CS 106X, Lecture 23
Dijkstra and A* Search

reading:

Programming Abstractions in C++, Chapter 18

Plan For Today

- **Recap**: DFS and BFS
- Dijkstra's Algorithm
- Announcements
- A* Search

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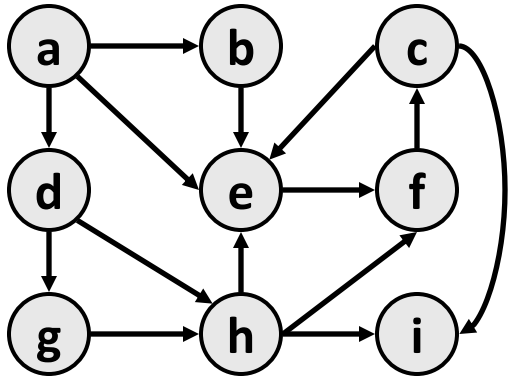
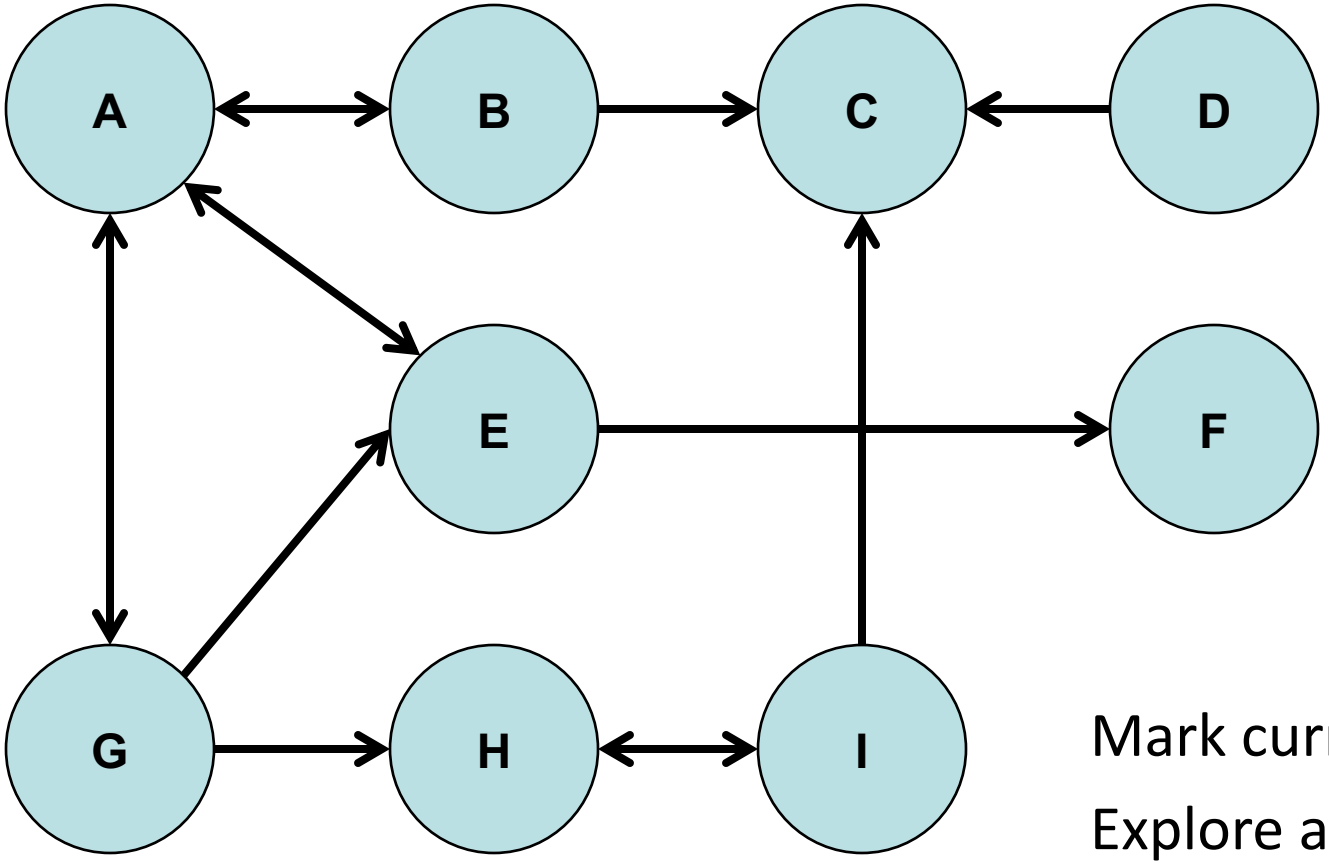
Depth-first search (18.4)

- **depth-first search** (DFS): Finds a path between two vertices by exploring each possible path as far as possible before backtracking.
  - Often implemented recursively.
  - Many graph algorithms involve *visiting* or *marking* vertices.

- DFS from *a* to *h* (assuming A-Z order) visits:
  - **a**
    - **b**
      - **e**
        - **f**
          - **c**
            - **i**
    - **d**
      - **g**
        - **h**
  - path found: **{a, d, g, h}**



DFS

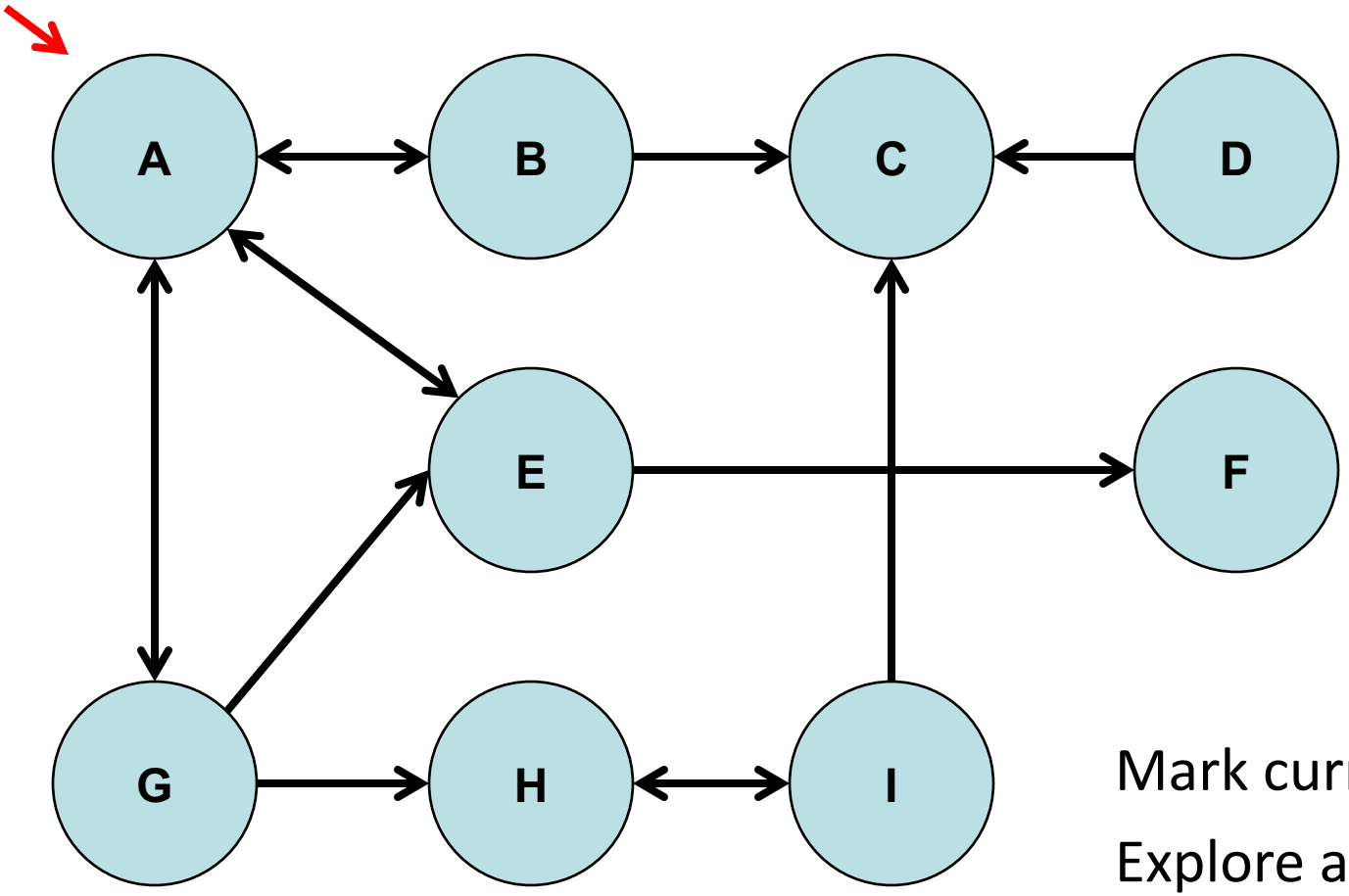


Let's say we are starting at A and searching for C.

Mark current as visited

Explore all the unvisited nodes from this node

DFS



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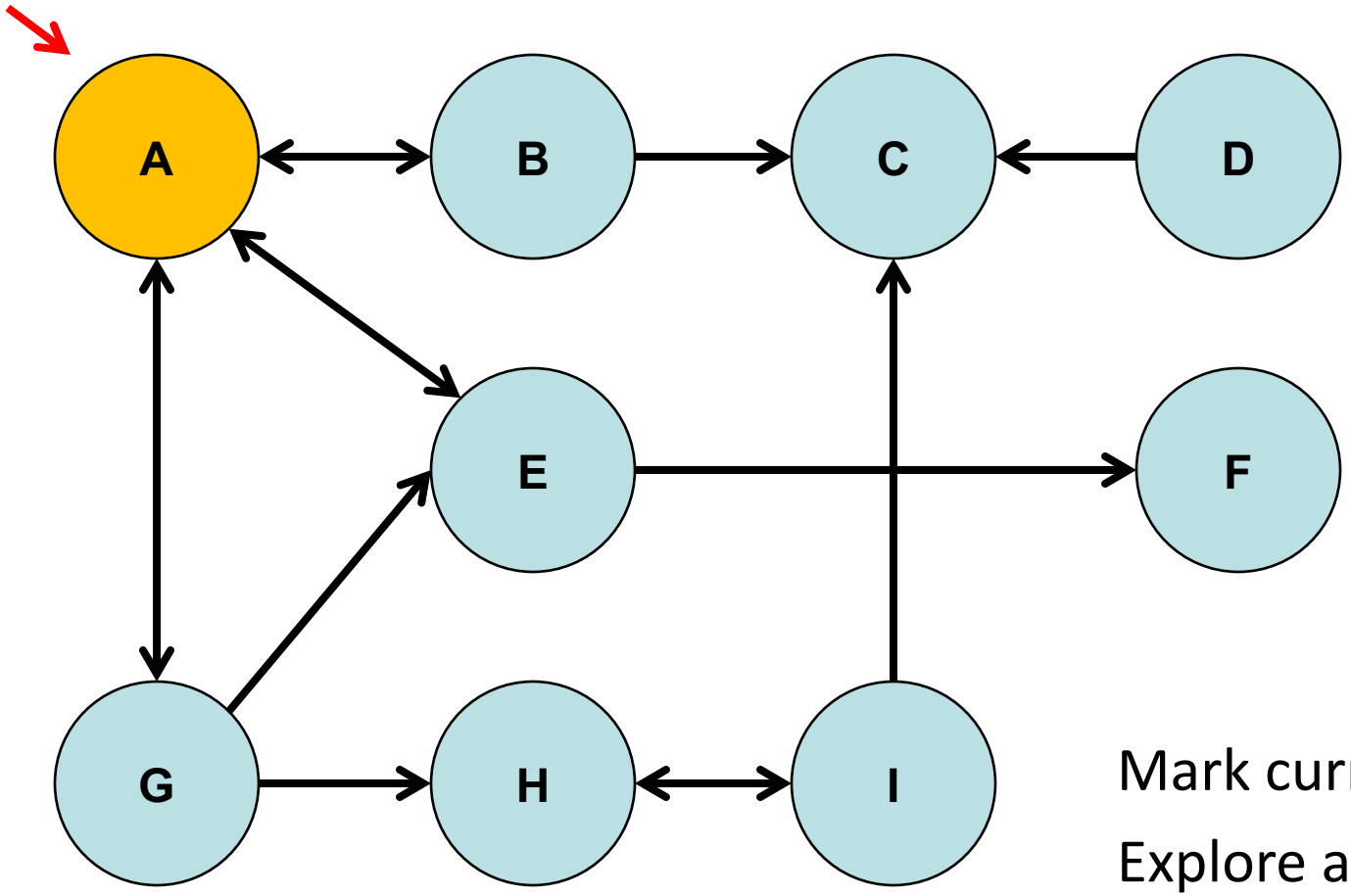

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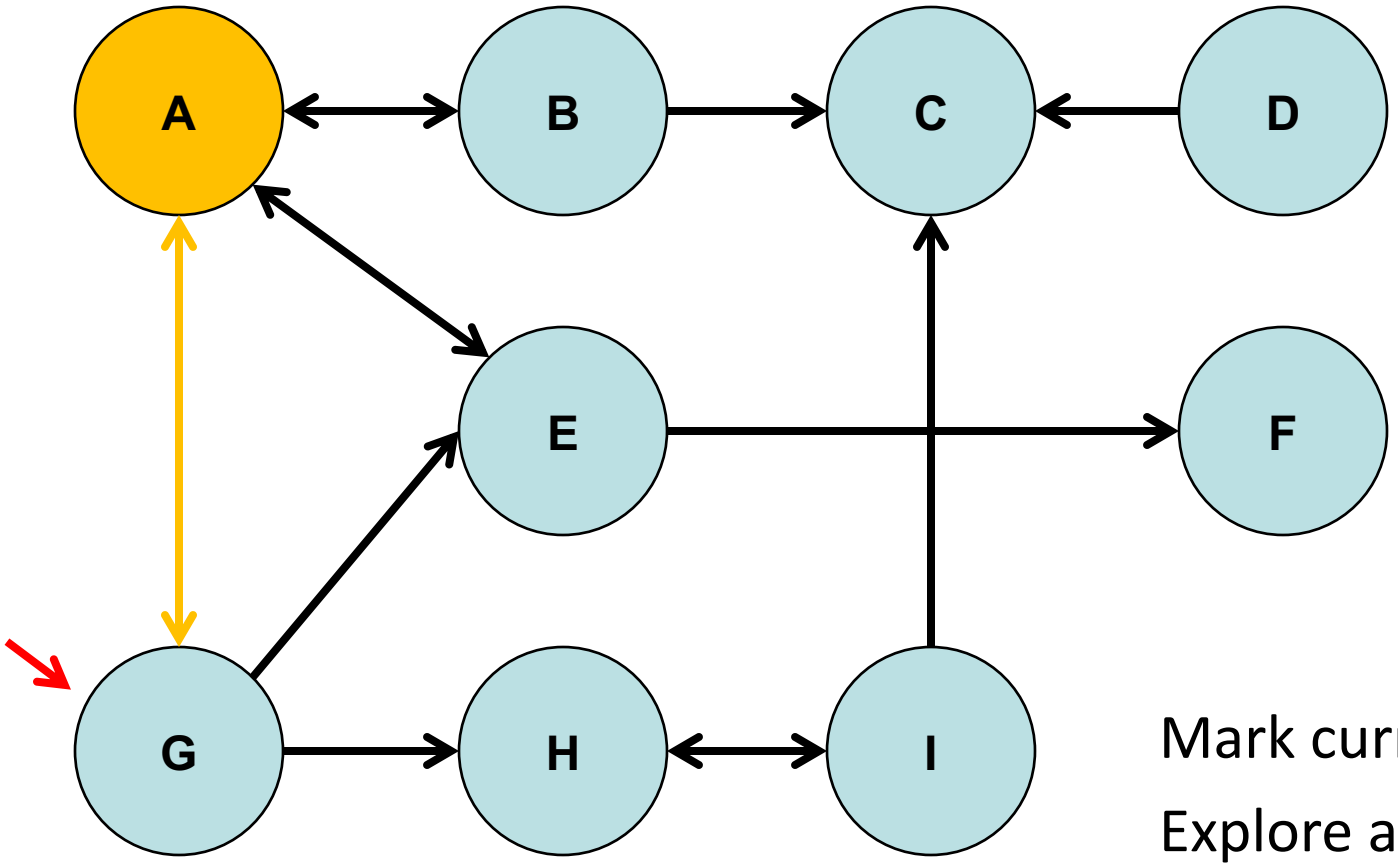

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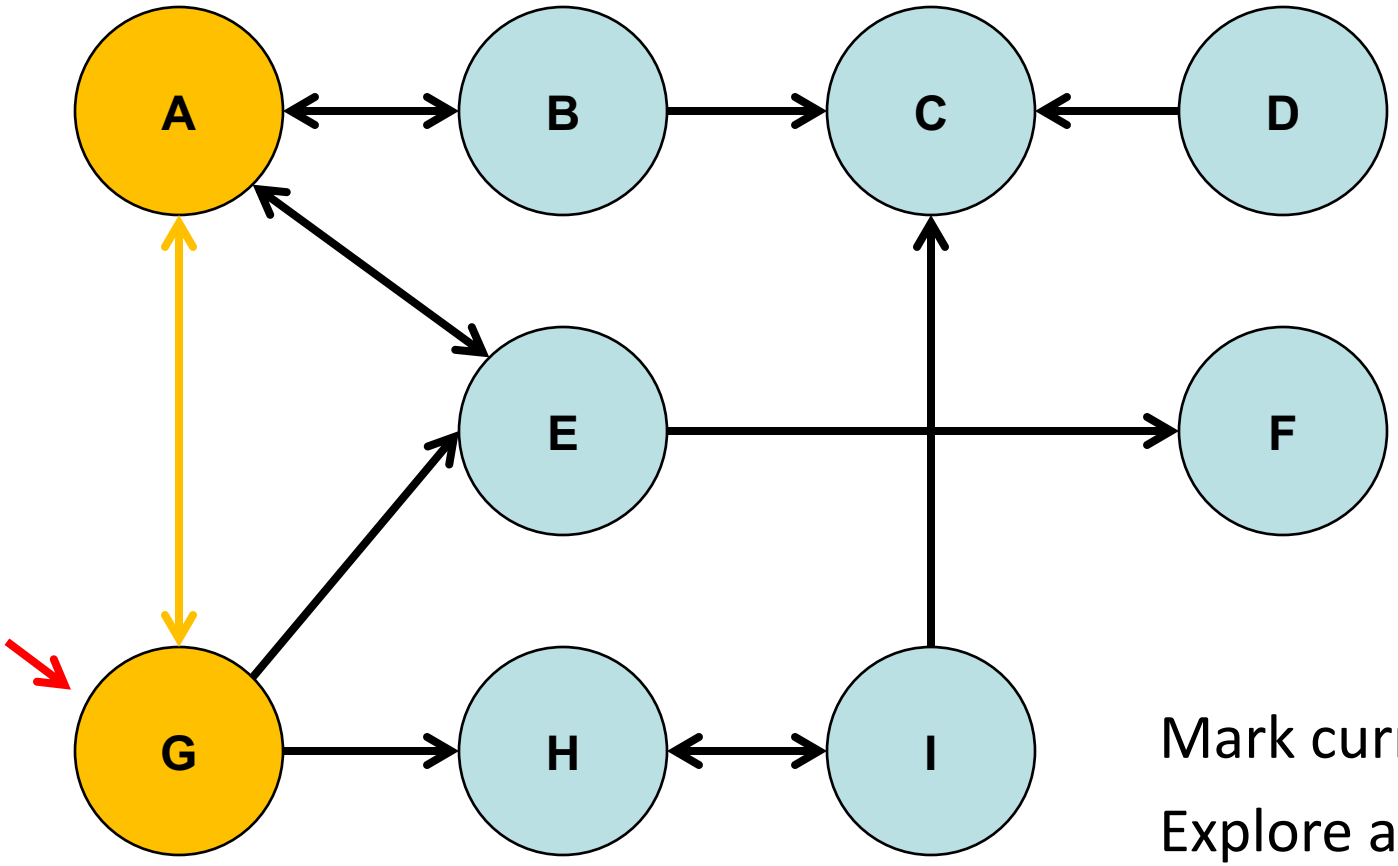

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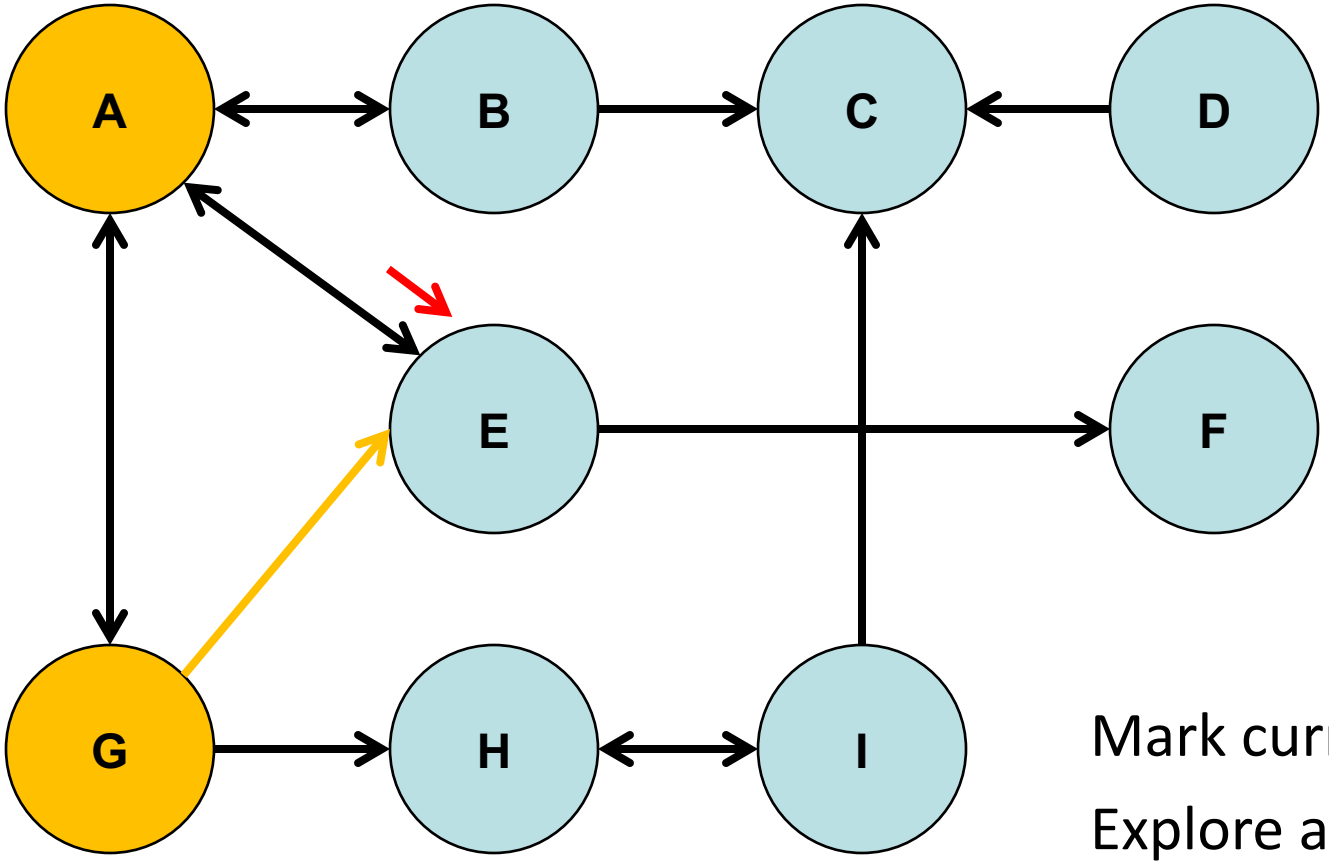

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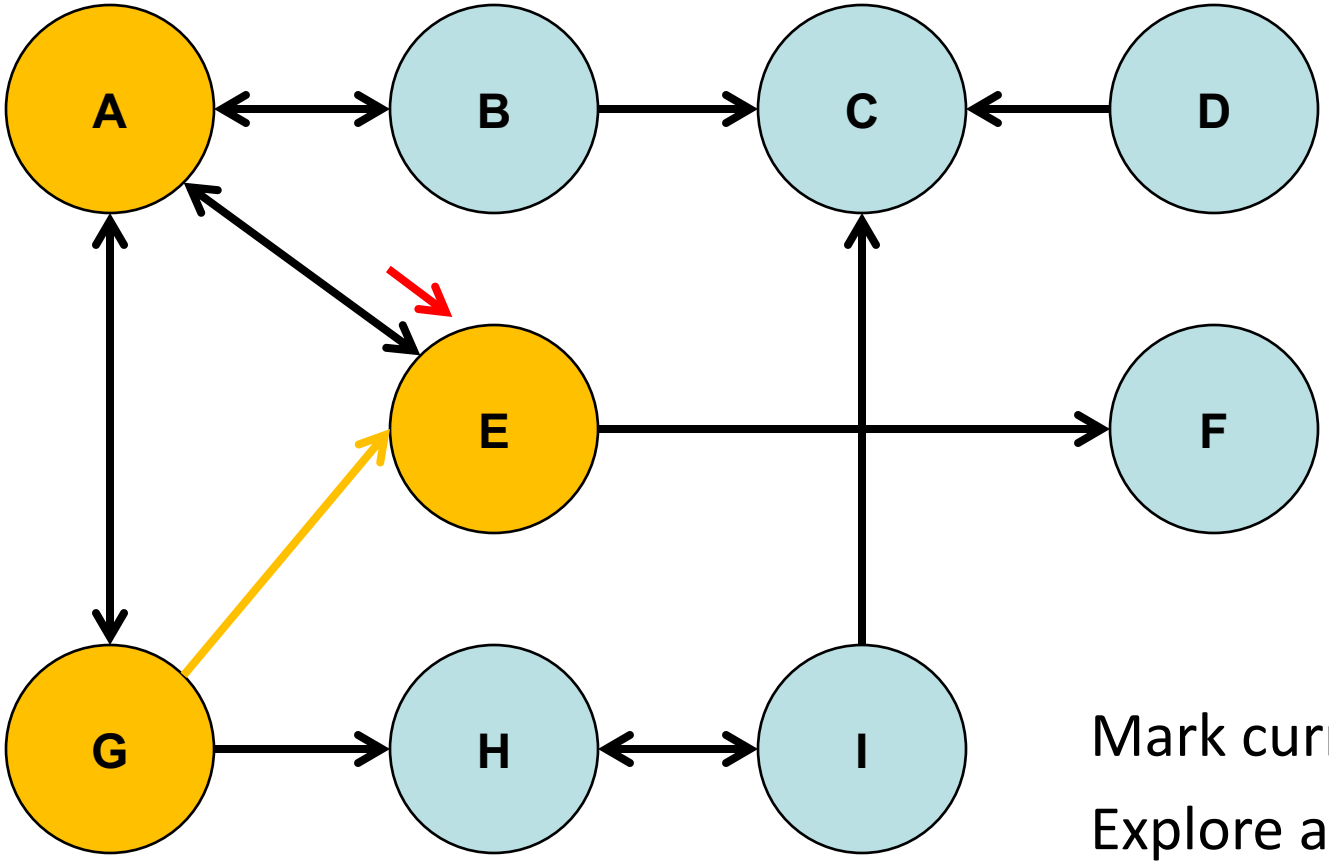

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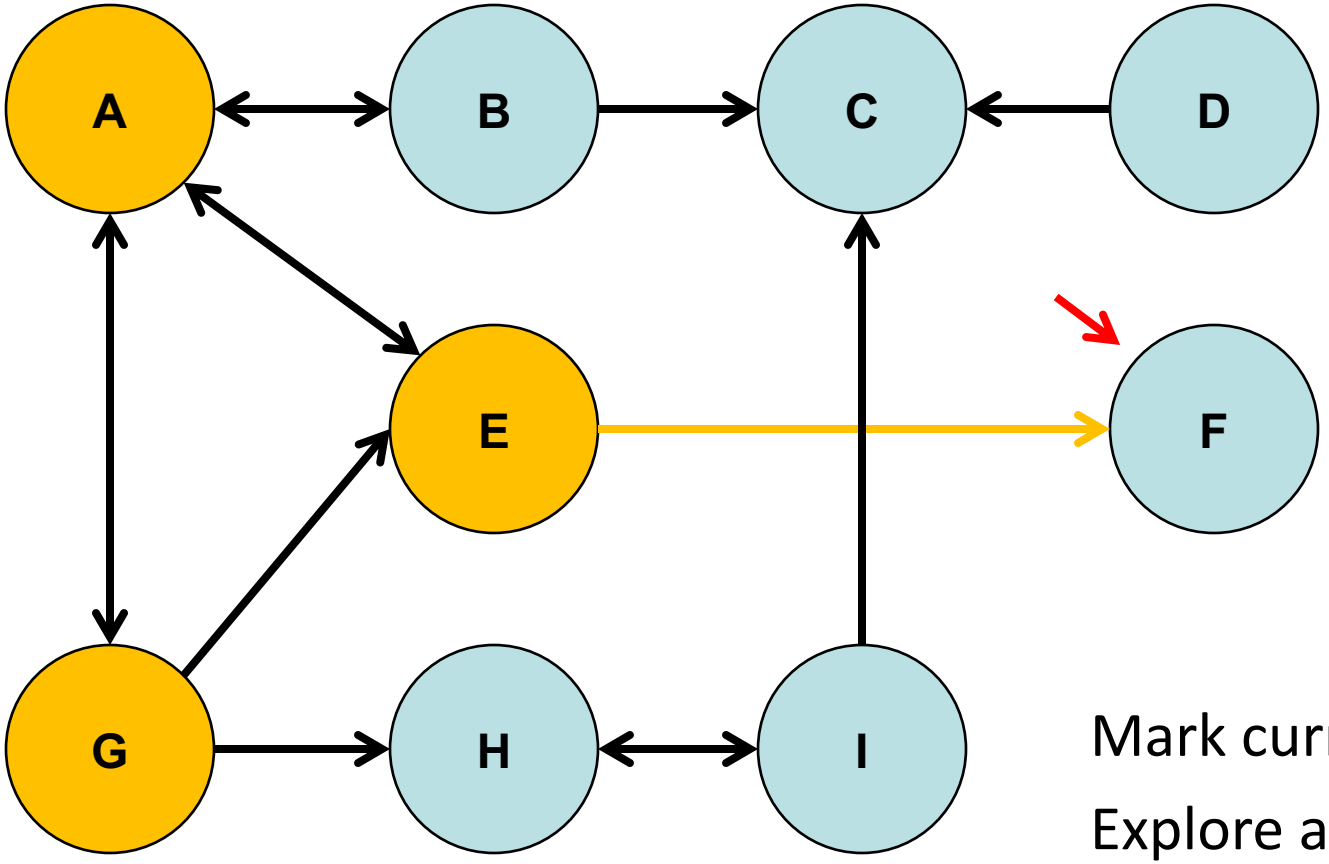

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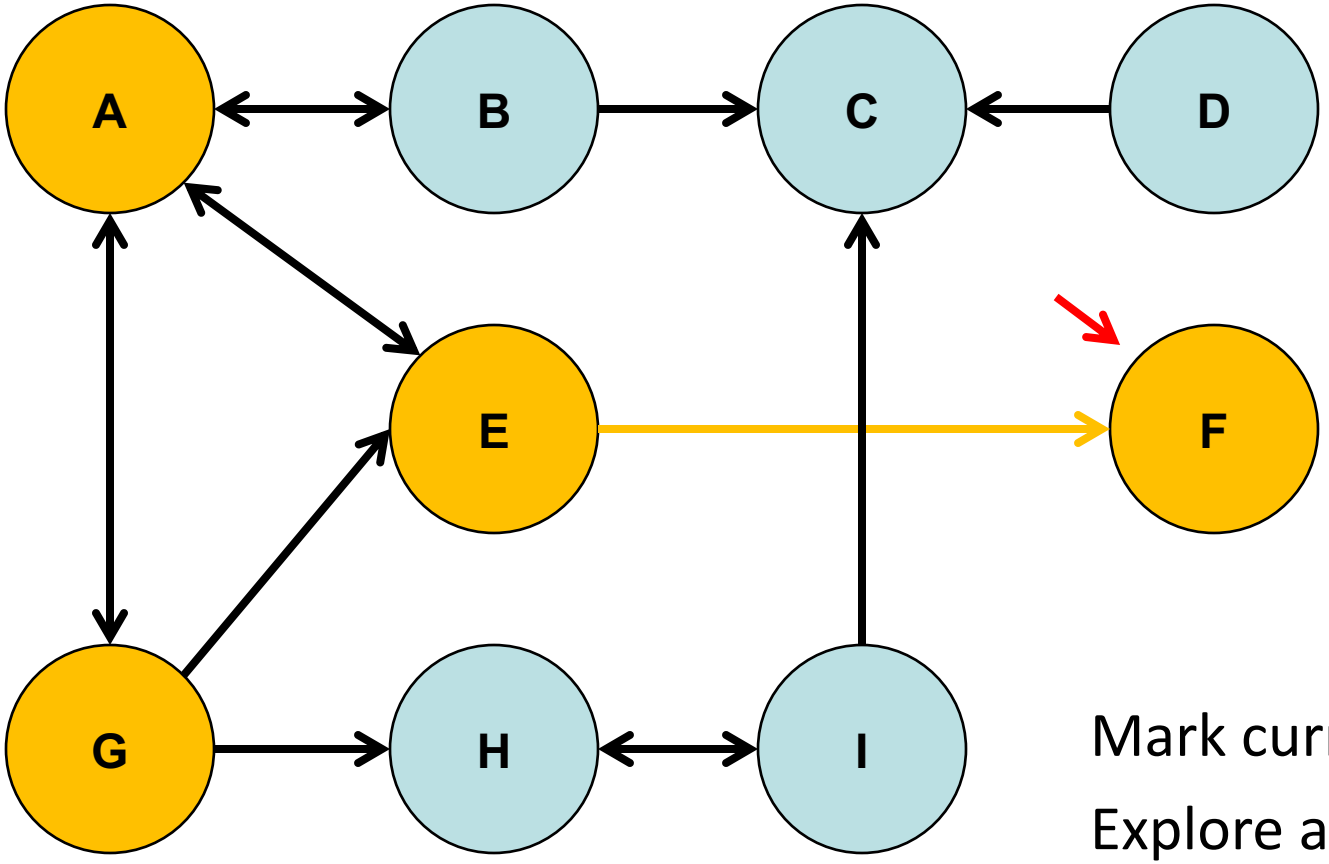

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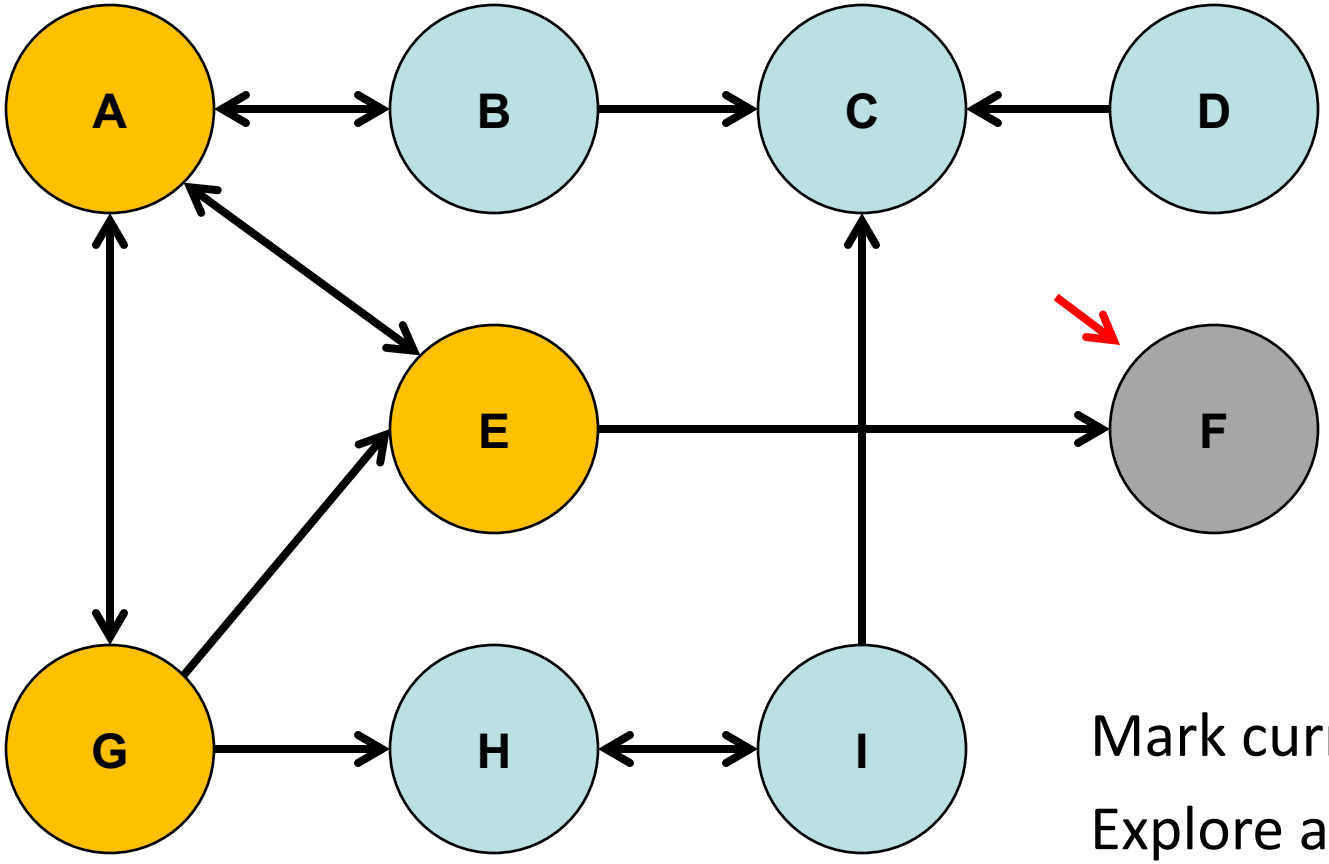

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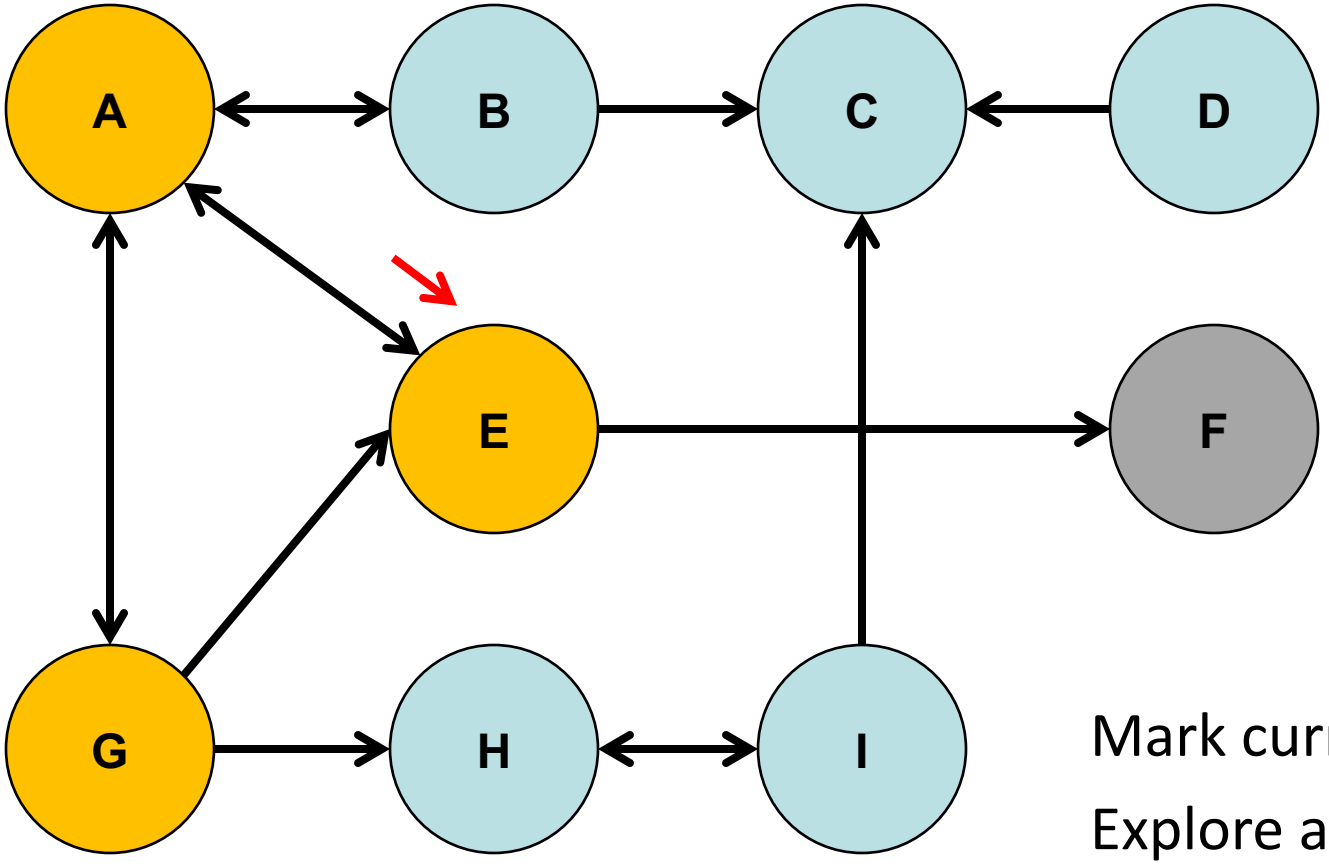

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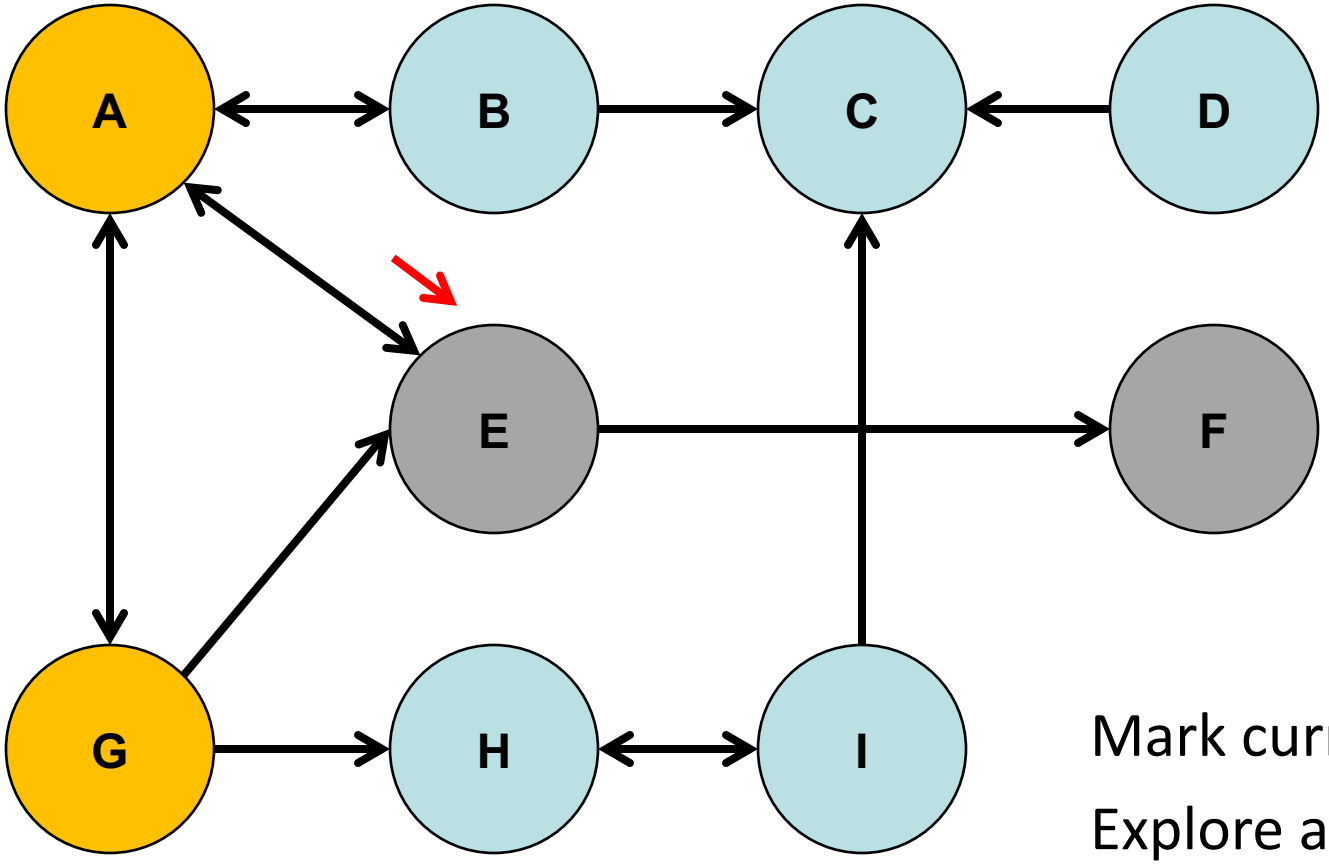

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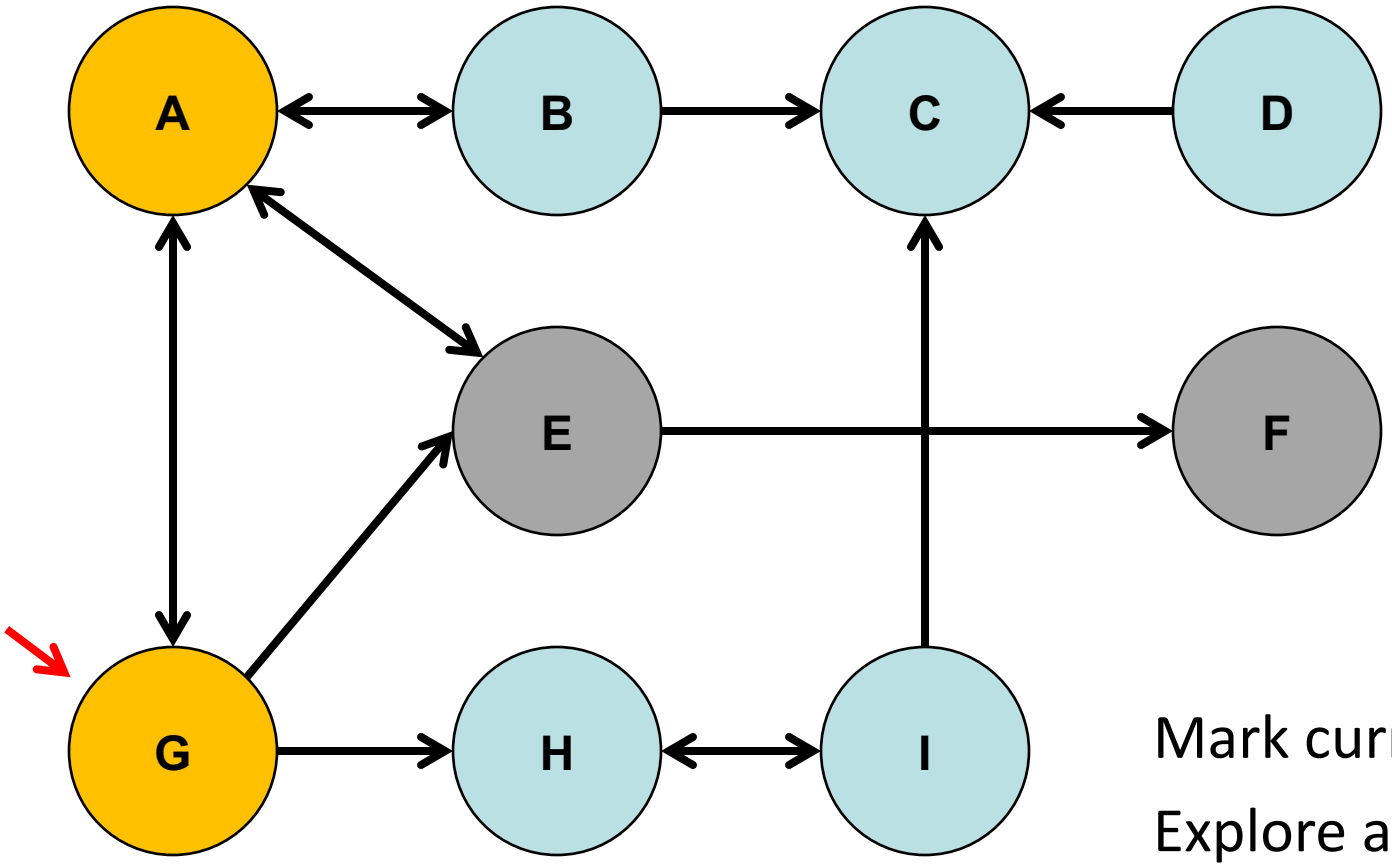

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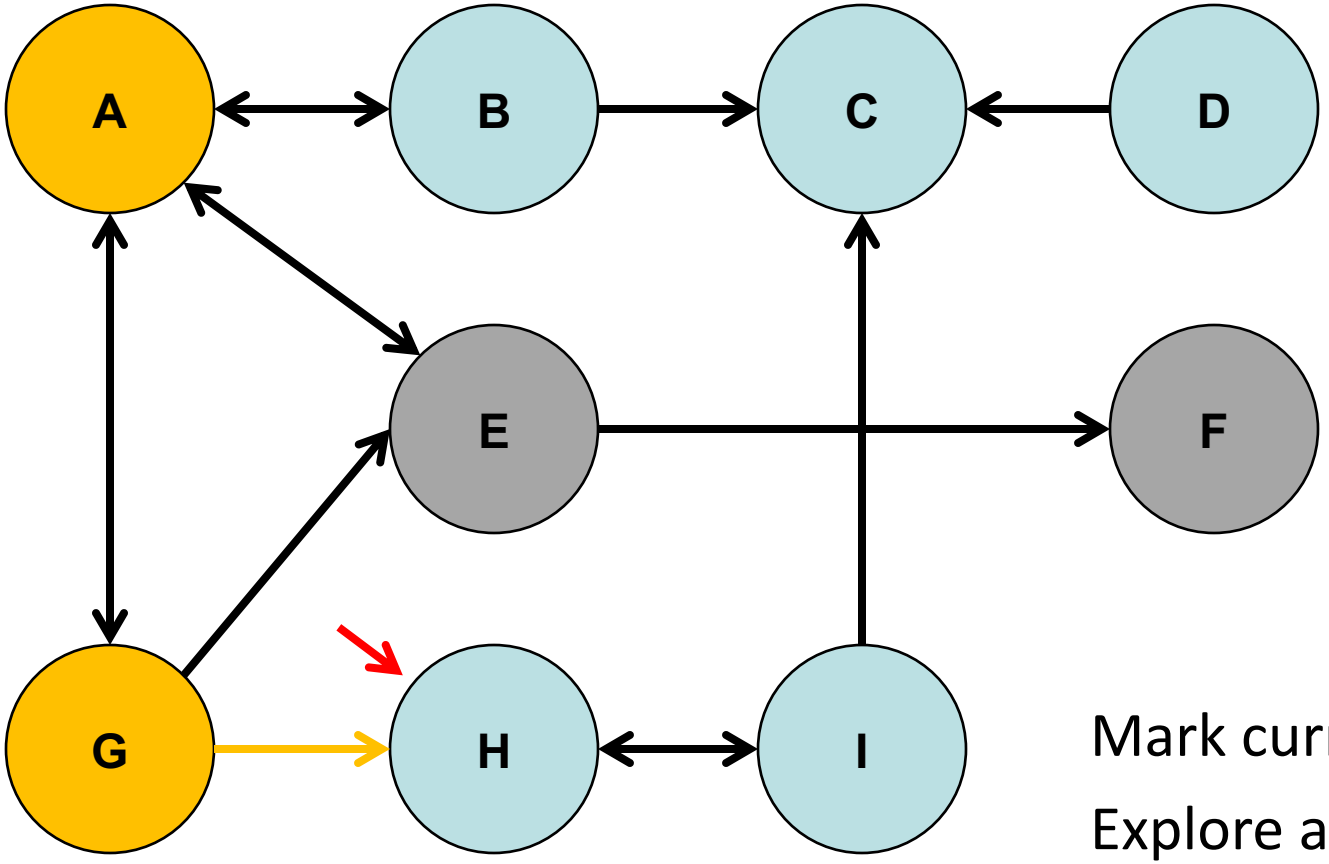

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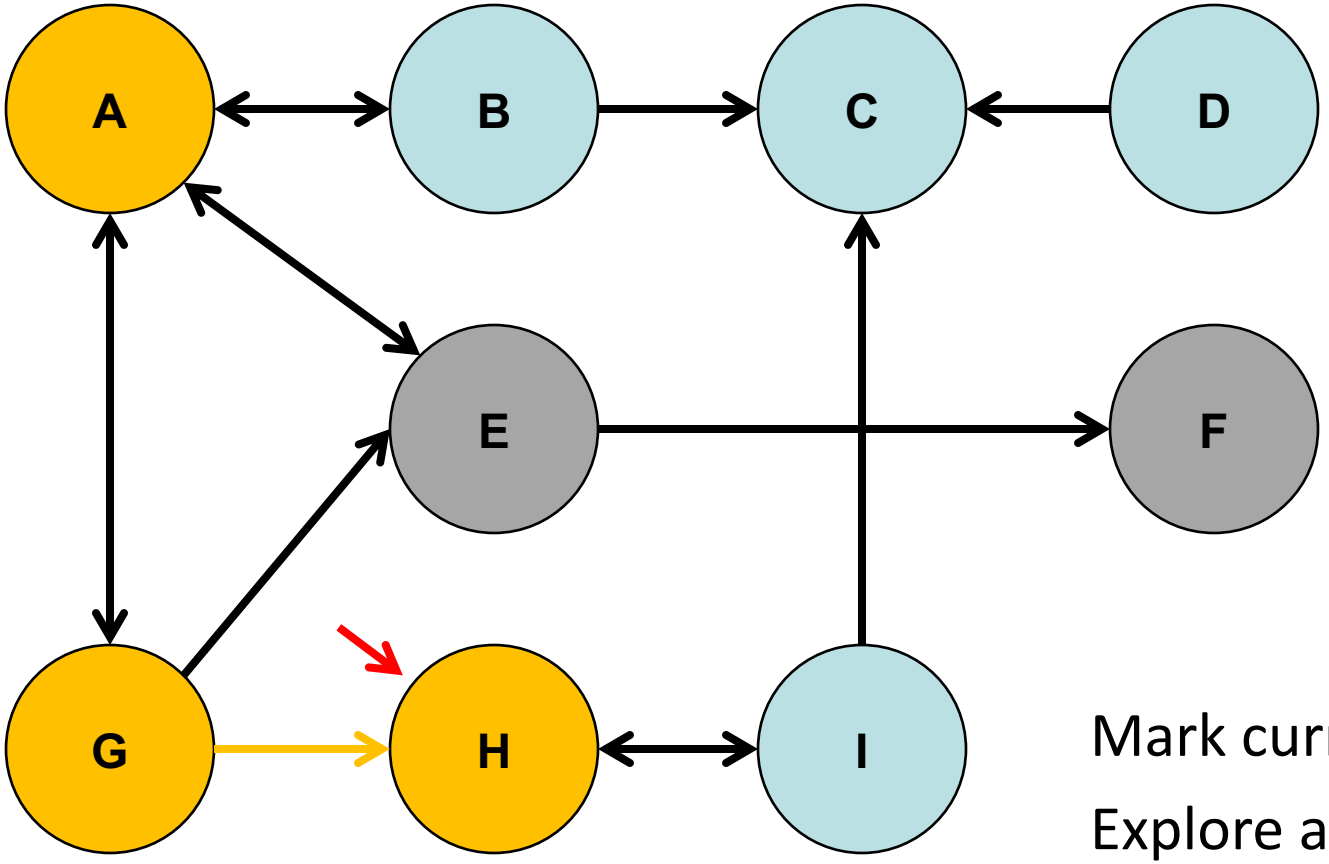

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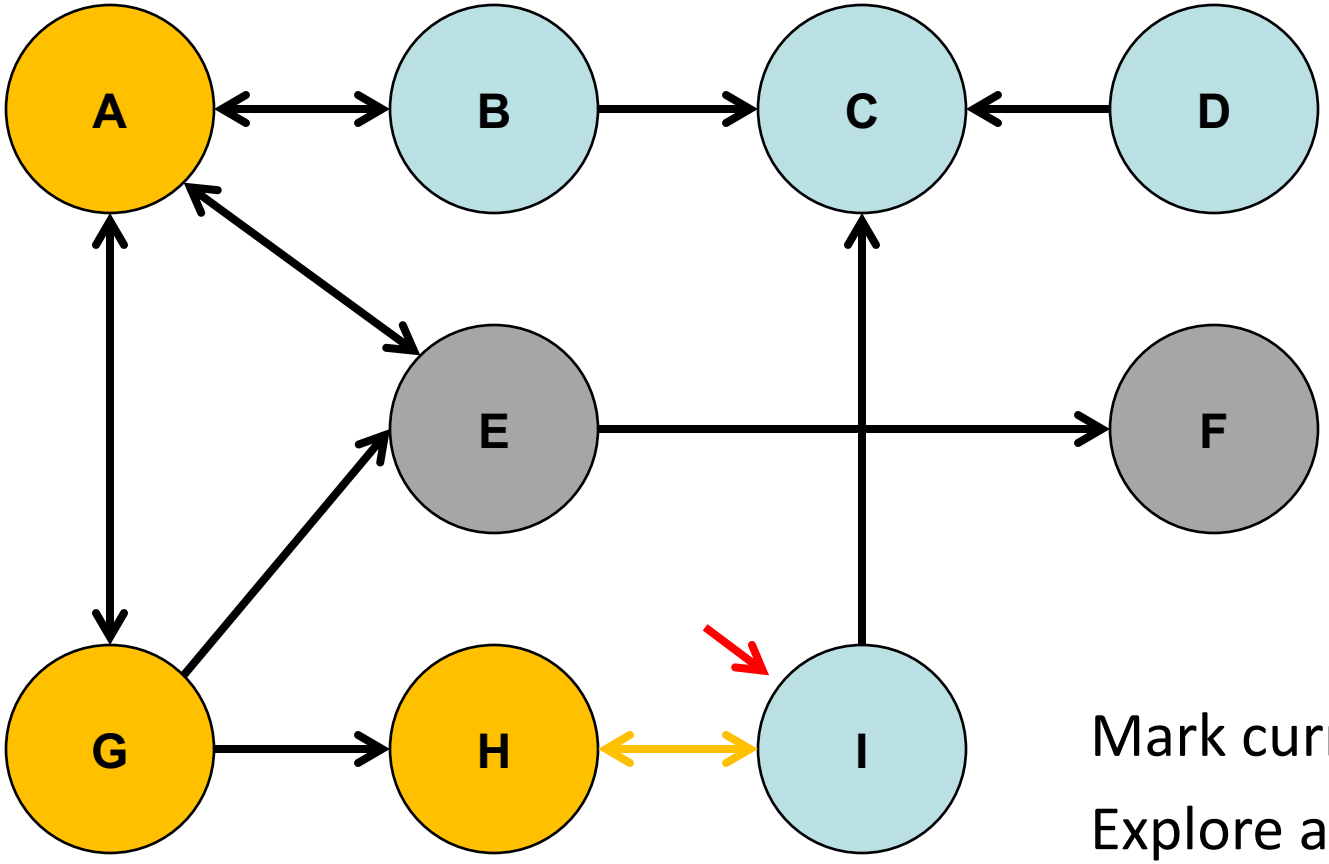

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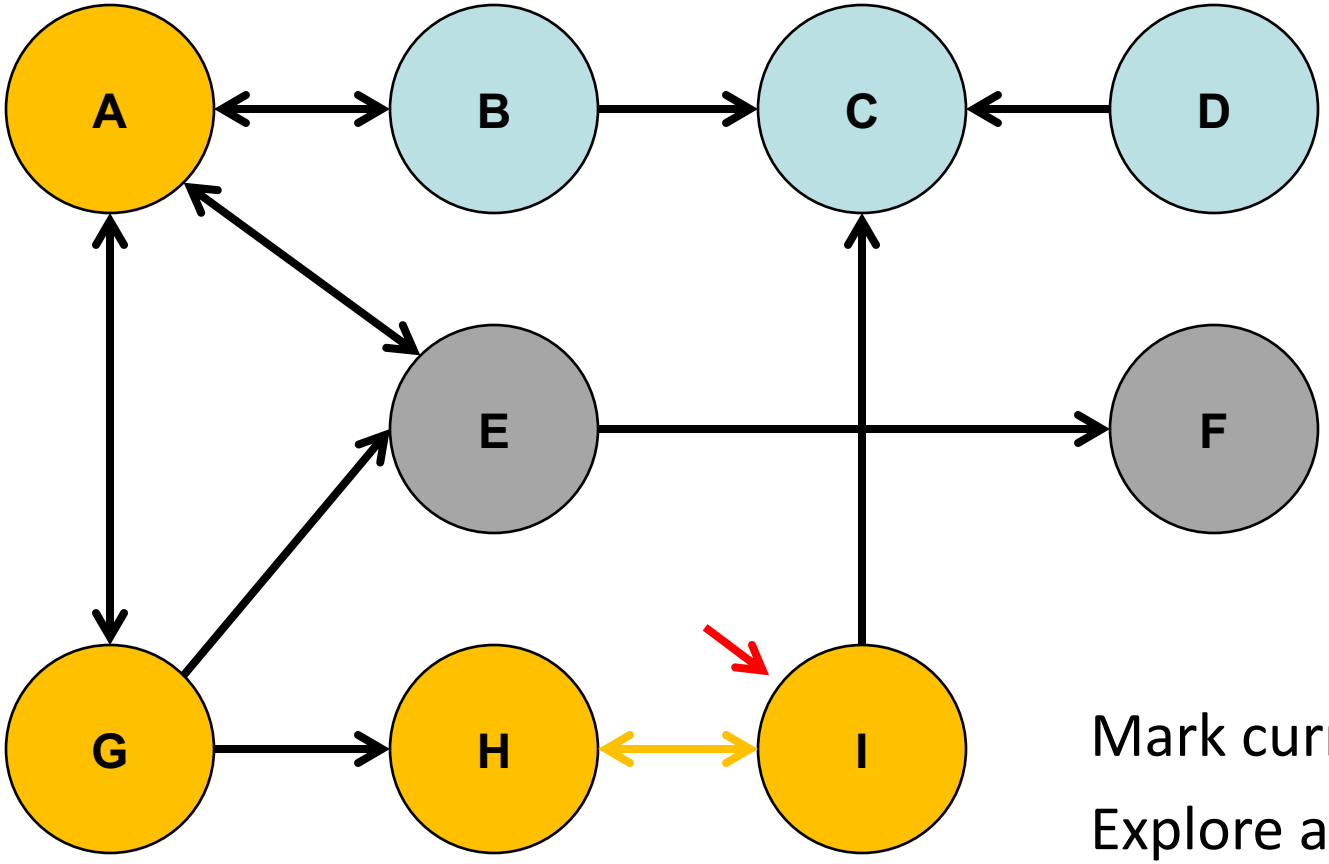

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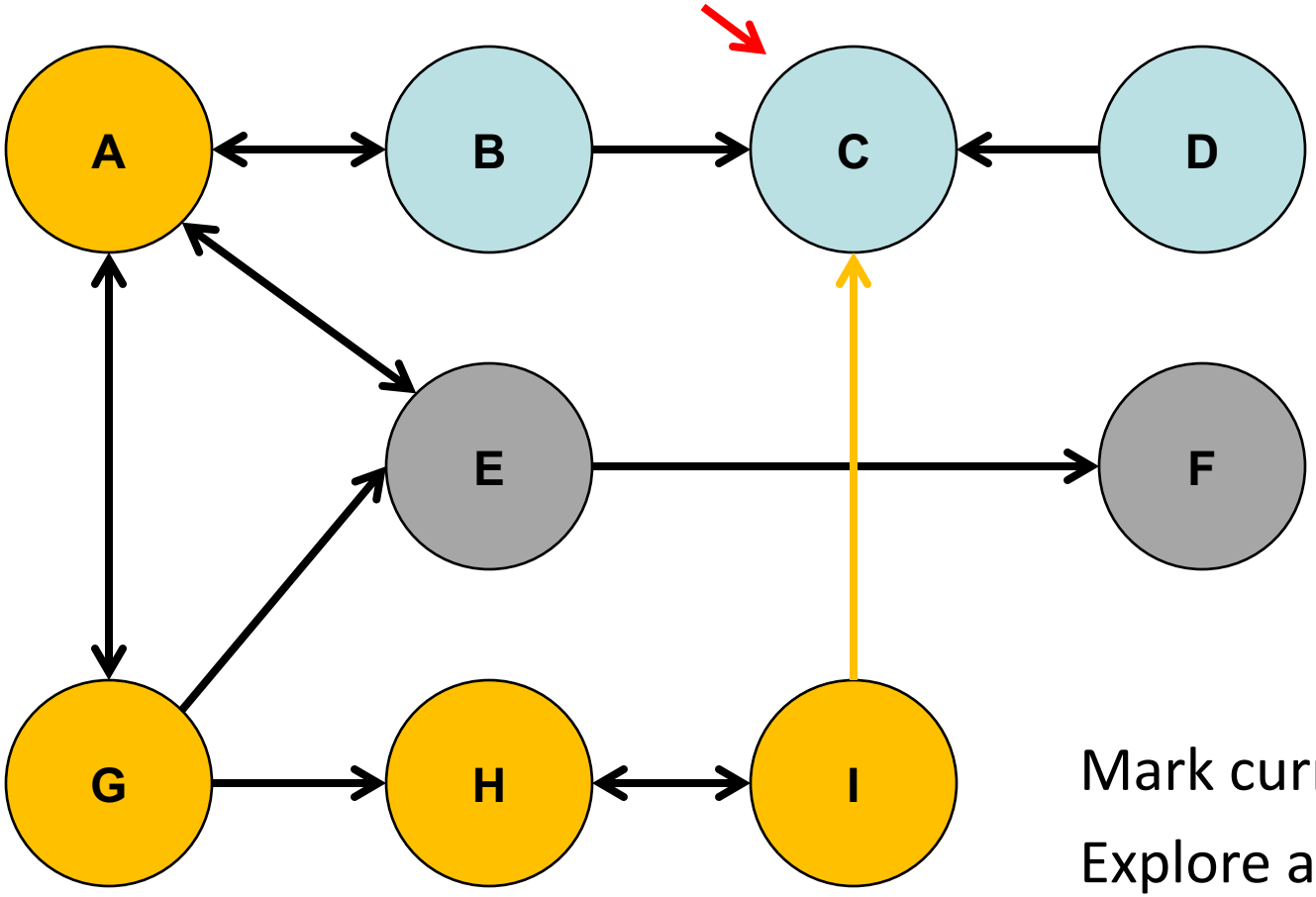

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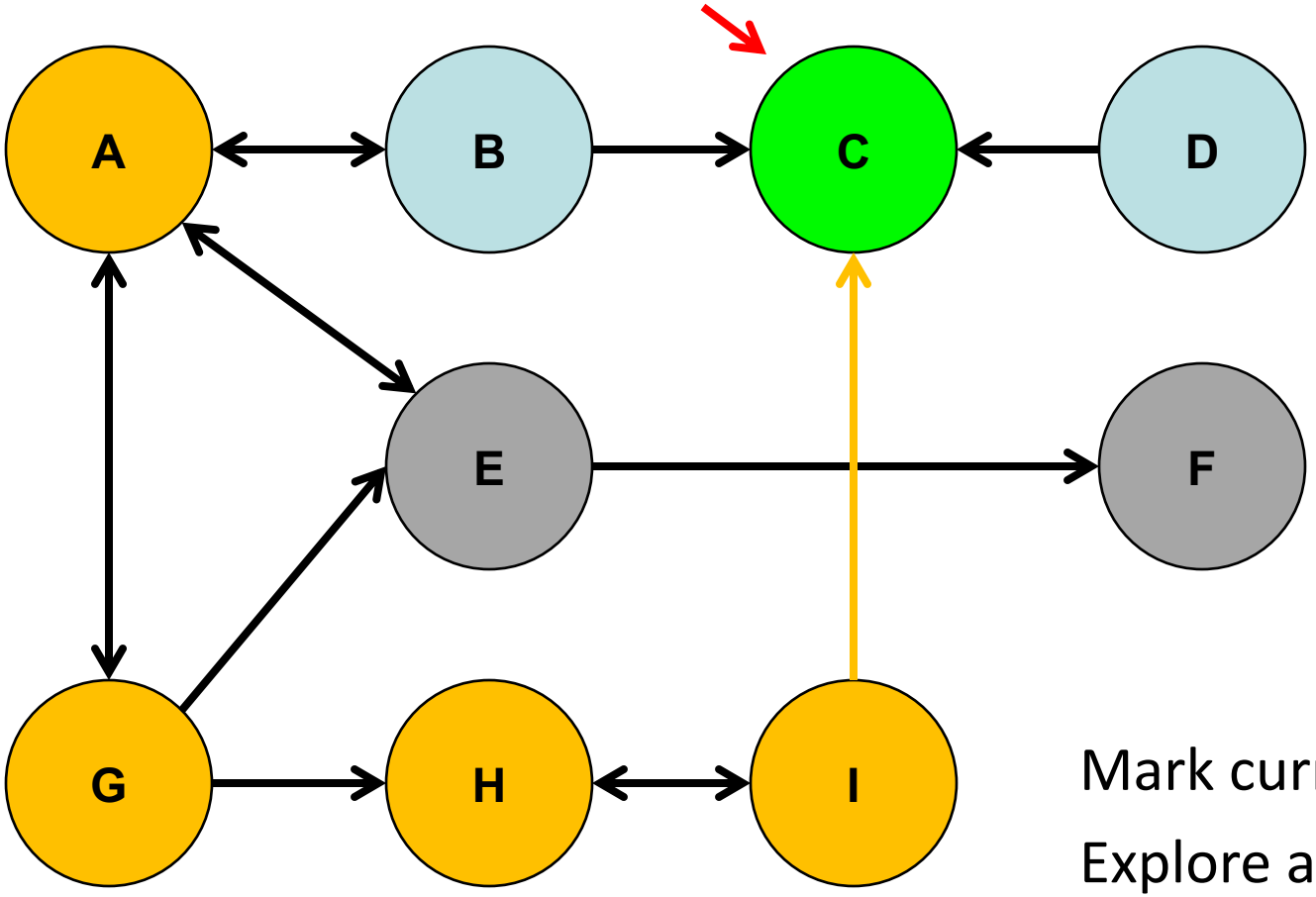

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Mark current as visited

Explore all the unvisited nodes from this node

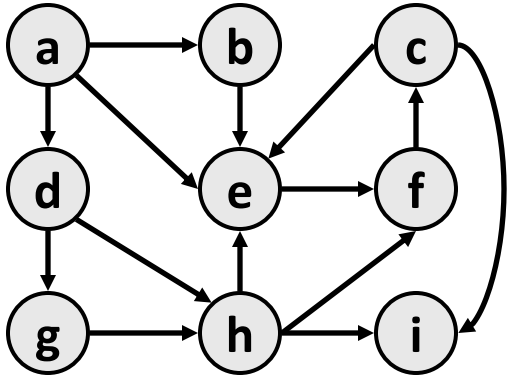
DFS Details

- In an *n*-node, *m*-edge graph, takes O(*m* + *n*) time with an adjacency list
  - Visit each edge once, visit each node at most once
- Pseudocode:

```
dfs from v1:
    mark v1 as seen.
    for each of v1's unvisited neighbors n:
      dfs(n)
```

DFS that finds path

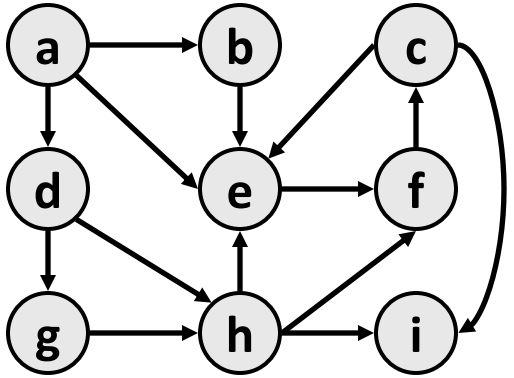
dfs from v_1 to v_2:
    mark v_1 as **visited**, and **add to path**.
    perform a **dfs** from each of v_1's unvisited neighbors n to v_2:
        if **dfs**(n, v_2) succeeds: a path is found! yay!
    if all neighbors fail: **remove v_1 from path**.



- To retrieve the DFS path found, pass a collection parameter to each call and choose-explore-unchoose.

DFS observations

- *discovery*: DFS is guaranteed to find *a* path if one exists.



- *retrieval*: It is easy to retrieve exactly what the path is (the sequence of edges taken) if we find it
  - choose - explore - unchoose

- *optimality*: not optimal. DFS is guaranteed to find a path, not necessarily the best/shortest path
  - Example: dfs(a, i) returns {a, b, e, f, c, i} rather than {a, d, h, i}.

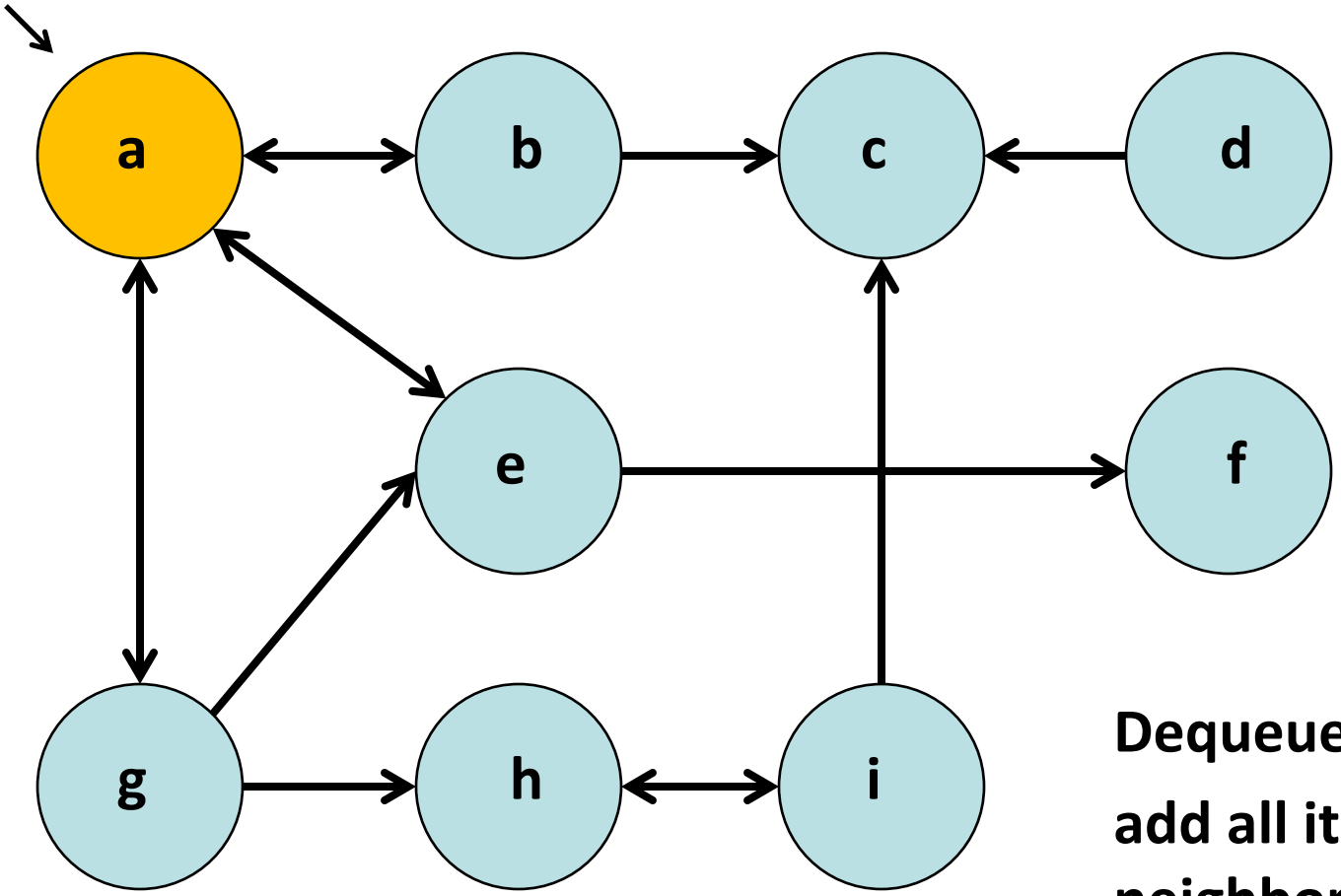
Finding *Shortest* Paths

- We can find paths between two nodes, but how can we find the **shortest** path?
  - Fewest number of steps to complete a task?
  - Least amount of edits between two words?

Breadth-First Search (BFS)

- Idea: processing a node involves knowing we need to visit all its neighbors (just like DFS)
- Need to keep a TODO list of nodes to process
- Keep a Queue of nodes as our TODO list
- Idea: dequeue a node, enqueue all its neighbors
- Still will return the same nodes as reachable, just might have shorter paths

BFS



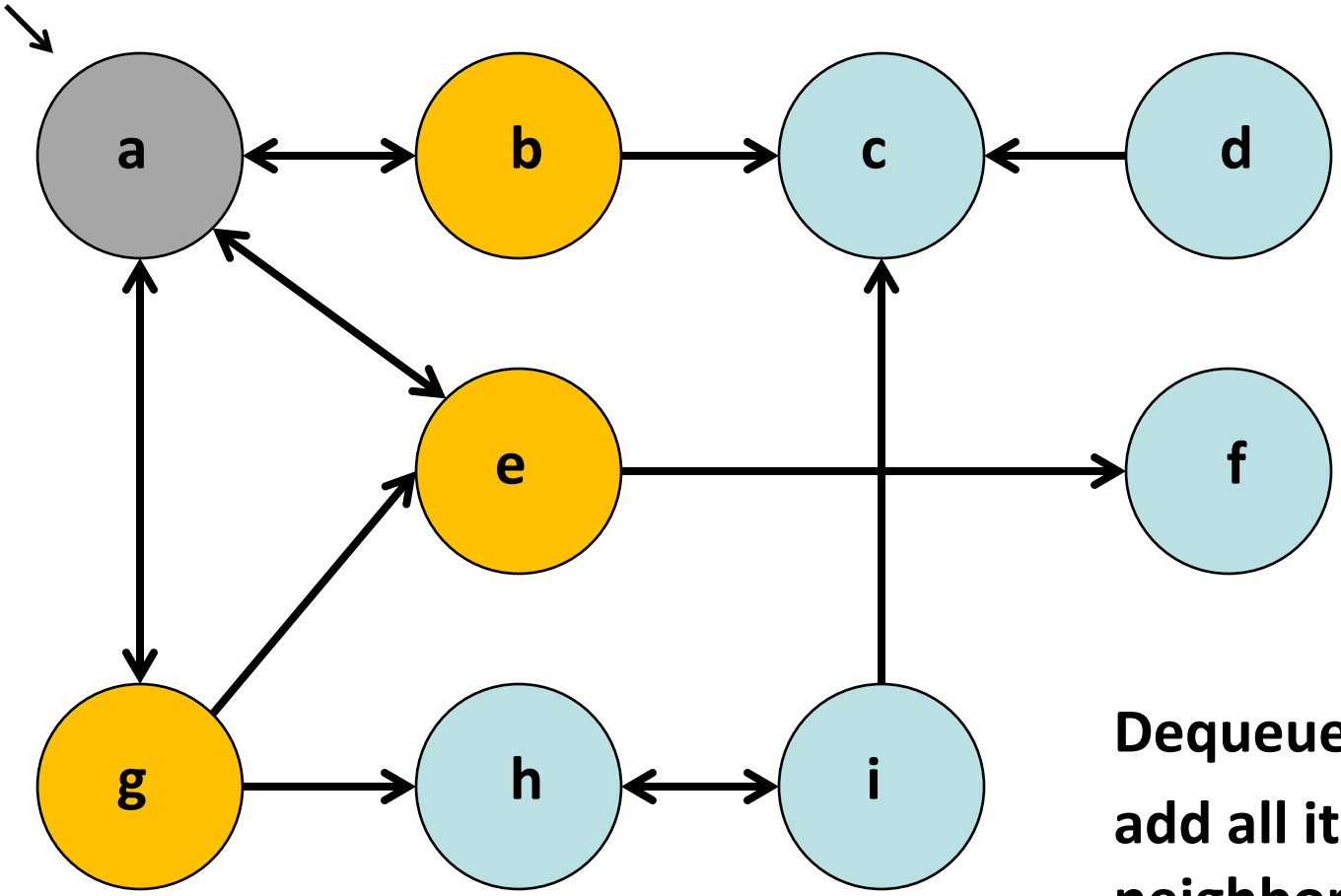
Let's say we are starting at a and searching for c.

Dequeue a node

add all its unseen neighbors to the queue

queue: a

BFS



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queue: e, b, g

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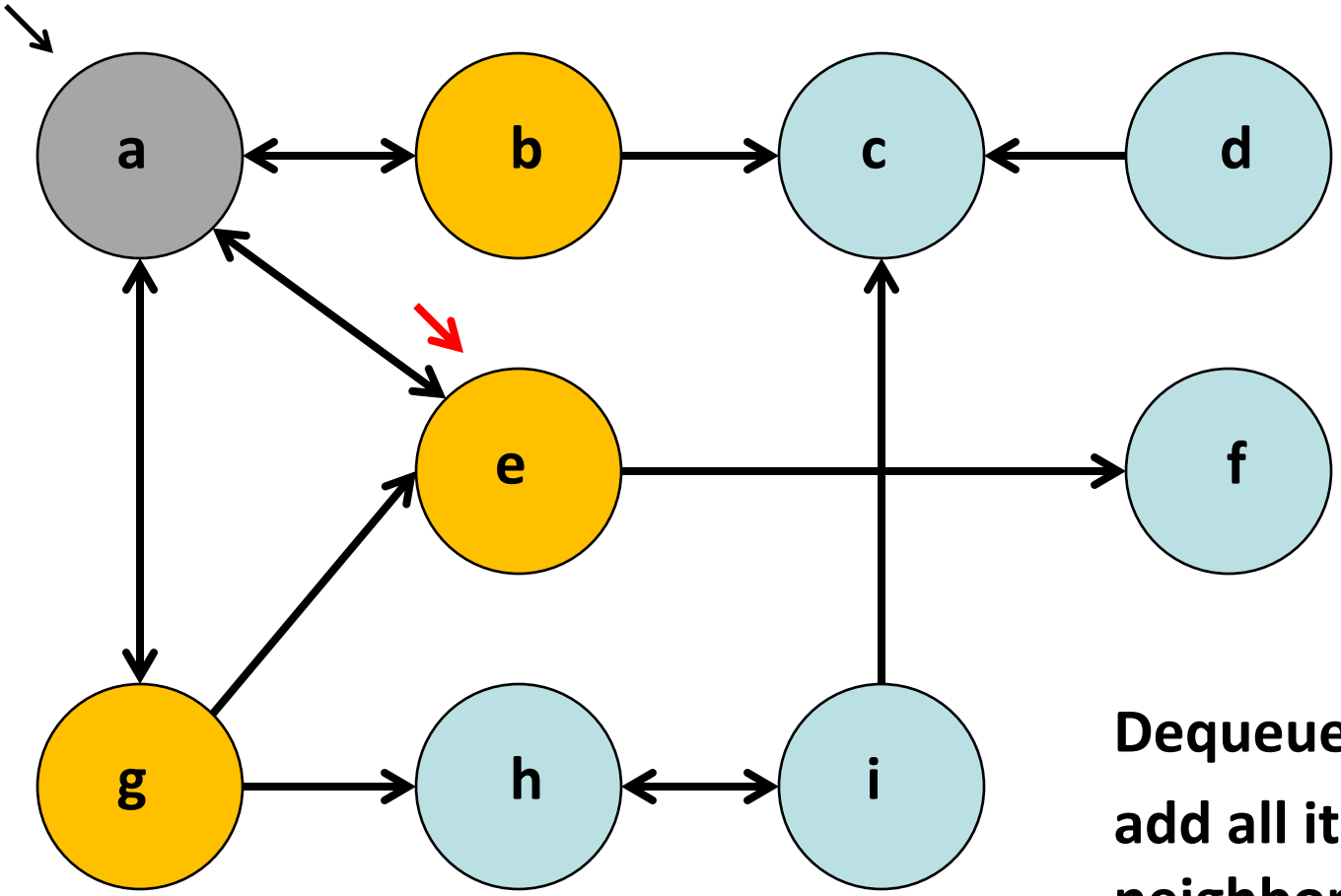

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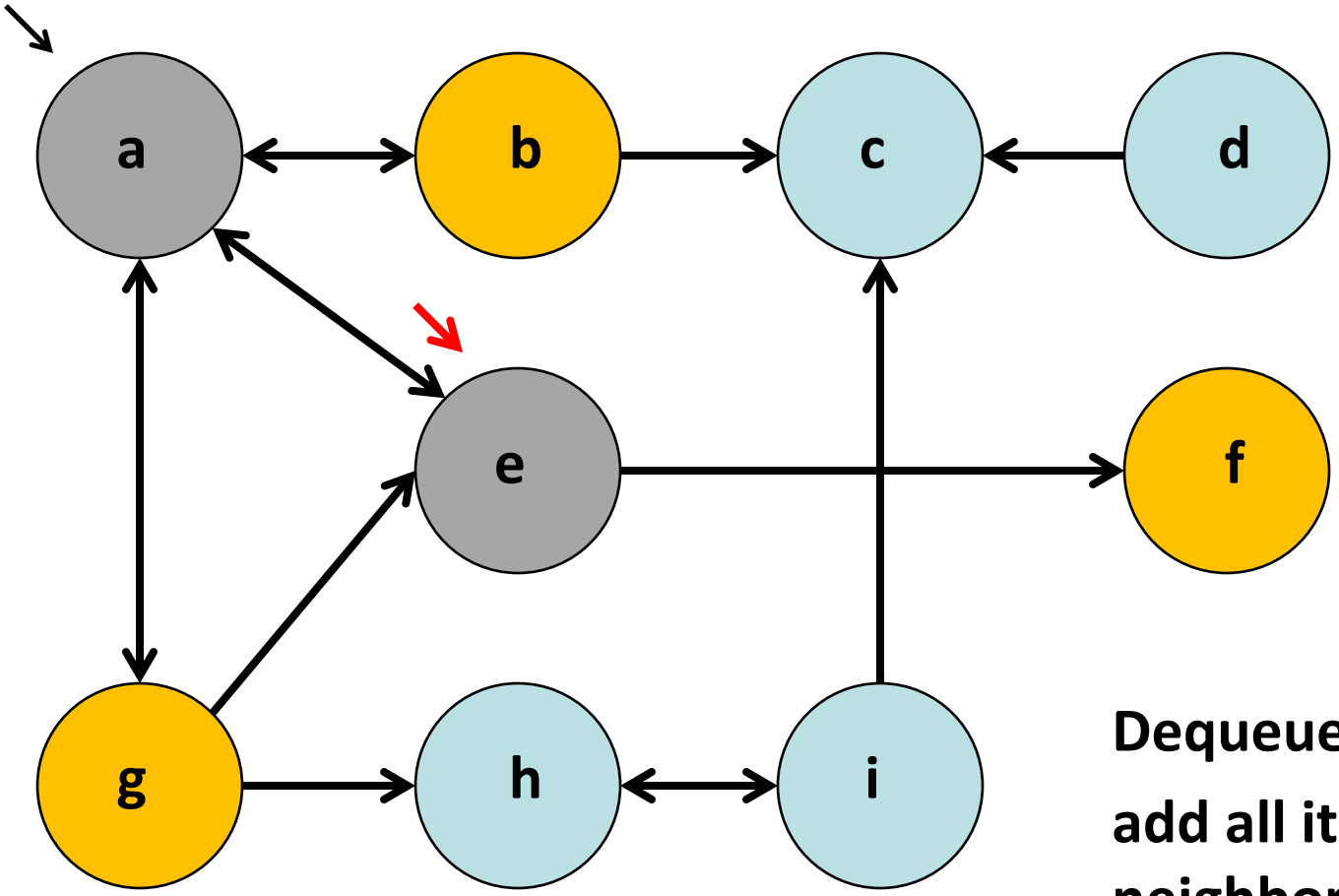

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queue: b, g, f

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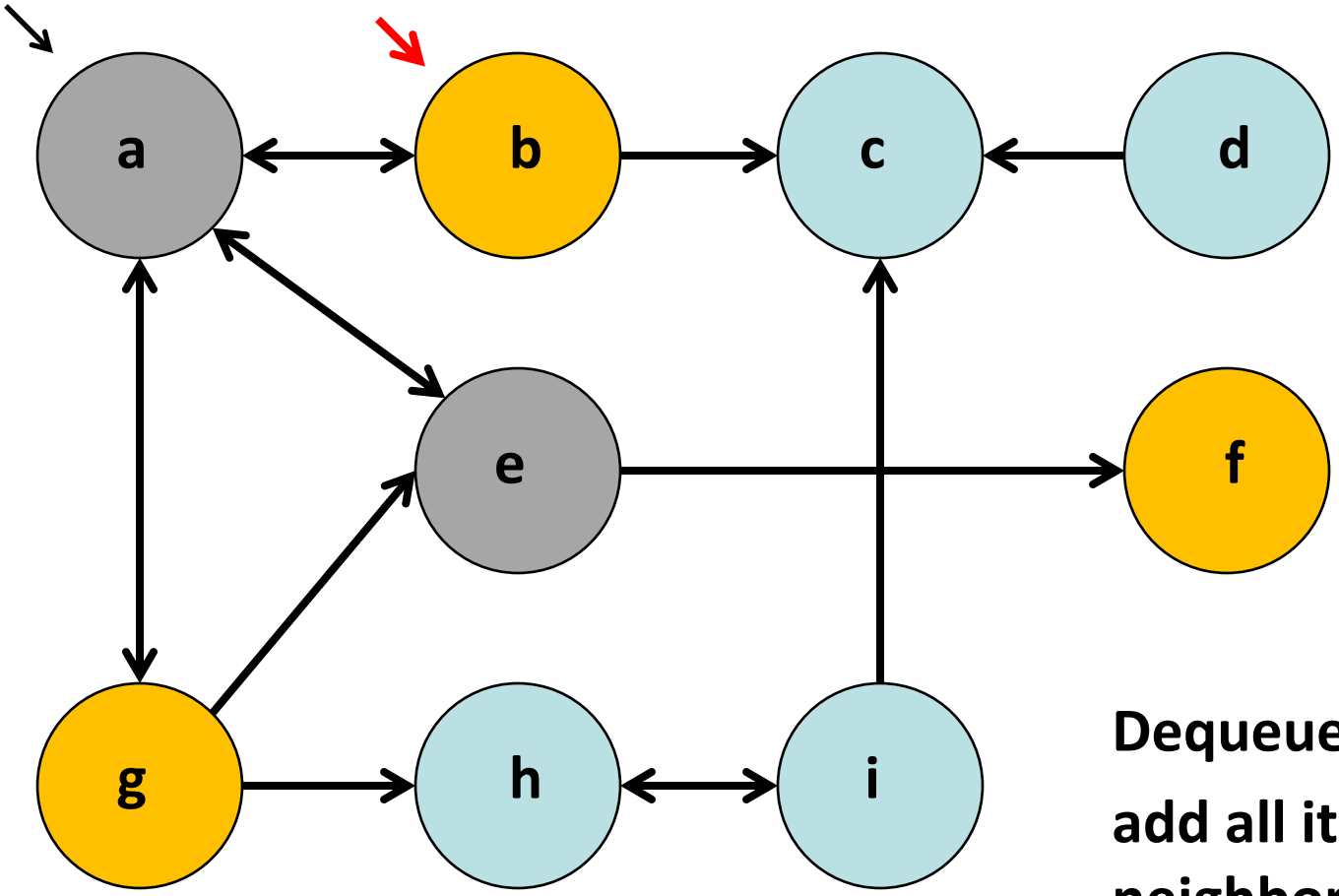

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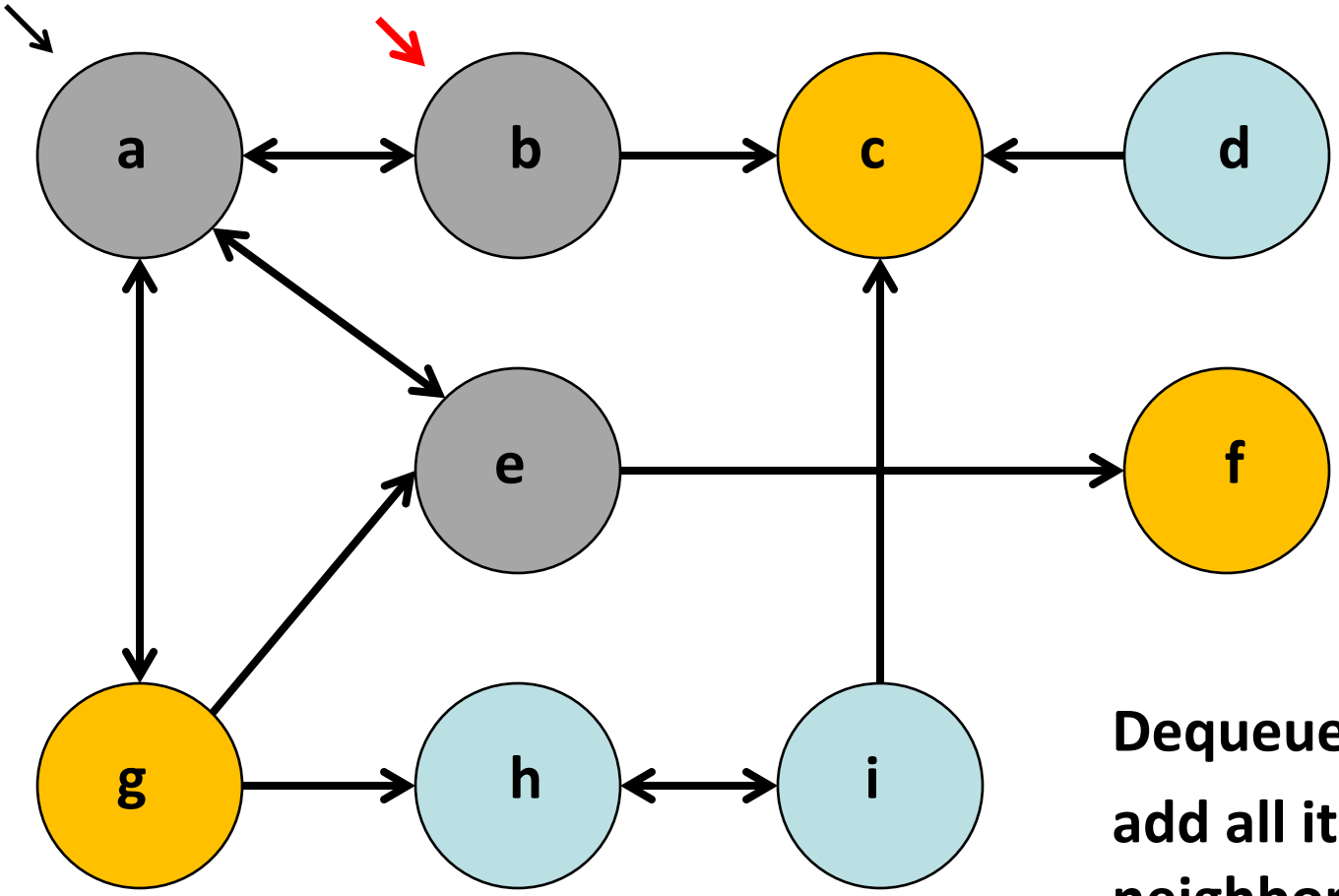

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queue: g, f, c

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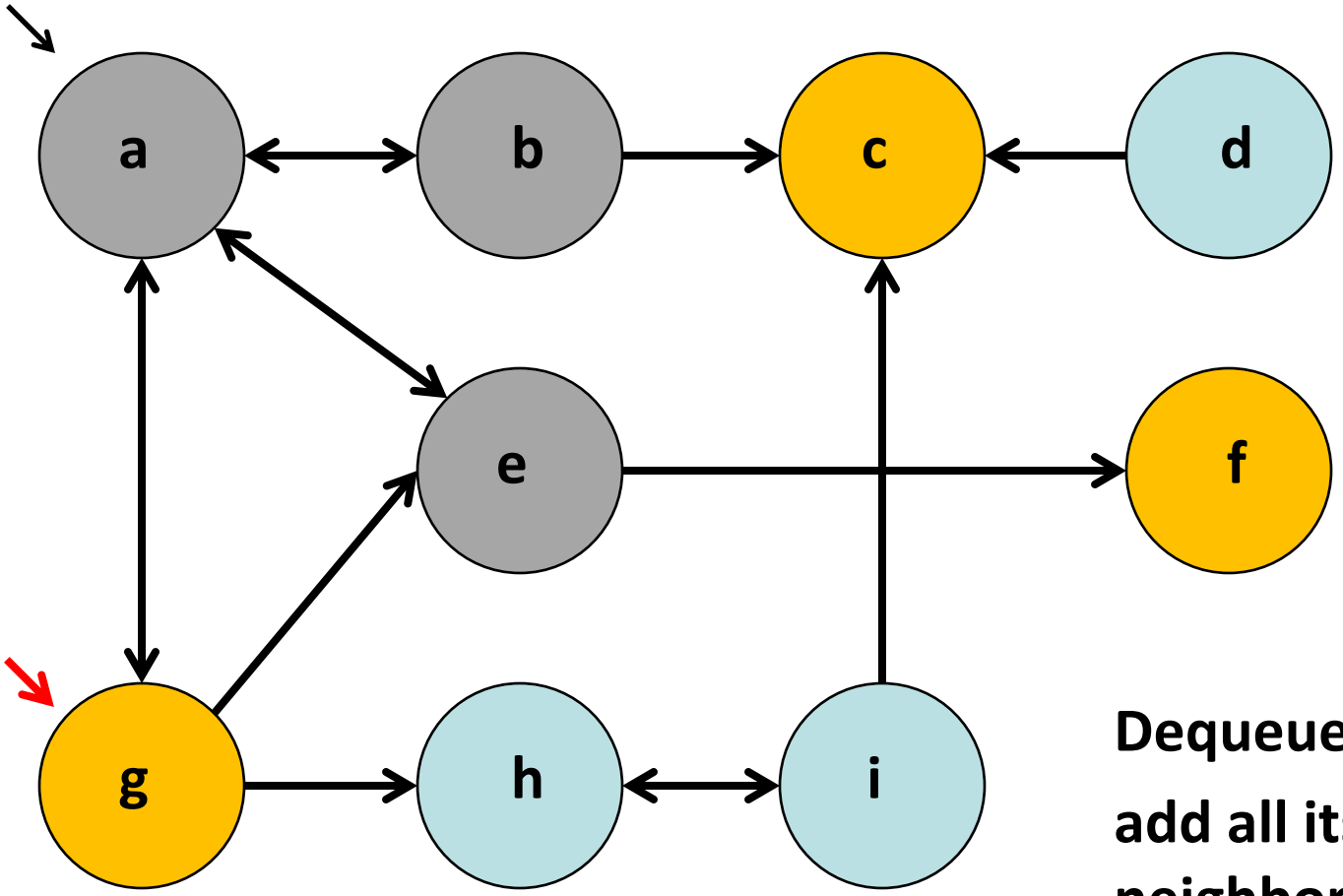

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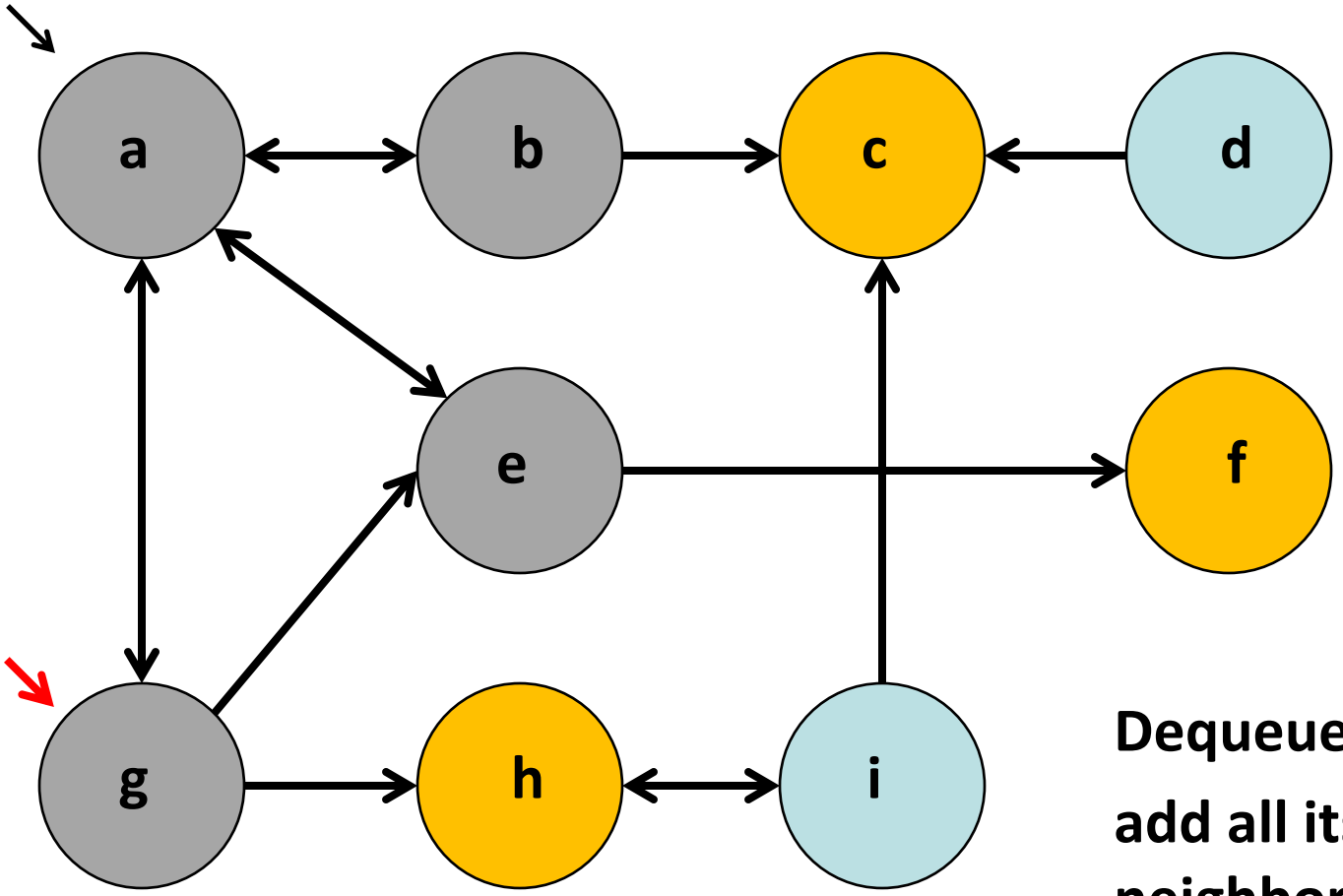

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Dequeue a node

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queue: f, c, h

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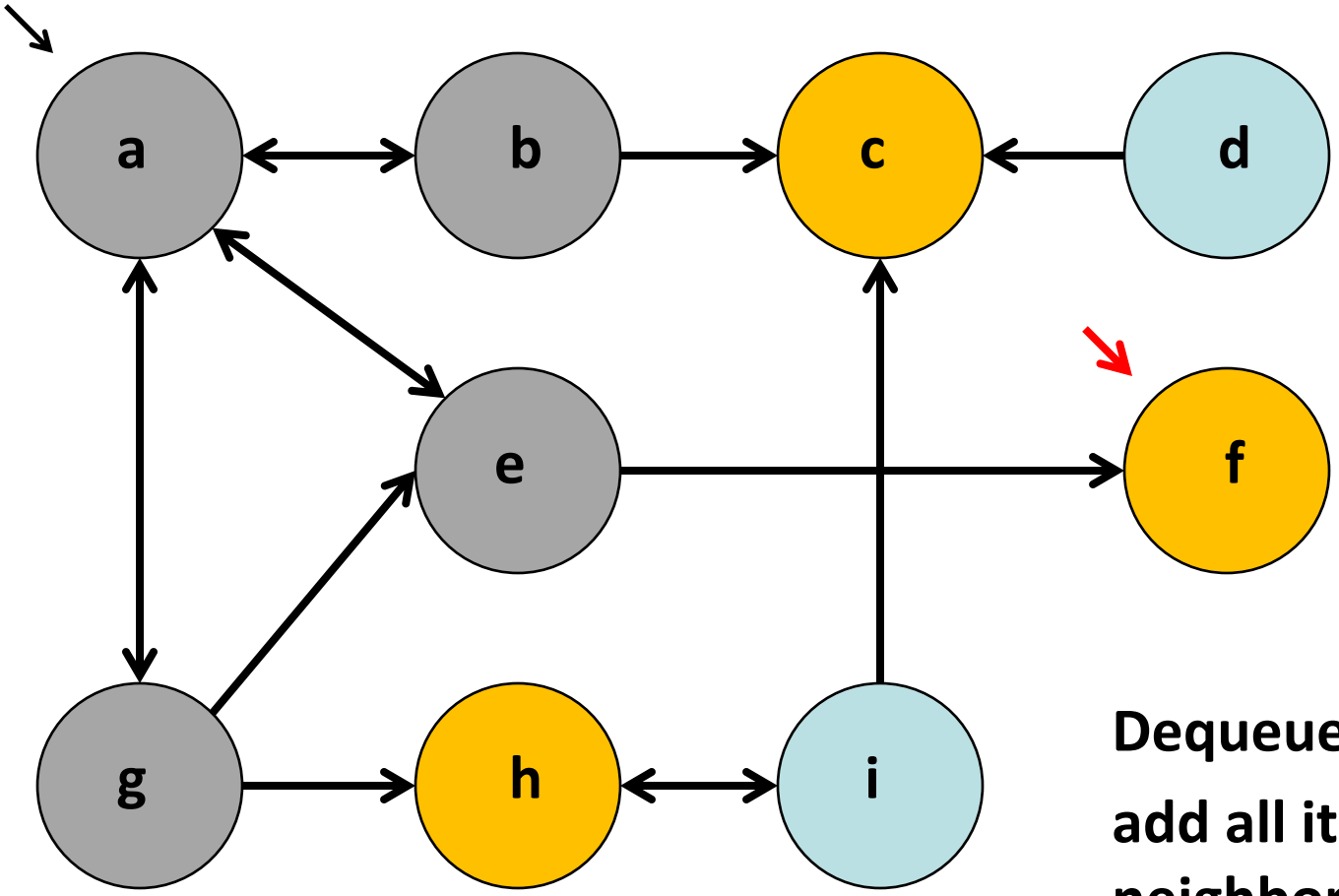

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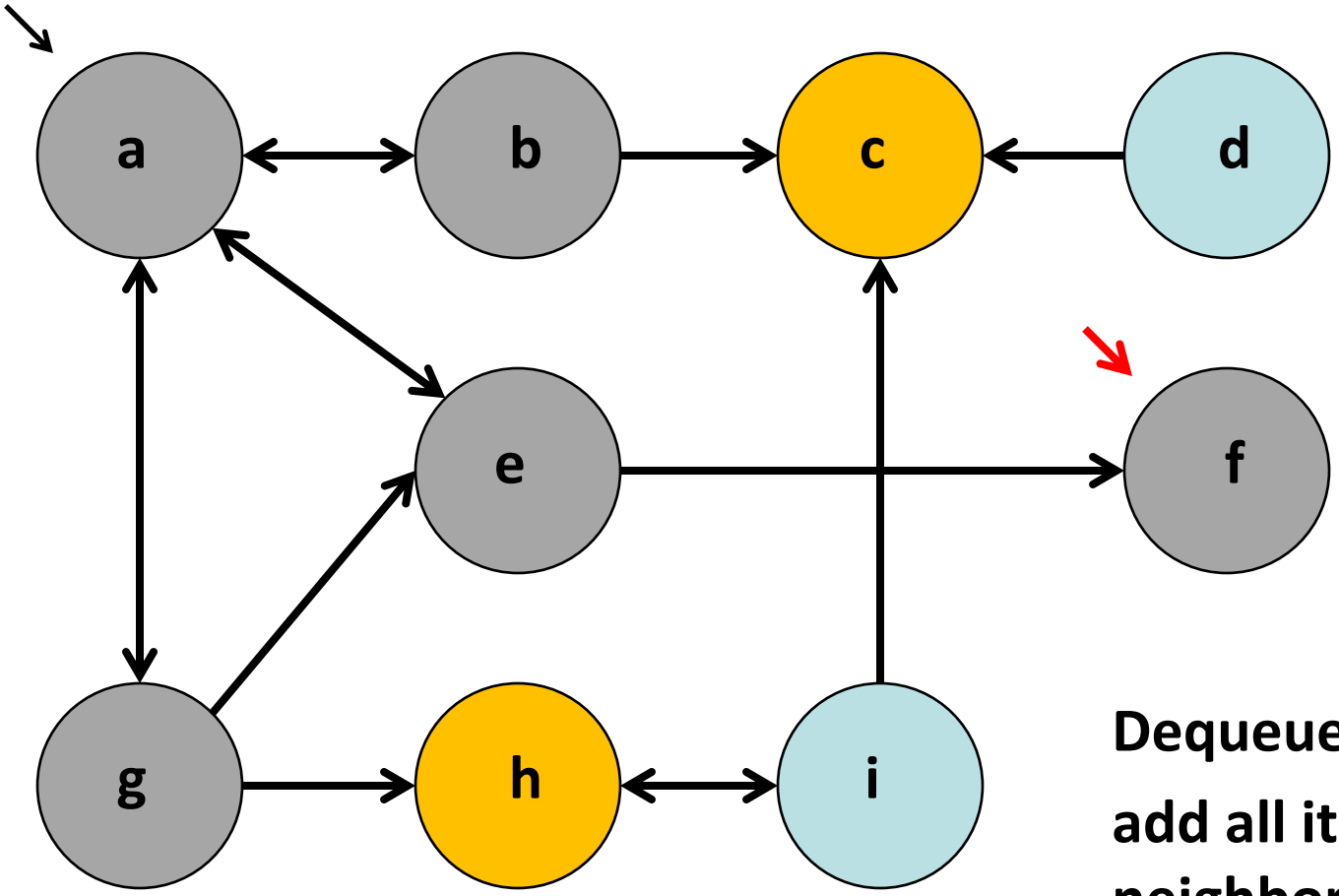

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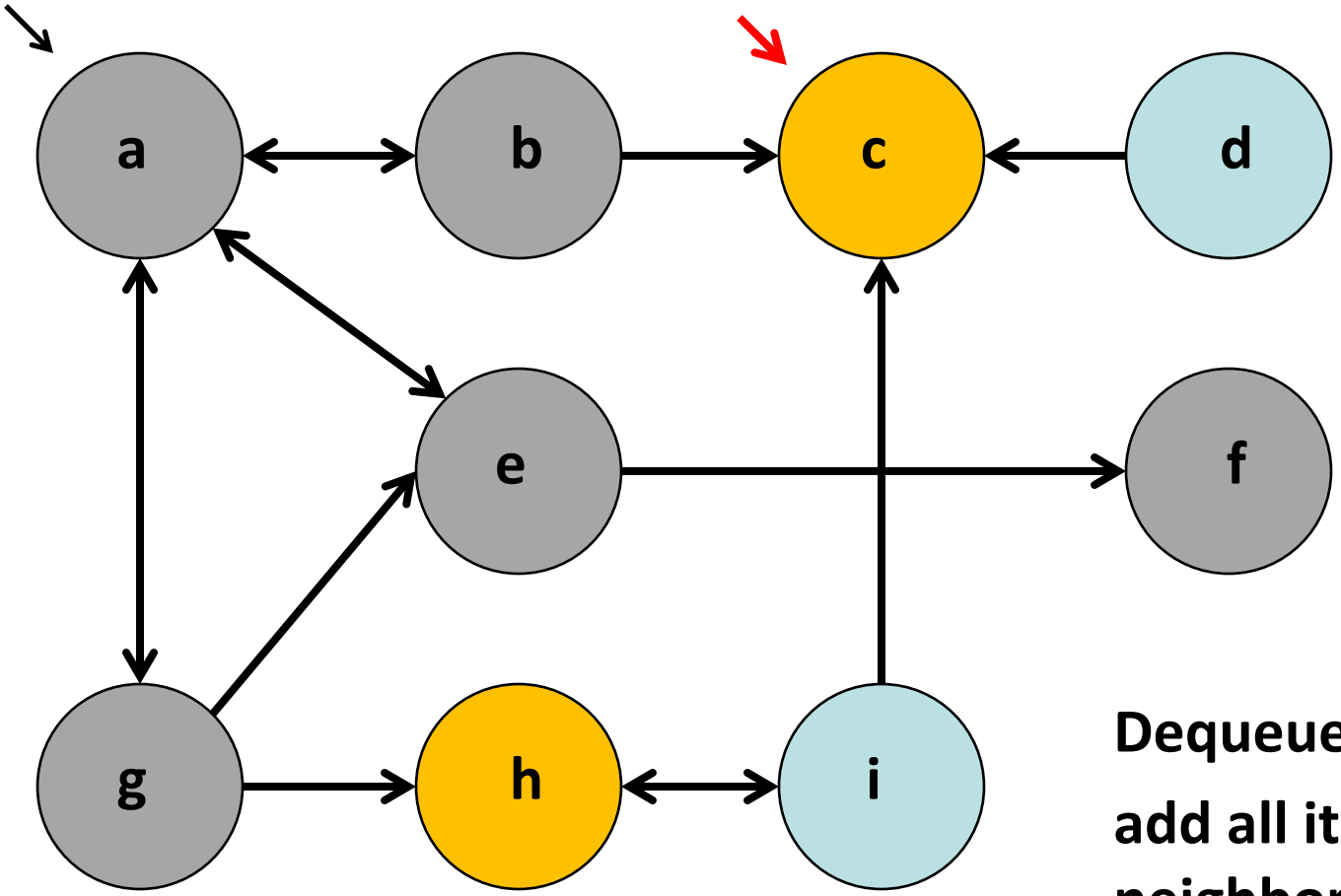

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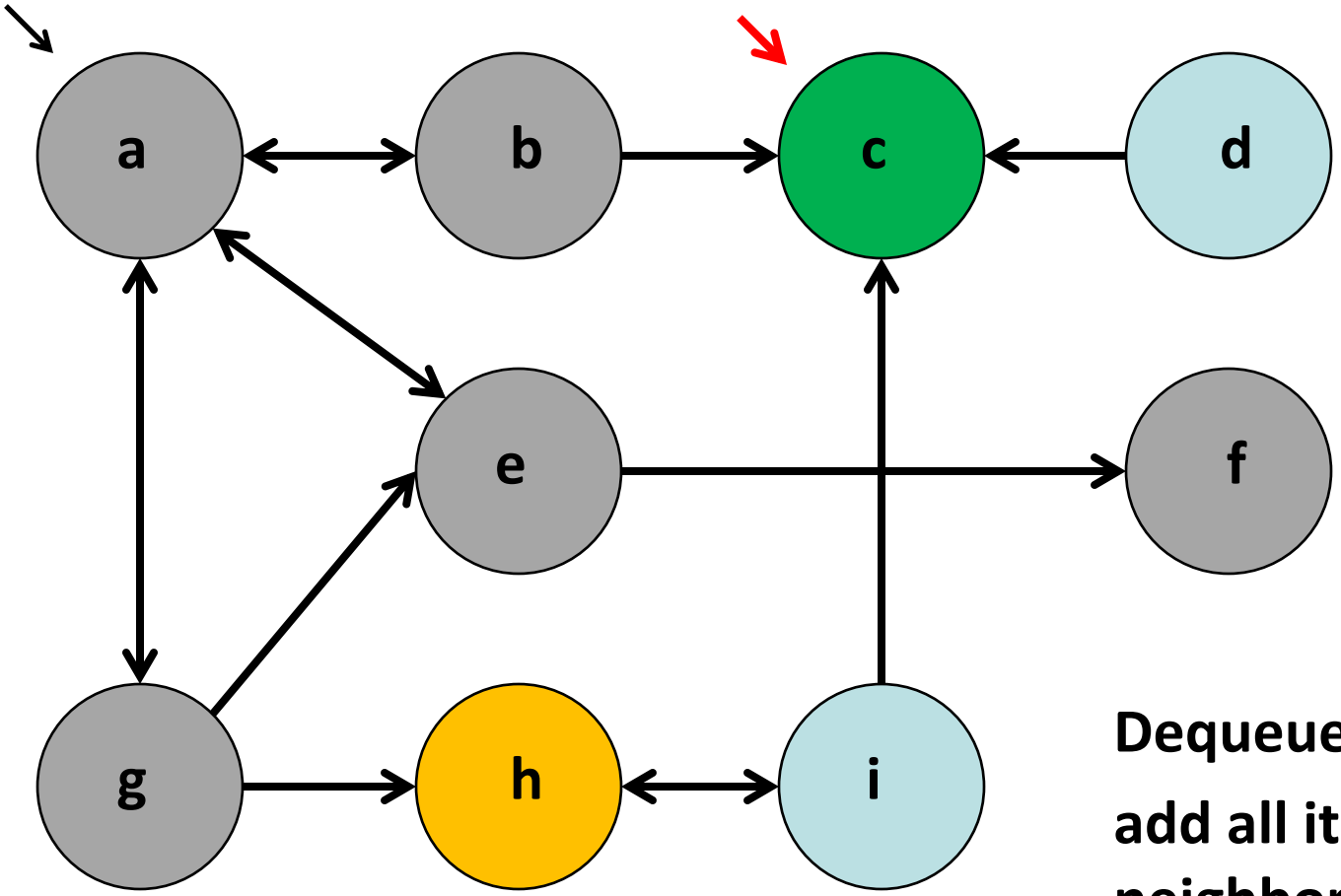

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queue: h

BFS Details

- In an *n*-node, *m*-edge graph, takes O(*m* + *n*) time with an adjacency list
  - Visit each edge once, visit each node at most once

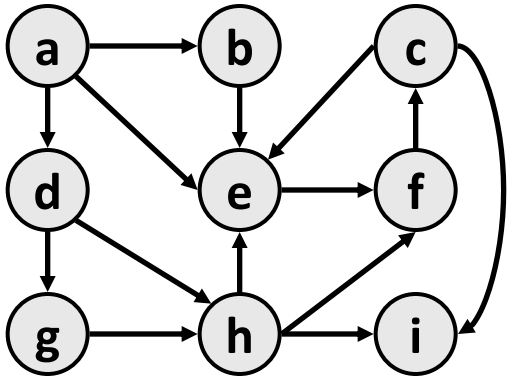
bfs from v_1 to v_2:
create a *queue* of vertexes to visit,
initially storing just v_1.
mark v_1 as **visited**.

while *queue* is not empty and v_2 is not seen:
dequeue a vertex *v* from it,
mark that vertex *v* as **visited**,
and add each unvisited neighbor ***n*** of *v* to the *queue*.

- How could we modify the pseudocode to look for a specific path?

BFS observations

- *optimality*:
  - always finds the shortest path (fewest edges).
  - in unweighted graphs, finds optimal cost path.
  - In weighted graphs, *not* always optimal cost.



- *retrieval*: harder to reconstruct the actual sequence of vertices or edges in the path once you find it
  - conceptually, BFS is exploring many possible paths in parallel, so it's not easy to store a path array/list in progress
  - solution: We can keep track of the path by storing predecessors for each vertex (each vertex can store a reference to a *previous* vertex).

- DFS uses less memory than BFS, easier to reconstruct the path once found; but DFS does not always find shortest path. BFS does.

BFS that finds path

bfs from v_1 to v_2:
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  mark v_1 as **visited**.

  while *queue* is not empty and v_2 is not seen:
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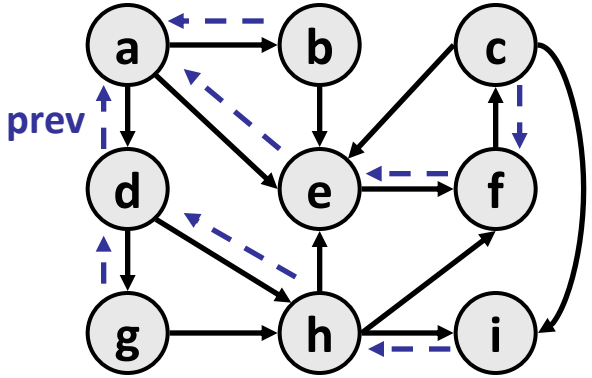

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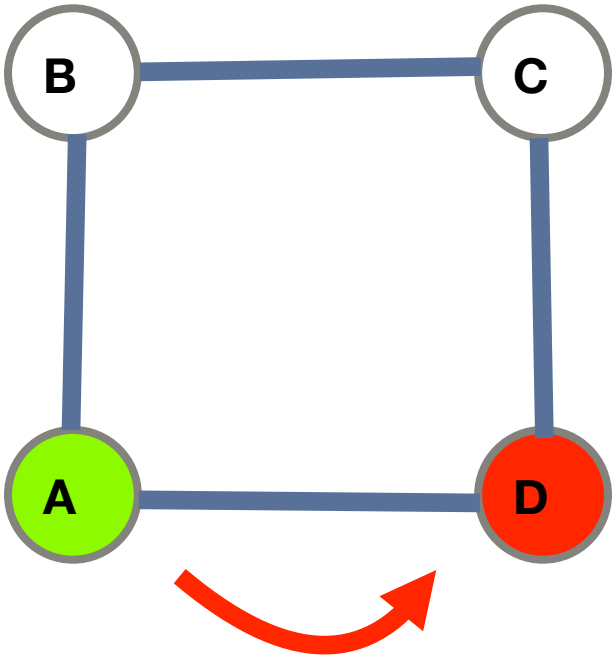
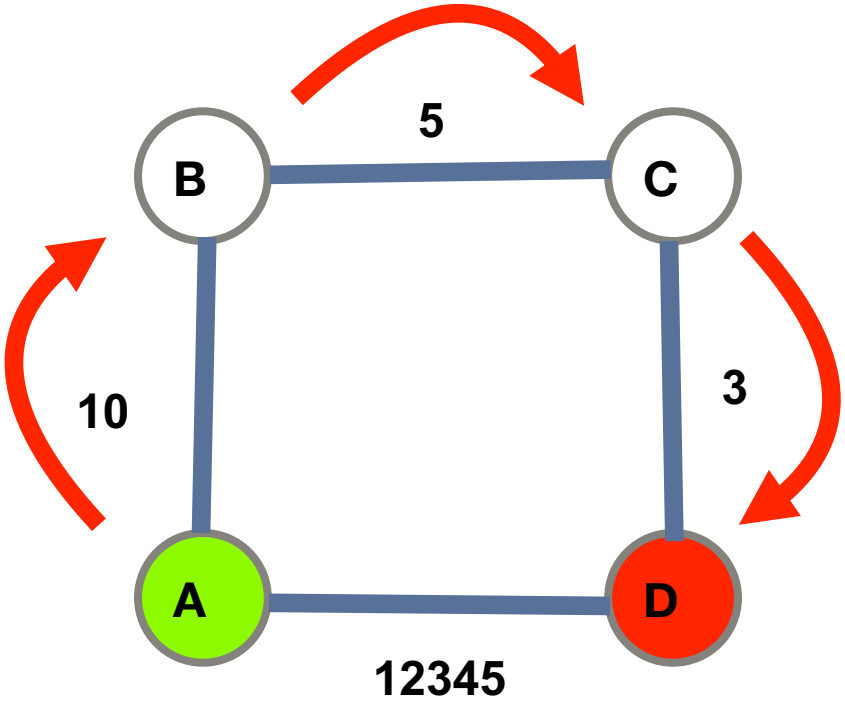
Search

- Search **without** weights: What is the shortest path from A to D?



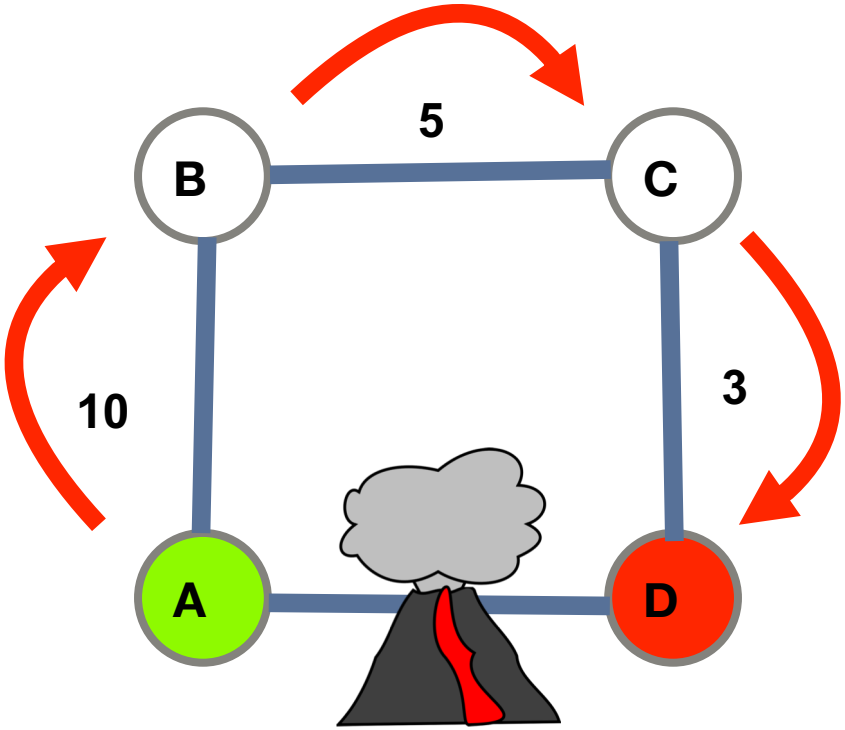
Search

- Search **with** weights: What is the shortest path from A to D?



Search

- Search **with** weights: What is the shortest path from A to D?



Least-Cost Paths

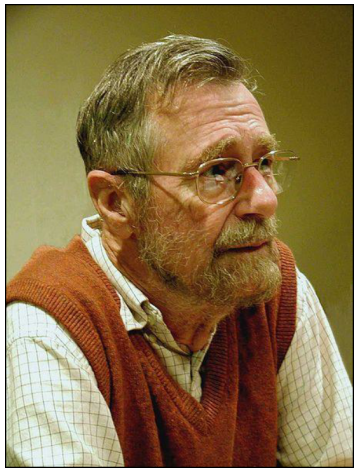
- BFS uses a **queue** to keep track of which nodes to use next
- BFS pseudocode:

```
bfs from v1:
    add v1 to the queue.
    while queue is not empty:
      dequeue a node n
      enqueue n's unseen neighbors
```

- How could we modify this pseudocode to dequeue the **least-cost** nodes instead of the **closest nodes**?
  - Use a **priority queue** instead of a queue

Edsger Dijkstra (1930-2002)

- famous Dutch computer scientist and prof. at UT Austin
  - Turing Award winner (1972)



- Noteworthy algorithms and software:
  - THE multiprogramming system (OS)
    - layers of abstraction
  - Compiler for a language that can do recursion
  - Dijkstra's algorithm
  - Dining Philosophers Problem: resource contention, deadlock
    - semaphores

- famous papers:
  - "Go To considered harmful"
  - "On the cruelty of really teaching computer science"

Dijkstra's Algorithm (18.6)

- **Dijkstra's algorithm**: Finds the minimum-weight path between a pair of vertices in a weighted directed graph.
  - Solves the "one vertex, shortest path" problem in weighted graphs.
  - *basic algorithm concept*: Create a table of information about the currently known best way to reach each vertex (cost, previous vertex), and improve it until it reaches the best solution.

- *Example:* In a graph where vertices are cities and weighted edges are roads between cities, Dijkstra's algorithm can be used to find the shortest route from one city to any other.

Dijkstra pseudocode

dijkstra(v_1, v_2):
consider every vertex to have a cost of infinity, except v_1 which has a cost of 0.
create a *priority queue* of vertexes, ordered by cost, storing only v_1 .

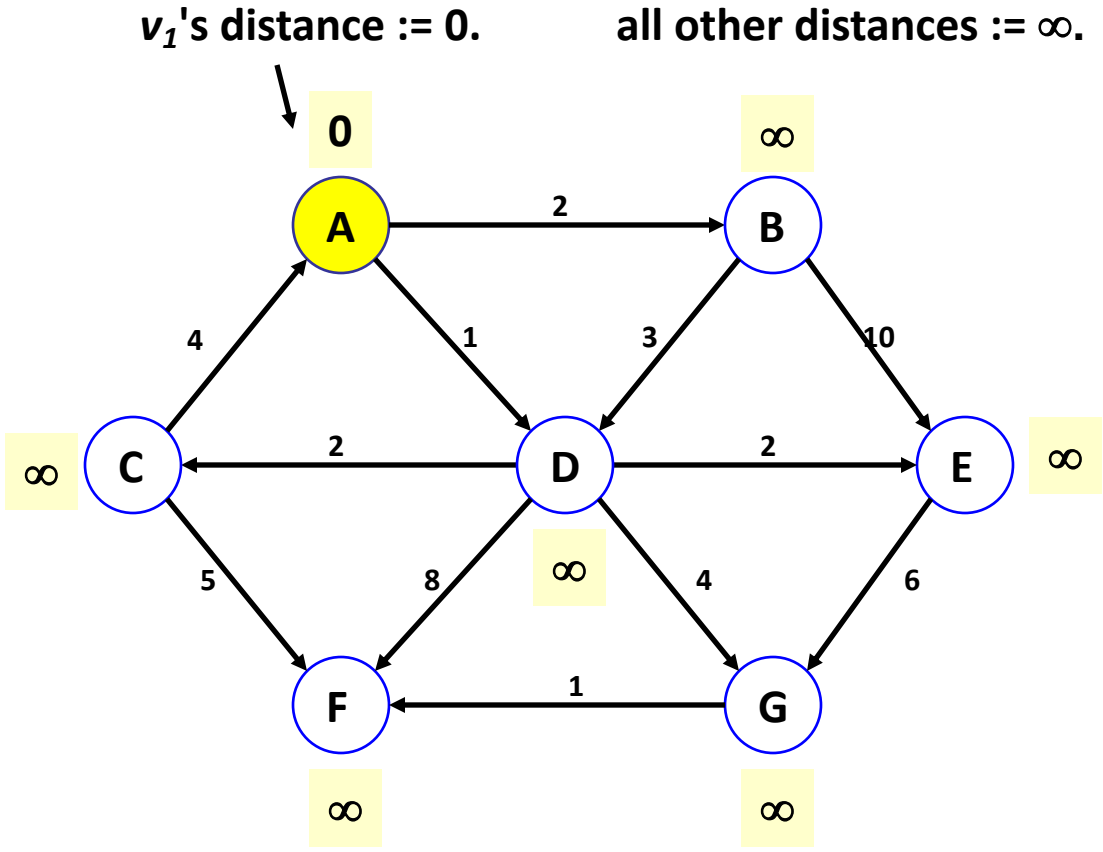
while the *pqueue* is not empty:
dequeue a vertex *v* from the *pqueue*, and mark it as **visited**.
for each of the unvisited neighbors *n* of *v*, we now know that we can reach this neighbor with a total **cost** of (*v*'s cost + the weight of the edge from *v* to *n*).
if the neighbor is not in the *pqueue*, or this is cheaper than *n*'s current cost, we should **enqueue** the neighbor *n* to the pqueue with this new cost, and with *v* as its previous vertex.

when we are done, we can **reconstruct the path** from v_2 back to v_1
by following the previous pointers.

Dijkstra example

dijkstra(A, F);

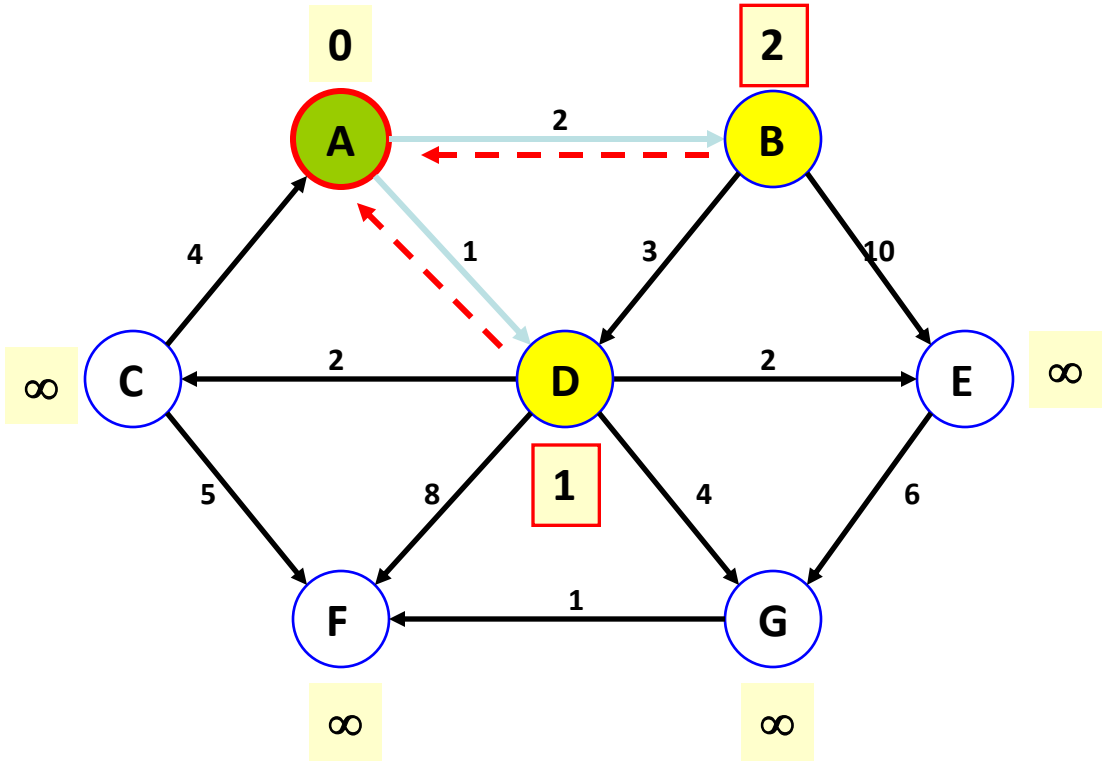
- color key
  - **white**: unexamined
  - **yellow**: enqueued
  - **green**: visited



pqueue = {A:0}

Dijkstra example

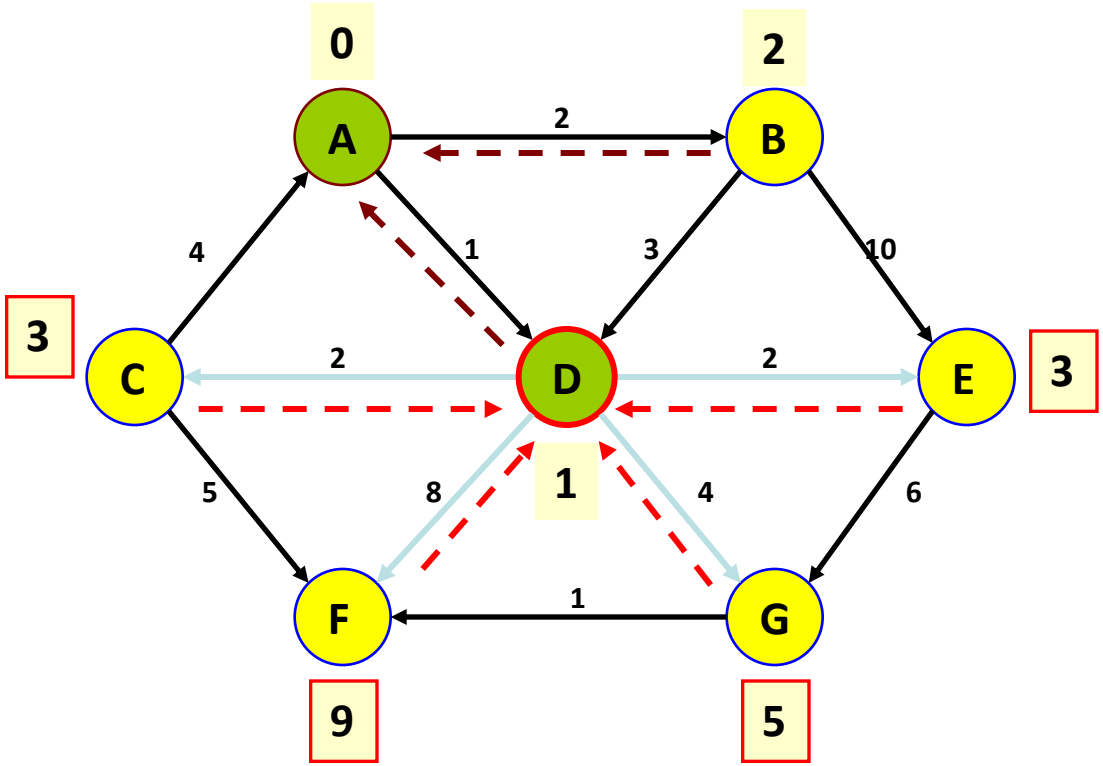
dijkstra(A, F);



pqueue = {D:1, B:2}

Dijkstra example

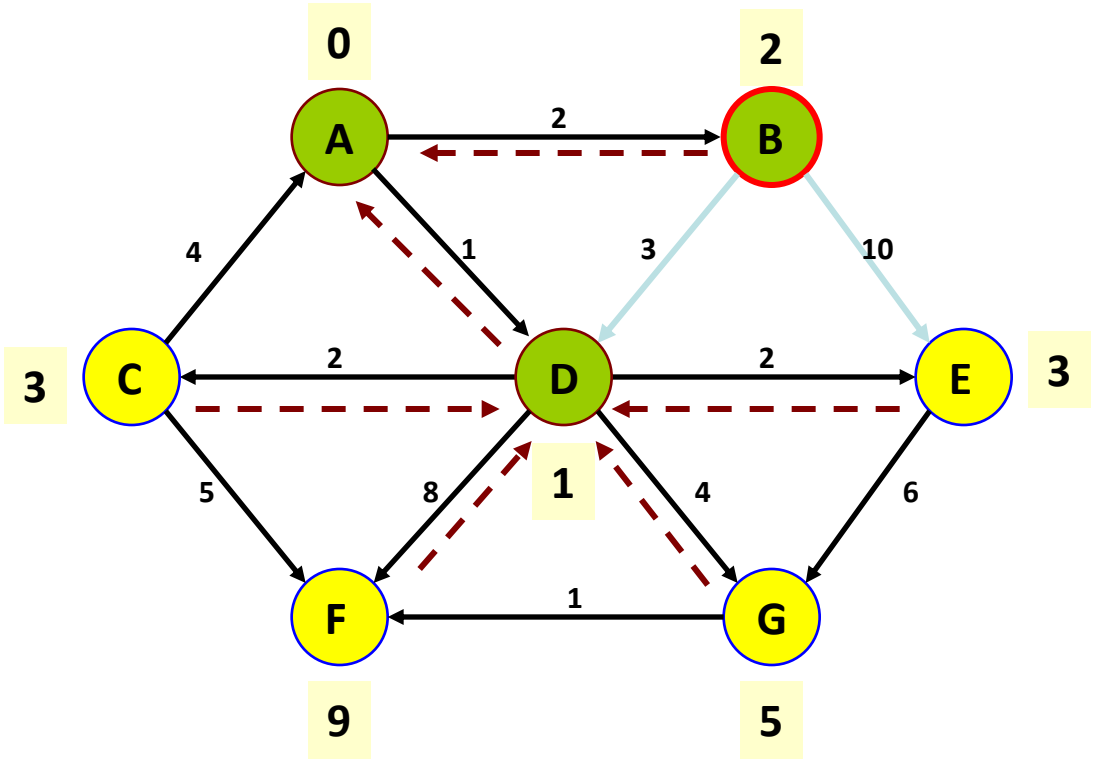
dijkstra(A, F);



pqueue = {B:2, C:3, E:3, G:5, F:9}

Dijkstra example

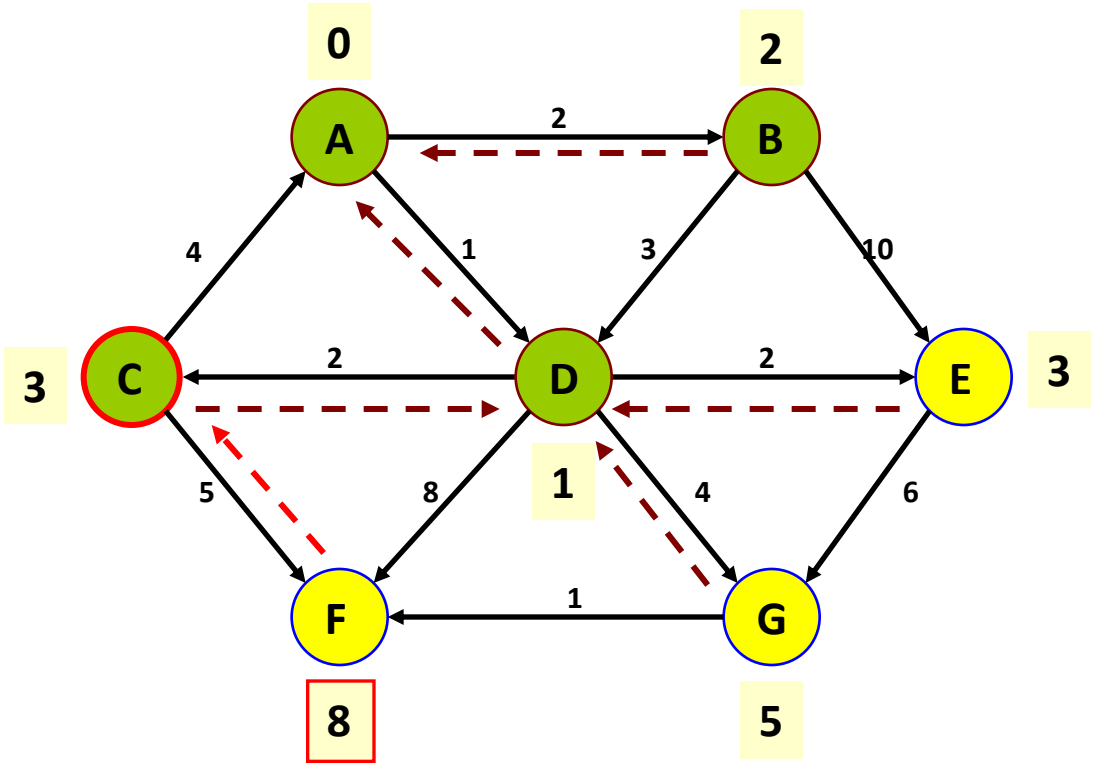
dijkstra(A, F);



pqueue = {C:3, E:3, G:5, F:9}

Dijkstra example

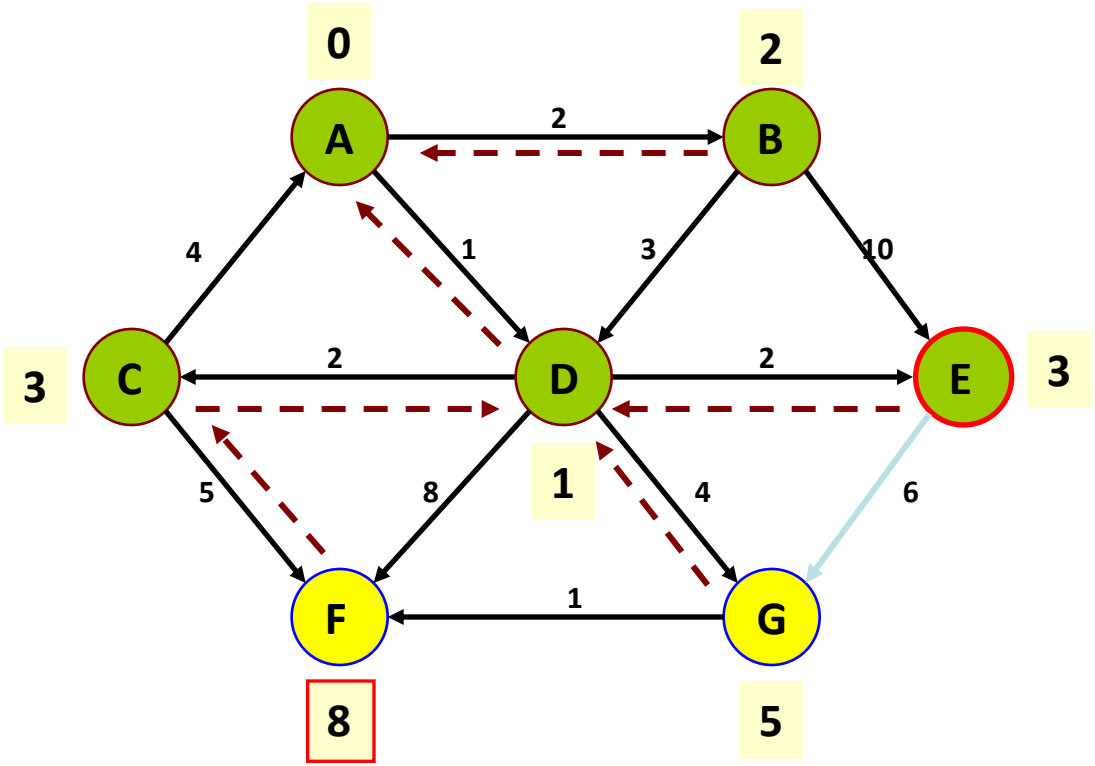
dijkstra(A, F);



pqueue = {E:3, G:5, F:8}

Dijkstra example

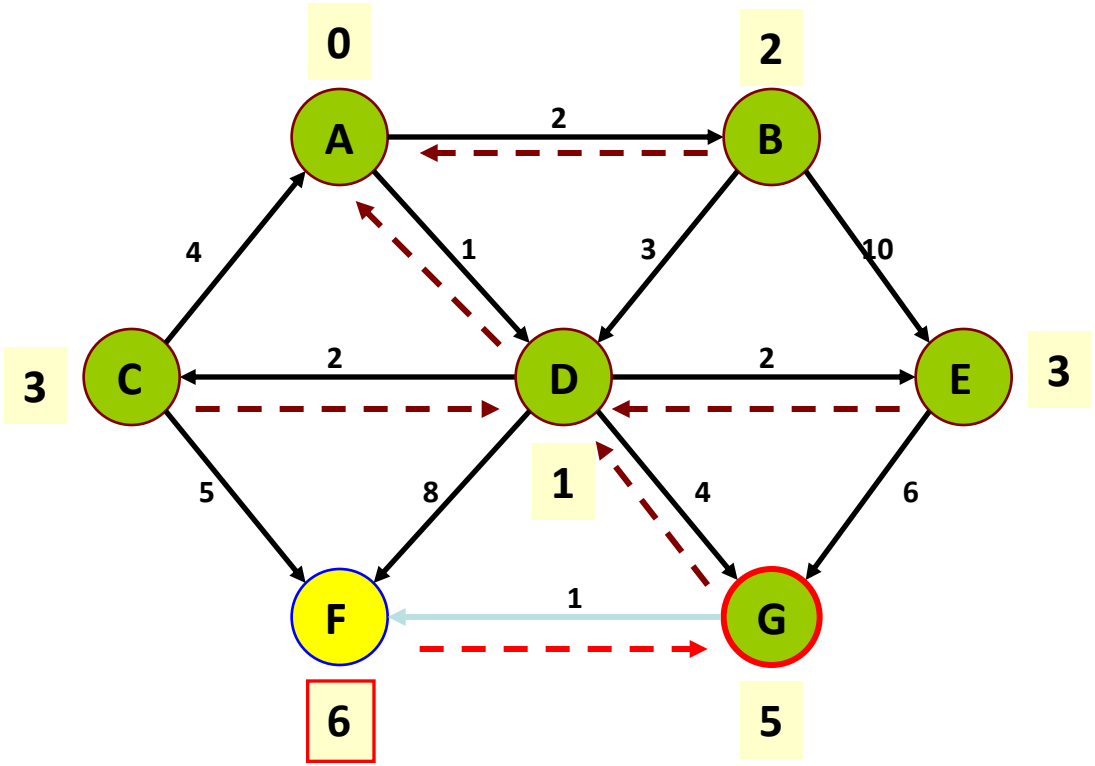
dijkstra(A, F);



pqueue = {G:5, F:8}

Dijkstra example

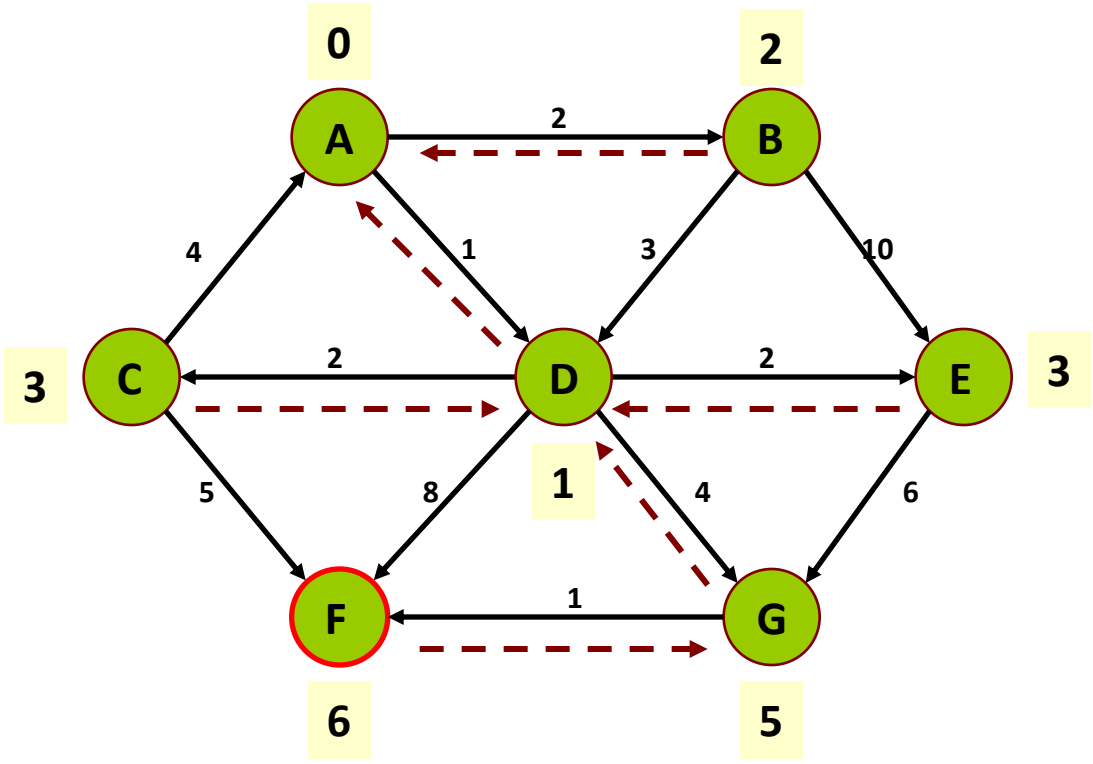
dijkstra(A, F);



pqueue = {F:6}

Dijkstra example

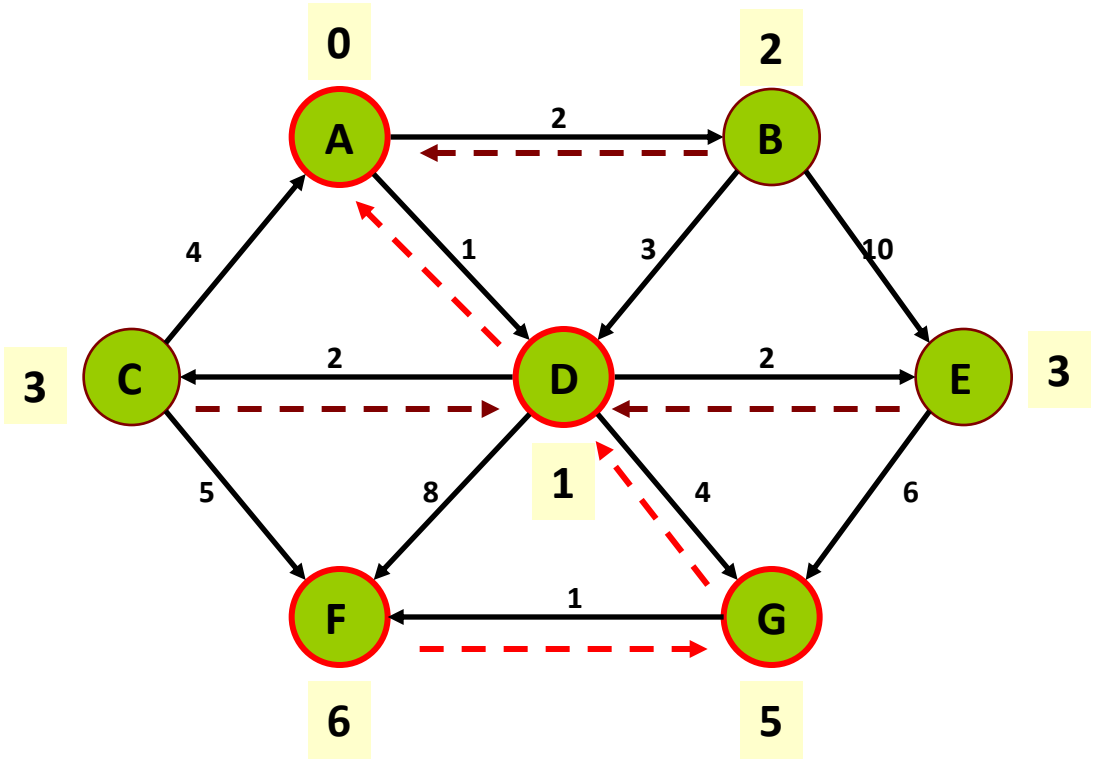
dijkstra(A, F);



pqueue = {}

Dijkstra example

dijkstra(A, F);



Algorithm properties

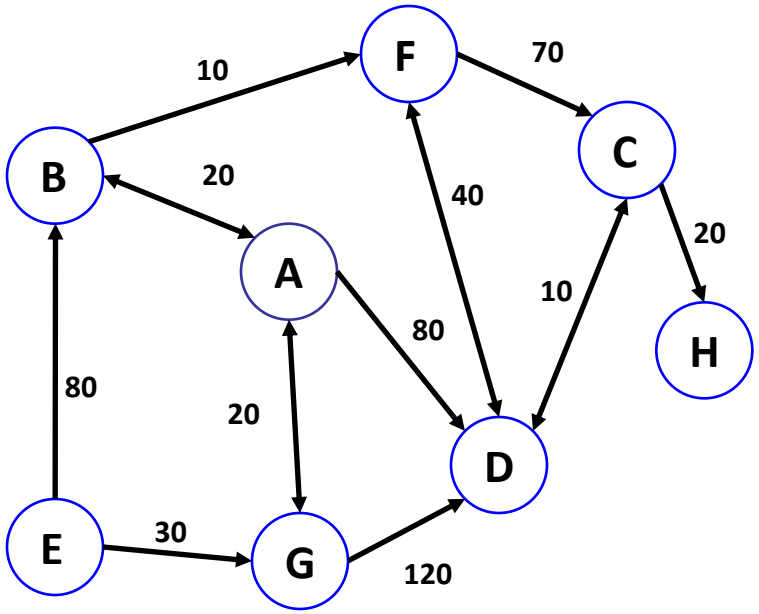
- Dijkstra's algorithm is a *greedy algorithm*:
  - Make choices that currently seem the best.
  - Locally optimal does not always mean globally optimal.

- It is correct because it maintains the following two properties:
  - 1) for every marked vertex, the current recorded cost is the lowest cost to that vertex from the source vertex.
  - 2) for every unmarked vertex *v*, its recorded distance is shortest path distance to *v* from source vertex, considering only currently known vertices and *v*.

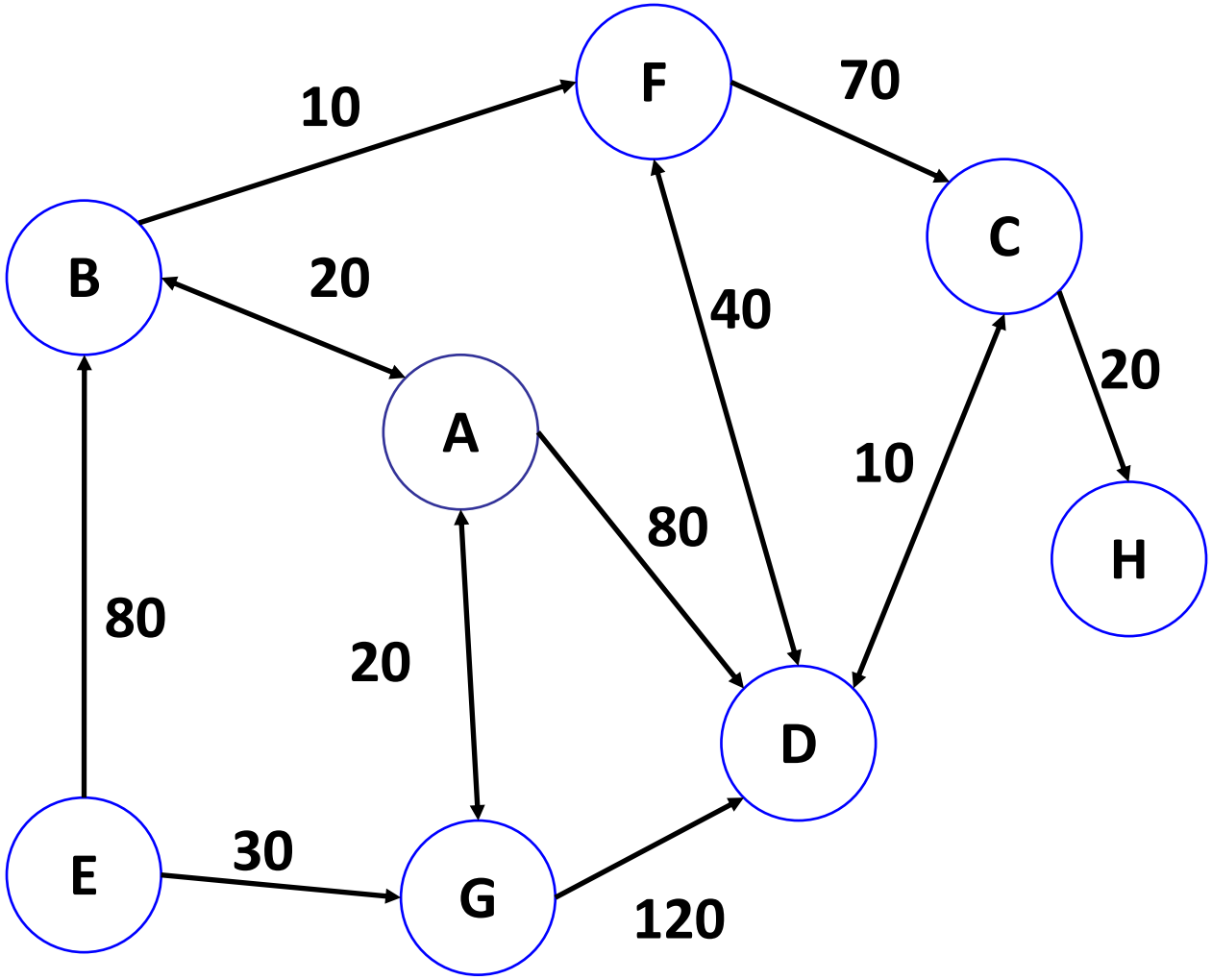
Dijkstra exercise

- Run Dijkstra's algorithm from vertex ***E***
  to all of the other vertices in the graph. *(see next slide)*
  - Keep track of previous vertices so that you can reconstruct the path.

  - **Q:** What path does it find from E to H?
    - **A.** {E, B, F, C, H}
    - **B.** {E, B, A, D, C, H}
    - **C.** {E, G, A, B, F, D, C, H}
    - **D.** {E, G, D, C, H}
    - **E.** none of the above



Dijkstra tracing



Plan For Today

- **Recap**: DFS and BFS
- Dijkstra's Algorithm
- Announcements
- A* Search

Announcements

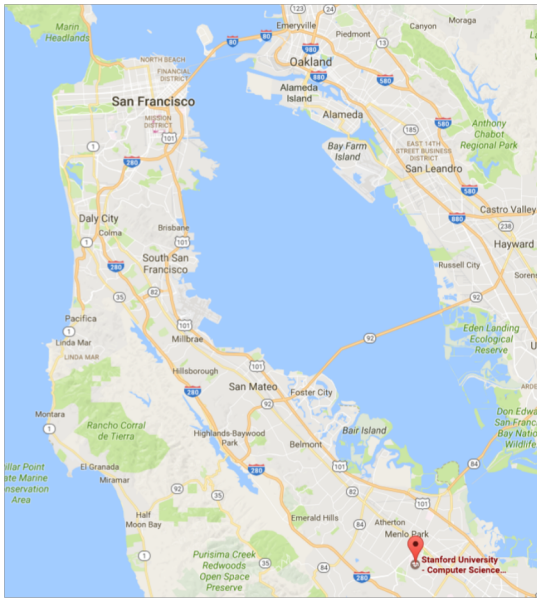
- Mid-quarter grade reports are available on the course website

Plan For Today

- **Recap**: DFS and BFS
- Dijkstra's Algorithm
- Announcements
- A* Search

Improving on Dijkstra's

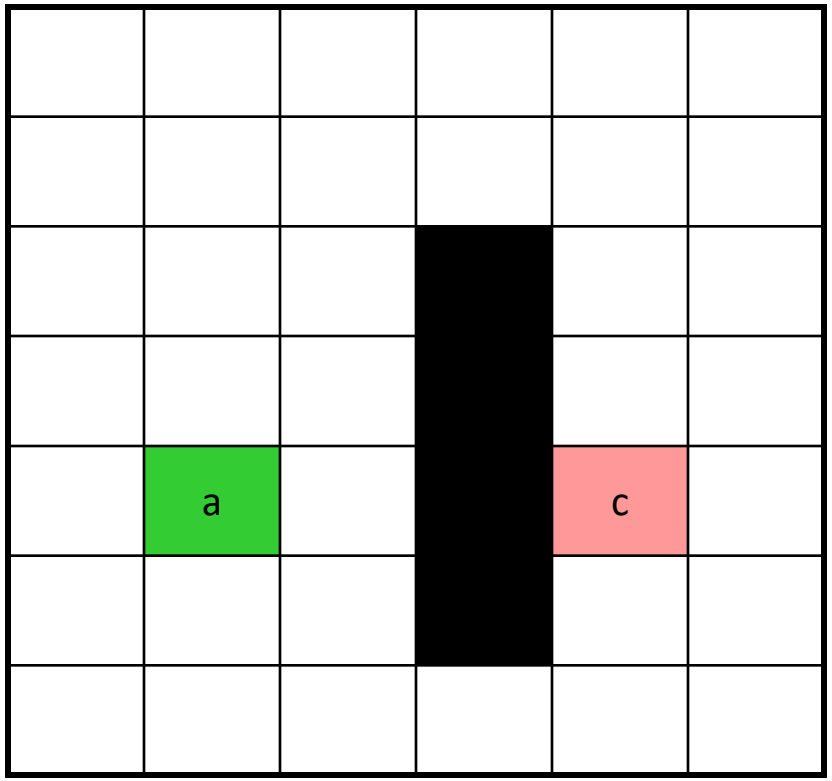
- If we want to travel from Stanford to San Francisco, Dijkstra's algorithm will look at path distances around Stanford. But, we know something about how to get to San Francisco -- we know that we generally need to go Northwest from Stanford.



- This is more information! Let's not only prioritize by weights, but also give some priority to the **direction** we want to go. E.g., we will **add more information based on a *heuristic*, which could be direction in the case of a street map.**

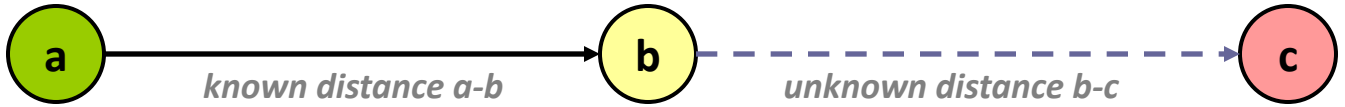
Finding a maze path

- Suppose we are searching for a path in a maze.
  - The 'cost' of a square is the min. number of steps we take to get there.
  - What does Dijkstra's algorithm do here? What "should" it do?



Dijkstra observations

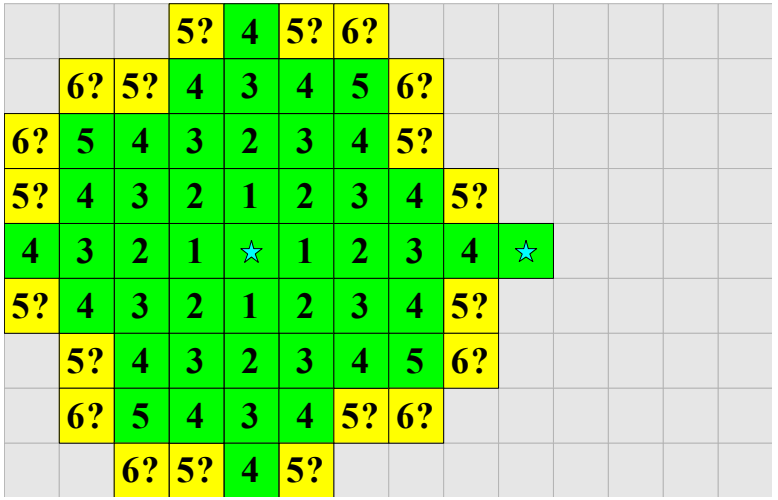
- Dijkstra's algorithm uses a priority queue and examines possible paths in increasing order of their known cost or distance.
  - The idea is that paths with a lower distance-so-far are more likely to lead to paths with a lower total distance at the end.



  - But what about the remaining distance? What if we knew that a path that was promising so far will be unlikely to lead to a good result?
  - Can we modify the algorithm to take advantage of this information?

Dijkstra observations

- Dijkstra's algorithm works by incrementally computing the shortest path to intermediary nodes in the graph in case they prove to be useful.
  - Some of these paths are in the "wrong" direction.



- The algorithm has no "big-picture" conception of how to get to the destination; the algorithm explores outward in all directions.
  - Could we give the algorithm a hint? Explore in a smarter order?
  - What if we knew more about the vertices or graph being searched?

Heuristics

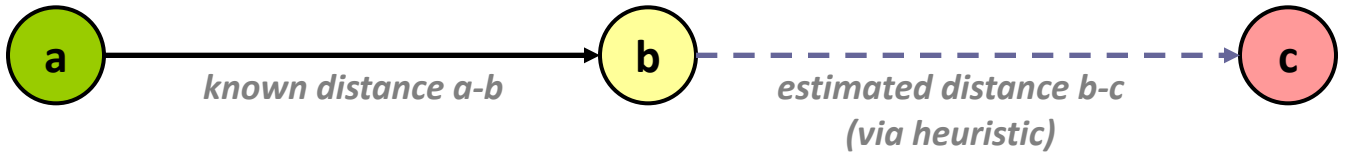
- **heuristic**: A speculation, estimation, or educated guess that guides the search for a solution to a problem.
  - *Example:* Spam filters flag a message as probable spam if it contains certain words, has certain attachments, is sent to many people, ...
  - In the context of graph searches: A function that approximates the distance from a known vertex to another destination vertex.
  - Example: Estimate the distance between two places on a Google Maps graph to be the direct straight-line distance between them.

very eager, AH <= real answer

- **admissible heuristic**: One that never overestimates the distance.
  - Okay if the heuristic underestimates sometimes (e.g. Google Maps).
  - Only ignore paths that in the *best case* are worse than your current path

The A* algorithm

- **A*** ("A star"): A modified version of Dijkstra's algorithm that uses a heuristic function to guide its order of path exploration.



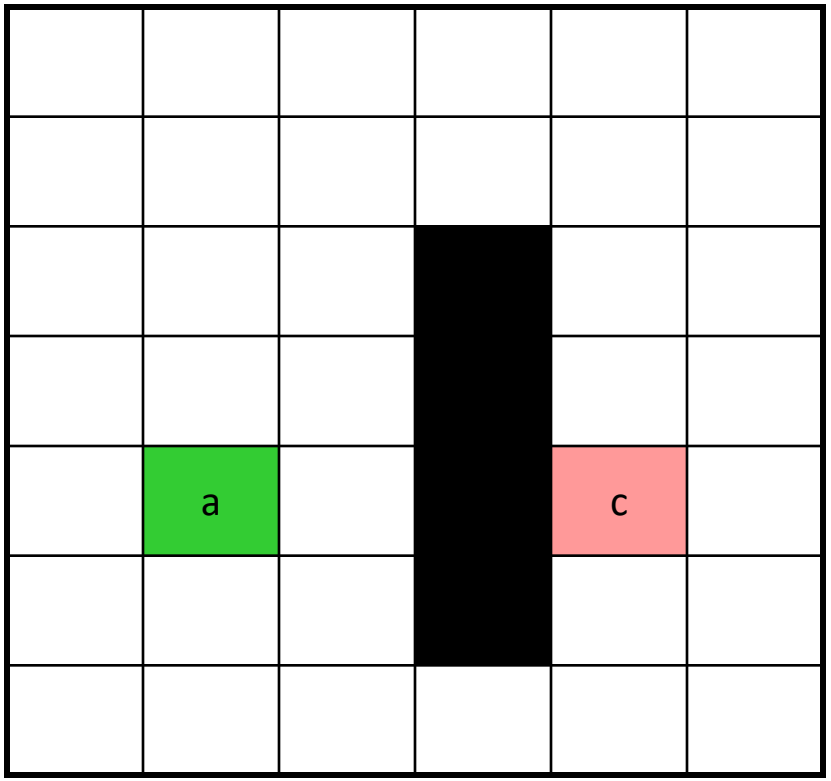
关键在于如何准确定义 heuristic

- Suppose we are looking for paths from start vertex *a* to *c*.
  - Any intermediate vertex *b* has two costs:
  - The known (exact) cost from the start vertex *a* to *b*.
  - The heuristic (estimated) cost from *b* to the end vertex *c*.

- *Idea:* Run Dijkstra's algorithm, but use this priority in the pqueue:
  - priority(*b*) = cost(*a*, *b*) **+ Heuristic(*b*, *c*)**
  - Chooses to explore paths with lower estimated cost.

Example: Maze heuristic

- Suppose we are searching paths in a maze.
  - The 'cost' of a square is the min. number of steps we take to get there.
  - What would be a good heuristic for the remaining distance?

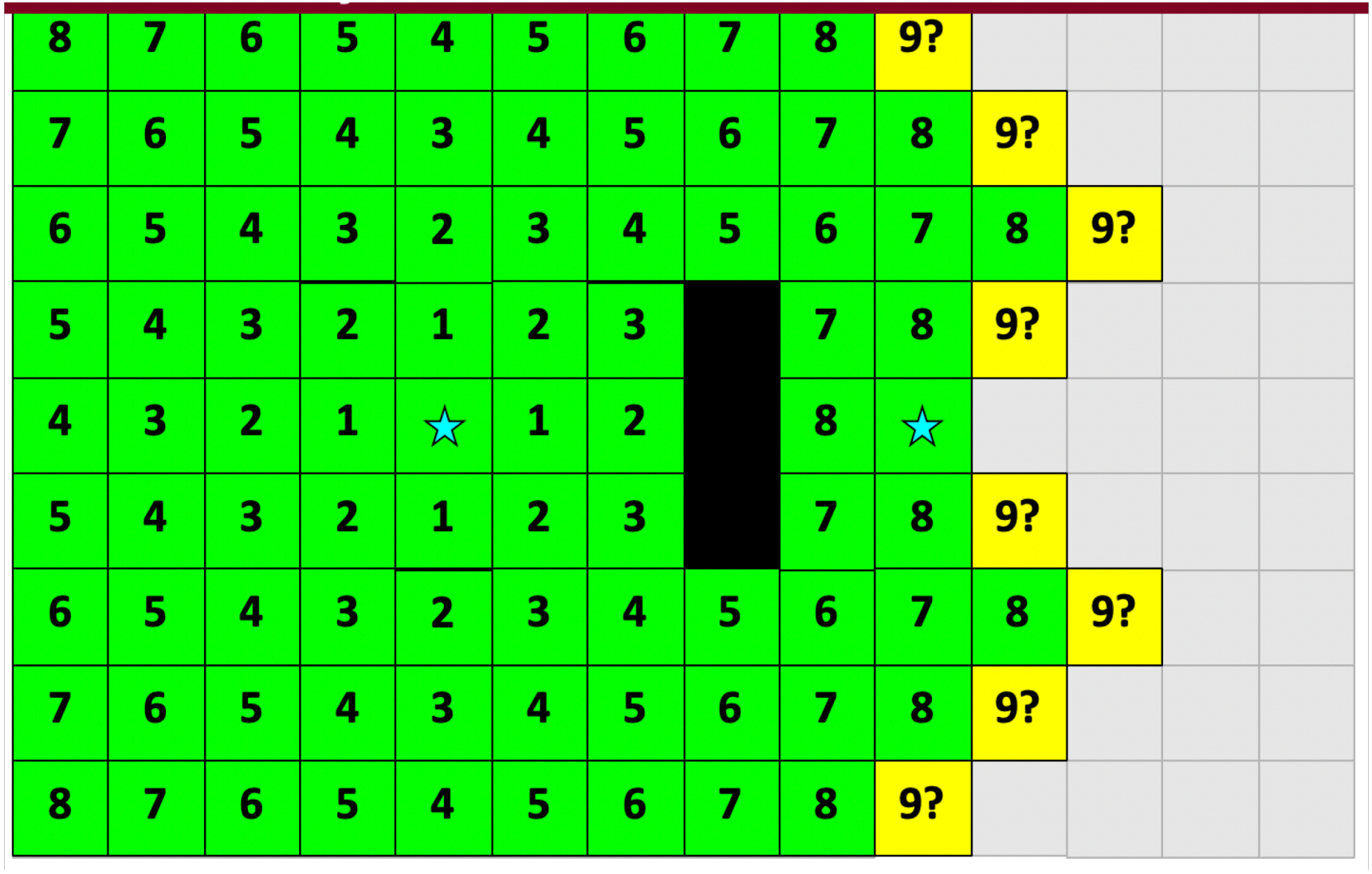


Maze heuristic

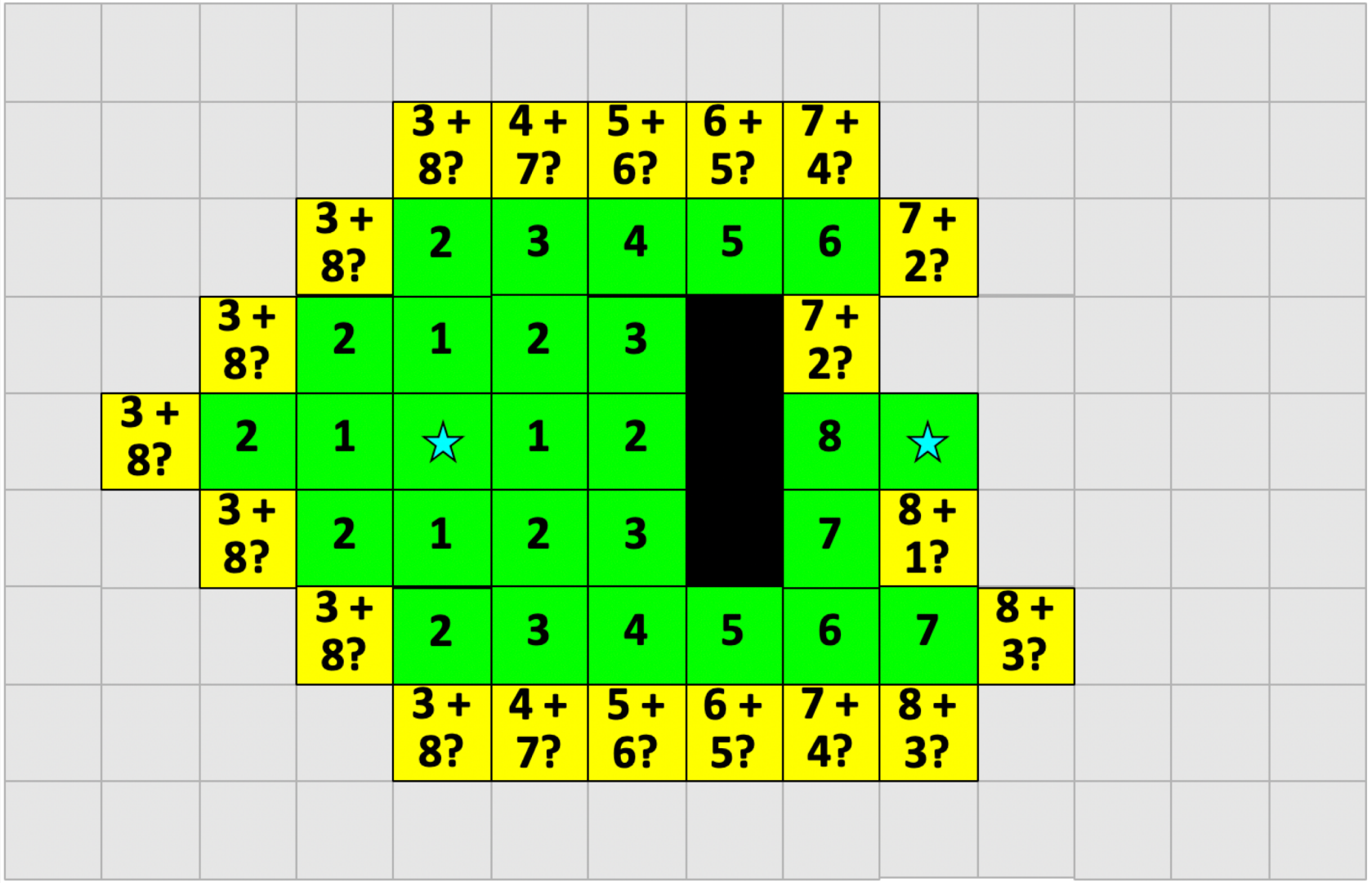
- Idea: Use "Manhattan distance" (straight-line) between the points.
  - H(p_1, p_2) = abs(p_1.x - p_2.x) + abs(p_1.y - p_2.y) // dx + dy
  - The idea: Dequeue/explore neighbors with lower (cost+Heuristic).

8	7	6	5	4	5
7	6	5	4	3	4
6	5	4	3	2	3
5	4	3	2	1	2
4	a	2	1	c	1
5	4	3	2	1	2
6	5	4	3	2	3

Dijkstra


8 7 6 5 4 5 6 7 8 9?
7 6 5 4 3 4 5 6 7 8 9?
6 5 4 3 2 3 4 5 6 7 8 9?
5 4 3 2 1 2 3 7 8 9?
4 3 2 1 1 2 8
5 4 3 2 1 2 3 7 8 9?
6 5 4 3 2 3 4 5 6 7 8 9?
7 6 5 4 3 4 5 6 7 8 9?
8 7 6 5 4 5 6 7 8 9?

A*


3 + 8?
4 + 7?
5 + 6?
6 + 5?
7 + 4?
3 + 8?
2
3
4
5
6
7 + 2?
3 + 8?
2
1
2
3
7 + 2?
3 + 8?
2
1
1
2
8
3 + 8?
2
1
2
3
7
8 + 1?
3 + 8?
2
3
4
5
6
7
8 + 3?
3 + 8?
4 + 7?
5 + 6?
6 + 5?
7 + 4?
8 + 3?

Recall: Dijkstra code

dijkstra(v_1, v_2):
consider every vertex to have a cost of infinity, except v_1 which has a cost of 0.
create a *priority queue* of vertexes, ordered by cost, storing only v_1 .

while the *pqueue* is not empty:
dequeue a vertex *v* from the *pqueue*, and mark it as **visited**.
for each of the unvisited neighbors *n* of *v*, we now know that we can reach this neighbor with a total **cost** of (*v*'s cost + the weight of the edge from *v* to *n*).
if the neighbor is not in the *pqueue*, or this is cheaper than *n*'s current cost, we should **enqueue** the neighbor *n* to the pqueue with this new cost, and with *v* as its previous vertex.

when we are done, we can **reconstruct the path** from v_2 back to v_1
by following the previous pointers.

A* pseudocode

astar(v_1, v_2):
consider every vertex to have a cost of infinity, except v_1 which has a cost of 0.
create a *priority queue* of vertexes, ordered by (cost+**heuristic**), storing only v_1
with a priority of H(v_1, v_2).

while the *pqueue* is not empty:
dequeue a vertex *v* from the *pqueue*, and mark it as **visited**.
for each of the unvisited neighbors *n* of *v*, we now know that we can reach
this neighbor with a total **cost** of (*v*'s cost + the weight of the edge from *v* to *n*).
if the neighbor is not in the *pqueue*, or this is cheaper than *n*'s current cost,
we should **enqueue** the neighbor *n* to the pqueue with this new cost
plus H(*n*, v_2), and with *v* as its previous vertex.

when we are done, we can **reconstruct the path** from v_2 back to v_1
by following the previous pointers.

** (basically, add H(...) to costs of elements in PQ to improve PQ processing order)*

Recap

- **Recap**: DFS and BFS
- Dijkstra's Algorithm
- Announcements
- A* Search

Next time: Minimum Spanning Trees