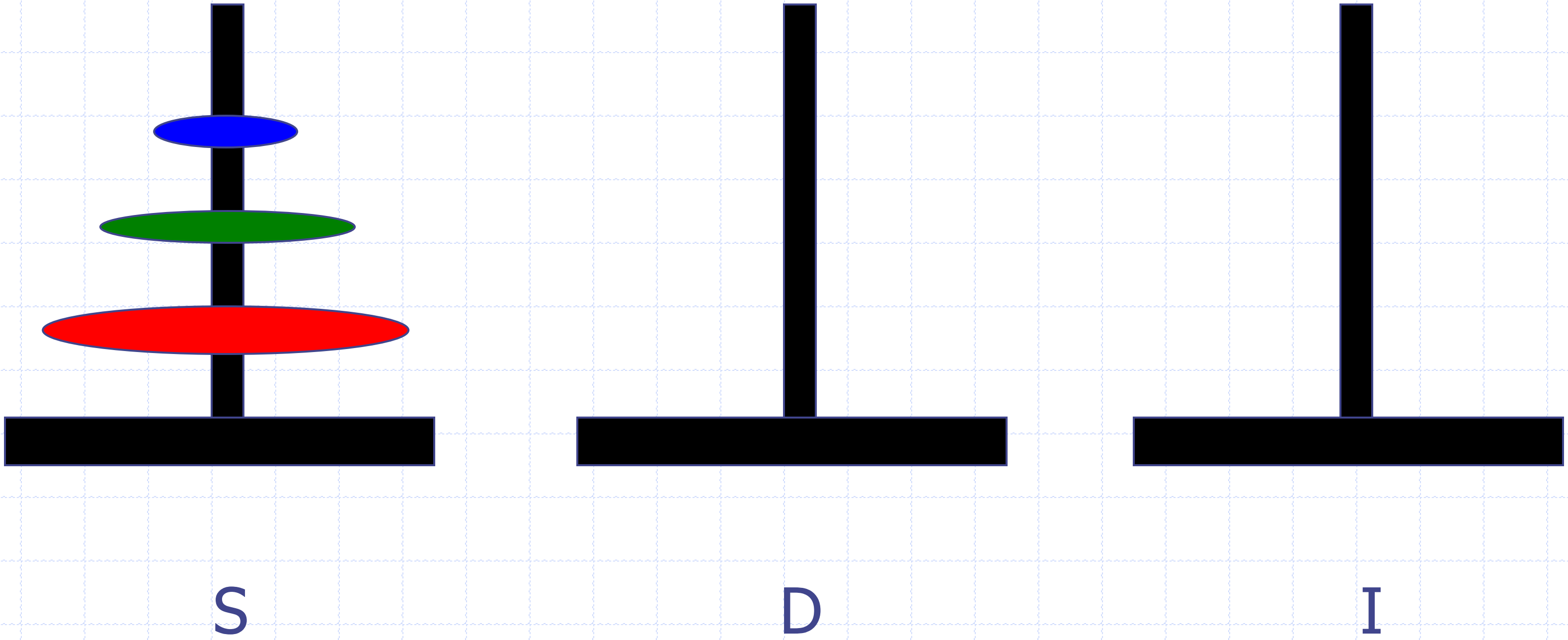
- DaC can be used with recursion as well as non recursion

Towers of Hanoi

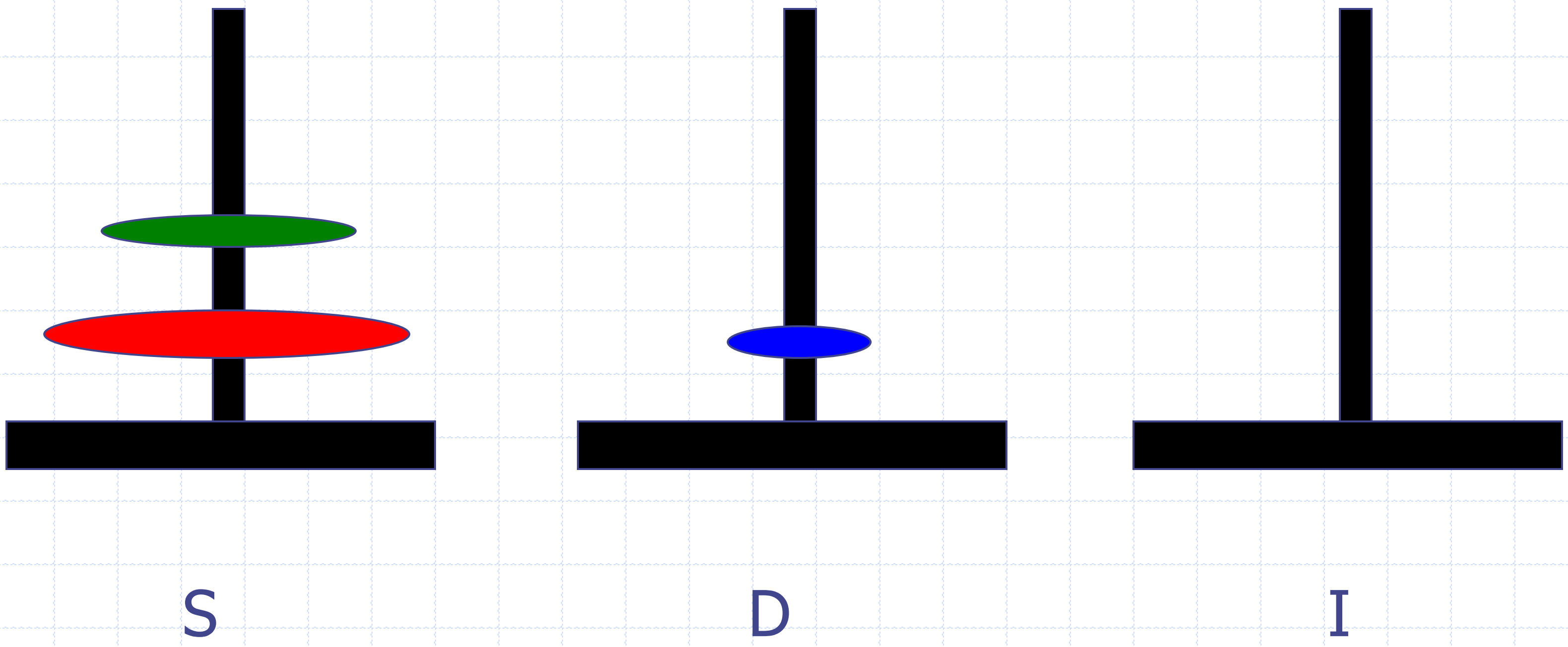
Invented by Edouard Lucas in 1883

- Three towers
- 64 gold disks (decreasing sizes) placed on the first tower
- All disks must be moved from the Source tower to the Destination Tower
- Larger disks can not be placed on top of smaller disks
- The third tower can be used to temporarily hold disks

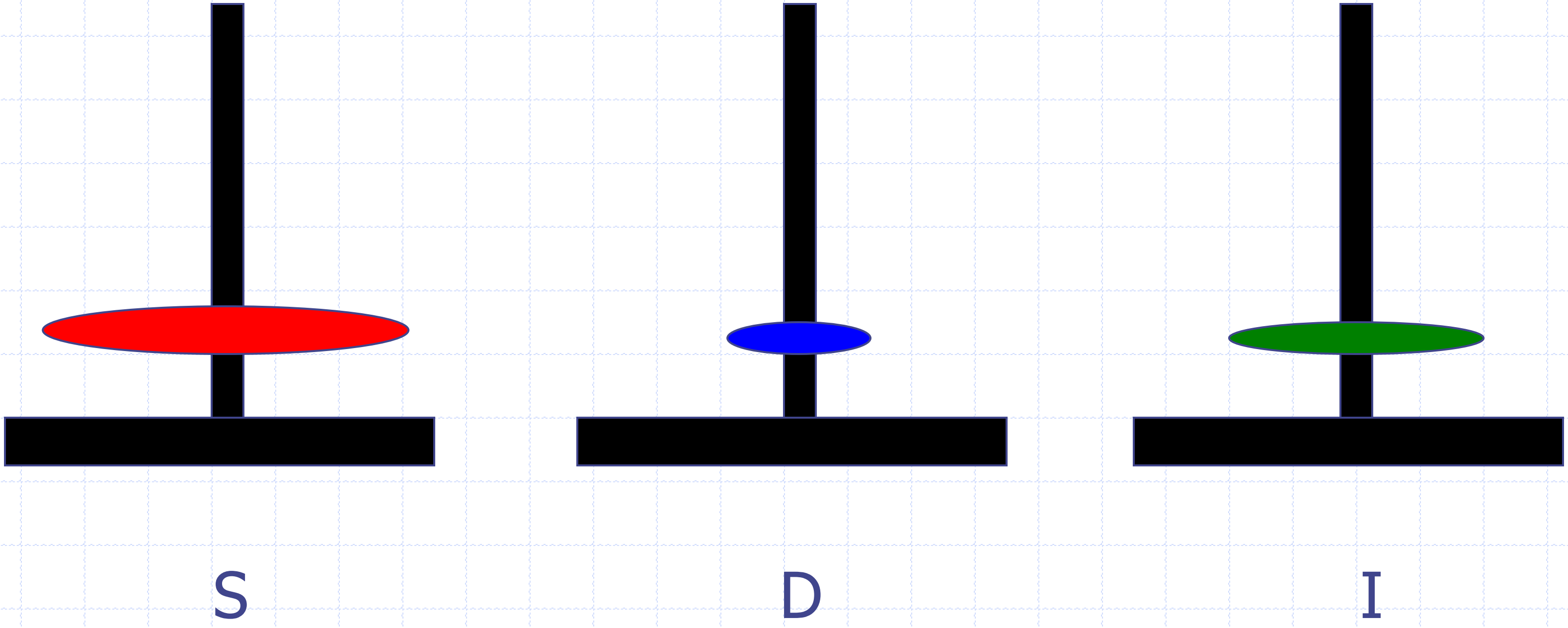
Towers of Hanoi



Towers of Hanoi



Towers of Hanoi



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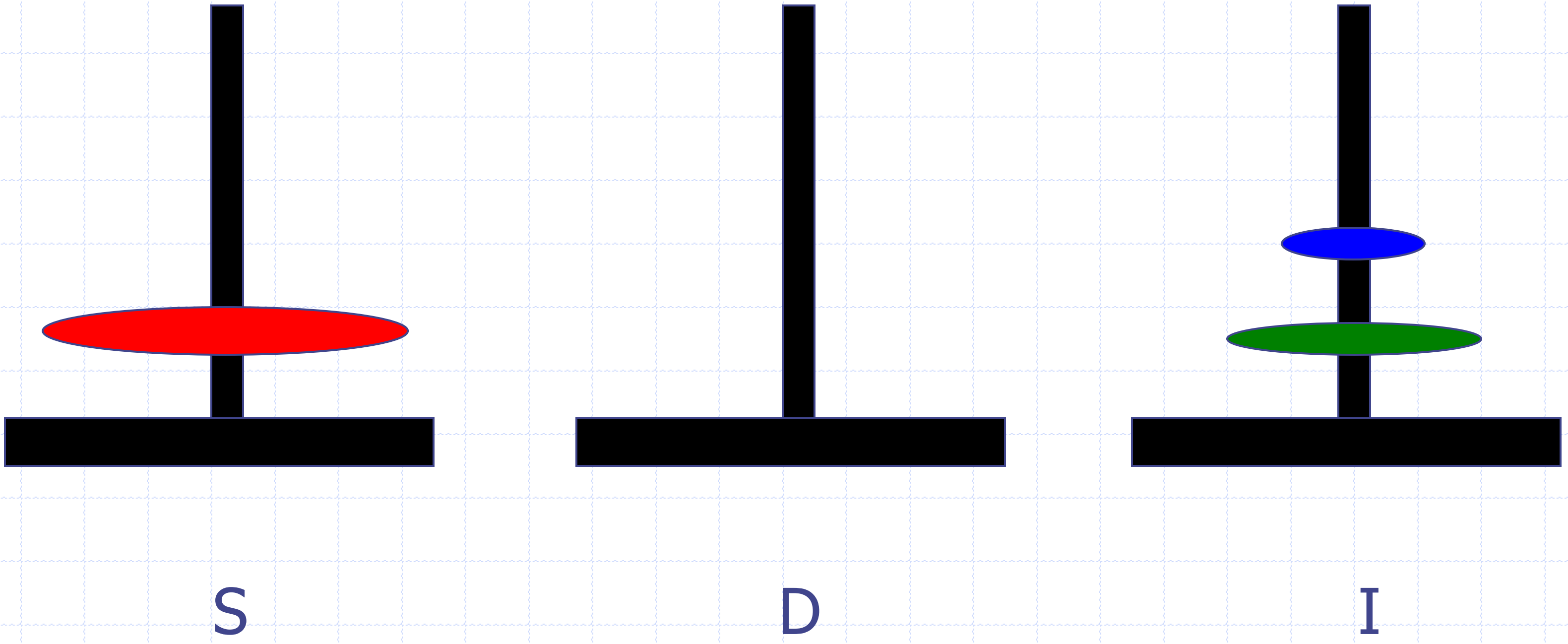

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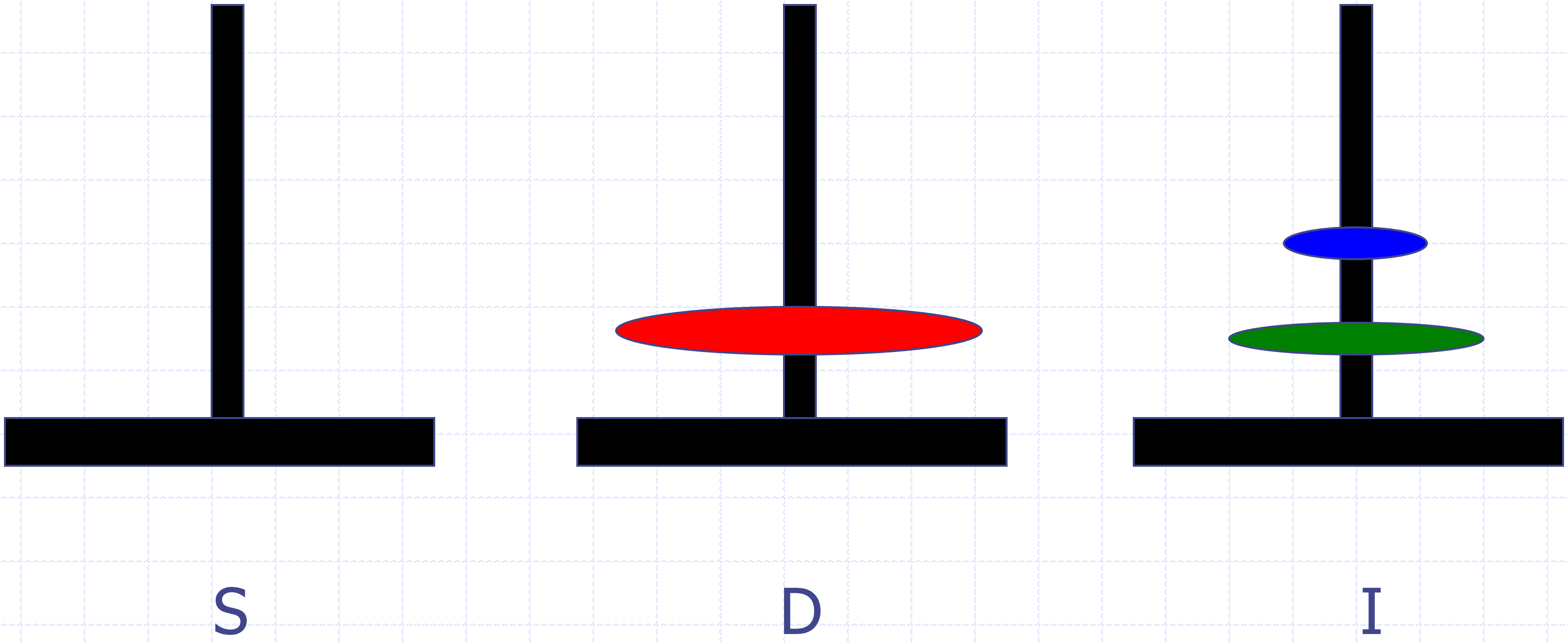

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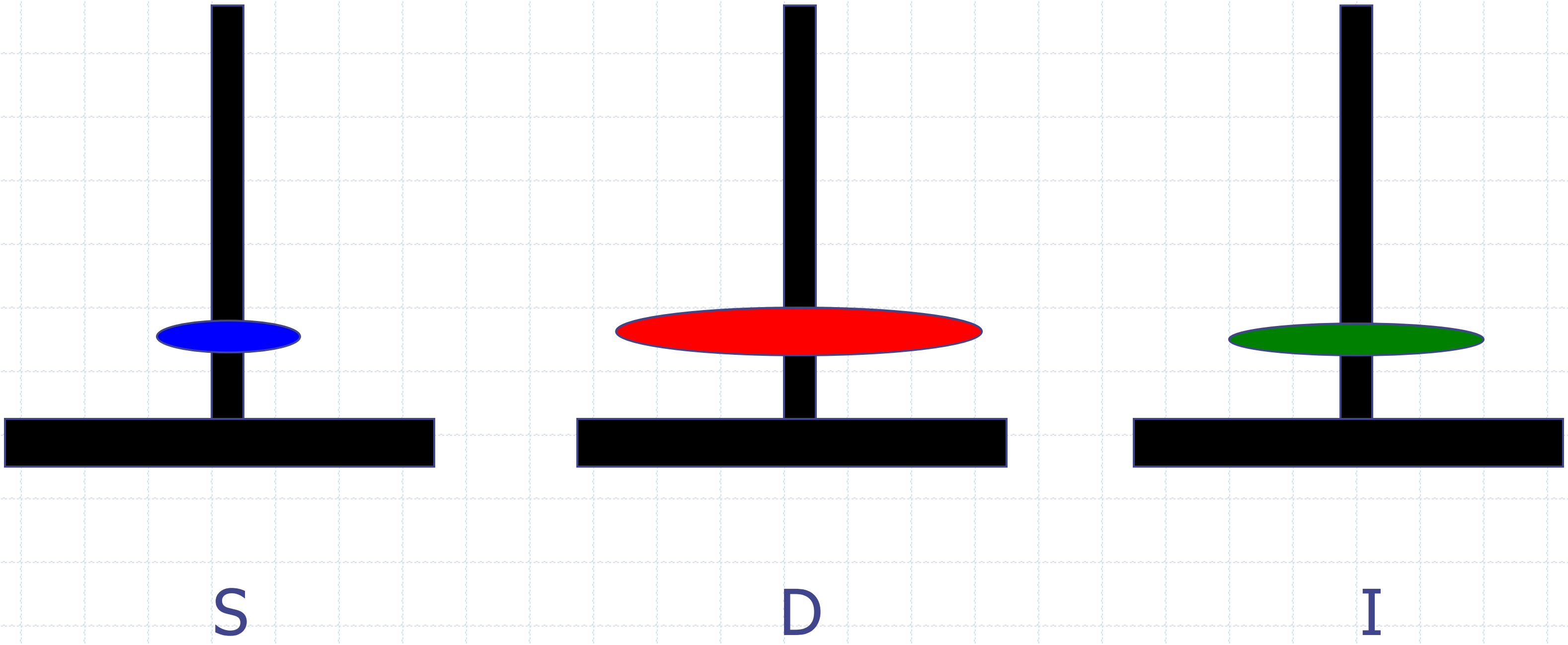

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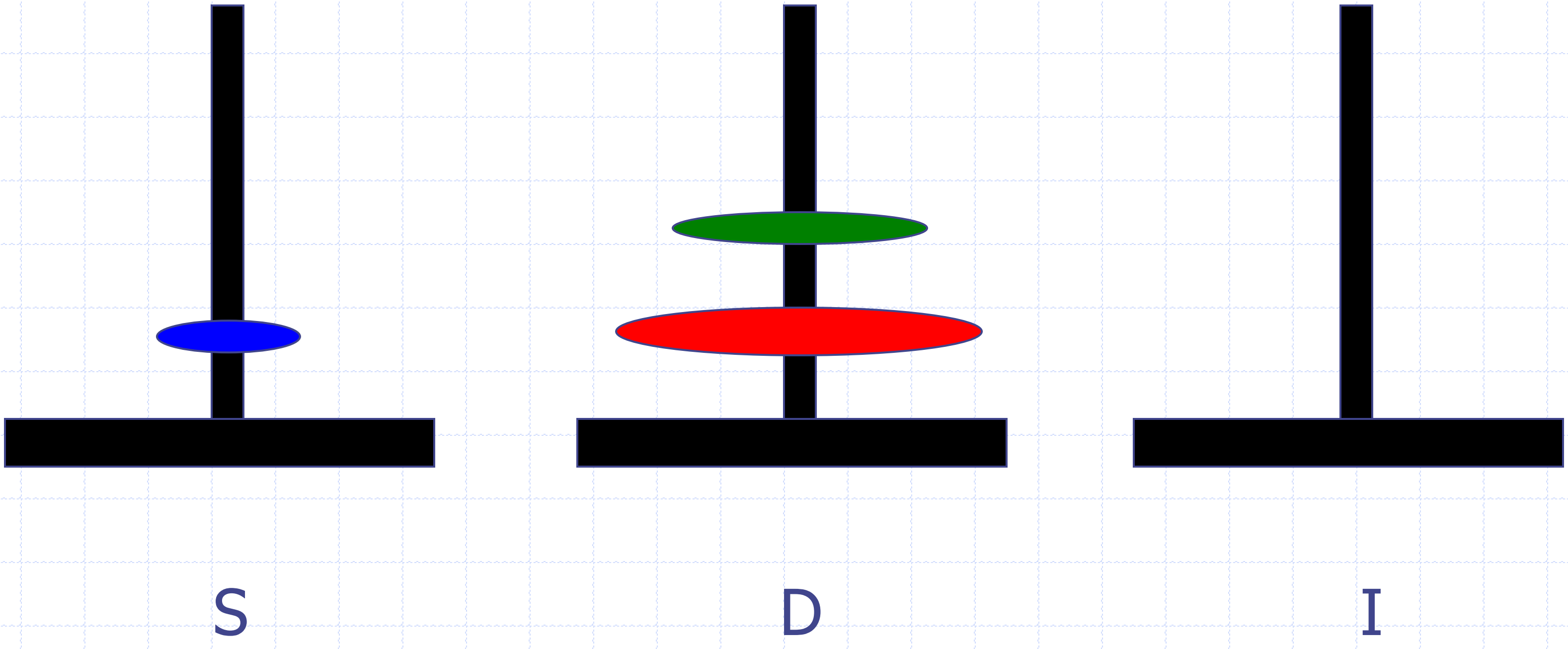
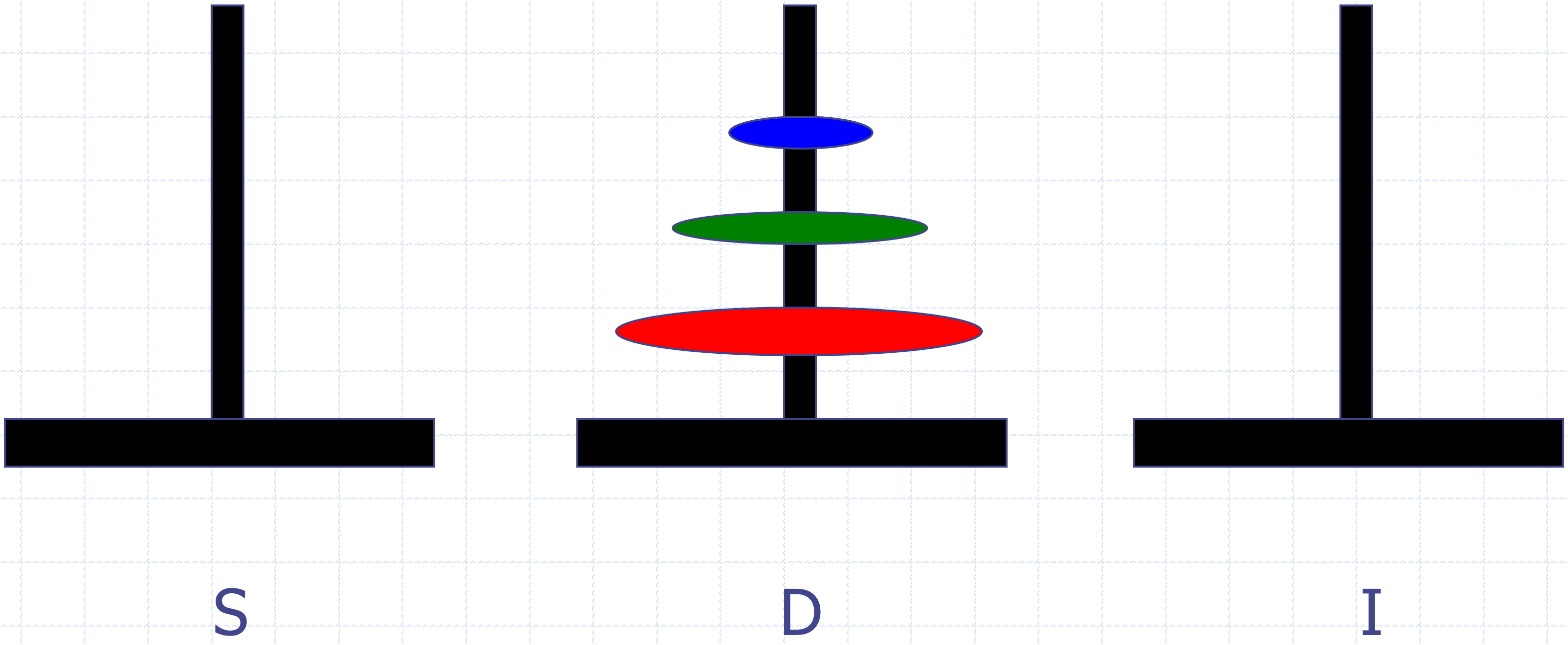


Towers of Hanoi



ToH – Recursive Solution

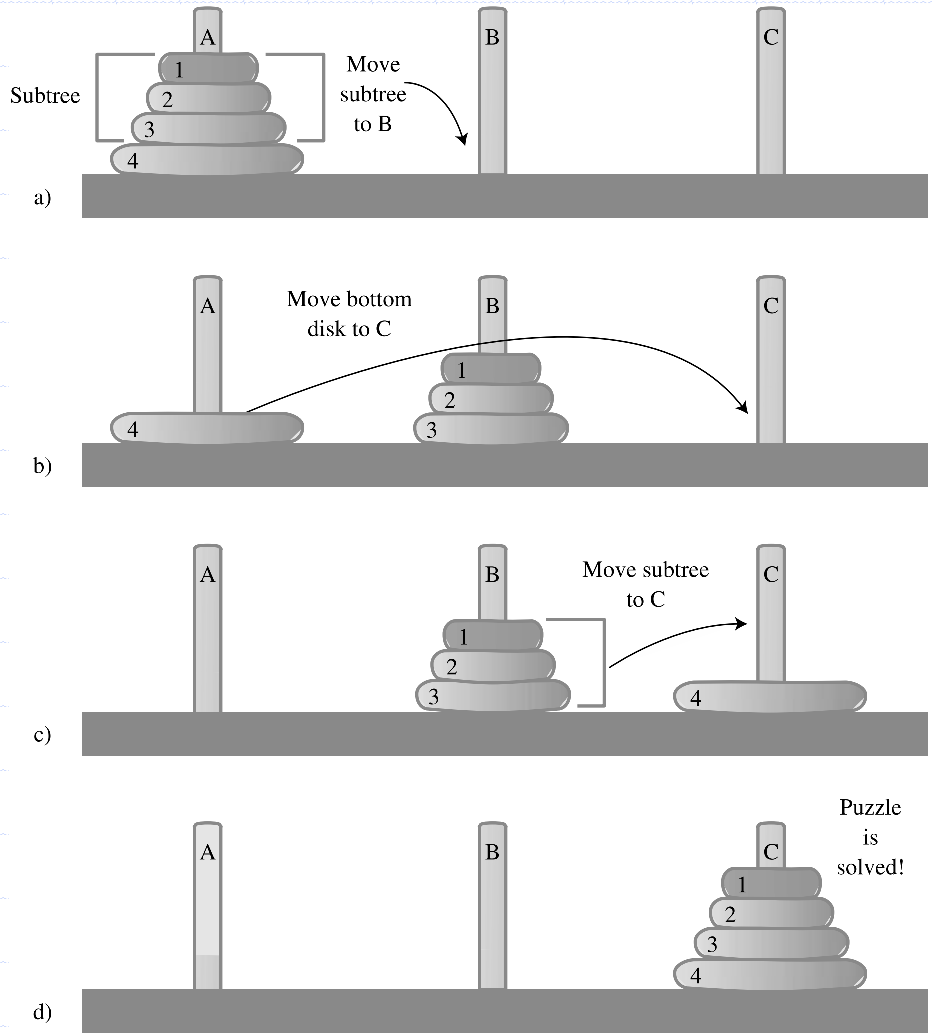
Model

- Source tower S
- Intermediate tower I
- Destination tower D

Assume n disks on S

1. Move subtree (top n-1) disks from S to I
2. Move the remaining (largest) disk from S to D
3. Move the subtree from I to D.

ToH – Recursive Solution (Moving disks from Tower A (Source) to C (Destination))



ToH – Recursive Algorithm

```
def Hanoi(topN, from, inter, to):

    if topN == 1:  // base case
      Move(1, from, to);
    else:  // recursion
      Hanoi(topN-1, from, to, inter); // from → inter
      Move(topN, from, to);
      Hanoi(topN-1, inter, from, to); // inter → to
```

Runtime of Recursive Functions

```
def recursiveFun1(n):
    if n<=0:
        return 1
    else:
        return 1+recursiveFun1(n-1)
```

Runtime of Recursive Functions

```
def recursiveFun2(n):
        if n<=0:
                return 1
        else:
                return 1+recursiveFun2(n-5)
```

Runtime of Recursive Functions

```
def recursiveFun3(n):
    if n<=0:
        return 1
    else:
        return 1+recursiveFun3(n//5)
```

Runtime of Recursive Functions

```
def recursiveFun4(n,m,o):
    if n<=0:
        print(m,",",o)
    else:
        recursiveFun4(n-1,m+1,o)
        recursiveFun4(n-1,m,o+1)
```

Runtime of Recursive Functions

```
def recursiveFun5(n):
    for i in range(0,n,2):
        print("hello")
    if n<=0:
        return 1
    else:
        return 1+recursiveFun5(n-5)
```

Divide and Conquer Master Method

$$f(n) = af(n/b) + g(n).$$

This is called a **divide-and-conquer recurrence relation.**

THEOREM 2 MASTER THEOREM Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

Complexity of Merge Sort In Example 3 we explained that the number of comparisons used by the merge sort to sort a list of n elements is less than $M(n)$, where $M(n) = 2M(n/2) + n$. By the master theorem (Theorem 2) we find that $M(n)$ is $O(n \log n)$, which agrees with the estimate found in Section 5.4.

Practice Master Method

1. $T(n) = 3T(n/2) + n^2$

2. $T(n) = 4T(n/2) + n^2$

3. $T(n) = T(n/2) + 2^n$

1. $T(n) = 3T(n/2) + n^2 \implies T(n) = \Theta(n^2)$
2. $T(n) = 4T(n/2) + n^2 \implies T(n) = \Theta(n^2 \log n)$
3. $T(n) = T(n/2) + 2^n \implies \Theta(2^n)$

Practice Master Method

5. $T(n) = 16T(n/4) + n$

6. $T(n) = 2T(n/2) + n \log n$

15. $T(n) = 3T(n/4) + n \log n$

16. $T(n) = 3T(n/3) + n/2$

5. $T(n) = 16T(n/4) + n \Longrightarrow T(n) = \Theta(n^2)$

6. $T(n) = 2T(n/2) + n \log n \Longrightarrow T(n) = n \log^2 n$

15. $T(n) = 3T(n/4) + n \log n \Longrightarrow T(n) = \Theta(n \log n)$

16. $T(n) = 3T(n/3) + n/2 \Longrightarrow T(n) = \Theta(n \log n)$

Practice Master Method (Can't apply always)

$$9.\ T(n) = 0.5T(n/2) + 1/n$$

$$4.\ T(n) = 2^n T(n/2) + n^n$$

$$9.\ T(n) = 0.5T(n/2) + 1/n \Longrightarrow \text{Does not apply } (a < 1)$$

$$4.\ T(n) = 2^n T(n/2) + n^n \Longrightarrow \text{Does not apply } (a \text{ is not constant})$$