Exercise

$$(x+e)^2 = f$$

$$x^2 + 2xe + e^2 = f$$

Set variables equal to

$$x^2 + bx = c$$

Solve

$$e = \frac{b}{2}$$

$$f = c + e^2 = c + \frac{b^2}{4}$$

Average

Remember how we can motivate the average. If we have

$$x_1, \dots, x_n$$

then we can try to find the nearest number c according to the expression

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - c)^2$$

Set $\bar{x} = \frac{1}{n}(x_1 + \dots + x_n)$. Note

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - c)^2$$

$$=$$

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x} + \bar{x} - c)^2$$

$$=$$

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 + 2(x_i - \bar{x})(\bar{x} - c) + (\bar{x} - c)^2$$

$$=$$

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 + \frac{2(\bar{x} - c)}{n}\sum_{i=1}^{n}x_i - \bar{x} + \frac{1}{n}\sum_{i=1}^{n}(\bar{x} - c)^2$$

$$\sum_{i=1}^{n}x_i - \bar{x} = \left(\sum_{i=1}^{n}x_i\right) - n\bar{x}$$

$$= 0$$

$$=$$

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 + \frac{1}{n}\sum_{i=1}^{n}(\bar{x} - c)^2$$

So to minimize this quantity take

$$c = \bar{x}$$

Derivatives

Let us find another way to solve for the value c. Note

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - c)^2$$

$$=$$

$$\frac{1}{n}\sum_{i=1}^{n}x_i^2 - 2x_i c + c^2$$

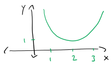
$$=$$

$$\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right) + \left(-\frac{2}{n}\sum_{i=1}^{n}x_i\right)c + \left(\frac{1}{n}\sum_{i=1}^{n}1\right)c^2$$

Therefore

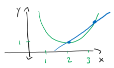
$$c^2 + (\text{some number})\,c = \text{some other number}$$

We can apply the exercise to deduce that this is the equation for a quadratic



$$y = (x-2)^2 + 1$$

Suppose we want to approximate the curve by a line near a point. We compute the slope of the secant line



$$y = (x-2)^2 + 1$$

$$\frac{\left((x_2-2)^2+1\right) - \left((x_1-2)^2+1\right)}{x_2 - x_1}$$

$$=$$

$$\frac{x_2^2 - x_1^2 - 4x_2 - 4x_1}{x_2 - x_1}$$

$$=$$

$$x_2 + x_1 - 4$$

Therefore as x_2 and x_1 get close together the slope of the secant line becomes $2x - 4$.

We say that the derivative of $(x-2)^2 + 1$ is $2(x-2)$

Average

We can use derivatives to solve for c.

By the rules of derivatives

$$\frac{d}{dx}\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - c)^2\right]$$

$$=$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{d}{dx}(x_i - c)^2$$

$$=$$

$$\frac{1}{n}\sum_{i=1}^{n}2(x_i - c)$$

Remember we want to find where the derivative is 0. We can solve

$$0 = \frac{1}{n}\sum_{i=1}^{n}2(x_i - c)$$

$$= \left(\sum_{i=1}^{n}x_i\right) - \left(\sum_{i=1}^{n}c\right)$$

Therefore

$$c = \frac{1}{n}\sum_{i=1}^{n}x_i$$