Introduction to Market Structures

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Cost Minimization

- Suppose a firm needs to produce y_o units of output.
- It produces y_o units of output using a two inputs, labour and capital (l, k).
- The technology is given by production function $y = f(l, k)$.
- Suppose the price of labour is w. It is called wage.
- The price of capital r. It is interest rate.

$$y_o = f(l, k)$$

- If l units of labour are used to produce y_o output, the cost on wages is wl.

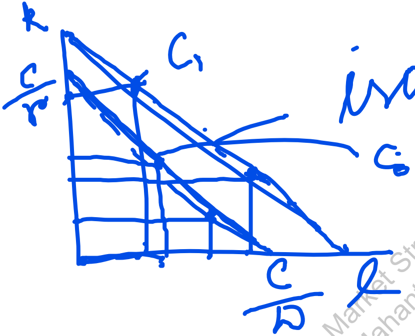
wl

- When k units of capital are used to produce y_o output. The cost on capital is rk.

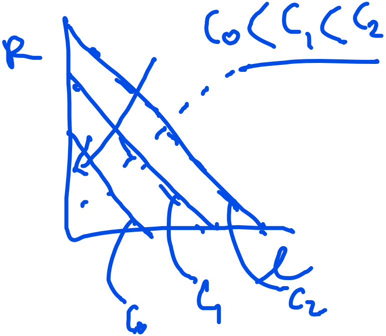
rk

- The total cost is $wl + rk$.
- $wl + rk = C$. Different combinations of (l, k) costing same total cost C gives us isocost lines. They are shown below.

$$\underline{wl + rk} = \underline{C}$$



isocost



The isocost line in the north east direction means higher total cost.

$$dC = \frac{\partial(wl + rk)}{\partial l} Dl + \frac{\partial(wl + rk)}{\partial k} dk.$$

$$dC = w.dl + r.dk$$

In the movement along an isocost curve, $dC = 0$

$\frac{dk}{dl} = -\frac{w}{r}$, the slope of isocost line

$$dC = \frac{\partial(wl+rk)}{\partial l} . dl + \frac{\partial(wl+rk)}{\partial k} dk$$

$$0 = w\,dl + r\,dk$$

y_0

- To produce output, the firm needs to hire these inputs. The firm takes the price of these inputs as given and only decide on the quantity of the inputs.
- We get demand for each inputs of a firm.
- While deciding on the amount of quantity of each input, a firm is solving the following problem;

Minimize $wl + rk$
subject to $y_o = f(l, k)$.

$k \leq \bar{k}$

The firm wants to hire that combination of labour and capital (l, k) which will cost minimum.

$wl + rk$

$y = l^{\alpha} k^{\beta}$

We assume that the production function is well behaved. It is similar to utility function. So we get the following type of isoquants.

$y = f(l, k)$

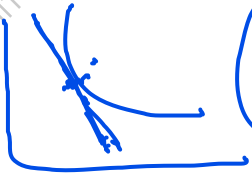
$y' < y'' < y'''$

Graphically, we solve the minimization problem in the following way;

When the isoquants are convex to the origin and differentiable, we get the following condition at the cost minimizing point.

$\frac{w}{r} = \frac{MP_l}{MP_k}$=slope of isocost line is equal to the slope of the isquant. The isoquant is tangent to the isocost line.

$\frac{w}{r} = \frac{MP_L}{MP_K}$



$\frac{MP_L}{w} = \frac{MP_K}{r}$

$$y_0 = f(l, k)$$

$$\overline{\quad w, r \quad}$$

$$\frac{C}{w}, \frac{C}{r}$$

$$\frac{dk}{dl} = -\frac{w}{r}$$

Iso cost line must be tangent to isoquant.

- Suppose the production function is $y = l^{\alpha} k^{\beta}$ and $0 < \alpha < 1$ and $0 < \beta < 1$.
- Suppose the wage rate is w and interest rate is r. The iso cost function is $wl + rk$.
- We solve the cost minimization problem subject to a given level of output through Lagrange method
- $L = wl + rk + \lambda(y_o - l^{\alpha} k^{\beta})$. Here λ is the Lagrange multiplier.
- We are minimizing $wl + rk$ and maximizing $(y_o - l^{\alpha} k^{\beta}) \leq 0$. So at the optimal point $y_o = l^{\alpha} k^{\beta}$.

- Since the production function is differentiable in (l, k). So we take the following derivatives;
- $\dfrac{\partial L}{\partial l} = w - \lambda\alpha l^{\alpha-1}k^{\beta}$ $= 0$
- $\dfrac{\partial L}{\partial k} = r - \lambda\beta l^{\alpha}k^{\beta-1}$ $= 0$
- $\dfrac{\partial L}{\partial \lambda} = y_o - l^{\alpha}k^{\beta}$ $= 0$
- First order condition gives
- $\dfrac{\partial L}{\partial l} = w - \lambda\alpha l^{\alpha-1}k^{\beta} = 0$
- $\dfrac{\partial L}{\partial k} = r - \lambda\beta l^{\alpha}k^{\beta-1} = 0$
- $\dfrac{\partial L}{\partial \lambda} = y_o - l^{\alpha}k^{\beta} = 0$

$$L = wl + rk + \lambda(y_o - l^{\alpha}k^{\beta})$$

$$\frac{w}{r} = \frac{\alpha l^{\alpha-1}k^{\beta}}{\beta l^{\alpha}k^{\beta-1}} = \frac{\alpha k}{\beta l}$$

Using first two equations we get

$$\frac{w}{r} = \frac{\alpha l^{\alpha-1}k^{\beta}}{\beta l^{\alpha}k^{\beta}} = \frac{\alpha k}{\beta l}.$$

$\Rightarrow k = \frac{w\beta l}{r\alpha}$

$\rightarrow \frac{w\beta l}{r\alpha} = k$. We substitute k in the third equation to get

$$y_o = l^{\alpha}\left(\frac{w\beta l}{r\alpha}\right)^{\beta} = l^{\alpha+\beta}\left(\frac{w\beta}{r\alpha}\right)^{\beta}$$

$y_o = l^{(\alpha+\beta)} \cdot \left(\frac{w\beta}{r\alpha}\right)^{\beta}$

$l = y_0^{\frac{1}{\beta+\alpha}}\left(\frac{r\alpha}{w\beta}\right)^{\frac{\beta}{\beta+\alpha}}$. This is the conditional demand function of labour. It is conditional on y_o. When we take a general y, we get $l = y^{\frac{1}{\beta+\alpha}}\left(\frac{r\alpha}{w\beta}\right)^{\left(\frac{\beta}{\beta+\alpha}\right)}$. The conditional demand function of capital is

$$k = y^{\frac{1}{\beta+\alpha}}\left(\frac{w\beta}{r\alpha}\right)^{\left(\frac{\alpha}{\beta+\alpha}\right)}.$$

$wl + rk$

The cost function function is
$wl^* + rk^*$ where l^* is the cost minimal value of labour and k^* is the cost minimal value of capital.
Substituting from above we get

$$C(y) = w.y^{\frac{1}{\beta+\alpha}}\left(\frac{r\alpha}{w\beta}\right)^{\left(\frac{\beta}{\beta+\alpha}\right)} + r.y^{\frac{1}{\beta+\alpha}}\left(\frac{w\beta}{r\alpha}\right)^{\left(\frac{\alpha}{\beta+\alpha}\right)}$$

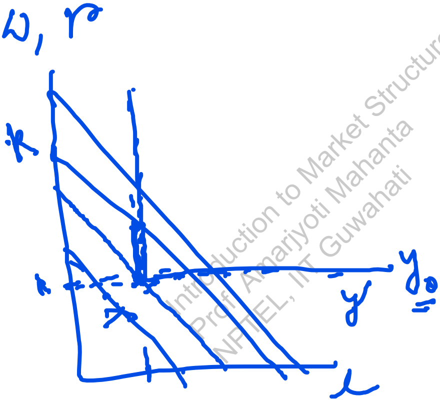
$$\Rightarrow C(y) = y^{\frac{1}{\beta+\alpha}}\left(w.\left(\frac{r\alpha}{w\beta}\right)^{\left(\frac{\beta}{\beta+\alpha}\right)} + r.\left(\frac{w\beta}{r\alpha}\right)^{\left(\frac{\alpha}{\beta+\alpha}\right)}\right)$$ Since w, r and α, β are fixed.

We can write the cost function as

$$C(y) = y^{\frac{1}{\beta+\alpha}}\Lambda, \quad \text{where } \Lambda = \left(w.\left(\frac{r\alpha}{w\beta}\right)^{\left(\frac{\beta}{\beta+\alpha}\right)} + r.\left(\frac{w\beta}{r\alpha}\right)^{\left(\frac{\alpha}{\beta+\alpha}\right)}\right)$$

We get that the cost function is a function of level of output and the price of inputs. We get the above cost function when we have Cobb-Douglas production function.

$$y = \text{Min}\{l, k\}$$



$$k = l = y_0$$

$$\min \; wl + rk$$

$$\text{s.t.} \; y_0 = \text{Min}\{l, k\}$$

$$y_0 = l = k$$

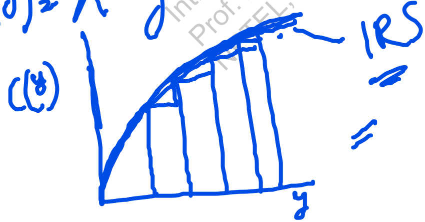
$$w y_0 + r y_0$$

$$C(y_0) = (w + r) y_0$$

$$C(y) = (w + r) y$$

If production function is increasing returns to scale, in the Cobb-Douglas production function $f(l,k) = l^{\alpha}k^{\beta}$, we must have $\alpha + \beta > 1$. The cost function in case of Cobb Douglas production function is $C(y) = \Lambda y^{\frac{1}{\alpha+\beta}}$. When, $\alpha + \beta > 1$, the cost function look like;

$C(y) = \Lambda \; y^{\frac{1}{\alpha+\beta}}$



$(\theta l)^{\alpha}(\theta k)^{\beta}$

$= \theta^{\alpha+\beta} l^{\alpha} k^{\beta}$

$\alpha+\beta > 1$

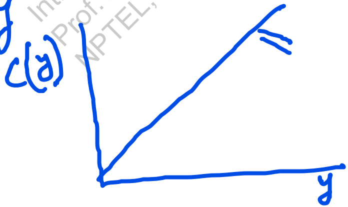
When production function exhibits constant returns to scale and it is Cobb Douglas we have $\alpha + \beta = 1$. So the cost function looks like

$$C(y) = A y^{\frac{1}{\alpha+\beta}}$$

$$= A y$$

$$a^{\alpha+\beta} l^{\alpha} k^{\beta} = a\, l^{\alpha} k^{\beta}$$

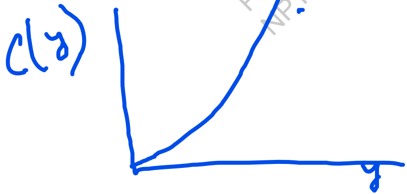
CRS



When production function exhibits constant returns to scale and it is Cobb Douglas we have $\alpha + \beta < 1$. So the cost function looks like

$$C(y) = \lambda \, y^{\frac{1}{\alpha+\beta}}$$

$$Q^{\alpha+\beta} \, l^{\alpha} k^{\beta} < Q \, l^{\alpha} k^{\beta}$$



In short run, some factor cannot be changed. They are fixed. Like capital is fixed at $\bar{k}$.

Now, the production function is $y = l^\alpha k^\beta, \quad$ if $k < \bar{k}$.

and $y = l^\alpha \bar{k}^\beta$, if $k \geq \bar{k}$.

We can use the above derivation to find the solution of this problem. Ideally we should be using Kuhn Tucker method. We can avoid it for simple problem like this. We know the demand curve of k is

$k = y^{\frac{1}{\beta+\alpha}} \left(\frac{w\beta}{r\alpha}\right)^{\left(\frac{\alpha}{\beta+\alpha}\right)}$. Plug in the value of y a firm wants to produce. If it is greater then $\bar{k}$, then $k = \bar{k}$.

Now use the production function to get the demand for labour

$y_o = l^\alpha (\bar{k})^\beta$

$\Rightarrow \left(\frac{y_o}{\bar{k}^\beta}\right)^{\frac{1}{\alpha}} = l.$

$y_0 \quad \bar{k}, \quad l_0 \left(\frac{y_0}{\bar{k}^\beta}\right)^{\frac{1}{\alpha}}$

$$k = \left[y_0^{\frac{1}{\beta+\alpha}} \left(\frac{w\beta}{\alpha r}\right)^{\frac{\alpha}{\alpha+\beta}} \right]$$

$k(y_0) \quad \bar{k}$

The cost function when $k \geq \bar{k}$ is

$$C(y) = w.\left(\frac{y_o}{\bar{k}^{\beta}}\right)^{\frac{1}{\alpha}} + r\bar{k}.$$

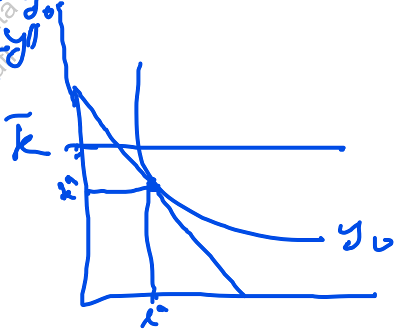
$$C(y) = \gamma y^{\frac{1}{\alpha}} + b.$$

$$wl^{\alpha} + rk^{\alpha}$$

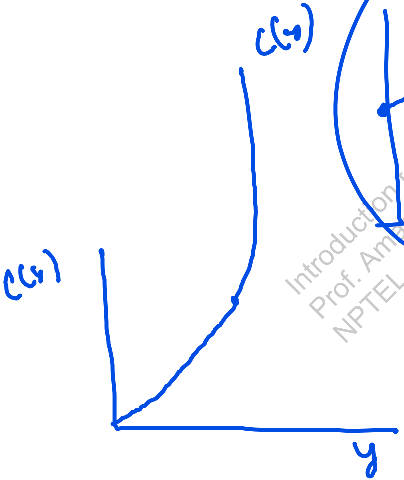
$$\gamma = \frac{w}{\bar{k}^{\frac{\beta}{\alpha}}}$$

$$w\left(\frac{y_0}{\bar{k}^{\beta}}\right)^{\frac{1}{\alpha}} + r\bar{k}$$

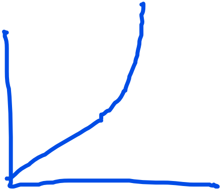
$\lambda y^{\frac{1}{\alpha+\beta}}$



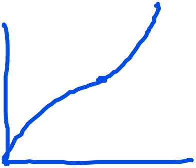
$$c(y) = \gamma y^{\frac{1}{\alpha}} + \beta$$



$\alpha + \beta < 1$



$\alpha+\beta=1$



When capital is fixed in short run. The total cost in the long run is going to be less than the cost in the short run.